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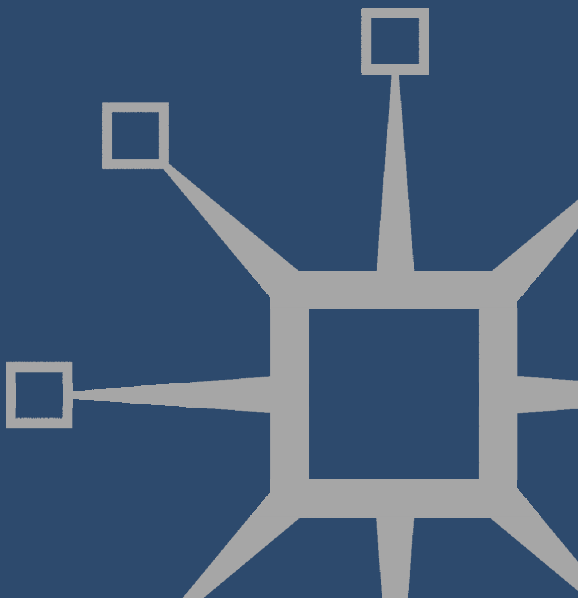
Bolzano's Theoretical Philosophy

An Introduction

Sandra Lapointe



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An Introduction

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Foreword

During the first half of the twentieth century, analytic philosophy gradually established itself as the dominant tradition in the English-speaking world, and over the last few decades it has taken firm root in many other parts of the world. There has been increasing debate over just what 'analytic philosophy' means, as the movement has ramified into the complex tradition that we know today, but the influence of the concerns, ideas and methods of early analytic philosophy on contemporary thought is indisputable. All this has led to greater self-consciousness among analytic philosophers about the nature and origins of their tradition, and scholarly interest in its historical development and philosophical foundations has blossomed in recent years, with the result that history of analytic philosophy is now recognised as a major field of philosophy in its own right.

The main aim of the series in which this book appears, the first series of its kind, is to create a venue for work on the history of analytic philosophy, consolidating the area as a major field of philosophy and promoting further research and debate. The *History of Analytic Philosophy* is understood broadly, as covering the period from the last three decades of the nineteenth century to the start of the twenty-first century, beginning with the work of Frege, Russell, Moore and Wittgenstein, who are generally regarded as its main founders, and the influences upon them, and going right up to the most recent developments. In allowing the 'history' to extend to the present, the aim is to encourage engagement with contemporary debates in philosophy, for example, in showing how the concerns of early analytic philosophy relate to current concerns. In focussing on analytic philosophy, the aim is not to exclude comparisons with other – earlier or contemporary – traditions, or consideration of figures or themes that some might regard as marginal to the analytic tradition but which also throw light on analytic philosophy. Indeed, a further aim of the series is to deepen our understanding of the broader context in which analytic philosophy developed, by looking, for example, at the roots of analytic philosophy in neo-Kantianism or British idealism, or the connections between analytic philosophy and phenomenology, or discussing the work of philosophers who were important in the development of analytic philosophy but who are now often forgotten.

Bernard Bolzano (1781–1848) occupies a unique place in the history of modern philosophy. Born in the year in which Kant's *Critique of Pure Reason* was published and dying in the year in which Frege was born, his philosophy – like his life – can be seen as offering a bridge between Kant's seminal work and the birth of analytic philosophy. In Bolzano's writings, one finds many of the characteristic themes of analytic philosophy anticipated. Like Frege and Russell after him, Bolzano was dissatisfied with Kant's account of mathematics and realised that a better conception of logic was required to do justice to mathematics. Bolzano's conception of logic was not Frege's or Russell's, but he did criticise traditional subject–predicate analysis, suggested that there was a fundamental form underlying all types of proposition and was insistent on the need to keep psychology out of logic. Like Frege, Bolzano construed existential statements as being concerned with the non-emptiness of appropriate 'ideas' ('*Vorstellungen an sich*' in Bolzano's terms) or 'concepts' ('*Begriffe*' in Frege's terms), and his conception of 'propositions' ('*Sätze an sich*') is similar in many respects to Frege's conception of 'thoughts' ('*Gedanken*'). Like Frege, too, Bolzano emphasised that there is a class of entities, including both 'ideas'/'concepts' and 'propositions'/'thoughts', which are objective but not actual ('*wirklich*'), in the sense of not existing in the spatio-temporal realm.

Despite these similarities, however, Bolzano had no direct influence on any of the acknowledged founders of analytic philosophy. He had an influence on other German-speaking philosophers such as Franz Brentano, Benno Kerry, Edmund Husserl, Alwin Korselt and Kazimierz Twardowski, who themselves had an influence on the early analytic philosophers, both through correspondence and in their own publications (even if, often, mainly as a target of criticism). Through Twardowski, the founder of the Lvov-Warsaw school, he also had an influence on a whole generation of Polish logicians and philosophers, including Jan Łukasiewicz, Stanisław Leśniewski and Alfred Tarski, who played an important role in the development of analytic philosophy. So a full account of the history of analytic philosophy must certainly pay attention to Bolzano's work. His significance, however, lies not just in these patterns of influence. The similarities and differences between his views and those of Frege, in particular, reveal much about the nature of analytic philosophy: the conceptions of analysis and logical form involved, for example, and key debates such as those about analyticity and other modal notions. These influences and connections are explored and elucidated by Sandra Lapointe in this book.

At the heart of Bolzano's logic – logic being understood in the traditional broad sense as including both methodology and theory of science (hence the title of Bolzano's major work, the *Wissenschaftslehre*) – lies his critique of Kant. As Lapointe explains in the first three chapters, Bolzano criticises Kant's theory of intuition and his decompositional conception of analysis. In doing so, Bolzano develops his own positive doctrines, concerning analyticity and logical consequence, in particular, based on a method of substitution, as Lapointe elaborates in Chapters 4–6. In the remaining chapters, further clarifying his semantic theory, she discusses his epistemological and ontological views and his connection with Frege and Husserl.

Over the last 20 years there has been a blossoming of interest in Bolzano's philosophy, led by German-speaking scholars active in the International Bernard Bolzano Society based at the University of Salzburg. More and more works – both translations of Bolzano's writings and discussions of his ideas – have also been appearing in English. Lapointe is at the forefront of this Bolzano renaissance, and her book is thus timely both in making Bolzano's theoretical philosophy accessible to a broader English-speaking audience and in contributing to a deeper understanding of the history of analytic philosophy.

Michael Beaney
August 2011

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Introduction

In the late 1790s, as he was finishing his studies in mathematics and philosophy at Charles University, Bernard Bolzano announced his intention of becoming a priest. Bolzano's father, an Italian art merchant who would have preferred to see his son go into business, initially opposed the plan and convinced him to postpone the decision for a year.[1] Bolzano spent that time deepening his knowledge of mathematics and reading Kant, a remarkable fact given that Kant had been banned in the Austrian Empire the previous year. The *Critique of Pure Reason* had a tremendous impact on Bolzano, and one that can be felt throughout his later work. If Bolzano persisted in his intention to join the priesthood, it was not for lack of enthusiasm or talent for the "speculative part of mathematics that belongs at once to philosophy". His reasons had, apparently, little to do with his opinion of Christianity about which he claims to have had his doubts and to fail to know whether it "be true or of a truly divine nature". Why then become a priest? Bolzano's reasons were in part sentimental – he believed it had been the wish of his deceased mother – in part ethical. Bolzano adhered to a distinct form of religious pragmatism: he assumed that a religious doctrine need not be true, but that it is justified if people's believing in it generates a greater sum total of happiness.[2] Bolzano began his theology studies in the fall of 1800 and simultaneously wrote a doctoral thesis in mathematics which he submitted in 1804 under the title: *Considerations on Some Objects of Elementary Geometry*. Two positions were then open at Charles University, one in mathematics and the other in "Science of the Catholic Religion". He came in second for the former, first for the latter, was hastily ordained and took up his functions directly. Bolzano's responsibilities consisted in teaching the "principles of Christian morality" to all philosophy students (which at the time

included all students of the arts and natural sciences) and to deliver "edifying speeches" on Sundays and holidays. He held his inaugural speech "On the Necessity of a Faith that Proceeds on the Basis of Reasons" on 1 May 1805. Students who were initially quite displeased with the creation of the new chair had planned to sabotage Bolzano's inauguration with "continuous thumping" whenever the new professor would express a phrase with which they would disagree. As it occurs, the plan fell through: one could apparently not find any objectionable passage and felt wondrously captivated by the unusual teacher. Bolzano's speech, in every respect representative of his moral and religious standpoint and of his idea of the role he should play as a teacher, was however far from the doctrines he had been appointed to profess. Three months after his appointment, a royal decree informed Bolzano that his office would end with the academic year. He was accused of being a "Kantian" and to be an adept of the *Zeitphilosophie*. It is clear that the fact that Bolzano taught according to his own liberal and tolerant views rather than according to the manual prescribed by the Royal chaplain, Jakob Frint, played a role. With the support of influential friends, Bolzano was able to vindicate himself and after agreeing to use the authorised textbook, he was able to resume his duties. He was granted tenure in 1806.

It is not hard to understand what led to Bolzano's eventual dismissal. Given the usual treatment of intellectual "dissidents" in Austria at the turn of the nineteenth century, the only puzzle is that it did not happen earlier. According to Bolzano himself – he suffered from respiratory illness since childhood and had been forced to interrupt his teaching between 1813 and 1815 – the termination of his professorial functions had at least one positive effect, that of allowing him to devote himself entirely to his scientific work. During the 1820s and 1830s, he received moral and financial support from Josef and Anna Hoffmann, living with them most of the time. In the course of this period, Bolzano realised his *opus magnum*, the *Theory of Science*, which he published anonymously in 1837. When Anna Hoffmann died in 1842, Bolzano returned to Prague to live with his brother. He took some pupils, among them Robert Zimmermann, Brentano's future colleague in Vienna. Bolzano passed away on 18 December 1848.

Though by all standards Bolzano's achievement is compelling, the interest he has generated in the English-speaking philosophical world has remained at best marginal. Of course, this is also the case for many other German-speaking authors of the turn of the nineteenth century. But no other had Bolzano's genius, and the situation is unnecessary

and unfair to a philosopher who was incontestably the best logician between Leibniz and Frege, and one of the best artisans of the profession since Aristotle. The scarcity of specialised work on Bolzano today is to a great extent a consequence of the fact that Bolzano was unable to create the kind of impact he was hoping for on his contemporaries and successors. This, in turn, can at least in part be explained by sociological factors that range from adverse imperial academic policy to bad editorial management. One ought to consider, for instance, the extent of Austria's deliberate intellectual wreckage over the century that extends from the beginning of Maria-Theresia's reign to the end of that of Franz II and beyond. Its consequences on university life and scientific research were dramatic and, in particular, it is no exaggeration to say that during this period, the Austrian government was largely successful in eradicating academic philosophy. By the time Bolzano was nominated to his position of professor of science of religion in 1805, a series of educational "reforms" motivated in part by the necessity to cut down on governmental expenses and explicitly aimed at tightening the grip on intellectual culture in the wake of the Enlightenment had managed to turn universities into more or less technology-oriented post-secondary institutions where the use of *textbooks* by instructors was mandatory and had to receive the approval of the royal board of education (see e.g., Valent 1997). There was a brief attempt during the reign of Joseph II in the 1780s to turn universities back into genuine academic institutions where intellectual freedom would be, as it should, the motor of moral and intellectual progress. But the scope of this attempt was limited – mostly because academic freedom is inconsistent with absolutism of any form, enlightened or not (see e.g., Sauer 1982). Austria's peculiar educational policies had ruinous effects on higher institutions of learning, and the latter soon came to lose their credibility in neighbouring German states where university tradition had been going strong ever since the fifteenth century – and where the nineteenth century might have been a golden age. The Hegelian Karl Rosenkranz explained in a speech he pronounced on the occasion of the 150th anniversary of the Prussian crown in 1851 and in which he sketched a picture of the "progress" of German philosophy thus:

> In Austria, philosophy does not exist at all. Despite the fact that a religious and political form of it is taught in Gymnasien and universities [...] the latter is reduced to the most extreme subjection [...] As soon as a philosopher gets away from medieval scholastics, he is either publicly or secretly persecuted or ousted by what is meant to

> appear as some kind of promotion. As regards philosophy, Austria is nothing less than a wilful moron.
>
> (Quoted in Künne 1997a, 16)[3]

Notwithstanding the condescending tone, Rosenkranz' assessment of Austrian philosophy is, all things considered, sadly accurate. For with Bolzano's exception and not counting Karl Leonhard Reinhold who moved from Austria to Prussia after the Austrian Jesuit-act of 1784–85, all significant professional German-speaking philosophers at the turn of the nineteenth century were indeed German.

Bolzano's lack of influence is also in some measure to be explained by his putatively "subversive" activities, which eventually led to his discharge from the University and to his indefinite ban from public scientific and clerical activities in 1824. In 1811, Bolzano was unofficially granted permission to use his own course notes, and his openness, for instance, on the question of sexual education provides a good illustration of his teaching style. As Bolzano sees it, the best approach when it comes to dealing with sexual curiosity is information. He does not hesitate to talk about pornography, which he considers to be natural, and recommends that instead of banning it, one discourage it – by promoting physical exercise and manual work. He talks about masturbation and denounces it only to say that it is a loss of time and energy. He advises young men against entertaining relationships with prostitutes but admits that it may be necessary to do so if one – and especially if one is pursuing an academic degree – can only marry late when one has effectively the means to start a family (cf. Lorenzová 1999).[4] In general, Bolzano's speeches filled students with enthusiasm: they proved to voice the views of an independent thinker and straightforward mind. Yet, under the zealous rule of Metternich at the end of the 1810s, conservatism was reaching new heights in Austria, and the views of a priest who also proposed his "utopic" vision of a society base on equality, criticized the Austrian Constitution, professed in favour of freedom of thought, expresses his opinions on the role of the Church within the State and even warned theology students against the austerity of celibate could not fail to infuriate his superiors. After he was linked to the presumed political intrigues of his student Josef Fesl Bolzano was accused of "heresy" and subjected to a particularly degrading investigation that lasted 5 years.[5] Bolzano refused to admit that he could have committed any other mistake than that of having offered "an incorrect scientific or rhetorical exposition" and did not retract himself. Bolzano was suspended and placed under police surveillance. He was eventually

forbidden to practice in quality of priest or teacher in Austria and, until the end of the 1830s, he was also banned from publication. In contrast to Fesl, however, he was not incarcerated.

Finally, the lack of success of Bolzano's publications, especially the *Theory of Science*, can be explained by his literary style. As a whole, Bolzano's work is dense, and though the fact that he systematically examines in detail the views of his predecessors makes the *Theory of Science* an outstanding source of historical analyses, it also makes it remarkably long-winded. But more significantly, perhaps, Bolzano's style of argumentation and his theoretical preoccupations were undeniably closer to that of pre-Kantian philosophy and were therefore judged obsolete by his German contemporaries. Bolzano's logical, epistemological and methodological investigations did not fit well within an intellectual context dominated, in Germany, by Fichte and Hegel's idealism and, increasingly, by psychologistic interpretations of Kant (e.g., Fries, Beneke and Herbart – who also had antipsychologistic tendencies). Besides, the management of Bolzano's unpublished writings – a considerable part of his work – was initially a remarkable failure. Though efforts were made, those Bolzano had chosen to take care of his scientific legacy were impeded by a series of unfortunate circumstances – death, theoretical shift and sheer indolence – and had little impact.[6] His posthumous writings did not, for instance, have the desired effect to "contain, through the diffusion of clear ideas, the terrible disorder Kant, without presuming it, created in Germany through his philosophy".

This said, it would be wrong to assume that Bolzano's work had no historical impact whatsoever. At the end of the nineteenth century, more than half a century after its publication, Bolzano's *Theory of Science* attracted the interest of Brentano's students. Brentano himself had no favourable disposition towards Bolzano's theories and forcefully criticised as "deplorable" his students' interest in the latter. Nonetheless Husserl, Kerry, Korselt and Twardowski – all of whom substantially refer to Bolzano in their work – are likely to have been spurred at least initially by a seminar on the *Paradoxes of the Infinite* in the winter of 1884–85 (cf. Brentano 1966). Husserl's first in-depth reading of the *Theory of Science*, between 1894 and 1896, considerably informs his argument against psychologism in the *Logical Investigations* as well as his conception of the role of logic within the theory of knowledge. The renewed interest Husserl came to take in the *Theory of Science* in the second half of the 1890s was triggered by Kazimierz Twardowski's habilitation thesis, written under the supervision of Bolzano's former student Robert Zimmermann[7] and published under the title *On the Content and Object of*

Ideas in 1894. In *Content and Object*, Twardowski takes over a Bolzanian distinction towards which a series of articles by Benno Kerry had drawn his attention, namely the distinction Bolzano makes in the *Theory of Science* between subjective ideas, objective ideas (or ideas "in themselves") and the object or referent of the latter (Twardowski 1991, 11). Twardowski used a modified version of this idea in order to criticise some aspects of the Brentanian theory of intentionality and, in particular, the putative lack of a distinction in Brentano between the content and the object of ideas. Through Twardowski, Bolzano's ideas will come to infuse the Polish School of Philosophy. Thus we find, for instance, a critique of Bolzano's notion of a variable in Jan Łukasiewicz (1913), a debate, in 1913, between Tadeusz Kotarbinski (1968) and Stanislaw Lesniewski (1991) on the notion of eternal truths, a Bolzanian version of the concept of analyticity in Kazimierz Ajdukiewicz (1958) and the application of a Bolzano-type substitutional method to the definition of logical consequence in Tarski.

The publication of the *Theory of Science* in 1837 was the outcome of philosophical research Bolzano had been pursuing since the beginning of his career. Two years after the publication of his first book, *The Contributions to a Better Founded Exposition of Mathematics* in 1810, Bolzano wrote in his philosophical diaries that he had resolved to publish a logic – the title of which he envisaged to be *Attempt at a New Logic Following which a Reform of All Sciences Should take Place*. Despite the fact that Bolzano set the stage early in his career, one ought not to underestimate the evolution of Bolzano's thought in the period that extends between the publications of the two books. There is a certain continuity to the extent that Bolzano's main concern remains the elaboration of a theory of proof whose immediate purpose is to provide the basis for mathematical practice. In this respect, Bolzano's contribution to the foundations of calculus attests to the importance he ascribes to the development of a rigorous methodology of demonstration that leaves no place to Kant-type putatively "pure" intuitions in mathematics. But Bolzano's conceptual resources will grow substantially richer: the *Theory of Science* put forward a series of theoretical innovations that concurred to shape his later approach to logical and epistemological questions.

While a contemporary of Hegel, Bolzano's ideas are closer to those of analytical philosophers of the turn of the twentieth century. He anticipated ground-breaking ideas such as the (Fregean) distinction between sense and reference, the (Tarskian) notion of logical consequence and the (Quinean) definition of logical truth. This alone should deserve him a comfortable place in the encyclopaedia of philosophical

knowledge. The existing literature is careful to highlight and document these achievements. Nonetheless, important aspects of Bolzano's thought, and aspects that are crucial to understanding his place within the history of analytical philosophy, have been neglected. Bolzano was not merely a great anticipator; he was also a formidable analyst whose knowledge and understanding of the context were accomplished and whose acumen goes far beyond what has been argued until now. In particular, Bolzano did not only offer a thorough criticism of Kant's theory of the construction of concepts in pure intuition (Chapter 1), a realist approach to semantics that largely foresees Frege's (Chapter 10), as well as an account of logical truth and consequence that anticipates the modern definitions (Chapters 5 and 6, respectively). He also provided the first criticism of the naïve form of representationalism that is associated with the "picture" theory of concepts, and he was first to offer an examination of and to reject the "decompositional" conception of analysis (Chapter 2). We find in his work the first account of a (semi-formal) language based on the idea of a finite vocabulary and set of compositional rules (Chapter 3) as well as the first theory of polyadic predication and multiple quantification (Chapter 4). His is the first theory of proof to have sought to catch up with the tremendous developments of mathematics at the beginning of the nineteenth century (Chapter 7) and, likewise, the first account of knowledge by virtue of meaning in axiomatic disciplines (Chapter 8). Besides, his analyses of series and natural number were first to be based on a workable theory of "parts and wholes" (Chapter 9).

Bolzano's philosophical project arose from the rejection of many of the views of his predecessors, the understanding of which requires an acute sense for the historical background. The most eminent of these figures ought to be Kant, and Bolzano's disagreement was considerable on certain fundamental issues. Yet Bolzano also learned a great deal from Kant and those who advocated Kantian views, and he sought to preserve some of Kant's most important contributions to epistemology (though whether he did or not is another question). Likewise his relationship to Frege (Chapter 10) and Husserl (Chapter 11) should not be neglected. The first because despite the absence of historical connections, the similarities between Bolzano's and Frege's views on meaning go much further than what is usually assumed and the second because the extent of Bolzano's influence on Husserl's thought has until now been insufficiently appreciated. More than half a century after the publication of the *Theory of Science*, both Frege and Husserl will resort to entities of the type of Bolzano's *Sätze and sich*, what they call "*Gedanke*" and "*Sätze*"

respectively, in order to explain the distinction between sense and reference. Just like what is the case in Bolzano, Fregean, Husserlian and in general more contemporary versions of "propositionalism" maintain four theses:

(1) Propositions are the primary bearers of truth.[8]
(2) Propositions are abstract entities.
(3) Propositions are to be distinguished from the mental states in which they are "grasped" and from the sentences of which they are the meaning.
(4) Propositions are ultimately composed of parts that are not themselves propositions.

The idea that logic is about the relations that hold among abstract entities of the type of Bolzanian propositions and that the latter subsist, so to say, in a "third realm" bears with it metaphysical and epistemological problems: Quine, for instance, argues that their identity conditions remain unspecified (Quine 1970, 3). It has become common place in the literature to question what it means for an abstract entity to be grasped by certain types of mental states. It would be wrong to assume that Bolzano defends a naïve form of semantic realism. Propositions are devices to which Bolzano resorts in order to solve a series of problems that arise in connection to the logical theories of the time. This, I take it, is what Bolzano means when he writes that logicians should be allowed to appeal to entities that may reveal themselves to be inconsistent with a naturalistic epistemology (1837, §25, 113–14) and that his logical theses could also be accepted by those who reject the concept of a proposition in itself (1841, 34–5, 50, 68). He writes:

> The usefulness of the distinction [between propositions in themselves and thought propositions] manifests itself in tens of places and in the most surprising way in that it allows the author to determine objectively a number of concepts that had not been explained before or that were explained incorrectly. For instance, the concept of experience, *a priori*, possibility, necessity, contingency, probability, etc.
>
> (1839, 128)

If we follow this line of interpretation, it occurs that what matters to Bolzano is ultimately not the ontological import of his theory. The motivation behind Bolzano's antipsychologism as well as the semantic

realism on which the latter is based were the actual needs of scientific practice – in mathematics in particular – which is essentially based on demonstration. Bolzano thought that only semantic investigations of the type he had in mind would provide a rigorous foundation to mathematics and deplored the fact that so few of his colleagues:

> concede that it would be a great benefit to their discipline if we succeeded in analysing the concepts we find therein and which we accept as obvious or without any definition; and to infer on the basis of their objective ground the great number of propositions we otherwise care little to establish or which we pose without any proof as evident in themselves.
>
> (1843, Preface)

In Bolzano's theory, properties such as analyticity and logical consequence are defined on the basis of a substitutional procedure that comes with a conception of logical form that prefigured contemporary treatments such as those of Quine and Tarski. I discuss this method at length in Chapters 4–6. Three results are particularly interesting: the elaboration of a calculus of probability, the definition of analyticity and the definition of what it is for a set of propositions to stand in a relation of "*Ableitbarkeit*" with another. The assessment of Bolzanian analyticity and *Ableitbarkeit* requires however many provisions, the main problem being that while they offer a genuinely original treatment of certain kinds of semantic regularities, they do not deliver an account of epistemic and/or modal necessity. This putative failure has often been misjudged to imply that Bolzano does not have an account of what it means to know that some propositions are true by virtue of the sole meaning of terms involved. In Chapter 8, I not only argue that he in fact does but also present Bolzano's theory of *a priori* knowledge in detail. Bolzano's views on knowledge by virtue of meaning rest in turn on a theory of grounding (*Abfolge*) and justification whose role in Bolzano's theory is to complement his views on logical consequence and, in general, provide the basis for a theory of scientific demonstration and explanation which I present in Chapter 7.

Bolzano's work is monumental, a fact to which the recent critical edition of his complete work, the *Bernard Bolzano Gesammtausgabe*, testifies: when completed, it will count more than 150 volumes. It will include Bolzano's published works, his scientific diaries, those of his lecture and tutorial notes that have been preserved, as well as the totality of those of his writings that remained unpublished or that were

not completed. It documents Bolzano's vast and diverse philosophical interests, his humanism and his devotion to the spirit of the Enlightenment. In his posthumous *On the Best State* (2007; written around 1831), Bolzano developed his political theory and, in particular, his view on property rights. In the *Edifying Speeches* – the notes from the speeches he was pronouncing in quality of teacher of science of religion – he exposed his social and ethical views as well as his views on racism and nationalism, for instance (see Morscher and Neumaier 1996). The *Science of Religion* (1834) contains the bulk of his theology and his philosophy of religion. In *Athanasia* (1827), his most successful publication, Bolzano in addition to seeking to demonstrate the eternity of the soul presents the foundations of his ontology of reality. Two opuscula "On the Concept of the Beautiful" and "On the Division of the Fine Arts" contain the highlight of his aesthetics (see Blaukopf 1996). In the 1830s–1840s, Bolzano developed his *Theory of Magnitudes*, and we find excerpts of the latter in the *Paradoxes of the Infinite* (1851) and in *On the Mathematical Method* (1981). Bolzano's mathematical studies and, in particular, his results concerning, for instance, the theorem of the intermediate value (1817) and infinite sets (1851) did not fail to catch the attention of some of his imminent contemporaries and successors: Cauchy, Cantor, Weierstrass, Dedekind and Peirce read them and freely borrowed from them.

Bolzano's increasing popularity in the course of the last decades is partly in debt to the work of the many philosophers such as Rudolf Haller, Jan Wolenski, Jan Sebestik, Peter Simons and Edgar Morscher who devoted themselves to writing a much-needed account of Central European analytical philosophy. However, even today, Bolzano remains at the margin of standard accounts of the development of analytical philosophy. This is hardly justifiable. While Bolzano's doctrines, in contrast to those of Hegel and Fichte, for instance, did not benefit from a wide diffusion, he had no reason to be jealous of them. The aim of this book is to provide a survey of Bolzano's philosophy of logic and epistemology, and my hope is that it will help make clear why Bolzano has long earned his place in standard accounts of the history of analytical philosophy. For, among the ranks of those history unjustly left out, there is no other character whose philosophical interest is so tremendous.

1
Kant and German Philosophy

In the introduction, I've sketched a landscape of the cultural context in which Bolzano's work evolved. I argued that different sociological factors such as the anti-intellectual spirit of educational policies in Austria partly explain the lack of attention his work received until recently. Because of its broad liberality and emphasis on autonomous thought, Bolzano's social and political philosophy – which he presented in part in his edifying speeches – was associated in the mind of the Austrian establishment with Kant's. This led in Bolzano's early career to a series of vexations. In Austria, being accused of "Kantianism" was not unusual and often served as a pretext to oust detractors of the State – mostly Jesuits and Free-Masons (see Sauer 1982, 267ff.).[1] These accusations were, in Bolzano's case, consequential enough to threaten his academic position. The number of those who were dismissed under the same pretext is significant. It included Lazarus Ben-David from Vienna in 1793, Anton Kreil from Pest in 1795, Stephan Tichy from Kaschau in 1795 and Benedikt Feilmoser from Innsbruck in 1820.

The reasons why Bolzano's early views were associated with Kant's are not especially conspicuous. His published work is evidence for the fact that whether or not he was sympathetic to Kant's positions on politics and religion, he also was a fierce opponent of critical philosophy as a whole. Bolzano's criticism of Kant's theoretical philosophy is one of the greatest contributions to Kant Studies of the last two centuries. This is not to say that every aspect of Bolzano's criticism is justified. Nonetheless, it is valuable for at least two reasons. On the one hand, as Rusnock (2000, 45) points out, though Kant's epistemology of mathematics is unlikely to have influenced the development of mathematical practice at the time, it shaped much of the subsequent philosophical views in intellectual circles in Germany. Bolzano's concern about its

consequences for epistemology was not unfounded. In this respect, the originality of Bolzano's endeavour is partly the result of its anachronistic character: he was alone, at the beginning of the nineteenth century to have attempted a criticism of Kant from the standpoint of logic and the philosophy of mathematics. On the other hand, Bolzano's criticism of Kant is also the starting point of his own positions, in particular, on the nature of mathematical knowledge and therefore a natural way into his own theories. Since Bolzano thought that Kant's doctrines were mistaken and indeed detrimental, he felt an obligation to respond to them. Though his interpretation of Kant might not have always been on target, it is a significant ingredient when attempting to reconstruct Bolzano's own views.

As far as dates go, and as far as his place within the reception of Kant's first *Critique* is concerned, Bolzano would seem to belong to Hegel's generation. But Bolzano refused to engage with Hegel's philosophy – the same holds for the other idealists who were his contemporaries. Bolzano did read and write caustic criticisms of Idealism at large (he also attempted to create an anti-Hegelian journal), but what he wrote are less philosophical analyses than essays devoted to the disparagement of what he called "Schwärmerei" (Bolzano 1976, 119–154). Despite the relative chronological distance, Bolzano considered that he was engaged in a dispute of his own with Kant, and his place in the history of German thought should be determined with that in mind. According to the standard narrative, two trends dominated the early reception of the first *Critique* in the German-speaking world: the Wolffians and the Lockeans (see Beiser 1987, 165–225). It could be tempting to consider Bolzano in the lineage of one or the other. But to do so would obscure Bolzano's motivations as well as the originality of his achievement. Besides, the standard narrative does not exhaust all the options. Despite the fact that he was a vastly superior thinker, Bolzano was mainly concerned with responding to the theories of a widely neglected "logical movement" to which Kant's critical philosophy gave its impulse in the years that followed the publication of the second edition of *Critique of Pure Reason*. In fact, Bolzano himself might have been the most thorough chronicler of this logical movement. The numerous sections of the *Theory of Science* where Bolzano discusses "Other Views" or seeks to show that a given concept was already present in others make copious reference to the "new logicians", as Bolzano often calls them. This puts Bolzano mostly in opposition to – and sometimes in agreement with – authors he discusses at length in the *Theory of Science*, some of them more than Kant himself:

Jakob, Kiesewetter, Hoffbauer, Krug, Twesten, Bachmann, Fries and Herbart.[2]

Bolzano very early on formulated original criticisms that do not find an equivalent in the writings of Kant's Lockean and Wolffian detractors. In turn, there are aspects of Lockean and Wolffian Kant-criticisms that are absent from Bolzano's. For instance, while this is a concern that is shared by almost all of Kant's empiricist opponents, Bolzano never addressed Kant's putative tendency towards idealism. This cannot be explained by the fact that Bolzano was unacquainted with the issue, or with the relevant criticisms of, to name only the most obvious, Feder and Garve. Bolzano had read the criticisms of Locke-inspired empiricists who almost invariably took Kant to task for his putative tendencies towards subjective idealism. Lockeans had felt an obligation to reply to Berkeley's criticism of Locke, and Kant, at least in the first edition of the *Critique*, was commonly perceived as putting forward an idealist position similar to that of Berkeley.[3] Yet, this is a concern that Bolzano, despite his natural sympathies for empiricism as a whole, does not share.[4]

Bolzano's relationship to the Wollfian school is somewhat complicated. Some commentators claim that Austria was still at the turn of the nineteenth century a stronghold of the Leibnizo-Wolffian school and that Bolzano's philosophy owes considerably to the latter. There are good grounds however to doubt that this was the case. For one thing, as we have seen in the introduction, Austrian intellectual culture at the beginning of the nineteenth century could hardly be compared to Germany's. It would be incorrect to think of the "academic" situation in Austria to have been fecund enough to give birth to an Austrian counterpart to what had been the German Wolffian philosophical tradition. Besides, while Wolffianism had dominated the German-speaking scene since the middle of the eighteenth century, with students and followers of Wolff teaching in most major German universities, by the time Bolzano started his university studies in the second half of the 1790s, the Wolffian school had already suffered its fatal blow in Germany (the French Revolution), and Kant had become the new philosophical authority. The decline of Wolffianism had not escaped Austrian intellectuals – and though an unwelcome influence in Austria, Kant nonetheless was seen as a new, powerful one. The civil servants who acted on the royal board of education and who were involved in deciding on the future of post-secondary education in Austria feared that Kant's ideas were a threat to civic stability within the Empire. They also believed that Wolffianism was much more amenable to conservatism.

Though the Emperor did not always implement their recommendations, the various existing reports of their deliberations indicate that in Austria the imposition of Wolffian textbooks on the philosophical curriculum was seen less as a philosophical statement than as an ideological tool.[5]

While Bolzano did have in Austria a training that was largely different from the one he would have received in Germany, this training left hardly any traces of sympathy for Wolffian philosophy in his work. Bolzano's criticism of Kant differs from that of the Wolffians in many ways. For instance, he disagrees with the Wolffian claim that Kant's distinction between analytic and synthetic truths is neither new nor useful. In particular, according to the Wolffians, Leibniz offers a putatively acceptable account of synthetic *a priori* knowledge based on the "principle of sufficient reason". While Bolzano does propose fundamental changes to Kant's definition of this notion – Bolzano's views on *a priori* knowledge are tightly linked to the way in which he sought to account for deductive or axiomatic knowledge at large – he nonetheless thought that Kant had made a genuine and useful discovery. There are of course certain somewhat incidental points of convergence. For instance, Bolzano refers to Wolff when he seeks to establish the claim that in order for a proposition to be true, the subject must denote at least one object (1837, §196, 328). More generally, Bolzano and the Wolffians share a commitment to the idea that the solution to epistemological problems should be obtained by "logical" means. Yet, Wolffians upheld their claims using tools – the Leibnizo-Wolffian logic – that were defective in Bolzano's eye. Bolzano thought that a reform of logic was necessary. While his theories might have been influenced by aspects of Leibniz's philosophy, the magnitude of the reform he proposed in the *Theory of Science* goes vastly beyond any of Leibniz' actual proposals. Bolzano in fact explicitly rejected virtually every aspect of Leibnizo-Wolffian logical and epistemological theories. For instance, one important aspect of Bolzano's criticism of the Leibnizo-Wollfian tradition concerns the inadequacy of the method of logical analysis on which their theories are based. As we will see in the next chapter, he discarded the decompositional conception of analysis entirely. Hence to suggest, for instance, that Bolzano's logic is an "expansion of Leibniz's logic" (Danek 1970, see also 1975) is misleading and is at any rate a gross understatement of Bolzano's accomplishment.

The *Contributions to a Better Founded Exposition of Mathematic* is a short, programmatic treatise Bolzano wrote in 1810. The *Appendix* of the latter, "On the Kantian Theory of Construction of Concepts through Intuition", offers an important contribution to Kant-scholarship – and,

regrettably, one that remains largely unknown to Kant specialists. Discussions of Bolzano's relation to Kant are typically based on the latter (e.g., Laz 1993; Rusnock 2000, 45ff.). While the import of Bolzano's criticism of Kant in this piece and in the two volumes of the *Contributions* more generally is all by itself considerable, one should not underestimate the importance of the evolution of Bolzano's thought after its publication. In particular, compared to the *Appendix*, the posthumously published *New Anti-Kant* (1850) – which recapitulates Bolzano's Kant-criticism as it is developed in the *Theory of Science* (1837) and in the *Science of Religion* (1834) – displays a scope and a level of sophistication that is lacking from the earlier analyses.[6] The conceptual tools Bolzano develops in *Theory of Science* are the result of a long and considerable maturation. It would be wrong to assume that Bolzano's criticism of Kant in the early 1810s was any more definitive than were Bolzano's own theories.

Consider Bolzano's criticism of the Kantian theory of pure intuition, for instance. As regards the latter, the evolution of Bolzano's thought is quite remarkable and while his general position remained constant – pure intuition must be eliminated – his reasons for thinking so changed quite radically. If we follow the *Contributions*, one of the main points to be made against Kant's doctrine of pure intuition is that it is "self-contradictory". Bolzano's criticism rests on a series of assumptions concerning the form of judgements. According to what Bolzano says there, whether a given truth is *a priori* or *a posteriori* does not depend, contrary to what Kant claims, on the nature of the concepts contained in the judgements but "on the copula" that is involved in the latter. What Bolzano has in mind is a theory according to which whether a judgement is *a priori* or *a posteriori* is to be established on the basis of the sentence-forming operator that connects the subject and the predicate. Depending on the sentence-forming operator involved, the predicate and the subject will belong to different semantic sub-categories. For instance, if we follow the *Contributions*, since the copula in judgements of perception is the concept of "perceiving", the subject is invariably designated by the first person pronoun 'I', and the predicate is invariably a term designating an intuition, that is, a term referring to an individual (1810, Appendix §4). On the basis of this conception of the copula, Bolzano argues that Kant's notion of pure intuition is "contradictory". On the Bolzanian account, if we ask whether *a priori* propositions may contain intuitions, the answer is negative: there cannot be *a priori* intuitions, that is, intuitions contained in *a priori* judgements since the semantic categories of the terms connected by the copula "is a kind of"

in *a priori* judgements precisely exclude that they contain singular terms referring to intuitions – they contain only pure concepts, that is, on Bolzano's early view, general terms. In the *Theory of Science*, however, Bolzano modified his views on the structure of propositions. He rejected his earlier claim that there are multiple types of copulas. All judgements are of the form 'A has B', where 'has' is a sentence-forming operator that can connect terms of virtually any non-logical semantic category (cf. 1837, §128, 18). Every proposition is composed of (i) a subject that refers to an object or a collection thereof – it can be designated by a deictic, a proper name, a common noun, a collective term, a mass term and so on; (ii) a predicate that refers to a property; and (iii) an operator 'has' (or, what amounts to the same in Bolzano: 'is') that connects the subject and the predicate (cf. Bolzano 1837, §27, 9ff.). Given this new conception, Bolzano is no longer in a position to claim that whether a proposition is *a priori* or *a posteriori* depends on the type of its copula. What distinguishes *a priori* propositions from *a posteriori* ones in the *Theory of Science* is that the former do not contain "intuitive" components. In the *Theory of Science*, Bolzano's definition of intuitions as simple and singular ideas that are caused by our actual interaction with the world excludes intuitions that have no empirical import and thus excludes that there be *a priori* intuition: propositions that contain an intuition are necessarily *a posteriori*.[7]

Bolzano's criticism of Kant's doctrine of pure intuition in the *Theory of Science* is also linked to his views on what it means for a truth to have a "ground" – a theory that stands in sharp contrast to Kant's. In the light of his distinction between analytic and synthetic truths, Kant identifies two types of "grounds". On the one hand, Kant claims, the "ground" of analytic truths is the "principle of contradiction" (Kant 1781, B10, B190): the analysis of the concepts they contain yields a logical contradiction every time the truth in question is negated. In this sense, showing that the negation of a truth implies a contradiction is a sufficient reason or "ground" to conclude to it being a truth. By contrast, such derivation is not possible for synthetic truths, and Kant suggested that synthetic truths must instead be "grounded" in an intuition. Of course, here, "grounding" – whatever it may be – is no longer a relation between truths and for this reason looks like a completely different kind of relation. And this is precisely Bolzano's point: what Kant has in mind when he speaks of intuitions grounding truths – in the context of a mathematical theory, for instance – is inadequate and misleading (see Bolzano 1837, §315, 240ff.). Bolzano disagrees with the idea that intuitions may be part of an explanation of what it means

for a cognition to have a ground. Though this might seem to be a trivial point to make for a logician, it is not a trivial point to make against Kant. As Bolzano conceived of it, mathematical knowledge is always purely conceptual – so that it excludes, by definition, appeal to intuition. According to Bolzano, in deductive disciplines such as arithmetic and geometry, the truth of a proposition is grounded in another truth in the sense that the former is an "objective consequence" – more on this in Chapter 6 – of the latter. For this reason, and though this is a trivial consequence of his definition, Bolzano cannot concede to Kant that intuitions can play a grounding role.[8] The idea that mathematicians could resort to non-conceptual cognitions in order to justify the truth of a proposition contradicts the aim of deductive practice. The latter ought to be founded on the study of the objective and purely logical relations that define the structure of axiomatic orders in which propositions are related as objective grounds to their objective consequences.

The main problem from Bolzano's standpoint with Kant's use of the notion of grounding is that it implies a mistaken account of what counts as an adequate justification in disciplines such as arithmetic and geometry. Since Kant appeals to pure intuitions as the putative "ground" of mathematical demonstrations, as Kant understands geometry, geometrical truths do not get demonstrated from the axioms via purely logical inferential steps. As Bolzano sees it, if Kant is right about mathematics, mathematics isn't actually deductive in his sense. Bolzano believed that had not Kant been confused about this, that is, had not Kant misunderstood what it means for a truth to have a ground, Kant would not have fallen pray to the "awkward" doctrine of pure intuition.[9] (Bolzano 1810, Appendix §8). It is not only that the idea that intuitions may play a grounding role short-circuits the deductive procedure by introducing non-logical inferential steps but according to Bolzano that this may lead – and effectively led some of Kant's successors such Krug and Fries – to turn to "supernatural" cognitive powers when attempting to establish certain truths. According to Bolzano, Kant may thus be considered to be responsible for the trend in German philosophy that accepts that "[scientists] be freed from their obligation to provide rigorous demonstrations and precise definition in their respective disciplines" (1837, §315, 257).

2
Decomposition

When asked to explain what conceptual analysis is, philosophers sometimes resort to the idea of decomposition: analysis is "the process of breaking a concept down into more simple parts, so that its logical structure is displayed" (Blackburn 2008, 14; see also Hanna 2007, 145). Recourse to this idea is however problematic. The idea of decomposition is unlikely to provide an adequate description of philosophical practice and, as Beaney (2009) has argued, one should rather think of it as one among many different historical conceptions of analysis – and one that no longer constitutes a paradigm.[1] At the time Bolzano wrote, however, most philosophers held theories that were based on this idea. In Kant, for instance, 'decomposition' is used in a quite literal sense. Notions such as '*Zergliederung*', '*Auflösung*', '*Inhalt*' and '*enthalten sein*' were intended to provide a straightforward description of formal features of and relations among concepts. While Bolzano's views on analysis were pioneering, he also often sought to preserve the terminology already available. This can at times be misleading. One likely mistake is to believe that Bolzano, because he assumes that ideas are complex (*zusammengesetzt*) and sometimes speaks of the "resolution" (*Zerlegung*) of an idea into its parts, also adopts a decompositional conception of analysis. Decompositional conceptions of analysis present certain variations. But they also share features that stand in sharp contrast with the views Bolzano put forward in the *Theory of Science*. Bolzano had an original account of what it means for a concept to be composed of other concepts. The type of structure he has in mind is both richer than that of his predecessors and closer to what contemporary philosophers have in mind. This is a point that has been vastly neglected in the literature. Bolzano's criticism of decompositional conceptions of analysis constitutes an important aspect of his theory and one whose examination

enables a more accurate assessment of his own views. The problem for Bolzano was to explain how the kind of knowledge we acquire in arithmetic and geometry, for instance, can both extend our knowledge and remain purely conceptual at once. The main obstacle to Bolzano's program was the limitation of the theory of logic that was available at the time. In order to provide an account of deductive knowledge that would not resort to extra-conceptual resources, Bolzano had to show that logic can effectively reflect the structure of deductive knowledge. The first step was to do away with the decompositional conception of analysis that provided the paradigm explanation of conceptual knowledge at the time.

The idea that in order to understand a concept one needs to decompose it is most eminently associated with Kant's views on analyticity. The problem as Kant conceives of it bears on the conditions under which a judgement is warranted, that is, the conditions under which it can be said to have the status of knowledge. As Kant puts it, what needs to be established is what guarantees the "connection" between the concepts in a judgement. When Kant spoke of the "connection" between concepts in a judgement, the issue he was considering was not syntactic and did not concern the question whether a given expression or judgement is well formed or not. The question whether an analytic judgement is well-formed did not arise in Kant who was considering only "categorical" judgements and assumed that all judgements of this form as well as their negation are well formed (cf. 1781, B10).[2] As Kant saw it, the fact that the concepts A and B are "connected" together in the sense that is relevant to him gives us a justification for the corresponding belief. If we follow Kant, there are two cases in which the concepts A and B are connected in his sense. Either there is a conceptual relation between A and B such that the judgement can be said to be "analytic", or the connection between A and B is warranted by an intuition – whatever this means – and the judgement is "synthetic". What Kant meant with the former is the key to understanding his views on analysis.

Kant used a number of expressions to describe what he took to be the connection between the concepts in analytic judgements. The first paragraph of section IV of the *Introduction* to the first *Critique* alone contains seven (supposedly equivalent) different ways of describing what it is for a judgement 'All As are Bs' to be analytic (cf. Kant 1781, B10, B11):

- B belongs to the subject A as something that is (covertly) contained in this concept;
- B lies in A;

- The connection of the predicate B with the subject A is thought through identity;
- The predicate B does not add anything to the subject A;
- B breaks A into its parts by means of analysis;
- One does not need to go beyond A to see that B is connected with it;
- One only needs to analyse A, that is, become conscious of the manifold one always think in A in order to encounter B therein.

