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CONTENIDO

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PREFACE

The Fifth Latin American Symposium on Mathematical Logic was held at the Universidad de los Andes, Bogotá, Colombia, from July 27 to July 31, 1981. The meeting was sponsored by the Universidad de los Andes and the Universidad Nacional de Colombia, under the auspices of the Association for Symbolic Logic. It received financial support from the two sponsoring universities, the Instituto Colombiano para el Fomento de la Educación Superior (ICFES), and the Fondo Colombiano de Investigaciones Científicas (COLCIENCIAS).

The Symposium was preceded by four schort courses and followed by two more. They were: Logica Modal by N.C.A. da Costa; El ε-câlculo de Hilbert y la Lógica intuicionista by E.G.K. López-Escobar; Complejidad de teorías lógicas by K. Madlener; Fundamentos del concepto de probabilidad by R. Chuaqui; Teoría de modelos de cuerpos y anillos locales by M.A. Dickmann; and Semántica de lenguajes de programación by K. Madlener.

The conference was dedicated to the memory of Antonio Monteiro, who contributed decisively to the development of Mathematics and Logic in Portugal, Brazil and Argentina. He died in this last country in 1980. An inaugural address about his life and work was given by R. Cignoli (opening lecture of this volume).

The program of the conference consisted of invited lectures delivered by A. I. Arruda (Brazil), R. Cignoli (Brazil), M. Corrada (Chile), R. Chuaqui (Chile), N.C. da Costa (Brazil), M.A. Dickmann (France), C.A. Di Prisco (Venezuela), C. Federici (Colombia), S. Feferman (USA), G. Hoyos (Colombia), J. Keisler (USA), M. Laserna (Colombia), E.G.K. López-Escobar (USA), J. Malitz (USA), K. Madlener (Germany), J. Mycielski (USA), L. Peña (Ecuador), W. Reinhardt (USA), G. Reyes (Canada), M. Sânchez-Mazas (Spain), and M.A. Sette (Brazil). In addition eight contributed papers were read and seven more papers were presented by title.(*)

This volume contains English versions of some invited lectures made available by their authors; many of them are expanded version of the talks actually given at the Symposium. It includes also two contributed papers as well three more papers presented by title (Bunder, da Costa & Wolf, and Ro Routley).

(*) A complete list of abstracts of the talks and papers presented at the Symposium appears in the **Journal of Symbolic Logic**, Vol.48, Number 3 (1983), pp. 884-892.

A person whose devoted labor and enthusiasm was indispensable to the success of this Symposium was the late Ayda I. Arruda, active member of the Organizing Committee. The editors want to express their deep regret for the untimely demise, occurred in October of 1983. She was a decisive force in the success of many an enterprise related to Logic in Latin America, specially in Brazil, and her absence will be felt by her many friends.

The editors are indebted to the Sociedad Colombiana de Matemáticas and the Universidad Nacional de Colombia which have made possible the inclusion of this volume among their publications. Were not for the interest shown by J. Lesmes, President of the Society, and J.M. Muñoz, former head of the Mathematics Department of the Universidad Nacional, it would have been very difficult to overcome the damage caused by the original publishers when they unexpectedly broke publishing contract, causing the subsequent delay in the appearance of this volume. Special thanks are due to Mrs. Nohora de Sânchez for her excellent typing of the manuscript.

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ANTONIO MONTEIRO
1907 - 1980

Roberto Cignoli

In the following pages, I will try to give at least a faint image of the personality of Antonio Monteiro and the important role he played in the development of Mathematics and Mathematical Logic in Latin America.

Monteiro was a person firmly convinced that the scientific development of a country was a necessary condition for its economical and social development. Consequently, the major task of his life was to arouse the consciousness of the people on this issue and to promote scientific research, first in Portugal and later on in Brazil and Argentina.

Some of the leading mathematicians of these three countries, as José Sebastião e Silva and Hugo Ribeiro in Portugal, Leopoldo Nachbin and Mauricio Peixoto in Brazil, and Orlando Villamayor in Argentina, among others, were initially trained by him.

He initiated the publication of several journals and series of monographs on mathematics. Some of this publicacions are *Portugalia Mathematicae*, *Gazeta Matematica* and the series *Cadernos de Análise Geral* in Portugal; the series *Notas de Matemática*, formely published by the Universidade Nacional do Brasil and now published as an international series under the guidance of Leopoldo Nachbin, the journal *Revista Cuyana de Matemática* (Mendoza, Argentina), and the series *Notas de Lógica Matemática* (Bahía Blanca, Argentina).

One of the best mathematical libraries of Latin America, at the Instituto de Matemática in Bahía Blanca, is a result of Monteiro's personal effort.

His strict moral statute and his compromise with the development of scientific research caused him serious personal problems in the three countries where he worked. At the begining of his professional life, Monteiro refused to sign a document supporting the corporate Portuguese constitution of 1933, and the government prevented him from occupying any position at the universities. Between 1939 and 1943 he did a tremendous effort to develop modern mathematics in Portugal, but he had to earn his living by giving private lessons and cataloguing the scientific bibliography existent in Portugal. He took this last job very seriously and learned a lot about the organization of libraries. Of course, those were years of great deprivations for his family. Almost at the end of his life, in 1974, the goverment-appointed rector of the University at Bahía Blanca fired him from his position of Emeritus Professor. Since then, he was explicitly forbidden to enter the university building, not even to consult books and journals in the library that had been built up through his effort.

His complete name was Antonio Aniceto Ribeiro Monteiro and he was born in Mossamedes, Angola, on May 31, 1907. After his father's death, when Monteiro was seven years old, his family returned to Portugal. He obtained his degree of Licenciado em Ciências Matemáticas from the University of Lisbon in 1930, and immediatly earned a fellowship from the Ministry of Education of Portugal that allowed him to study at the Institut Henri Poincaré, Paris. In 1936, he got his "Docteur en Sciences" degree from the Sorbonne, with the thesis "Sur l'additivité des noyaux de Fredholm" written under the guidance of Maurice Fréchet.

Monteiro returned to Portugal, where he organized the "seminário de Análise Geral" in Lisbon, and later the "Junta de Investigação Matemática" in Porto, with the aim of training researchers in mathematics. As explained before, these activities had no support from the university.

In December of 1945 Monteiro was appointed professor at the Universidade Nacional do Brazil (now Universidade Federal de Rio de Janeiro). Later he was also a professor at the Fundação Getulio Vargas and participated in the organization of the Centro Brasileiro de Pesquisas Físicas.

In 1949 he had to leave Brazil, and was appointed professor at the Universidad Nacional de Cuyo, in San Juan, Argentina. At this university he contributed greatly to the creation of an Institute of Mathematics, entirely devoted to research, which played a very important role in the development of mathematics in Argentina. Two important events, of a wide latin-american projection, were organized by this Institute, with UNESCO support: a course for trainning latin-american professors in mathematics (February-March, 1955) and a Symposium (July, 1955) in which almost all the mathematicians that were at the time in Latin-America participated. Unfortunately, at the end of 1955 the Institute was deactivated by the university authorities. In 1956 Monteiro moved to Bahía Blanca, where, at the Universidad Nacional del Sur, he organized the Institute of Mathematics and developed the programs of "Licenciatura" and "Doctorado". In 1969-1970 he received a grant from the Consejo Nacional de Investigaciones to visit several European universities. This was his first visit to Europe after 25 years in Latin-America. In 1970, at the age of 65, he was appointed Emeritus Professor; he was fired in 1974. In 1977 he went to Portugal to return later to Bahía Blanca in 1979, where he died on October 29, 1980.

During his years in Paris, Monteiro was in touch with some of the leaders of the classical French School of Analysis, like E. Borel, H. Lebesgue, H. Hadamard and, simultaneously, was a witness to the moderns trends in the study of algebraic and topological structures. He participated in the Julia's Seminar which was the kernel of the N. Bourbaki group. At that time (late thirties), the papers by M. Stone on the topological representacion of Boolean algebras and distributive lattices, those by G. Birkhoff on the foundations of lattice theory and universal algebra, and those by A. Tarski on Boolean algebras and the relations between deductive systems and closure operators appeared. They strongly influenced Monteiro and, according to his own words, he devoted his work to the study of topological spaces, lattices, and the relations among them.

The first papers Monteiro wrote, after his return to Portugal, were on the foundations of general topology. His research on the characterization of closure operators and continuous functions naturaly led him to the theory of partially ordered sets and lattices.

The integers **Z** form a lattice, under the order relation of divisibility. The filters of this lattice are precisely the ideals of **Z** as a ring. The maximal filters are the sets of multiples of prime numbers and the prime filters are the sets of multiples of prime powers. The basic arithmetic properties of **Z** can be expressed in terms of filters. For instance, the decomposition of an integer into prime factors is equivalent to the fact that each filter in the lattice **Z** is a finite intersection of prime filters. Thus lattices can be considered as generalization of the integers, and the study of the properties of the filters of a lattice can be considered as an "arithmetic" for this lattice.

The filters of a lattice, ordered by inclusion, from a new lattice. Monteiro and his coworker's research in Brazil was mainly devoted to the study of the relationship between a lattice and the lattice of its filters. For instance, they characterized the lattice of filters and prime filters of several classes of lattices. An important property proved by Monteiro (1947) is that a lattice L is distributive if and only if each filter of L is an intersection of prime filters. Consequently, we can say that distributive lattices are exactly those lattices in which the analogue of the factorization of an integer holds.

Real numbers generalize the integers, and topological spaces, in a sense, generalize the real line. Then we may consider topological spaces as generalizationsof the integers, and it then appears natural to look for some type of arithmetic properties in topological spaces. This is the point of view adopted by Monteiro in his paper *Arithmétique des espaces topologiques* (1950)*. Given a topological space X, the closed sets of X, ordered by inclusion, form a dis-

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* Submitted to the French Mathematical Society for a contest in honor of Maurice Fréchet, it was among the four best papers presented(Bull.Soc.Math.France, 1951, XXXIX-XL). Twenty years later, published in "Notas de Lógica Matemática" of Bahía Blanca, Nº 29 (1974), under the title: *L'Arithmétique des filtres et les espaces topologiques - I.*

tributive lattice L(X). Monteiro viewed the filters of L(X) as generalized integers. He showed how some of the separation axioms currently considered in topology can be interpreted in terms of arithmetic properties of L(X). For instance, X is a normal space if and only if each prime filter of L(X) is contained in a unique maximal filter. In the lattice **Z**, this property is obvious because prime filters correspond to prime powers, and maximal filters to prime numbers. Moreover, X is a completely normal space if and only if given a prime filter P of L(X), the set of filters F of L(X) such that $P \subseteq F$ is totally ordered by inclusion. This property is stronger than the previous one, and it is also a property of the lattice **Z**. These two examples illustrate Monteiro's idea: the spaces X such that L(X) has arithmetic properties closer to those of the lattice **Z**, should be considered better generalizations of the intergers.

The lattices of closed sets are more than distributive lattices: they are Brouwerian algebras. Their duals, Heyting algebras, are the algebraic counterparts of the intuitionistic propositional calculus, and they play for intuitionistic logic the same role as that of Boolean algebras for classical logic. In the previously mentioned paper, Monteiro considered Brouwerian and Heyting algebras in detail. For instance, he gave the interesting algebraic result that the class of Boolean algebras coincides with the class of semi-simple Heyting algebras.

During his first years in Argentina, Monteiro continued his research on Brouwerian and Heyting algebras. Around 1955, Halmos introduced polyadic algebras as a tool for the algebraic analysis of quantifiers, and immediately Monteiro showed that it was possible to give a non-trivial generalization of the theory of monadic Boolean algebras to monadic Heyting algebras.

Throughout the study of Boolean and Heyting algebras, Monteiro became interested in the algebraic aspects of logic, this interest having been aroused by direct contact with Roman Sikorski and Helena Rasiowa when they visited Bahía Blanca in 1958. Monteiro applied his mathematical experience to the study of algebraic systems related to non-classical logics. He was convinced that the algebraic methods in logic would have important technological applications in the future, as a consequence of the development of computers. In view of such prospective applications, he tried to use finitistic and combinatorial methods in studying classes of algebras, whenever possible.

Given a class of algebras K, it was a basic problem for him to decide if the finitely generated free algebras in K were finite, and if so, to find explicitly the number of its elements as a function of the number of free generators. In general, to achieve this goal it is necessary to have a deep understanding of the structure of the algebras in K. As an example, let me reproduce the formula giving the number of elements $L_n(r)$ of the free algebra with r free generators in the class K_n corresponding to the n-valued Łukasiewicz propositional calculus (n an integer ≥ 2). Here, D is the set of divisors of n-1, M(d) is the set of maximal divisors of d, Λ denotes the greatest common divisor and $|X|$ is the number of elements of X:

$$L_n(r) = \prod_{d\in D}(d+1)^{\left\{\sum_{X\subseteq M(d)}(-1)^{|X|}\left(\bigwedge_{m\in X}m+1\right)^r\right\}}$$

This formula was obtained by Monteiro in 1969, solving in this way a problem which was open since 1930.

In the same vein, he determined the structure of the finite De Morgan and Nelson algebras. Monteiro investigated in depth the structure of Nelson algebras (introduced by H. Rasiowa as the algebraic counterparts of the constructive logic with strong negation considered by Nelson and Markhoff), and he proved that the class of semi-simple Nelson algebras coincides with the class of algebras corresponding to three-valued Lukasiewicz logics. Thus he showed that, from the algebraic point of view, the three-valued Lukasiewicz logic stands in the same relation to constructive logic with strong negation as classical logic does to intuitionistic logic.

Monteiro also proved that each three-valued Lukasiewicz algebra can be represented as a suitable monadic Boolean algebra. He liked this result very much, because it meant that Lukasiewicz three-valued propositional calculus had an interpretation in the classical monadic functional calculus. This results is analogous to the construction of Euclidean models of non-Euclidean geometry.

It is worthwhile to mention here that Monteiro also characterized the class of monadic Boolean algebras as the class of semi-simple closure algebras.

Monteiro won the 1978 prize for scientific and technological achievements from the Gulbenkian Foundation in Lisbon, for a manuscript on symmetric Heyting algebras, where he gave a rather detailed account of some of his work done in the previous years*

I have tried to give here a brief indication of what I consider Monteiro's main contributions to mathematics, but there are many more. Usually, he just published short summaries or abstracts of his main results. The details as well as the underlying ideas of his work were given in his courses and seminars, and they are dispersed in the notes taken by his students.

I am sure that Monteiro will live for ever in the memory of all who, like myself, have had the privilege of receiving his teaching on both, moral conduct and mathematics.

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* To appear in Portugalia Mathematicae.

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IMECC
Universidade Estadual de Campinas
Campinas, S.P., Brazil.

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REMARKS ON DA COSTA'S PARACONSISTENT SET THEORIES

Ayda Ignez Arruda*

ABSTRACT. In this paper we analyse da Costa's paraconsistent set theories, i.e., the set theories constructed over da Costa's paraconsistent logics $C_n^=$, $1 \leqslant n \leqslant \omega$. The main results presented here are the following. In any da Costa paraconsistent set theory of type **NF** the axiom schema of abstraction must be formulated exactly as in **NF**; for, in the contrary, some paradoxes are derivable that invalidate the theory. In any da Costa paraconsistent set theory with Russell's set $\mathbf{R} = \hat{x}\neg(x \in x)$, $\mathbf{R} \cup \mathbf{R}$ is the universal set. In any da Costa paraconsistent set theory the existence of Russell's set is incompatible with a general (for all sets) formulation of the axiom schemata of separation and replacement.

1. INTRODUCTION.

A set theory is *paraconsistent* if it is inconsistent but non-trivial, i.e., at least one contradiction is derived but still there are formulas that are not theorems. Thus, the underlying logic of a paraconsistent set theory must be a *paraconsistent logic*, i.e., a logic in which there is a symbol of negation $\neg$, such that, from a formula A and its negation $\neg A$, it is not possible in general to obtain any formula B whatsoever.