Furthermore, in the same passage of the *Introduction* and at other places in the *Critique*, Kant suggests that analytic judgements are merely "explicative":

- cognition is not amplified: a concept that one already has is set out and made intelligible (cf. Kant 1781, Introduction A, B11)

and, in turn, that the truth of analytic judgements ought to be established, not merely on the basis of their analysis, but following in addition certain deductive principles:

- I merely draw B out of A in accordance with the principle of contradiction (cf. Kant 1781, B10; B190).

As it turns out, while the criteria that a judgement must fulfil in order to be analytic may appear to vary widely, Kant's many formulations all refer back explicitly or implicitly to two narrowly connected notions that form the ground of the decompositional conception of analysis: inclusion and decomposition. Assessed in the historical context, the idea that there are among concepts relations of inclusion and that they can be decomposed into simpler parts is to be expected. It is the point of convergence of a number of traditional doctrines. Kant can of course be criticised for not having done away with the decompositional conception of analysis, but contrary to what Quine (1953, 21) – and Bolzano (1837, §148, 87) – suggests, the notion of decomposition to which he resorts is not a mere metaphor and should not be represented as such.

For Kant, analytic judgements are "purely conceptual": if 'All As are Bs' is analytic, then 'A' and 'B' designate concepts, and the truth of 'All As are Bs' can, at least in principle, be established on the sole basis of their decomposition. Concepts contain components that represent certain aspects or properties of the objects they denote. Conceptual analysis aims at determining the content of a concept, and (completely) determining the content of a concept requires that we decompose it,

that is, that we establish the list of (all) the components it contains. Kant writes:

> Every given concept can be defined through analysis. For one can only clarify given concepts insofar as one makes the characters (*Merkmale*) of the latter clear. – If all characters of a given concept are made clear, then the concept is fully distinct. If it does not entail too many characters, it is moreover precise and there arises at the same time a definition of the concept.
>
> (Kant 1800, §104; see also 1781, B755)

It is not easy to situate the origin of the decompositional conception of analysis with precision. To complicate things, the historical source of the theory were at best unclear to its proponents by the time it became popular in the eighteenth century. The idea that concepts have components and that there exist among concepts inclusion relations that determine (a great part, if not all) their relevant logical properties was introduced in the second half of the seventeenth century. It was one of the innovations of Antoine Arnauld's and Pierre Nicole's *Ars Cogitans*, that is, the *Port Royal Logic*. Arnauld and Nicole call "*compréhension*" of a concept the set of concepts the latter comprises or "*enferme en soi*" – in what follows I will use the term 'content'. Concepts have a content, a determinate set of components. But they also have an extension (*étendue*): the set of objects they represent. In the *Port Royal Logic*, the notions of content and extension are introduced in conjunction with two closely related ideas: (i) the idea that concepts are always part of a conceptual hierarchy; and (ii), what is presupposed by (i), the traditional method of division.[3] The connection between the notions of content and extension, on the one hand, and (i)–(ii) on the other is not insignificant. It plays an important role when it comes to explaining the origin of the idea that there exists a relation of inverse proportion between the content and the extension of an idea – a central tenet in decompositional approaches, and one Bolzano rejects. Arnauld and Nicole's account of the relation between content and extension resorts to the two technical notions of superiority and inferiority of concepts. They write:

> But in these abstractions, one always sees that the inferior degree comprises the superior one with some particular determination [...]; but that the superior degree since it is less determined can represent more things.
>
> (Arnauld and Nicole 1993, 57)

Recourse to the notion of superiority and inferiority of concepts in this context implies reference to both (i) and (ii). (i) and (ii) were popularised by Porphyrus, a late Antiquity commentator of Aristotle whose "trees" were considerably influential. As such, the method of division consists in defining classes of objects starting with a very general class and dividing it into two – or more (cf. Arnauld and Nicole 1993, 163, 164) – smaller, mutually exclusive classes. For instance:

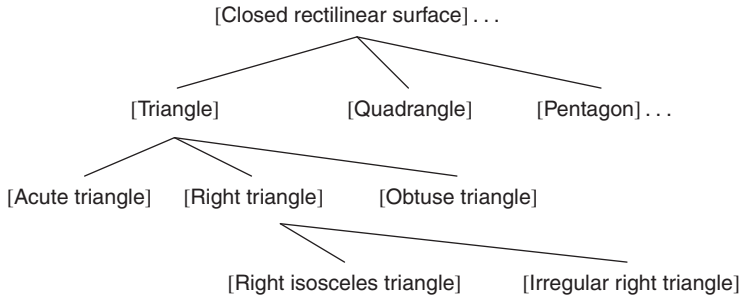


Figure 1

A conceptual chain is any part of a conceptual hierarchy – the hierarchic nature of the outcome of a division is clearly visible in the previous figure – formed by concepts that are subordinated to one another. A concept A is subordinated to another concept B if, for any object x, if x is part of the class of As, it is also part of the class of Bs (or what amounts to the same if the extension of A is a proper part of the extension of B). When a concept is subordinated to another, the former is the "species" and the latter the "genus". The notions of species and genus are relative: any concept – except for obvious reasons those that are situated at the very top or the very bottom of a conceptual hierarchy – may in turn be species or genus, depending on their relative position in the conceptual chain under consideration. Two species that are subordinated to the same genus differ from one another by virtue of a determinate component, what is traditionally called the "specific difference". For instance, the property of having a right angle is what distinguishes right triangles with respect to other species of triangles that fall under the same genus. The property of having a right angle is thus in this case the specific difference of right triangles.

Arnauld and Nicole are assuming (i) and (ii) when they describe the relation between two concepts that are situated at different "levels". On the one hand, from the standpoint of extension, the superior

concept or genus "comprises" the inferior concept. This amounts to saying in agreement with what the method of division predicts that the superior concept subordinates the inferior one. On the other hand, from the standpoint of the content, the inferior concept or species "includes" the superior: the superior concept is a component of the inferior one and is therefore "less determined" than the latter (cf. Arnauld and Nicole 1993, 57). If the concept of a triangle is composed of the concept of a figure and of that of having three angles, these concepts must in turn be simpler than the concept they compose together, and so on for every concept found in the same conceptual chain. The assumption of the method of division and of the conceptual hierarchies it yields – if only not altogether explicitly as is the case in most authors for whom this remains a tacit assumption – bears with it the idea that is at the root of the decompositional conception of analysis, namely that the concept to be defined is always "more complex" than those that are used to define it – its components – and that the components of a concept, if they are not themselves primitive can all be decomposed into concepts that are absolutely "simple".

Interesting for the purpose of a historical reconstruction is the fact that the application of the decompositional method does not require a deep insight into the conceptual apparatus on which it was originally based. If one has a good enough grasp of the idea that concepts have a content and an extension and that the latter vary in inverse proportion, one only requires a workable independent criterion for the determination of a concept's content to be in a position to "decompose" concepts in the relevant sense. As it occurs, Bolzano paid considerable attention to the question as to how, in a decompositional account of analysis, a concept's components are established. As he understands it, the idea that forms the basis of the decompositional method of analysis is the following:

> While some thought of the conformity between an idea and the corresponding object as a sort of resemblance in their respective composition, others believed that it was found in the idea that the idea of an object must contain as its parts the ideas of all its properties. In the same way as an object is the collection of all its properties, it was believed that the idea which corresponds to it could be nothing else but the collection of all the ideas of these properties.
>
> (1837, §64, 269)

In other terms according to Bolzano, the decompositional conception of analysis rests implicitly on the idea that the set $\{[\beta_1] \,\&\, [\beta_2] \,\&\, \ldots \,\&\, [\beta_n]\}$ of

components of a concept [α] – I introduce square brackets to designate concepts – is determined by the application of the following rule:

> (1) If β_1 & β_2 & … & β_n are essential properties of object α, then the concepts $[\beta_1]$ & $[\beta_2]$ & … & $[\beta_n]$ are included in the concept of α, i.e. $[\alpha] = \{[\beta_1]$ & $[\beta_2]$ & … & $[\beta_n]\}$

The question whether we should take only the essential properties or all properties into consideration is not inconsequential. The latter is the case in Leibniz whom Bolzano takes to be the paradigm in this respect (cf. 1837, §64, 272). But as Bolzano sees it, if it is to be plausible at all, (1) must be restricted to the *essential* properties of objects.[4] (1) implies, for instance, that if being rational and being animal are essential properties of men, as was traditionally assumed, then the concepts of rationality and animality are comprised in the concept of humanity. When we decompose the concept of humanity, we find therein these two concepts. Likewise, Kant's idea that the concepts of extension, impenetrability and form are included in the concept of a body (cf. Kant 1781, B12) constitutes an application of the criterion defined in (1), or of some other criterion similar to (1).

(1) is one element in a possible account – unacceptable, according to Bolzano – of the relation between concepts and objects. Whoever subscribes to (1) conceives of this relation in terms of "resemblance". (1) expresses a naïve form of representationalism: a concept is in some substantial way determined by the features of the object it represents, and it represents things by virtue of "resembling" them. (1) supposes that the structure of (components in) concepts is in some systematic manner analogous to the structure of (properties in) objects. The kind of analogy adherents to (1) had in mind was typically based on the idea that concepts are "pictures" of the objects they represent; the idea that they are "pictures" of the objects they represent was meant to explain how they represent the latter, namely by virtue of their components' corresponding to some features of the objects to which they refer. While it was discarded by some authors such as Descartes, for instance, the idea that concepts are "pictures" of objects was a widespread tenet for most of the modern period and thus constituted an obvious target for Bolzano.[5]

In the *Theory of Science*, Bolzano attacks the decompositional conception of analysis on several fronts. For one thing, the kind of naïve representationalism that underlies the decompositional conception of analysis rests, if we follow Bolzano, on a misunderstanding concerning

the nature of pictures as well as the nature of ideas. On Bolzano's account pictures are (i) objects that subsist independently of the thing of which they are images both with respect to their origin and to their persistence, (ii) objects that may "stand for" the objects of which they are pictures by virtue of their resemblance with them and (iii) that may also be used for given cognitive purposes. A picture of my son, for instance, may be used to remind me of him while I'm travelling; or, I can compare photos of him with photos of his father at the same age to trace hereditary features such as the colour of their hair or the shape of their eyes. But concepts or more generally ideas can fulfil none of conditions (i)–(iii). Bolzano concludes:

> [...] it results that one should not call the idea of an object in the stronger sense an image of the latter; for it is indeed no object that we can consider instead of it, but it is that which arises in our minds, when we consider this object itself.
>
> (Cf. 1837, §52, 230)

Bolzano's criticism of the decompositional conception of analysis also rests on his rejection of (1). According to (1), the content of a concept comprises other concepts, and the latter correspond to properties of the things they represent. For someone who subscribes to (1) there is a correlation between concepts and objects which according to Bolzano amounts to a positive answer to the questions whether or not *the components of an idea are identical to the idea of the properties of its object(s)*. (1837, §64, 269) Bolzano denies the latter as well as the more simplistic version of this correlation he attributes to Johann Heinrich Abicht who claims that:

> The more particular the object of a concept is, the more parts of ideas must also be distinguishable in this object [...] The completeness of a complex is recognized as true when we can see on the basis of reasons that an object present these and only these parts.
>
> (Quoted in Bolzano 1837, §63, 267)

The latter amounts to saying that:

> (2) If γ_1 & γ_2 & ... & γ_n are parts of object α, then the concepts of $[\gamma_1]$ & $[\gamma_2]$ & ... & $[\gamma_n]$ are included in the concept $[\alpha]$, i.e. $[\alpha] = \{[\gamma_1]$ & $[\gamma_2]$ & ... & $[\gamma_n]\}$

Bolzano puts forward a series of objections designed to invalidate (1) and (2). These objections include the following:

- There are ideas such as [nothing] and [round square] which do not have objects. If (2) were true – the same holds for (1) – these ideas could not have even one constituent (1837, §63, 267). But these ideas, according to Bolzano, must obviously have at least one component. As Bolzano sees it, the same argument can be made for all ideas designated by syncatgoremata: 'has', 'is', 'not', 'which has', for instance.
- Neither (1) nor (2) can be generalised since they do not apply whenever a concept contains a component that refers to a property or a part the object lacks, for example, [A landscape without mountains], [a book without copper binding] and so on. In these examples, the concepts [moutains] and [copper binding] contained in the subordinate clause represent neither a part of the object nor one of its properties. On the contrary, they represent something that is absent from the object.
- (2) cannot account for ideas implying a part-relation (e.g., [the gable of the house]) but whose reference is not the whole (e.g., the house), but the part (e.g., the gable). In order for the relevant instance of (2) to be true, that is, in order for the concept of a house be included in the concept of a the gable of the house, it would have to be the case that the house is a part of the gable of the house, which is absurd.
- Finally, (2) fails to account for cases in which the object itself does not have parts (e.g., God) but where the corresponding idea is nonetheless complex (e.g., the being who has the ground for his existence in itself).

Bolzano wants neither to deny that an object may have some properties essentially nor that a concept may contain the representation of an essential property of its object.[6] But Bolzano considers that the case in which an idea contains the representation of an essential property of its objet is relatively unimportant for the purpose of logic. Interesting logical relations between concepts in deductive discipline such as arithmetic and geometry do not reside in relations of inclusion as conceived by the proponents of the decompositional conception of analysis. Rather, they reside in "inferential" ones. As Bolzano conceived of them, in deductive theories the properties of an object may be "inferred" from the concept of the latter. Taken on its own, the concept of a triangle that is

equilateral tells us nothing about the triangle except that it is equilateral. This is what Bolzano meant when he wrote:

> It is known that every equilateral triangle has the property of equiangularity; but one must admit however that the concept of this equiangularity does not lie in the concept of an equilateral triangle taken as such. For this concept arises when the concept "triangle" is connected with the proposition: "which is equilateral". Now it is obvious that the concept of equiangularity occurs neither in the concept "triangle" nor in the concept "which is equilateral", therefore certainly no more in the whole which is indeed composed of nothing else but these two parts [...] Otherwise one would merely have to say that it is in itself impossible to connect the concept "triangle" and the propositions "which is equilateral" without enclosing a number of other parts and among others also those which entail the concept of equiangularity. This however would be quite false [...].
>
> (1837, §64, 273, 274)

Taken within the deductive structure that defines geometry, however, it is possible to establish a connection between the concept of an equilateral triangle and that of equiangularity, for instance: it *follows* from a triangle's being equilateral that it also be equiangular. Bolzano assumes – more on this in Chapter 8 – that conceptual relations in deductive disciplines are contingent on the deductive relations that subsist among the propositions that contain the concepts in question. On Bolzano's view, once a theory of logic is equipped with a satisfactory account of deductive orders, an account of inclusion relation of the type described in decompositional conceptions of analysis becomes redundant. One clear indication of the redundancy is the fact that in Bolzano's theory the principle of inverse relation between content and extension collapses. As Bolzano sees it, on the one hand, the content of an idea can be increased without implying that the extension decreases. Bolzano explains:

> [...] It is only required that we add a component [...] from which no new property of the object represented follows [...] Thus the content of the concept of a round sphere is greater than the concept of a sphere in general though the extension of both concepts is the same.
>
> (1837, §120, 569)

Conversely, the content of an idea can be decreased without implying that the extension increases:

> But as we have already seen in §64 the objects that fall under a given idea A can have properties whose idea do not appear as components in the idea of an A. If we assume that b is the property of all objects that fall under the concept A, then all the objects that belong under A also fall under the idea of a B or of something that has the property b. thus either A is equivalent to B or [...] subordinated to it; nonetheless A does not need to be composed of b [...].
>
> (1837, §120, 569, 570)

Interestingly, despite the radical character of his theory in other respects, the rejection of the decompositonal conception of analysis is an option Kant never considered. In order to account for the kind of knowledge we acquire in mathematics, Kant resorted instead to the idea that knowledge that is not analytic needs to be grounded in non-conceptual resources, and this assumption led him to develop his theory of pure intuition. But as Bolzano saw it:

> The idea that synthetic propositions are not deducible from mere concepts without consulting intuition is precisely the *proton pseudos* of the *Critique*, and Kant nowhere demonstrates it. Kant appeals here in order to support his claim to mathematics, namely to geometry. It seems to him that its principles can only be obtained through pure intuition. But we have seen how awkward his doctrine of pure intuition happens to be [...].
>
> (Příhonský 1850, 69)

The chapters that follow flesh out Bolzano's views on analysis, quantification, analyticity, deductive knowledge and demonstration. They aim at providing an overall picture of Bolzano's philosophy of logic and theory of knowledge, and with it a broad picture of what should be considered to have been the most important development of the discipline since Aristotle.

3
Meaning and Analysis

We say of a sentence or a sentence utterance that it is context-sensitive when its truth or falsehood depends on certain elements that only the context of utterance can reveal. Sentence utterances that contain indexical expressions such as 'here', 'now', 'yesterday', 'she' or 'this' are such that what they mean – and what they refer to – can shift from context to context. What 'It is windy today' refers to, that is, the weather on the day it is uttered, at the place it is uttered, will sometimes make it false, sometimes make it true – since it can be uttered on different more or less windy days or in different more or less windy places. By contrast, an utterance of '$2+2=4$' is typically considered not to be context-sensitive since it is true no matter what the context in which it is uttered. On Bolzano's account, context-sensitive sentences are semantically underdetermined: they do not express the proposition that forms their content "completely". Only sentences that are "eternally" or "immutably" true (or false), that is, only sentences that are not context-sensitive express their content completely in the sense that is relevant to Bolzano. The purpose of analysis in his theory is to reveal the "*Sinn*" of expressions (cf. 1837, §285, 67). Since propositions are the "*Sinn*" of sentences (1837, §28, 12), analysis aims at making them explicit, which given the immutability requirement appears to require that we eliminate context-sensitive elements and replace them by non context-sensitive ones. The idea that context-sensitive elements must be eliminated is not unproblematic. In the context of Bolzano's theory, however, it plays an important role. Analysis as Bolzano conceives of it is designed to support the elaboration of deductive disciplines by fixing the semantic properties of the sentences they involve. To this purpose, signs that are epistemically underdetermined are to be replaced with other signs that are "better known" or that belong altogether to

"another symbolic system" (cf. 1837, §387, 543). In other terms, Bolzano requires that we paraphrase sentences of ordinary language that are often vague or ambiguous by using sentences which, in so far as they express a propositional content completely, aren't.[1] Bolzano's tacit aim with this paraphrastic procedure is – at least inasmuch as axiomatic disciplines are concerned – the construction of (semi-formal) deductive languages in which sentences are *complete* in the sense that they are neither context-sensitive, nor ambiguous nor vague and therefore not liable to changing their truth or falsehood.

One interesting aspect of Bolzano's theory is the fact that it treats indexicals and inarticulate constituents as equally problematic. As regard the latter, he writes:

> When one ascribes to some proposition in itself, e.g. religious, mathematical or metaphysical truths the predicate 'eternal' [...] one only wants to say that the latter are propositions that express a constant (eternal), perpetual relation. Meanwhile, other propositions, e.g. the propositions: 'A bushel of grain costs 2 talers", or 'It is snowing' and so on, express a passing relation (taking place only at a given time in a given place). Hence in order to be true, they require the addition of the corresponding temporal (and likewise locative) determinations.
>
> (1837, §25, 112)

The idea that different kinds of "determinations" (*Bestimmungen*) must be added to sentences follows from the idea that all sentence utterances express a proposition which is "immutably" or "eternally" true (or false). The number and nature of determinations that must be added to a sentence in order to express its propositional content completely is a function of the conditions under which the latter is to be said to be true *immutably*. This implies that sentences be not context-sensitive. They must be completed with the adequate determinations, and the context-sensitive components they contain must be replaced by non-context-sensitive ones. My utterance of 'It is raining', for instance, must veil elements that are bound to the context since without them, so the argument goes, nothing would exclude that what I am saying be both true and false at the same time: it may be true, for instance, that it is raining at the time of utterance, say, in Manhattan, KS while false that it is raining at the same time in New York, NY. But Bolzano thinks that contradictions of this type are to be avoided if we are to give a consistent (and true) account of the causal world. He writes:

> We may assert that every proposition of the kind: The real object A has (the attribute) b can express a complete truth only insofar as we incorporate in its subject idea a time determination. Thus, for instance, the proposition: 'I have a sensation of pain', or: 'The earth is a planet', are not completely true unless we include a time specification in their subject idea, e.g., 'I, at the present moment', or 'The earth in the present time span'. And if we examine the matter more closely, it becomes apparent, as I believe, that by the word 'time' we mean nothing but that particular determination in a real thing which is the condition for correctly attributing to it a given attribute.
>
> (1837, §79, 365)

Whenever the subject of a sentence 'A has b' designates an object located in time, it should include an adverbial expression of the form 'at t' – a temporal determination – so as to express its content completely. As Bolzano sees it, the addition of determinations also makes it possible to avoid contradiction in such cases as "This man (on my right) is drunk" and "This man (on my left) is not drunk" or "Caius (as a carpenter) is very apt" and "Caius (as a cook) is inept."

Though there are arguably problems with this view – one may doubt for instance that determinations in fact do much genuine explanatory work or one may suspect that Bolzano misunderstood the syntax of temporal determinations (see Textor 2003) – Bolzano's reasons for maintaining that they are needed should not be downplayed. The same holds of Bolzano's views on indexicals. In Bolzano's theory, "intuitions" are indexical components of our beliefs. They are the simple, singular, subjective – as opposed to objective – cognitive events whose object is the immediate "modification of the mind" caused by what is perceived, that is, a certain kind of phenomenal properties. It is worthwhile to quote the passage in its entirety:

> Whenever we direct our mind's attention on the modification that some external body brought to our senses produces in our soul, for instance a rose, the *next* an *immediate* effect of our attention is that an *idea* of this modification originates in us. This idea is *objectual*; its object is indeed the modification that happens in our mind at this moment and nothing else; it is therefore a singular object so that we may say that this idea is a *singular idea*. Of course, on such occasion and through the continual activity of our soul many other ideas are engendered, and among them also ideas that are not singular and

> likewise also whole judgements, namely about the very modification which is appearing at this moment in us. We can thus, for instance, say: This (which I am seeing right now) is the sensation or the idea red; this (which I smell now) is a pleasant smell [...] In these judgements the ideas: red, pleasant smell, [...], etc... have many objects. The only authentic singular idea (cf. §68) is the subject-idea that is produced and which we designate by the word 'this'. For, by 'this', we merely understand the single modification which is actually appearing and none of the others which may take place at some other point no matter how very subjective they may be. It is no less certain that these ideas are all *simple*. For if they were composed of parts, then it would not be the *next* and *immediate* effect arising from the examination of the modification happening in our soul at this very moment; rather, the individual ideas that constitute these parts would be produced earlier and more immediately. We have already explained at §59 and §69 why we can't conclude from the fact that we use the many words: 'this (which I am seeing now)' [...] to designate this subject-idea that it must itself be complex. [...] every time we examine the modification that happens in our soul, ideas arise in us that are simple and have only one object, namely the examined modification itself to which it relates like the next and immediate effects relate to their cause.
>
> (Bolzano 1837, §72, 326)

For Bolzano, an intuition is always the result of an agent's causal interactions within the causal world. Whenever we produce a report on a perception, for example, 'This is (a) red (sensation)', an intuition is involved, and the same holds when we report an experience we have of a particular object, for example, 'This is a rose.' Like the demonstratives 'this' and 'that', all other types of indexical expressions, all proper names (1837, §75, 335), as well as empirical and natural kind terms (1837, §75, 338, 339) always involve at least one constituent – the latter may be unarticulated – designating an intuitive idea in Bolzano's sense. On Bolzano's account, in order to be in a position to assert truly that this smells good or that gold is yellow, I must be acquainted with determinate elements of the context in which I find myself. In particular, I must stand in an immediate cognitive relation with a "modification of the mind" that is caused by the relevant aspects of the object to which I have phenomenal access. This implies a relatively idiosyncratic conception of what the "context" of an utterance consists in. What is picked out by a Bolzanian intuition is a certain aspect of my phenomenal experience,

not the object that causes it. When I say 'This smells good!' what 'this' picks out in context is the "modification of the mind", not the rose though the rose is the cause of the modification and in this sense the object that makes the intuition possible in the first place. Given this conception of what an intuition is, there is no such thing as an intuition that would not have empirical import. One cannot have an intuition of something that does not exist (in the causal world). Since intuitive cognitive events are, just like anything causal, indexed on the time-continuum, they always necessarily differ from one another in at least this respect. In principle – we will see that this is not unproblematic – since every subjective idea corresponds to an objective idea, the propositional content of each utterance of a sentence containing an indexical term (that designates a Bolzanian intuition) will differ from all the others expressed by other utterances of that same sentence.

Bolzano's immutability requirement implies that in order to complete a sentence, we must both make all indexical components explicit and *eliminate* them. It is a standard objection to Bolzano-type views that some indexical expressions cannot be eliminated and replaced by non-indexical expressions because they constitute an essential aspect of the meaning of a sentence or of the content of our beliefs. To put it briefly, according to this objection, Bolzano would be wrong to assume that my belief – or the correlative utterance – that it is snowing now, when this belief occurs in Kansas City on the eve of Christmas 2009 at 22:30, has the same meaning as my belief that it is snowing in Kansas City on 24 December 2009 at 22:30. The reason for this, following the well-known argument by Perry (1979), is that it is possible for me to believe that the former is true and have good reasons to believe it – I perceive the snow and perception reports of other agents coincide with mine – while denying mistakenly the latter. (It could be that I have the date wrong or, distracted by the unusually cold weather, I've come to be mistaken about which city I am in.) This can however only be the case, so the argument goes, if the two beliefs are different. Assuming that we are concerned with the preservation of meaning, that is, the synonymy of the two sentence utterances, this implies that indexical expressions are not simply expressions that can be eliminated but expressions that express components of our beliefs that are essential to them.

In the context of a "theory of science" as Bolzano conceived of it, the elimination of context-sensitive components makes sense: deductive sciences are purely conceptual and analysis amounts precisely to eliminating non-conceptual components. Note that inasmuch as Bolzano's aim is the elaboration of deductive languages – which it is

for a substantial part – he is not liable to the reproach that his treatment of indexicality is inadequate. While true that an account of meaning preservation in natural language cannot eliminate indexicals, this does not hold for the elaboration of the kind of languages which are used in disciplines such as arithmetic, geometry and logic. As Bolzano conceives of the latter, they are "purely conceptual" in the sense that they contain no intuitions (remembering here the point made earlier that for Bolzano all indexicals are associated with intuitions).[2] More generally, as traditionally conceived, deductive languages are precisely designed to avoid, for the purpose of the discovery of formal or semantic properties of sentences, the kind of equivocation that may arise, among other things, from the use of indexical, ambiguous or vague terms. This is what Bolzano effectively sets out to do in the relevant sections of the *Theory of Science*.

As Bolzano conceives of it, analysis is not and need not be meaning preserving. When Bolzano considers the interplay between sentences and propositions, his aim is to provide what he calls an "interpretation" (*Auslegung*) of sentences. On Bolzano's view 'It is snowing in Kansas City on 24 December 2009 at 22:30' is the correct interpretation of the sentence 'It is snowing now' when uttered in Kansas City on the eve of Christmas 2009 at 22:30. While the two sentences do not have the same linguistic meaning, they do share certain aspects, namely their propositional content (in Bolzano's term, their *Sinn* or the proposition they express). According to Bolzano, one ought to distinguish "what words are designed to convey" from what a speaker "intends to convey with them" on a particular occasion (cf. 1837, §285, 68 – see also Chapter 10). To analyse an expression on Bolzano's account consists in establishing what the speaker intends to convey with them – and we may do so by using expression tokens that are designed to convey different things and that do not therefore have the same linguistic meaning. More generally, if we follow Bolzano, a sentence utterance u expresses one proposition [p] in a given context. Let us designate by 'p' the sentence token – be it itself an utterance or another type of linguistic equivalent – that expresses the propositional content [p] of u *completely*. If we follow Bolzano, to interpret u consists in making the propositional content [p] of u explicit by using another sentence token 'p' which (i) is better known than u or which pertains to a different (and presumably also clearer) language; (ii) which is at least "equivalent" (*gleichgeltend*) to u, that is, true whenever u is true and vice versa (cf. 1837, §§387, 542); and (iii) – what follows from what we've discussed above – that present no ambiguities, no vagueness and no context sensitive components (i.e., no

intuitions). While u pertains to, say, ordinary language, p pertains to an "ideal language" in which all sentence tokens express their propositional content completely and immutably. Let us call this ideal language L_B. To analyse or interpret a sentence in Bolzano is to find the sentence p that makes the following schema true:

(B) p expresses "completely" the propositional content of u in L_B.

That p need not preserve the linguistic meaning of u is a claim Bolzano makes explicitly. On Bolzano's account, the utterance u and its paraphrase p must be merely "equivalent" (*gleichgeltend*).[3] In order to provide an interpretation of my utterance of 'It is snowing here', for instance, my interlocutor needs to consider the contextual elements it involves. If she is to interpret what I say correctly, she needs to know that if 'It is snowing here' when uttered at time *t* in location l is true, then any utterance of the form 'It is snowing at *t* in *l*' will be true, and vice versa. What is interesting is the fact that Bolzano does not require synonymy. Bolzano himself makes it clear that he takes the mere equivalence of u and p to be necessary for an adequate paraphrase and rejects the stronger condition that they be synonymous:

> Were someone to find that it is improbable that the interpretation of this linguistic expression we attempted renders the thought that we connect with it since it is composed of completely different words [...], I would not want to contest it; but I only demand that he at least concede that the *Sinn* that one connects with this formulation and the *Sinn* which I gave in my interpretation be *equivalent* (*gleichgeltend*) as regards their *Bedeutung*, i.e. that every time one of the proposition is true, the other one is as well.
>
> (Bolzano 1837, §137, 53)

The idea that equivalence is sufficient to insure the adequacy of the interpretive procedure has made some commentators perplexed, and some have argued that Bolzano needs synonymy (cf. Textor 1997, 194). One could worry, for instance, that equivalence is too weak to insure the adequacy of *Auslegung*. Note however that if *Auslegung* is a matter of making a sentence "more precise", paraphrases effectively entail information that the sentences paraphrased do not. Indeed, one important factor to take into account when considering Bolzano's views about the way in which we establish the propositional content of an utterance is that the latter is only possible with respect to a language

whose resources are constrained, namely L_B. In this sense, Bolzano's interpretive procedure turns out to consist, just like in Carnap in:

> [...] the task of making more exact a vague or not quite exact concept used in everyday life or in an earlier stage of scientific or logical development, or rather of replacing it by a newly constructed, more exact concept [...].
>
> (Carnap 1947, 7–8)

Carnap calls this paraphrastic procedure "explication". As Quine notes, Carnap's explicative procedure does not consist simply in providing a synonymous expression but in improving the expression by refining or completing its meaning (Quine 1953, 55). This, as I understand him, is also what Bolzano has in mind. Besides, since logical equivalence here is relativised (to a given axiomatic) structure, what we get is a notion which, while it is not equipped to account for our intuitions about sameness of meaning in natural language, does provide an understanding of what it amounts to in deductive languages: two sub-sentential expressions are synonymous if they can be substituted for one another in any given theorem *salva veritate*. More on the Bolzanian notion of synonymy in Chapter 5.

Bolzano's views on analysis are informed by his views on what makes for deductive languages in which sentences express their content completely. It is difficult to exaggerate the import of this idea and the novelty of the resources Bolzano deployed at the time in order to bring it to fruition. Bolzano's predecessors typically adhered to the picture theory of ideas (see Chapter 2). This naïve form of representationalism is based on the assumption that the relation between meaning (concepts) and referent (objects) ought to be explained by resorting to the notion of resemblance: the structure of meaning is in some determinate way defined by analogy with the structure of the referent. Bolzano found it necessary to establish a strict distinction between the two types of entities: we must distinguish the "constituents" of ideas from the "properties" of objects (1837, §64, 287). Of course, concepts and objects are not strictly independent of one another in Bolzano's theory: objects are the referent of the expressions that designate ideas and propositions (1837, §66, 296ff). But the conception of analysis that supports Bolzano's criticism of the picture theory of ideas, instead of focussing the investigation on providing an account of the relation between concepts and reality – which Bolzano assumed is primitive – redirects the concern towards an investigation of the relation between expressions and their meaning.

One important feature of Bolzano's conception of analysis and one which marks a sharp departure from the views of his predecessors is the assumption that (complex) ideas and propositions, just like the terms and sentences that express them, are syntactically articulated: they consist in the connection of ideas according to determinate rules. Bolzano suggests in many places that ideas are structured according to syntactic patterns – as opposed to, say, mereological ones. For one thing, Bolzano argues that ideas may contain fully fledged constituents whose function is purely syntactic. He writes:

> In order to represent the object which has in itself the properties b, b′, b″ … one must build the idea of: "something, which has the property b, b′, b″ …". But aside from the ideas of the properties b, b′, b″ … many other ideas occur in this idea, namely that of *something*, the idea of the relative pronoun *which* and the idea of *having*.
>
> (1837, §64, 271)

Bolzano's emphasis, at this place and others, on the fact that the ideas for which, among others, 'which' and 'has' stand are constituents in their own right might appear to be banal. At the time, however, it happened to be both original and considerably fruitful. By rejecting the view that "syncategorematic", that is, logical terms should be seen to play a merely incidental role in the analysis of concepts he put into question the idea that then prevailed and according to which ideas contain only "categorematic" components, that is, concepts that have a reference when taken on their own.

Bolzano's approach is partly borrowed from grammatical analyses as they apply to sentences of natural languages. Bolzano assumes that the structure of ideas and propositions is similar to the structure of the linguistic items that compose natural languages. As Bolzano sees it, the analysis of ideas and propositions is based on the analysis of their proxy, that is, the words and sentences that express them. He writes:

> Every word in language serves to designate an idea and some of them even complete propositions. Therefore, it is only natural to suppose that each idea is composed of at least as many parts as there are words in its expression.
>
> (1837, §57, 246)

Two things must be emphasised. On the one hand, one ought not to be misled by the fact that Bolzano speaks of concepts being "included"

in other concepts or in propositions or, at certain places, that we must "decompose" or "resolve" ideas into their parts. It would be wrong to assume that Bolzano adopts the decompositional conception of analysis. On his account, the idea $[\beta_1]$ is included in the idea $[\alpha]$ if and only if $[\alpha]$ is a construct that results from the connection of $[\beta_1]$ with other ideas $[\beta_2], \ldots [\beta_n]$ according to determinate rules. Bolzano claims, for instance, that the idea designated by 'Erdengeschöpf' is complex. We know that it is complex, Bolzano explains, because we think by 'Erdengeschöpf' precisely what we think by the words 'Ein Geschöpf, das auf der Erde wohnt' (cf. *WL* §56, 243), that is, a name that is formed by the apposition of an attributive clause. On the other hand, it must be stressed that though Bolzano reiterates at many places that all propositions have the form 'A has b', he also allows for intrapropositional articulation of various types. Bolzano's propositions are built on the basis of a (in principle finite) primitive vocabulary and of determinate recursive rules. For instance, Bolzano uses uppercase to designate object variables and lower case to indicate property variables; he thus clearly shows an understanding for the importance of representing syntactic subcategories in the vocabulary and marks a rupture with classical Aristotelian notations that do not. In Bolzano's notation, 'A', 'B' and so on stand for ideas representing objects, and denote individuals or collections thereof; 'a', 'b' and so on stand for ideas that denote properties cf. *WL* §60. According to him:

- There are simple and complex object-ideas. Simple ideas are treated as unstructured. (Cf. 1837, §56, 243ff.)[4]
- A complex object-idea is typically attributive i.e. of the form 'Something which has b' where 'which has' is a name-forming operator (cf. 1837, §60, 259ff.). If, for instance, the subject-idea 'A' of a proposition 'A has b' is complex, Bolzano proposes that we analyse it according to the following pattern: 'Something which has a, has b', and so forth. (Cf. 1837, §§58–59, 251ff.)
- A property-idea, if it is complex, is typically "conjunctive". Such is the predicate-idea 'b, b′, b″' in 'A which has b, b′, b″'. (Cf. 1837, §64, 270ff.)

Furthermore, as we have already discussed, Bolzano makes use of what he calls "determinations" in order to specify the conditions under which a sentence utterance is true immutably. One of the most common types of determinations is temporal: if the object of a proposition in itself 'A has b' is real (*wirklich*), then in order to exclude possible contradictions the subject of that proposition must include a temporal

completer and is in fact of the form: 'A *at t*'. Hence, all propositions whose subject-idea refers to a real object have the form 'A *at t* has b' (1837, §79, 364, 365, §127, 15). In addition to formulating a set of syntactic rules, Bolzano makes extensive use of logico-grammatical subcategories: metaphysical distinctions among the objects denoted also make for logico-grammatical distinctions between the expressions that denote them. 'A' may designate a real object, but it may also denote an idea in which case it is a meta-logical idea, what Bolzano calls a "symbolic idea" (*symbolische Vorstellungen*) (1837, §90, 426). 'A' may designate a collection-idea (*Inbegriffvorstellungen*) – of which there are different types: sums, series, unities, pluralities, totalities and so on that define objects presenting different structural properties. (1837, §§82–86, 393ff.; see also Chapter 10) 'b', on its part, may designate either a causal property or a logical one. The latter explains frequent use of "metalinguistic" analyses. For instance, Bolzano argues that (true or false) existential propositions such as 'There are triangles' should be interpreted as 'The idea of a triangle has objectuality'; where the subject-idea 'The idea of a triangle' refers to an idea and where 'objectuality' refers to a second order predicate-idea, a property of a concept, namely the property of having a non-empty extension (1837, §172, 215).

Bolzano sees a great deal of significant, articulate structure within propositions, and he is committed to an elaborate theory of such structure. It is strictly true that Bolzano constrains the theory by adhering to the idea that all propositions have the form 'A has b' and that his doing so is the source of some intricate contortions through which he has to go in order to paraphrase various ordinary language expressions into L_B. Nonetheless, we should not accept that, in Bolzano, "every proposition is structured in the same way" (cf. Textor 1997, 182) if this is supposed to imply that Bolzano's theory of syntax comes to nothing more than the claim that all propositions have a subject–predicate structure. Given the fact that ideas can be complex and that, if they are, they are themselves syntactically structured and can be so in a number of different ways, Bolzano's claim that all propositions are composed of the three ideas 'A' 'has' and 'b' does not imply that the only form of syntactic articulation he recognises is predication. Propositions can have considerably more discernible syntactic structure beyond the 'A has b' form they all share.

As we have seen above, Bolzano suggests that each idea is composed of at least as many parts as the expression used to designate it in ordinary language (cf. 1837, §57, 246; see also 1837, §69, 312). But this principle admits of at least one important exception, that of quantificational

terms. One of Bolzano's favoured examples is 'All' in 'All As are Bs' (1837, §57, 247ff.). If we follow Bolzano, a concept should always be taken in its fullest extension. Under this assumption, according to him, the two expressions 'all As' and 'A' designate the same idea, namely [A]. What makes a proposition universal in the relevant understanding of generality is not indicated by a quantificational particle – the same holds for existential propositions which, as we have seen above, are conceived as ascribing a second order predicate 'is objectual' to a concept. Rather, universality is determined through the application of a method of substitution we will detail in the next chapters.

As Bolzano sees it, a word may designate a complex idea – as in the case of 'Erdengeschöpf' above. It would seem natural to assume that in an adequate paraphrase every term expresses exactly one idea and that every idea is expressed by exactly one term. This, with an important caveat, is what Bolzano has in mind. Bolzano thinks that expressions of everyday language may designate "redundant" (*überfüllte*) ideas. But according to Bolzano, redundant ideas are "mistaken" (*fehlerhaft*) and should not be included in a "properly scientific exposition" (cf. 1837, §69, 310); hence, the conditions of an adequate paraphrase also include that in:

p expresses "completely" the propositional content of u in L_B

'p' must not be redundant in Bolzano's sense. A redundant constituent of x is a constituent whose omission does not affect the extension of x (1837, §69, 309). Redundancy occurs when, in an attributive construction, the appositive does not modify the extension of the main clause. In 'This, which is an A', if 'this' is used as an indexical, then A is redundant (cf. 1837, §59, 258). Likewise, in the sentence: 'A being, who does not have a ground for its existence and who is also perfect, omniscient and omnipotent', for instance, each of the constituents: 'who is perfect', 'who is omniscient', 'who is omnipotent', is redundant. Bolzano's constraint on redundancies allows him to systematically eliminate constructions in which a constituent is idle as regards its role in determining the extension of the concept: the only cases in which the inclusion of a property-concept is acceptable is when the latter also modifies (either restricts or increases) the extension of the concept of which it is a part. If a and b are essential properties of the object x, then 'x, which has a and b' is systematically redundant. Strictly speaking, redundant constructions are well-formed: they do not violate any of the compositional rules Bolzano sets out. Hence, one may wonder why Bolzano seeks to

eliminate them. The most plausible reason – Bolzano never makes the point explicitly – is that redundant ideas do not do any genuine work in deductive disciplines as Bolzano conceives of them. Consider 'A triangle, which is equilateral and equiangular'. On Bolzano's account, taken in the context of geometry, there is nothing I can infer from this concept, which I cannot infer from either 'A triangle which is equilateral' or 'A triangle which is equiangular' taken individually. Admittedly, the idea that we may infer something from a concept is odd, but Bolzano is committed to the idea (more on this in Chapter 8). To admit the idea 'A triangle, which is equilateral and which is equiangular' would amount to accepting the proposition 'X which is a triangle is equilateral and equiangular'. But a proposition such as 'X which is a triangle, is equilateral and equiangular' is nothing more than the conjunction of the two propositions 'X which is a triangle, is equilateral' and 'X which is a triangle, is equiangular'. Hence, an argument that takes as a premise the former, say:

> X which is a triangle, is equilateral and equiangular
> Therefore, X is neither isosceles nor right.

Should read as:

> X which is a triangle, is equilateral
> X which is a triangle is equiangular
> Therefore, X is neither isosceles nor right

But on Bolzano's account, this argument contains one premise too many – it is itself "redundant" (*überfüllt*) – since the conclusion follows from any of the two premises taken individually (see our discussion of "exact *Ableibarkeit*" in Chapter 6).

Bolzano's approach to the analysis of meaning is in sharp contrast to Kant's approach. On Bolzano's view, inclusion relations of the type Kant has in mind do not point to semantically interesting features of propositions: use of the notion of 'inclusion' and other cognates are "mere metaphors (*bildliche Redensarten*) that do not analyse the concept to be defined or expressions which allow for too broad an interpretation" (1837, §148, 87). In general, it seems to him that definitions of logical notions such as analyticity that rest on the idea of inclusion:

> do not emphasise sufficiently what makes this type of propositions *important*. It is my opinion that the latter consists in the fact that their

> truth or falsehood does not depend on the particular Ideas of which they are composed but remains the same whatever the modifications we undertake with the latter, granted only that one does not destroy the objectuality of the proposition itself.
>
> (*WL* §148, note 4, 88)

In the light of what precedes, it is safe to read 'does not depend on the particular ideas of which they are composed' as implying that, in Bolzano's eyes, a good definition of analyticity should not rest on decomposition. The next chapters detail Bolzano's conception of logical form, his criticism of Kantian definitions of analyticity and his presentation of his alternative: a substitutional approach to the analysis of semantic notion that anticipated both Quine's definition of logical truth as well as Tarski's definition of logical consequence.

4
A Substitutional Theory

Considering later developments in the field, two aspects of Bolzano's semantics are particularly significant: his definition of *Ableitbarkeit* and his definition of "logical analyticity". The first – Bolzano's attempt at an analysis of statements of the form 'if . . . , then . . .' – has often been compared to Tarski's notion of logical consequence, and as we will see in Chapter 6, there are good reasons to maintain the comparison. The second anticipates the Quinean definition of logical truth and will be discussed in some detail in the next chapter. Both notions are common themes in the literature. Yet, important features of both have been neglected. Consequently, their role in Bolzano's theory has often been misunderstood with the upshot that crucial aspects of Bolzano's theory as a whole have been completely overlooked. The concern that is at the core of virtually every discussion of Bolzano's substitutional method is that the latter does not deliver the kind of results one would reasonably expect when it comes to defining analyticity and consequence and, in particular, that it does not account for the kind of epistemic and metaphysical necessity those notions are assumed to carry with them. This putative failure and the underlying confusion should not be overstated for this tends to obscure the fact that Bolzano's views on deductive rationality and *a priori* knowledge are not exhausted by his discussion of these notions and indeed lie somewhere else, as we will see in Chapters 7 and 8. Likewise one should not downplay the potential of the substitutional method as a whole. For one thing, combined with other aspects of his theory, the substitutional resources Bolzano puts forward reveal a rich understanding of generality, conditional statements and multiple quantification. More generally, they account for the kind of structural features and semantic regularities that underlie an account of the latter. In this respect, they can be seen to have

been one of the most important development of logic in the nineteenth century.