Paraconsistent set theory appeared as an application of paraconsistent logic. The pioneering effort to construct a paraconsistent set theory was made by N.C.A. da Costa, in 1963, in [12], the same work in which he presented his hierarchy of paraconsistent logics (see also [14] and [15]). Further attempts can be found in Arruda and da Costa [6] and [8], Asenjo and Tamburino [9], Brady [10], Brady and Routley [11], and Goodman [17]. Except for da Costa's, and Asen-

* This paper is the text, now considerably expanded, of our lecture in the Fifth Latin American Symposium on Mathematical Logic (Bogotá, July of 1981). We thank IBM of Brazil for the financial help given to attend the meeting. This paper was written when the Author was Visiting Professor at the University of Warsaw (October of 1981), with a partial grant (81/0522-4) of the 'Fundação de Amparo à Pesquisa do Estado de São Paulo' (FAPESP), Brazil.

jo and Tamburino's set theories, the others are proved to be non-trivial. Those paraconsistent set theories already proven to be non-trivial may be called *weak* paraconsistent set theories for, because of their underlying logic, many of the basic results of *classical set theories* (the usual set theories based on classical logic) are not valid in them. The others, supposing that they are nontrivial, may be called *strong* paraconsistent set theories, for all or almost all results of classical set theories are valid in them. Of course, although useful in this paper, the above characterization of strong and weak paraconsistent set theories is not precise.

Let us call a set *non-classical* if it may exist in a paraconsistent set theory but not in a classical set theory, and *contradictory set* to a nonclassical set $\mathbf{X}$ such that $\mathbf{X} \in \mathbf{X}$ and $\neg(\mathbf{X} \in \mathbf{X})$. Thus, Russell's set, $\mathbf{R} = \hat{x}\neg(x \in x)$, is a contradictory set. In the above mentioned attempts to construct a paraconsistent set theory, Russell's set is, in general, introduced through a widening of the scope of validity of the axiom schema of abstraction, but its properties (as well as the properties of other nonclassical sets) have not been studied well. However, these properties may be very important, because they characterize the behavior of the non-classical sets in a paraconsistent set theory.

In this paper we restrict ourselves to da Costa's paraconsistent set theories analysing mainly two problems: first, the widening of the scope of validity of the axiom schemata of abstraction and separation, and second some properties of Russell's set and their consequences. The results presented here may be valid in other paraconsistent set theories, as will be shown in [5].

A *da Costa* paraconsistent set theory is a paraconsistent set theory whose underlying logic is one of da Costa's paraconsistent logics $C_n^=$, $1 \leqslant n \leqslant \omega$.

The postulates of $C_\omega^=$ are those of the positive intuitionistic first-order logic with equality, axiomatized as in Kleene [18], plus:

(1) $\neg\neg A \supset A$,

(2) $A \vee \neg A$.

The postulates of $C_n^=$, $1 \leqslant n \leqslant \omega$, are those of $C_\omega^=$, plus:

(3) $B^{(n)} \& (A \supset B) \& (A \supset \neg B) \supset \neg A$,

(4) $A^{(n)} \& B^{(n)} \supset (A \supset B)^{(n)} \& (A \& B)^{(n)} \& (A \vee B)^{(n)}$,

(5) $(x)(A(x))^{(n)} \supset ((x)A(x))^{(n)} \& ((Ex)A(x))^{(n)}$,

where $A^{(n)}$ is defined as follows: $A^1 = A^0 = \neg(A \& \neg A)$, $A^{n+1} = (A^n)^0$, $A^{(n)} = A^1 \& A^2 \& A^3 \& \ldots \& A^n$.

In each $C_n^=$, $1 \leqslant n \leqslant \omega$, $\neg^* A$ is defined as $\neg A \& A^{(n)}$, and it is proved that $\neg^*$ satisfies all the properties of the classical negation. Then classical logic can be obtained inside these systems; consequently, they are finitely trivializable. For, from any formula of the form $A \& \neg A \& A^{(n)}$ we can deduce any formula whatsoever. Nonetheless, $C_\omega^=$ is not finitely trivializable. Moreover, each systems in the hierarchy $C_1^=$, $C_2^=$, ..., $C_n^=$,... $C_\omega^=$ is strictly stronger than the following ones. Thus, we may construct a hierarchy of da Costa's paraconsistent set theories in which, at least intuitively, it seems that each system may admit more nonclassical sets than the preceding ones.

For a long time, in fact since 1964, we have been studying from time to time da Costa's paraconsistent set theories of type **NF** and $\mathbf{NF}_n$, $1 \leqslant n \leqslant \omega$. We have proved that da Costa's formulation of the axiom schema of abstraction for the systems $\mathbf{NF}_n$, $1 \leqslant n < \omega$, leadsto the trivialization of the systems (see [1], [3] and [4]). Thus, we have proposed to formulate the axiom schema of abstraction in these systems exactly as in **NF**.

The main objetive of this paper is to present some results (in fact some paradoxes) that we believe are very important in the construction of strong paraconsistent set theories, particulary, da Costa's paraconsistent set theories. In Section 2, we prove that tha Costa's formulation of the axiom schema of abstraction for $\mathbf{NF}_\omega$ leads to the *paradox of identity*, $(x,y).x = y$, and conclude with some argumentation showing that this axiom in any da Costa paraconsistent set theory of type **NF** should be formulated exactly as in **NF**. In Section 3, we summarize the syntactical development or our version of $\mathbf{NF}_\omega$ showing that this system may be considered as a strong paraconsistent set theory. In Section 4, we prove that if Russell's set is introduced in any da Costa set theory based on $C_n^=$, $1 \leqslant n < \omega$, then UUR is the universal set. In Section 5, we analyse some limitation in the construction of da Costa's set theories of type **ZF** where Russell's class is a set. Finally, Section 6 is a conclusion where we call attention to some open problems concerning paraconsistent set theory.

2. ON DA COSTA'S SET THEORIES OF TYPE NF.

In this section we analyse da Costa's formulation of the axiom schema of abstraction for the set theories $\mathbf{NF}_n$, $1 \leqslant n \leqslant \omega$. In [1] and [4], we proved that da Costa's formulation of the axiom schema of abstracion for his systems of type **NF** and $\mathbf{NF}_n^C$, $1 \leqslant n < \omega$, leads to their trivialization. Here we prove that da Costa's formulation of the axiom schema of abstraction for $\mathbf{NF}_\omega^C$ leads to the paradox of identity.

The systems $\mathbf{NF}_n^C$ are constructed, similarly to **NF**, over the respective calculus of descriptions $\mathbf{D}_n$. The calculi of descriptions $\mathbf{D}_n$, $1 \leqslant n \leqslant \omega$, are constructed as usual (see Rosser [20]) from the respective $C_n^=$.

The specific postulates of $\mathbf{NF}_\omega^C$ are the following:

EXTENSIONALITY. $(\alpha,\beta):.(x).x\in\alpha \equiv x\in\beta:\supset:\alpha=\beta$.

ABSTRACTION. $(E\alpha)(x):x\in\alpha\cdot\equiv\cdot F(x)$, *where* x *and* α *are different variables*

α does not occur free in F(x), F(x) *is stratified or it does not contain any subformula of the form* $A \supset B$.

In $\mathbf{NF}^C_\omega$, the restrictions regarding the use of non-stratified formulas obstruct a direct proof of the paradox of Curry. Russell's set **R**, defined as $\hat{x}\neg(x \in x)$, exists as well as many other non-classical sets. The paradox of Russell in the form $\mathbf{R} \in \mathbf{R} \& \neg(\mathbf{R} \in \mathbf{R})$ is derivable but, apparently, it causes no harm to the system.

Due to its weakness, the primitive negation of $\mathbf{NF}^C_\omega$, $\neg$, is almost useless for set-theoretical purposes. Thus, let us define

$$\sim A \quad \text{for} \quad A \supset (x,y).x \in y \,\&\, x = y.$$

The universal set **V** is defined as $\hat{x}.x = x$, the empty set **Λ** as $\hat{x} \sim (x = x)$, and the complement of a set α, $\bar{\alpha}$, as $\hat{x} \sim (x \in \alpha)$. From now on the set theoretical notations and terminology in this and the next sections are mainly those of Rosser [20].

TEOREM 2.1. *In* $\mathbf{NF}^C_\omega$, $\sim$ *is a minimal intuitionistic negation.*

Proof. It is enough to prove $(A \supset B) \& (A \supset \sim B) \cdot\supset\cdot \sim A$. Let us suppose that $A \supset B$ and $A \supset \sim B$. Then, we obtain $A \supset (x,y).x \in y \,\&\, x = y$, i.e., $\sim A$.

COROLLARY 1. $\vdash A \supset (\sim A \supset \sim B)$.
$\vdash (A \supset B) \supset (\sim B \supset \sim A)$.

COROLLARY 2. *All the theorems of* **NF** *whose proofs depend only on the laws of the minimal intuitionistic first-order logic with equality and on the postulates of extensionality and abstraction of* **NF** *are valid in* $\mathbf{NF}^C_\omega$.

THEOREM 2.2. (Cantor's Theorem) $\vdash (\alpha) \cdot \sim (\alpha \text{ sm } SC(\alpha))$.

Proof. Let us suppose that α sm $SC(\alpha)$. Then, there exists a relation **S** such that $\mathbf{S} \in 1\text{-}1$, $Arg(\mathbf{S}) = \alpha$ and $Val(\mathbf{S}) = SC(\alpha)$. Now, let us take

$$\mathbf{X} = \hat{x}(x \in \alpha \cap \overline{\mathbf{S}(x)}). \tag{1}$$

As the formula $x \in \alpha \cap \overline{S(x)}$ is non-stratified and does not contain any sub-formula of the form $A \supset B$, then **X** is a set. Moreover, as $\mathbf{X} \subseteq \alpha$, then there exists $y \in \alpha$ such that $S(y) = \mathbf{X}$. Then, by (1), we have:

$$y \in \mathbf{X} \cdot\equiv\cdot y \in \alpha \cap \overline{S(y)}.$$

However, $y \in \alpha \cap \overline{\mathbf{S}(y)} \cdot\equiv\cdot y \in \alpha \,\&\, \sim(y \in \mathbf{S}(y))$. Then,

$$y \in \mathbf{X} \cdot\equiv\cdot y \in \alpha \,\&\, \sim(y \in \mathbf{X}). \tag{2}$$

From (2) we have $\sim(y \in \alpha)$, but, as $y \in \alpha$, then we obtain

$$(\alpha \text{ sm } SC(\alpha)) \supset y \in \alpha, \quad \text{and} \quad (\alpha \text{ sm } SC(\alpha)) \supset \sim (y \in \alpha).$$

Thus, by Theorem 2.1, the desired result follows.

COROLLARY. (Cantor's Paradox) $\vdash (\mathbf{V} \text{ sm } SC(\mathbf{V})) \& \sim (\mathbf{V} \text{ sm } SC(\mathbf{V}))$.

Proof. As $\mathbf{V} = SC(\mathbf{V})$, then we obtain $\mathbf{V}$ sm $SC(\mathbf{V})$. On the other hand, by the theorem, we have $\sim(\mathbf{V} \text{ sm } SC(\mathbf{V}))$. □

Apparently, Cantor's paradox does not trivialize $\mathbf{NF}_\omega^C$. For, from A and $\neg A$ we cannot obtain any formula B whatsoever. For instance, apparently, we cannot obtain any formula of the form $\neg B$, where B is a nonatomic formula.

THEOREM 2.3. I. $\vdash (\alpha,\beta).\alpha = \beta \& \sim (\alpha = \beta)$

II. $\vdash (\alpha,\beta).\alpha \in \beta \& \sim (\alpha \in \beta)$

III. $\vdash (\alpha).\alpha \in \alpha \& \sim (\alpha \in \alpha)$.

Proof. I. By the corollaries of theorems 2.1 and 2.2, we obtain $x = x \supset (\alpha,\beta).\alpha \in \beta \& \alpha = \beta$. Thus, as $x = x$, then $(\alpha,\beta).\alpha = \beta$. By the same corollaries we also obtain $(\alpha,\beta).\sim(\alpha = \beta)$. The proof of part II is similar to that of part I. Part III is an immediate consequence of part II. □

By Theorem 2.3, it could seem that $\mathbf{NF}_\omega^C$ is trivial. Nonetheless, apparently this is not the case. However, though it is nontrivial, $\mathbf{NF}_\omega^C$ is without interest, for not only are every two sets identical, but also every set belongs and "does not belong" to itself.

In order to avoid the results mentioned in Theorem 2.3, one could think of introducing more restrictions in da Costa's formulation of the axiom schema of abstraction when $\mathbf{F}(x)$ is non-stratified. Nonetheless, we belive that this is a worthless effort. For:

(i) The only non-stratified formula used in the proof of Cantor's Theorem (which is fundamental in the proof of Theorem 2.3) is a non-stratified formula of the form $\alpha \in \beta$. Then, the new restrictions must avoid those nonstratified atomic formulas of the form $\alpha \in \beta$ which determine a set.

(ii) A new proof of Theorem 2.3 may be obtained in the following way: in $\mathbf{NF}$ the formula $y = \{x\}$ cannot determine a relation because $\langle x,y\rangle = \langle x,y\rangle \& y = \{x\}$ is non-stratified. But, such a formula does not contain any subformula of the form $A \supset B$; then, in $\mathbf{NF}_\omega^C$ it determines a relation $\mathbf{S}$ such that $\mathbf{S} \in 1\text{-}1$. With such a relation we prove that $(\alpha).\alpha$ sm $USC(\alpha)$. In $\mathbf{NF}_\omega^C$ we also prove that $(\alpha).\sim(USC(\alpha) \text{ sm } SC(\alpha))$. Then, these new restrictions must also avoid that those non-stratified formulas whose atomic subformulas are of the form $\alpha = \beta$ determine a set.

From the above remarks we conclude that, in order to avoid the counterintuive results mentioned in Theorem 2.3, the axiom schema of abstraction in $\mathbf{NF}_\omega$

should be formulated as in **NF**.

Due to the paradoxes obtained in $\mathbf{NF}^{c}_{\omega}$, $1 \leqslant n \leqslant \omega$, we conclude that in these systems the axiom schema of abstraction should be formulated as in **NF**. Thus, if we want these theories to be paraconsistent set theories, we need to postulate directly the existence of contradictory sets. Apparently, we may postulate the existence of Russell's set without any problem. Nonetheless, due to the two above considerations about the non-stratified formulas that lead to the proof of the paradox of identity, we believe that, besides Russell's set, very few other non-classical sets may exist in $\mathbf{NF}_n$, $1 \leqslant n \leqslant \omega$.

3. THE CLASSICAL PART OF $\mathbf{NF}_\omega$.

In this section we summarize the development of our version of $\mathbf{NF}_\omega$, in order to show that it can be considered a strong paraconsistent set theory, i.e., almost all the results of **NF** can be obtained in $\mathbf{NF}_\omega$. In our version of $\mathbf{NF}_\omega$, the axiom schema of abstraction is formulated as in **NF**, an axiom for the complement of a set is introduced, and no postulate is considered concerning the existence of non-classical sets.

The specific postulates of $\mathbf{NF}_\omega$ are extensionality and abstraction (both formulated as in **NF**), and the following one for the complement:

$$(\alpha,x).x \in \alpha \vee x \in \bar{\alpha}.$$

This axiom is fundamental if we want to preserve in $\mathbf{NF}_\omega$ at least the same properties of the algebra of classes of **NF**. Moreover, if we want to prove many other results of **NF** for instance, some of those in whose proof in **NF** it is necessary to use the principle of excluded middle or the law of double negation both for non-atomic formulas, this axiom is needed.

Observe that the universal set, the empty set, and the complement of a set are defined without using negation as follows, and the these definitions in **NF** are equivalent to the usual ones.

$\mathbf{\Lambda}$ for $\hat{x}(\alpha).x \in \alpha$

$\mathbf{V}$ for $\hat{x}(E\alpha).x \in \alpha$

$\bar{\alpha}$ for $\hat{x}(E\beta).x \in \beta \,\&\, \alpha \cup \beta = \mathbf{V} \,\&\, \alpha \cap \beta = \mathbf{\Lambda}$.