It is easy to forget that what philosophers might have understood at any given historical juncture when they spoke of logical form (of a proposition, of an inference) varies greatly. Following the publication of Kant's *Critique of pure reason*, the views that had become standard in Germany rested on the idea that "logic concerns not the matter but the form of thought [...] for it abstracts from all the differences in the objects" (Jakob 1791, §62), that "the form of a thought [...] is that which is produced through thinking" (Hoffbauer 1794).[1] As Kant put it:

> General logic abstracts, as we know, from all content of cognition, that is, from all relation of the latter to the object and considers only the logical form of though in general.
>
> (Kant 1781, B79)

> Logic is a rational science not of the matter but of the mere form, an *a priori* science of the necessary rules of thought, but not with regards to particular object, but of all objects in general.
>
> (Kant 1800, A9)

When it comes to dealing with the notion of form, one interesting aspect of Bolzano's theory is that he keeps the idea that logic concerns structure but changes the conception of what structure amounts to. As Bolzano saw it, the idea that logic is concerned with the "form of *thought*" is wrong because it involves the idea that logic is somehow concerned with thoughts. Conceptions that resort to the idea of an investigation of thought when it comes to determining the structure of concepts and propositions rest typically on a psychologistic understanding of logic. Authors who defend this view suppose that logic is primarily concerned with the description of certain aspects of specific types of cognitive processes or their products. While Bolzano thought that philosophers should not neglect the investigation of cognitive processes – a fact to which the detailed investigations of the third volume of the *Theory of Science* attests – he also believed that the latter is irrelevant to logic. As Bolzano puts it, logic should be seen to pertain "to the rules [...] that hold not for thought [...] but for truths in general" (1837, §12, 47), that is, for *Sätze an sich*. On the other hand, many of the conceptions that were popular at the time took for granted that considerations of the "matter" of particular judgements – those aspects that are not strictly linked to their logical structure – should be seen as irrelevant

to logic. As Bolzano conceived of it, logic is not concerned exclusively with the definition of properties propositions and inferences have by virtue of their form, but with a general study of the properties propositions and ideas have, which includes also the determination of such notions as truth, meaning, apriority and necessity, for instance. On this, he agrees with Kant: any conception of logic on which *only* formal – as opposed to semantic and modal – relations are taken into consideration is bound to be too narrow.[2] As Bolzano writes:

> By contrast [to the distinction between affirmative and negative propositions] one declares the distinction between *a priori* and *a posteriori* propositions to be a *material* one since it can [...] only be decided by considering its entire content. But were this to be really the opinion of our logicians [...] then I would have to declare the restriction of the domain of logic to the mere form to be arbitrary and detrimental. For [...] is not the distinction between *a priori* and *a posteriori* propositions important enough that almost every logician [...] brings it up?
>
> (1837, §12, 51)

In Bolzano's theory, most properties of propositions and ideas come to the fore through the application of a subsitutional procedure. The substitutional procedure consists in considering as "variable" determinate components in a proposition so as to establish how this "affects its truth or falsity", and then derive from the latter different types of semantic regularities. A proposition may, for instance, be such that the substitution of some or all of its non-logical components leaves its truth-value unchanged, or the members of two sets of propositions S and T may share components such that any substitution of the latter that makes S true also makes T true. In the first case, Bolzano calls the propositions "universally *gültig*" – more on this directly. In the second case, Bolzano says that T is "*ableitbar*" from S (see Chapter 6).

Since on Bolzano's account substitution occurs at the level of propositions, not linguistic expressions, talk of propositional form is cashed out in terms of sets of propositions.[3] The primary bearers of logical properties are not individual propositions but "species" thereof:

> The clearest definitions say hardly more than that we consider the form of propositions and ideas when we keep an eye only on what they have in common with many others, that is, when we speak of entire *species* or *genera* of the latter. [...] one calls a species or

> genus of proposition *formal* if in order to determine it one only needs to *specify* certain *parts* that appear in these ideas or propositions while the rest of the parts which one calls the *stuff* or *matter* remain arbitrary.
>
> (1837, §12, 51)

> Logic – at least as far as its principles are concerned, it can be otherwise with examples – should never consider an individual, fully fixed proposition, i.e. one such that its subject, predicate and copula are already fully fixed, but a whole genus (*Gattung*) of propositions, i.e. at once, all propositions such that though some of their components are fixed, the others can be made out in this way or other.
>
> (1837, §12, 48)

Logical notions are defined for entire sets of propositions: when Bolzano says that a given proposition has a certain property with respect to determinate components, he means that the associated set of propositions contains members all sharing some fixed vocabulary. If we follow Bolzano, sets of propositions sharing all ideas but those identified as arbitrarily exchangeable are represented by linguistic expressions containing one or more variables:

> When I speak of ideas, propositions and inferences falling under this or that form, I understand a determinate connection of words or signs in general through which a certain species of ideas, propositions or inferences can be represented.
>
> (1837, §81, 393)

If we arbitrarily vary 'Caius' in the proposition:

> The man Caius has mortality

that is, if we replace 'Caius' by other ideas, whatever they may be, for instance 'Titus', 'Semprionus', 'triangle' and so on, we obtain a set of variants of the initial proposition. This set contains the propositions:

> The man Titus has mortality
> The man Semprionus has mortality
> The man triangle has mortality

and so on. All variants of the proposition 'The man Caius has mortality' with respect to 'Caius' "fall under" (Bolzano's term) or have the same propositional form, which can in this case be expressed by:

X who has humanity, has mortality

In this example, 'X who has humanity, has mortality' does not stand for an individual proposition but for a propositional form. 'X who has humanity, has mortality' is not itself a complete sentence. It is only by substituting a name for 'X' that one gets a substitution instance, that is, a sentence that expresses a proposition.

When applying substitution, Bolzano is looking for propositional forms that present certain features. Take universal *Gültigkeit*[4] (cf. 1837, §147, 77ff.). In order to have the property of being universally *gültig* a propositional form must be such that all objectual interpretations, that is, all propositions expressed by substitution instances whose subject is denotative – more on the objectuality condition in what follows – are true. More specifically:

> The propositional form S is universally *gültig* with respect to the variables i, j, ... if and only if every objectual substitution instance of S is true.
>
> The propositional form S is universally *ungültig* with respect to the variables i, j, ... if and only if every objectual substitution instance of S is false.

Following this definition, Bolzano would say that the propositional form:

The man X is mortal

is universally *gültig* since the set of its objectual substitution instances that would be false is empty. Likewise:

The man X is immortal

is universally *ungültig* since the set of true substitution instances of the latter is empty.

Universal Gültigkeit and *Ungültigkeit* are comparatively rare and propositional forms typically only have a certain "degree" of *Gültigkeit*.

As Bolzano sees it, the degree of *Gültigkeit* – that is, the probability – of a propositional form relative to a given set of variables is the ratio of the number of true substitution instances to the number of all objectual substitution instances; it is a fraction between 0 and 1. Bolzano writes:

> If we introduce a constraint and for instance establish that ideas that are equivalent to i, j, . . . can never be put in the place of the latter, then the set of ideas that may still to be chosen and therefore the set of propositions that can be generated is considerably diminished [. . .] if, for instance, we take only the idea '8' to be variable in the proposition: "that the ball designated by the number 8 will find itself among the those that will be drawn in the next lottery", but demand that this idea not be exchanged with others that are equivalent to it and in general that only ideas that form an objectual proposition be exchanged with it, then if the draw contains 90 balls, then number of proposition that can be generated in this way will be =90. For in the given proposition, only the numbers 1 to 90 can be put in the place of the 8 since any other idea that would be put in its place would transform the subject-idea (Unterlage) of the proposition or the idea "the ball designated by the number X" objectless. [. . .] It is useful for us to know how many different propositions are produced when one works with the adopted constraint and in particular in which relation the set of true propositions that thus appear stands to the total set. This relation indeed determines the degree of probability of a given sentence in certain circumstances [. . .] I wish to call it the *Gültigkeit* of the proposition [. . .] The degree of *Gültigkeit* of a proposition will be represented by a fraction whose numerator stands to the denominator as the former set stands to the latter. So, for instance, if we draw 5 balls, the degree of *Gültigkeit* of the previous propositions is 5/90 = 1/18, for there are only 5 propositions among the total 90 that are produced that are true.
>
> (1837, §147, 80, 81)

When it comes to determining the degree of *Gültigkeit* of a propositional form, another condition beside the objectuality and the equiveridicality of the substitution instances must be fulfilled. Logically equivalent terms must be excluded from the range of substitution. Take 'The die will roll a 4 or the die will roll a 5.' The degree of *Gültigkeit* of this proposition corresponds to the ratio of objectual and true substitution instances of 'The die will roll an X' to the number of all objectual substitution instances of the latter, that is, the number of substitution instances that

will turn out to be true (in this case 2). If we do not exclude equivalent ideas, the number of all possible substitution instances of 'The die will roll an X' is in principle infinite so that it is impossible to determine its degree of *Gültigkeit*. If we exclude equivalent ideas, however, the number of all possible substitution instances that fulfil the condition of objectuality is 6: 'The die will roll a 1', 'The die will roll a 2', ..., 'The die will roll a 6.' Of the six objectual substitution instances that can be generated, two will turn out to be true, which gives us a degree of *Gültigket* of $2/6 = 1/3$.

The objectuality constraint stands out and justifiably so. In its absence, it would be impossible to establish properties defined on the basis of the substitutional method. This is a consequence of Bolzano's conception of truth. According to Bolzano, propositions that violate the objectuality constraint are false as a matter of triviality.[5] The point might be contentious, but it is not insignificant for Bolzano's logic. Without the stipulation that objectless propositions be excluded from the range of acceptable substitution instances of the relevant form, it would be possible in virtually every case, to generate false variants of a true proposition. Take for instance 'X which is square is square'. This is just the kind of propositional form Bolzano would want to see as universally *gültig*. But if the interpretation of 'X' is to be arbitrary, 'A circle that is square is square' ought to be ranked among its substitution instances. Since there is no such thing as a round square, the latter however is trivially false and without further stipulation 'X which is square, is square' turns out not to be universally *gültig*. The role of the objectuality constraint consists in insuring that variants that are trivially false be rejected from the range of acceptable substitution instances of a propositional form, thus reconciling Bolzano's views on truth with his method for the definition of logical relations.

Some authors think that the objectuality constraint should be seen to have at least one other important role to fulfil in Bolzano's theory. Morscher (1997a, 142f.), for instance, argues that the objectuality constraint is a device that allows Bolzano to introduce a "category marker" in propositions. In the proposition:

The number 7 is prime

for instance, 'number' would be a category marker whose function would consist in restricting the domain of variation. Since the subject 'The number 7' is objectual only if the term substituted for '7' belongs to the category of objects that fall under 'number', the

range of variation is restricted to objects that belong to that category. If we follow Morscher, by introducing such a constraint, Bolzano is presumably in a position to avoid systematically many "defective" grammatical constructions. A category marker would prevent that the arbitrary interpretation of 'The number X is prime' yield constructions of the type of:

The number Gottlob is prime

There is at least one good reason to doubt that Bolzano would have endorsed Morscher's interpretation. If we assume with Morscher that Bolzano was concerned with "category mistakes", it is difficult to explain why Bolzano did not seek a more general solution. The idea that a proposition is objectual only if its subject is objectual implies that all putative category markers have to be found in the subject so that only defective constructions that occur in the subject can be avoided. But, as Morscher himself notes, nothing in Bolzano prevents other types of "category mistakes", for instance, those that occur in the predicate:

7 is a virtuous number

or those occurring, not in a sub-propositional part but at the level of the proposition itself:

Humanity is red

The fact that Bolzano neither raises nor deals anywhere explicitly with the problem of "category mistakes" seems to indicate that he adopted lax views on the type of restrictions that are to be imposed on grammatical constructions. Many presumably defective constructions are included among the examples he provides. He systematically emphasises the idea that the substitution has to be arbitrary and that in 'The man Caius is mortal', for instance, we may not only substitute 'Caius' for 'Semprionus' and 'Titus' that all belong to the category of men, but also for 'rose' and 'triangle' that do not (see 1837, §147, 78). While some think that one ought to consider expressions such as 'The number Gottlob' to be either agrammatical or grammatical but devoid of a sense, Bolzano assumed that though they are not referential, such expressions are nonetheless meaningful (cf. 1837, §70, 315ff.). As Bolzano puts it, an "imaginary idea" contains components that represent properties that "contradict" one other. While such ideas do not have an object, they

nonetheless have a meaning. By contrast, Bolzano explains, there are constructions that are entirely devoid of meaning such as 'Abracadabra' (cf. 1837, §70, 317). On Bolzano's account, complex ideas that contain concepts that represent "contradictory" properties (such as that of being round and square at the same time) just like those that represent objects that do not exist in reality (such as golden mountains and winged horses) are grammatical: they are constructed according to the rules he sets out concerning the formation of ideas and propositions (see Chapter 3). They are also equipped with meaning. If this were not the case, it would be impossible to determine that they indeed involve concepts that represent properties excluding each other or that they designate objects that do not exist (cf. 1837, §196, 328ff.).

While the notion of degree of *Gültigkeit* plays a noteworthy role in Bolzano's *Theory of Science* – it provides the basis for Bolzano's account of probability – the notion of universal *Gültigkeit* is usually taken to have little comparative import. When it is not entirely ignored, it is introduced perfunctorily or as a relatively inconsequential instance of Bolzano's application of the substitutional method. The idea that the notion of universal *Gültigkeit* is comparatively unimportant is, however, somewhat misleading. Universal *Gültigkeit* plays a substantial if underappreciated role in Bolzano's theory: it provides him with an account of universal quantification based on the substitutional procedure. This is not a difficult point to make, and one ought to be surprised that universal *Gültigkeit* was never presented as such. In Chapter 3, we saw that Bolzano rejected the traditional Aristotelian account of universal quantification. Bolzano disqualifies use of quantificational words of the sort of 'each' and 'every' and 'all' to indicate universality. What most commentators seem to assume is that Bolzano did not seek to deal further with the topic.[6] Universal *Gültigkeit* however is at very first glance the candidate that should be, if any notion should be called upon for the task of providing an analysis of generality in Bolzano's theory, first in rank.[7] Universal *Gültigkeit* and *Ungültigkeit* are ascribed to propositional forms on the basis of their being equiveridical for any objectual interpretation of the variable(s) they contain. Whenever it is the case that all objects that are As are also Bs, these two conditions are fulfilled by all members of the relevant substitution set: if all As are Bs, any propositional form establishing a connection between one of the objects to which 'A' refers and the property b will be both objectual and equiveridical. In other words, although this in principle requires that a language contain a considerable number of awkward expressions for abstracta, examples of universally *gültig* propositions

have a determinate form in Bolzano. The following belong to Bolzano's preferred examples:

> 'The man Caius has mortality' is universally *gültig* with respect to 'Caius'.
>
> (1837, §147, 78)

> 'A morally depraved man does not deserve respect' is universally *gültig* with respect to 'man'.
>
> (1837, §148, 83)

> 'A morally depraved man deserves perpetual happiness' is universally *ungültig* with respect to 'man'.
>
> (1837, §148, 83)

> 'This triangle has three sides' is universally *gültig* with respect to 'this'.
>
> (1837, §147, 81)

> 'In this triangle, the sum of the angles is equal to two rights' is universally *gültig* with respect to 'this'.
>
> (1837, §197, note)

If we generalize on these examples, though not every proposition of this form needs to be universally *gültig* – 'X who is a bachelor, is a philosopher', for instance, is not universally *gültig* – all (non relational) universally *gültig* propositions can be expressed in a statement ascribing the property of being universally *gültig* to a sentence containing a variable:

> 'X which has a, has b' is universally *gültig* with respect to X.

If we follow what Bolzano says in the *Theory of Science*, a statement ascribing universal *Gültigkeit* to a given propositional form, say 'X who is a man is mortal' if it is true, is true because every substitution instances of 'X who is a man is mortal' that also has objectuality is true:

> When, for instance, in the proposition "The man Caius is mortal" we consider the idea 'Caius' as arbitrarily variable and thus put in its place whatever other idea, e.g. Semprionus, Titus, rose, triangle, etc.: it occurs that all new propositions that appear are universally true, so long as they actually have objectuality.

One remarkable aspect of Bolzano's theory is that despite the fact that his definition of universal *Gültigkeit* offers a fairly clear description of substitutional quantification – to say that a propositional form is universally *gültig* is to say that all its substitution instances are true – since he deals not primarily with sentences and words but with their meaning, that is, ideas and propositions in themselves, there is in principle a name for every idea – and since there is (at least) one idea for every object, a "name" for every object.[8] For this reason, though Bolzano's approach to quantification is substitutional, he is not liable to the reproach according to which his interpretation of the universal quantifier cannot account for every state of the world: the resources he assumes are at his disposal are in principle as rich as necessary to provide a complete description of the domain the theory is about.

As is clear in retrospect when we consider that this is the one capital insight of Frege's *Begriffsschrift*, any perspicuous account of universality rests on a strictly regimented syntax. Given the unusual character of the means Bolzano had at his disposal, it is crucial to show that the logical syntax he put forward and which is captured in the semi-formal language whose rules we have described in Chapter 3 is apt to support such an account. For instance, it would be incorrect to assume that Bolzano's resources are too limited to express universally quantified conditional statements and relational statements or, in general, that Bolzano's account of quantification cannot be generalised. Bolzano did not himself grasp the power of his theory to its whole extent. Nonetheless, Bolzano's treatment of generality can not only be extended to conditional statements and statements of relations: combined with other aspects of his logic, it also accommodates an analysis of statements that involve multiple quantifiers.

For every statement of universal *Gültigkeit* – to the exclusion of those that express a relation; more on this in what follows – there is in Bolzano a corresponding equivalent statement of the form 'if…, then…' – a "statement of *Ableitbarkeit*" – that also involves a quantifier that binds one variable.[9] In Bolzano, if it is the case that all As are Bs, that is, if it is the case that a given proposition of the form 'X which has a, has b' is true, then the concept of an X which is A is subordinated to the concept of an X which is B. If 'X who is a philosopher, is a man' is true, then the concept of a philosopher is subordinated to the concept of a man (1837, §196, 330). On the other hand, Bolzano claims that:

> If two propositions 'X has a', and 'X has b' have the same subject, which is also the only variable idea in them, then the second

> proposition is *ableitbar* from the first if the idea B (the concretum belonging to b) includes the idea A[...]. For if idea B includes A, hence if every A is also a B, then every idea whose substitution for X makes the propositions 'X has a' true, also makes the proposition 'X has b' true.
>
> (*WL* §155.36, 127)

It follows that, whenever all As are Bs – and therefore whenever 'X which has a, has b' is universally *gültig*:

> 'X has b' is *ableitbar* from 'X has a' with respect to 'X'

is also true, that is, for all X, if X has a, then X has b.

One noticeable difference between the definition of the conditions under which a universally quantified conditional statement is true in Bolzano and the conception that was initially put forward by Frege and which remained paradigmatic for the greater part of the twentieth century is the fact that while the latter is defined extensionally, the former is defined intensionally on the basis of Bolzano's notion of *Ableitbarkeit*. This allows Bolzano to introduce a comparatively rich notion of generality. For instance, if we follow Bolzano's interpretation, a conditional statement of the form:

> 'X has b' is *ableitbar* from 'X has a' with respect to 'X'

is true only if there is at least one substitution of X that makes both the antecedent and the consequent true, that is, only if 'X has a' and 'X has b' are what Bolzano terms "compatible" (more on this in Chapter 6). As Bolzano sees it, if there is *not* at least one interpretation of 'X' that makes both the antecedent and the consequent true, then the relation of *Ableitbarkeit* does not hold. With respect to Bolzano's theory of quantification, one remarkable consequence of this constraint is that by contrast with the standard (Fregean) conception of generality, it insures the validity of the principle of subalternation. Bolzano's logic is devised so as to accommodate the classical Aristotelian intuition according to which the truth of a universal affirmative implies the truth of the corresponding particular affirmative. If there is at least one true substitution instance of 'X which has a, has b', then there are indeed some As that are Bs. This means that Bolzano is in a position to infer, for instance, from 'All As are Bs' and 'All A are Cs' that some Bs are Cs – since either of the two premises imply that there be at least one A. Although

Bolzano did not say so explicitly this turns out to be significant, for instance, in his account of empirical knowledge since it excludes that there be true general laws that refer to non-existing objects, for example, it excludes that 'If x is a raven, then x is black' be true even if there were no ravens.

Let us introduce the following notation. I use '[' and ']' to form designations for propositions and ideas, for example, '[A]' reads as 'the idea (of an) A' and '[A, which has b]' as 'the idea of an A which has (the property) b'. I introduce the symbols '{' and '}' to form designation for collections of objects. '{A}' reads 'the collection of As'. Note here that the brackets may contain names for objects as well as names for ideas and propositions. '$\{A, B\}_R$' reads 'the objects A and B form a collection of type R', where R determines the type of the collection (see Chapter 9); '[{A, which has b}]' reads 'the idea of the collection of As which have b'; '[{A, A_1}]' reads the idea of a collection of two objects A and A_1; and finally '{[S], [T]}' reads as 'the collection composed of the proposition that S and the proposition that T'. The following summarises Bolzano's interpretation of quantifiers and logical connectives:

'There is an A'	'[A] is objectual' (1837, §172, 215)
'Some As are B'	'[A which is B] is objectual' (1837, §189, 266)
'It is not the case that S' (negation)	'[S] has lack of truth' (1837, §141, 63)
R (A, B)	$\{A, B\}_R$ (1837, §80.3, 381)
'Either S or T' (inclusive disjunction)	{[S], [T]} has lack of (the property) of being a collection of falsehoods (1837, §166.3, 205[10])
'Either S or T' (exclusive disjunction)	'{[S], [T]} contains exactly one truth (1837, §166, 204)
'If S, then T' (conditional)	'[T] has the property of being *ableitbar* from [S]' (1837, §155,113ff.)
'S and T' (conjunction)	'{[S], [T]} has (the property) of being a collection of true propositions (1837, §192, 300, 301).

Take the following statement of relation:

John kisses every tall man

Following the classical Aristotelian analysis, 'John' would be the subject, 'is' the copula, and 'kissing every tall man' the putative predicate. But, as it turns out, the copula in this proposition introduces a special case of predication for which traditional syllogistic cannot account. As Bolzano puts it, the property of kissing tall men is not an "internal", but an "external" property of John, that is, a "relation" (*Verhältnis*) in which he stands to all men who are tall (cf. 1837, §80). What this means for Bolzano is the following:

> It is easy to see that every object will have its own properties. A whole that has several objects A, B, C, D . . . as parts is, as such, a special object which is essentially different from its parts. It is obvious that each whole will have certain properties, which its parts will not have. If I am not mistaken, these properties are what we call *relations between those parts*. In particular, this holds when we think of the objects A, B, C, D . . . on the one hand and the property x of the whole on the other as variable, i.e. if we think that other objects A′, B′, C′, D′ . . . which are of the same kind as A, B, C, D . . . have a property that is, although not the same, yet of the same kind as x.
>
> (1837, §80, 381)

> [. . .] According to this definition, a relation x that holds between the objects A, B, C, D . . . is a property that actually belongs only to the whole composed of A, B, C, D . . . as such. In spite of this we can at least say of any individual part, e.g. A, that it has the property of 'forming a whole with B, C, D, . . ., which has the property x.
>
> (1837, §80, 382)

If we apply the general claim Bolzano makes above to the analysis of propositions such as 'John kisses every tall man,' the latter should be seen as a statement that ascribes a property to a collection – in this case an empirical collection. On Bolzano's account, the difference between John's kissing every tall man and every tall man kissing John is determined by the type of the collection in question. Let us call the type of the collection in question "kissing" (see Chapter 9 for a detailed account of Bolzano's views on collections). Among other things, the type of a collection determines the structure, that is, the order in which the members of a collection stand to one another. Unless the type of a collection is such that the order of the elements is arbitrary, the order in which the terms appear in a collective expression is relevant to determining the structure of the latter. Given our understanding of the term, a collection of the type "kissing" is a collection whose members

can only be agents of some kinds; the relation it instantiates is not transitive – but whether it is reflexive and/or symmetric again depends on what "kissing" is supposed to mean in the context. "Kissing" is not a property of the collection composed of John on the one side and all things that are both tall and men taken collectively on the other. Rather, it is a property of any whole composed by John and *any one* part of the collection of all things that are both tall and men. To say that John kisses every tall men, in other terms is to say that:

> [{John, X who is a men, who is tall}$_{\text{kissing}}$] is universally *gültig* with respect to X

Combined with other analytic tools that are to be found throughout the *Theory of Science*, Bolzano's notion of collection not only allows him to provide a systematic, if unfamiliar, account of statements of relations but also accommodates a systematic treatment of quantified statements in general. This holds for universally quantified statements as well as for existentially quantified ones, for instance:

> John kisses someone

The latter, according to the analysis of existential statements we introduced above, would be expressed in the following manner:

> [{John, X}$_{\text{kissing}}$] is objectual at the X position

Bolzano can also in principle account for multiply quantified statements such as, 'Everybody loves someone':

> [[{X, Y}$_{\text{loves}}$] is objectual at position Y] is universally *gültig* with respect to X.

Or 'There is someone that everybody hates'

> [[{X, Y}$_{\text{hates}}$] is universally *gültig* with respect ot X], is objectual at the Y position.

Similarly:

> Everybody loves someone who hates someone:
>
> [[{[{X,Y}$_{\text{loves}}$], [{Y, Z}$_{\text{hates}}$]} is a collection of truths] is objectual at the Z position] is objectual at the Y position] is universally *gültig* with respect to X.

Which is equivalent to saying that:

> For every X, there is a Y and there is a Z such that X loves Y and Y hates Z.

Bolzano is also in a position to provide an analysis of multiply quantified conditional statements that involve a relational predicate, such as:

$\forall x\ [N(x) \rightarrow \exists y\ (N(y)\ \&\ \text{Precedes}\ (x,\ y))]$

Namely:

> [[{[Y has N], $[\{X,Y\}_{\text{precedes}}]\}$ is a collection of truths] is objectual at position Y] is *ableitbar* from [X has N] with respect to X.

Compared with contemporary accounts of relational statements, quantificational statements and statements involving propositional connectives, it is hard not to realise that Bolzano's treatment is cluttered with semantic and ontological commitments: the fact that conjunction is about pairs of propositions or that 'If X is a man, then X is mortal' is about a somewhat awkward intensional relation between two propositions, for instance, does not fare well when compared to the elegant and uniform treatment of first order predicate logic. This set aside, there is no ground however to believe that a Bolzanian analysis does any worse than modern predicate logic when it comes to representing various quantificational claims. In this respect, and though he might not have been aware of it himself, Bolzano was the first, well before Frege, to provide a syntax whose expressive powers are rich enough to allow for an analysis of generality and multiple quantification.

5
Analyticity

Bolzano's theory of analyticity is a favoured topic in the literature – cf. Bar-Hillel 1950; Etchemendy 1988; Künne 2006; Lapointe 2000, 2008; Morscher 2003b; Neeman 1970; Proust 1981, 1989; Textor 2001. This should be no surprise, and there are many reasons why Bolzano's theory should be seen to be significant and to deserve a thorough treatment. For one thing, it rests on a substitutional procedure that had until the end of the nineteenth century no equivalent. Besides, it anticipated in great detail the definition of logical truth we find in the works of Quine. Bolzano's theory of analyticity in fact offered the first alternative to the Kantian idea that analyticity is defined by the inclusion of the predicate in the subject. The connection to Kant is not unimportant. At least until 1812, Bolzano seemed to have thought that he was retaining the letter of the Kantian definition:

> It remains one of Kant's great merits to have for the first time drawn our attention to the important difference between the analytic part of our knowledge and the synthetic part, though we cannot accept nor take into account everything our philosopher otherwise says about the intrinsic nature of our synthetic judgements. Assuredly the truth of analytic judgements rests on an entirely different ground (*Grund*) than that of synthetic judgements. If they do deserve the name of genuine judgements (what I grant them not without reserve), then they all rest in this unique universal proposition which is expressed by the following formula: '(A cum B) is a kind of A' [...] and we can say that the principle of contradiction is the universal source of all analytic judgements.
>
> (Bolzano 1810, *Appendix*, §1)

> The division of judgements into analytic and synthetic is however very important. a. An analytic judgement is such that its predicate is a component of the concept of the subject. For instance, the judgement: "A pocket mirror is a mirror" – or "A pocket mirror is something one can carry in their pocket". b. A synthetic judgement is any other judgement whose predicate is not a component in the subject, such as for instance: "The Sun warms".
>
> (Bolzano 1812, §30)

In the *Theory of Science* however Bolzano came to think of Kant's conception as inadequate for at least three reasons. First, as Bolzano sees it, the notion of analyticity should not be understood as a property some propositions have by virtue of the analysis of their concepts. As Bolzano puts it:

> In general, it seems to me that none of these definitions stresses sufficiently what makes this type of proposition *important*. Their importance, in my opinion, consists in the fact that their truth or falsehood does not depend on the ideas of which they are composed but rather on the fact that these propositions remain true or false whatever the modifications one endeavours some of their ideas, assuming that the objectuality of the proposition is not destroyed.
>
> (1837, §148, 88)

Whether a proposition is analytic is to be determined through the same substitutional procedure that is at play in the definition of the notion of universal *Gültigkeit* (see previous chapter). It consists in considering sets of propositions that differ from one another with respect to some determinate component(s) and to establish a certain kind of semantic regularity. In the case of analyticity, the property in question is the equiveridicality of the objectual members of a set with respect to a fixed vocabulary.

Secondly, it is not only that Bolzano resorts to a different kind of procedure from his predecessors when it comes to defining analyticity but, as we have seen in some detail in Chapter 2, that he came to reject altogether the conception of conceptual analysis on which the Kantian notion of analyticity rests. Kant defines analyticity in terms of "inclusion" of the predicate in the subject. In the *Theory of Science*, Bolzano argued that this notion is improperly defined in Kant – it is a "metaphor" (*bildliche Redensarte*; cf. 1827, §148, 87). As we have seen in

Chapter 3, when Bolzano spoke of concepts being part of the content of other concepts, what he had in mind was substantially different from Kant. Bolzano assumed that to say of a concept or a proposition that it is complex (*zusammengesetzt*) amounts to saying that it is a structured whole that results from the connection of concepts according to determinate compositional rules. Consequently, to be part of a concept (or a proposition) consists in being connected in a certain way with the other concepts that make out its content.

Thirdly, given Bolzano's views on what it means for a concept to be included in another concept, Kant's definition turned out to be at once too broad and too narrow. It is too narrow to the extent that it cannot be applied to constructions that rest on syntactic resources that are more sophisticated than those Kant had at his disposal. Since Kant adopts the view that all analytic propositions are of the form 'All As are Bs' and that only propositions of this form are analytic, his notion of analyticity is underdetermined: it does not allow us to establish whether hypotheticals, disjunctions, conjuntions and so on can be said to be analytic. Likewise, the Kantian criterion is too broad since it turns out to include as analytic propositions which Kant himself would not have considered to be such. In the *New Anti-Kant*, Bolzano spells out the problem:

> Note that while we take the division of judgements in analytic and synthetic to be one of the most opportune and influential discoveries that were ever made in the fields of philosophical research, it seems to us that it was not grasped by Kant with a satisfactory degree of clarity. For though we may want to turn a blind eye on the fact that when he uses the expressions: "explicitly" and "outside of the concept", he contradicts a well known logical rule and admits metaphors among his definition, we cannot help blaming him for the fact that his definition of analyticity is too broad. Propositions that nobody would rank among analytic propositions, for instance, among others: "The father of Alexander, King of Macedonia, was king of Macedonia"; "A triangle isomorphic to an isosceles triangle is itself isosceles". On might want to prevent this mistake by stating that analytic judgements are those whose predicate are essential components of the subject. But this definition would have another kind of fault, namely that of being too narrow. It would of course suit analytic judgements of the form "A which is B is B" but it would not suit judgements of the form [...] "Every object is either B or not B" which is manifestly analytic. The Kantian definition of analytic judgements [...] cannot satisfy us, all the more that it completely ignores what is

> essential, the distinction which is most important to the philosopher who seeks to establish this division, namely that the truth or falsehood of some propositions does not depend on the particular ideas of which they are made, but rather on the fact that they remain true or false whatever may be the changes we endeavour with the one or the other of their ideas as long as the proposition itself remains objectual [...] i.e. that its subject-idea is not empty.
>
> (Příhonský 1850, 35, 36, see also 1837, §148, 87, 88;)

It might be tempting to write off Bolzano's counterexamples. The use of relative clauses is in general eliminable, and it could be argued that Bolzano is wrong to assume that Kant would have been bound to declare 'The father of Alexander, who was King of Macedonia, was King of Macedonia' analytic. Kant could have replied that the sentence in question expresses not one, but two propositions, namely 'Alexander was King of Macedonia' and 'The father of Alexander was King of Macedonia' and that neither is analytic in his sense. But this would be beside Bolzano's point. What makes 'The father of Alexander, who was King of Macedonia, was King of Macedonia', an interesting example in Bolzano's eye is the fact that it presents just the kind of syntactic features that show the inadequacy of Kant's conception of conceptual inclusion. In this respect, Bolzano's argument anticipates one important intuition that will be emphasised by a number of philosophers of the first half of the twentieth century: a good definition of analyticity must allow us to determine not only whether a grammatical construction of the form subject–predicate is analytic, as Kant has it, but whether *any* construction is analytic or not. This includes hypotheticals, disjunctions, conjunctions, but also, in general, any proposition such as the one Bolzano has in mind above that present a syntactic complexity that is foreign to traditional (Aristotelian) logic. More generally, we usually assume that a good definition of analyticity should not presuppose any given syntax. The reason for this is that analyticity is not a syntactic notion but a semantic one, that is, a relation pertaining to the truth of propositions and not merely to their form or structure. The merit of Bolzano's account is that it is not tied to any "syntactic" conception of "logical form". Bolzano's definition was the first to be grounded in a clearly semantic criterion.

Bolzano's theory of analyticity is twofold. On the one hand, Bolzano presents a broad notion of analyticity: a propositional form is analytic if all its substitution instances have the same truth-value:

> If however there is even only one single idea in a proposition that may be exchanged arbitrarily without altering its truth or falsity; i.e. if all propositions which turn up through the exchange of this idea with any other are all true or all false provided only that they have objectuality, then this property of the proposition is remarkable enough to be distinguished from all others for which this is not the case.
>
> (1837, §148, 83)

On the other hand, Bolzano presents a narrow notion of "logical" analyticity: a propositional form is logically analytic in Bolzano's sense if and only if its logical vocabulary is fixed and all its substitution instances have the same truth-value.

Let us consider the wider notion first. I take it that what Bolzano means in the previous quotation is the following:

> The propositional form S is analytically true in the broader sense with respect to i, j, . . . if and only if every objectual substitution instance of S with respect to i, j, . . . is true.
>
> The propositional form S is analytically false in the broader sense with respect to i, j, . . . if and only if every objectual substitution instance of S with respect to i, j, . . . is false.

There are four important consequences of Bolzano's definition of analyticity. Firstly, in Bolzano, propositions may be analytically true as well as analytically false. When Bolzano claims for instance that the proposition:

> A depraved man does not deserve respect

is analytically true with respect to 'man' (1837, §148, 83, he means, on the one hand, that the arbitrary substitution of 'man' in this proposition produces a set of variants such that, if they are objectual, they are also true. On the other hand, he means that the set of objectual variants of the latter that are false is empty. By contrast, the proposition:

> A depraved man deserves abiding happiness

is analytically false with respect to 'man' since the set of true variants with respect to 'man' is empty.

Secondly, though Bolzano's terminology is often sometimes sloppy, analyticity ought to be seen as a property of sets of propositions that share some fixed components. Individual propositions can be said to be analytic derivatively by virtue of having the relevant form. It is, in other terms, just like *Gültigkeit*, a property of propositional forms. As Bolzano puts it, the proposition:

A virtuous man deserves abiding happiness

is analytically true "with respect to 'man' ". While talk of "analytic propositional form" is unusual, the point is nonetheless important and interesting. Strictly speaking, there is no such thing as a proposition that would be "simply" analytic in Bolzano's sense. Since analyticity is always relative to a given constituent, it would be misleading to say that 'Caius, who is a bachelor, is unmarried' is analytic. While analytic with respect to 'Caius' the proposition is not analytic with respect to 'bachelor', and this is important for Bolzano. Individual propositions are analytic inasmuch as they are substitution instances of a given form. At the linguistic level, this can be cashed out in terms of expressions containing variables: 'A virtuous X deserves abiding happiness' is true under any interpretation of 'X'.[1]

Thirdly, Bolzano's definition of analyticity requires that only *objectual* substitution instances be taken into account. The objectuality constraint is indispensible to Bolzano's definition. Just as is the case with universal *Gültigkeit* (see previous chapter), since the substitution is supposed to be arbitrary, we need only replace the variable in the above example by a term that denotes an object or a genus of objects that does not fall under the concept of depravity in order to produce a substitution instance of 'A virtuous X deserves abiding happiness' that is a falsehood:

A virtuous *triangle* deserves abiding happiness

The latter proposition does not satisfy the objectuality condition since the idea expressed by 'virtuous triangle' is objectless. This is an upshot of Bolzano's views on truth, and the objectuality constraint is meant to harmonise the latter with the substitution procedure.

Finally, universal propositions as conceived traditionally, that is, propositions of the form 'All As are Bs' or 'No A is B' cannot strictly speaking be analytic in Bolzano's sense. One should worry about this inasmuch as it would seem to make it impossible for Bolzano to

accommodate our intuitions concerning some eminently analytic statements, for instance:

No bachelor is married

'No bachelor is married' does not appear to satisfy Bolzano's definition of analyticity. If we take the sentence at face value, it would seem that the proposition it expresses does not contain at least one idea that can be varied arbitrarily without changing its truth-value. But it seems reasonable to ask from whoever claims to have a theory of analyticity that he should be in a position to account for the analyticity of the proposition expressed by a sentence such as 'No bachelor is married' – or to explain in a convincing manner why Quine's famous example is putatively not analytic. Bolzano in fact did not think that a sentence such as 'No Bachelor is married' expresses a proposition of the form 'No As are Bs'. Though English sentences may present themselves in this form the proposition expressed by 'No bachelor is married' on its part cannot have this form. The reason for this is to be found in Bolzano's analysis of expression of the form 'No A'. According to what Bolzano says (1837, §89, 426) – whether he is right is to be left open – the interpretation of an expression of the form:

No A has b

is

All As have not-b

Given Bolzano's analysis of traditional quantificational expressions such as 'All' – which he eliminates – (1837, §57, 247ff.; see intra, 10) and of "concrete" expressions such as 'Bachelor' (1837, §60, 259ff.; see intra, 38), which he construes as expressing ideas of the form 'Something, which has bachelorhood', 'No bachelor is married' amounts to saying that:

'Something, which has bachelorhood, has not-marriedness'

and the latter is analytic in Bolzano's sense, namely with respect to 'something'. Likewise:

All vixen are female foxes

should be understood as expressing a proposition of the form:

> 'X which is a vixen, is a female fox' is analytic with respect to 'X'

The upshot of these analyses, which can be generalised, is that there is in Bolzano, for any proposition analytic in Quine's sense, an equivalent which is also analytic in Bolzano's broader sense.

While it offers a systematic procedure for the identification of certain types of semantic regularities that is worthy of interest, Bolzano's broader definition of analyticity is also problematic. For one thing, contrary to what we usually expect from a definition of analyticity, it fails to account for what it means for a proposition to be true by virtue of the meaning of the terms. Analytic propositions in Bolzano's broader sense are analytic by virtue of what is the case: they are not meant to describe necessary states of affairs, and to do so *a priori*. If, for instance, all children of Mr and Mrs Dion learned to sing before they were 7 years old, then:

> X who is a child of Mr and Mrs Dion learned to sing before the age of 7

is analytically true in Bolzano's broader sense with respect to 'X' since all objectual substitution instances of the corresponding form are also true. But in addition to knowing the meaning of the terms, in order to know that this propositional form is analytically true in Bolzano's sense, one must know whether the objectual substitution instances of this form express true propositions or false ones, and this requires one to have certain cognitions about the world, such as whether 'the *second* child of Adhémar Dion and Thérèse Tanguay' is objectual, and assuming that this is the case, whether it has the property that is ascribed to it in that sentence.

By admitting "analytic" propositions that would be known *a posteriori*, Bolzano seems to have missed an important point about what we usually take to be the nature of analytic knowledge. The notion of analyticity should aim at providing an objective criterion on the basis of which one may account for cognitions whose justification is entirely independent of empirical data. But this is precisely the insight which Bolzano's broader notion of analyticity would seem unable to capture. Note that even if the diagnosis is correct, the news is not entirely disappointing. It is not difficult to see the connection between analyticity and universal *Gültigkeit* (see Chapter 5). While Bolzano does not make the link explicitly, the definition of universal *Gültigkeit*

precedes immediately the definition of analyticity in the *Theory of Science*, and Bolzano presents the latter emphasising again the peculiarity of propositional forms whose substitution instances are all true. Besides, if the notion of analyticity does not point to the same property, it is at least logically equivalent to that of universal *Gültigkeit*. A distinction between the two notions would be warranted if there were conditions universal *Gültigkeit* fulfils that were not fulfilled by analyticity or vice versa. This however is not the case.[2] Just like universal *Gültigkeit*, analyticity is designed to establish the kind of semantic regularities we normally associate with universally quantified statements and thus, in some cases, with empirical generalities. Naturally, if universal *Gültigkeit* and Bolzanian analyticity define the same property, it follows that the latter just like the former is a non-modal notion. For this reason, and quite understandably, Bolzano's theory compares poorly to standard contemporary accounts of analyticity that endeavour to provide an explanation of *a priori* knowledge. The mistake would be to assume that since Bolzano's notion of analyticity does not account for knowledge by virtue of meaning, it is bound to be the case, on the one hand, that the notion of analyticity serves a relatively insignificant purpose and, on the other, that Bolzano does not have an account of *a priori* knowledge. As for the latter, Bolzano has a sophisticated account of knowledge by virtue of the meaning of terms we will present in Chapter 8. As for the former, the idea that the problem Bolzano is endeavouring to solve when he redefines the Kantian notion of analyticity is in fact that of generality has the merit of making sense of an aspect of his theory that turned out to have bridged an important gap in the logic of the time. It is usually assumed that prior to Frege, no satisfactory account of generality and quantification in general was available. As it turns out, Bolzano was first to have had the resources to articulate a systematic one.

This said, diagnosing a failure might also be too hasty. Bolzano makes an important distinction between two notions of analyticity: the "broader" concept we have just examined and the "narrower" concept of *logical* analyticity. The latter turns out to be an eminent precursor to Quine's notion of logical truth, and the connection to Quine is not unimportant. At (1837, §148) Bolzano offers four examples of analytic propositions that, according to him, present a peculiar interest:

A is A
A, which is B, is A
A, which is B, is B
Every object is either B or non B

Bolzano writes:

> The examples of analytic propositions that I introduced here differ from those of [alinea] no 1 in that to judge the analytic nature of the former absolutely no other cognitions than logical ones are necessary since the concepts that form the unchangeable part in these proposition all belong to logic. By contrast in order to judge the truth or falsehood of the propositions of the kind of no 1 entirely different cognitions are required, for here concepts that are alien to logic are incorporated. This difference is of course unsteady, because the domain of the concepts that pertain to logic is not so sharply delineated that there could be no quarrel about it. It can nonetheless sometimes be useful to pay attention to this distinction. Thus, one could call the proposition of the kind of [alinea] no 2 logically analytic or analytic in the narrow sense, those of [alinea] no 1 by contrast analytic in the broader sense.

Bolzano explains the difference between the latter four propositions and analytic propositions in the broader sense by making a series of claims:

(i) In logically analytic propositions, all non logical concepts are considered to be arbitrarily variable, that is, only logical concepts occur in them "essentially".
(ii) We can know that logically analytic propositions are analytic by virtue of mere logical knowledge.
(iii) We may know that logically analytic propositions are true or false on the basis of logical cognitions alone since they contain only logical concepts essentially.[3]
(iv) The distinction between logically analytic and analytic proposition rests on the distinction between logical and non-logical components.

It is worth noting at once with respect to (iv) that Bolzano's definition of logical analyticity anticipates a considerable problem usually associated with Tarski, namely that of determining what makes a term a logical term (cf. Simons 1987). Bolzano does not provide a definition of the latter and explains that it is most likely impossible to delimit the domain of purely logical concepts exclusively and exhaustively. But without such a demarcation between logical and non-logical terms, a definition of analyticity that rests on this demarcation is inapplicable. Here, I will abstract from the problem: on the one hand, the definition

Quine considers does not in this respect present an advantage; on the other hand, the acceptability of Bolzano's position is not ultimately contingent on the resolution of this difficulty.

(i) provides a definition of logical analyticity in Bolzano's sense and can be put in the following terms:

> A propositional form S is logically analytically true if and only if S contains only logical terms essentially and all objectual substitution instances of S are true

Note that when it comes to defining the relationship between their respective conceptions of the broader and the narrower notion of analyticity, Bolzano and Quine proceed in reverse of each other. While Quine attempts to establish the notion of a proposition analytic in the broader sense bringing into play the notion of synonymy, Bolzano defines logical analyticity as a special case of analyticity in the broader sense. In Bolzano, an analytic proposition in the broader sense contains (at least) one non-logical term inessentially, whereas in a logically analytic proposition, *all* non-logical terms are inessential. The apparent advantage of Bolzano's theory would consist in the fact that he defines both the broader and the narrower notion, without recourse to synonymy, on the basis of the same eminently acceptable resources: the notion of truth and the substitutional method (more on this in what follows).