To express that α and β are different (or distinguishable) sets we use the symbol $\neq$ defined as follows:

$$\alpha \neq \beta \text{ for } (Ex).x \in \alpha \cap \bar{\beta} \vee x \in \bar{\alpha} \cap \beta.$$

LEMMA 3.1. I. $\vdash (Ex).x \in \mathbf{\Lambda} : \supset : (y).y \in \mathbf{\Lambda}$.
II. $\vdash (Ex).x \in \mathbf{\Lambda} : \supset : (y,z).y \in z \,\&\, y = z$.

LEMMA 3.2. I. $\vdash (\alpha,\beta).\alpha = \beta \vee \alpha \neq \beta$.

II. $\vdash (\alpha,\beta).\alpha = \beta \& \alpha \neq \beta :\supset: (Ex).x \in \Lambda$.

DEFINITION. $\sim A$ for $A \supset (Ex).x \in \Lambda$.

THEOREM 3.1. *In* $\mathbf{NF}_\omega$, $\sim$ *is a minimal intuitionistic negation.*

THEOREM 3.2. *For atomic formulas of* $\mathbf{NF}_\omega$, $\sim$ *is a classical negation.*

Proof. Let P and Q be variables for atomic formulas. Thus, by theorem 2.1, it suffices to prove that (i) $P \supset (\sim P \supset Q)$ and (ii) $P \vee \sim P$.

(i) From P and $\sim P$ we obtain $(Ex).x \in \Lambda$. Thus, from part II of Lemma 3.1, we obtain any atomic formula whatsoever.

(ii) P is of the form $\alpha \in \beta$. Supposing that $\alpha \in \beta$, we obtain $\alpha \in \beta \vee \sim(\alpha \in \beta)$. On the other hand, supposing that $\alpha \in \bar{\beta}$, we obtain $\alpha \in \beta \supset (Ex).x \in \Lambda$, i.e., $\sim(\alpha \in \beta)$; consequently, $\alpha \in \beta \vee \sim(\alpha \in \beta)$. Finally, using the axiom for the complement, we obtain the desired result.

P is of the form $\alpha = \beta$. Supposing that $\alpha = \beta$ we obtain $\alpha = \beta \vee \sim(\alpha = \beta)$. On the other hand, supposing that $\alpha \neq \beta$, then there exists an x such that $x \in \bar{\alpha} \cap \beta$ or $x \in \alpha \cap \bar{\beta}$. Taking the additional supposition that $\alpha = \beta$, in both cases we obtain $(Ex).x \in \Lambda$. Thus, from $\alpha \neq \beta$ we obtain $\alpha = \beta \supset (Ex).x \in \Lambda$, i.e., $\sim(\alpha = \beta)$, and so, by part I of Lemma 3.2, we have $\alpha = \beta \vee \sim(\alpha = \beta)$.

COROLLARY. I. $\vdash (x): x \in \Lambda \cdot\equiv\cdot \sim(x = x)$.
II. $\vdash (x): x \in \mathbf{V} \cdot\equiv\cdot x = x$.
III. $\vdash (x): x \in \bar{\alpha} \cdot\equiv\cdot \sim(x \in \alpha)$.
IV. $\vdash (x): x \in \bar{\bar{\alpha}} \cdot\equiv\cdot \sim\sim(x \in \alpha)$.
V. $\vdash (\alpha,\beta): \alpha \neq \beta \cdot\equiv\cdot \sim(\alpha = \beta)$.

THEOREM 3.3. *For positive formulas of* $\mathbf{NF}_\omega$ *(i.e., formulas in which no subformula is of the form* $\neg A$*)* $\sim$ *is an intuitionistic negation.*

Proof. Due to Theorem 3.1, we only have to prove that, if A^+ and B^+ denote positive formulas of $\mathbf{NF}_\omega$, then $A^+ \supset (\sim A^+ \supset B^+)$. In fact, let us suppose A^+ and $\sim A^+$; thus we obtain $(Ex).x \in \Lambda$. Then, using Lemma 3.1, part II, we have $(\alpha,\beta).\alpha = \beta \& \alpha \in \beta$. Now, by induction on the length of B^+ we conclude the proof. □

In the rest of this section, we follow Rosser [20], chapters IX to XIII. As almost all results of chapters IX and X that are valid in **NF** are proved in [19], the summary of these chapters is very short. The other chapters will be summarized section by section. From now on when we say that a theorem (or an exercise) of **NF** is valid in $\mathbf{NF}_\omega$ we are saying that the theorem (or exercise) is valid with the same formulation as in [20]; of course, negation is understood as the defined negation of $\mathbf{NF}_\omega$. When we say that the proofs are similar to Rosser's proofs, we are saying, in fact that either the proofs are exactly the same given in [20] or that small changes in Rosser's proofs are made in order to use the axiom for the complement and the above lemmas and theorems.

(CHAPTER IX) CLASS MEMBERSHIP. Except for part II of Theorem IX.4.11 and its corollaries, all the other theorems are valid in $\mathbf{NF}_\omega$, and the proofs are similar to Rosser's.

Let A^+ denote a positive formula of $\mathbf{NF}_\omega$, then part II of Theorem IX. 4.11 and its corollaries are proved in $\mathbf{NF}_\omega$ with the following formulations respectively:

$\vdash (x) . \sim A^+ :\equiv: (x).x \in \Lambda \equiv A^+$

$\vdash (x) . \sim A^+ : \supset : \exists \hat{x} A^+ .$

$\vdash (x) . \sim A^+ : \supset : \Lambda = \hat{x} A^+ .$

(CHAPTER X) RELATION AND FUNCTION. All the theorems of this section are valid in $\mathbf{NF}_\omega$ and the proofs are similar to Rosser's proofs.

(CHAPTER XI) CARDINAL NUMBERS.

1. ***Cardinal similarity.*** All the theorems of this section are valid in $\mathbf{NF}_\omega$, and the proofs are similar to Rosser's proofs.

2. ***Elementary properties of cardinal numbers.*** All the theorems of this section are valid in $\mathbf{NF}_\omega$; concerning the exercises, only XI.2.12 apparently is not valid. This exercise guarantees in **NF** that if **S** is a relation not belonging to 1-1, then there exist x and y such that $x,y \in \mathrm{Arg}(\mathbf{S})$, $\mathbf{S}(x) = \mathbf{S}(y)$ and $x \neq y$. We call attention to the apparent nonvalidity of this exercise because it is used by Rosser to prove the 'Pigeonhole Principle'.

3. ***Finite classes and mathematical induction.*** Let us discuss first the theorems about mathematical induction. Weak Induction (Theorem XI.3.18) is proved in $\mathbf{NF}_\omega$ as in Rosser; Strong Induction is valid as in Theorem XI.3.19 and its Corollary, but apparently it is not valid in the form of Theorem XI.3.20. Nonetheless, a restricted form of Theorem XI.3.20 is valid in $\mathbf{NF}_\omega$; to wit, when the formula $\mathbf{F}(x)$ appearing in it is a positive formula. Apparently, Theorem XI.3.22 (Principle of Infinite Descent) is not valid in $\mathbf{NF}_\omega$. Rosser's proof does not obtain because he uses the principle of double negation ($\sim\sim A \equiv A$) for non-atomic formulas and we could not find another way to prove this theorem.

All the other theorems of this section are valid in $\mathbf{NF}_\omega$ and, except for the proof of Theorem XI.3.21; the proofs are similar to Rosser's ones. To prove Theorem XI.3.21 (every non-empty subset α of **Nn** has a minimum), it is enough to prove the lemma used in Rosser's proof. The proof of this lemma runs as follows:

Case 1. Supposing that $n+1 \leqslant m$, we obtain the desired result.

Case 2. Supposing that $m < n+1$, by the axiom for the complement, we obtain $m < n+1 \,\&\, (m \in \alpha \vee m \in \bar{\alpha})$, and from this formula the desired result follows.

Now, the lemma follows from cases 1 and 2, because in $\mathbf{NF}_\omega$ the following formula is provable: $(m,n).m \leqslant n \vee n < m$.

4. Denumerable classes. All the theorems of this section are valid in $\mathbf{NF}_\omega$ and the proofs are similar to Rosser's proofs, except for the proof of Theorem XI.4.4 ($(\alpha).\alpha \subseteq \mathbf{Nn} \supset \alpha \in \mathrm{Count}$). This proof in **NF** runs as follows:

Case 1. If $\alpha \subseteq \mathbf{Nn}$ and $\alpha \in \mathrm{Fin}$, then, obviously, $\alpha \in \mathrm{Count}$.

Case 2. Let us supposse that $\alpha \subseteq \mathbf{Nn}$ and $\alpha \in \mathrm{Infin}$, then it is easily proved that

$$(n): n \in \mathbf{Nn} \;.\supset.\; \hat{z}(z \in \alpha \,\&\, z > n) \neq \Lambda. \tag{1}$$

Consequently, we obtain

$$(n): n \in \mathbf{Nn} \;.\supset.(Ey).y \in \alpha \,\&\, y > n. \tag{2}$$

Now, by (2) and the corollary of Theorem XI.4.3, we obtain $\alpha \in \mathrm{Den}$, consequently $\alpha \in \mathrm{Count}$.

Finally, as it is easy to prove that $(\alpha).\alpha \in \mathrm{Fin} \vee \alpha \in \mathrm{Infin}$, the desired result follows from cases 1 and 2.

5. The cardinal number of the continuum. All the theorems of this section are valid in $\mathbf{NF}_\omega$, and the proofs are similar to Rosser's, except for the proof of Theorems XI.5.5 ($c = 2^{\mathrm{Den}}$). The proof of this theorem runs as follows. Let α be defined as in Rosser's proof.

LEMMA 1. $\vdash \alpha \;\mathrm{sm}(\mathrm{SC}(\mathbf{Nn}) \cap \mathrm{Fin})$.

Proof. Like in Rosser's Lemma 3.

LEMMA 2. $\vdash \mathrm{NTBX} \;\mathrm{sm}(\mathrm{SC}(\mathbf{Nn}) \cap \mathrm{Infin})$.

Proof. Similar to the proof of Lemma 1, but taking

$$\mathbf{W} = \hat{S}\hat{\beta}(\mathbf{S} \in \mathrm{NTBX} \,\&\, \beta = \hat{m}(m \in \mathbf{Nn} \,\&\, \mathbf{S}(m) = 1)).$$

LEMMA 3. $\vdash \alpha \cap \mathrm{NTBX} = \Lambda$.

LEMMA 4. $\vdash (\alpha \cup \mathrm{NTBX}) \mathrm{sm}(\mathrm{PI} \restriction \{0,1\})$.

The rest of the proof is similar to the rest of Rosser's proof.

(CHAPTER XII) ORDINAL NUMBERS.

1. Ordinal similarity. All the theorems of this section are valid in $\mathbf{NF}_\omega$, and the proofs are similar to Rosser's proofs.

2. Well-ordering relations. Except for Theorems XII.2.10 to XII.2.13 (all about definitions and proofs by transfinite induction), all the others are valid in $\mathbf{NF}_\omega$. Rosser's proofs of Theorems XII.2.10 to XII.2.13 do not obtain in $\mathbf{NF}_\omega$, and we could not find another way to prove them. It is worthwile to mention that the proof of Theorem XII.2.14 (two well ordered sets either are similar or one is shorter than the other) is obtained without using transfinite induction (as mentioned in [20], p.462).

3. Elementary properties of ordinal numbers. All the theorems of this section are valid in $\mathbf{NF}_\omega$. The proof of theorem XII.3.4 ($\leqslant_0 \in$ Word) is a little different from Rosser's proof. To wit: suppose that $\beta \cap \mathbf{NO} \neq \Lambda$, then there exists ϕ such that $\phi \in \beta \cap \mathbf{NO}$. Case 1: $\beta \cap \mathbf{NO} = \{\phi\}$. Then there exists a minimal element in β. Case 2: $\beta \cap \mathbf{NO} \neq \{\phi\}$. Then, as in Rosser's proof, there exists a minimal element in β. Now, using Lemma 3.2, part I, and the fact that $\leqslant_0 \in$ Sord, we conclude the proof.

4. The cardinal number associated to an ordinal number. All the theorems of this section are valid in $\mathbf{NF}_\omega$, and the proofs are similar to Rosser's proofs.

(CHAPTHER XIII) COUNTING. The additional results about natural numbers given in Section 1 are valid in $\mathbf{NF}_\omega$ and, adding the *axiom of counting*, we also proved the Theorems of Section 2. Nonetheless, the main result of this chapter, the *pigeonhole principle*, apparently is not valid in $\mathbf{NF}_\omega$. Rosser's proof does not obtain because, as mentioned above, Exercise XI.2.12 apparently is not valid in $\mathbf{NF}_\omega$.

4. RUSSELL'S SET IN DA COSTA SET THEORIES.

In this section we show that in any da Costa set theory based on $C_n^=$, $1 \leqslant n < \omega$, UUR is the universal set; the same holds in a da Costa set theory based on $C_\omega^=$ when stregthened with some additional suppositions.

Let us denote by $\mathbf{DC}_n$ any da Costa set theory based on the respective $C_n^=$, where Russell's class is a set. Thus, in $\mathbf{DC}_n$, $1 \leqslant n < \omega$, the defined negation $\neg^*$ ($\neg^* A =_{df} \neg A \,\&\, A^{(n)}$) is a classical negation; and in $\mathbf{DC}_\omega$ the defined negation $\sim$ ($\sim A =_{df} A \supset (x,y).x \in y \,\&\, x = y$) is a minimal intuitionistic negation.

Let us denote by ϕ the empty set, defined in $\mathbf{DC}_n$, $1 \leqslant n < \omega$, as $\hat{x}\neg^*(x = x)$, and in $\mathbf{DC}_\omega$ as $\hat{x} \sim (x = x)$. Thus, in $\mathbf{DC}_n$, $1 \leqslant n < \omega$, we prove that $(x)\neg^*(x \in \phi)$, and in $\mathbf{DC}_\omega$ we prove that $(x). \sim (x \in \phi)$.

LEMMA 4.1. $\vdash \phi \in \mathbf{R}$.

Proof. In $\mathbf{DC}_n$, $1 \leqslant n < \omega$, if $\phi \in \phi$ then $\phi \in \phi \,\&\, \neg^*(\phi \in \phi)$. As this formula trivializes the system, then $\neg(\phi \in \phi)$. Consequently, $\phi \in \mathbf{R}$.

In $\mathbf{DC}_\omega$, if $\phi \in \phi$ then $\phi \in \phi \,\&\, \sim(\phi \in \phi)$; and so $(x,y).x \in y$. Thus, $\phi \in \mathbf{R}$. On the other hand, if $\neg(\phi \in \phi)$ then $\phi \in \mathbf{R}$.

LEMMA 4.2. $\vdash x \in \mathbf{R} \supset \{x\} \in \mathbf{R}$.

Proof. If $\neg(\{x\} \in \{x\})$, then $\{x\} \in \mathbf{R}$. On the other hand, if $\{x\} \in \{x\}$ then $\{x\} = x$; thus, by the hypothesis $x \in \mathbf{R}$, we obtain $\{x\} \in \mathbf{R}$.

LEMMA 4.3. I. $\vdash \mathbf{R} \subseteq \mathrm{U}\mathbf{R}$

II. $\vdash \mathbf{R} \subseteq \mathrm{UU}\mathbf{R}$.

Proof. I. If $x \in \mathbf{R}$ then, by Lemma 4.2, $\{x\} \in \mathbf{R}$. Now, as $x \in \{x\}$, then $x \in \mathrm{U}\mathbf{R}$.
II. By part I, we have $\mathbf{R} \subseteq \mathrm{U}\mathbf{R}$ and $\mathrm{U}\mathbf{R} \subseteq \mathrm{UU}\mathbf{R}$. Consequently, $\mathbf{R} \subseteq \mathrm{UU}\mathbf{R}$.

LEMMA 4.4. I. $\vdash (x).\neg^*(x \in \mathbf{R}) \supset x \in \mathrm{UU}\mathbf{R}$, *in* $\mathbf{DC}_n$ $(1 \leqslant n < \omega)$.