(ii) states that while it is possible to know that a logically analytic propositional form is analytic by virtue of purely logical cognition, this is not the case for propositional forms analytic in the broader sense. What Bolzano means is that there are propositional forms such that we can know without considering any of their particular substitutional instances that all their (objectual) instances are true, and hence that they are analytic. Such are 'A is A' and, assuming the validity of the law of excluded middle, 'Every A is either B or not B.' By contrast, the logical form of analytic propositions in the broader sense does not allow us to judge whether they are analytic or not. For instance, I cannot know that 'X who is a bachelor is unmarried' is analytic by considering only its logical form: 'X which is a, is b', since both analytic and non-analytic propositions have this form. In order to know whether 'X who is a bachelor, is unmarried' is analytic, we must, for each substitution instance, establish whether the proposition it expresses fulfils the

two conditions of objectuality and of equiveridicality, which requires us to have cognitions about determinate individuals.

The claim Bolzano makes in (iii) is particularly significant. To say of a propositional form that it contains only logical components "essentially" is to say that it is a logical truth in the contemporary (Quinean) sense. If Bolzano is right, and if one can putatively know the truth of the relevant substitution instances on the basis of purely logical cognitions, logical analyticity would seem not to be liable to the same modal and epistemological shortcomings as the broader notion of analyticity.[4] The claim is all the more significant that if we follow what Bolzano says at (1837, 148, 84ff.), not only can we find a Bolzanian equivalent for every proposition such as 'No bachelor is married' analytic in Quine's sense, but any proposition analytic according to Quine turns out to be logically analytic in Bolzano's sense. On Bolzano's account, 'X who is a bachelor, is unmarried' would not, in fact, be the complete analysis of 'All bachelors are unmarried'. According to what Bolzano says in the first note to (1837, §148), propositions such as 'All bachelors are unmarried' express logically analytic propositions "in a covert manner". As Bolzano sees it, interpretation – in the technical sense of an *Auslegung*, see Chapter 4 – "requires more than a cursory reading" since a proposition "may be analytic or even logically analytic without its verbal expression immediately showing it" (cf. 1837, §148, 84, 85). Let us call sentences that display this feature "quasi-logically analytic".[5] If Bolzano is right, sentences that are analytic in Quine's sense are quasi-logically analytic in Bolzano's sense: they can be reduced to logically analytic propositions. Since Bolzano considers logically analytic propositions to be knowable *a priori* – his being right would have considerable import.

Bolzano gives the following examples of quasi-analytic propositions:

Every effect has a cause
If A is greater than B, then B is lesser than A
If $P = M \bullet m$, then $M = P/m$

Bolzano explains that the reason why we must consider 'Every effect has a cause', for instance, to be quasi-logically analytic is that, on the one hand, 'effect' means 'what is caused by something else" and that, on the other, 'to have a cause' means the same as 'to be caused by something'. If we generalise the comment, this means that to say of a sentence that it is quasi-logically analytic in Bolzano's sense is to say that while it is not a logical truth, it can be turned into one if we substitute some

of the expressions it contains by other expressions that have the same meaning. The question, therefore, is whether Bolzano's theory presents us with a criterion for sameness of meaning that can do the job where Quine's fails. Some assume that he does not (see Sebestik 1992, 220). Some others assume that he does, and I am inclined to side with the latter, though for different reasons. If I am right about what Bolzano has to say about the way in which terms are defined in deductive systems, namely implicitly, on the basis of the axioms (see Chapter 9), there is room for a notion of synonymy (relative to a given deductive system) that would also lend its support to his analyses of propositions such as 'All bachelors are unmarried' or 'All vixen are female fox.' As Bolzano puts it, two expressions x and y are synonymous if I think and must think with 'x' precisely what I think with 'y' (cf. *WL* §56, 243) In the kind of deductive languages, Bolzano has in mind, to say for instance that I think and must think with 'square' precisely what I think with 'a plane figure with four equal straight sides and four equal angles' is to say that whether or not the first is the abbreviation of the other – I leave the question open – I can substitute one for the other in any of the relevant theorems without affecting the truth value of the latter in that axiomatic system. In that context, and though Bolzano himself is not explicit about it, synonymy would reduce to logical equivalence *relative to an axiom system*. Such an interpretation is not trivial. And while it does not provide an account of what one might want to call the "intuitive" notion of synonymy (e.g., for natural language), it does allow us to make sense of an important aspect of Bolzano's theory. Bolzano's views on analyticity are not as precarious as what they might seem to be. Nonetheless, since the notion of synonymy is relativised, a full account of knowledge by virtue of meaning requires a detailed excursion into Bolzano's views of deductive knowledge as a whole.

6
Ableitbarkeit and *Abfolge*

Bolzano's conception of what it is for a conclusion to follow from a set of premises has been a popular topic in recent scholarship as well as in more general work on logical consequence – (cf. van Benthem 1985, 2003; Etchemendy 1990; Siebel 1996, 2002; Tatzel 2002). Most commentators agree that Bolzano's account fails, and my position is not essentially different.[1] However, Bolzano's views are both more complex and more interesting than one might assume. Bolzano's account of consequence comprises not one but three distinct aspects: his conception of *Ableitbarkeit* (1837, §155), his conception of statistical inference (1837, §161) and his conception of *Abfolge* (1837, §162). On the one hand, the notion of *Ableitbarkeit* delivers a sophisticated and plausible semantic account of truth-preservation, and his generalisation of it to an account of statistical inference is also worthy of consideration. On the other hand, Bolzano's attempt at a definition of *Abfolge* constitutes the basis of an account of *a priori* knowledge (see Chapter 8) that remains underappreciated and of a theory of justification and demonstration (Chapter 7) whose interest has been noticed by some authors, and in some cases even vindicated.

As Bolzano conceived of it, *Ableitbarkeit* is a special case of a more primitive notion he calls "compatibility" (*Verträglichkeit*) (see Bolzano 1837, §155, 113), a notion he defines as follows:

> We already know that any proposition, if we set in the place of given ideas which are considered to be variable in it some other arbitrary ones, can be made sometimes true, sometimes false. But if we compare many propositions A, B, C, D, . . . with one another and consider as arbitrary given ideas i, j, which occur conjointly in them, then arises the question whether there are some ideas that could be set in

> the place of i, j, . . . that are such that these propositions would thus become *all true at the same time*. If the answer is yes: then I wish to call the relation that holds among the propositions A, B, C, D, . . . a relation of compatibility (*Verträglichkeit*) or agreement (*Einstimmung*) [. . .] Thus I call the three following propositions compatible with one another: 'This flower has a red bloom'; 'This flower smells good' and 'This flower belongs to the 12th class of Linneaus' system' [. . .].
>
> (1837, §154, 100)

Bolzano seems to have thought of his account of compatibility as the first natural step in the generalisation of the method of substitution he applies to the notions of *Gültigkeit* (see Chapter 5) and analyticity (see Chapter 6) to inferential notions.[2] Bolzano explains:

> When we claim that some propositions A, B, C, D, . . . M, N, O stand in the relation of compatibility and indeed with respect to the idea i, j, . . . then we claim [. . .] nothing more that there are determinate ideas which in the place of i, j, . . . transform all these propositions into truths. [. . .] But let us think first about the case in which the following relation between the compatible propositions A, B, C, D, . . . M, N, O, . . . : every idea which can be put in the place of the variable i, j, . . . that make a determinate part of the latter, namely A, B, C, D, . . . true also have the property of making another part thereof, namely M, N, O, . . . true [. . .] I therefore call the relation that holds between the propositions A, B, C, D . . . on the one hand, and M, N, O, . . . on the other the relation of *Ableibarkeit*; and I say that the propositions M, N, O, . . . are *ableitbar* from the propositions A, B, C, D . . . with respect to the variable parts i, j, . . . if every set of ideas that can be put in the place of i, j, . . . and that makes all of A, B, C, D, . . . true, also make all of M, N, O, . . . true. For the sake of variety and since it is already so common, I will also at times say that the propositions M, N, O, . . . follow (*folgen*) or can be inferred (*gefolgert*) or concluded from the propositions A, B, C, D, . . .".
>
> (1837, §155, 114)

Let '$A_{ij\ldots}$ (T, T′, T″, ... S, S′, S″ . . .)' stand for 'The propositions T, T′, T″ . . . are *ableitbar* from S, S′, S″ with respect to ideas i, j, . . .':

$A_{ij\ldots}$ (T′ T′, T″, . . .; S, S′, S″, . . .) iff:

(i) i, j, . . . can be varied so as to yield at least one true variant of S, S′, S″, . . . and T, T′, T″, . . . (compatibility)

(ii) whenever S, S′, S″ . . . is true, T, T′, T″, . . . is also true (truth-preservation)

Bolzano's discussion of *Ableitbarkeit* extends over 36 paragraphs in the course of which he states a series of theorems, the most significant of which are asymmetry, transitivity and reflexivity:

- $(A_{ij\ldots}(T, T', T'' \ldots; S, S', S'') \rightarrow A_{ij\ldots}(S, S', S'' \ldots; T, T', T'' \ldots,))$ (asymmetry)
- $(A_{ij\ldots}(T, T', T'' \ldots; S, S', S'') \& A_{ij\ldots}(R, R', R'' \ldots; T, T', T'' \ldots) \rightarrow (A_{ij\ldots}(R, R', R'' \ldots; S, S', S'' \ldots)))$ (transitivity)

In addition (assuming that S, S′, S″ . . ., share at least one variable that make them all true at the same time)[3]:

- $A_{ij\ldots}(S, S', S'' \ldots; S, S', S'')$ (reflexivity)

Take:

Caius is rational

is *ableitbar* with respect to 'Caius', 'man' and 'rational' from

Caius is a man
Men are rational

As Bolzano conceives of it, if 'Caius', 'man' and 'rational' are considered arbitrarily "variable" in the above propositions and if we accordingly replace them by schematic letters, we obtain an argument of the form:

X is Z

is *ableitbar* from

X is Y
Y is Z

If, in turn, 'Caius' is considered to be arbitrarily variable in:

Caius is rational

is *ableitbar* from

Caius is a man

the argument's logical form is:

X is rational

is *ableitbar* from

X is a man

On Bolzano's account, both examples are fully fledged cases of *Ableitbarkeit*: every interpretation of the variable components that yields a true substitution instance of the premise(s) also yields a true substitution instance of the conclusion. It is however customary to draw a distinction between arguments of the former kind and arguments of the latter. Arguments of the former kind are considered "formally valid". Assuming a satisfactory account of logical form, in order to know that the conclusion follows from the premises, one only needs to consider their structure or form. No other kind of knowledge is required. In the latter kind of arguments, however, in order to infer from the premise to the conclusion, one must know more than its form. In order to conclude from 'Caius is a man' that 'Caius is rational', one also needs to understand the signification of 'man' and 'rational': one can only conclude that Caius is rational on the basis of the premises if one also knows, in addition to the fact that Caius is a man, that all men are rational. There is good evidence that Bolzano was aware of some such distinction between arguments that preserve truth and arguments that do so by virtue of their "form":

> [There are] propositions that are *ableitbar* from other propositions by virtue of their sole form (that is, that are *ableitbar* insofar as we consider all the parts that do not belong to their form as variable).
>
> (1837, §29, 141)

Unfortunately, Bolzano's definition of *Ableitbarkeit* does not systematically uphold the distinction. *Ableitbarkeit* applies across the board to all inferences that preserve truth from premises to conclusion (with respect to a given set of ideas) and it does not of itself warrant a distinction between instances of merely materially valid arguments and formally

valid ones. The problem is that without such a guarantee the notion of *Ableitbarkeit* turns out to be flawed. It makes it impossible to extend our knowledge in the way we would expect it. If we know, for instance, that all instances of *modus ponens* are logically valid, we can infer from two propositions whose truth we've recognised, for example:

If Caius is a man, then he is mortal
Caius is a man

A new proposition:

Caius is mortal

whose truth we might not have previously known. Unfortunately, Bolzano's account of *Ableitbarkeit* does not allow one to extend one's knowledge in this way since in order to know that truth is preserved from the premises to the conclusion one has to know, on his account, that the premises are true *and* that the conclusion is true.

As Etchemendy has argued, similar problems arise from Tarski's definition of consequence (Etchemendy 2008, 270). This is not to say that Bolzano's notion of consequence does not present an originality of its own. As it turns out, Bolzano's assumption that *Ableitbarkeit* ought to be conceived as a special case of compatibility has a number of distinctive consequences. In order for a conclusion to be *ableitbar* from a given set of premises, it is not only that the conclusion must be true every time the premises also are but that there must be at least one substitution that makes both the premises and the conclusion true at once, a requirement that is not reflected in classical conceptions of consequence. As a result, Bolzano's program converges with many contemporary attempts at a definition of non-classical notions of logical consequence. Given the compatibility condition while a logical truth may follow from any (set of) true premises (with respect to certain components), *nothing* as opposed to everything is *ableitbar* from a contradiction. The compatibility condition invalidates the *ex contradictio quod libet* or explosion principle. The idea that contradictions imply everything is a largely accepted theorem in contemporary logics – to the exclusion of paraconsistent ones, for instance. The classical argument in favour of the *ex contradictio quod libet* principle is straightforward:

(i) from the conjunction 'p and non-p', one may infer 'p' (elimination of conjunction);

(ii) from p, one may infer 'p or q' (introduction of disjunction);
(iii) from (i) and (ii) one may conclude to q (disjunctive syllogism).

(iii) is a principle Bolzano was inclined to reject:

> One sees that the rule applies at best to disjunctive arguments in which all disjuncts have one and the same subject and whose major therefore has the form: 'A is either B or C' [...] But how could one want to apply the principle to disjunctive arguments whose major contains disjuncts that have different subjects, for instance 'Either Caius says the truth or Semprionus is already dead'?
>
> (1837, §267, 564)

But more significantly, the compatibility constraint compels Bolzano to reject (i). The reason for this is that since 'p and non-p' is a contradiction, no substitution of 'p' in " 'p' is *ableitbar* from 'p and non-p' " can ever fulfil the compatibility constraint; no intepretation of 'p' in 'p and non-p' can yield a true variant and hence there are no ideas that can be varied so as to make both the premises and the conclusion true at once. In general, any inference involving a contradiction is invalid, a point on which Bolzano was explicit and adamant (see, e.g. Bolzano 1837, §530, 269ff.). This has at least two remarkable consequences. Firstly, the compatibility constraint invalidates the law of contraposition: contraposition holds only in case the premise (or the conclusion) do not contain a logical truth. Bolzano writes:

> If certain propositions M, N, ... *only some of which are true* [and some false – SL] are *ableitbar* from an individual proposition A, then the contradiction of A must also be *ableitbar* from the contradiction of M, N, ... all with respect to the same ideas. For since M, N, ... are *ableitbar* from A, all ideas that make A true also make M, N, ... true. Therefore all ideas that makes M, N, ... false and hence that make the contradiction of the latter true (and there are such), must also make the proposition A false, therefore the contradiction of the latter true. Therefore the contradiction of the latter is *ableitbar* from the contradiction of the former.
>
> (1837, §159, 157, my emphasis)

By contrast, given the compatibility condition, whenever the set of premises S, S′, S″ ... contain a proposition that is universally *gültig*, that

is, a proposition such that all its substitution instances are true, we cannot infer from:

$A_{ij\ldots}(T, T', T'', \ldots; S, S', S'', \ldots)$

to

$A_{ij\ldots}(S, S', S'', \ldots; T, T', T'' \ldots)$

For instance, while:

Caius is a physician who specialises in the eyes.

is deducible (with respect to 'Caius') from:

Caius who is an ophthalmologist is an ophthalmologist.

But:

It is not the case that Caius who is an ophthalmologist is an ophthalmologist

is not deducible with respect to the same component from:

It is not the case that Caius is a physician who specialises in the eyes

The compatibility condition is not fulfilled; no substitution could make the conclusion true since it is a logical falsehood.

Secondly, consider monotonicity. Monotonicity is a feature of classical notions of consequence that guarantees that if a conclusion follows from a given set of premises, it still follows from these premises plus some other arbitrary claims, whether they are true or false or even contradictory. Adding a new premise does not, in monotonic logics, affect the logical import of the initial *modus ponens*. One widespread motivation for rejecting monotonicity is the fact that it does not accommodate our intuitions when it comes to dealing with contradictory information. On Bolzano's account, given the compatibility condition, whenever the premise added contains contradictory information, the conclusion no longer follows. More generally, while compatibility does not allow him to deal with all cases of defeasible inference, it allows him at least to account for cases that imply typicality considerations. It is typical of

crows that they be black. Hence from the fact that x is a crow, we can infer that x is black. On the classical account, adding new information does not invalidate the conclusion. On Bolzano's account adding a premise that describes a new case that contradicts previous observation invalidates the conclusion:

> This crow is black

is not *ableitbar* from

> All crows are black
> This is a crow
> This crow is not black

since the inference does not fulfil the compatibility condition: no substitution can make both the premises and the conclusion true at the same time.

Note that in addition the compatibility constraint also guarantees the validity of the principle of subalternation: If $(x)\ F(x) \rightarrow G(x)$, then $\exists x\ F(x)\ \&\ G(x)$. If 'X is G' is *ableitbar* from 'X is F' (with respect to X), then there is at least one individual that makes both 'X is F' and 'X is G' true, and hence there is at least one individual that is both F and G.

At many places, Bolzano suggests that *Ableitbarkeit* is a type of probabilist inference, namely the limit case in which the probability of a proposition T relative to a set of premises S, S′, S″ . . . = 1. Bolzano also calls 'perfect inference' (*Schluss*) inferences of this type:

> An inference in which we wish to deduce (*ableiten*) from the propositions A, B, C, . . . not only the propositions that M is probable but M itself is a probable inference or probability inference in the narrower sense; and by contrast with the latter we call all other inferences (propositions that assert a true relation of *Ableitbarkeit*) proper or perfect inferences.
>
> (1837, §253, 510)

The value of a probability inference from S, S′, S″, . . . to T with respect to a set of variable ideas i, j, . . . is determined by comparing the number of cases in which the substitution of i, j, . . . yields both true instances of S, S′, S″ . . . and T, to the number of cases in which S, S′, S″, . . . are true (with respect to i, j, . . .) (1837, §161, 171ff.). Let's assume, for instance, that Caius is to draw a ball from a container in which there

are 90 black and 10 white and that the task is to determine the degree of probability of the conclusion 'Caius draws a black ball'. On Bolzano's account, in order to determine the probability of the conclusion, one must first establish the number n of admissible substitution instances K_1, K_2, …, K_n of the premises. n is a function of the following considerations: (i) the probability of each of K_1, K_2, …, K_n with respect to 'Caius draws a ball' (with respect to 'ball') is the same; (ii) only one of K_1, K_2, …, K_n can be true at once; (iii) taken together, they exhaust all objectual substitution instances of 'Caius draws a ball' (with respect to 'ball'). Establishing the number of admissible substitution instances of the premise is indispensable in order to determine the probability of the conclusion. In our example, since there are 100 balls in the container, there are also 100 admissible substitution instances of the premises, namely K_1: 'Caius draws ball number 1', K_2: 'Caius draws ball number 2', …, K_{100}: 'Caius draws ball number 100'. If the set of K_1, K_2, …, $K_n = k$ and the number of cases in which 'Caius draw a black ball' is *ableitbar* from 'Caius draws a ball' is m, then the probability μ of 'Caius draws a black ball' is the fraction $m/k = 90/100 = 9/10$ (1837, §161, 175, 176). In the case of *Ableitbarkeit*, the number of cases in which the substitution yields both true variants of the premises and the conclusion is identical to the number of true admissible variants of the premises, $\mu = 1$. If there is no substitution that makes both the premises and the conclusion true at the same time, then the degree of probability of the inference is 0, that is, the conclusion is not *ableitbar* from the premises. In every other case, the conclusion will have a determinate degree of probability with respect to the premises. Bolzano goes on to specify for instance that (cf. 1837, §253, 509ff.):

> 1) If the degree of probability of a proposition T, relative to S, S′, S″, … (with respect to $i, j, \ldots$) is μ, then the degree of probability of $T = 1 - \mu$.
>
> 2) If T and T′ are equivalent, that is, if $A_{ij\ldots}(T'; T)$ and $A_{ij\ldots}(T; T')$ and the degree of probability of T relative to S, S′, S″, … = μ, then the degree of probability of T′ relative to S, S′, S″, … = μ.
>
> 3) If T is probable relative to S, S′, S″, … and an agent holds S, S′, S″, … to be true, then if the agent knows that T is probable relative to S, S′, S″, …, then T is probable for that agent. T is probable if the probability of $M < {}^1/_2$. If the degree of probability of M is = 1, then M is certain.

> 4) Induction 1: If b is a property of A in all Xs we have observed until now, then 'A has b' is probable. The greater the number of Xs we have observed, the greater the degree of probability of 'A has b'.
>
> 5) Induction 2: If x is a property we always or almost always encounter in connection with the properties a, b, c, and if we find a, b, c in a given object Y, then Y probably has x.

Bolzano did not think that his account of truth preservation exhausted the topic of inference since it does not account for what is specific to knowledge we acquire in mathematics. Such knowledge he considered to be necessary and *a priori*, two qualities relations that are defined on the basis of the substitutional method do not have. Bolzano called "grounding" (*Abfolge*) the relation that defines axiomatic structures in which propositions relate as "grounds" to their "consequences" (1837, §162, 191). As Bolzano conceived of it, my knowing that 'p' grounds 'q' has explanatory virtue: grounding aims at epitomising certain intuitions about scientific explanation and seeks to explain, roughly, what, according to Bolzano, the truly scientific mind ought to mean when, in the conduct of a scientific inquiry, she answers the question 'why . . .?'.[4] Since in addition the propositions that pertain to "grounding" orders such as arithmetic and geometry are invariably true and purely conceptual, then grasping the relations among propositions in the latter invariably warrants knowledge that does not rest on extra-conceptual resources – a move that allowed Bolzano to debunk the Kantian theory of pure intuition.

The notion of grounding Bolzano has in mind is explicitly inspired by the passage of *Metaphysics Θ* where Aristotle writes:

> It is not because we think truly that you are white, that you are white but it is because you are white that, when we say it, we speak the truth.
>
> (10: 1051 b6-9)

The standard interpretation of this passage of the *Metaphysics* is however different from Bolzano's. On most contemporary accounts, grounding relates propositions or statements, on the one hand, and the entities that make them true: certain facts or states of affairs (Armstrong 1997) or tropes (Mulligan et al. 1984), for instance. That Bolzano disagrees with the standard interpretation is explicit:

> Does a thing X have the property x because the proposition 'X has x' is true or is on the contrary this proposition true because the thing X holds this property? The right answer, in my opinion is: neither. The ground for the truth of a proposition is to be found, if this truth has a ground, in another truth and not in the thing it is about. And it is even less allowable to say that the reason why the thing X has the property x lies in the truth that X has x. If X is indeed a real (*wirklich*) thing, then there cannot be a ground by virtue of which it has the property x, but there can be a cause that explains that it has the property x, and this cause lies in another thing.
>
> (Bolzano, *Gesamtausgabe* 2 A 12/2, 60)

Bolzano's denying relevance to the idea that propositions are true because they are made true by the entities they describe in this context makes sense: Bolzano sought, with the notion of grounding, to define the structure of axiomatic orders, and he understood – correctly – that in an axiomatic system what makes a propositions true is the fact that it is grounded in more primitive truths.

In the same passage, Bolzano also claims that causality is to things, what grounding is to proposition thus implying that the semantic and the ontological (causal) order are to be strictly separated.[5] Bolzano suggests in many places that grounding can hold between empirical propositions (as well as between *a priori* propositions), and in particular between true empirical propositions that express relations of causality. Considering the relation between grounding and causality, Bolzano is inclined to think of a model in which the two relations work so to say in parallel:

> The most plausible way to [conceive of the relation between grounding and causality] is that somehow those truths that state the existence of the properties of a cause be considered as the *ground*, and those that concern the existence and the properties of the effect be considered as the *consequence*. The truth : 'God is', could be considered as the ground of the truth: 'The world is' because the existence of God is the cause of the existence of the world. But in this way the relation of grounding would only hold between truths that relate to something causal (*wirklich*).
>
> (1837, §201, 349)

It must be emphasised that Bolzano puts relatively little weight on this connection and this for two reasons. On the one hand, Bolzano is clear

that the relation of grounding is not restricted to expressing causal relations since, for instance, mathematical truths also stand in a grounding order; but mathematical truths obviously do not concern causal relations. On the other hand, it is also clear from the text that Bolzano's main preoccupation is with grounding relations that hold precisely between conceptual propositions of the kind we find in mathematics and axiomatic systems in general.

Bolzano's notion of grounding is defined by a set of distinctive features:

1) Grounding is a unique relation: for every true proposition that is not primitive, there is a unique tree-structure that relates it to the axioms from which it can be deduced. Uniqueness follows from two distinctions Bolzano makes. On the one hand, Bolzano distinguishes between simple and complex propositions. A ground (consequence) may or may not be complex. A complex ground is composed of a number of different truths that are in turn composed of a number of different primitive concepts. On the other hand, Bolzano distinguishes between the complete ground or consequence of a proposition and a partial ground or consequence of the latter. On this basis, he claims that the complete ground of a proposition is never more complex than is its complete consequence, that is, propositions involved in the complete ground of a proposition are not composed of more distinct primitive concepts than is the complete consequence (1837, §205, 357). Given that Bolzano thinks that the grounding order is ultimately determined by a finite number of simple concepts, this restriction implies that the regression in the grounding order from a proposition to its ground is finite: ultimately, the regression leads to primitive propositions, that is, axioms whose defining characteristic is their absolute simplicity.

2) Bolzano resorts to the distinction between complete consequence and partial consequence to strengthen his case against the idea that different grounds may have exactly the same consequences:

> Following the concept we presented of the relation of *Ableitbarkeit*, a number of conclusions can flow from the same premises and the same conclusion can be deduced from different premises. This is however not the case with *grounding*. [...] One may believe to be able to introduce examples in which from different grounds the same consequence follows [...] Thus the command: you shall not

> lie can be derived from the supreme moral law in very different ways, namely each time we consider one of the many disadvantages lying effects on general happiness and each of these deduction should deserve the name of a grounding [...] But what these examples prove, when one pays closer attention to them, is nothing else than that different grounds sometimes have common partial consequences, they do no show that the complete consequence is the same.
>
> (1837, §206, 359)

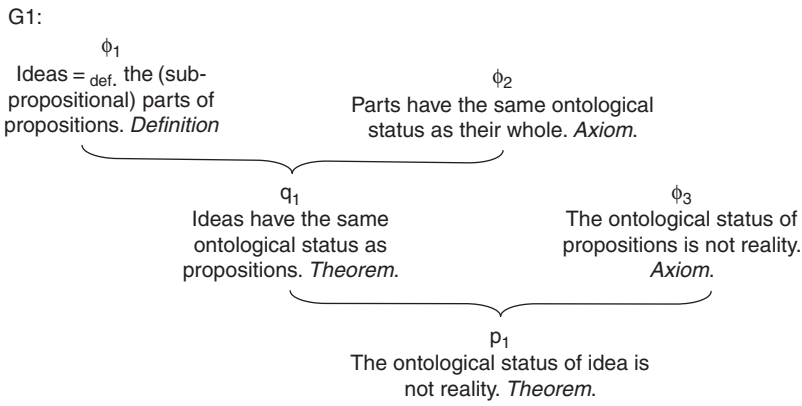
That lying may deprive me of my credibility and that lying may cause inconvenience to the person who is lied to are both partial grounds on the basis of which, given the utility principle – to which Bolzano adheres – I can infer that I ought not to lie. But it would be wrong to think that these truths have the exact same complete consequence. As Bolzano explains:

> To the complete consequence of given truths A, B, C, D, ... belongs, among others, the truth that the propositions A, B, C, D, ... are all true. But this is a consequence (namely a partial consequence) that belongs to no other collection of propositions except this one.
>
> (Bolzano 1837, §206, 359)

Bolzano's point is that each ground yields at least one partial consequence that is the consequence of no other ground. Though the example Bolzano gives in the passage above may seem trivial, it lends support to his view concerning the complexity of consequence.

3) Bolzano draws a distinction between cases in which what we have is the immediate ground for the truth of a proposition and cases in which the ground is mediated (implicitly or explicitly) by other truths. When Bolzano speaks of grounding, what he has in mind is *immediate* grounding, and he understands the notion of mediate grounding as a derivative notion: it is the transitive closure of the more primitive notion of immediate grounding that relates propositions to their immediate objective ground. p is the mediate consequence of $\phi_1, \ldots, \phi_n$ if and only if there is a chain of immediate consequences starting with $\phi_1, \ldots, \phi_n$ and ending with p. p is the immediate consequence of $\phi_1, \ldots, \phi_n$ if there are no intermediate logical step between $\phi_1, \ldots, \phi_n$ and p. Let us assume that the

following is an adequate partial representation of a given grounding order G1:



In G1, p_1 is immediately grounded in q_1 and ϕ_3 which, in this segment, taken together constitute its *complete* ground (each of q_1 and ϕ_3 is a partial ground for p_1). p_1 is however only mediately grounded in the axioms ϕ_1 and ϕ_2, namely via q_1, which, on its part is immediately grounded in ϕ_1 and ϕ_2. Given Bolzano's distinction between mediate and immediate grounding, it must be concluded that the relation of grounding is not transitive.

4) Grounding is anti-reflexive. p cannot be its own ground, whether mediate or immediate. The anti-reflexive character of grounding can be inferred from its anti-symmetry:

> It is clear that if the truths A, B, C, . . . form the complete ground of the truth M, N, O . . . [. . .] they cannot also be considered to be their consequence, for to the complete consequence of A, B, C, . . . belong [not only M, N, O, . . . but – SL] also the proposition 'A, B, C, are all true'; but this proposition would then itself have to be considered to be the partial ground of the truths A, B, C, . . . which is absurd.
>
> (1837, §209, 362)

If grounding were to be reflexive, then the truth that A could be grounded on itself, but given that if A grounds B, it is not the case that B grounds A, this would imply a contradiction since, by substitution, A could at once ground itself and not ground itself. If nothing else,

> anti-reflexivity allows Bolzano to deny the traditional tenet according to which some propositions such as axioms are grounded in themselves (1837, §204, 356, 357). Bolzano explains that this is a loose way of speaking, that those who maintain this idea are unaware of the putative absurdity of saying that a proposition is its own consequence and that the main motivation behind this claim is the attempt to maintain the idea that every proposition has a ground across the board.[6]

One important distinction to be made between *Ableitbarkeit* and grounding as Bolzano conceives of them rests in the fact that while grounding is meant to support the idea that *a priori* knowledge is axiomatic – that there are (true) primitive propositions from which all other propositions in the system follow as consequences – *Ableitbarkeit* does not have such an implication. Whether or not a proposition q is *ableitbar* from another proposition p is not contingent on q's being ultimately derivable from the propositions φ, ψ, ... from which p is derivable. That:

> Socrates is mortal

is logically *ableitbar* with respect to 'man', 'Socrates' and 'mortal' from

> Socrates is a man
> Men are mortal

can be established without consideration to any other proposition from which it could in turn follow, for example:

> Men are finite beings

One question that arises at this point is whether there are any significant connections between *Ableitbarkeit* and grounding. For instance, one might want to consider if q's being *ableitbar* from p implies that p grounds q, or *vice versa*, if p's grounding q implies that q be *ableitbar* from p. The possibility that *Ableitbarkeit* be a special case of *Abfolge* is unacceptable for Bolzano. Not all cases of *Ableitbarkeit* are cases of *Abfolge*. For instance, while:

> It is warm in the summer

is *ableitbar* with respect to 'summer' from

Thermometers, if they function properly, are high in the summer

it is not a consequence of the latter in Bolzano's sense (cf. 1837, §162, 192, 193). On the contrary, the reason why thermometers are high in the summer is that it is warm so that, in the previous example, the order of grounding is reversed. There are cases in which true propositions that stand in a relation of *Ableitbarkeit* also stand in a relation of *Abfolge* and Bolzano did not consider such cases – what he terms "*formale Abfolge*" – to be entirely irrelevant:

> Not every relation of *Ableitbarkeit* is such that it also expresses a relation of *Abfolge* when the propositions it contains are all true. Without any doubt however a relation of *Ableitbarkeit* that has this property is remarkable enough to deserve its own designation. I want therefore to call the latter a relation of formal *Abfolge* [...] I say that the propositions M, N, O, ... stand in a relation of formal *Abfolge* to the propositions A, B, C, ... with respect to the ideas i, j, ... [...] if every set of ideas that we put in the place of i, j, ... and that make all of A, B, C, ... true also turn all of M, N, O, ... into truths and indeed transform the latter in truths that comport themselves towards the truths A, B, C, ... like a real consequence to its ground.
>
> (1837, §162, 193)

Bolzano does not explain what exactly makes the relation of "formal" *Abfolge* important, but it is not difficult to see what could have been its interest: in an inference that fits the notion of formal *Abfolge*, the conclusion follows both *necessarily* (by virtue of its being a relation of *Abfolge*) and as a matter of *truth-preservation* (by virtue of its being an instance of *Ableitbarkeit*) from the premises. It would be easy however to overplay the significance of Bolzano's notion of formal *Abfolge*. It would be interesting if it allowed Bolzano to define a new type of relation that systematically picks out inferences in which the necessity of a conclusion can be established formally on the basis of true premises. But this is not the case. Formal *Abfolge* is not an additional resource of Bolzano's logic but a designation for types of inferences that present the specificity of suiting two definitions at once: I can only know that an inference fits the definition of formal *Abfolge* if I know that it fits both that of *Abfolge* and that of *Ableitbarkeit*. Once I know that it fits both, to say that it is a case of formal *Abfolge* does not teach me much I did not already know.

On the other hand, it could be tempting to think that *Abfolge* is a kind of *Ableitbarkeit*, namely the case in which the premises are systematically

simpler than the conclusion. Bolzano suggests something similar when he claims that *Abfolge* might not, in the last instance, be more than:

> An ordering of truths by virtue of which we can deduce (*ableiten*) from the smallest number of simple premises, the largest possible number of the remaining truths as conclusion.
>
> (1837, §221, 388)

Bolzano gives the impression that he is seeking a notion to bridge the gap between the two notions when he defines the notion of "exact" (*genaue*) *Ableitbarkeit*. On the one hand, exact *Ableitbarkeit* imposes a series of constraints whose main purpose is to insure that no premise be unnecessary to draw the conclusion: Bolzano writes:

> If the premises A, B, C, D, ... from which the propositions M is *ableitbar* with respect to the ideas i, j, ... have the property that it is not possible to leave out one of the propositions A, B, C, D, ... indeed any of the parts they contain, if M is still *ableitbar* from the rest with respect with the same ideas i, j, ...: then I call the relation of *Ableitbarkeit* of M from A, B, C, D ... *exact*, *exactly measured* or *adequate*. Such is the relation of *Ableitbarkeit* between the premises: 'All α are β', All 'β are γ' and the conclusion: 'All α are γ' when the ideas α, β, γ are considered as variable [...] By contrast, I call *redundant* (*überfüllte*) the relation of *Ableitbarkeit* between the same premises and the conclusion: 'Some β are α' for it holds even when we keep only the first premise. Equally redundant is the inference from the premises: 'All α are γ', 'All β and γ are δ' to the conclusion: 'All α are δ', for the latter also follows if, instead of the premise: 'All β and γ are δ' we choose the simpler one: 'All β are δ '.
>
> (1837, §155, 123)

One consequence of this definition is that an inference from exact *Ableitbarkeit* cannot contain an analytic truth (be it in Bolzano's broader or narrower sense) since such propositions are systematically "redundant". For instance:

> Caius is a physician who specialise in the eyes

is not exactly *ableitbar* (with respect to Caius) from:

Every ophthalmologist is a physician who specialise in the eyes
Caius is an ophthalmologist

since one may omit the second premise without changing the conclusion. Likewise, an argument containing the proposition:

Caius who is virtuous deserves abiding happiness

would be redundant in Bolzano's sense. It can be neither the premise nor the conclusion of an inference from exact *Ableitbarkeit* (with respect to Caius) since it is true for any interpretation of Caius.

As Rusnock (2000, 148) notes, one other consequence of the definition is that the premises must be formally independent. None may be *ableitbar* (with respect to determinate components) from the others. If a premises is *ableitbar* from the other premises, an inference from *Ableitbarkeit* is in any case redundant in Bolzano's sense.

In the *Theory of Science*, Bolzano's definition of exact *Ableitbarkeit* remains incidental. By contrast in his later (mathematical) work, Bolzano adopted the stricter notion (see 1840, §8, 40–44). At any rate, the notions of exact *Ableitbarkeit* and grounding remain distinct. In order to establish a substantial tie between the two notions, Bolzano would have had to ignore basic differences between the two kinds of relations as he conceives of them. One such difference and one on which Bolzano is adamant concerns the fact that while the relation of *Ableitbarkeit* can "hold amongst falsehoods" the relation of *Abfolge* by definition cannot (1837, §162, 192). The point is important. What Bolzano means to emphasise when he claims that falsehoods can be *abgeleitet* from other falsehood is the fact that *Ableitbarkeit* is defined for inferential forms, not for individual inferences. By contrast, *Abfolge* holds between individual truths. When I say that 'The thermometer is high in the summer' is *ableitbar* from 'It is warm in the summer' with respect to 'summer' I am making a claim about *the fact* that every time 'It is warm in X' yield a true substitution instance, 'The thermometer is high in X' yields one as well. When I say that 'The thermometer is high in the summer' is grounded in 'It is warm in the summer', I am making a claim about determinate conceptual relations within a given theory: I am saying that given what it means to be warm and what it means to be a thermometer, it cannot be the case that it be warm and that the thermometer not be high. Of course the theory can be wrong,

but assuming that it is true, the conclusion is necessary since it follows from the axioms of the theory.

Bolzano's views on *Ableitbarkeit* and *Abfolge* can be criticised in various respect. One could worry that an epistemological theory based on the latter may present little interest. There is at least one reason to shake the worry off. There is a sense in which what Bolzano has to say when he tackles epistemological questions does not depend on what he has to say about grounding and *Ableitbarkeit per se*. The significance of what he has to say depends on the import of his views about the role of any comparable notion in an account of *a priori* knowledge, justification and demonstration. What makes Bolzano's views on justification in deductive disciplines interesting, for instance, is not the peculiar nature of the notion of grounding on which they are based but the idea that there must be some such notion on which they are based. While they remain idiosyncratic in certain respects, Bolzano's views on *a priori* knowledge – that is, knowledge by virtue of the meaning of terms – objective justification and scientific rationality as a whole present a vast historical interest. They are rooted in theoretical assumptions he shares with his successors, Frege, Russell and Husserl, for instance, for which the latter have come to be praised, and Bolzano should at least be considered to have anticipated their concerns and solutions. In retrospect, it is rather surprising that so little attention should have been paid to Bolzano's views on *a priori* knowledge and demonstration *per se*, and in the next chapters, I will document what I take to be the most interesting part of Bolzano's epistemology.

7
Justification and Proof

Bolzano's theory of objective ground and consequence is part of a general account of *a priori* knowledge and a theory of demonstration or proof – note here that I do not use the term 'proof theory' which Bolzano's theory is not – that has attracted some attention in recent years. When it comes to explaining the nature of deductive knowledge at large, Bolzano distinguishes between three notions: grounding (*Abfolge*), objective justification (*objective Erkenntnisgrund*) and what we may call objective demonstrations or proofs, which Bolzano calls *Begründungen*. This tripartite distinction in itself testifies to Bolzano's refined sense for the differences between logical, epistemological and pragmatic concerns: grounding is a relation between true propositions; objective justification is a relation between beliefs or cognitions (i.e., certain types of epistemic states); and *Begründungen* are linguistic objects that are meant, according to Bolzano, to reliably cause in agents objectively justified knowledge of the type we find in *a priori* sciences such as mathematics. Roughly, the structure of the theory is the following: (i) grounding is a relation that subsists, according to Bolzano, between *true* propositions independently of epistemic access to them. We may grasp grounding relations and (ii) our grasping the latter is also the condition for our having objective justifications for our beliefs, as opposed to, say, merely subjective or evidential ones. Finally, Bolzano thinks that (iii) *Begründungen* are meant to reliably cause agents to have objective justifications in the latter sense. With respect to (ii), Bolzano's idea is explicitly Aristotelian: Bolzano believes that whenever an agent grasps p and grasps the grounding relation between p and q, she also knows the ground for the truth of q and therefore putatively *why* q is true – namely *because* p. Bolzanian *Begründungen*, because they reflect the grounding order and proceed by minimal inferential steps, cause the justification to

be in principle epistemically immanent: *Begründung* should in principle cause agents who effectively grasp the inference to gain full awareness of its necessity – and thus certainty. Bolzano's way of putting this is the following:

> We will call certain (*gewiss*) or assured (*sicher*) judgements that arise in a manner that excludes the danger of error. [...] Certainty (*Gewissheit*) in the sense in which we take it here is [...] a property that can only be ascribed to judgements, and the latter only in relation to a thinking being that is posing them [...] Only true propositions in themselves can deserve to be called certain with respect to a thinking being. For if the judgement that contains this proposition arises in a way that excludes the possibility of error, then it must be true. Since true judgements deserve the name of cognitions, one may in general call cognitions propositions that are true [...] As regards us humans [...] only truths which we recognise immediately, or those we deduce (*ableiten*) immediately through an inference which is not a probability inference make up the collection of what we recognize as certain. (1837, §317, 264, 265)

If we follow Bolzano, any cognition that is inferred as an objective consequence from other propositions – any cognition that is objectively justified in his sense – will be certain since it excludes the possibility of error. Objective justification, if it holds, holds necessarily, between cognitions – that is, beliefs whose content is a true proposition – that derive axiomatically from primitive cognitions. Hence, from the epistemic standpoint, any objective justification will have, because it reflects the immediate inferential steps that are defined by the relation of grounding, certainty.

As regards (iii), the role of a typically linguistic or schematic representation of the proposition q and its objective ground p is to cause agents to have an objective justification, namely p, for the belief that q. On Bolzano's account, demonstrations or proofs are devices that are meant to reliably cause us to have a justification for certain beliefs: a demonstration is (typically) the linguistic representation of a set of propositions in a given order or diagrammatic composition we may use in order to bring our interlocutor (or ourselves) to bestow confidence on the truth of a given proposition. Bolzano writes:

> I say that [...] a set of sentences[1] is a proof that belongs to M if these sentences present themselves with the determinate intention, or in

> fact are such as if they presented themselves thereat with the determinate intention to produce, in the mind of the reader, the judgement M with a degree of confidence (*Zuversicht*) with which he would not have made it beforehand.
>
> (1837, §512, 237)

In particular, *Begründungen* succeed in providing agents with an objective justification for their relevant beliefs because they make the objective ground of the propositions that form the content of these beliefs epistemically accessible to that agent – with the result that she should form the relevant true belief with confidence.

It would be incorrect to think that Bolzano conceives of a *Begründung* as the mere "subjective" grasping of a given grounding relation between true propositions. Confusion on this point may occur if objective *justification* and *Begründungen* are not sufficiently discriminated. When I stand in some cognitive attitude to the grounding relation between p and q, what I have according to Bolzano is an objective justification for the truth of p. But a *Begründung* is something different. *Begründungen* are ordered sets of – typically – written propositions that are used in science with the intention of bringing about or causing a belief in the mind of an agent – and organised in such a way that she will have reasons to credit this belief with a higher degree of confidence than she did before or that she would have been likely to do had the demonstration not been available. A *Begründung* of q is an epistemically accessible representation of q, its (complete) ground and the grounding relation between q and its ground that is such as to reliably cause the agent to believe that q with confidence: it should lead her to be fully confident in its necessity since she also putatively has insight into why it is certain.[2]

It seems reasonable to think that the availability of a *Begründung* is not always sufficient to cause an agent to have objective knowledge. And this is also Bolzano's view. But it also seems reasonable to think that in some cases, the availability of a *Begründung* is not optimal for the purpose of causing an agent to acquire a determinate belief. While a *Begründung* is meant to cause objective knowledge of, say, the intermediate value theorem in a given situation of scientific exchange, it is not always the best way to bring someone to recognise the truth of a particular proposition. Upon surveying a *Begründung*, I may fail to have the relevant beliefs: the proof might be too technical given my background knowledge of the topic, or I may simply be unable to concentrate that morning. Likewise I may have gained epistemic access to the linguistic object thus acquiring the cognition that p and the cognition that

p grounds q, but because of some epistemic shortcoming remain incapable of concluding to q, say, because the list of premises is so long that I can only ever grasp at once a subset of the premises required to draw the conclusion. The *Begründung* is not fool proof since whether I do acquire objective knowledge upon surveying the proof in question depends in part on my background knowledge, in part on my overall ability to process the relevant inferences. The latter according to Bolzano's theory of cognition is mostly a function of my having been previously acquainted with many inferences of different types. The more accustomed I am to drawing inferences, the more reliably the *Begründung* is likely to cause in me the relevant objective justification.