II. $\vdash (x).\sim(x \in \mathbf{R}) \supset x \in \mathrm{UU}\mathbf{R}$, *in* $\mathbf{DC}_\omega$.

Proof. I. Let us suppose that $\neg^*(x \in \mathbf{R})$. Thus, if $x = \phi$, we obtain $\neg^*(\phi \in \mathbf{R})$, and, using Lemma 4.1, we have $\phi \in \mathbf{R} \,\&\, \neg^*(\phi \in \mathbf{R})$. Consequenlty

$$\neg^*(x = \phi). \tag{1}$$

Let us suppose that $\{\{x,\phi\}\} \in \{\{x,\phi\}\}$. Then, we obtain $\{\{x,\phi\}\} = \{x,\phi\}$. Consequently, $x = \phi$. Now, by (1), we have a contradiction that trivializes the system. Thus, $\neg(\{\{x,\phi\}\} \in \{\{x,\phi\}\})$, and so

$$\{\{x,\phi\}\} \in \mathbf{R}. \tag{2}$$

But, $\{x,\phi\} \in \{\{x,\phi\}\}$. Then, by (2), we have

$$\{x,\phi\} \in \mathrm{U}\mathbf{R}. \tag{3}$$

However, $x \in \{x,\phi\}$. Then, by (3), we obtain $x \in \mathrm{UU}\mathbf{R}$.

II. Let us suppose that $\sim(x \in \mathbf{R})$. Thus, if $x = \emptyset$, by Lemma 4.1, we obtain $\sim(\phi \in \mathbf{R})$ and $\phi \in \mathbf{R}$. Consequently, $(x.y).x \in y \,\&\, x = y$. Thus, $x = \phi \supset (x,y).x \in y \,\&\, x = y$. Then,

$$\sim(x = \phi). \tag{1}$$

Supposing that $\{\{x,\phi\}\} \in \{\{x,\phi\}\}$, as in part I, we obtain $x = \phi$. Thus, by (1), we have $(x,y).x \in y \,\&\, x = y$. Consequently, $(x,y).x \in y$, and so $\{\{x,\phi\}\} \in \mathbf{R}$. On the other hand, supposing that $\neg(\{\{x,\phi\}\} \in \{\{x,\phi\}\})$, we obtain $\{\{x,\phi\}\} \in \mathbf{R}$. consequently, $\{\{x,\phi\}\} \in \mathbf{R}$.

The rest of the proof follows as in part I.

THEOREM 4.1. *In* DC_n $(1 \leqslant n < \omega)$, $\mathrm{UU}\mathbf{R}$ *is the universal set.*

Proof. It follows from Lemmas 4.3 and 4.4. □

The proof of Theorem 4.1 does not obtain in any $\mathbf{DC}_\omega$, but it obtain in any $\mathbf{DC}^u_\omega$, i.e., any $\mathbf{DC}_\omega$ with universal set $\mathbf{V}$, defined as $\hat{x}(x = x)$. To have a proof of Theorem 4.1 in any $\mathbf{DC}^u_\omega$ it is necessary to say what it means for a set to be different or distinguishable from the universal set. Thus, let us define

$$x \neq \mathbf{V} \quad \text{for} \quad (Ey).\sim(y \in x).$$

Moereover, if there exists a universal set it is obvious that every set must be equal to or different from the universal set. If this is not a theorem, it must be introduced as postulate:

P1. $(x).x = \mathbf{V} \vee x \neq \mathbf{V}$.

THEOREM 4.2. *In* $\mathbf{DC}^u_\omega$ *plus* **P1**, *we prove that* $\mathrm{U}\mathbf{UR} = \mathbf{V}$.

Proof. By **P1** we have $\mathrm{U}\mathbf{UR} = \mathbf{V}$ or $\mathrm{U}\mathbf{UR} \neq \mathbf{V}$. If $\mathrm{U}\mathbf{UR} = \mathbf{V}$, we have already the desired result. If $\mathrm{U}\mathbf{UR} \neq \mathbf{V}$ then, by the above definition, we have $(Ey).\sim(y \in \mathrm{U}\mathbf{UR})$. Thus, by Lemma 4.3, part II, it follows that $\sim(y \in \mathbf{R})$, and by Lemma 4.4, part II, $y \in \mathrm{U}\mathbf{UR}$. Consequently, $(x,y).x \in y \,\&\, x = y$, and so $(x).x \in \mathrm{U}\mathbf{UR}$. Thus, $\mathrm{U}\mathbf{UR} = \mathbf{V}$.

REMARK. In the sistema $\mathbf{NF}_n$ $(1 \leqslant n \leqslant \omega)$ we prove that $\mathrm{U}\mathbf{UR} = \mathbf{V}$. For $n < \omega$, the proof is the same as in Theorem 4.1; for $n = \omega$, the proof is the same as in Theorem 4.2, since **P1** is a theorem of $\mathbf{NF}_\omega$.

We have introduced some conditions in order to prove that $\mathrm{U}\mathbf{UR}$ is the universal set in $\mathbf{DC}^u_\omega$. Thus, it could seem possible to construct a $\mathbf{DC}_\omega$ without universal set. In the next section we prove in such a system the paradox of identity is derivable.

5. ON DA COSTA'S SET THEORIES OF TYPE ZF.

In sections 2 and 3 we have analysed da Costa's set theories with universal set, constructed according to the pattern of **NF**. Now we analyse the possibility of constructing da Costa's set theories following the pattern of classical set theory without universal set. We choose to analyse da Costa's set theories of type **ZF**, denoted by $\mathbf{ZF}_n$, $1 \leqslant n \leqslant \omega$.

Firstly, we show that if **R** is a set in $\mathbf{ZF}_n$, $1 \leqslant n \leqslant \omega$, then the supposition of non-existence of a universal set leads to some paradoxes that invalidate these theories. Such a result may already be intuitively inferred from the results presented in Section 4. Secondly, we show that the axiom schema of separation, formulated for all sets, is incompatible with the existence of Russell's set. Consequenlty, the axiom schema of replacement is also incompatible with the existence of Russell's set.

Le us consider the set theories $\mathbf{ZF}_n$, $1 \leqslant n \leqslant \omega$, in which the axioms of pairing and union are postulated in general, and in which we also postulate the existence of the empty set and of Russell's set. Moreover, let us suppose that there is no universal set, i.e.,

Sn. $(x)(Ey).\neg^*(y \in x)$, in $\mathbf{ZF}_n$, $1 \leqslant n < \omega$;

Sω. $(x)(Ey).\sim(y \in x)$, in $\mathbf{ZF}_\omega$

Let us observe that the lemmas of the preceding section are provable in $\mathbf{ZF}_n$, $1 \leqslant n \leqslant \omega$.

THEOREM 5.1. *The set theories* $\mathbf{ZF}_n$ $(1 \leqslant n < \omega)$ *plus* **Sn** *are trivial.*

Proof. By Sn there exists y such that $\neg^*(y \in \mathrm{U}\cup\mathbf{R})$. By part II of Lemma 4.3, and part I of Lemma 4.4, we obtain $(x).x \subset \mathrm{U}\cup\mathbf{R}$. Consequently, $y \in \mathrm{U}\cup\mathbf{R} \,\&\, \neg^*(y \in \mathrm{U}\cup\mathbf{R})$, and this formula trivializes the system.

THEOREM 5.2. *The paradox of identity is derivable in* $\mathbf{ZF}_\omega$ *plus* $\mathbf{S}_\omega$.

Proof. By Sn, there exists a y such that $\sim(y \in \mathrm{U}\cup\mathbf{R})$. Using part II of Lemma 4.3, we obtain $y \in \mathrm{U}\cup\mathbf{R}$. Consequently, by the definition of $\sim$, $(x,y).x \in y \,\&\, x = y$. Thus, the paradox of identity, $(x,y).x = y$, follows. Moreover, we also obtain the other results mentioned in Theorem 2.3.

THEOREM 5.3. *The systems* $\mathbf{ZF}_n$ $(1 \leqslant n < \omega)$ *with Russell's set and the axiom schema of separation postulate for all sets are trivial.*

Proof. If the axiom schema of separation is postulated for all sets then there exists a subset α of **R** such that

$$(x) : x \in \alpha \cdot\equiv\cdot x \in R \,\&\, (x \in x)^{(n)}. \tag{1}$$

From (1) we obtain

$$\alpha \in \alpha \cdot\equiv\cdot \neg(\alpha \in \alpha) \,\&\, (\alpha \in \alpha)^{(n)}. \tag{2}$$

Consequently, we have $\alpha \in \alpha \,\&\, \neg^*(\alpha \in \alpha)$, and this formula trivializes the system.

THEOREM 5.4. *In* $\mathbf{ZF}_\omega$ *with Russell's set and the axiom schema of separation postulated for all sets, the paradox of identity is derivable.*

Proof. From the axiom schema of separation and Russell's set we obtain $(x) : x \in \alpha \cdot\equiv\cdot x \in R \,\&\sim(x \in x)$. Thus,

$$\alpha \in \alpha \cdot\equiv\cdot \neg(\alpha \in \alpha) \,\&\sim(\alpha \in \alpha). \tag{1}$$

Case 1. Let us suppose that $\alpha \in \alpha$. Then, by (1), we obtain $\sim(\alpha \in \alpha)$. Thus, $\alpha \in \alpha \,\&\sim(\alpha \in \alpha)$. Consequently, $(x,y).x = y$.

Case 2. Le us suppose that $\neg(\alpha \in \alpha)$. By (1) we obtain $\sim\neg(\alpha \in \alpha)$, i.e., $\neg(\alpha \in \alpha) \supset (x,y).x \in y \,\&\, x = y$. Thus, $(x.y).x = y$.

From cases 1 and 2, the paradox of identity follows. Moreover, the other results mentioned in Theorem 2.3 are also derivable. □

As a consequence of Theorems 5.3 and 5.4 we conclude that the existence of Russell's set is incompatible with a general (for all sets) formulation of the axiom schema of replacement. For, on the one hand, a general formulation of the axiom schema of replacement implies a general formulation of the axiom schema of separation. On the other hand, using the axiom schema of replacement we prove Cantor's Theorem. But, as UUR is the universal set, then Cantor's paradox is derivable. Consequently the $\mathbf{ZF}_n$, $1 \leq n < \omega$, are trivial and the paradox of identity is derivable in $\mathbf{ZF}_{\omega}$.

6. CONCLUDING REMARKS.

The main results presented in this paper are the following: (i) in any da Costa paraconsistent set theory with Russell's set the scope of validity of the classical formulations of the axiom schemata of abstraction, separation and replacement cannot be enlarged; (ii) it is not possible to construct da Costa's set theories with Russell's set and without universal set. These results may be obtained in many other strong paraconsistent set theories.

In a certain sense, these results may be considered as limitative ones. By (i), Russell's set as well as other non-classical sets have to be introduced by specific postulates. Thus, in each case, we must investigate if the non-classical set we want to introduce does not lead to a paradox that invalidates the theory. Still by (1), Russell's set is incompatible with a general formulation of the axiom schema of replacement. This fact makes it impossible to prove some interesting thing about some contradictory sets generated by **R**. For instance, let us define $SC^1(\mathbf{R})$ as $SC(\mathbf{R})$ and $SC^{n+1}(\mathbf{R})$ as $SC(SC^n(\mathbf{R}))$. If we could apply the axiom schema of replacement to these sets, we would prove that they are universes. This is the most interesting property of contradictory sets we have already devised. But, unfortunately, up to now we have not found any paraconsistent set theory in which this property is valid.

Set theories without universal set may be considered richer and more interesting than the ones with universal set. Moroeover, it is natural to guarantee the existence of Russell's set in paraconsistent set theories. But, by (ii) it seems that we cannot construct a strong paraconsistent set theory with Russell's and without universal set.

A natural question one may ask is if the above limitative results are valid in weak paraconsistent set theories. In [5] it is proved that Russell's set implies the existence of universal set in weak paraconsistent theories in whose underlying logic the law of excluded middle is valid. In [8] it is proved that Russell's set is not incompatible with a general formulation of the axiom schema of abstraction in some weak and non-trivial paraconsistent se theories. Nonetheless, it has not been investigated whether the paradox of identity is derivable or not in them. However, in [7] it is shown that the paradox of identity is derivable in some other weak and non-trivial paraconsistent set theories.

The weak paraconsistent set theories have the advantage of being non-trivial. But, even if they are free from the paradox of identity, they seem to be weak concerning the set-theoretical operations. Thus, it is interesting to know if they may be strengthened in a way similar to that used by Griss to construct his logic of species (see [2]). An idea of how to proceed in this direction is given in Section 3 above.

To finish, we mention some open problems whose solution we believe are important in the development of paraconsistent set theories. In da Costa paraconsistent set theories, is **R** different from the universal set? If the answer is affirmative, is U**R** different from the universal set? What is the meaning of the defined negation in **NF** (see Section 3, Theorems 3.1-3.3)? Is it possible to construct a paraconsistent set theory with Russell's set and without universal set? Apart from Russell's set what other non-classical sets may be introduced in paraconsistent set theory?

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IMECC
Universidade Estadual de Campinas
Campinas, S.P., Brazil.

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CHUAQUI'S DEFINITION OF PROBABILITY IN SOME STOCHASTIC PROCESSES

Leopoldo Bertossi*

ABSTRACT. Models for Markov Dependent Bernoulli Trials, Markov Chains, Random Walks and Brownian Motion are constructed in the framework of Chuaqui's Definition of probability.

Chuaqui 1980 and 1981 explains how a semantical definition of probability can be applied to random experiments that give rise to compound outcomes. In order to do this, he introduces what he calls "compound probability structures" (CPS). These CPS are based on causal trees of the form (T,R) where T is a nonempty set and R is a partial order in T which reflects the causal dependence relation between the simple outcomes which make up the compound outcome.

In the applications we are interested in, the elements of T are time moments and R is a the natural linear order $\leqslant$.

A compound outcome is a function f with domain T for which f(t) is an outcome in a simple probability structure (SPS) (see Chuaqui 1977 and 1981). Starting with known probability measures on these SPS, he defines a probability measure on the set of compound outcomes (see Chuaqui 1980).

In what follows we show how this definition works for some known stochastic processes.

1. MARKOV DEPENDENT BERNOULLI TRIALS (MDBT)

We repeat n times an experiment which has only two possible outcomes, s and f (for success and failure). We assume that $p_{s,f}$ is the probability of f on the $(k+1)$-st trial, given that the outcome was s on the k-th trial, and that the analogously defined probabilities $p_{s,s}$, $p_{f,s}$, $p_{f,f}$ are known and independent of k. We also assume the initial probabilities p_s, p_f to be known.

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Here $T = \{t_0,\ldots,t_{n-1}\}$, and we consider in T the natural order relation: $t_i \leqslant t_j$ if and only if $i \leqslant j$. We associate with each $t_i \in T$, $i > 0$, two simple probability structures $\mathbf{K}^s$ and $\mathbf{K}^f$. The models in $\mathbf{K}^s$ are of the form $\langle A^s, S^s, F^s, \{a\}\rangle$, with A^s a non-empty set, $\{S^s, F^s\}$ a fixed partition of A^s where the proportion of elements of A^s which are in S^s is $p_{s,s}$ and that of the elements which are in F^s is $p_{s,f}$, and a an element of A^s.

The appearance of an element of S^s gives outcome s and the appearance of an element of F^s gives outcome f.

We write $\mathcal{A}^s_s$ for $\{\langle A^s, S^s, F^s, \{a\}\rangle : a \in S^s\}$ and $\mathcal{A}^s_f$ for $\{\langle A^s, S^s, F^s, \{a\}\rangle : a \in F^s\}$ and simplifying, we write $\mathbf{K}^s = \{\mathcal{A}^s_f, \mathcal{A}^s_s\}$. Analogously, we define $\mathbf{K}^f$ and write $\mathbf{K}^f = \{\mathcal{A}^f_f, \mathcal{A}^f_s\}$; $\mathcal{A}^f_f$ and $\mathcal{A}^s_f$ give outcome f; $\mathcal{A}^f_s$ and $\mathcal{A}^s_s$ give outcome s.

To complete our formulation, we associate with t_0 the simple probability structure $\mathbf{K}_0 = \{\mathcal{A}^0_s, \mathcal{A}^0_f\}$, where the definition of $\mathcal{A}^0_s$ and $\mathcal{A}^0_f$ is analogous to that of $\mathcal{A}^s_s$ and $\mathcal{A}^s_f$, respectively.

The following probabilities follow immediately from the above definition:

$$\mu(\mathcal{A}^s_f) = p_{s,f} \qquad \mu(\mathcal{A}^s_s) = p_{s,s}$$
$$\mu(\mathcal{A}^f_f) = p_{f,f} \qquad \mu(\mathcal{A}^f_s) = p_{f,s}$$
$$\mu(\mathcal{A}^0_s) = p_s \qquad \mu(\mathcal{A}^0_f) = p_f\,.$$

A *compound outcome* is a function f which satisfies the following conditions:

a) Domain of $\mathrm{f} = T$,

b) $\mathrm{f}(t_0) \in \mathbf{K}_0$,

c) $\mathrm{f}(t_k) \in \mathcal{A}^*_f$ implies $\mathrm{f}(t_{k+1}) \in \mathbf{K}^f$ and $\mathrm{f}(t_k) \in \mathcal{A}^*_s$ implies $\mathrm{f}(t_{k+1}) \in \mathbf{K}^s$,

with $* \in \{s,f\}$ for $k > 0$ and $* = 0$ for $k = 0$.

The compound probability structure corresponding to these MDBT is $\mathbf{H} = \langle T, \leqslant, \mathbf{H}\rangle$ where $\mathbf{H}$ is the set of all functions that satisfy (a)-(c). On the basis of the probabilities assigned above and the relation $\leqslant$ in T, we define a probability measure μ on $\mathbf{H}$. In the case of MDBT, it is interesting to calculate the probabilities:

$$p^k_s = \text{probability of } s \text{ on the } k\text{-th trial}$$
$$p^k_f = \text{probability of } f \text{ on the } k\text{-th trial.}$$

To do this, it is enough to solve a difference equation whose derivation is based on the "total probability theorem" which can be formulated and proved in this context in the usual fashion.

Clearly, the situation corresponding to Markov Chains can be formulated in a form completely analogous to that of MDBT. In considering Markov Chains, it is merely necessary to choose a greater number of simple probability structures associated with each moment of time and a greater number of transition probabilities.

2. RANDOM WALKS.

Let us consider a one dimensional random walk which, starting from the origin, is controlled by a coin thrown n times, where the step size is constant and equal to 1.

Here $T = \{t_0, \ldots, t_n\}$, T ordered as in §1. We associate with t_0 the simple probability structure $\mathbf{K} = \{\mathfrak{A}_0\}$ with $\mathfrak{A}_0 = \langle\{0\},\{0\}\rangle$. For each t_k, $k > 0$, define

$$\boldsymbol{u}_{t_k} := \{-k, -k+2, \ldots, k-2, k\} \text{ and } \boldsymbol{u}_{t_0} := \{0\}.$$

A *random walk* (a compound outcome) is a function f which satisfies the following conditions:

a) Domain of $f = T$,

b) $f(t_0) = \mathfrak{A}_0$,

c) $f(t_k)$ is a model of the form $\langle \boldsymbol{u}_{t_k}, \{\lambda\}\rangle$ with $\lambda \in \boldsymbol{u}_{t_k}$,

d) $f(t_k) = \langle \boldsymbol{u}_{t_k}, \{\lambda\}\rangle$ implies $f(t_{k+1}) \in \mathbf{K}_{k+1,\lambda} := \{\langle \boldsymbol{u}_{t_{k+1}}, \{\lambda+1\}\rangle, \langle \boldsymbol{u}_{t_{k+1}}, \{\lambda-1\}\rangle\}$.

With each $t_k \in T$, $k > 0$, we associate a family of SPS with the same similarity type and a common universe, namely the family $\{\mathbf{K}_{k,\lambda} : \lambda \in \boldsymbol{u}_{t_{k-1}}\}$.

The probability in $\mathbf{K}_{k,\lambda}$ is uniformely distributed if the coin which controls the walk is unbiased, but, in general,

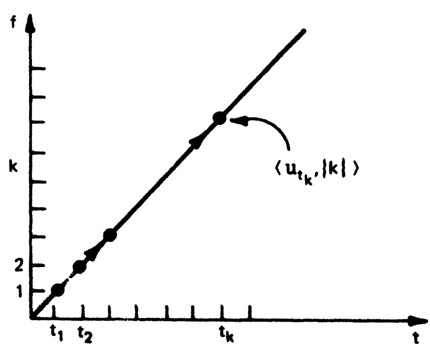
$$\mu(\langle \boldsymbol{u}_{t_k}, \{\lambda+1\}\rangle) = p \quad \text{and}$$

$$\mu(\langle \boldsymbol{u}_{t_k}, \{\lambda-1\}\rangle) = 1-p \quad \text{for each } k \text{ and for each } \lambda \in \boldsymbol{u}_{t_{k-1}}.$$

we assign to $\mathfrak{A}_0$ the probability 1.

The CPS corresponding to this kind of random walk is $\mathcal{H} = \langle T, <, \mathbf{H}\rangle$, where **H** is the set of all functions which satisfy (a)-(d). On **H** one obtains a probability measure μ determined by $\mathcal{H}$ and the probabilities assigned above to the SPS $\mathbf{K}_{k,\lambda}$.

We can calculate probabilities according to Chuaqui 1980, such as, for example, the probability of the path $f \in \mathbf{H}$ shown in the given figure.



We show that $\mu(f) = p^n$, as expected. In Chuaqui 1980, the measure μ on **H** is defined by induction on ordinals.

Let $g \in \mathbf{H}$, $t \in T$ and $T_t := \{s \in T : s \leqslant t, s \neq t\}$, then $\mathbf{H}(g,t) := \{h(t) : h \in \mathbf{H}, g \upharpoonright T_t = h \upharpoonright T_t\}$ is an SPS where a probability measure with values p and $1-p$ is defined. Denote this measure by $\mu_{g,t}$. We need some definitions from Chuaqui 1980:

T'_α is the set of all minimal elements of $T - \cup\{T'_\beta : \beta \subset \alpha\}$

$T_\alpha := \cup\{T'_\beta : \beta \subset \alpha\}$; $\bar{T}_\alpha := \cup\{T'_\beta : \beta \subseteq \alpha\}$, α ordinal.

$\bar{T}_t := T_t \cup \{t\}$, $t \in T$.

$\mathbf{H}(S) := \{f \upharpoonright S : f \in \mathbf{H}\}$, $S \subseteq T$.

$\mathbf{A}(S) := \{f \upharpoonright S : f \in \mathbf{A}\}$, $S \subseteq T$, $\mathbf{A} \subseteq \mathbf{H}$.

Then we have $T'_i = \{t_i\}$, $T_i = \{t_0, \ldots, t_{i-1}\}$, $\bar{T}_i = \{t_0, \ldots, t_i\}$.

We have to find the measure μ on **H**. Clearly $\mathbf{H} = \mathbf{H}(\bar{T}_{t_n})$. Then the measure on **H** is $\bar{\mu}_{t_n}$ which is defined for $\mathbf{A} \subseteq \mathbf{H}$ by

$$\bar{\mu}_{t_n}(\mathbf{A}(\bar{T}_{t_n})) = \int_{\mathbf{A}(T_{t_n})} \mu_{g,t_n}(\mathbf{A}(g,t_n))\, d\mu_{t_n}$$

where $\mathbf{A}(g,t_n) := \{h(t_n) : h \in \mathbf{A} \text{ and } h \upharpoonright T_{t_n} = g \upharpoonright T_{t_n}\}$. In our case $\mathbf{A} = \{f\}$, so that

$$\mu(f) = \bar{\mu}_{t_n}(\{f\}) = \int_{\{f \upharpoonright T_{t_n}\}} \mu_{g,t_n}(\mathbf{A}(g,t_n))\, d\mu_{t_n}$$

$$= \mu_{f,t_n}(f(t_n)) \cdot \mu_{t_n}(f \upharpoonright T_{t_n})$$

$$= p \cdot \mu_{t_n}(f \upharpoonright T_{t_n}).$$

The measure μ_{t_n} is defined by

$$\mu_{t_n} = \pi\langle \bar{\mu}_s : s \in T'_{n-1} \text{ and } s < t_n \rangle = \bar{\mu}_{t_{n-1}}.$$

Thus, $\mu(f) = p \cdot \mu_{t_{n-1}}(f \upharpoonright T_{t_n})$. If we calculate $\bar{\mu}_{t_{n-1}}$ as we calculated $\bar{\mu}_{t_n}$, we have, upon iteration, $\mu(f) = p^n$.

Within this formulation we can prove all the results of Probability Calculus involving random walks.

3. BROWNIAN MOTION.

Our formulation is motivated by the known fact thay by speeding up a random walk it is possible to obtain a good model of Brownian Motion. We avoid this explicit acceleration process using non-standard techniques. Let us consider a Brownian Motion during a unit of time and a ω_1-saturated non-standard extension $V(^*\mathbb{R})$ of the superstructure $V(\mathbb{R})$ of the real numbers.

Let $\eta \in {}^*\mathbb{N} \setminus \mathbb{N}$ be an infinite natural number and $T = \{0, 1/\eta, 2/\eta, \ldots, 1\}$ ordered in the natural way. We associate to each $t_\lambda = \lambda/\eta \in T$ the set

$$\mathcal{u}_{t_\lambda} := \{-\frac{\lambda}{\sqrt{\eta}}, \frac{-\lambda+2}{\sqrt{\eta}}, \ldots, \frac{\lambda-2}{\sqrt{\eta}}, \frac{\lambda}{\sqrt{\eta}}\}, \quad \lambda \geqslant 1$$
$$\mathcal{u}_0 := \{0\}.$$

Now, if $\alpha \in \mathcal{u}_{t_{\lambda-1}}$, $\lambda \geqslant 1$, then $\mathbf{K}_{\lambda,\alpha} := \{\mathcal{a}_{\lambda,\alpha+}, \mathcal{a}_{\lambda,\alpha-}\}$, with $\mathcal{a}_{\lambda,\alpha+} := \langle \mathcal{u}_{t_\lambda}, \{\alpha + \frac{1}{\sqrt{\eta}}\}\rangle$ and $\mathcal{a}_{\lambda,\alpha-} := \langle \mathcal{u}_{t_\lambda}, \{\alpha - \frac{1}{\sqrt{\eta}}\}\rangle$ is a simple probability structure with $\mu_{\lambda,\alpha}(\mathcal{a}_{\lambda,\alpha+}) = \mu_{\lambda,\alpha}(\mathcal{a}_{\lambda,\alpha-}) = 1/2$.

A possible path of Brownian Motion is a function f such that:

a) Domain of f = T,

b) $f(0) = \langle \mathcal{u}_0, \{0\}\rangle$,

c) $f(t_{\lambda-1}) = \langle \mathcal{u}_{t_{\lambda-1}}, \{\alpha\}\rangle$ implies $f(t_\lambda) \in \mathbf{K}_{\lambda,\alpha}$, $\lambda \geqslant 1$.

Let **H** be the set of all possible trajectories. On **H** one obtains a probability measure μ induced by the $\mu_{\lambda,\alpha}$'s. As indicated in Chuaqui 1980, μ is defined by induction on ordinals which in this situation may be hyperfinite.

We define random variables $(X_{t_\lambda})_{\lambda=0}^{n}$ on **H** by $X_{t_\lambda}(f) := \mathrm{Var.}(f(t_\lambda))$, where $\mathrm{Var.}(f(t_\lambda)) \in {}^*\mathbf{R}$ is the real number that belongs to the variable part of $f(t_\lambda)$. For example, if $f(t_\lambda) = \langle \mathcal{u}_{t_\lambda}, \{\alpha\}\rangle$, then $\mathrm{Var.}(f(t_\lambda)) = \alpha$.

Using some results of Anderson 1976, it may be shown that this is a good model for Brownian Motion. Indeed, if $f \in \mathbf{H}$, we define $X_s(f)$ for each $s \in {}^*[0,1]$ by

$$X_s(f) := X_{t_{[\eta s]}}(f) + (\eta s - [\eta s]) \cdot (X_{t_{[\eta s]+1}}(f) - X_{t_{[\eta s]}}(f)).$$

In this way we have a set **H** that contains all possible trajectories, a measure μ defined on **H**, or more precisely, on a family $\boldsymbol{A}$ of subsets of **H** and a family $(X_s)_{s \in {}^*[0,1]}$ of random variables. Furthermore, all these objects (T, the functions f, **H**, $\boldsymbol{A}$, the X_s's) are internal. This is also the case for the measure μ, because it is defined in terms of standard measures and internal ordinals. $(\mathbf{H}, \boldsymbol{A}, \mu)$ is an internal probability space.

Now, we consider Loeb's standard probability space $(\mathbf{H}, \mathrm{L}(\boldsymbol{A}), P)$ associated with $(\mathbf{H}, \boldsymbol{A}, \mu)$ (see Loeb 1975). $\mathrm{L}(\boldsymbol{A})$ is the σ-algebra generated by $\boldsymbol{A}$, and P is the probability measure defined on $\mathrm{L}(\boldsymbol{A})$ and generated by the standard part ${}^o\mu$ of μ. If we now define

$$Y_s(f) := {}^oX_s(f), \quad s \in [0,1],$$

then

$$P(Y_s \leqslant \alpha) = \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{\alpha} \exp(-y^2/2s)dy, \quad \alpha \in \mathbf{R}.$$

In fact,

$$\begin{aligned} P(Y_s \leqslant \alpha) &= P({}^oX_s \leqslant \alpha) \\ &= P({}^oX_{t_{[\eta s]}} \leqslant \alpha) \\ &= ({}^o\sum_{k=0}^{[\eta s]-1} (X_{t_{k+1}} - X_{t_k}) \leqslant \alpha) \end{aligned}$$

$$= P\left(\frac{\sum_{k=0}^{[\eta s]-1}(X_{t_{k+1}}-X_{t_k})}{\sqrt{[\eta s]}\frac{1}{\sqrt{\eta}}} \leq \frac{\alpha}{\sqrt{\frac{[\eta s]}{\eta}}}\right).$$

Because the random variables $X_{t_{k+1}}-X_{t_k}$ are independent with mean 0 and variance $1/\eta$, by a non-standard version of central limit theorem (Anderson 1976) one has that the last expression equals

$$\lim_{m\to\infty} {}^{o}(^{*}\psi)\left(\sqrt{\frac{\eta}{[\eta s]}}\,(\alpha+\frac{1}{m})\right) = \lim_{m\to\infty} \psi\left(\frac{\alpha+\frac{1}{m}}{\sqrt{s}}\right) = \psi\left(\frac{\alpha}{\sqrt{s}}\right),$$

where ψ is the distribution function of the normal probability law with mean 0 and variance 1. Thus, Y_s has normal distribution $N(0,s)$, with mean 0 and variance s.

It is known (Anderson 1976) that P is an extension of the Wiener measure on $C[0,1]$.

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Facultad de Matemáticas
Pontificia Universidad Católica de Chile.
Santiago, Chile.

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NATURAL NUMBERS IN ILLATIVE COMBINATORY LOGIC

M.W. Bunder

1. INTRODUCTION. In this paper we attempt to develop natural numbers using the second order predicate calculus and the three axioms also used to obtain the set theory of [3]. Of these three axioms, two give us that the natural numbers, as we define them, are elements of the class of individuals A, the other says that all individuals are sets under the definition given in [3]. Besides the induction property and some elementary arithmetical properties of natural numbers, we can prove that the class of all natural numbers forms a set which is a subset of A.

It has been thought that this theory would be strong enough to contain all of first order arithmetic, and hence that it would be subject to Gödel's incompleteness results; this however seems not to be the case. In order to obtain the remaining Peano type axioms for arithmetic, we need to assume a restricted form of substitution of equality and a "type" for the paradoxical combinator **Y**.