In order to stand occurrently (and not a merely dispositionally) in a cognitive attitude towards a given propositional content, an agent must somehow be causally affected. This may be brought about in many ways. As Bolzano sees it, beliefs and ideas arise in our mind in a more or less sophisticated, chaotic and spontaneous way, on the basis of mental associations and/or causal interactions with the world (see, for instance, Bolzano 1837, §283 where Bolzano discusses the topic at length). The availability of a linguistic object that represents the grounding relation is meant to reliably cause objective knowledge, that is, to bring one's interlocutor to have occurent objective knowledge of a certain truth. But this may not be the best way to cause the given belief *per se*. It might be that in order to convince me of the truth of the intermediate value theorem, my interlocutor needs to resort to a more or less intuitive diagrammatic explanation, which is precisely what *Begründungen* are designed to avoid. In order to have an objective justification for a proposition, an agent must minimally be caused to recognise at least part of the deductive structure in which the proposition in question is related to its premises. This would seem to place strict constraints on mathematical demonstration. It would be wrong however to assume that Bolzano thought that mathematical knowledge can only be achieved via *Begründungen*. On Bolzano's account, I would still know that $2+2=4$ if the justification for my believing that $2+2=4$ were that I've been told so (under the assumption that we can acquire knowledge by testimony – I leave the question open) or that after putting two bottles of champagne in the fridge and then two more, I found that there were four. *Begründungen* are, in Bolzano, one type of demonstration, that is, one way to bring me to have a justification for one of my beliefs. But *Begründungen* are not the only type of demonstration in Bolzano's theory of knowledge, nor indeed the only *bona fide* one. There are two kinds of demonstrations in Bolzano's theory, that is, two general types of devices whose purpose

is to reliably cause agents to acquire greater confidence in the truth of a given proposition. On the one hand, there are those that reflect the objective deductive order and hence also provide an objective justification for a truth: *Begründungen*. On the other hand, there is the panoply of those that don't reflect the order of grounding. Bolzano calls the latter *Gewissmachungen* or certifications. He writes:

> Until now, one has not always distinguished distinctly enough the objective ground of a truth from the subjective means through which [we gain it] and it follows automatically that one could not distinguish *Begründungen* from mere certifications. Aristotle (Anal. Post. Book I, c.2 & 13) and the Schoolmen introduce a division between proofs that show only that (hoti), and other that also give the why (dioti) that something is in a very diligent manner but claim somewhat *exaggeratedly that only the latter produce true cognition*.
>
> (1837, §525, 262; my emphasis)

When an agent is caused to know that something is true on the basis of a certification, the agent has a subjective – as opposed to an objective – justification for his belief (cf. 1837, §313, 232). Bolzano's theory of certification and subjective justification is an indispensible element of his account of empirical knowledge. He explains:

> Almost every demonstration should also be here [in the pure conceptual sciences and ethics] authentic *Begründungen*. In others, in particular in the empirical sciences, by contrast, in chemistry, medicine, history, etc., it is only rarely possible to derive the proof of a truth from it objective ground alone [. . .].
>
> (1837, §525, 261)

Certifications are ubiquitous in empirical sciences such as medicine – medical diagnosis, for instance, relies on certifications in Bolzano's sense. Symptoms are typically visible effects – direct or indirect – of diseases that allow us to recognise them: when we rely on symptoms to identify a disease, we thus never know this disease through its objective ground.

The latter remark points to a crucial aspect of Bolzano's theory of certifications: if we follow Bolzano, whenever the warrant we have for the truth of a proposition is not its objective ground, we would seem to acquire the conviction for the truth of the proposition that is demonstrated on the basis of one of its consequences. He explains, for instance,

that while my belief that it is warmer in the summer than in the winter is the objective justification for my belief that, all other things being equal, the thermometer is higher in the summer than in the winter, my belief that the thermometer is high is, all other things being equal, the subjective warrant of my belief that it is warm. Note – this is a case Bolzano does not consider – that I may fail to form the belief that it is warm when the thermometer is high. This could be due to some defect of my background knowledge, say, I've never seen a thermometer or, more relevantly perhaps, because of a cognitive shortcoming: I happen not to have formed the relevant connection between the two events. Note also that while it seems to be the case that at least some instances of subjective justification are the converse of a relation of objective justification, it would be wrong to assume that in order to recognise p on the basis of one of its objective consequence q, I must also know that p grounds q. I may have formed the belief that when there is smoke there is fire without knowing that fire causes smoke – say I ignore everything about the phenomenon of combustion and I simply take the events to be co-occurrent. Likewise, my son may have formed the belief that whenever the light is on in the living room, the light is also on in the hallway, and the former may for him be a subjective justification for the latter (and vice versa, but not necessarily). But the two events do not stand in a direct causal relation, they are merely related by their having a common cause of which he may have no knowledge, namely that they are on the same electric circuit.

Another important difference between *Begründungen* and certifications concerns their respective cognitive status. Contrary to what is the case in empirical sciences that may involve non-conceptual cognitive states, deductive practices, at least on Bolzano's account, exclude non-conceptual cognitions. While certain types of certifications, for instance, those we find in theoretical disciplines – for example, my acquiring the belief that two is greater than 3 by supposing the opposite and deriving a contradiction – are eminently based on linguistic as well as conceptual resources, other types appear to rest on pre-linguistic, pre-conceptual knowledge. In order for my belief that this is smoke to be a certification of my belief that there is fire – as it should be – it seems unnecessary that there be a linguistic object, in addition to the state of affairs themselves, to cause me to have the relevant beliefs. Though Bolzano does not make this explicit in his discussion of certifications, other passages of the *Theory of Sciences*, for instance, the passages where he discusses the role and nature of perceptions and experience make it reasonable to assume that he would agree to say that what causes me to have the

belief that there is smoke (and therefore fire) – if we exclude the case in which I base my self on someone else's report – is typically the smoke itself (cf. 1837, §300, 131). Assuming that for Bolzano, states of perceptions are pre-doxastic, it would seem that subjective justification could in some cases be pre-doxastic. By contrast, given the nature of deductive disciplines as Bolzano conceives of them, that is, arithmetic and geometry, for instance, it seems unlikely that I could cause my interlocutor to know (objectively) *why*, for instance, the sum of the internal angles is equal to that of two right otherwise than by providing him with a means to recognise the truth of the propositions from which it derives. But it is hard to imagine how I could cause my interlocutor to recognise the truth of the premises without the support of some intersubjective device, and I am inclined to think that linguistic support is always required. One could defend the idea that strictly speaking an epistemically accessible linguistic representation of the grounding relation between p and q, while it may be useful, is not always required for the purpose of our having knowledge of q on the basis of its ground p. What causes me to know q on the basis of p is the knowledge of q and the knowledge of q's being grounded by p itself, and I may be in a position to acquire this knowledge independently of any available public artefact meant to cause that knowledge in me. This amounts to arguing that I may acquire an objective justification in the absence of a *Begründung*, and on this Bolzano would agree. The case is however different when it comes to *proving* objectively a given proposition, that is, when it comes to engaging precisely in the exercise of devising objects that are intended to cause agents to know given propositions with more confidence. In such cases, the endeavour is invariably both linguistic and conceptual. Indeed, it would not do to argue that my knowing that q on the basis of p may be non-conceptual and that objective justifications in sciences like mathematics do not systematically require conceptual support. Of course, I may ask myself, "why q?", have an "insight" or make a guess and "invent" a theory that explains its truth and then realise that this theory indeed provides me with the actual ground for its truth. On Bolzano's view, however, the production of the objective justification and of the relevant *Begründung*, my establishing that p is the ground for q would still involve my grasping propositions that are eminently conceptual.

One question Bolzano's theory raises is whether mathematicians, in particular, should rely only on *Begründungen* and seek only to cause objective knowledge. Since the purpose of demonstrations is primarily to cause the interlocutor to have a higher degree of confidence in one

of his beliefs, and that Bolzano emphasises the effectiveness of proofs over their providing objective justifications (see 1837, §§526–535), *Begründungen* should not be seen as the only canonical or scientifically acceptable means to bring an agent to bestow confidence on a judgement. If we follow (1837, §525), Bolzano is of the opinion that "*almost* every demonstration in the pure conceptual sciences should be authentic *Begründungen*" and therefore not all of them. It might be, for instance, that in order to convince my interlocutor of a given truth, the most effective means is to use a *reductio* – which for Bolzano does not provide us with an objective justification since, for one thing, the relation does not hold between truths as grounding requires but consists in deriving a contradiction from a falsehood (cf. 1837, §§329, 530). Besides, Bolzano warns us against the idea that one ought to use only logical or formal demonstrations that might end up boring the interlocutor to distraction and have a rather adverse epistemic effect. Although Bolzano claims that we ought to use *Begründungen* as often as possible, he also recognises that we sometimes have to take shortcuts or simply use creativity to cause our interlocutor to bestow confidence on the truths of mathematics, especially when they have only partial and scattered knowledge of the discipline.

What is important is the following: although Bolzano considered that *Begründungen* should be favoured in mathematical demonstration and despite the fact that he thought that only *Begründungen* have the advantage of letting us *understand why* a given proposition is indeed true, he did not think that mathematical demonstrations ought to be *Begründungen*. This may seem odd, but Bolzano has good reasons to avoid requiring that all our mathematical proofs provide us with objective (and if some interpretations of Bolzano are correct explanatory) knowledge. For one thing, asking that all mathematical proofs be *Begründungen* would not be a reasonable requirement and, in particular, it would not be one that is always epistemically realisable. Given the nature of grounding, it would often require us to engage in the production of linguistic objects that have immense proportions. On the other hand, Bolzano thinks that there are situations in which it is legitimate to accept, even within mathematics, proofs that deliver only evidential knowledge that something is true. Bolzano emphasises the importance of pragmatic considerations in mathematical practice. He also stresses the fact that when it comes to exposing a mathematical theory, the main objective should be to cause the agent to have more confidence in the truth of the proposition to be demonstrated than he would

have otherwise or even merely to incite him to look for an objective justification by himself. Hence, given certain circumstantial epistemic constraints, Bolzano is willing to concede that it be allowed to merely provide a brief justification of one's opinion:

> Since it is often the case that an intricate development of these grounds would take up too much place; since it is also often the case that in order to assess the latter previous knowledge is required of our reader, which we can't assume he has; since the nature of these grounds is sometimes such that they do not have in our own eyes a very high degree of reliability, we must in fact be excused if we often only indicate our opinion without entering into a complex discussion of its grounds. For even an opinion that is not accompanied by a demonstration [...] can sometimes be useful. All by itself, it will already prompt the reader to think further, and if the hint that we give him is not incorrect the discovery of truth will be considerably alleviated.
>
> (1837, §577, 387)

Furthermore, and this would deserve to be investigated further, it is worth mentioning that Bolzano is not averse to reverting to purely evidential means, for instance, when it comes to mathematical demonstration:

> Besides, it is in no way absurd to accept, from time to time, in geometry as well as in the other mathematical disciplines, mere proofs of probability. Who will not recall those admirable theorems of the theory of prime number which Fermat managed to discover through experimentation and to make probable through examples.
>
> (Příhonský 1850, 183)

Since they are merely probable, Bolzano does think that evidential proofs need to be supplemented by "decisive" ones. But there are various types of non-evidential proofs that could fulfil that role, and here Bolzano does not specify which types of proof are decisive. One could want to argue that the latter reduce to *Begründungen*. If upon surveying an objective proof I acquire an objective justification, I cannot doubt the truth of the conclusion, and it is therefore decisively true. But it is hard to imagine that Bolzano would have thought that my recognising the truth of a proposition on the basis of the linguistic representation

of an inference from *Ableitbarkeit* would be any less decisive. Take, for instance:

Triangles have two dimensions

is *ableitbar* from

Figures have two dimensions
Triangles are figures

Not only is the inference truth preserving, but the conclusion is also a conceptual truth – it is composed only of concepts – which, according to Bolzano, means that its negation would imply a contradiction and is therefore necessary (1837, §133, 37). *Ableitbarkeit* and grounding, in Bolzano's theory, both have the epistemic particularity of yielding an (inferred) propositional content that can be asserted or judged with certainty. As Bolzano sees it, certainty is a property of judgements that are indefeasible, that is, not liable to error (1837, §317, 264). The objective consequence of a set of *a priori* propositions cannot be defeated if only because, if I know its ground, I also know why it is true and necessarily so. Similarly, if p is true and if I know that q is *ableitbar* from p – and this holds a fortiori in the case in which p and q are conceptual truths – I have a warrant, namely the fact that *Ableitbarkeit* preserves truth from premises to conclusion, and I cannot be mistaken about the truth of q.

Nonetheless *Begründungen* remain a crucial part of Bolzano's epistemology of *a priori* knowledge. *Begründung* is a means to introduce pragmatic constraints on demonstration that are meant to steer actual demonstrative practices in deductive science and not as an adequacy condition for demonstrations *per se*. Seen in these terms, the idea that mathematical demonstrations "ought to" (in a practical sense, not in the sense of a necessary condition) reflect the grounding order would insure, in Bolzano's theory, two things. First, it would insure that an agent does not deny that a proposition has an objective ground and is thus inferable from more primitive propositions every time this agent, perhaps owing to her medical condition or limited recognitional means, fails to recognise that the proposition has an objective ground. Consequently, it would insure that the demonstration procedure is not short-circuited by criterion such as intuition, evidence or insight. The requirement that mathematical demonstrations be *Begründungen* forbids that the agent's inability to derive a proposition from more primitive ones be compensated by a non-grounding-related feature. In this relation, Mancosu

speaks of the heuristic fruitfulness of Bolzano's requirement on scientific exposition (Mancosu 1999, 436).

Bolzano's methodology and didactics of scientific knowledge is based on a theory of logic that presents a number of semantic innovations. Commentators often favour interpretations that stress Bolzano's commitment to semantic realism and this is important: half a century before Frege he laid down a conception of philosophical logic that would inform a significant part of the theories for decades to come. The importance of Bolzano's contribution to semantics can hardly be overestimated. But the same holds for his contribution to the theoretical basis of mathematical practice. Far from ignoring epistemic and pragmatic constraint, Bolzano discusses them in detail, thus providing a comprehensive basis for a theory of mathematical knowledge that was aimed at supporting work in the discipline. As a mathematician, Bolzano was attuned to philosophical concerns that escaped the attention of most of his contemporaries and many of his successors. His theory presents a historical interest and a philosophical one as well, and one that deserves to be investigated further.

8
A priori Knowledge

What Bolzano had to say about the Kantian conception of *a priori* knowledge in his early essay on the philosophy of mathematics, the *Contributions to a Better Founded Exposition of Mathematics*, is valuable on many accounts. In the *Appendix* to the latter – the title is *On the Kantian Doctrine of the Construction of Concepts in Intuition* – Bolzano criticises Kant's doctrine of pure intuition. The views he puts forward in the latter have been discussed in some detail (Laz 1993; Rusnock 2000; Sebestik 1992).[1] In fact, commentators tend to rely exclusively on this short text when it comes to assessing Bolzano's criticism of Kant's views on *a priori* knowledge. In 1810, however, Bolzano's theory remained overall tentative. In particular, Bolzano did not provide a substantial alternative to Kant's views. By contrast, in the *Theory of Science*, Bolzano not only offered a thorough and mature criticism that became the basis for the comprehensive assessment of Kant's philosophy later published under the title *The New Anti-Kant* (Příhonský 1850) but also developed his own alternative conception of *a priori* knowledge.

As Bolzano sees it deductive, that is, axiomatic disciplines such as arithmetic and geometry are "purely conceptual". Being purely conceptual as Bolzano conceived of it does not amount to being analytic (in his or Kant's sense). In particular, on Bolzano's account, Kant's idea that logic is analytic is misleading and the kind of resources Kant puts into play in his account of "general logic" were far too unsophisticated to account for the richness of the kind of knowledge one acquires in disciplines such as arithmetic and geometry. The main problem with the logic Kant and his followers adopted was the fact that it relied on the decompositional conception of analysis (see Chapter 2) thus maintaining a definition of what counts as purely conceptual that could not support an adequate account of *a priori* knowledge. Kant's idea that some

types of *a priori* cognitions, because they are not analytic, are not conceptual either constituted a questionable move whose effects pervaded Kant's views on the nature of deductive disciplines as a whole. Kant had been wrong to think that the truths of arithmetic and geometry, because they are not analytic, could not be purely conceptual and needed to be "grounded" in non-conceptual, that is, "intuitive" yet *a priori* cognitions. The result was a theory which, according to Bolzano in addition to perpetuating a defective conception of conceptual analysis, introduced a theory of "synthetic *a priori*" knowledge that could not account for what is specific to disciplines such as arithmetic and geometry, for instance, namely the fact that the truth of the propositions they set out can be known by virtue of meaning alone.

The idea that Bolzano had an account of knowledge by virtue of meaning is likely to seem controversial in light of the existing literature. Oddly, the passages where Bolzano explicitly puts forward his views have been neglected and, as far as I can see, for no good reasons. As a consequence in the literature on Bolzano's philosophy of logic and mathematics when Bolzano's distinction between *a priori* and *a posteriori* knowledge is not simply overlooked, authors seem to assume that Bolzano had in fact little to say about the topic and that his views are exhausted by the distinction he made between intuitive (*a posteriori*) and conceptual (*a priori*) propositions in (1837, §133, 33, 34). This omission is regrettable on many accounts. Failure to take note of Bolzano's discussion of knowledge by virtue of meaning in the relevant sections of the *Theory of Science* curtails Bolzano's epistemology of one of its most important aspects. According to Bolzano, it is possible to know the truth of propositions in disciplines such as arithmetic and geometry by virtue of the meaning of the terms they contain because these terms are implicitly defined by the axioms of the theory. Bolzano was first to put forward a theory of implicit definition and while this has not remained unnoticed (see Sebestik 1992, 139ff.) the role the latter plays within Bolzano's theory of deductive knowledge is largely undervalued. Assuming a substantial commitment to the idea that primitive terms are defined implicitly by the axioms in a deductive discipline, to know that a proposition is true by virtue of the meaning of the terms amounts to knowing that it is or is deducible from the presumably true primitive propositions that define the terms it contains.[2] This, as I will argue, was Bolzano's view.

Bolzano did not think that analytic knowledge, whether in his or in Kant's sense, plays an important role in deductive knowledge. In fact, as Bolzano conceived of it, none of the disciplines of which we usually

say that they are *a priori* are said to be analytic. This includes arithmetic and geometry as well logic itself. To be fair, Bolzano's own terminology is somewhat confusing. As Bolzano sees it, the truths of disciplines such as arithmetic and geometry are synthetic *a priori*, and this is confusing inasmuch as the term is often associated with Kant's doctrine of pure intuition. Yet, in the context of his theory, Bolzano's claim that deductive knowledge is synthetic *a priori* is not especially problematic. It amounts to saying that logic, just like arithmetic and geometry, is an axiomatic discipline whose structure is defined on the basis of the notion of *Abfolge* (see Chapter 6) and in which analytic propositions play a negligible role. Bolzano explains:

> In my opinion not even one principle in logic, or in any other science, should be a merely analytic truth. For I look upon merely analytic propositions as much too unimportant to be laid down in any science as a proper theorem of it. Who would want to replenish geometry, for example, with propositions like: an equilateral triangle is a triangle, or is an equilateral figure, etc.?
>
> (1837, §12, 51, 52)

> We must however agree with [Kant] when he claims that "in all the theoretical sciences of reason synthetic *a priori* judgements are involved as principles" But we find judgements of this sort not only in mathematics, in the pure natural sciences and in metaphysics, as Kant proves it incontestably, but they are also to be found in logic, namely not merely among the theorems that belong to this discipline if we understand it, with Bolzano, according to a wider concept, but in the very part of it which one calls analytic and which has been worked on since Aristotle.
>
> (Příhonský 1850, 42, 43)

Disciplines such as logic, arithmetic and geometry, as Bolzano sees it, present two determining features. On the one hand, they are ordered sets of propositions that stand to one another as objective grounds to their consequences, that is, systems defined through the relation of *Abfolge*. On the other hand, they contain no non-conceptual elements: the propositions they contain are "purely conceptual".

Bolzano thought that Kant had been wrong to assume that the answer to the question he famously raised in the *Critique of Pure Reason* when he asked what warrants the "connection" of the predicate and the

subject in non-analytic truths required him to appeal to non-conceptual knowledge. As Bolzano sees it, it does not make sense to think that:

> [...] this unknown X on which the understanding must base itself when it believes to have found outside the concept of A a predicate B which is alien to the latter and which it considers to be nevertheless connected with it
>
> (Bolzano 1837, §305, 180)

is at once *a priori* and non-conceptual. Given Bolzano's definition of the distinction between conceptual (*a priori*) and non-conceptual (*a posteriori*) knowledge at §133, this remark may appear somewhat trivial. But the point is not merely terminological. For one thing, the notion of an idea that would be at once *a priori* and imply reference to the empirical world – in Bolzano, intuitions are indexical components of our thoughts that invariably imply reference to the context and thus to empirical considerations (see Chapter 3) – would make it obviously impossible to maintain the distinction between empirical and non-empirical knowledge. On the other hand, Bolzano considered Kant's idea that "this unknown X" on which the truth of a proposition is "grounded", when it is not primitive, could be anything less than another proposition to be absurd. Grounding is a relation between truths and Kant, he thought, misunderstood the concept by applying it to intuitions. To be fair, Bolzano's answer to the question is not especially conspicuous either. But it is nonetheless worth a thorough examination. When it comes to providing an answer to the question as to the ground of *a priori* knowledge, Bolzano proposes the following explanation:

> ... [i] Nothing else [is required], I say, than that the understanding *has* and *knows* the concepts A and B. In my opinion, from the mere fact that we have certain concepts, we must also be in a position to judge about them. For to say that someone has certain concepts A, B, C, D, ... is indeed to say that he knows and differentiates them. But to say that he knows and differentiates them is again only to say that he asserts something about the one that he does not want to assert about the other; this amounts therefore to saying that he judges about them. [ii] Since this holds universally, it holds as well in the case in which these concepts are perfectly simple. But in this case, the judgements we make are certainly synthetic ... and it seems

> to me therefore that we must be in a position to make a synthetic judgement about all objects of which we have a concept.
>
> (Bolzano 1837, §305, 180)

Consider again:

> [iii] If a given proposition consists of mere concepts, such as, for instance, the proposition that virtue deserves respect or that two sides of a triangle taken together are bigger than the third, etc.; [iv] then the truth or falsity of the latter depends only on the properties of these concepts; and, [v] at least in many cases, nothing else will be required in order to convince yourself of its truth that you examine attentively the concepts themselves of which it is composed. Thus, it will be possible for you to recognise the truth that virtue deserves respect from the mere fact that you have the concepts virtue, to deserve and respect. [vi] One could not say that you have a concept if you could not differentiate it from another one, that is, if you did not know that certain other concepts can be connected with it to form true propositions which cannot be connected with another. [vii] You cognise truths of this kind (purely conceptual truths) by virtue of the fact that you know the concepts of which they are composed. [viii] Things are different with judgements that [. . .] contain intuitions
>
> (Bolzano 1837, §42, 180, 181)

Much of what Bolzano has to say about *a priori* knowledge rests on what he means by "having" or "knowing" a concept. The general claim – in (i) – is that in axiomatic disciplines such as arithmetic and geometry, knowing the concepts propositions contain is sufficient for knowing that these propositions are true. Let us confine the claim to theorems for now. If I know the concept A and I know the concept b in the sense of knowing relevant to Bolzano, then according to Bolzano I should also know that that the proposition that A has b is true. Since Bolzano takes it that concepts are the sense of expressions (cf. e.g., 1837, §285, 67), what this means is that in an *a priori* discipline, I can know that 'A has b' is true if I know the meaning of both 'A' and 'b'. More on this directly.

(ii) and (iii) emphasise the fact that what Bolzano says about knowledge by virtue of meaning applies to propositions that are both synthetic and conceptual, that is, synthetic *a priori* in his sense. The latter include axioms, (at least some kinds of) definitions, theorems and hence the greater number of truths we find in disciplines such as arithmetic and

geometry. Admittedly, the expression '*a priori*' does not occur in the quote. One could argue that the reason for this is that Bolzano ought to have recognised that '*a priori*' and 'conceptual' are not co-extensive. One could draw attention, for instance, to Bolzano's claim that "the truth of *most* conceptual propositions can be decided through mere reflection . . ." (1837, §133, 36) and that this suggests that he did not mean to imply that all conceptual propositions are knowable *a priori*. This however would not only be odd but mistaken. What Bolzano means to convey here concerns what can be known at all: the qualification is not aimed at implying that we may need experience, in some cases, in order to know whether a certain conceptual proposition is true. Rather, what Bolzano has in mind is the fact that it may not be possible at all to know, given the limitation of our recognitional capabilities, whether a conceptual proposition is true or false and hence to know it, be it *a priori* or *a posteriori*. We do not know, for instance, the mathematical and therefore purely conceptual truth from which we can infer, of every number, whether it is prime or not; and though this truth would in any case be purely conceptual, since we don't know it, we do not know it, according to Bolzano, *a priori* (cf. Bolzano 1837, §133, 37). In order to avoid confusion as to Bolzano's views of the relationship between the property of being conceptual and that of being *a priori*, it is important to emphasise that purely conceptual truths, as Bolzano conceives of them, need not be logical truths in the contemporary sense – they only need to be conceptual in his sense.[3] In Bolzano's view, if a conceptual proposition is knowable at all, then it is knowable *a priori*. By contrast, as Bolzano emphasises in (viii) we know propositions that contain intuitions in another manner: their truth depends also on the nature of the objects that are represented and therefore, since these objects are perceptual, it depends on our experience.

As Bolzano sees it, to know the meaning of a term x in a discipline D is to have some beliefs about the meaning of x, what Bolzano calls, in the terminology of his time, "being in a position to judge about" the meaning of x. This sounds odd. It would seem more natural to say that to know the meaning of x in D is to be in a position to make judgements about the corresponding object(s). The claim however can be sustained in Bolzano's theory as it stands. As Bolzano explains:

> . . . to the complete ground why we formulate our judgement in this precise way, why we claim that to any object represented by the concept A pertains the property represented by the concept b [. . .] belongs, in particular, the specific nature of the two concepts

> themselves. If the latter were different, we might have deemed it necessary, instead of connecting them in a judgement: A has b, perhaps, to separate them.
>
> (1837, §302, 140)

To know the specific nature of a concept in his sense is to know what properties of the corresponding objects can be inferred from it. According to Bolzano, we can "infer" from pure concepts the "essential properties" of the objects to which they refer. This talk of "inference" of properties from concept may sound somewhat disconcerting, but it is not a lapse and more importantly it is not insignificant either. Bolzano makes this claim at a number of places: a property can be "inferred" (*gefolgert*; e.g., Bolzano 1837, §65, 287), it may "follow" (*folgen*) or "ensue" (*sich ergeben*) from a concept (e.g., Bolzano 1837, §114, 531). What it means for a property to be inferred from a concept is made clear at (1837, §111, 520) in connection with Bolzano's discussion of essential and inessential properties. There, Bolzano explains that from the concept of an object it is possible to infer all the "essential" properties of the corresponding object. Bolzano in fact defines the "essence" of an object as the set of properties that can be inferred from its concept (Bolzano 1837, §111, 521 – I will come back on the scare quotes directly). The property b can be inferred from the concept of A and, correlatively, b is an essential property of the object A iff (Bolzano 1837, §111, 520):

(i) 'A has b' is true
(ii) 'A' designates a concept

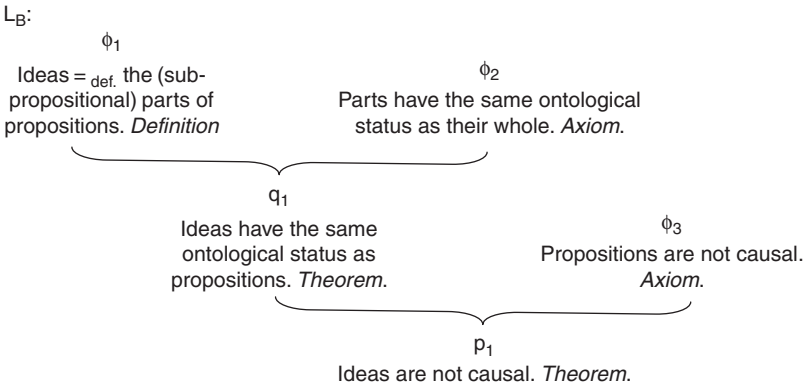
Given that we are here considering systems such as arithmetic and geometry that contain only conceptual truths, in a theorem 'A has b', 'A' and 'b' designate, by definition, pure concepts. Following what Bolzano says (the property), b can thus be inferred from the concept A, and the object A has therefore the property b "essentially". On Bolzano's account, to know a concept A is to know which properties can be inferred from it – and which cannot – that is, to know the propositions of the form 'A has x' that are true. Since every such proposition will be either itself an axiom or grounded in axioms, to know a concept in a deductive system as Bolzano understands them amounts to knowing the axioms of the system: to know a concept is to know how its object is (implicitily) defined by the axioms.

Bolzano has an extensive theory of implicit definitions.[4] One reading of what it is to have or cognise the meaning of the primitive terms in Bolzano's view is the following: we know the meaning of the terms involved in primitive sentences when we have determined what properties the objects to which they refer would have if the latter (and all their relevant consequences) were true. This, I take it, is what Bolzano means by the following passage:

> [i] It is known that, when we come across a sign which is unknown to us in connection with other signs whose meaning we know, we are more than often in a position, if we also merely suppose that the writer does not want to express something manifestly absurd, to determine with more or less exactness what he represented himself with that sign. In such cases, we know the meaning of the term *on the basis of the use or context*. [. . .] [ii] Understandably, not every sentence in which our sign occurs in whatever connection with other known signs is equally apt to its determination and many sentences are often necessary to determine it completely. [iii] Without doubt, sentences that state a truth and a truth which is known and familiar to the reader are much more useful for this purpose than others.
>
> (Bolzano 1837, §668.9)

(i) states the general idea behind implicit definitions. (ii) states a condition for this definitional procedure to succeed: it must be the case that the sentence(s) used to define a term implicitly *determine it fully* or at least as fully as possible. As far as (ii) is concerned, "to determine a sign" can be read as a semantic matter or as an epistemic one. The semantic concern is easily answered: since Bolzano assumes that the essence of an object is determined by its concept and given that, for him, a concept is what makes up the meaning of a sign, then a set of axioms necessarily determines any sign it contains as much as possible (for that axiom system) and hence must be in the relevant sense a "definition" of the sign. On the other hand, from the epistemic standpoint, the full determination of, say, 'A' and 'b' in case they both are primitive (and 'A has b' is an axiom) rests on what Bolzano says in (iii). (iii) states the epistemic condition under which the reader will be in a position to understand the meaning of the terms defined implicitly, and thus the conditions under which implicit definitions fulfil their epistemic role: that their truth be known, or familiar to the reader. In other terms, what warrants *a priori* knowledge of the meaning of a term we find in a given axiomatic order

is minimally our recognising that the axioms that contain this term are true. Take:



If we follow Bolzano our having and examining the concepts, in this case, 'idea' and 'causality' should be sufficient for us to know *a priori* whether the theorem 'Ideas are not causal' is true. In this context, to know the concepts 'idea' and 'causality' is to know that the axioms that define them are effectively true. On Bolzano's view once I reach the upper limit of the system, assuming that I also both recognise the axiomatic status of the relevant propositions and their truth, I can claim to know the meaning of the terms by virtue of implicit definitions. My knowledge of the axiomatic structure may, as it stands, be imperfect and partial and unsatisfactory but, according to Bolzano, full knowledge can at least in principle be attained and my belief that ideas have non-reality is thus ultimately justified by the axioms: since I know that these beliefs find their justification within the axiomatic order, I know that they are grounded in more primitive and ultimately fundamental conceptual truths, namely those I would reach once I had regressed in the axiomatic order to the primitive propositions.

The question whether an axiom is true is problematic for Bolzano who believes that axioms, just like the other propositions involved in an axiomatic order, are *in fact* true and should be known as such: axiomatic orders such as the ones we find in arithmetic and geometry are systems of true propositions: if p grounds q, then both p and q are truths (see Chapter 7). This assumption leads Bolzano to develop an account of epistemic warrant for axioms that constituted

a philosophically interesting alternative to theories based on evidence. Likewise, the question whether a given proposition is in fact an axiom is problematic for Bolzano since he does not consider axioms to be mere postulates. In both respects, Bolzano's theory may be seen to be somewhat antiquated but, be it as it may, Bolzano considered both problems at length and came up with solutions which, in the context, were both original and resourceful.

In the *Contributions*, Bolzano introduced a metadeductive procedure he calls '*Herleitung*' whose purpose is the determination of the axiomatic status of a proposition. On the one hand, in order to establish the axiomatic status of a proposition, it must be shown that the concepts involved in the propositions under consideration are "simple", which according to Bolzano requires us to follow the standard definitional procedure (cf. 1837, §554, 330, 331). The latter consist in determining – to the extent that this is possible given our epistemic limitations – whether the ideas contained in a given proposition are simple or composed of parts, and in the latter case to establish what these parts are as well as their connection. On the other hand, it must be shown that there is no other proposition such that the proposition under consideration would be a consequence of the latter. The two tasks are complementary. Given Bolzano's concept of grounding, a proposition such that it contains only primitive concepts cannot at the same time be liable to being inferred from a more primitive set of premises: the consequence of a proposition must always be more complex than its ground. But there is no proposition that is less complex than a primitive truth. In the *Theory of Science*, though the terminology changes superficially to accommodate his rich epistemology, Bolzano retains the metadeductive procedure, emphasising the intrinsic difficulty there is in achieving it successfully:

> [...] there are many judgements which others see as non-mediated [i.e. primitive – SL] which belong to the class of those that are [mediated – SL], and I hold it to be absolutely wrong for the aims of science to declare a judgement to be immediate if one has not previously demonstrated that it could not, in any case, be mediated.
>
> (1837, §316, 260, 261)

Though he assumed that the *Herleitung* of an axiom is in principle always possible, Bolzano also thought that the procedure typically remains underdetermined from the epistemic standpoint. As he sees it, our recognitional capacities are limited and we are rarely, if ever, in a

position to know for sure that the propositions under consideration do not contain a concept that would be liable to further definition – and therefore that it is not grounded in a (more) primitive truth.

Herleitung alone, even if successful, cannot cause agents to recognise, in addition to their axiomatic status, the *truth* of primitive propositions – and by way of consequence cannot cause them to fulfil the condition under which they would be in a position to understand the meaning of the terms defined through the axioms. What a *Herleitung* ultimately shows, if it succeeds, is that the truth of the proposition in question cannot be justified on the basis of more primitive premises since there are none. But a proposition that has been shown to be primitive in Bolzano's sense – though it might appear to be certain to the agent – could be false. In this respect, Bolzano's insistence on a verification procedure for the axioms reflects a feature of Bolzano's views on axiomatisation that is at the heart of other classical models of science: it is not only that our system must be sound and complete but that it must in effect reflect the (unique) way the (e.g., mathematical) world is and of which we may have only very partial and unsystematic knowledge. Bolzano's concern with the truth of axioms was therefore not particularly original. It could be argued that he was an apt pupil of Aristotle, but that this was precisely the reason why he did not become the precursor of modern axiomatic formalism.

In order to provide a warrant for axioms, Bolzano reverts to a "meta-inductive" procedure. In order to establish the truth of a proposition whose axiomatic status has been demonstrated, according to Bolzano, an agent must ascertain whether it generates (only) true consequences. A proposition whose axiomatic status has been shown (via *Herleitung*) and whose logical consequences are also known to be true – some of the consequences may be and indeed often are more epistemically accessible than the axioms themselves – is very likely to be itself true (Bolzano 1837, §577, 388).[5] The truth of an axiom is to be "inferred" from the truth of its consequences. Of course, in order for the meta-inductive procedure to succeed, there has to be some independent warrant for the truth of the consequences themselves. When it comes to explaining what warrants the truth of a given non-primitive proposition, Bolzano speaks of a "sentiment for truth". On the face of it, what is odd about this solution is the fact that Bolzano's overall criticism of epistemic warrants in deductive practice would seem to leave little if any place for an appeal to the kind of epistemic device – neither quite an "intuition", nor quite a mere 'insight" – the notion of a sentiment for truth seems to imply. If we follow what Bolzano says on this topic at different

places in the *Theory of Science*, one of the dangers in introducing non-logical epistemic devices whose role is to offer a warrant for cognitions is that it creates a slippery slope that may lead to the postulation of "supernatural" powers which have no place within science: the ultimate consequence of this type of intuitionism in a theory of *a priori* knowledge is likely to be the eventual degradation of the requirements of rigor in demonstrative and definitional practices altogether (cf. 1837, §315, 257). The question is thus whether Bolzano's notion of a sentiment for the truth of a proposition is not liable to the same reproach.

Bolzano defines the sentiment for truth as the "power to pose a true judgement or (what amounts to the same) to recognise its truth without at the same time being conscious of the ground on the basis of which one knows it" (1837, §316, 259).[6] According to Bolzano, an inference may be hidden to us, and in such case, we may entertain a proposition – hold a belief – without being aware of the fact that it is a conclusion. According to Bolzano, the number of cognitions such that we are not aware of their being inferred is indefinitely large and depends to a great extent on our individual cognitive make up. He suggests, for instance, that my considering that a proposition is immediate may be the result of the speed at which I draw the inference of which it is the conclusion. It is possible to pose the judgement that q without remembering or having occurrent knowledge that q is inferable from p. (1837, §300, 127, 128) Though Bolzano is not explicit on this point, he would agree that the fact that I may not be conscious of the premises p, p′... that implicitly lead me to judge q must at least imply the possibility of their eventually surfacing at the conscious level. In other terms, it implies having a disposition to infer q on the basis of p, p′,... If this is correct, then it is reasonable to assume that what causes the sentiment for truth in Bolzano's account is at least in part contingent on this disposition. The latter would explain why Bolzano thinks that, though we are unable to say exactly why, we cannot doubt the truth of the proposition in question. According to Bolzano:

> [...] the fact that one, despite all disagreement about its ground, still cannot doubt the proposition is a proper mark of its truth.
>
> (1837, §315, 249)

Note that on Bolzano's view we need not and, in the case of primitive propositions, cannot feel the truth of the antecedent in the relevant sense: the sentiment for truth is not something a cognition inherits

from its antecedent. In fact, supposing the latter leads to a contradiction. If the grounding relation between p, p′... and q is explicit, then by definition I don't have the sentiment for the truth of q: I know q necessarily on the basis of p, p′.... And if the inference is implicit, I cannot have a sentiment for the truth of the premises since the latter are unconscious. This supposition also makes clear why Bolzano does not resort to the idea of the sentiment for truth when it comes to providing a warrant for primitive propositions. We cannot have a sentiment for the truth of primitive propositions on Bolzano's account: the fact that they are primitive, that is, the fact that they lack a ground makes it impossible for me to have a disposition to infer them in the relevant sense.

Let us sum things up. According to Bolzano, we are able to cognise the meaning of the terms involved in the type of implicit definitions we find at the upper limit of axiomatic systems when we have determined what properties the objects to which they refer would have if they (and all their relevant consequences) were true. It is not enough for Bolzano that we assume that axioms be true, we need to know that they are. According to Bolzano, if we know that the objective consequences of a proposition are true, we can infer inductively that it itself is true. Hence in order to know that a proposition that has been shown to be primitive is true, we must establish a warrant for its consequences, and this warrant consists in our having a feeling for their truth. It is important to emphasise that, according to Bolzano, we are never in a position to feel the truth of an axiom. In other words, are we to find ourselves in a situation where we are inclined to think that a proposition is true without having an objective ground for it, we must assume, either that we find ourselves feeling the truth of the proposition in the relevant sense and therefore that it is not itself a primitive proposition, or that the proposition is primitive and that the feeling we have is not a feeling for the truth of the proposition in the relevant sense – it is merely a hunch and epistemically idle. In the former case, the proposition is not an axiom and one is to inquire after its ground. In the latter case, the proposition may well be an axiom, but in order to establish this, we cannot merely trust the feeling we have and must proceed to its metadeduction, that is, its *Herleitung* by showing that it is composed of primitive concepts and that there is no other propositions on which it can be grounded. We must then determine whether its truth can be inferred on the basis of the truth of its consequences. It is only when we have established the latter that we can claim to know that the meaning of the terms they contain is fully determined or as determined as possible for that axiom

system. It is only at this point that we know the meaning of the terms and know that we do *a priori*.

Though Bolzano's concern is not reflected in contemporary theories of axiomatic knowledge, one should not underestimate the import of Bolzano's views on epistemic warrants. If the condition for knowing that an *a priori* proposition is true is that one grasp (part of) the axiomatic structure in which it is embedded, then the truth of *a priori* cognitions cannot be a mere function of evidence or intuition. In general, one may understand one of the core motivations of early twentieth-century analytic philosophy as consisting precisely in explaining what warrants us to hold as true *a priori* propositions without however appealing to the idea of evidence or other similar epistemic criteria such as certitude or conviction. Bolzano's aim was not to rid philosophy of epistemological concerns altogether but to insure that demonstrations do not rest on evidence. One ought to look for the objective ground of a proposition as long as one has not shown its axiomatic status. This (pragmatic) constraint ensures that justificatory procedures are not short-circuited by epistemic breakdowns: in the absence of a *Herleitung*, Bolzano directs us to assume that the proposition in question has an objective ground and to find out what it is. At any rate, once we do away with the superficial terminological confusion which may arise from the fact that he claims that deductive knowledge is "synthetic *a priori*" in his sense, one finds in Bolzano an intuition that will prove to be fruitful in a number of his successors: *a priori* knowledge is always deductive and cannot be explained without the support of a theory of logical consequence.

9
Things, Collections and Numbers

While Bolzano's theory of collections has been compared to set-theory and Lesniewskian-type mereology, it can be reduced to neither.[1] His analyses present a vast historical interest and arguably a philosophical one as well – the latter as long as they are considered for what they are: an investigation in the nature of collective entities whose results are put to work in Bolzano's semantics as well as in his philosophy of mathematics. Bolzano's collections (*Inbegriffe*) are neither sets, nor mereological sums, nor classes. What they are follows from the following tentative definition:

> One very important genus of complex ideas that we encounter everywhere are those in which the idea of collection (*Inbegriff*) appears. There are many types of the latter [...] I must first determine with more precision the concept I associate with the word *collection*. I use this word in the same sense as it is used in the common usage and thus understand by a collection of certain things exactly the same as what one would express by the words: a combination (*Verbindung*) or association (*Vereinigung*) of these things, a gathering (*Zusammensein*) of the latter, a whole (*Ganzes*) in which they occur as parts (*Teile*). Hence the mere idea of a collection does not allow us to determine in which order and sequence the things that are put together appear or, indeed, whether there is or can be such an order. [...] A collection, it seems to me, is nothing other than something complex (*das Zusammengesetztheit hat*).
>
> (1837, §82, 393)

As Bolzano presents it in the *Theory of Science* – he expands on the theory in the first sections of the *Paradoxes of the Infinite* (1851) – there

are two main species of collections. On the one hand, there are those that belong to the species to which "masses"[2] (*Mengen*) and sums (*Summen*) belong (1837, §84, 399ff.). On the other hand, there are those that belong to the species to which "series" (*Reihen*) belong (1837, §85, 401ff.). What distinguishes the two species of collections, as Bolzano understands them, is the fact that while in the latter the "mode of combination" of the parts is relevant, in the former it is not. The distinction is significant: in order to determine the nature of a given collection and, more importantly, its particular "type", it is necessary to establish (at least) two things. On the one hand, one must determine the nature of its parts, that is, their natural or social (or otherwise) kind and, on the other hand, when the latter are combined or ordered in a certain way, the mode of their combination. These two parameters concur to determine a considerable number of different types of collections. Bolzano's collections are collections of things, concrete or abstract, the types of which come in an indefinite and potentially infinite number. Contrary to what is the case in classical mereology or set theory where the notion of part or of set remains a purely formal one, Bolzano's theory affords for a manifold of different kinds of parts and collections and is therefore the basis for a rare exercise in material ontological analysis. It involves the description of a wide range of sub-types of collections whose identity conditions depend on the actual material features of the objects they contain as part. While gold coins can form a *heap*, for instance, H_2O molecules cannot. And while H_2O molecules can form an object denoted by a mass-term (water, mist, condensation, fog, etc.) the coins, unless they are melted and hence no longer coins, cannot: objects that fall under the sub-type 'heaps' have specific features that distinguish them from objects that fall under the sub-types designated by mass-terms. Collections of human beings cannot form abstract structures or swarms, but they can form armies, states and university faculties. Here 'heap', 'mass', 'structure', 'armies' and 'university faculties' all stand for sub-types of collections that are at least in part determined by the nature of their parts.[3]

The nature of its parts is typically not sufficient to determine the subtype to which a given collection belongs entirely as the examples above show. Whether a collection of human beings is a football team or a university faculty depends on a number of other factors such as membership conditions, conditions of existence and conditions of identity. Take for instance a determinate collection – a plurality – of 10 people whose existence overlap over time. The plurality of these 10 people subsists as long as they all live. Let's assume that at a given time, these 10

people are all employed in the same philosophy department. In the case of a department, the way in which the people are related to one another is a determinant part of their being a department, but the conditions of existence of the department are different from those of its parts. The department is not identical to the plurality of its members since it can gain or loose new faculty without stopping to be the same department. Likewise, the plurality may persist even if the department is dissolved.[4]

While the number of sub-types to which collective entities may belong is indefinite, the fundamental categories under which the latter fall are limited. Bolzano does not assign a name to every formal type of collection, but his theory is rich enough to account for at least the following – I indicate the Bolzanian terminology where appropriate:

> Collections where the mode of combination is relevant: series (*Reihen*), ideas, propositions, deductive inferences, empirical collections (e.g. football teams, forests, religious orders), others.
>
> Collections where the mode of combination is not relevant: masses (*Mengen*), sums (*Summen*), quantities, lists, unities (*Einheiten*), pluralities (*Vielheiten*), totalities (*Allheiten*), others.