2. THE SECOND ORDER LOGIC.

The primitive constants that we require, besides the combinators K and S (or λ-abstraction), are: A, H (the class of propositions) and Ξ (restricted generality). ΞUX (or $Uy \supset_y Xy$) expresses the fact that XY holds for all Y in U. $\supset$ can be defined in terms of Ξ as in [1].

The rules of the logic are:

Eq If X = Y then $X \vdash Y$.

P $X \supset Y,\ X \vdash Y$.

Ξ $\Xi UX,\ UY \vdash XY$ where U is A, H or FAH.(1)

DTP If Δ, $X \vdash Y$ then Δ, $HX \vdash X \supset Y$.

DTΞ If Δ, $UY \vdash XY$ then $\Delta \vdash \Xi UX$ where U is A, H or FAH.

H $X \vdash HX$.

PH $HX,\ X \supset HY \vdash H(X \supset Y)$.

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(1) $F = \lambda x\lambda y\lambda z\Xi x(Byz)$ and FXY can be interpreted as the class of functions from X to Y. FAH can therefore be interpreted as the class of first order predicates.

ΞH FUHX $\vdash$H(ΞUX) where U is A, H or FAH.

(The last five of these could be replaced by Axioms as in [1]).

Quantification over H is not used directly in this paper, but it is needed to define the connectives Γ, $\vee$ and $\wedge$, and the existential quantifier ΣA and to prove their appropriate properties (see [2]).

3. ARITHMETIC.

Natural numbers were defined in terms of λ-abstraction by Church in [4] and equivalently by Curry and Feys in [5] in terms of combinators. They have:

$$Z_0 = KI \qquad (= \lambda x\ \lambda y.y)$$
$$Z_{n+1} = SBZ_n \qquad (= \lambda x\ \lambda y.x(Z_n xy))$$

(Church starts with $Z_1 = \lambda x\ \lambda y.xy$ (=BI)). These combinators, called *iterators*, have the property:

$$X^n = Z_n X$$

where

$$X^0 = I \qquad (= \lambda x.x)$$

and

$$X^{n+1} = BXX^n \ (= \lambda u.X(X^n u)).$$

It is easy to prove that:

$$Z_{m+n} = \Phi BZ_m Z_n$$
$$Z_{mn} = BZ_m Z_n$$

and

$$Z_{n^m} = Z_m Z_n.$$

Thus the iterators themselves can be taken as numbers and addition, multication and exponentiation can be defined so that the appropriate commutative, associative and distributive laws hold.

If however we want to represent numbers in a predicate calculus, a symbol for equality has to be introduced into that system; combinatory equality (=) is a primitive predicate.

Church, and also Kleene in his development of Church's system in [7], use

$$Q = \lambda x \lambda y(ux. \supset_u uy),$$

which is satisfactory when Church's strong deduction theorem for Ξ is used. Church's system is however inconsistent because of this deduction theorem, and our weaker theorem (or rule) requires a different equality. The one we use is an extensional equality given by:

$$Q_1 = \lambda x\ \lambda y.\quad FAHx \wedge FAHy \wedge Au \supset_u (xu \leftrightarrow yu)^{(2)}$$

In [3] sets and first order predicates were identified so that "FAHX" stands

(2) "$\leftrightarrow$" stands for "if and only if". We will often write $X =_1 Y$ for $Q_1 XY$.

for "X is a set".

The main axiom of [3] asserts that all individuals are sets i.e.

(A) $$\vdash \text{Au} \supset_u \text{FAHu}.$$

The empty class, based on Q_1, given by:

$$0 = \lambda x.\Gamma(Q_1 xx)$$

could then be shown to be a set (i.e. $\vdash$ FAHO), as could the class containing only 0 ($Q_1 0$), the class containing only that ($Q_1(Q_1 0) = Q_1^2 0$) etc.

Now 0, {0} (= $Q_1 0$), {{0}} (= $Q_1^2 0$)... have been used as definitions for 0,1,2,... in naive set theory and are very suitable here. Thus we define:

$$1 = Q_1 0,\ 2 = Q_1^2 0, \ldots, n = Q_1^n 0, \ldots$$

As $n = Q_1^n 0 = Z_n Q_1 0$, we can define the arithmetical operations in the following way:

$$n+m = \Phi B Z_m Z_n Q_1 0 = Q_1^m(Q_1^n 0)$$

$$n \cdot m = B Z_m Z_n Q_1 0 = (Q_1^n)^m 0$$

and

$$n^m = Z_m Z_n Q_1 0 = Q_1^{n^m} 0.$$

As before the appropriate commutative, associative and distributive laws hold over = ; this fact is independent of our definitions of 0 and Q_1 and of our axiom (A). Given (A) these laws also hold over Q_1 as we then have $\vdash$ FAH($Q_1^i 0$) for $i \geqslant 0$.

To prove that for $i \neq j$, $Q_1^i 0$ and $Q_1^j 0$ are unequal in the Q_1 sense we need, as was shown in [3]:

$$\vdash \text{AO} \tag{1}$$

and

$$\vdash \text{FAA}Q_1, \tag{2}$$

which guarantee that all our numbers are individuals.

These axioms, as was also shown in [3], allow us to prove the cancellation law for addition.

Thus we have all the simpler "Peano axioms" in the following form:

$$\vdash \Gamma(0 =_1 n + 1)$$
$$\vdash n + 1 =_1 m + 1 \supset n =_1 m, \quad \text{etc.},$$

where n and m are natural numbers, but we do not have

$$\vdash (\forall x)(x \in N \supset \Gamma(0 =_1 x + 1) \quad \text{etc.}$$

Also to prove the mathematical induction axiom we need a class N of natural numbers. This we define as follows:

$$N = \cap(\lambda x(x0 \wedge Ay \supset_y (xy \supset x(Q_1 y))))$$

where

$$\cap Z = \lambda u.\ \text{FAHv} \supset_v Zv \supset vu.$$

We then prove that N is a set:

THEOREM 1. $\vdash$ FAHN.

Proof. By (1) $\quad$ FAHx $\vdash$ H(x0)

and $\quad$ FAHx, Ay $\vdash$ H(xy)

also by (2) $\quad$ FAHx, Ay $\vdash H(x(Q_1y))$

so $\quad$ FAHx, Ay $\vdash H(xy \supset x(Q_1y))$

and $\quad$ FAHx $\vdash H(x0 \wedge Ay \supset_y (xy \supset x(Q_1y)))$

(this step uses rules PH and PΞ and results from [2]).

Let $W = \lambda x.x0 \wedge Ay \supset_y (xy \supset x(Q_1y))$

then, by DTΞ, $\quad \vdash$ F(FAH)HW

Now, $\quad$ Au, FAHz $\vdash$ H(zu)

and $\quad$ FAHz $\vdash$ H(Wz)

so $\quad$ Au, FAHz $\vdash H(Wz \supset zu)$

and so $\quad$ Au $\vdash H(FAHz \supset_z Wz \supset zu)$

$\therefore \quad \vdash$ FAH(⋂W)

i.e. $\quad \vdash$ FAHN. □

We could note that (A) is not needed in this proof so that this theorem holds in alternative set theories where (1) and (2) hold but (A) does not.

We now show that N is, in a sense, a subclass of A; again this does not require (A).

THEOREM 2. Nu $\vdash$ Au.

Proof. By (2) we have $\quad Ay \vdash A(Q_1y)$

and so $\quad Ay \vdash Ay \supset A(Q_1y)$

and by DTΞ $\quad \vdash Ay \supset_y Ay \supset A(Q_1y)$.

Then by (1), taking W as above we have:

$$\vdash WA$$

Now $\quad Nu = \bigcap Wu = FAHv \supset_v Wv \supset vu$,

so as $\quad \vdash$ FAHA

we have $\quad$ Nu $\vdash$ Au. □

Note that because DTΞ holds only for U = A, H and FAH we cannot conclude $\vdash Nu \supset_u Au$ from this. Using Theorem 1 we can get no more than $Au \vdash Nu \supset Au$, which is not very useful. If we use the stronger deduction theorem for Ξ given in [1]:

$$\text{If } \Delta, Xu \vdash Y \text{ then } \Delta, \text{ FAHX} \vdash \Xi XY,$$

we could obtain $\vdash$ΞNA, but this rule makes the system substantially stronger.

The theorem however means that $Au \supset_u (Nu \supset Xu)$ has all the properties that $Nu \supset_u Xu$ could have, namely:

$$Au \supset_u (Nu \supset X),\ Nu \vdash X$$

and if Δ, $Nu \vdash X$ then $\Delta \vdash Au \supset_u (Nu \supset X)$.

We now prove that mathematical induction holds for N.

THEOREM 3. *If* $\vdash XO \wedge Ay \supset_y (Xy \supset X(Q_1y))$ (a)

$\vdash Au \supset_u Xu \supset Nu$ (b)

$\vdash FAHX$, (c)

then $\vdash Q_1XN$,

Proof. With W as above we have:

$$Nu \vdash FAHv \supset_v Wv \supset vu$$

so by (c) $Nu \vdash WX \supset Xu$

and by (a), $Nu \vdash Xu$.

By Theorem 1, $Au \vdash H(Nu)$

so $Au \vdash Nu \supset Xu$.

Also by (b) $Au \vdash Xu \supset Nu$

so $Au \vdash Xu \backsim Nu$

$\therefore$ $\vdash Au \supset_u Xu \backsim Nu$

i.e. $\vdash Q_1XN$. □

The following restricted form of this theorem will be all we need in many cases:

COROLLARY. *If* (a) *and* (c) *hold then* $Nu \vdash Xu$.

We now look at some further basic properties of the natural numbers as we have defined them.

THEOREM 4. $Nx \vdash N(Q_1x)$.

Proof. Nx, $FAHv$, $[vo \wedge Ay \supset_y vy \supset v(Q_1y)] \vdash vx$.

By Theorem 2,

$$Nx \vdash Ax,$$

$\therefore$ Nx, $FAHv$, $[vo \wedge Ay \supset_y vy \supset v(Q_1y)] \vdash v(Q_1x)$.

Thus as $FAHv \vdash H[vo \wedge Ay \supset_y vy \supset y(Q_1y)]$

$Nx \vdash FAHv \supset_v [vo \wedge Ay \supset_y vy \supset v(Q_1y)] \supset v(Q_1x)$

i.e. $Nx \vdash N(Q_1x)$.

THEOREM 5. $Nx \vdash x =_1 0 \vee \Sigma A(\lambda z . x =_1 Q_1z \wedge Nz)$.

Proof. Let $X = \lambda x . x =_1 0 \vee \Sigma A(\lambda z . x =_1 Q_1z \wedge Nz)$

then (3)

$$\vdash FAHX$$

Now as $\vdash FAHO$,

$Az \vdash FAH(Q_1 z)$,

and $Az \vdash H(Nz)$

$\vdash H(\Sigma A(\lambda z \cdot 0 =_1 Q_1 z \wedge Nz))$.

Also $\vdash AO$

$\therefore$ $\vdash 0 =_1 0$

and so $\vdash XO$. (4)

Now $Ax \vdash x =_1 0 \supset Q_1 x =_1 Q_1 0 \wedge NO$,

so $Ax \vdash x =_1 0 \supset \Sigma A(\lambda z \cdot Q_1 x =_1 Q_1 z \wedge Nz)$. (5)

Also $Ax, Az, x =_1 Q_1 z \wedge Nz \vdash Q_1 x =_1 Q_1(Q_1 z)$

and by Theorem 4, $Ax, Az, x =_1 Q_1 z \wedge Nz \vdash N(Q_1 z)$

so also by Theorem 2, $Ax,\ x =_1 Q_1 z \wedge Nz \vdash \Sigma A(\lambda w \cdot Q_1 x =_1 Q_1 w \wedge Nw)$

$\therefore$ $Ax \vdash \Sigma A(\lambda z \cdot x =_1 Q_1 z \wedge Nz) \supset \Sigma A(\lambda z \cdot Q_1 x =_1 Q_1 z \wedge Nz)$ (6)

$\therefore$ by (5) and (6)

$$Ax \vdash Xx \supset X(Q_1 x)$$

and using (4) $\vdash XO \wedge Ax \supset_x Xx \supset X(Q_1 x)$.

Thus by (3) and the Corollary to Theorem 3,

$$Nx \vdash x =_1 0 \vee \Sigma A(\lambda z \cdot x =_1 Q_1 z \wedge Nz).$$

THEOREM 6. $Nx \vdash Au \supset_u xu \supset \Sigma A(\lambda z \cdot Q_1 zu \wedge Nz)$.

Proof. $Nx, Au, xu \vdash \Gamma(x =_1 0)$

$\therefore$ $Nx, Au, xu \vdash \Sigma A(\lambda z \cdot x =_1 Q_1 z \wedge Nz)$

$Nx, Au, xu \vdash \Sigma A(\lambda z \cdot Q_1 zu \wedge Nz)$. □

If we have the following axiom of extent (mentioned in [3]):

(E2) $\vdash Au \supset_u Az \supset_z FAHt \supset_t Q_1 zu \supset tz \leftrightsquigarrow tu$.

We also have as $\vdash FAHN$:

THEOREM 7. $Nx \vdash Au \supset_u xu \supset Nu$.

Note that this axiom for the weakest form of substitution for equality is suggested in [3]. The first rule:

$$Q_1 zu,\ tz \vdash tu$$

leads to anomalous (though not inconsistent results). The second, the axiom:

(E1) $\vdash FAHu \supset_u FAHz \supset_z FAHt \supset_t Q_1 zu \supset tz \leftrightsquigarrow tu$

gives $\vdash RR \leftrightsquigarrow RR$

where R is the Russell class (which is a *set* in this paper). We also have $RR = \Gamma(RR)$ and so by Eq, $\vdash RR \leftrightsquigarrow \Gamma(RR)$, but as we do not have $\vdash AR$ and hence not $\vdash H(RR)$ we cannot prove a contradiction from this anomalous result.

The third possibility, the axiom (E2) mentioned above seems to be free of anomalies and is sufficient for what we require above.

Note also that Theorem 7 says that N is a transitive set.

Going back now to arithmetical properties in the more general form we find that we can easily prove:

THEOREM 8. (i) $Nz, Nx, Ny \vdash x =_1 y \supset (x =_1 z \supset y =_1 z)$

(ii) $Nx, Ny \vdash x =_1 y \supset Q_1x =_1 Q_1y$

(iii) $Nx \vdash \Gamma(0 =_1 Q_1x)$

(iv) $Nx, Ny \vdash Q_1x = Q_1y \supset x =_1 y$.

Proof. By Theorem 2, (A)

$$FAHx, FAHy \vdash H(x =_1 y)$$

and

$$FAHx \vdash FAH(Q_1x)$$

it is easy to show that on the basis of the assumptions all the formulas to the right of $\vdash$ are propositions; (i) then follows by the definition of Q_1 and (ii) and (iv) follow directly from (i); (iii) follows by the definitions of Q_1 and 0.□

The four parts of Theorem 8 correspond to the first four Peano type axioms given by Mendelson [8]. To prove the next one:

$$Nx \vdash x + 0 =_1 x,$$

however seems to be impossible with + defined contextually as it is.

The alternative is to define addition (and also multiplication) by recursion. The recursion operator has been defined in [6] in terms of an ordered pair operator, which is also defined in [6] and a predecessor function which is also definable in terms of combinators. We can however define the predecessor relation in terms of terms definable using Ξ, A and H, in a much simpler fashion.

DEFINITION. $[\pi] = \lambda z\ \lambda x \cdot \Sigma A(\lambda y \cdot yx \wedge zy)$.

We can then prove:

THEOREM 9. (i) $FAHz \vdash FAH([\pi]z)$

(ii) $FAHt \vdash [\pi]\ (Q_1t) =_1 t$

(iii) $\vdash [\pi]0 =_1 0$.

Proof. (i) By (A), $FAHz, Ax, Ay \vdash H(yx \wedge zy)$

so $FAHz, Ax, \vdash H(\Sigma A(\lambda y \cdot yx \wedge zy))$

i.e. $\qquad FAHz \vdash FAH([\pi]z).$

(ii) $[\pi](Q_1t) = \lambda x \cdot \Sigma A(\lambda y \cdot yx \wedge Q_1ty)$, so by (A) and Theorem 2

$$Ay, Ax \vdash H(yx)$$

and $\qquad FAHt, Ay, Ax \vdash H(Q_1ty)$

$\therefore \qquad FAHt, Ax, [\pi](Q_1t)x \vdash tx$

i.e. $\qquad FAHt, Ax \vdash [\pi](Q_1t)x \supset tx.$

Also $\qquad Ax, tx, FAHt \vdash tx \wedge Q_1tt,$

so $\qquad Ax, tx, FAHt \vdash \Sigma A(\lambda y \cdot yx \wedge Q_1ty),$

and so $\qquad Ax, FAHt \vdash tx \supset [\pi](Q_1t)x.$

$\therefore \qquad Ax, FAHt \vdash [\pi](Q_1t)x \leftrightarrow tx$

so $\qquad FAHt \vdash Ax \supset_x [\pi](Q_1t)x \leftrightarrow tx$

i.e. $\qquad FAHt \vdash [\pi](Q_1t) =_1 t.$

(iii) $[\pi]Ox = \Sigma A(\lambda y \cdot yx \wedge Oy).$

But $\qquad \vdash Ay \supset_y \Gamma(Oy)$

so $\qquad \vdash Ax \supset_x \Gamma([\pi]Ox)$

and so $\qquad \vdash Q_1([\pi]O)O.$ □

The ordered pair operator D can now be defined by:

DEFINITION D. $D = \lambda x \lambda y \lambda z \lambda u.(\Gamma(\Sigma Az) \supset xu) \wedge (\Sigma Az \supset y([\pi]x)u).$

To prove all the expected results for this however, we need the following stronger form of (E2), which is however still weaker than (E1) and seems to avoid the anomalies mentioned earlier.

AXIOM (B). $\vdash Au \supset_u FAHz \supset_z FAHt \supset_t Q_1zu \supset tu \supset tz.$

This gives in particular:

THEOREM 10. (i) $Au, FAHz, Q_1zu \vdash Az$
(ii) $Nu, FAHz, Q_1zu \vdash Nz,$

and also the basic properties of D:

THEOREM 11. (i) $F_2(FAH)AHy, FAHx \vdash F_2(FAH)AH(Oxy)$
(ii) $FAHx, F_2(FAH)AHy \vdash Dxy0 =_1 x$
(iii) $FAHx, F_2(FAH)AHy, At \vdash Dxy(Q_1t) =_1 yt.$

Proof. (i) By Theorem 9 (i) and (A),

$$F_2(FAH)AHy, Au, FAHx \vdash H(y([\pi]x)u)$$

also $\qquad FAHz \vdash H(\Sigma Az),$

so $\qquad F_2(FAH)AHy, Au, FAHx, FAHz, \vdash H(Dxyzu)$

$\therefore \qquad F_2(FAH)AHy, FAHx \vdash F_2(FAH)A(Dxy)$

(ii) $Au, FAHx, F_2(FAH)AHy, DxyOu \vdash xu$

and by (i)

$$FAHx,\ F_2(FAH)AHy,\ Au \vdash DxyOu \supset xu.$$

Also $$FAHx,\ F_2(FAH)AHy,\ Au \vdash xu \supset DxyOu$$

so $$FAHx,\ F_2(FAH)AHy \vdash Au \supset_u xu \leftrightarrow DxyOu$$

i.e. $$FAHx,\ F_2(FAH)AHy \vdash Dxy0 =_1 x.$$

(iii) As $$At \vdash Q_1tt$$

$$At \vdash \Sigma A(Q_1t) \qquad (7)$$

$$\therefore \quad Au, FAHx,\ F_2(FAH)AHy, At, Dxy(Q_1t)u \vdash y([\pi](Q_1t))u. \qquad (8)$$

Now as $$Ar \vdash FAHr$$

we have $$F_2(FAH)AHy, Ar, Au \vdash H(yru)$$

i.e. $$F_2(FAH)AHy, Ar, Au \vdash H(Cyur)$$

and $\therefore$ $$F_2(FAH)AHy,\ Au \vdash FAH(Cyu).$$

By (A) and Theorem 9 (i) $$At \vdash FAH([\pi](Q_1t))$$

and by Theorem 9 (ii) $$At \vdash [\pi](Q_1t) =_1 t$$

$\therefore$ by (B) and (S):

$$Au, FAHx, F_2(FAH)AHy, At, Dxy(Q_1t)u \vdash ytu.$$

Similarly using (B)

$$Au, FAHx,\ F_2(FAH)AHy, At, ytu \vdash y([\pi](Q_1t))u$$

so by (7),

$$Au, FAHx,\ F_2(FAH)AHy,\ At, ytu \vdash Dxy(Q_1t)u,$$

and the result can be proved. □

Now we define the recursion operator:

DEFINITION R. $R = \lambda x\lambda y.\mathbf{Y}(B(Dx)(Sy))$.

A result such as that in parts (i) of Theorems 9 and 11, about the funcionality (or type) of R seems to be impossible. We have

$$Rxy = Dx(Sy(Rxy)),$$

so to determine, for given x and y, the type of Rxy from that of D we need first the type of Rxy.

Alternatively, we need to know a type for $\mathbf{Y}$; however, this is known to be not derivable from the types for K and S. (These "types" in fact constitute the two basic axioms of the kind of system that we are dealing with - they allow the proof of the deduction theorem for Ξ - see [1], and for a discussion of the relation between axioms and types see [6]).

We can however postulate a type for $\mathbf{Y}$ that does not conflict with those for K and S and which will lead to a type for R.

($\mathbf{Y}$) $\vdash F\{F[F(FAH)(FAH)][F(FAH)(FAH)]\}[F(FAH)(FAH)]\mathbf{Y}.$

($\vdash F(FTT)TY$ is reasonable for any T as then $FTTZ \vdash T(YZ)$ and $FTTZ \vdash T(Z(YZ))$. Below we only need the above special case).

We can now prove:

THEOREM 12. (i) $F_2(FAH)(FAH)(FAH)y, FAHt, FAHx \vdash FAH(Rxyt)$
(ii) $F_2(FAH)(FAH)(FAH)y, FAHx \vdash Rxy0 =_1 x$
(iii) $F_2(FAH)(FAH)(FAH)y, FAHx, At \vdash Rxy(Q_1t) =_1 yt(Rxyt)$.

Proof. $F(FAH)(FAH)u, FAHv \vdash FAH(uv)$

and $F_2(FAH)(FAH)(FAH)y, FAHv \vdash F(FAH)(FAH)(yv)$

$\therefore$ $F_2(FAH)(FAH)(FAH)y, F(FAH)(FAH)u, FAHv \vdash FAH(yv(uv))$

and so $F_2(FAH)(FAH)(FAH)y, F(FAH)(FAH)u \vdash F_2(FAH)AH(Syu)$

$\therefore$ by Theorem 11 (i):

$$F_2(FAH)(FAH)(FAH)y, F(FAH)(FAH)u, FAHx \vdash F_2(FAH)AH(Dx(Syu))$$

and so $F_2(FAH)(FAH)(FAH)y, FAHx \vdash F[F(FAH)(FAH)][F_2(FAH)AH](BDx(Sy))$.

Now $F_2(FAH)AH = F(FAH)(FAH)$, so by (Y) and Definition R:

$$F_2(FAH)(FAH)(FAH)y, FAHx \vdash F(FAH)(FAH)(Rxy)$$

so the result follows.

(ii) As $\vdash FAH0$, we have by (i) that

$$F_2(FAH)(FAH)(FAH)y, FAHx \vdash FAH(Rxy0).$$

Now
$$\begin{aligned} Rxy0 &= Y(B(Dx)(Sy))0 \\ &= B(Dx)(Sy)(Rxy)0 \\ &= Dx(Sy(Rxy))0. \end{aligned}$$

Now $F_2(FAH)(FAH)(FAH)y, FAHu \vdash F(FAH)(FAH)(yu)$

so by (i) $F_2(FAH)(FAH)(FAH)y, FAHx, FAHu \vdash FAH(yu(Rxyu))$

$\therefore$ $F_2(FAH)(FAH)(FAH)y, FAHx \vdash F_2(FAH)A(Sy(Rxy))$. (9)

So by Theorem 11 (ii)

$$F_2(FAH)(FAH)(FAH)y, FAHx \vdash Dx(Sy(Rxy)0) =_1 x,$$

and the result holds.

(iii) This holds by (9) and Theorem 11 (iii). □

Now we can define addition.

DEFINITION +. $+xy = x + y = Rx(KQ_1)y$.

THEOREM 13. (i) $FAHx \vdash x + 0 =_1 x$
(ii) $FAHx, Ay \vdash x + Q_1y =_1 Q_1(x+y)$
(iii) $Nx, Ny \vdash N(x+y)$.

Proof. (i) By (A), $FAHv, Aw \vdash H(Q_1vw)$

i.e. $FAHu, FAHv \vdash FAH(KQ_1uv)$

and so $\vdash F_2(FAH)(FAH)(FAH)(KQ_1)$ (10)

∴ By (A) and Theorem 12 (ii)

$$FAHx \vdash Rx(KQ_1)0 =_1 x$$

i.e. $$FAHx \vdash x+0 =_1 x.$$

(ii) By (10) and Theorem 12 (iii),

$$FAHx, Ay \vdash x+Q_1y =_1 KQ_1y(x+y)$$

so $$FAHx, Ay \vdash x+Q_1y =_1 Q_1(x+y).$$

(iii) Theorem 12 (i) $\vdash$ FAH(x+0), so by (i) and Theorem 10 (ii)

$$Nx \vdash N(x+0)$$

∴ $$\vdash Ax \supset_x Nx \supset N(x+0).$$

Now $Ax \supset_x Nx \supset N(x+y), Ax, Nx \vdash N(x+y)$

so by Theorem 4,

$$Ax \supset_x Nx \supset N(x+y), Ax, Nx \vdash N(Q_1(x+y)),$$

also then $$Ax \supset_x Nx \supset N(x+y), Ax, Nx \vdash A(Q_1(x+y)).$$

Now $$Ay \vdash FAH(Q_1y)$$

so by Theorem 12 (i) $Ay, FAHx \vdash FAH(x+Q_1y)$

∴ by (ii) and Theorem 10 (ii)

$$Ax \supset_x Nx \supset N(x+y), Ax, Ay, Nx \vdash N(x+Q_1y)$$

so $$\vdash Ay \supset_y [Ax \supset_x Nx \supset N(x+y)] \supset [Ax \supset_x Nx \supset N(x+Q_1y)].$$

So by the corollary to Theorem 3

$$Ny \vdash Ax \supset_x Nx \supset N(x+y)$$

so $Nx, Ny \vdash N(x+y)$. □

Clearly (i) and (ii) of this theorem have as special cases:

$$Nx \vdash x+0 =_1 x$$

$$Nx, Ny \vdash x+Q_1y =_1 Q(x+y).$$

We now define the multiplication:

DEFINITION X. $Xxy = x \cdot y = R0(K(+x))y$.

THEOREM 14. (i) $FAHx \vdash x \cdot 0 =_1 0$

(ii) $FAHx, Ay \vdash x \cdot Q_1y =_1 x + x \cdot y$

(iii) $Nx, Ny \vdash N(x \cdot y)$

Proof. (i) By Theorem 12 (i), $FAHx, FAHy \vdash FAH(x+y)$

so $$FAHu, FAHx, FAHy \vdash FAH(K(+x)uy) \qquad (11)$$

∴ $$FAHx \vdash F_2(FAH)(FAH)(FAH)(K(+x))$$

then by Theorem 12 (ii) $FAHx \vdash R0(K(+x))0 =_1 0$

so (i) follows.

(ii) By (11) and Theorem 12 (iii)

$$FAHx, Ay \vdash x \cdot (Q_1y) =_1 K(+x)y(x \cdot y)$$

so $$FAHx, Ay \vdash x \cdot (Q_1y) =_1 x+x \cdot y.$$

(iii) By (i) and Theorem 10 (ii)

$$Nx \vdash N(x \cdot 0)$$

so

$$\vdash Ax \supset_x Nx \supset N(x \cdot 0),$$

$$Ax \supset_x Nx \supset N(x \cdot y), Ax, Nx \vdash N(x \cdot y)$$

∴ by Theorem 13 (iii)

$$Ax \supset_x Nx \supset N(x \cdot y), Ax, Nx \vdash N(x \cdot y + x).$$

Now by Theorem 12 (i), (2) and Definition X

$$FAHx, FAHy \vdash FAH(x \cdot Q_1 y)$$

∴ by Theorem 10 (ii) and (ii)

$$Ax \supset_x Nx \supset N(x \cdot y), Ax, Nx, Ay \vdash N(x \cdot Q_1 y)$$

∴

$$\vdash Ay \supset_y [Ax \supset_x Nx \supset N(x \cdot y)] \supset [Ax \supset_x Nx \supset_x N(x \cdot Q_1 y)]$$

so by the corollary to Theorem 3

$$Ny \vdash Ax \supset_x Nx \supset N(x \cdot y)$$

∴

$$Nx, Ny \vdash N(x \cdot y). \ \square$$

Thus given the extra axioms (B) and (Y) which we have had to introduce, we can develop all the Peano type axioms of [8], and hence Mendelson's development of formal number theory can be carried out here.

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Department of Mathematics
University of Wollongong
Australia.

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FACTUAL AND COGNITIVE PROBABILITY

Rolando Chuaqui*

ABSTRACT. The paper presents a modification of the definition of probability presented in earlier papers of the author (e.g. *A semantical definition of probability*, in Non-Classical Logics, Model Theory and Computability, North Holland, pp.135-167). This modification separates the two aspects of probability: probability as a part of physical theories (factual), and as a basis for statistical inference (cognitive). Factual probability is represented by probability structures as in the earlier papers, but now built independently of the language. Cognitive probability is interpreted as a form of "partial truth". The paper also contains a discussion of the Principle of Insufficient Reason and of Bayesian and classical statistical methods, in the light of the new definition.

This paper presents a modification of the semantical definition of probability introduced in Chuaqui 1977 and 1980. The new definition presented here brings forth the two aspects of probability: as a basis for statistical inference and as a part of physical theories.

The main modification introduced is making independent of the language the definition of the group of transformations that preserve the laws of the phenomenon. Thus, the determination of the probability measure for the simple probability structures of Chuaqui 1977 becomes independent of linguistic elements, and the simple probability interpretations $\langle \mathbf{K}, \mathcal{B}, \mu \rangle$ of Chuaqui 1980 may be considered as models of reality.

The connection with cognitive elements is established via the concept of probability as degree of partial truth, introduced in earlier papers.

The first section analyzes the different uses of probability, while in the second, I give a brief account and a classification of theories on probability. Section 3 introduces the modification of the definition in Chuaqui 1977 and 1980 that permits to consider probability structures as models of reality.

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The fourth section justifies the cognitive uses of probability by considering it as a logical concept, while the fifth applies these remarks to an analysis of the *Principle of Insufficient Reason*. The last section complements the study in Chuaqui 1980 of classical and Bayesian statistical methods.

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1. USES OF PROBABILITY.

Probability is used as a basis for statistical inference and as a part of scientific theories. In its first use, I shall distinguish two different applications. One application is to use probability as the very guide of life in the face of uncertainty. Thus, probability is the basis of decision theory in its classical and Bayesian forms. The need for statistical inference arises from our uncertainty as to how we ought to behave under circumstances where we are ignorant concerning the state of the world. As Kyburg 1974 says, "we attempt to develop rules of behavior which we may follow, whatever the state of the world, in the expectation that we are following rules whose characteristics are generally (1) desirable, and (2) attainable". The most desirable rule is one that tells us how to discover the true state of the world; the most attainable and simplest is to forget about the arithmetic and act as we feel like it. A compromise between these two extremes is to follow what we may know about the probabilities of the different possible states of the world, i.e. about a measure of the degree of truth of each possible state of the world.