Different more or less explicit rules follow from the general structural principles that underlie the identification of the formal categories in Bolzano's theory:

- Whenever (i) the mode of combination of the parts is irrelevant and (ii) the whole and its parts fulfil a general condition of ontological homogeneity, that is, the parts of the immediate parts are of the same kind as the immediate parts of the whole, then (iii) the parts of the immediate parts are also automatically parts of the whole and the collection is invariably a "sum". As Bolzano conceives of it, this is the sense in which mathematicians use the term:

  > In my opinion, the meaning with which mathematicians associate the word 'sum' when they connect the expressions with the sign '+' by which they acknowledge a sum is indeed no other than the one given here: hence they only ever apply the concept to *quantities* (*Grössen*). They definitively take the word in a totally different sense when they consider the determination of the sum of a given set of expression, e.g. $1+1/2+1/4+$ *in infinit.*, as a task of its own. It seems that they are then thinking under a sum a certain expression, equivalent to the

> given collection, and which among all the others that can be thought is the simplest.
>
> (1837, §84, 401)

- If the principle of homogeneity is not fulfilled – a case that seems to cover the greater range of theoretical possibilities, including that of all empirical collections – and, whether or not the type of combination is relevant to the collection, the nature of the whole is contingent on the nature of its parts:

> The example of a heap of money shows that there may be reasons that forbid that even in a collection in which the mode of combination of the part is insignificant we consider the parts of the parts as parts of the whole itself or to exchange them for the latter. For if we wanted to exchange the gold coins we find in this or that heap of money for the parts in which the latter can be mechanically or chemically broken, the worth of the whole would certainly change. Nonetheless, we don't lack collections that afford both, that is, in which the mode of combination is absolutely insignificant and in which the parts of the parts must be considered to be part of the whole. This is the case, for instance, with the length of a line. For when we, in a line, consider only its length, then we observe that the line is composed of smaller lines whose mode of combination is insignificant and we can consider the part of which they are composed (as long as they are themselves lines) as part of the whole itself.
>
> (1837, §84, 400)

- The same object can fall under different collective descriptions. For instance, a deck of cards can be seen as a collection of 52 cards, or as a collection of four suits, or as a collection of 13 sets of equinumerical cards.

As we have seen in Chapter 4, Bolzano makes significant use of the notion of collection in his logic in order to introduce relations and operators and thus account for determinate types of syntactic complexity. In Bolzano, relational statements are statements about collections of a certain type. For instance, 'Louis, Eric and Julie form a trio' should be understood to mean that the collection formed by the three individual is of a kind (let us call it: "being a trio") that determines a given structural

property which they have collectively – but which none of them has individually. Likewise, conjunction and disjunction are defined as collections of propositions that have certain properties namely, in the former case, that of containing only truths and, in the latter, that of containing at least one.[5] Combined with other resources of his theory, these analyses provide Bolzano with a means to define, for instance, multiply quantified statements involving relational predicates (see Chapter 5):

> [[{[Y has N], $[\{X, Y\}_{\text{precedes}}]$} is a collection of truths] is objectual at position Y] is *ableitbar* from [X has N] with respect to X

that is:

$$\forall x[Nx \rightarrow \exists y(Ny \ \& \ Precedes(x, y))]$$

The notion of collection is used in Bolzano's philosophy of mathematics for the purpose of defining series and numbers. Bolzano gives the following explanation of natural numbers:

> Let us build a series whose first member is a unit of some type A and in which all other members are sums that are generated by combining an object that is identical to the previous member with a new unit of type A: I call any member of this series a [concrete] number (*Zahl*) as long as I think it along with the idea that give us the way in which it is generated. [...] I call the property through which each of these members becomes a number (and which keeps it no matter how we change the object we take as units) a number in the abstract sense of the word or an abstract number, and by contrast to abstract numbers I call the members themselves concrete numbers or numbers in the concrete sense of the word. Finally I call the complete series the series of numbers or, in order to distinguish it from other series whose members are also numbers, the natural series of numbers or, following others, the series of natural numbers.
>
> (1931, §1)[6]

By a series, Bolzano means the following:

> [...] a collection of things [...] *A*, *B*, *C*, *D*, *E*, *F*, ... *L*, *M*, *N*, ... such that there is for every part M one and only one other part N such that, for each part of the collection, either N is determined through

> its relation to M or M through its relation to N and this according to the same rule.
>
> (Bolzano 1851 §7, 5; cf. 1837, §85, 401ff.)

As Bolzano sees it, the notion of an (abstract) number is to be derived from the notion of a "concrete" number, and the latter is therefore primitive. A concrete number is a concrete sum of a certain type, which is to be found in determinate place within a series. The procedure by which we arrive at this ordering consists, according to Bolzano, in "combining an object that is identical to the previous member with a new unit" of the same type. A sum of three units is a concrete three only when it is the third member of a series that starts with one unit of type A, then a sum composed of A plus another member of type A, then the sum of these two As plus a third object of type A and so on. Such an ordering, if we follow Bolzano's instruction to "think [each member] along with the idea that give us the way in which it is generated" looks like this: {A, {A, A_1}, {{A, A_1}, A_2}, {{{A, A_1}, A_2}, A_3}, . . .}. Were we not to think the manner in which the number is generated, and simply add to the previous sum a new member, the result would look like this: {A, {A, A_1}, {A, A_1, A_2}, {A, A_1, A_2, A_3}, . . .}. What we would have, in other terms, would not be a series but a "list" of "sums" that differ from each other by virtue of their cardinality. The reason why {A, {A, A_1}, {A, A_1, A_2}, {A, A_1, A_2, A_3}, . . .} is a mere list as opposed to a series is that a collection of this type does not fulfil Bolzano's requirement that every part of the collection be determined through its relation to another member according to the rule "add one unit to the previous member" – and by way of consequence, it does not provide an account of ordinality. When Bolzano writes that we should form a new sum composed from the previous member, say {A, A_1} and a new unit of the same type, say A_2, he requires that we build a new collection whose members are {A, A_1} and A_2, respectively. Since in a sum, the parts of the members are parts of the whole, the new member of the series {{A, A_1} A_2} still has three elements – though it has only two immediate members. On Bolzano's account in order to become a number, the sum {{A, A_1} A_2} must be found in a given ordering that is determined by a rule (in this case, 'add one unit') that fixes, in addition to its cardinality, its ordinality. The series of concrete natural numbers is such that its *n*th member always has n parts.

Given this understanding, it follows that the same parts of a sum of concrete As, say of the three books A, A_1, A_2, in my study – which also contains, say, A_3, A_4 A_5, A_6, A_7, A_8 – may be

the parts of different members in a series of concrete natural numbers. They can be the parts of the third member of a series {A, {A, A_1}, {{A, A_1}, A_2}, {{{A, A_1}, A_2}, A_3}, ...} or of the concrete three in the concrete series {A, {A, A_2}, {{A, A_2}, A_1}, {{{A, A_2}, A_1}, A_3}...}. The series of *abstract* natural numbers is obtained, on Bolzano's account, by abstracting from the type of the units involved in the series. By "abstracting" from the fact that {A, {A, A_1}, {{A, A_1}, A_2}, {{{A, A_1}, A_2}, A_3}, ...} is a concrete series, Bolzano means that we should no longer consider the members as being of a certain type A but of any type ϕ, so that {ϕ, {ϕ, ϕ_1}, {{ϕ, ϕ_1} ϕ_2}, ...} would give me the number 3 in the abstract sense. Presumably, we then proceed to map each member of the series onto the relevant signs – '1', '2', '3', for instance – that will allow us to designate them unequivocally.

According to Simons (1999), a problem arises from a restriction Bolzano putatively imposes on collections in general, namely a non-redundancy constraint that requires that they always contain *all* new members. Such a restriction is indeed crucial to Bolzano's account of list-type collections: in its absence, I might never be able to complete the list of my books, that is, make an inventory of the books there are in my study since I could include each of them twice or more. According to Simons, however, the same restriction makes it impossible to reconcile Bolzano's account of the way in which we arrive at the series of natural numbers with the simple procedure of enumeration, much less the account of number just offered. In particular, Simons claims, it makes it impossible for Bolzano to generate a series of the form: {ϕ, {ϕ, ϕ_1}, {{ϕ, ϕ_1} ϕ_2}, ...} since the non-redundancy restriction requires that each new member of the series contain nothing but members that occur in no earlier sums in the series. Assuming that there be only nine books in my study, this would mean that I would be in a position to form at most three such members and thereby only be able to count to three.

Simons' conclusion however does not follow and that for at least two reasons. First, assuming that Bolzano does impose a non-redundancy constraint on collections in general, one ought to consider that in a series both the order in which the parts appear as well as the fact that we need to think each member of the series "along with the rule that generates it" are relevant. On this account, each (immediate) member {ϕ, {ϕ, ϕ_1}, {{ϕ, ϕ_1} ϕ_2}, ...} of the series of natural number is new: while the second member is composed of ϕ and ϕ_1, the third member is composed of the sum {ϕ, ϕ_1} and of ϕ_2: every immediate part of the series must be considered as an object that is different from all

the others. Second, Simons bases his interpretation on the following passage:

> Even when an idea A is objectual, the idea of a collection of the form (*A, A*) or (*A, A, A*) may be objectless, namely if the idea of an A has only one object; and when and if the idea A has only two object, the idea (*A, A, A*) is not objectual.
>
> (BBGA, 2 A vol. 7, §20, 108)

But what Bolzano is discussing here does not apply to collections in general nor indeed to series. What Bolzano is considering is the fact that the idea of a sum of three As is objectual only if there are at least three As. In other terms, Bolzano is stipulating the conditions under which a quantificational statement of the form '[{A, A_1, A_2}] is objectual' is false. Again, the condition that is expressed here applies quite naturally to *lists* of objects. 'There are at least nine books in my study', that is, 'The idea of a list of nine books in my study is objectual' is false in case, for instance, the collection of all the books in my study is {A, A_1, A_2, A_3, A_4 A_5}, i.e. if there are less than nine books in my study – it is objectual however if there are nine or more. But the passages to which Simons refers says noting about series in particular which are not mere lists of objects but orderings of sums.

Bolzano's ontological analyses are not exhausted by his treatment of collections. Following a classical Aristotelian line, Bolzano also attempted to provide an account of the type of ontological categories that define reality, ultimately restricting their numbers to two: substances and adherences.[7] He writes:

> Everything that exists, that is, everything that is real (wirklich) [...] belongs to one of two species: either it subsists in something other as a property of this thing, or it is not a mere property of something other but subsists, as one usually says, in itself [...]. Philosophers usually call realities of the first kind adherences, and those that belong to the second kind substances.
>
> (1838, 21; 1976, 235)

Bolzano is here confining his definitions to what is causal (*wirklich*), that is, what can have effects (*Wirkungen*). This is an important point to make: as Bolzano sees it, talk of substances and adherences concerns things of the world we live in and understand according to the category of causality. Whenever causal objects are concerned, Bolzano is

consistently talking of their being described by propositions connecting substances-ideas to adherence-ideas (via the copula) (e.g., 1837, §119, 562; §142, 65; §192, 303) – conversely, he never applies these categories to objects that are not causal. If we assume that Bolzano meant these remarks to apply across the board, as we should, then it follows that in any proposition of the form 'A has b' where 'A' denotes a causal object, 'b' denotes an adherence and is also itself causal. This is a simple application of an uncontroversial principle of ontological homogeneity Bolzano states as follows:

> [...] we can and must ascribe [causal] existence (*Daseyn*) to a property when [...] the object in which the latter is to be found itself has [causal] existence (*Daseyn*).
>
> (Bolzano 1837, § 80, 387)[8]

Accordingly, we can include in the definition of adherences their being causal properties.

Not surprisingly, Bolzano's metaphysical views have attracted some attention as regards the debate between realism and nominalism. One question that ought to be raised is whether Bolzano's metaphysics of reality commits him to realism about universals and, in particular, about the kind of universals properties are often taken to be. If we follow the greater part of Bolzano scholars who tackled the question (e.g., Künne 1998, 239, 240; Textor 1996, 63, 64), the answer to this question is putatively affirmative. The latter interpretation is supported by the claim that Bolzano presumably uses the term 'property' ambiguously. On the one hand, Bolzano would sometimes use the term to denote universal abstract entities – say, redness or virtue – that can be instantiated in real things. On the other hand, Bolzano would sometimes use 'property' to refer to actual instances of the latter in causal objects, that is, "adherences" – for instance, the redness of this rose or Caroline's virtue. On this interpretation, objects instantiate a given attribute only if there is an adherence that (i) falls under this attribute and (ii) is to be found in the relevant object. Attributes so conceived would mediate, so to say, the relation between a causal object and the causal properties it instantiates. The notion of adherence would be relevant to Bolzano's semantics only by virtue of his postulating an instantiation-relation in which adherences stand to attributes. The latter would explain the putative interchange, in some passages, of 'adherence' (*Adhärenz*) for 'property' (*Beschaffenheit*) and, in others, of 'attribute' (*Eigenschaft*) for 'property'. However, there need not be mediating universal entities between causal objects and their causal properties. An alternative

interpretation, the one put forward by Schnieder (2003), for instance, is to conceive of Bolzanian metaphysics as committed only to the existence of adherences as abstract entities that are individuated by their (particular) bearers. Such an interpretation can be supported by the following passage, for instance:

> This red (*numero idem*) cannot be found in two roses. The red that is to be found in a second rose may appear to us, if you wish, as similar or perfectly similar, but it cannot be the same because it is not the same rose: two roses require two reds.
>
> (Bolzano and Exner 1931, 32, 33)

As it occurs, if the latter account is to hold, one must eliminate misunderstandings about what Bolzano means to say when he writes, for instance:

> Thus an idea that appears as the assertive part [the predicate-idea – SL] in a propositions is in no case taken in its whole extension: we must rather say the proposition leaves undecided (i) which of the many (in case there are many) *properties that fall under this idea* pertains to the one or the many objects that fall under the idea A. The propositions: 'A has b' has no other meaning than that *one of the properties grasped through b* belongs to one or the other of the objects falling under A; but, whenever there are a manifold of the latter, leave undetermined which it is that belongs to A.
>
> (Bolzano 1837, § 131, 26; my emphasis)[9]

We do not need to interpret this passage as implying that Bolzano uses property-terms ambiguously, in this case to refer to attributes – a term that does not occur here – that is, certain kinds of universals, and sometimes to their particular instances. Given what Bolzano says about the truth conditions of propositions, there is no need to assume that predicate-concepts in propositions about real object ever refer to universals. It is sufficient to assume that whenever 'A' in the proposition 'A has b' refers to a causal object, Bolzano uses "*Beschaffenheit*" (he also sometimes uses "*Eigenschaft*") to designate *sortal concepts* under which certain causal and therefore particular properties fall. It would be inaccurate to think that Bolzano assumes a relation of instantiation between attributes and adherences or indeed that he ascribes any role at all to attributes in his metaphysics of the causal world.[10] Bolzano has a consistent account of the conception of the relation between property-terms and adherences that avoids commitment to the latter.

In the passage above, Bolzano makes it clear that the causal properties that belong to a causal object "fall under an idea", and not under a given attribute or property. Hence one alternative interpretation of what Bolzano means to say in the above passage is that a proposition 'A has b' is true if and only if the causal object A realises one of the causal properties falling under a *sortal concept* designated by 'b'. The proposition 'Caroline is virtuous', on this account, is true because (i) Caroline's virtue – which belongs only to Caroline and is individuated by her – is one of the causal properties that fall under the sortal concept 'virtuous' and (ii) Caroline indeed has Caroline's (particular) virtue. Similarly, 'This rose is red' is true because this rose's red falls under the sortal concept 'red' and this rose has this rose's red. Neither 'virtuous' nor 'red' need to be considered as universals that have instances. They are semantic devices – not metaphysical entities – that are used to pick out abstract particular properties objects have.

The nominalist interpretation of Bolzano's account of properties in the causal world – in addition to be based on the principle of charity since it understands Bolzano as making a consistent use of the notion of property – preserves some important intuitions about the nature of properties in the causal world. Causal properties are individuated by their bearers and are thus numerically distinct: while they are not concrete in the sense that they themselves occupy a point in space and time, they remain particular. The redness of this rose is located in the rose and exists as long as the rose exists; Peter's smile occupies a space between his nose and chin and exists only during the interval of time when the relevant muscles tighten and loosen again. Interestingly, this interpretation finds support in Bolzano's philosophy of mind. When Bolzano discusses the individuation conditions of cognitive processes such as beliefs (*Urteile*) or pre-doxastic states (*subjective Vorstellungen*), he resorts to the idea that the latter are invariably individuated by their bearers, that is, the "mental substance" or mind in which they are to be found and therefore by the person whose mind it is (cf. 1837, §272, 10):

> It is equally obvious that the idea I have of this tree, and the idea my companion has of the same object should not be considered to be the same unique object, but rather as two different ideas notwithstanding all the coincidences that may be found in them.
>
> (1837, §273, 12)

This claim has important consequences for Bolzano's epistemology. It stresses the distinction Bolzano establishes between objective ideas

and the subjective mental states that grasp them – the same can be said about propositions and beliefs (cf. 1837, §292, 112) – *because* it makes clear that mental states are at least in part individuated by the particular mind to which they adhere at a certain time and over a certain period. As Bolzano explains, cognitive processes and mental states are discrete events; they have a finite (and a more or less brief) duration:

> It thus becomes clear that [i] it would not be appropriate to take for granted that there are as many subjective ideas as there are times or moments of which it can be said that one and the same objective idea appeared to us, yet in different manners. It is best to consider the entire time in which the same objective idea appears *uninterrupted* in us, even if it is in different manners, as the duration of one particular subjective idea; [ii] and to speak of *many* subjective ideas when either the same objective idea appears in the same mind at [different – SL] uninterrupted times, or when the appearance of the latter proceeds in the minds of *different* beings, or when the very objective ideas that build the stuff of the subjective ones are different.
>
> (1837, §273, 13)

What Bolzano means by "different manners" in (i) relates to his views on the strength and intensity of subjective ideas (cf. 1837, §275, 17). There are different manners of ideas, namely more or less strong and more or less intense ones. The latter considerations, though they might seem out of fashion from a contemporary standpoint, are not insignificant for Bolzano philosophy of action. Bolzano thinks that different strength and/or intensity will affect the causal powers of a mental event, hence marking their individuality. According to him, a starker mental event is likely to have increased effects, for example, is likely to generate more associations or, say, cause the agent to act upon a belief as opposed to remain passive.[11] What is relevant is the fact that his views rest on the assumption that though they may have the same objective content, different subjective ideas, as real events in the mind of real agents, will have different (kinds of) causal powers. For this reason, mental states are considered to be causal properties in the sense of (ii) and this makes clear that in addition to their being individuated by their objective content, namely the objective ideas or propositions that make-up their *Stoff* or meaning, they are individuated by both their bearers (the agents' minds) and/or their temporal features as events.[12]

10
Frege, Meaning and Communication

In *Dialogue de connexione inter verba et res*, Leibniz was concerned with the question as to what entity should bear the predicate 'is true'. The discussion pertains to the question whether 'is true' should be ascribed to "things" or to "thoughts". The property of being true, Leibniz observed, cannot be ascribed to thoughts since a truth, for instance, that the surface described by a fixed length on a plane is a circle does not depend on the fact that it be thought. But the predicate 'is true' cannot be ascribed to things either. As Leibniz sees it, whatever can be said to be true can also in principle be called false but, according to Leibniz, it does not make sense to say that things are false. Leibniz takes the solution to consist in ascribing truth to neither the former nor the latter but to what he calls *propositio* or *cogitationes possibile* (possible thoughts). Notwithstanding certain definitional qualifications,[1] it is to this Lebnizian idea – which he also ascribes to the Stoics – that Bolzano appeals when he asserts that he was not the first to have put forward the notion of a *Satz and sich* (cf. Bolzano 1837, §21, 84, 85).

Bolzano's characterisation of propositions as abstract is meant to epitomise a common intuition about logic: that the laws of logic are universal and the scope of logical properties, relations and rules should be wide enough to apply even to truths – or falsehoods – that will never be expressed in words or grasped in cognitive attitudes. As Bolzano puts it in his correspondence with Franz Exner, Bolzano (1931, 62) – recall that I use square brackets to designate propositions and ideas:

> [p] is a *Satz an sich*, that is, a "proposition in itself" or, in short, "proposition" iff:
>
> > [p] is either true or, when it is not true, false[2]
> > [p] is something, but p is not *wirklich*.

It follows from this and from what Bolzano says of propositions at different places in the *Theory of Science* that propositions fulfil the following set of criteria:

(1) Propositions are the primitive bearers of the property of being true or false. Other entities of which we customarily say that they are true or false, sentences and beliefs, for instance – Bolzano speaks of uttered propositions (*ausgesagte Sätze*) and of thought propositions (*gedachte Sätze*) – are true or false derivatively by virtue of determinate relations they entertain with propositions (cf. 1837, §19, 77).

(2) Propositions are the *Sinne* of sentences (cf. 1837, §28, 121; §285, 67). As we have seen in Chapter 3, to analyse an expression is to establish that its *Sinn* is such and such.

(3) Propositions are objective as opposed to subjective entities. They are namely the objective content of cognitive attitudes such as beliefs. When I assert something, that is, when I judge, say, that two and two are four is true, I effectively commit myself to the truth of a proposition that I "grasp" (cf. 1837, §34, 154ff.).

(4) Propositions are abstract entities in the following sense: they do not belong to the causal world and cannot therefore have effects or stand in real relations with causal objects. Propositions are acausal to the extent that they are not "*wirklich*", for only *wirkliche* things can cause or act upon (*bewirken*) something and hence be said to have causal powers (cf. Bolzano 1837, §122, 4).

(5) Propositions are structured, they have subpropositional parts which Bolzano calls *'Vorstellungen an sich'*, that is, 'ideas in themselves' – or in short, 'ideas' (cf. 1837, §48, 216).

In essence, (1)–(3) can be seen to correspond to a minimal view that is shared by all proponents of propositions: propositions are the primitive bearers of truth and falsity and the objective communicable content of cognitive attitudes. While he did not speak of "propositions" but of "Thoughts" (*Gedanken*), Frege also eminently put a similar conception forward in his articles "*On Sinn and Bedeutung*" (1892) and "The Thought" (1918). In this respect, those who have noticed the parallels between Bolzano's and Frege's respective theories are understandably

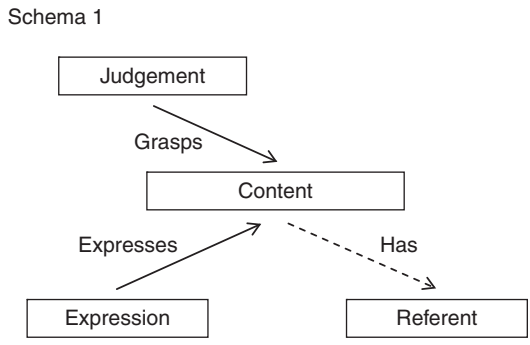
perplexed by the fact that the latter apparently never read the former. As Michael Dummett puts it:

> The only nineteenth-century philosopher of whom it would be reasonable to guess, just from the content of his writings and those of Frege, that he had *influenced* Frege, is Bernard Bolzano, who died in the year Frege was born; but there is no evidence whatever that Frege ever read Bolzano
>
> (Dummett 1991, vii)

Dummett is right. It is unlikely that Frege ever opened the *Theory of Science*. Although some mediated (via Husserl and Korselt) and belated (around 1903) historical connections may be made plausible with respect to specific aspects of their respective theories of consequence, meticulous archival work and scholarship have been unable to document the claim that Frege's *Gedanken* are somehow indebted to Bolzano's theory of *Sätze an sich*.[3] This is an important point to keep in mind. In Bolzano and Frege, the introduction of propositions carries with it a series of important distinctions on which they ostensibly agree: the distinction between the content of a cognitive attitude and the cognitive attitude itself or the distinction between an expression, its content and its referent, for example.[4] But there are other points of comparison, their respective conceptions of what it is for a proposition to be "grasped" in a cognitive attitude for instance. Likewise – though this is an aspect of Frege's and Bolzano's theories that has been vastly neglected – their views on linguistic communication and understanding also contribute to show that their account of meaning is much richer than what is often assumed.

Bolzano and Frege have the same reasons to think of the content of our beliefs and other cognitive attitudes as both objective and abstract. If I had an occurrence of the belief that all bachelors are unmarried upon reading Quine yesterday and it occurs to me again today reading the same lines that all bachelors are unmarried, I have had on both occasions, if we follow their account, beliefs whose content is identical. Similarly, if I believe that $2 + 2 = 4$ and Marie believes that $2 + 2 = 4$, then what we believe is the same (objective, abstract) true proposition. There is a vast consensus on the idea that Bolzano and Frege conceive of the relation between the abstract and objective propositional content of epistemic attitudes and the attitudes of which they are the content

in similar terms.[5] Both putatively adopt a simple model that can be represented as follows:



In both cases, a relation – what both Frege and Bolzano term "grasping" – is posited between the content of expressions and certain types of cognitive attitudes, and this relation is supposed to play a role when it comes to explaining cognition and behaviour. Unfortunately, what it means for something a causal to be "grasped" in a judgement – that is, something causal – remains in both case at best unclear. According to Frege:

> [i] Even the timeless, if it is to be anything for us, must somehow be implicated with the temporal [...] How does a *Gedanke* act? By being grasped and taken to be true. This is a process in the inner world of a thinker which may have further consequences in this inner world, and which may also encroach on the sphere of the will and make itself noticeable in the outer world as well. If, for example, I grasp the *Gedanke* we express by the theorem of Pythagoras, the consequence may be that I recognize it to be true, and further that I apply it in making a decision, which brings about the acceleration of masses. This is how our actions are usually led up to by acts of thinking and judging. [ii] And so *Gedanken* may indirectly influence the motion of masses. The influence of man on man is brought about for the most part by *Gedanken*.
>
> (Frege 1918–19, 76–77)

A cursory reading of (ii) may suggest that Frege was committed to the idea that *Gedanken* have properties that make them apt to cause us to act.

This reading however implies an inconsistent view of what it is for an entity to be abstract since abstract entities have no such causal powers. Whether this is the correct reading of Frege is open to different interpretations. In defence of Frege, one may note for instance that nothing in what Frege says about the relation between a judgement and a *Gedanke* in (i) commits him to the idea following which *Gedanken* must have consequences on our behaviour. One charitable interpretive line thus consists in saying that in (ii) it is not the *Gedanke* itself that acts but *the judgement grasping it*, that the relation of grasping is not itself a real relation and that this is emphasised by the fact that Frege specifies that *Gedanken* act only "indirectly".

By contrast to Frege, Bolzano nowhere suggests that there could be a causal interaction between the content of cognitive attitudes and these attitudes themselves. He is explicit on the fact that while judgements have causal powers, propositions don't (Bolzano 1837, §19, 78) and, more generally, he is clear on the fact that there can only be a causal relation between causal things (cf. 1837, §168, 208). In fact, Bolzano suggests in some places that talk of a relation of grasping is in fact metaphorical.[6] On the one hand, he explains, the meaning of an expression such as 'a thought grasps a proposition' should be understood in the context of his theory (cf. Bolzano and Exner 1931, 84–85). On the other hand, he also argues that to say of a judgement that it "contains a true proposition" is equivalent to saying of this judgement that it "is correct" or that it "conforms to truth" (cf. 1837, §34, 154; §36, 163). But while the notion of grasping appears to have existential import – it suggests that there is indeed a proposition such that it is contained in the judgement – the notion of a judgement's being correct or conform to truth only implies that the judgement has a certain property, not that it be related to another individual in some particular way.

Although Bolzano's characterisation of propositions quite consistently suggests that he conceives of propositions as mind-independent entities that subsist in a "third realm", there are reasons to doubt that his commitment to semantic realism was uncompromising. This does not amount to saying that Bolzano definitely treated propositions as a mere *façon de parler*. Nonetheless, as Bolzano sees it:

> [T]he logician must have the same right to speak of truths in themselves as the geometer who speaks of spaces in themselves (i.e. the mere possibilities of certain locations) without thinking of them as filled with matter, although it is perhaps possible to

> give metaphysical reasons why there is not, and cannot be any, empty space.
>
> (1837, §25, 113, 114)

If Bolzano's point is that logicians must have the right to speak of truths in themselves although there may not be truths in themselves, it is not impossible that he conceived of propositions as a kind of ontological "dummies" that could – though he does not do so himself – eventually be eliminated. This interpretation finds support in the fact that in the *Wissenschaftslehre und Religionswissenschaft in einer beurtheilenden Übersicht*, Bolzano claims repeatedly that almost all his logical theses can be accepted even by someone who rejects propositions since every thesis about propositions in themselves can be transposed to thought propositions (cf. Bolzano 1841 34, 35, 50, 68). He writes, for instance:

> Everything the author asserts of propositions in themselves in the first section – with the exception of what he says at §122, namely that they don't exist – holds of thought proposition; likewise, in the second section, the "Differences amongst propositions as regards their internal properties" are all such that whoever admits of thought propositions can also admit of them.
>
> (Bolzano 1841, 50)

And again:

> But now the question arise whether someone who rejects the concept of propositions in themselves and accepts only that of thought propositions could nonetheless admits of a connection amongst the latter more or less like the one Bolzano describes as objective. And this, we think, should be answered in the affirmative.
>
> (Bolzano 1841, 68)[7]

While I am inclined to take Bolzano's claim at face value, the *Wissenschaftslehre und Religionswissenschaft in einer beurtheilenden Übersicht* was written for a larger public and Bolzano's concession might have been motivated by his awareness of the fact that his theory of propositions was susceptible to criticism and harsh opposition (see Morscher 1973, 39). But there is at least one other reason to be wary of the idea that Bolzano's theory of meaning is entirely determined by his commitment to semantic realism. Bolzano's views on communication and linguistic understanding indicate a rich and sophisticated

comprehension of the types of cognitive processes that are involved in asserting something or in understanding – or misunderstanding – what someone says.

To be fair, the same holds for Frege. Though very little has been written on the topic, Frege expressed views on linguistic signs and how they function in linguistic understanding that enrich his epistemology considerably. For instance, in the second volume of the *Grundgesetze* (1903) Frege appeals to our ability to recognize certain types of mental states – what Frege terms 'intentions' (*Absichte*) – as the basis of successful linguistic understanding. According to Frege (1903, §99), signs would be useless if they could not be understood to denote the same thing at different times and in different contexts. While Frege thought that in order to fulfil this purpose different occurrences of the same sign type must have "sufficiently similar figures", he also assumed that we cannot recognise two instances of a sign as being of the same type solely on the basis of their physical characteristics.[8] According to Frege, in order to recognise that signs denote the same thing at different times and in different contexts:

> . . . nothing else is required but that there be present the intention of producing a sign which is similar to the one that had been made previously and this need only to succeed inasmuch as the reader correctly recognises this intention. In what follows, we understand by "signs of same figure" those which, following the intention of he who writes, are supposed to be the same so as to designate the same thing. (1903, §99)

As Frege sees it, we recognise that two signs are similar because we recognise that the authors' intention of producing similar signs in order to denote the same thing is in each case the same. If I am in a position to recognize the relevant intention of my interlocutor when she uses a sign, according to Frege, I also know what she means. I take it that what Frege suggests is that Marie can recognise that Pierre uses a sign only if she recognises, minimally, that Pierre's intention is to communicate something. The reason for this, as Frege puts it is that "figures which we write or print or which, in general, are produced on the surface of a physical object" (Frege 1903 §98, 105) need not be used for the purpose of linguistic communication – Pierre could be doodling or practicing sol-fa. In order to recognise them as occurrences of signs, we need to recognise that they were produced with the intention to communicate. If this idea is to be taken seriously, and though Frege is not

explicit, it cannot exclude considerations of the way in which in practical interactions we can tell what other people are trying to do prior to understanding the particular signs they use – and this seems right. However, this alone does not explain how we understand each other, and this minimal interpretation of what Frege means with 'intention' would in any case be too weak. It is unlikely that Marie can recognise Pierre's intention of producing signs that denote the same thing if Marie does not recognise Pierre's intention to denote the same thing. Arguably Marie can only know that the two occurrences of 'Frege was a logician' in Pierre's last book refer to the same state of affairs if she knows that the first occurrence of the relevant signs is in some substantial way similar to the second and this would be presupposing precisely what Frege is seeking to explain. Nonetheless, what the relevant passages of the *Grundgesetze* show is that Frege was aware that an account of the way in which we "grasp" meaning in situations of communication cannot leave out considerations to the way in which we process linguistic information and while his account is incomplete, it is also richer than what is usually supposed.

At (1837, §285, 67) Bolzano suggests that there is a distinction between what a sentence utterance means – its "*Bedeutung*" – and what the speaker intends to convey when she uses it – its "*Sinn*".[9] Bolzano writes:

> The *Bedeutung* of a sign is the idea for whose arousal it [the sign] is already determined, and which it also indeed usually arouses; the *Sinn* or *Verstand* of the latter however is the idea whose arousal we intend in individual cases [...] To consider given signs in order to extract what ideas he who produces them wants to arouse in us is to *read* them in the widest sense. To extract from them the ideas he who produces them wants to produce is to *understand* them.
>
> (1837, §285, 67)

Bolzano's discussion of the relation between the *Bedeutung* and the *Sinn* of an utterance is terse, and one may only speculate as to what Bolzano might have added. Nonetheless, at least four points can be made. First, just like what is the case with Frege, in Bolzano the intention to use a sign is what determines it as a sign. In this sense, as Bolzano puts it, a sign remains a sign even if it is misunderstood or remains unnoticed (1837, §285, 77). Second, it would seem natural to interpret what Bolzano is saying as implying that there is a difference between the literal meaning of words – what the signs are "designed to convey" – and

the content they express, that is, what the speaker intends to communicate when she uses them. When Marie says something to Pierre, in order to understand what she says, Pierre must have the recognitional capacities necessary to figure out the *Bedeutung* of her utterance. If Marie says 'Die Katze ist auf der Fussmatte', Pierre must have the minimal linguistic competences in German that enable him to understand the meaning of each term and the way in which they are connected. But understanding the literal meaning of an utterance is not sufficient for determining what "the speaker intends to convey" with them, that is, the *Sinn* of the utterance used in a particular situation. Determining the *Sinn* of an utterance requires the hearer to engage in an analytic procedure Bolzano calls 'Auslegung' whose purpose is the elucidation of context-sensitive, ambiguous and vague terms from the sentence in question (see Chapter 3). Though this might not be a conscious process, Pierre can understand the proposition Marie expresses when she says "Die Katze ist auf der Fussmatte" only if in addition to figuring out the literal meaning of these words, he also finds in context the elements that allow him to determine to which cat is on which mat, at which time and in which location Marie refers on this occasion if her utterance is to be true. As Bolzano sees it, interpretation should be subject to the principle of charity, that is, our interlocutor should assume that the literal meaning of what we say is such that the corresponding proposition "expresses something true and reasonable" (1837, §285, 79). When I use a metaphor, for instance, the meaning of my words is not the literal meaning but a determinate "borrowed" meaning (1837, §285, 71), and it is the interlocutor's task to realise this – though he may fail. If the interlocutor fails, then the proposition he will take me to express will be "absurd".

Third, since determining the speaker's intention is a condition for determining the proposition the speaker expresses through her utterance, what the signs are designed to convey never coincides with what the utterance means in context – even in the case of utterances that do not contain ambiguities, indexicals or vague terms. If I know that Marie does not speak German, when she utters 'zwei un zwei sind vier', I know that she is not expressing a proposition since I cannot ascribe her the intention to effectively communicate the fact that two and two are four. Even if I believe that Marie speaks German, in order to know that she refers to the arithmetic theorem and does not rather wish to express that she feels overwhelmed by the fatality of a particular event, I must be in a position to ascribe her that intention and can again only do so in context.

Fourth, Bolzano considered that linguistic communication invariably occurs in practical contexts in which what the speaker wants us to believe is relevant to our understanding it. As Bolzano puts it, typically:

> someone acts with the determinate intention that we [...] conclude from perceiving his action that he wants us to accept the proposition [...] because he also himself holds it to be true [...].
>
> (Bolzano 1837, §306, 203)

More generally, as Gieske's (1997) rich analysis of Bolzano's views on what is it to say something to someone, that is, to "give a testimony" (*Zeugnis*) suggests Bolzano is likely to have subscribed to the idea that an agent A expresses a proposition p with the aim of communicating it to agent B if:

> A is performing an action with the specific intention that B, if he follows his best insight will conclude from perceiving it that A wants B to believe that p because A himself believes that p.
>
> (Gieske 1997, 259)

According to this interpretation, understanding what a speaker says consists in understanding a complex and sophisticated form of voluntary action that must be perceived *as voluntary* in order to be understood. Marie's perceiving of Pierre's linguistic behaviour as voluntary is a necessary condition to her understanding what Pierre wants her to believe. When on a given occasion Pierre asserts "Alonzo is an admirable logician", he is not simply producing sounds that are meant to be perceived by Marie. He is seeking to engage Marie to have a determinate belief. Typically, what Pierre demands from Marie when he asserts something is, on Bolzano's account, that Marie "co-believe" the proposition he expresses. Of course, as Bolzano explains, Marie does not effectively need to understand the proposition Pierre is expressing:

> It does not matter whether [he who speaks] has really achieved the intention of his action; i.e. whether the hearer has noticed his action or not, or could guess its point or not, and whether he has come to believe [the proposition] or not. Depending on the circumstances [what he says ...] is understood or not understood, believed or not believed, but in any case [something has been said].
>
> (Bolzano 1834, volume I, 84)[10]

More generally, Marie is free to believe or not to believe the proposition expressed: she can doubt it, deny it or simply not notice it. Bolzano's point is that in order to understand what Pierre asserts and act upon it, Marie must at least recognise that he produces the sounds in order to arouse in her a disposition to co-believe.

By making the intention of the speaker relevant to linguistic understanding in a situation of communication, and despite the fact that their views diverge – for one thing Bolzano's are vastly richer than Frege's – Bolzano and Frege expand the scope of their respective theory far beyond the epistemology of "grasping" they seem committed to. In particular, it would be a mistake to assume that Bolzano's study of language is solely determined by a strong commitment to semantic realism and antipsychologism. His investigation of what it means for an agent to understand the proposition her interlocutor expresses rests on substantial views on the use of language in situations of communication. His understanding of linguistic practices is good evidence for the fact that his views on truth and meaning extend beyond what is generally assumed. This is not a trivial point to make. Though his theory deals to a large extent with logic and its epistemology, and while this book is mostly involved with the latter, it would be wrong to assume that Bolzano's achievements in the philosophy of language are exhausted by his "construction of a systematical semantics, which anticipates the modern semantics of Taski and other thinkers of the past [60] years [...]" (Føllesdal 1997, 6) For Bolzano, language is something we use to communicate and while it is clear that he considered formal analyses to play an important part in the philosophy of language, his *Theory of Science* encompasses a rich pragmatic that extends from the analyses of the role played by signs in linguistic communication and thought (1837, §285; §334) to the way in which they should be used in scientific expositions (1837, §§637–699).

11
Husserl, Logical Psychologism and the Theory of Knowledge

It is often claimed that Frege was the one to have steered the "early" Husserl away from his presumed psychologism: Frege according to a widespread opinion was the one who prompted Husserl's criticism of psychologism in the *Logical Investigations* (cf. Bell 1994; Follesdal 2001). It is clear that there were intellectual connections between Husserl and Frege at the time Husserl was first coming to grips with the issue in the early 1890s (see for instance the Frege–Husserl correpondence, in Bernet et al. 2005, 20–31). But in light of Husserl's 1896 lectures on logic, it is much clearer that the real impetus behind Husserl's criticism was Bolzano's *Theory of Science*. Husserl's antipsychologistic position rests on a meta-epistemological reflection that has two aims. The task is to fix the respective domains of the sub-disciplines that belong to the theory of knowledge in order to explain how these sub-disciplines are connected. In this regard, Husserl considerably diverges from Frege whose treatment of metaphilosophical questions of this type is insubstantial. Directly and indirectly – "indirectly" because Twardowski played an important role in Husserl's rediscovery of Bolzano around 1894 – Husserl is indebted to Bolzano for many of his views on the nature of logic, its relation to psychology and their respective role within the theory of knowledge. While the historical connection between Husserl and Bolzano is no longer altogether ignored, it is still nonetheless inadequately documented. In particular, most of the research focuses on the role Bolzano might have played in Husserl's adopting a position akin to semantic realism in logic. Bolzano's influence is to be found in Husserl's *Logical Investigations*, especially the first book, in a much more substantial manner, and this is what I will try to argue in what follows.

Consider the following questions:

(1) What is a valid inference?
(2) What types of mental states are involved in a valid inference?
(3) What types of neurophysiological processes are involved in a valid inference?
(4) How can I avoid inferential mistakes?
(5) What types of inferences provide scientifically acceptable knowledge?
(6) What types of inferences are most likely to convince one's hearer?

These questions are all independent of one another. The question raised in (1) relates to semantics. One answers this question by providing, for instance, an objective criterion on the basis of which one is to establish systematically which inferences are such as to necessarily preserve truth from the premises to the conclusion. (2) pertains to the philosophy of mind and (3) to the cognitive sciences. One provides an answer to these questions by describing, respectively, either in mental terms (i.e., in terms of beliefs, knowledge, desire, etc.), or in neurophysiologic terms, the processes that underlie the realisation of an inference by an epistemic agent. The problems brought up by (4) relate to the various heuristic procedures that support the discovery of scientific truths. (5) concerns what, in epistemology, we call justification. And, finally, (6) pertains to pragmatics and relates to the practical rules to which in a scientific presentation an agent must conform in order to insure, for instance, that his interlocutor be persuaded that the proposition she seeks to demonstrate is true.

This brief survey seeks to emphasise that the notion of inference or reasoning or, more generally, of rationality, may arise in a variety of theoretical contexts that must be discriminated. At the turn of the twentieth century however questions (1)–(6) were often not distinguished and, for the most part, the disciplines to which they pertain did not exist under separate headings. At any rate, little attention was paid to providing a clear conception of the respective tasks of or of the relations between semantics, the philosophy of mind, cognitive sciences, etc. Traditionally, the task of putting together a "doctrine of inference" or reasoning or, more generally, of rationality belonged to what philosophers called in a somewhat undifferentiated manner "logic" or "*Vernunftlehre*". However, in the absence of sufficiently clear distinctions between (1), (2), etc… logic encompassed *de facto* a set of heterogeneous concerns that co-existed in a more or less confused manner.

In the period that extends from the seventeenth century until the turn of the twentieth century a large number of philosophers started to call into question the traditional Aristotelian scholastic conception of logic. The most common criticisms consisted in saying that the logical niceties of syllogistic are inapt to account for the "laws of thought" or for human reasoning. Locke (1690) and Hume (1740, 1748) and Arnauld and Nicole (1662), for instance – and even the young Kant – were to contribute to the initial development of the idea that an investigation of the "soul" or thought unavoidably, if not exclusively, also involves psychological considerations (cf. George 2003, 99, 100). Of course, in the eighteenth century, what was called rational psychology remained an essentially philosophical discipline and its object – the soul, thoughts, the mind, reason; the terminology varies – is studied *a priori*. But in the nineteenth century, the field evolved rapidly, it became more autonomous and took a resolutely experimental turn (cf. Kusch 1995, 2). Hence the question of its relation to traditional logic and philosophy in general became more and more pressing.

One of the most influential doctrines in this respect was that of John Stuart Mill (cf. Mill 1843, 1865). Mill's position as regards the relation between logic and psychology is the following:

> Logic is not a Science distinct from, and coordinate with, Psychology. So far as it is a Science at all, it is a part of Psychology; differing from it, on the one hand, as a part differs from a whole, and on the other, as an Art differs from a Science. Its theoretical grounds are wholly borrowed from Psychology [...].
>
> (Mill 1843, 359)

By the end of the nineteenth century, Mill was the leading figure of the trend called "psychologism" – a term that he himself did not use – and a Millian-type theory as found in the work of such thinkers as Christoph Sigwart, Benno Erdamnn and Theodor Lipps, to name a few is also Husserl's main target in the *Logical Investigations*. Roughly speaking, according to the proponents of psychologism, the object of logic is thought and the laws of logic are generalisations over certain types of mental states. For someone who adopts this view, one can only answer the question:

(1) What is a valid inference?

through an empirical study of inferential processes. It requires an examination – putatively via introspection or experimentation – of the nature

and structure of the mental processes and states that are involved in an inference. The procedure is supposed to be inductive and is aimed at obtaining general propositions that state the empirical laws that underlie these mental processes. These empirical laws – that is, the "laws of thought" – are said to justify the definitions and rules of logic. As used throughout the literature, the term "psychologism" is not unambiguous. Nor is it always very well defined (cf. Kusch 1995, 4). In what follows, "psychologism" refers to the Millian thesis about the relation between logic and psychology, namely the thesis according to which logical laws are psychological generalisation. As understood here, logical psychologism claims that the rules of logic are true because they are justified by general empirical laws that concern the nature of mind or thought.

In the *Prolegomena to pure logic*, Husserl puts forward a clever and sophisticated argument that aims at showing that logic is a "normative" discipline whose "theoretical" foundation is independent of psychology and that the object of the latter is not thought in the sense of individual cognitive processes that can be observed empirically. As far as Husserl's conception of the relation between logic and psychology and their respective place within the theory of knowledge is concerned, Husserl owes a great deal to Bolzano (cf. Husserl 1913, 12). In a letter to Brentano – and one that did not please Husserl's former teacher – Husserl writes:

> In the first two (and only significant) volumes of the *Theory of Science*, a book I admire greatly, Bolzano offered extremely fruitful approaches to the treatment of a pure logic.
>
> (3 January 1905; Husserl 1994, 29)

In addition to the numerous annotations one finds in Husserl's personal copy of *Theory of Science*, the manuscript of Husserl's logic lectures of 1896 testifies to the assiduous study Husserl made of Bolzano at the time that preceded immediately the composition of the *Prolegomena*. In particular, Husserl's 1896 logic lectures contain a long introduction in the course of which Husserl attempts to determine the relation between two conceptions of logic, – logic as a "theory of science" and "pure logic" – that is the key to Husserl's criticism of psychologism.[1] Husserl paid close attention to Bolzano's division of the *Theory of Science* into five distinct parts (cf. 1837, §15, 60ff.) and seems to have (correctly) understood Bolzano's endeavour as consisting in one of its most important respects in an attempt to situate the pure part of logic, that is, the theory of

ideas and propositions in themselves within the wider framework of a general theory of knowledge that includes, according to Bolzano, in addition to investigation into the philosophy of mind (*Erkenntnislehre*) and the discovery of truth (*Heuristik*) the rules according to which one should proceed in the exposition of a scientific discipline (in textbooks) (cf. 1837, §1, 3ff.), that is, logic in the wider sense of a theory of science. Bolzano's distinction between the *theory of science* and *pure logic* was crucial for Husserl. According to what Husserl says in the introduction of his logic lectures, his task consists in the first place in defining the relation between:

> [...] a *pure* or *formal logic* by contrast with a theory of science, and we deem that it is our task to develop the latter [pure logic – SL] independently so as to preserve it from any psychological impurity.
> (Husserl 1896, §1, 43)

Husserl read attentively the passages where Bolzano makes this distinction (cf. 1837, §16, 61), especially those where Bolzano points out the fact that pure logic – what Bolzano terms '*Elementarlehre*' – is about propositions and ideas *in themselves* as opposed to thought processes and where Bolzano criticises his predecessors for having overlooked this distinction. For instance, the following passage of Husserl's copy of the *Theory of Science* is highlighted:

> Since the publication of the *Critique of Pure Reason* it has become [...] standard, especially in Germany, to bring the diverse doctrines of general logic under two headings which one usually calls the pure and applied or also the empirical part of this science. [...] My plan does not diverge considerably from the one that has become so common. [...] Nonetheless I cannot conceal that a fundamental difference between my plan and that of others consists first and foremost in the fact that I endeavour to speak of ideas, propositions and truths in themselves, while in all other logic textbooks (to the extent of my knowledge) these objects are dealt with as if they were (real or possible) appearances in the mind of a thinking being, that is only as manners of thinking [...] When undertaking to deal, in the first or pure part of logic, only with the rules of thought that hold for *all* beings (and also for God), one assumes (and not incorrectly) that these rules in some respect are no others than the conditions of truth itself, that is, that everything that must be considered to be true according to a rule of thought that holds for all rational beings is also

> objective and vice versa. But precisely for this reason, it is superfluous that one speak of the rules of conceivability. For one could instead deal only with the conditions of truth itself.
>
> (Bolzano 1837, §16, 60, 61)

It is Bolzano's distinction between pure logic and the theory of science that allowed Husserl to come up with his solution to the controversy around logical psychologism as he conceives of it. As Husserl puts it, the debate concerns the question whether logic is a normative discipline (an art or a *Kunstlehre*) or whether, on the contrary, it is a descriptive, and thus non-normative, discipline (a science or *Wissenschaft*). Husserl had first considered this question in his 1896 logic lectures where Bolzano's mark is again very clear. Husserl closes the introduction to his lectures by considering a series of "logical disputes": (i) whether logic in independent; (ii) whether is it an art (*Kunst*) or a science (*Wissenschaft*) and (iii) whether it is a purely formal discipline. Though Husserl does not agree with Bolzano on every point, and though his discussion of (ii) goes far beyond what Bolzano writes on the topic, what Husserl had in mind when he wrote the first chapters of the *Prolegomena* were the questions Bolzano had raised in the *Theory of Science* (cf. 1837, §§13, 11 and 12, respectively). In particular, as regards (ii), Husserl agrees with Bolzano when the latter writes:

> When the Schoolmen claim that *Logica est scientia, et quidem speculative*, they don't mean to dispute the fact that logic also contains practical prescriptions, but they merely say that this science does not consists *solely* of rules, but contains also many theoretical principles, from which these rules follow as corollaries. By the name *scientia speculative* [...] they only want to draw attention on the existence and the importance of these theoretical principles.
>
> (1837, §11, 45)

When Husserl resumes his discussion in the *Prolegomena*, what he conceives to be at issue in this controversy can be explained as follows. Take:

> (Barbara) If A is B and if B is C, then A is C

All would agree to say that (Barbara) is a logical "law". But the use of the word "law" is here ambiguous. (Barbara) can be said to be a law in the sense of a generalisation to which we arrive by observing actual inferential practices of real agents. In this sense, (Barbara) is an empirical

generalisation that describes the manner in which agents actually infer (or reason). Secondly, (Barbara) can be understood as an entity that is not mind-dependent and whose logical properties can be determined objectively and independently of agents' actual cognitive states. In this sense, (Barbara) is an abstract entity whose objective properties correspond to regularities in the logical domain. Finally, (Barbara) may also be understood to express a rule or a norm such that, if an agent conforms to it, she will necessarily realise correct inferences and, if she violates it, this inevitably leads her to commit fallacies. Asking whether logic is a normative or descriptive discipline thus amounts to asking which status one must ascribe to logical laws such as (Barbara). Are they descriptions (be they psychological or logical) or rules? In the first sense logic is a "science", and in the second it is an "art".

This alternative is Husserl's starting point. According to him when understood in terms of a theory of science logic undeniably aims at providing the rules, precepts, methods or norms which, when they are complied with, necessarily yield valid inferences. In other terms, that logic is a normative discipline is unquestionable and as far as Husserl is concerned it is "hard to imagine that one can dispute [...] that it is possible to ascribe to logic practical aims." (Husserl 1900, 30). In this, he agrees with Bolzano (cf. 1837, §11, 44). Yet, according to Husserl, this observation alone does not solve the problem since the task of the logician is not restricted to stating these rules. On the contrary, Husserl again echoes Bolzano (1837, §11, 45) when he writes:

> every normative and likewise every practical discipline rests on one or more theoretical disciplines, inasmuch as its rules must have a separable theoretical content [...] whose scientific investigation is the duty of these theoretical disciplines. [...]
>
> (Husserl 1900, 40)

Hence, Husserl argues, the true controversy around logical psychologism is the one that concerns the nature of the discipline that fulfils this foundational role. The psychologistic logician claims that this discipline is psychology. Husserl, on the contrary, seeks to show that this is not the case and following Bolzano that the discipline that is the basis of logic understood as a theory of science is *pure logic*.

Both Husserl and Bolzano agree that the antipsychologistic position in logic, that is the idea that logic in the wider sense of a theory of science is grounded in pure logic, is not incompatible with the idea that the theory of science proper – logic in the wider sense of a normative

discipline – is itself grounded in psychological investigations. This is the claim Bolzano makes at (1837, §13, 54) when he writes[2]:

> Logic is meant to teach us in which way we can unite our cognitions in a properly scientific whole, it is therefore precisely meant to teach us how to fin truth and uncover mistakes, etc… All this, it cannot achieve without consideration of the way in which the human mind comes to its ideas and cognitions. It must therefore necessarily include, in order to demonstrate the propositions and rules which it gives, propositions that deal with, for instance, our faculty of representation, memory, association of ideas, imagination etc. But we already have an independent science, empirical psychology, in which the object of these propositions, namely the human soul with its powers, is examined. It follows therefore that logic, if it does not depend from any other science, depends at least from psychology.

In Husserl's copy of the *Theory of Science*, the passage is highlighted, and while his thought might have evolved on this point between 1896 and 1900 – I leave the question open – in his lectures on logic, Husserl makes his agreement with Bolzano explicit:

> If logic is understood as the theory of science or method of knowledge, there is of course nothing to be said of its independence. On the one hand, logic presupposes the objective science that we have briefly designated as the science of inferences, on the other hand, psychology, since the methodological arrangements that we use in order to make the cognition of the peculiarities and laws of the objective useful for the progress of human cognition are obviously based on the psychology of intellectual activities.
>
> (1896, 32)

In this regard, Bolzano and Husserl agree, in contrast to Frege for instance, to maintain a connection between logic and psychology. While the type of investigations involved in psychology do not pertain to pure logic, they belong to logic in the wider sense of a theory of science. And this explains why, in both Bolzano and Husserl, in contrast to Frege, we find rich psychological and epistemological investigations.

Bolzano's account of what it means for logic to be an art, that is, a practical discipline as opposed to a science, that is, a theoretical one at (1837, §11) is laconic. Nonetheless, what Bolzano and Husserl have to say converges. According to Husserl, every normative discipline is

defined by three factors. First, a normative discipline is defined by its domain: it consists in a set of normative propositions about a given class of objects. The class of objects that determines the domain of ethics, for instance, is that of voluntary actions; that which defines the art of war – this is Husserl's example – is the class of military actions and virtues. Each of the propositions that pertain to a given domain states of an object or of a set of objects in the class in question that they must conform to a certain rule, they state that this object must be such and such, for instance:

A soldier must be brave

A normative discipline is defined, secondly, by a pair of evaluative predicates 'good' and 'bad'.[3] According to Husserl:

> Every normative proposition presuppose a certain kind of evaluation (approbation or appreciation) by virtue of which we form the concept of "good" [...] or, depending on the case "bad", with respect to a certain class of objects.
>
> (Husserl 1900, 43)

According to Husserl, a normative proposition is always grounded in an evaluative proposition. 'A soldier must be brave', for instance, is grounded in the evaluative propositions:

A brave soldier is a good soldier

or what amounts to the same, according to Husserl:

A soldier who is not brave is a bad soldier

More generally, Husserl (1900, 42) assumes that normative propositions of the form:

An A must be B

must be understood as implying evaluative propositions of the form:

An A which is B is a good A

or

An A which is not B is a bad A.

The idea that normative propositions imply an evaluation is an uncontroversial thesis. As Husserl puts it, an evaluation requires a criterion that will allow us to determine which objects of the domain qualify as good and which do not so as to provide a complete evaluation of the relevant domain. Without such a general and unique demarcation criterion between what is good and what is not a normative discipline would be at best a collection of propositions expressing more or less coherent beliefs about what is or is not a good object of that kind, but it would neither allow us to formulate systematically new evaluations, nor gain new practical cognitions, nor justify the beliefs we have. According to Husserl, a normative discipline is thus also defined, by a non-normative predicate that plays the role of fundamental criterion of demarcation – what Husserl calls in a somewhat confused matter a "fundamental norm". Husserl explains:

> If, for instance, we consider the production and conservation, the increase and intensification of pleasure to be the good [in ethics], then the matter will be to determine what objects generate pleasure or, depending on the case, under which subjective or objective conditions they do so; more generally, what are the necessary and sufficient condition for the generation of pleasure, for its being maintained, increased, etc. These questions [...] give us [...] normative ethics in the hedonist understanding. The evaluation of the pleasure that is created provides here the fundamental norm that determines the unity of this discipline and distinguishes it from all other normative disciplines. Thus, each discipline has its particular fundamental norm and the latter constitutes, each time, the principle that gives this discipline its unity.
>
> (Husserl 1900, 46)

If we follow Husserl, we obtain the predicate that serves as fundamental criterion when we are in a position to eliminate the term 'good' in a proposition of the form:

> An A which is B is a good A

and to replace it by a term that expresses, for every object of the class of As, the condition or criterion it must fulfil in order to be evaluated positively. Once we have identified this condition, let us call it 'C', we are in a position to formulate propositions of the form:

An A which is B is a C

for instance:

A brave soldier contributes to his troops' military success

Following this example (no longer Husserl's) and supposing that the fundamental criterion that separates good from bad soldiers is indeed the ability or propensity to contribute to his troops' military success, the question whether 'A soldier must be brave' is true depends on the truth of 'A brave soldier contributes to his troops' military success'. Yet, the latter is not a normative proposition: it specifies the necessary and/or sufficient conditions – Husserl's examples are not clear – for ascribing the evaluative predicate 'good' to a given soldier. But the question whether valour is indeed a factor of success in military combat is a question whose truth we can and must be in a position to determine on the basis of the objective facts about valour, the art of war and so on.

If we follow what precedes, Husserl's claim that the theory of science is a normative discipline implies that he must have some views about the objects that pertain to its domain, namely the method of correct inference, and the conditions the latter must fulfil in order to be good objects of that kind. He explains:

> When establishing that such and such method, for instance M_1, M_2, ... conform to the supreme aim of science [...] the theory of science states propositions of the following form: any group of mental acts of type α, β, ... that proceed according to the form of the complexes M_1 (or M_2,...) provides and instance of a correct method; or what amounts to the same: every methodological procedure of the form M_1 (or M_2,...) is correct. If we succeeded in establishing all possible and valid propositions of this or similar type, the normative discipline would then comprise all the adequate rules for all possible methods in general...
>
> (Husserl 1900, 27)

According to what Husserl says there, the evaluative propositions that ground normative statements that pertain to the theory of science have the following form:

The inferential processes α, β, ... of the form M_1, M_2,... are correct

By a "group of mental acts α, β,…", Husserl understands actual inferences realised by actual agents and individuated by some formal feature. For instance:

> Every A is B, X is A, therefore X is B (Husserl 1900, 18)

Given Husserl's conception of normativity, in order to establish the truth of a proposition of the latter form, one must determine the predicate that serves as a demarcation criterion for what form a correct inference must have. The latter will make it possible, for every similar evaluative proposition, to formulate a non-normative proposition of the form:

> The inferential processes α, β, … of the form M_1, M_2,… are C

where C expresses the "supreme aim" of science. According to Husserl (1900, §§6–7) in order to correspond to "what makes a science a science", inferences must fulfil the condition – to be specified – of having the character of "*Begründungen*". The notion of *Begründung* is also at the heart of Bolzano's *Theory of Science* (see 1837, §525; see also Chapter 7). Despite the terminological proximity, however, it is uncertain that Husserl measured the whole extent of Bolzano's views on what makes for objectively justified knowledge. Indeed, Husserl read very little of the fourth volume of the *Theory of Science* and while he annotated §524, there is no evidence that he paid special attention to the section that follows it and in which Bolzano introduces his own views on what he terms *Begründung*. Nonetheless, what Husserl says about what makes for a *Begründung* in his sense is consistent with what Bolzano writes at other places in the *Theory of Science* about the nature of scientific rationality. In particular, Husserl's views on what affords the character of *Begründung* and what does not were at least in part informed by what Bolzano has to say about the nature and structure of science. Husserl, for instance, highlighted all the passages of (1837, §1) where Bolzano explains that a science is not a mere sum of truths – whereby we have to understand an unstructured collection of truths in themselves in Bolzano's sense – that pertain to a given domain, but a collection of truths that are ordered and connected in a certain way.[4] In the *Prolegomena*, Husserl on his part writes:

> […] [i] in the concept of science (*Wissenschaft*) and its aim, there is more than mere knowledge (*Wissen*). When we experience internal perceptions, be they as a group or on their own, and we recognise

> them as present, we have knowledge, but we are far from having a science. And, in general, this holds for any groups of cognitive acts that have no relation with one another. [ii] Of course, science must give us a manifold (*Mannigfaltigkeit*) of cognitions, but not only the latter [...] there must be something more, namely a systematic ordering in the theoretical sense of the word, and [iii] this is what a *Begründung* consists in.
>
> (Husserl 1900, 13/[14s])

In this passage, Husserl explain that (i) though it is possible to attain individual cognitions and more or less eclectic sets thereof through different means, (ii) only cognitions that are ordered in a systematic way – in a deductive theory – may be said to be properly scientific and (iii) that, for a cognition, to belong to such a system and to have the character of a *Begründung* are one and the same. As Husserl explains in 1896, it is this property that defines the fundamental demarcation criterion in a theory of science:

> Once one grasps with evidence an inferential rule, one has the intuitive evidence that all conceivable reasoning of the same form and whose premises are true must also be true. *One thus obtains a valid norm that allows us to measure the correctness of all Begründungen* whose objective content has the prescribed form. We know *a priori* that when someone grounds [a conclusion – SL] following this form, he grounds [it – SL] correctly [...] If one is to understand science, one must thus understand what *Begründung* is and this means to be clear on the fact that to *Begründungen* correspond objective inferences, that these inferences are subject to laws which, when we fully recognize them, imply the totality of all forms of possible correct *Begründungen*.
>
> (Husserl 1896, 21, 22; my emphasis)

In the *Prolegomena*, the passages that deal with Husserl's notion of a *Begründung* remain programmatic. If we follow the last paragraph of the book where Husserl specifies the type of deductive structures he has in mind when he is speaking of manifolds of truths connected in a systematic manner, one must revert to the third and fourth *Investigations* which constitute a major departure from what Bolzano has in mind when he speaks of grounding. Nonetheless, what is relevant here is the fact that in both Husserl and Bolzano, the definition of the structure of a deductive system requires at least truth preservation and, in both cases, the crucial step in the elaboration of a theory of deductive theory is the

definition of the relation that corresponds to relations of dependence. For Husserl as for Bolzano, a theory of science is a theory of deductive theories, that is, a theory of axiomatisation. And whether a science is a good science depends on whether is can be axiomatised in this sense.

The question that opposes Husserl – and Bolzano – to psychologistic logicians is thus the issue as to which discipline statements such as:

> Only inferential processes α, β, ... of the form M_1, M_2,... have the character of *Begründungen*

belong. Is the psychologistic logician right to claim that they belong to psychology? According to Husserl, the psychologistic position is not utterly implausible. He explains:

> If we ask for the justification of such views, a most plausible line of argument is offered, which seems to cut off all further dispute *ab initio*. However one may define logic as an *art* [*Kunstlehre*] – as the *art* of thinking, judging, inferring, knowing, proving, of the courses followed by the understanding in the pursuit of truth, in the evaluation of the grounds of proof etc. – we find invariably that mental activities and product are the object of practical regulations. [...] The scientific investigation of the rules according to which this stuff should be worked over, naturally leads back to the scientific investigation of these properties. Psychology therefore provides the theoretical basis for constructing a logical art, and, more particularly, the psychology of cognition.
>
> (Husserl 1900, 52)

But Husserl nevertheless thinks that this argument is wrong (cf. Husserl 1900, 62). The psychologistic argument consists in saying the following:

(AP)

(i) A theory of deductive theories (a theory of science) contains the laws that underlie the realisation of valid inferences.
(ii) Inferences, proofs and other inferential procedures are cognitive processes realised by real epistemic agents.

Therefore:

(iii) Logical laws describe the cognitive processes of real epistemic agents.

Husserl accepts the premises (i) and (ii), but rejects the conclusion (iii). In doing so, Husserl makes an important point: the premises of (AP) are not only acceptable, but it would be hard to interpret them in a way that would make them false. Inasmuch as they are human productions, it is undeniable that, to put it in Husserl's terms:

> A science is, in the first place a unified item in anthropology: it is a unity of acts of thinking, of thought dispositions, as well as of certain external devices pertinent hereto.
>
> (Husserl 1900, 227, 228)

Of course, Husserl also believes that the theory of science should provide an explanation of cognitive processes of this type. But the latter does not concern logicians. What the logician is concerned with are not the real connections that exist among the cognitive acts realised by epistemic agents. What the logician is concerned with, according to Husserl, are the ideal or formal connections among the objective truths and the states of affairs that correspond to these cognitions. He insists:

> What makes this unified whole [...] psychological is not here our concern. We are rather interested in what makes a science science, which is certainly not its psychology, not any real context in which acts of thought are embedded, but a certain objective or ideal interconnection [...].
>
> (Husserl 1900, 227, 228)

In Husserl, just like in Bolzano, formal inferential relations are explicitly defined, not for mental processes or products of real or possible epistemic agents, but for entire systems of "propositions in themselves". For Husserl as for Bolzano, the primary bearers of semantic properties – truth and validity, for instance – are not judgements and inferences conceived as individual psychological or mental processes but, as Husserl explains to his logic students, what is thought in them objectively:

> Logical rules do not concern phenomena and mental dispositions [....] but propositions (*Sätze*) characterised as members of general classes of propositions. One might think that propositions are, as such, mental phenomena. However, one must distinguish the representation (*Vorstellen*) or thought of a proposition from the proposition itself. When I think $2 \times 2 = 4$, this thought is a conscious phenomenon. But this phenomenon, this mental act vanishes as

> soon as I turn toward another object. If I think again about the fact that 2 times 2 make 4, it is a new act; the mental phenomenon is not the same but what I thought, that is the same. Hence, countless acts, whether they are mine or that of others, rest objectively on the same thing and that identical which is not itself an individual phenomenon, is here the proposition: $2 \times 2 = 4$. But when I say that our logical laws do not concern mental phenomena, I do not mean that they have no relation to phenomena or the mental whatsoever. On the contrary, one may at once add: it is because this law exists that whoever is normally constituted, each time he is presented with two premises of this form, will also experience the evidence of the relevant conclusion [...].
>
> (Husserl 1896, 19, 20; cf. 1901, 114ff.)

Although Husserl's theory presents its own sophistication, it is ultimately on the Bolzanian distinction between a mental act and its content that Husserl's antipsychologism rests. This approach, and Husserl's conception of the relations between truths in themselves and the mental states in which they are "grasped" or "thought", allows us to understand why Husserl is able to reject the conclusion of (AP) while maintaining the premises. The psychologistic logician is mistaken to define the object of logic as thought. The object of logic is not thought or individual thoughts but rather the ideal, abstract and communicable content of the mental states in which objects and states of affairs are thought.[5] The latter are also, as we have seen above, the objects of pure logic. Husserl's antipsychologistic argument can be formulated as follows:

(AaP)

(i) A theory of deductive theory (a theory of science) contains the laws that underlie the realisation of valid inferences.
(ii) Inferences, proofs and other inferential procedures are cognitive processes realised by real epistemic agents.
(iii) Inferential procedures are individuated by their objective content.

Therefore;

(iv) Logical laws describe the formal properties of objective content or propositions in themselves.

Later in the *Logical Investigations*, Husserl makes explicit the fact that he took over his notion of objective content from Bolzano (1901 IV, §4, 302)

Husserl agrees with Bolzano to say that while logicians are concerned with the elaboration of a pure logic, the overall aim of a theory of science is to provide an account of the place of pure logic within a general theory of knowledge that also involve psychological investigations. Contrary to the psychologistic logician, Bolzano does not claim that pure logic is directly dependent on psychology but rather that the theory of science of which pure logic is one aspect must account for questions such as:

(4) How can I avoid inferential mistakes?
(5) What types of inferences provide scientifically acceptable demonstrations?
(6) What types of inferences are most likely to convince one's hearer?

The psychologistic logician, by constrast, assumes that it is the task of psychology to provide and answer to all questions that pertain to the theory of science in general.

Despite this similarity, Husserl thought that Bolzano's approach to the philosophy of mind was misguided. Husserl's conception of the tasks of philosophers and logicians, and his disagreement with Bolzano, can best be seen in the passage of the *Prolegomena* where Husserl states his overall appreciation of Bolzano's philosophy. For despite his considerable admiration for the semantic theory put forward in the two first of the four volumes of the *Theory of Science* – and it is clear that he considered Bolzano's theories them to be much superior to Frege's – Husserl also thinks that Bolzano, following the empiricist tradition, neglected to develop an acceptable philosophy of mind and thus ignored the philosopher's ultimate task. I quote the passage at length:

> While Lange warmly supported the idea of a purely formal logic, he had no notion that this idea had already been realised to a relatively hight degree [...] I am referring to Bernhard [sic!] Bolzano's *Wissenschaftslehre*, published in 1837, a work that, in its treatment of the logical 'theory of elements', far surpasses everything that world literature has to offer in the way of a systematic sketch of logic. Bolzano did not, of course, expressly discuss or support any independent demarcation of pure logic in our sense, but he provided one

> *de facto* in the first two volumes of his book, in his discussion of what underlies a *Wissenschaftslehre* or theory of science in the sense of his conception; he did so with such purity and scientific rigor, and with such a rich store of original, scientifically confirmed and ever fruitful thoughts, that we must count him as one of the greatest logicians of all time [...] In each line of his wonderful book, Bolzano show himself to be an acute mathematician, who lets the same spirit of scientific rigor rule in logic which he himself first introduced into the theoretical treatment of the basic concepts and propositions of mathematical analysis, which thereby acquired a new foundation [...] Logic as a science must be built upon Bolzano's work, and must learn from him its need for mathematical acuteness in distinctions, for mathematical exactness in theories [...]
>
> Much as Bolzano's achievement is "cast in one piece", it cannot be regarded [...] as in any way final. To mention only one point, one particularly feels his defects in epistemological directions. There are either no investigations, or else only quite insufficient one, which give genuine philosophical intelligibility to logical thought-achievements, and so provide a philosophical estimate of logic as a discipline. Such questions can be evaded by a thinker who, like the mathematician, is building theories upon theories, without having to bother himself about questions of underlying principle. They cannot be evaded by someone who undertakes to make clear to those who either fail to see or to admit a discipline's validity, or who mix up essential tasks with quite inessential ones, what the inherent justification of such a discipline really is, and what the nature of its tasks and objects may be.
>
> (Husserl 1900, 224ff.)

When Husserl claims that a theory of science must rely on psychology, the kind of investigations he has in mind, at least in the *Logical Investigations* pertain more precisely to what he calls 'descriptive psychology' and which corresponds the type of studies one pursues in the philosophy of mind. Bolzano, though his approach to the philosophy of mind differs largely from Husserl's, was engaged in the third volume of the *Theory of Science* in precisely the type of investigation that belongs to the study of consciousness and the nature of judgements and actual inferential processes. Husserl however goes one step further – he takes the step that will later open the way to what will come to be known as phenomenology. Husserl's theory does not merely aim at explaining

what is the relation between a mental act, its (intentional) content and the object it "intends", it also aims at providing an *a priori* description of the type and structure of mental states involved in cognitive processes that belong to all spheres of human activities, be they linguistic, practical or theoretical. Even more crucially, Husserl seeks to answer the traditional metaphysical questions of the relation between mind and world. Although, the idea that semantic reflections concerning, for instance, the nature of meaning, its cognitive accessibility or the question of reference require that we have recourse to considerations that pertain to the philosophy of mind has been criticised by a large number of analytic philosophers throughout the last century, Husserl and Bolzano are right to assume that it is not incoherent to maintain both that logic is independent from all psychological considerations and that, just like any other philosophical discipline, it ultimately rests on a (metaphysical) theory of mind. On the contrary, they should be seen to have anticipated much of the contemporary approach in the field.

Notes

Introduction

1. See Bolzano (1976, 21ff.).
2. Bolzano (1976, 27) attributes the view to Marian Mika, one of his teachers.
3. All translations are from the author.
4. The English text of a conference on the topic by the same author was presented at the 20th World Philosophy Congress in Boston in 1998.
5. See Winter (1970, 41ff.).
6. See Künne (1997a).
7. Zimmermann was the author of one of the compulsory philosophy textbooks used in Austrian Lyceums in the 1850s. The latter followed closely Bolzano's logic and to such an extent that Zimmermann was accused of plagiarism. Some have speculated that Zimmermann's *Philosophische Propädeutik* was the source of a clandestine Bolzanian philosophical subculture in Austria in the second half of the 1800s, and one whose traces one would find even in Wittgenstein (see Sebestik 1989). However, very early on in his career, Zimmermann became a disciple of Herbart and rewrote his *Propädeutik* entirely for the second edition (1860), removing from it much of what it owed to Bolzano. On the role of Zimmermann in the diffusion of Bolzano's work, see (Morscher 1997b).
8. While Frege did not think of truth as a predicate, he nonetheless assumed that only *Gedanken* could be related to "the True" or "the False".

1 Kant and German Philosophy

1. Kant was the main target of Austrian academic censorship at the turn of the nineteenth century. The *Allgemeine Litteratur Zeitung*, an important vehicle for Kant's ideas in the German-speaking world, was banned in Austria in 1792, and a ban on Kant's writings on religion and political philosophy – along with a general interdiction of the works Fichte and Schelling – was imposed in 1798. Their work could still be borrowed in university libraries but only by professors and only for the purpose of producing refutations (cf. Sauer 1982, 278). Attacks against Kant were often used for the purpose of reactionary propaganda: the aim of Kant's detractors was not to refute his theories but to convince a vastly untrained public of their putative ridicule. For instance, in one of the 1802 issues of the *Eipeldauerbriefe* – a "satirical" magazine funded by the secret police – one finds a piece in which a fictive 13 year old writes: "Ich habe mir jetzt das Buch vom berühmter Kant 'kauft, und da steht die ganze reine Vernunft drin, in die kost' nicht mehr als 30 Kreuzer; und da soll mir einer sagen, ob man vor Zeiten d'reinen Vernunft um so ein wohlfeiles Geld

gekriegt hat. Ich habe schon über 50 lateinische Wörter auswendig g'lernt, die ich nicht versteh, und wenn ich jetzt wo in ein' G'sellschaft komm, so werf' ich mit'n *transcendental*, und der *Tendenz*, und den *subjecten objectiv* und zwanzig andern so türkischen Wörtern herum, und da reissen alle Maul' und Aug'n auf, und gratulier'n meiner Frau Mama zum jungen Philosophen, der sich selbst nicht versteht." (The quote is from Sauer 1981, 282f. For a detailed landscape of the state of philosophy in Austria at the turn of the nineteenth century, see Sauer 1981). Some of the most telling material documenting the fate of Kantian philosophy in Austria is to be found in the report of a July 1798 meeting of the Royal Commission on the Reform of Education. During this meeting, the members discussed recommendations that had been put forward by F.S. Karpe, a disciple of Kant's critic J.G.H. Feder – the author, with C. Garve, of a famous review of the *Critique of Pure Reason* (1782) – concerning the teaching of Kant's philosophy. Their task was to decide whether the ban on Kant should be lifted. This meeting, given that all university professors were required to have their textbooks and lecture notes approved by the Ministry of Education, was meant to have a crucial impact. Karpe's recommendations are lost, but we know that he did not think that Kant's philosophy was appropriate for the classroom and as his own textbook shows – *Erklärung der Logik, Metaphysik und praktischen Philosophie nach Feders Leitfaden und dem Geiste der öffentlichen Vorlesungen an der Wiener hohen Schulen* (1804) (*Definition of Logic, Metaphysics and Practical Philosophy following Feder's Guidelines and the Spirit of Public Lectures at Viennese Universities*) – he believed that it was more beneficial to teach Wolff and Leibniz and to mention Kant only in passing (cf. Wotke 1903, 290). The recommendations of the Commission are themselves compelling when it comes to understanding the attitude of Austrian academic establishment toward Kant and the general hostility to which his writings were subject. For instance: "[one did] not think that it would be prudent to proscribe Kantian philosophy explicitly and thus betray the fear it inspires; this would only make young men even more concupiscent toward it *(lüsterner nach ihr*) and could only promote evil (*das Übel befördern*)." (Quoted in Ortner 1903, 718). It was suggested by the commission that one wait and see that the first *Critique*'s "usefulness and endurance be demonstrated" before authorising its addition to the curriculum (cf. Ortner 1903, 718). The main impediment to a straightforward interdiction against Kant, though this remains implicit in the report, was its popularity in neighbouring Germany, which made it implausible to declare it completely unfit for study – some members of the Commission considered that it would be retrograde and intellectually irresponsible to ban it entirely. For this reason, the Commission suggested a series of indirect measures meant to minimise the impact of Kant's – mainly political and religious – doctrines and his overall standing in intellectual circles. Among the Commission's recommendations were that Kant's work should be taught only at the very end of the philosophy curriculum, that it should be the object of an elective course, that it be examined from a purely historical standpoint and that the treatment of it remain synoptic. The Commission also recommended that it be taught by a carefully chosen teacher – one selected on the basis of his lack of enthusiasm (!) – without a salary, that is, by a "Dozent" to whom fees were paid directly by students. Despite the obvious reluctance of the members of the Commission to allow the teaching of Kant's philosophy,

their recommendations, had they been implemented, might have had significant influence on Austrian education. But they were not, and the ban was not lifted.

2. According to Ueberweg (1871, 61), the "logic of Kant's school" brings together authors "who essentially share its tendency [but] refrain from entering upon deeper problems and do not make up for this want by perfect accuracy, sufficiency, and clearness in the problems to which they have limited themselves." Ueberweg does not rank Bolzano among the Kantian logicians: there is no mention of Bolzano in the latter's history of logic.
3. See Sassen (2000, 139ff.).
4. Bolzano mentions Berkeley twice in the *Theory of Science* – when he considers alternatives to his explanation of the way in which empirical beliefs come about (1837, §304, 168) and the sources of error (1837, §310, 220). But Berkeley in these passages is considered among and with no greater emphasis than a considerable number of other philosophers.
5. See above, note 1.
6. The *New Anti-Kant* endeavours to assess Kant's work section by section, and to provide what despite the reverent tone of the introduction turns out to be a thorough refutation of critical philosophy, a fact to which the elaborate and detailed table of contents attests. For a discussion of the work, see the introduction to the French translation (Lapointe 2006). See also (Morscher 2003a).
7. Note that Bolzano's views on intuitions so defined are not unproblematic. Given that intuitions are invariably causally related to an object, propositions in themselves, that is, entities that have by definition no causal relations to the world, cannot properly speaking contain intuitions. While this is a real problem for Bolzano as Textor (1996, 142ff.) has argued, it does not need to be addressed here. It is sufficient for now to assume that it is consistent with Bolzano's doctrine that sentences that contain a term denoting an intuition – an individual event in the mind of the agent that implies a causal relation with an object – may be said to have empirical import.
8. Bolzano's concern for the definition of the notion of grounding spans from the *Contributions* to the *Theory of Science*. In the *Contributions*, the elaboration of this theory receives the name of "*Grundlehre*": a science that "deals with the universal laws (forms) to which things must conform in their existence" (1810, I §8). The *Contributions* approach the problem from an ontological standpoint, considering relations between things or state of affairs. By contrast, in the *Theory of Science*, Bolzano sought with the notion of grounding to determine the necessary relations that subsist among true propositions about these states of affairs in a deductive structure.
9. Bolzano's criticism rests on the assumption that mathematical knowledge is purely conceptual and that Kant was wrong to assume otherwise. This is not uncontroversial and some authors have sought to show that Kant's claims about the role of intuition in mathematics were substantial and plausible. Although I am inclined to side with Bolzano (and thus with Friedman 1992; Hintikka 1966) on this issue, I have nothing to add at this point to the debate. See Carson (2006) and Majer (2006) for a defence of the alternative "phenomenological" interpretation of Kant's views on intuitions.

2 Decomposition

1. See also Lapointe (2008).
2. While Kant takes it that all analytic judgements are of the form 'All As are Bs', he did not think that all judgements have this form. (See Kant's table of judgements at 1781, B95). But, Kant did not consider analyticity for other forms of judgement than categorical ones. This is the ground for one of the most common criticism of Kant's definition of analyticity. More on this in Chapter 5.
3. See Lapointe (2000, 2008); see also Anderson (2004). Arnauld and Nicole were the first to introduce the idea of an inverse relation between content and extension explicitly. Most of their successors however, Leibniz (1960, 469) and Kant amongst them, came to subscribe to the idea. Kant writes for instance: "As a component, each concept is contained in the idea of things; As [...] character (*Merkmal*), these things contained under it. In the first respect, each concept has a content, in the second an extension. The content and extension of a concept stand to one another in an inverse relation. Namely: the greater the number of things that fall under a concept, the smaller the number [of character] it contains and vice versa" (Kant 1800, §7, A148, A149).
4. While this might not be a relevant criticism of Leibniz whose views on infinite analysis were more articulate than what Bolzano credits him for, Bolzano is right to think that whoever holds that the analysis of a concept should provide us with the ideas of the properties of the correlative object should not include the properties particular to *one* individual thing falling under the general concept but those that belong to all of them (cf. 1837, §63, 268).
5. Kant was well acquainted with the kind of representationalism philosophers held at the time: for Meier, the author of the manual Kant was using in many of his lectures, ideas were like "paintings" (*Gemälde*) or images (*Bilder*) of things (cf. Coffa 1991, 9). Kant himself suggests that concepts are images of the things they represent when he claims that they are structured, namely like the objects which they represent. He writes for instance: "An idea is composed of its conceptual parts in the same manner as the whole thing represented is composed of its parts. Just like, for instance, we can say of the notes on a musical score that they are a representation of the harmonic connection of tones, not because each note is similar to each tone but because the notes are connected together just like the tones are" (Ak 16, 78, my translation). Likewise, when Kant discusses the role of schemata in mathematical knowledge not only does he explicitly adopt the view that concepts are in some systematic way connected to images but that they are so precisely by virtue of their relation to the schemata that make them possible in the first place: "We can only say that much: the image is a product of the empirical powers of the productive faculty of imagination, the schema of a sensuous concept a product and at the same time a monogram of the pure faculty of imagination *a priori* through which the images become possible at all [...]" (Kant 1781, B180). While Kant is clear that schemata themselves are not pictures, he also thought that they are the "representation of a general procedure of the imagination for providing a concept with its image" (Kant 1781, B180). However, there are passages in Kant that suggest that the kind of structure we find in concepts is not such as to provide an image of the object. According

to Hanna (1997, 143, 144), for instance, the kind of microstructure one finds in Kantian concepts is defined on the basis of relations of "inferiority" and "superiority" that come out as a result of standard definitional procedure – Hanna refers to (Kant 1781, B40; 1800, §9, 58) in particular. Yet again, it is uncertain that Kant had the intention of pursuing this: when Kant discusses the conditions of an adequate analysis (cf. Kant 1800, §104; 1781, B755), he says nothing that would imply that components in a concept are structured in any way. In fact, it is unclear whether Bolzano himself did take Kant to have held any of these views in particular, and Bolzano did not attack Kant's views on analysis directly – in the relevant passages, Bolzano mentions only Abicht and Herbart.

3 Meaning and Analysis

1. For an alternative interpretation, see Textor (1996, 37ff.).
2. To be fair, Bolzano does claims that it is a consequence of his view on the relation between subjective ideas and objective ideas that propositions may contain what he calls "intuitions in themselves", that is, the objective correlate to subjective intuitions (cf. 1837, §72, 326f). Assuming that the view is consistent, this would imply that indexicals cannot be eliminated. In what follows, I focus on Bolzano's views on conceptual languages in which, by definition, no intuitions can be contained. For a thorough discussion of Bolzano's theory of intuition – and intuition in themselves – see (Textor 1996, 78ff., 90–124, 142ff.).
3. Bolzano's use of the notion of equivalence is technical: equivalent propositions are "inter-*ableitbar*": p is equivalent to q in the relevant sense if and only if p is *ableitbar* from q and q is *ableitbar* from p (with respect to a fixed vocabulary) (cf. 1837, §156, 133). If p and q are inter-*ableitbar* (with respect to a fixed vocabulary), they also have the same consequences (with respect to that vocabulary; cf. 1837, §156, 134). Since *Auslegung* is to be understood in terms of *Ableitbarkeit* – about which we will say more in Chapter 6 – and since *Ableitbarkeit* is always relative to a vocabulary, there would seem to be no absolute fact of the matter as to what constitutes an adequate interpretation of a given utterance. Nonetheless, in the light of what we've said above – and though Bolzano is not himself clear on the question – it is easy to see that the vocabulary that is held fixed is the non-contextual one.
4. This requires us to provide an account of what counts as a primitive term or an adequate definition in Bolzano. Though this question has not been settled in the literature, there is good evidence that, at least in sciences such as arithmetic, geometry and logic, Bolzano assumes that all primitive terms are defined implicitly by the axioms and all non-primitive terms are definable on the basis of the primitive terms within the axiomatic structure. I develop his view in Chapter 8.

4 A Substitutional Theory

1. Jakob and Hoffbauer are quoted by Bolzano.
2. Bolzano disagreed with Kant however as to the nature of the investigations required and in particular with the idea that the definition of notions such

as apriority and necessity pertain to a putative "transcendental" logic, that is, an investigation of the "faculties" of knowledge. In this respect, Bolzano's approach to logic is closer to that of Leibniz and Wolff, though again the parallel ought not to be overdramatized (see Chapter 1).

3. Bolzano's terminology does not always reflect this point. Most of the time in fact Bolzano talks of our "modifying" certain components in propositions or of our considering these components to be "exchangeable". It has been amply argued in the literature that this way of speaking is problematic for Bolzano: propositions are, according to Bolzano's own conception, abstract entities that are not subject to modifications. When they occur in Bolzano, such formulations must be taken as metaphors (1837, §147, 77).
4. The standard translation of '*Gültigkeit*' by 'validity' is misleading. In what follows, I will use the German expression and its derivatives.
5. As Bolzano puts it, just like Wolff, he assumes that truth presupposes the "possibility" of the subject (cf. §196, 328; see also 1837, §130, 24, 25).
6. Proust (1981, 220) is an exception.
7. Less obvious is what is implied: since in Bolzano analyticity, on the one hand, and universal *Gültigkeit*, on the other, define the same property – or properties that are in any case indiscernible as we will see in more detail in the next chapter – the same role would have to be ascribed to analyticity. This sounds odd and Bolzano's account of analyticity arguably rests on a misunderstanding concerning the notion he took over from Kant: the propositions Kant terms 'analytic' are not merely universal but, as Kant makes clear in the *Critique* (cf. Kant 1781, B10) also necessary and *a priori* (more on this in the next chapter.)
8. See Simons (1987) for a related problem.
9. For Bolzano's definition of the notion of *Ableitbarkeit*, see Chapter 6.
10. Bolzano's actual paraphrase is: *"Die Vorstellung von einem Inbegriff wahrer Sätze unter den M, N, O, . . . hat Gegenstandlichkeit"* i.e., [[{[M], [N], [O], . . .}, which has the property of having a part, which has truth] is objectual]. The above formalisation is equivalent, yet simpler.

5 Analyticity

1. To be fair, it must be stressed that Bolzano is not always consistent and that while he claims that logical properties are to be defined for propositional forms – for "entire genera of propositions" (Bolzano 1837, §12, 48) – he also often speaks of a determinate proposition having this or that property (with respect to a determinate set of variable ideas). It would nonetheless be mistaken to conclude that he ascribes properties such as analyticity to individual propositions. At most one may says that he ascribes analyticity to proposition by virtue of their containing certain variable constituents.
2. Firstly, like universal *Gültigkeit* analyticity is defined as the property of the set of substitution instances of a propositional form that fulfil a condition of equiveridicality: all objectual substitution instances of a propositional form must have the same truth-value. Secondly, there are in Bolzano, both analytically true (universally *gültige*) as well as analytically false (universally *ungültige*) propositional forms. Finally, just like universal *Gültigkeit*, analyticity is subject to the objectuality constraint: only substitution instances whose

subject-idea denotes an object are to be admitted as acceptable variants. There are no further conditions analytic propositions need to fulfil. Nor are there any additional conditions universally *gültige* propositions need to fulfil either.

3. By contrast, as Bolzano points out, since analytic propositions in the broader sense contain non logical terms that are not varied at all, the recognition of their analyticity and of their truth requires non logical cognitions as well as logical ones.
4. Bolzano writes: "Whoever concedes what precedes will also easily admit that there are not only analytic propositions but also synthetic ones in both classes of propositions, in the class of intuitive propositions as well as in the class of purely conceptual propositions." (1837, §197.3, 337) Take 'X, which is a figure that has three angles, is a figure'. The latter is logically analytic. However since the variation is arbitrary there are substitution instances of the latter such that the reference will be carried by a demonstrative, as in: 'This, which is a figure that has three angles, is a figure. The problem is that given the objectuality constraint, it would seem that in order to know that the substitution instances in question are true, we need to know that whatever is substituted for 'This' makes the proposition objectual, i.e. that the reference of 'this' is effectively a particular triangle and not Caius, for instance. This would requires us to revert to the context and hence to some aspect of our experience. This is problematic inasmuch as it would put into question Bolzano's claim that logically analytic propositions can be known by virtue of conceptual knowledge alone. I think that the problem can be put aside for at least two reasons. On the one hand, since one can know a priori that a logically analytic proposition is analytic, one can know a priori that if the proposition is objectual, it is also true. One does not need to verify the objectuality of each instance in order to decide whether a proposition is logically analytically true (false): since one knows that it is logically analytic, one knows that if it is objectual, it is also true (false). On the other hand, the problem does not arise in purely conceptual disciplines which, by definition, do not involve demonstrative or other terms "intuitive" in Bolzano's sense. Since deductive disciplines are purely conceptual it would seem that the latter are not subject to the difficulty. Hence, while he may not be in a position to define logical analyticity for natural languages that contain intuitive components, Bolzano is in any case still in a position to do so for conceptual disciplines such as arithmetic and geometry, for instance. More on this in what follows.
5. At the same place, Bolzano notes that there are also sentences that are "pseudo-analytic", that is, sentence such that while they appear to fulfil the conditions of logical analyticity, they in fact express a proposition whose meaning is entirely. He gives as examples truisms such as 'What is bad is bad' (cf. *WL* §148, note 1, 85).

6 *Ableitbarkeit* and *Abfolge*

1. See Rusnock and Burke (2010) for an argument to the contrary.
2. Bolzano thought that the type of class relations he had defined for ideas – also conceived in terms of compatibilities – had an equivalent at the propositional level. Bolzano writes: "This similarity between the relations among

propositions and the relations among ideas that I designated with the same name at §94 and which I extend [to objectless ideas] at §108 is clear. The fact that certain objects be effectively represented is to ideas what the fact that truth is to proposition." (1837, §154, 101) Two ideas A and B are compatible if they have one object in common; they are incompatible if they have no object in common. Likewise, a set of propositions is compatible if there is at least one substitution with respect to a given variable that makes all its members true at the same time. Similar equivalences hold for the relations of subordination (*Unterordnung*), intersection (*Verschlungenheit*) and equivalence (*Gleichgültigkeit*). See Sebestik (1992, 170–173) for the theoretical background of Bolzano's theory of extensional relations between ideas. Sebestik claims that the possibility to transfer extensional relations among ideas to proposition is "one of Bolzano's most important discovery" (1992, 233). Bolzano, at least seems to have thought so too (cf. 1837, §155, 113) though it is difficult to say what this tells us about his logic.

3. As regard reflexivity, the assumption that the S, S′, S″… must share at least one variable follows from the fact that every time S, S′, S″… contain a falsehood that does not share at least one variable idea i, j, with the conclusion T, T′, T″,…, then there is no substitution that can make both the premises and the conclusion true at the same time, and the compatibility constraint is not fulfilled. See (1837, §154, 107, 108).
4. Tatzel (2002, 1) suggests that the notion of grounding provides an account of the relation expressed by statements of the form 'q, because p'.
5. In some theories, what comes under the heading of 'grounding' is taken to belong to formal ontology. In Husserl, for instance, the notion of grounding is conceived as a relation of ontological dependence used to define what he takes to be the many ways in which parts of a whole relate to one another and to the whole of which they are part (see Correia 2004; Fine 1995). Bolzano does not have an account of relations of ontological dependence so conceived.
6. Note that the regression to primitive propositions is not in principle affected by the fact that the same proposition may appear at different levels of the hierarchy. While the grounding order is structured vertically and cannot have infinitely many distinct immediate antecedents, in order to conduct basic inductive mathematical demonstration, the horizontal structure needs on its part to allow for recursions. Provided that the recurring propositions do not appear on the same branch of the tree, Bolzano is in a position to avoid loops that would make it impossible to guarantee that we ever arrive at the primitive propositions or that there be primitive propositions in the first place.

7 Justification and Proof

1. The term Bolzano uses is '*Satz*', but in the context, he is referring to the kind of *Sätze* that can "present themsleves" to an agent and hence to linguistic entities.
2. For a qualification of this idea, see Bolzano (1837, §525, 261, 262; §527, 263, 264).

8 *A priori* Knowledge

1. See also, intra, 15–17.
2. The fact that Bolzano assumed that axioms are not mere postulates but that they are in effect true – more on this in what follows – is not unproblematic. If I am to know that 'Equilateral triangles are equiangular' is true because I know that it can be deduced from the axioms that define 'equilateral', 'triangle' and 'equiangular', I must in return know that the axioms are true. But what warrants the truth of the axioms cannot in turn be the meaning of the terms they contain since the axioms define the latter. Bolzano sought to find a solution to this problem, and while his solution may be seen to be unsatisfactory, it has at least the merit of not relying on the idea that the truth of axioms rests in their evidence, a move that would have been inconsistent with Bolzano's views on epistemic warrants as a whole.
3. For Bolzano, the proposition expressed by the sentence 'There is something', that is, the proposition that the concept of something has objectuality is purely conceptual and can be known *a priori* despite the fact that it is synthetic in his sense. In order to decide whether 'There is something' is true one does not need to resort to beliefs about the world. Bolzano ought to have believed that his own demonstration that there is at least one truth at (1837, §31, 145) implies that there is something and that it implies it *a priori*. More significantly, one may also point to (1837, §99, 459) where Bolzano proceeds to a metalogical analysis of the concept of a concept's extension (*Weite*) that allows us to conclude that there is something without recourse to extra-conceptual considerations. He writes: "I believe in any case that there is an idea that is the absolute widest and the highest; I believe namely the concept of something or of an object in general to be this very idea." (1837, §99, 459) If it is a postulate of Bolzano's semantics that the concept of something is that whose extension is the widest, then it has an extension – in Bolzano ideas that are objectless do not have an empty extension, they have no extension at all (1837, §66, 298) – and I do not need to rely on experience to come to this conclusion.
4. Paul Rusnock (2000, 54) makes a convincing case that the conceptual resources Bolzano has at his disposal in 1810 were not sufficient to allow him to say that axioms define implicitly the primitive terms occurring in them. I think that this is right as regards the *Contributions*. But I also think that this no longer holds in the *Theory of Science*.
5. One ought to keep in mind that Bolzano's views on *Abfolge* (see Chapter 6) are eminently different from contemporary conceptions of consequence. One could suggest that the meta-inductive procedure Bolzano has in mind is bound to fail since, for instance, "0 = 1" has as a consequence "0 = 1 or 1 = 1". The latter is obviously true. But surely this gives us no reason at all to think that "0 = 1" is true.
6. Bolzano goes on to explain that cognitions that are accompanied by a sentiment for truth, while they can be considered to be cognitions, are however "indistinct" cognitions. Only judgements that are known through their grounds acquire the status of "distinct" cognitions. As Anita Konzelmann-Ziv pointed out (in discussion) the idea that only truths that are known through their ground are also known distinctly creates at least one difficulty for Bolzano. Since a distinct cognition always requires that we reach it through its

ground, this would seem to imply that primitive truths can never be known distinctly. The difficulty is however merely apparent. Since we cannot sense the truth of primitive judgements in the sense here relevant to Bolzano, but only that of mediated cognitions we acquire through unconscious inference (1837, §316, 260) whether or not primitive truths can be known distinctly is a question that must be decided otherwise.

9 Things, Collections and Numbers

1. See Simons (1997). My aim here is not to argue for this point since Simons' account is sufficiently convincing on its own.
2. I adopt Simons' translation.
3. Krickel provides an extensive list of Bolzano's own examples (1995, 71, 72). They include the following: sciences as collections of truth, collections of numbers (series), classes of readers, religion as a collection of opinions (*Glaube*), a human being as a collection of cells, a clockwork as a collection of wheels, springs, etc., a glass, a body as a collection of atoms, a state as a collection of people, an army, a language as a collection of signs, a book as a collection (of a different type of) signs, the solar system, time as the collection of all moments, the world as the collection of all finite beings.
4. Simons (1997) gives a similar example.
5. This would be a definition of inclusive disjunction. Exclusive disjunction is represented as a collection that contains exactly one truth.
6. Berg (1962, 165) quotes the entire passage.
7. See Schnieder (2003) for a detailed discussion. Here, I agree with Schnieder's interpretation.
8. See also Bolzano (1837, §142, 65).
9. See also Bolzano and Exner (1931, 90).
10. To be precise, Bolzano uses "Beschaffenheit" as a generic term that covers "Eigenschaften", that is, "internal" properties and "Verhältnisse", that is, external properties (cf 1837, §80, 378ff.).
11. See Siebel (2004) for a presentation of Bolzano's theory of judgement.
12. One apparent consequence of Bolzano's standpoint, but the same applies to any theory that conceives of mental events as particular properties of the mind that are individuated by their causal properties, is that it seems to make it impossible to say that two agents may have the same pain, or that the headache I have now is qualitatively identical to the headache I had yesterday. In order to be able to say that they are, one may argue, I would have to assume that they instantiate one and the same property and as we have seen above, given Bolzano's understanding of the truth conditions of propositions about real objects, we should not assume that the predicate concept in a proposition refers to a universal of this sort.

10 Frege, Meaning and Communication

1. A Leibnizian *propositio*, at least as Bolzano understood it, is "something that can be thought, i.e. that can constitute the content of a thought" (cf. Bolzano

1837, §21, 84, 85). But while Bolzano was ready to concede that the possibility of being thought is a property of propositions, he also assumed that the concept of a proposition does not entail the idea of this property. The concept of being "thinkable" is not necessary for our understanding the concept of a proposition and, hence, does not belong to its definition, assuming that it can be defined, a point on which Bolzano remains unclear (cf. Bolzano 1837, §128, 18).

2. On Bolzano's account, this excludes that there be vague propositions. See Chapter 3.
3. See Lapointe (2004). For an argument to the contrary, see Sundholm (1999) and Künne (1997a).
4. See Künne (1997b, 203). Künne presents a thorough study of the similarities and differences between Bolzano's *Gedanken* and Bolzano's propositions.
5. See Künne (1997b, 213ff.).
6. The suggestion is awkward to the extent that Bolzano repeatedly uses the notion. A judgement "grasps" (*auffasst*, *erfasst*) or "seizes" (*ergreift*) a proposition (cf. 1837, §19, 78; §24, 109; §27, 120; §34, 154; §36, 163; §122, 4) Conversely, a proposition "is" (*ist*), "makes up" (*macht . . . aus*, *bildet*) or "gives" (*gibt*) the "matter" (*Stoff*) of a judgement (cf. 1837, §22, 90; §34, 154; §122, 4).
7. See also Bolzano (1841, 34, 35).
8. Frege argues that considering the imperfect nature of human perception and the fact that two tokens of the same sign are seldom if ever exactly the same physically, we cannot rely on two signs being physically identical to decide whether they are instances of the same type – indeed, they may have quite different physical properties. Frege also assumes that "different things cannot be made to coincide by abstraction" (1903, §99, 106, 107).
9. It is important to keep the Bolzanian distinction between "Sinn" and "Bedeutung" apart from Frege's, and it would be wrong to assume that they are somehow related.
10. See Gieske (1997, 255).

11 Husserl, Logical Psychologism and the Theory of Knowledge

1. Likewise, Husserl's discussion of the referential properties of terms as well as the nature and structure of their meaning is largely inspired by Bolzano's considerations on the topic (in particular at 1837, §56). Finally, and perhaps more surprisingly, Husserl' reading of the *Theory of Science* seem to coincide with the redefinition of the notion of form in terms of "species" which is the basis of his conception of logic as a purely formal discipline. While Husserl often diverges from Bolzano, most of his comments about what he calls "traditional logic" are pronouncements on one or other of Bolzano's doctrine.
2. On the other hand, Bolzano writes:

> If one does not already know the different relations of *Ableitbarkeit* and *Abfolge* which are to be found among propositions; if one has never heard of the quite distinctive modes of connection that subsist among truths

> when they behave like grounds and consequences; if one has no knowledge of the different types of propositions [...], then one is certainly not in a position to determine the rules [that state] how to recognise new truths on the basis of truths already known, how to test the truth of a given proposition, how to judge whether these propositions belong to this or this science, in which connection they should be presented in a textbook if their truth is to be fully evident to everyone, etc.
>
> (Bolzano 1837, §15, 58)

Husserl also highlighted this passage.

Bibliography

Cited Works

Ajdukiewicz, Kazimierz (1958) "Le problème du fondement des propositions analytiques" *Studia logica* 8, 259–81.

Anderson, Lanier (2004) "It Adds Up After All: Kant's Philosophy of Arithmetic in Light of Traditional Logic" *Philosophy and Phenomenological Research* 69/3, 501–40.

Aristotle (1984) "Posterior Analytics" in *Complete Works of Aristotle*, vol. I, Jonathan Barnes (ed.) Princeton, Princeton University Press.

Armstrong, David (1997) *A World of States of Affairs*, Cambridge, Cambridge University Press.

Arnauld, Antoine and Nicole, Pierre (1662) *La logique ou l'art de penser* (1993 Edition), Paris, Vrin.

Bar-Hillel, Yehoshua (1950) "Bolzano's Definition of Analytic Propositions" *Methodos*, 32–55 [republished *Theoria* 16, 1950, 91–117; reprinted in *Aspects of Language. Essays and Lectures on Philosophy of Language, Linguistic Philosophy and Methodology of Linguistics*, Jerusalem, The Magnes Press, 1970, 3–28.]

Beaney, Michael (2009) "Analysis" in *The Stanford Encyclopedia of Philosophy* (2009 Edition), Edward N. Zalta (ed.) http://plato.stanford.edu/entries/analysis/s6.html.

Beiser, Frederick (1987) *The Fate of Reason. German Philosophy from Kant to Fichte*, Cambridge, Harvard University Press.

Beiser, Frederick (1992) "Kant's Intellectual Development: 1746–1781" in *The Cambridge Companion to Kant*, Paul Guyer (ed.) Cambridge, Cambridge University Press, 26–61.

Bell, David (1994) "Reference, Experience, and Intentionality" in *Mind, Meaning and Mathematics*, Leila Haaparanta (ed.) Dordrecht, Kluwer, 185–209.

Belnap, Nuel (1993) "On Rigorous Definitions" *Philosophical Studies* 72, 115–46.

Benthem, Johan van (1984) *Lessons from Bolzano*. Stanford, Center for the Study of Language and Information, Leland Stanford Junior University.

Benthem, Johan van (1985) "The Variety of Consequence, According to Bolzano" *Studia Logica* 44/4, 389–403.

Benthem, Johan van (2003) "Is There Still Logic in Bolzano's Key?" in *Bernard Bolzanos Leistungen in Logik, Mathematik und Physik*, Edgar Morscher (ed.) Sankt Augustin, Academia, 11–34.

Berg, Jan (1962) *Bolzano's Logic*, Stockholm, Almquist Wiksell.

Bernet, Rudolf, Welton, Donn, Zavota, Gina (eds) (2005) *Edmund Husserl: Critical Assessments of Leading Philosophers*, London, Routlege.

Blackburn, Simon (2008) *Oxford Dictionary of Philosophy*, Oxford, Oxford University Press.

Blaukopf, Kurt (1996) *Die Ästhetik Bernard Bolzanos. Beiträge zur Bolzano Forschung*, 8.

Bolzano, Bernard (1810) *Beyträge zu einer begründeteren Darstellung der Mathematik*; Widtmann, Prague.
Bolzano, Bernard (1812) *Etwas aus der Logik*, Bolzano *Gesamtausgabe*, Gesamtausgabe, Stuttgart, Frohmann-Holzboog, vol. 2, B 5, 140ff.
Bolzano, Bernard (1834) *Lehrbuch der Religionswissenschaft*, Sulbach, Seidel.
Bolzano, Bernard (1837) *Wissenschaftslehre*, Sulzbach, Seidel.
Bolzano, Bernard (1838)*Athanasia oder Gründe für die Unsterblichkeit der Seele*, Sulzbach, Seidel.
Bolzano, Bernard (1839) *Dr. Bolzano und seine Gegner*, Sulzbach, Seidel.
Bolzano, Bernard (1841) *Bolzano's Wissenschaftslehre und Religionswissenschaft in einer beurtheilenden Uebersicht. Eine Schrift für Alle, die dessen wichtigste Ansichten kennen zu lernen wünschen*, Sulzbach, Seidel.
Bolzano, Bernard (1843) *Versuch einer objektiveren Begründung der Lehre von den drei Dimensionen des Raumes*, Prag.
Bolzano, Bernard (1851) *Paradoxien des Unendlichen* (reprint) Wissenschaftliche Buchgesellschaft, 1964. [*Dr Bernard Bolzano's Paradoxien des Unendlichen herausgegeben aus dem schriftlichem Nachlasse des Verfassers von Dr Fr. Přihonský*, Leipzig, Reclam (Höfler et Hahn (eds), Leipzig, Meiner, 1920)]
Bolzano, Bernard (1851) *Paradoxien des Unendlichen*, Leipzig, Reclam.
Bolzano, Bernard (1931) *Reine Zahlenlehre*, *Spisy Bernada Bolzana*, Prague, Royal Bohemian Academy of Science.
Bolzano, Bernard (BBGA) (1969) *Bernard Bolzano Gesamtausgage*, Eduard Winter, Jan Berg, Friedrich Kambartel, Jaromír Louzil, Edgar Morscher, and Bob van Rootselaar (eds), Stuttgart, Fromann-Holzboog.
Bolzano, Bernard (1976) *Ausgewählte Schriften*, Edward Winter (ed.), Berlin, Union Verlag.
Bolzano, Bernard (2007) *Selected Writings on Ethics and Politics*, Paul Rusnock and Rolf George (trans), Amsterdam, Rodopi.
Bolzano, Bernard and Exner, Franz (1931) *Der Briefwechsel B. Bolzanos mit Franz Exner*, Prague, Königliche Böhmische Gesellschaft der Wissenschaften.
Brentano, Franz (1966) "Letter to Hugo Bergmann" *Archiv für Geschichte der Philosophie* 48, 306–8.
Carnap, Rudolf (1947) *Meaning and Necessity: A Study in Semantics and Modal Logic*, Chicago, University of Chicago Press.
Carson, Emily (2006) "Locke and Kant on Mathematical Knowledge", in Carson and Huber 2006, 3–20.
Carson, Emily and Huber, Renate (eds) (2006) *Intuition and the Axiomatic Method*, Dordrecht, Springer.
Coffa, Alberto (1991) *The Semantic Tradition Fro Kant to Carnap*, Cambridge, Cambridge University Press.
Correia, Fabrice (2004) "Husserl on *Foundation*" Dialectica 58, 349–67.
Danek, Jaromir (1970) *Weiterentwicklung der Leibnizschen Logik bei Bolzano*, Meisenheim am Glan, Anton Hain.
Danek, Jaromir (1975) *Les projets de Leibniz et de Bolzano. Deux sources de la logique contemporaine*, Québec, Presse de l'Université Laval.
Dörn, Georg (1987) "Zu Bolzanos Wahrscheinlichkeitslehre" *Philosophia Naturalis* 24, 423–41.
Dubucs, Jacques and Lapointe, Sandra (2006) "On Bolzano's Alleged Explicativism" *Synthese* 150/2, 229–46.

Dummett, Michael (1991) *Frege and Other Philosophers*, Oxford, Clarendon.
Dummett, Michael (1994) *The Origins of Analytical Philosophy*, Cambridge, Harvard University Press.
Etchemendy, John (1988) "Models, Semantics, and Logical Truth" *Linguistics and Philosophy* 11, 91–106.
Etchemendy, John (1990) *The Concept of Logical Consequence*, Cambridge, Harvard University Press.
Etchemendy, John (2008) Reflections on Consequence, in (Patterson 2008), 263–99.
Feder, G.H. and Garve, C. (1782) "Kritik der reinen Vernunft von Immanuel Kant. 1781" (anonymously published) *Zugabe zu den Göttingischen Anzeigen von gelehrten Sachen*, January, 40–48. [Eng. trans. Brigitte Sassen, in *Kant's Early Critics: The Empiricist Critique of the Theoretical Philosophy*, Cambridge, Cambridge Univerity Press, 2000 53–59.]
Fine, Kit (1995) "Part-Whole", in *Cambridge Companion to Husserl*, Barry Smith and David Woodruff Smith (eds) Cambridge, Cambridge University Press.
Føllesdal, Dagfinn (1997) "Bolzano's Legacy" *Grazer Philosophische Studien* 53, 1–11.
Føllesdal, Dagfinne (2001) "Bolzano, Frege, and Husserl on Reference and Object" in *Future Pasts*, Juliet Floyd and Sanford Shieh (eds) Oxford, Oxford University Press, 67–79.
Frege, Gottlob (1818–1819) "Der Gedanke", *Beiträge zur Philosophie des deutschen Idealismus* 1, 58–77.
Frege, Gottlob (1892) "Über Sinn und Bedeutung" *Zeitschrift für Philosophie und philosophische Kritik* 100, 25–50.
Frege, Gottlob (1903) *Grundgesetze der Arithmetik II*, Hildesheim, Olms.
Frege, Gottlob (1918) "Der Gedanke" *Beiträge zur Philosophie des deutschen idealismus* 1, 58–77.
Friedman, Michael (1992) *Kant and the Exact Sciences*, Cambridge/London, Harvard University Press.
George, Rolf (2003) "Intuitions" *Philosophiques* 30/1, 19–46
Gieske, Carsten Uwe (1997) "Bolzano's Notion of Testifying" *Grazer Philosophische Studien* 53, 249–66.
Hanna, Robert (2007) "Kant, Wittgenstein and the Fate of Analysis" in *The Analytic Turn*, Mike Beaney (ed.) London, Routledge 145–67.
Hintikka, Jaakko (1966) "Kant Vindicated" in *Deskription, Existenz und Analytizität*, P. Weingartner (ed.) Pustet, München.
Hoffbauer, Johann Christoph (1794) *Anfangsgründe der Logik*, Halle.
Hume, David (1740) *Treatise of Human Nature*, Oxford, Clarendon.
Hume, David (1748) *An Enquiry Concerning Human Understanding*, London.
Husserl, Edmund (1894a) «Intentionale Gegenstände» in *Aufsätze und Rezensionen (1890–1910)* [Husserliana XXII] B. Rang (dir.), The Hague, Martinus Nijhoff, 1979, 303–48.
Husserl, Edmund (1894b) «Besprechung von K. Tawrdowski, Zur Lehre vom Inhalt und Gegenstand der Vorstellungen», in *Aufsätze und Rezensionen (1890–1910)* [Husserliana XXII] B. Rang (dir.), The Hague, Martinus Nijhoff, 1979, 349–56.
Husserl, Edmund (1896) *Logik. Vorlesung 1896*. [Husserliana Materialienbände 1], Dordrecht, Kluwer, 2001.

Husserl, Edmund (1900) *Logische Untersuchungen: Prolegomena zur reinen Logik* [Husserliana XVIII], The Hague, Martinus Nijhoff, 1975.
Husserl, Edmund (1901) *Logische Untersuchungen: Untersuchungen zur Phänomenologie und Theorie der Erkenntnis* [Husserliana XIX], The Hague, Martinus Nijhoff, 1984.
Husserl, Edmund (1911) "Philosophie als strenge Wissenschaft", *Logos* 1, 289–341.
Husserl, Edmund (1913) "Entwurf einer 'Vorrede' zu den Logischen Untersuchungen" *Tijdschrift voor Filosophie* 1/1939, 106–33, 319–39.
Husserl, Edmund (1929) *Formale and transzendentale Logik. Versuch einer Kritik der logischen Vernunft*. [Husserliana XVII] The Hague, Martinus Nijhoff, 1974.
Husserl, Edmund (1994) *Briefwechsel*, Dordrecht, Kluwer.
Jakob, Ludwig Heinrich (1791) *Grundriss der allgemeinen Logik*, Halle, Hemmerde und Schwetschke.
Kant, Immanuel (1781) *Kritik der reinen Vernunft*; in *Werke in zehn Bänden vol. 3–4*; Darmstadt, Wissenschaftliche Buchgesellschaft, 1959, 1983. [Eng. Trans. Guyer, Paul & Wood, Allen W., Cambridge, Cambridge University Press, 1997.]
Kant, Immanuel (1783) *Prolegomena zu einer jeden künftigen Metaphysik. Die als Wissenschaft wird auftreten können*, in *Werke in zehn Bänden vol. 5*, Darmstadt, Wissenschaftliche Buchgesellschaft, 1959, 1983.
Kant, Immanuel (1800) *Logik*; in *Werke in zehn Bänden vol. 3–4*, Darmstadt, Wissenschaftliche Buchgesellschaft, 1959, 1983.
Kotarbinski, Tadeusz (1968) "The Problem of the Existence of the Future", "Prolegomena to Three-valued Logic" *The Polish Review* 13, 7–22.
Krickel, Frank (1995) *Teil und Inbegriff. Bernard Bolzanos Mereologie*, in *Beiträge zur Bolzano-Forschung*, 5, Sankt-Augustin, Academia.
Künne, Wolfgang (1997b) "Propositions in Bolzano and Frege" *Grazer Philosophische Studien* 53, 203–40.
Künne, Wolfgang (1997a) "'Die Ernte wird erscheinen'…: Die Geschichte der Bolzano-Rezeption (1849–1939)" in *Beiträge zur Bolzano-Forschung* vol. 6, Heinrich Ganthaler and Otto Neu- maier (eds) Sankt Augustin, Akademia Verlag, 9–82.
Künne, Wolfgang (1998) "Substanzen und Adhärenzen – Zur Ontologie in Bolzanos Athanasia" *Philosophiegeschichte und logische Analyse* 1, 233–50.
Künne, Wolfgang (2006) "Analyticity and Logical Truth: From Bolzano to Quine" in *The Austrian Contribution to Analytic Philosophy*, Textor Mark (ed.) London, Routledge, 184–249.
Kusch, Martin (1995) *Psychologism*, Londres, Routledge.
Lapointe, Sandra (2000) *Analysis and Analyticity. Bolzano and Kant*, Dissertation, University of Leeds.
Lapointe, Sandra (2002) "Bolzano et la réception de Kant en Autriche" in *Années 1781–1801. Kant. Critique de la raison pure. Vingt ans de réception*, Piché Claude (ed.) Paris, Vrin, 263–71.
Lapointe, Sandra (2004) "Why Frege never read Bolzano" in *Logica Yearbook 2003*, Behounek Libor (ed.) Prague: Filosofia, 183–94.
Lapointe, Sandra (2006) *Nouvel Anti-Kant* (translation and introduction), Paris, Vrin.

Lapointe, Sandra (2007) "Bolzano's Semantics and His Criticism of the Decompositional Conception of Analysis" in *The Analytic Turn*, Michael Beaney (ed.) London, Routledge, 219–34.
Lapointe, Sandra (2008) *Qu'est-ce que l'analyse*, Paris, Vrin.
Laz, Jacques (1993) *Bolzano critique de Kant*; Paris. Vrin.
Leibniz, Gottfried Wilhelm (1960) *Nouveau Essais*, *Philosophische Schriften*, V. Gerhardt Band (ed.) Hildesheim, Olms.
Lesniewski, Stanislaw (1991) "Is all Truth only True Eternally?" in *Collected Works*, Dordrecht/Warsaw, Kluwer/PWN, vol. I, 86–114.
Locke, John (1690) *An Essay Concerning Human Understanding*, London.
Lorenzová, Helena (1999) "Dra Bolzana Řeči (sexuálně) vzdělávací", in *Sex a tabu v české kultuře 19. století*, dir. V. Petrbok, Prague, Academia, 23–9.
Łukasiewicz, Jan (1913) *Die logischen Grundlagen der Wahrscheinlichkeitsrechnung*, Cracow, Akademie der Wissenschaft.
Majer, Ulrich (2006) "The Relation of Logic and Intuition in Kant's Philosophy of Science, Particularly Geometry", in (Carson & Huber 2006), 47–66.
Mancosu, Paolo (1999) "Bolzano and Cournot on Mathematical Explanation" *Revue d'Histoire des Sciences* 52, 429–55.
Mill, John Stuart (1843) *A System of Logic, Ratiocinative and Inductive, Being a Connected View of the Principles of Evidence and the Methods of Scientific Investigation*, Londres, Parker.
Mill, John Stuart (1865) *An Examination of Sir William Hamilton's Philosophy and of the Principal Philosophical Questions Discussed in His Writings*, Londres, Longmans.
Moese, Henryk (1981) "System Aufbau in Bolzanos Wissenschaftslehre und ihre Akntischen Quellen" in Bernard Bolzano 1781–1848. Studien und Quellen, Berlin, Akademie Verlag, 81–92.
Morscher, Edgar (1973) *Das Logische An-sich bei Bernard Bolzano*, München, Anton Pustet.
Morscher, Edgar (1997a) "Bolzano's Method of Variation: Three Puzzles" *Grazer Philosophische Studien* 53, 139–65.
Morscher, Edgar (1997b) "Robert Zimmermann – der Vermittler von Bolzanos Gedankengut? Zerstörung einer Legende", in *Bolzano und die Österreichische Geistesgeschichte. Beiträge zur Bolzano-Forschung* 6, Sankt Augustin, Academia, 145–236.
Morscher, Edgar (ed.) (2003a) "Einleitung" in Bernard Bolzano: *Neuer Anti-Kant*, Sankt Augustin, Academia, xxi–lxxxiv.
Morscher, Edgar (2003b) "La définition bolzanienne de l'analyticité logique" *Philosophiques* 30/1, 149–69.
Morscher, Edgar and Neimaier, Otto (eds) (1996) Bolzano's Kampf gegen Nationalismus und Rassismus, Sankt-Augustin, Academia.
Mulligan, Kevin, Simons, Peter, and Smith, Barry (1984) "Truth-Makers" *Philosophy and Phenomenological Research* 44, 287–321.
Neeman, Ursula (1970) "Analytic and Synthetic Propositions in Kant and Bolzano" *Ratio* 12, 1–25.
Ortner, Max (1903): "Kant in Österreich" dans *Zeitschrift für österreichischen Gymnasien* 54, 713–21
Patterson, Douglas (ed.) (2008) *News Essays on Tarski's Philosophy*, Oxford, Oxford University Press.

Perry, John (1979) "The Problem of the Essential Indexical" *Nous* 13/1, 3–21.
Příhonský, František (1850) *Neuer Anti-Kant: oder Prüfung der Kritik der reinen Vernunft nach den in Bolzanos Wissenschaftslehre niedergelegten Begriffen*, Bautzen, Hiecke.
Proust, Joëlle (1981) "Bolzano's Analytic Revisited" *The Monist* 64, 214–30.
Proust, Joëlle (1989) *Questions of Form. Logic and the Analytic Proposition from Kant to Carnap*, Minneapolis, University of Minnesota Press.
Quine, W.V.O. (ed.) (1953) "Two Dogmas of Empiricism" in *From a Logical Point of View*, Cambridge, Harvard University Press, 20–46.
Quine, W.V.O. (1960) *Word and Object*, Cambridge, M.I.T. Press.
Quine, W.V.O. (1970) *Philosophy of Logic*, Englewood Cliffs, Prentice Hall.
Rollinger, Robin (1999) *Husserl's Position in the School of Brentano*, Dordrecht, Kluwer.
Rusnock, Paul (2000) *Bolzano's Philosophy and the Emergence of Modern Mathematics*, Amsterdam, Rodopi.
Rusnock, Paul and Burke, Mark (2010) "Etchemendy and Bolzano on Logical Consequence" *History and Philosophy of Logic* 31, 3–29.
Sassen, Brigitte (2000) *Kant's Early Critics: The Empiricist Critique of the Theoretical Philosophy*, Cambridge, Cambridge Univerity Press.
Sauer, Werner (1982) *Österreichische Philosophie zwischen Aufklärung und Restauration*, Amsterdam, Rodopi.
Schnieder, Benjamin (2003) "Bolzano sur la structure des propositions et le rôle sémantique des propriétés" *Philosophiques* 30/1, 83–103.
Sebestik, Jan (1989) "The Archeology of the Tractatus: Bolzano and Wittgenstein" in *Wittgenstein – Towards a Re-Evaluation*, R. Haller and J. Brandl (eds) Vienna, Verlag Hölder-Pichler-Tempsky, 112–18.
Sebestik, Jan (1992) *Logique et sémantique chez Bernard Bolzano*, Paris, Vrin.
Sebestik, Jan (2003a) "La dispute de bolzano avec Kant" *Philosophiques* 30/1, 47–66.
Sebestik, Jan (2003b) "Husserl, reader of Bolzano" in *Husserl's Logical Investigations Reconsidered*, D. Fisette (ed.) Dordrecht, Kluwer, 71–92.
Siebel, Mark (1996) *Der Begriff der Ableitbarkeit bei Bolzano*, Sankt Augustin, Academia Verlag.
Siebel, Mark (2002) "Bolzano's Concept of Consequence" *Monist* 85, 580–99.
Siebel, Mark (2004) "Bolzanos Urteilslehre" *Archiv für Geschichte der Philosophie* 86, 56–87.
Simons, Peter (1987) "Bolzano, Tarski and the Limits of Logic" *Philosophia Naturalis* 24/4, 378–403.
Simons, Peter (1992) *Philosophy and Logic in Central Europe from Bolzano to Tarski. Selected Essays*, Dordrecht, Kluwer.
Simons, Peter (1997) "Bolzano on Collections" *Grazer Philosophische Studien* 53, 87–108.
Simons, Peter (1999) "Bolzano über Zahlen", Bernard Bolzanos geistiges erbe für das 21. Jahrhundert, Beiträge zur Bolzano-Forschung 11, Sankt-Augustin, Akademia, 217–30.
Simons, Peter (2003) "Bolzano sur les nombres" *Philosophiques* 30/1, 127–35.
Sundholm, Göran (1999) "When and Why Frege Read Bolzano", *Logica Yearbook*, Prague, Filosofia, 164–74.
Tatzel, Armin (2002) "Bolzano's Theory of Ground and Consequence" *Notre Dame Journal of Formal Logic* 43, 1–25.

Textor, Mark (1996) *Bolzanos Propositionalismus*, Berlin, De Gruyter.
Textor, Mark (1997) "Bolzano's Sententialism", *Grazer philosophische Studien* 53, 181–202.
Textor, Mark (2000) "Bolzano et Husserl sur l'analyticité" *Les Études Philosophiques* 4, 435–54.
Textor, Mark (2001) "Logically Analytic Propositions '*a posteriori*'?" *History of Philosophy Quarterly* 18, 91–113.
Textor, Mark (2003) "Caius-at-noon or Bolzano on Tense and Persistence" *History of Philosophy Quarterly* 20, 81–102.
Textor, Mark (ed.) (2006) *The Austrian Contribution to Analytic Philosoph*, New York, Routledge.
Twardowski, Kazimierz (1991) "Selbstdarstellung" *Grazer Philosophischen Studien* 39, 1–24.
Ueberweg, Friedrich (1871) *System of Logic and History of Logical Doctrines*, Bristol, Thoemmes, 2001.
Valent, Jutta (1997) "Die Grazer Universität zur Zeit Josef II" in *Bausteine zu der Geschichte der Philosophie an der Universität Graz*, Graz, 91–116.
Winter, Eduard (ed.) (1970) *Der Bolzanokreis: 1824–1833*, Vienna, Böhlau.
Wotke, Karl (1903) "Kant in Österreich vor 100 Jahren"; dans *Zeitschrift für österreichischen Gymnasien* 54, 289–305.

Bernard Bolzano's Collected Works

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Band (E1): Einleitungsband 1: Eduard Winter: *Bernard Bolzano. Ein Lebensbild*, 1969.
Band (E2/1): Einleitungsband 2: *Bolzano-Bibliographie und Editionsprinzipien der Gesamtausgabe*. Erste Abteilung, 1972.
Band (E2/1, Suppl. I): Einleitungsband 2, Erste Abteilung, Supplement I: *Ergänzungen und Korrekturen zur Bolzano-Bibliographie von Jan Berg und Edgar Morscher (Stand: Ende 1981)*, 1982.
Band (E2/1, Suppl. II): Einleitungsband 2, Erste Abteilung, Supplement II: *Ergänzungen zur Bolzano-Bibliographie von Jan Berg und Edgar Morscher (Stand: Anfang 1987)*, 1988.
Band (1, 2): Reihe I, Band 2: E*rbauungsreden für Akademiker. Herausgegeben von Jaromír Louzil*, 1985.
Band (1, 6/1): Reihe I, Band 6, Teil 1: *Lehrbuch der Religionswissenschaft, Erster Teil, §§ 1–85. Herausgegeben von Jaromír Louzil*, 1994.
Band (1, 6/2): Reihe I, Band 6, Teil 2: *Lehrbuch der Religionswissenschaft, Erster Teil, §§ 86–177. Herausgegeben von Jaromír Louzil*, 1995.
Band (1, 7/1): Reihe I, Band 7, Teil 1: *Lehrbuch der Religionswissenschaft, Zweiter Teil, §§ 1–54. Herausgegeben von Jaromír Louzil*, 1996.
Band (1, 7/2): Reihe I, Band 7, Teil 2: *Lehrbuch der Religionswissenschaft, Zweiter Teil, §§ 55–83. Herausgegeben von Jaromír Louzil*, 1997.
Band (1, 8/1): Reihe I, Band 8, Teil 1: *Lehrbuch der Religionswissenschaft, Dritter Teil, §§ 1–109. Herausgegeben von Jaromír Louzil*, 1999.
Band (1, 8/2): Reihe I, Band 8, Teil 2: *Lehrbuch der Religionswissenschaft, Dritter Teil, §§ 110–166. Herausgegeben von Jaromír Louzil*, 2000.

Band (1, 8/3): Reihe I, Band 8, Teil 3: *Lehrbuch der Religionswissenschaft, Dritter Teil, §§ 167–234. Herausgegeben von Jaromír Louzil*, 2004.
Band (1, 8/4): Reihe I, Band 8, Teil 4: *Lehrbuch der Religionswissenschaft, Dritter Teil, §§ 235–303. Herausgegeben von Jaromír Louzil*, 2006.
Band (1, 11/1): Reihe I, Band 11, Teil 1: *Wissenschaftslehre, Band I, §§ 1–45. Herausgegeben von Jan Berg*, 1985.
Band (1, 11/2): Reihe I, Band 11, Teil 2: *Wissenschaftslehre, Band I, §§ 46–90. Herausgegeben von Jan Berg*, 1987.
Band (1, 11/3): Reihe I, Band 11, Teil 3: *Wissenschaftslehre, Band I, §§ 91–120. Herausgegeben von Jan Berg*, 1987.
Band (1, 12/1): Reihe I, Band 12, Teil 1: *Wissenschaftslehre, Band II, §§ 121–163. Herausgegeben von Jan Berg*, 1987.
Band (1, 12/2): Reihe I, Band 12, Teil 2: *Wissenschaftslehre, Band II, §§ 164–222. Herausgegeben von Jan Berg*, 1988.
Band (1, 12/3): Reihe I, Band 12, Teil 3: *Wissenschaftslehre, Band II, §§ 223–268. Herausgegeben von Jan Berg*, 1988.
Band (1, 13/1): Reihe I, Band 13, Teil 1: *Wissenschaftslehre, Band III, §§ 269–306. Herausgegeben von Jan Berg*, 1989.
Band (1, 13/2): Reihe I, Band 13, Teil 2: *Wissenschaftslehre, Band III, §§ 307–348. Herausgegeben von Jan Berg*, 1990.
Band (1, 13/3): Reihe I, Band 13, Teil 3: *Wissenschaftslehre, Band III, §§ 349–391. Herausgegeben von Jan Berg*, 1992.
Band (1, 14/1): Reihe I, Band 14, Teil 1: *Wissenschaftslehre, Band IV, §§ 392–481. Herausgegeben von Jan Berg*, 1994.
Band (1, 14/2): Reihe I, Band 14, Teil 2: *Wissenschaftslehre, Band IV, §§ 482–578. Herausgegeben von Jan Berg*, 1999.
Band (1, 14/3): Reihe I, Band 14, Teil 3: *Wissenschaftslehre, Band IV, §§ 579–718. Herausgegeben von Jan Berg*, 2000.
Band (1, 16/1): Reihe I, Band 16: *Vermischte Schriften 1839–1840*, Erster Teil. *Herausgegeben von Jaromír Louzil*, 1989.
Band (1, 16/2): Reihe I, Band 16: *Vermischte Schriften 1839–1840*, Zweiter Teil. *Herausgegeben von Jaromír Louzil*, 1989.
Band (1, 18): Reihe I, Band 18: *Mathematisch-physikalische und philosophische Schriften 1842–1843. Herausgegeben von Gottfried Gabriel, Matthias Gatzemeier und Friedrich Kambartel*, 1989.
Band (1, 19/1): Reihe I, Band 19: *Ueber die Perfectibilität des Katholicismus*, Erster Teil. *Herausgegeben von Zdenék Kalista*, 1979.
Band (1, 19/2): Reihe I, Band 19: *Ueber die Perfectibilität des Katholicismus*, Zweiter Teil. *Herausgegeben von Zdenék Kalista*, 1979.
Band (2A5): Reihe II A, Band 5: *Mathematische und philosophische Schriften 1810–1816. Herausgegeben von Jan Berg*, 1977.
Band (2A7): Reihe II A, Band 7: *Einleitung zur Größenlehre; Erste Begriffe der allgemeinen Größenlehre. Herausgegeben von Jan Berg*, 1975.
Band (2A8): Reihe II A, Band 8: *Reine Zahlenlehre. Herausgegeben von Jan Berg*, 1976.
Band (2A10/1): Reihe II A, Band 10: *Functionenlehre. Verbesserungen und Zusätze zur Functionenlehre*, Erster Teil. *Herausgegeben von Bob van Rootselaar*, 2000.
Band (2A11/1): Reihe II A, Band 11: *Vermischte mathematische Schriften 1832–1848*, Erster Teil. *Herausgegeben von Jan Berg*, 2001.

Band (2A12/1): Reihe II A, Band 12: *Vermischte philosophische und physikalische Schriften 1832–1848*, Erster Teil. *Herausgegeben von Jan Berg*, 1977.
Band (2A12/2): Reihe II A, Band 12: *Vermischte philosophische und physikalische Schriften 1832–1848*, Zweiter Teil. *Herausgegeben von Jan Berg*, 1978.
Band (2A12/3): Reihe II A, Band 12: *Vermischte philosophische und physikalische Schriften 1832–1848*, Dritter Teil. *Herausgegeben von Jan Berg und Jaromír Louzil*, 1978.
Band (2A14): Reihe II A, Band 14: *Sozialphilosophische Schriften*. *Herausgegeben von Jan Berg und Jaromír Louzil*, 1975.
Band (2B2/1): Reihe II B, Band 2: *Miscellanea mathematica 1 + 2* [1803–1811], Erster Teil. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 1977.
Band (2B2/2): Reihe II B, Band 2: *Miscellanea mathematica 1 + 2* [1803–1811], Zweiter Teil. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 1979.
Band (2B3/1): Reihe II B, Band 3, Erster Teil: *Miscellanea mathematica 3* [1812–1813]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 1983.
Band (2B3/2): Reihe II B, Band 3, Zweiter Teil: *Miscellanea mathematica 4* [1813–1814]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 1989.
Band (2B4/1): Reihe II B, Band 4, Erster Teil: *Miscellanea mathematica 5* [1814]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 1990.
Band (2B4/2): Reihe II B, Band 4, Zweiter Teil: *Miscellanea mathematica 6* [1814]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 1991.
Band (2B5/1): Reihe II B, Band 5, Erster Teil: *Miscellanea mathematica 7* [1814]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 1993.
Band (2B5/2): Reihe II B, Band 5, Zweiter Teil: *Miscellanea mathematica 8* [1814]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 1993.
Band (2B6/1): Reihe II B, Band 6, Erster Teil: *Miscellanea mathematica 9* [1814]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 1994.
Band (2B6/2): Reihe II B, Band 6, Zweiter Teil: *Miscellanea mathematica 10* [1814]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 1995.
Band (2B7/1): Reihe II B, Band 7, Erster Teil: *Miscellanea mathematica 11* [1814–1815]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 1996.
Band (2B7/2): Reihe II B, Band 7, Zweiter Teil: *Miscellanea mathematica 12* [1815]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 1997.
Band (2B8/1): Reihe II B, Band 8, Erster Teil: *Miscellanea mathematica 13* [1815]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 1998.
Band (2B8/2): Reihe II B, Band 8, Zweiter Teil: *Miscellanea mathematica 14* [1815–1816]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 1999.
Band (2B9/1): Reihe II B, Band 9, Erster Teil: *Miscellanea mathematica 15* [1816–1817]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 2000.
Band (2B9/2): Reihe II B, Band 9, Zweiter Teil: *Miscellanea mathematica 16* [1818]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 2001.
Band (2B10/1): Reihe II B, Band 10, Erster Teil: *Miscellanea mathematica 17* [1819]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 2002.
Band (2B10/2): Reihe II B, Band 10, Zweiter Teil: *Miscellanea mathematica 18* [1820]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 2004.
Band (2B11/1): Reihe II B, Band 11, Erster Teil: *Miscellanea mathematica 19* [1820–1821]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt*, 2005.

Band (2B11/2): Reihe II B, Band 11, Zweiter Teil: *Miscellanea mathematica 20* [1821–1826]. *Herausgegeben von Bob van Rootselaar und Anna van der Lugt †*, 2006.
Band (2B16/1): Reihe II B, Band 16: *Philosophische Tagebücher 1811–1817*, Erster Teil. *Herausgegeben von Jan Berg*, 1981.
Band (2B17): Reihe II B, Band 17: *Philosophische Tagebücher 1817–1827. Herausgegeben von Jan Berg*, 1980.
Band (2B18/2): Reihe II B, Band 18: *Philosophische Tagebücher 1827–1844*, Zweiter Teil. *Herausgegeben von Jan Berg*, 1979.
Band (2B19): Reihe II B, Band 19: *Zur Physik I (1827–1840). Herausgegeben von Jan Berg*, 1995.
Band (2B20): Reihe II B, Band 20: *Zur Physik II (1841–1847). Herausgegeben von Jan Berg*, 2003.
Band (3, 3/1): Reihe III, Band 3, Erster Teil: *Briefe an František Příhonský 1824–1835. Herausgegeben von Jan Berg*, 2005.
Band (3, 3/2): Reihe III, Band 3, Zweiter Teil: *Briefe an František Příhonský 1836–1845. Herausgegeben von Jan Berg*, 2005.
Band (3, 3/3): Reihe III, Band 3, Dritter Teil: *Briefe an František Příhonský 1846–1848. Herausgegeben von Jan Berg*, 2005.
Band (4, 1/1): Reihe IV, Band 1: *Bilder und Dokumente zur Lebensgeschichte Bolzanos*, Erster Teil: *Bildnisse Bolzanos. Von Lubomír Srsen mit Photographien von Dagmar Landová. Übersetzt von Jitka Jílková und Peter Michael Schenkel*, 1986.
Band (4, 2): Reihe IV, Band 2: *Biographie Bolzanos. Von Gregor Zeithammer. Herausgegeben von Gerhard Zwerschke*, 1997.

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