The second application of statistical inference is the evaluation of scientific hypotheses. This application can be thought of as a case of a decision whether to accept or not a scientific hypothesis, and thus it can be assimilated to processes of the first kind. However, I believe with R.A. Fischer that "... such processes have a logical basis very different from those of a scientist engaged in gaining from his observations an improved understading of reality". (Fisher 1956 p.5).

Besides these statistical uses, probability statements appear as part of physical theories such as Statistical Mechanics or Quantum Mechanics. Also, most of the theory of stochastic processes serves as basis for scientific theories of particular phenomena, such as Brownian motion and radioactive decay.

Although the word "probability" itself might not occur in a sientific theory, probability concepts are present as general statements expressing stochastic relations among random quantities. For instance, there may be functions expressing distributions or densities of certain quantities under certain circumstances. Anyway, probabilities of events are obtained from them and used in applications.

Intuitively, there are other evidences of probabilities as independent of our knowledge or belief. For instance, if we toss a coin 100 times and in 60 obtain "heads", it seems natural to believe that this is a property of the coin or rather of the coin together with the mechanism for tossing it.

Since the statistical applications stem from our ignorance of the true state of the world, I shall call them *cognitive uses*. On the other hand, the other uses will be the *factual uses*.

2. COGNITIVE VERSUS FACTUAL INTERPRETATIONS.

The interpretations of the concept of probability which have been offered stress either the cognitive or the factual uses of probability. Among the first, I would put the subjectivist and logical views. Among the second, the frequentist and propensity views. The subjectivist do not attempt to explain how we get our probabilities while the holders of the logical views do. On the other side, the propensity theories do not attempt to define probabilities but only to measure them, while the frequentists build models of reality where probability is defined in terms of other concepts.

Probability, however, has both cognitive and factual aspects. Thus, any interpretation should give an account of both. In order to do this, many scholars hold a dual view: there is an interpretation of probability as *degree of belief* or credence (cognitive) and another as *chance or propensity* (factual). The connection between the two should be given by a principle of the following form (a similar principle was formulated in Lewis 1980): let X be the proposition that the chance, at time t, of A holding equals x, where x is a real number of the unit interval. Let C_X be any reasonable "degree of belief" function of a person that believes X at time t. Then $C_X(A) = x$.

In this principle, X is supposed to contain the statement about the factual probability of A. $C_X(A)$ is the cognitive probability of A. The two are supposed to be connected by the principle.

I believe that any interpretation of probability should explain both the factual and cognitive uses of probability and justify a principle such as the one above. I shall attempt to provide such an account by modifying the views espoused in Chuaqui 1977 and 1980.

3. PROBABILITY STRUCTURES AS MODELS OF CHANCE.

3.1. Simple probability models. In Chuaqui 1977, the theory of simple probability structures was presented. From the simple probability structures $\mathbf{K}$, there were obtained probability interpretations of languages, constituted by triples $\langle \mathbf{K}, \mathcal{B}, \mu \rangle$, where $\mathcal{B}$ is a field of subsets of $\mathbf{K}$ and μ a probability measure.

We want to consider these probability structures as models of reality which determine the probability measure μ. However, in my original presentation, the definition of μ depended heavily on the language used. I believe this to be a feature that precludes their use as models of reality. I shall now offer a modification of the concept of simple probability structures that will give us a definition of μ, independent of the language.

There are two elements in $\langle \mathbf{K}, \mathcal{B}, \mu \rangle$ that are language dependent in Chuaqui 1977. The first is the definition of $\mathcal{B}$ as the field of sets generated by all the sets $Mod_{\mathbf{K}}(\phi)$ where ϕ is a sentence of the appropriate language (recall that $Mod_{\mathbf{K}}(\phi)$ is the set of models of $\mathbf{K}$ in which ϕ holds). However, $\mathcal{B}$ is the algebra of events and there is no need of choosing it in this way, it can be taken to be just the algebra of events that are necessary for an adequate description of the situation. In case $\mathbf{K}$ is finite for instance, $\mathcal{B}$ will generally be the power set of $\mathbf{K}$. A more detailed explanation of $\mathcal{B}$ will be given in Section 4.

The other linguistic element is the definition of the group $G_{\mathbf{K}}$, namely conditions (1)(a) and (1)(b). This we have to get rid of.

In order to make the situation clear, I shall begin with the same example as in Chuaqui 1977, the choosing of a sample S of size m from a finite population of balls P. When we say "S has n red balls" we mean that one of the properties of the outcome was that the sample had n red balls. The same outcome has many different properties. We can think of an ideal approximation of an outcome, namely a relational system that represents a possible model of the situation involved. In the case we are looking at, we can schematize the possible outcomes as systems $\mathfrak{A}_S = \langle P, R_0, \ldots, R_{m-1}, S \rangle$, where P is the finite set of balls, $R_0, \ldots, R_{m-1}$ are fixed subsets of P that represent the properties we are interested in (for instance, red), and S is any subset of P of m members (the sample). For each subset S of m members there is a corresponding system $\mathfrak{A}_S$; the set of possible outcomes $\mathbf{K}$, consist of all models $\mathfrak{A}_S$ of the form described above.

Let us analyze a possible outcome $\mathfrak{A}_S = \langle P, R_0, \ldots, R_{m-1}, S \rangle$. The properties $R_0, \ldots, R_{m-1}$ are intrinsic properties of the balls in P. That is, when we move the balls around or choose a sample, their properties remain. Also, these properties are fixed in all $\mathfrak{A}_S$. We may thus call $\langle P, R_0, \ldots, R_{m-1} \rangle$ the *intrinsic part of* $\mathbf{K}$. On the other hand, $\langle P, S \rangle$ gives the structure of the experiment, and is variable in each $\mathfrak{A}_S$. $\langle P, S \rangle$ is called the ***structural*** part of $\mathfrak{A}_S$ and denoted $\mathfrak{A}_{S\,st}$. We may have different experiments performed on the same set of balls P. The probability structures for these different experiments might have the same intrinsic part but different structure. For instance, if the experiment consists of the choosing of a sample with two R_0-balls, the outcome will be of the form $\langle P, R_0, \ldots, R_{m-1}, Q \rangle$ where Q is a subset of P with two R_0-balls.

The second example I shall give is a modification of Example 3 of Chuaqui 1977. Suppose we have a circular roulette with a point for each real number.

For simplicity, a fixed force is applied but the roulette starts from a variable position. Each outcome results from beginning at a particular position. The systems in $\mathbf{K}$ may be taken to be of the form

$$\mathfrak{A}_I = \langle C,r,f,I \rangle$$

where C is the set of points in the circle, r represents translations in the circle (a ternary operation, $r(a,b,c)$ is c rotated by the angle from a to b), f is the continuous unary function that associates each initial position with a final position, and I is the set containing the initial position (I contains one element of C). Here, $\langle C,f \rangle$ constitutes the intrinsic part of $\mathbf{K}$, because it is an intrinsic property of each point x in C that it yield a final position $f(x)$. With this f we can express the asymmetry of the roulette, if it is asymmetric. We can model symmetric roulettes without this f. The structure of the experiment is given by $\langle C,r,I \rangle$, I is variable in the different $\mathfrak{A}_I$. However r is fixed. Thus, we cannot distinguish between the intrinsic part and the structure by just looking at the variable part of $\mathbf{K}$. Notice that C is not enough for defining a circle. It is necessary to add an operation between the elements of C. For instance, I have chosen in this paper r. This operation r should be part of the structure of the experiment, because it really consists of a rotation of the circle. If we had just $\langle C,I \rangle$ in the structure, the experiment would be the choosing of a point in a set C and not in a circle $\langle C,r \rangle$.

From these two examples, we see that in order to describe the simple probability structure we need to specify, besides the class of possible outcomes $\mathbf{K}$, its intrinsic part. Thus, we define a simple probability structure as a pair $\mathcal{K} = \langle \mathbf{K},\mathfrak{A} \rangle$ where $\mathbf{K}$ is a set of relational structures of a fixed similarity type (called the *set of outcomes* of $\mathcal{K}$) and $\mathfrak{A}$ is a relational structure (called the *intrinsic part* of $\mathcal{K}$) such that $\mathfrak{L} \restriction J = \mathfrak{A}$ for every $\mathfrak{L} \in \mathbf{K}$, where J is the index set of the similarity type of $\mathfrak{A}$ (thus, all structure $\mathfrak{L} \in \mathbf{K}$, have the same universe say A).

The group of functions $G_{\mathbf{K}}$ is now determined by $\mathcal{K}$ and $\mathcal{B}$ without reference to the language. For $\mathfrak{L} \in \mathbf{K}$, let $\mathfrak{L}_{st} = \mathfrak{L} \restriction (I-J)$ where I is the index set of the similarity type of K. That, is $\mathfrak{L}_{st}$ represents the structure of the experiment. Also, for $\mathbf{B} \subseteq$ K, $\mathbf{B}_{st} = \{\mathfrak{L}_{st} : \mathfrak{L} \in \mathbf{B}\}$.

The group of functions $G_{\mathbf{K}}$ that preserve the "laws of the phenomenon" contains all permutations f of the universe A such that

(1) For any $\mathfrak{L} \in$ K, $f^*(\mathfrak{L}_{st}) \in \mathbf{K}_{st}$ and $f^{-1*}(\mathfrak{L}_{st}) \in \mathbf{K}_{st}$

(2) For any $\mathbf{B} \in \mathcal{B}$, $\mathbf{B}^f \in \mathcal{B}$ and $\mathbf{B}^{f^{-1}} \in \mathcal{B}$, where $\mathbf{B}^f$ is the unique $\mathbf{C} \subseteq \mathbf{K}$ such that $\mathbf{C}_{st} = \{f^*\mathfrak{L} : \mathfrak{L} \in \mathbf{B}_{st}\}$.

Condition (1) can be expressed simply by $\mathbf{K}^f \subseteq$ K and $\mathbf{K}^{f^{-1}} \subseteq \mathbf{K}$. The measure μ is a measure invariant under $G_{\mathbf{K}}$.

In our first example $G_{\mathbf{K}}$ consists of all permutations of the universe P. In the second, $G_{\mathbf{K}}$ contains all automorphisms of $\langle C,r\rangle$. In general, if $\langle A,P_0,\ldots,P_{n-1}\rangle$ is constant in all elements of **K** and none of the relations $P_0,\ldots,P_{n-1}$ are in the intrinsic part of **K**, then $G_{\mathbf{K}}$ is a subgroup of the automorphisms of $\langle A,P_0,\ldots,P_{n-1}\rangle$.

In order to extend the situation to compound structures, we define a symmetry relation between subsets A, B of **K**:

$$\mathbf{A} \underset{\mathbf{K}}{\sim} \mathbf{B} \text{ iff } \mathbf{A}^f = \mathbf{B} \text{ for a certain } f \in G_{\mathbf{K}}.$$

The measure μ is now invariant under $\sim_{\mathbf{K}}$, i.e. $\mathbf{A} \underset{\mathbf{K}}{\sim} \mathbf{B}$ implies $\mu(\mathbf{A}) = \mu(\mathbf{B})$.

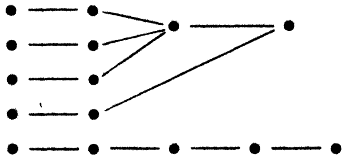
Summarizing, we have obtained a model of reality $\langle \mathcal{K},\mathcal{B}\rangle$ that gives a probability measure μ invariant under $G_{\mathbf{K}}$. If this μ were the unique invariant measure, then $\langle \mathcal{K},\mathcal{B}\rangle$ would be a sufficient description of the model. Uniqueness is not a rare phenomenon. As a matter of fact, all situations I have analyzed yield a unique measure. However, the general conditions for uniqueness that I know of are rather technical and, hence, they are not natural to add as requirements for $\langle \mathcal{K},\mathcal{B}\rangle$. Thus, the *simple probability models* have to be specifed by the triple $\mathcal{J} = \langle \mathcal{K},\mathcal{B},\mu\rangle$ where $\mathcal{K} = \langle \mathbf{K},\mathfrak{A}\rangle$ is a simple probability structure, $\mathcal{B}$ a σ-algebra of subsets of **K**, and μ a probability measure invariant under $G_{\mathbf{K}}$; **K** is called the *set of outcomes of* $\mathcal{J}$.

3.2. Compound probability models. The compound probability structures introduced in Chuaqui 1980 need only minor modifications.

However, it may be noted that with the new definition of simple probability structures, the compound structures seem much more natural, since now the symmetries are independent of the language.

An outline of the definition of compound probability structure will help to understand the situation. These structures are determined by three elements.

1. ***The causal structure.*** The basic elements of the causal structure are the (causal) trees $\langle T,\leqslant_T\rangle$. T is a set and $\leqslant_T$ is a well founded partial ordering on T such that the successors of any $t\in T$ are countable and well ordered by $\leqslant_T$. A graphical representation of an example of a tree is:



where $\leqslant_T$ is in the horizontal direction from left to right.

A tree is a generalization of the notion of causal dependence. Thus, what happens at $t \in T$ influences what happens at a succeding s, i.e. at $s_T \geqslant t$. If

s and t are not related by $\leqslant_T$, then they are independent moments. The elements of T can be considered in most cases as time moments, but this is not necessarily so.

A causal structure **F** consists of a family of trees that includes all its subtrees. ($\langle T,\leqslant_T\rangle$ is a subtree of $\langle S,\leqslant_S\rangle$ if $\leqslant_T \subseteq \leqslant_S$, and $t \in T$, $s \in S$ and $s \leqslant_S t$ imply $s \in T$.) In most cases, it is enough to consider all the subtrees of a given tree $\langle T,\leqslant_T\rangle$.

2. *The set of outcomes.* The outcomes are functions f with domain a certain $T \in$ **F**. For any $t \in T$, $f(t)$ is what happens at t, and is a member of the set of outcomes $\mathbf{H}(f,t)$ of a simple probability structure $\mathcal{H}(f,t)$. This simple probability model is determined by the preceding values of f, i.e. f restricted to $T_t = \{s : s \leqslant_T t$ and $s \neq t\}$. Calling $\mathbf{H}_T$ the set of all outcomes with domain T, the events are subsets of $\mathbf{H}_T$ for $T \in$ **F**. In fact, $\mathrm{H}(f,t) = \{j(t) : j \in \mathbf{H}_T$ and $j \restriction T_t = f \restriction T_t\}$ is the sure event determined by $f \restriction T_t$, i.e. the set of outcomes that are possible if $f \restriction T_t$ has occurred.

3. *The symmetry relation.* On each simple probability model $\mathcal{H}(f,t)$ we obtain a symmetry relation $\sim_{f,t}$ as explained above. From these relations, the symmetry relation $\sim$ between compound events is obtained. Since now $\sim_{f,t}$ is independent of the language, $\sim$ will also be. The relation $\sim$ is defined in several stages.

a) We first define isomorphism between two simple probability models $\mathcal{J} = \langle\langle \mathbf{K},\mathcal{Ol}\rangle,\mathcal{B},\mu\rangle$ and $\mathcal{J}' = \langle\langle \mathbf{K}',\mathcal{Ol}'\rangle,\mathcal{B}',\mu'\rangle$. Suppose A and A' are the universes of $\mathbf{K}$ and $\mathbf{K}'$ respectively.

$\mathcal{J} \cong_g \mathcal{J}'$ if and only if g is a one to one function from A onto A' satisfying:

(i) $\mathbf{K}' = \mathbf{K}^g$ (i.e. $\mathbf{K}'_{st} = \{g^*\mathfrak{L} : \mathfrak{L} \in \mathbf{K}_{st}\}$).

(ii) For any $\mathrm{B} \in \mathcal{B}$, $\mathrm{B}^g \in \mathcal{B}'$.

(iii) For any $\mathrm{B} \in \mathcal{B}$, $\mu(\mathrm{B}) = \mu'(\mathrm{B}^g)$.

In case μ is the unique $G_{\mathbf{K}}$-invariant measure, condition (iii) is implied by the other two, since then μ' is also the unique $G_{\mathbf{K}'}$-invariant measure. This is so because the groups $G_{\mathbf{K}}$ and $G_{\mathbf{K}'}$ are related by the isomorphism g as follows:

$$f \in G_{\mathbf{K}} \text{ iff } g \circ f \circ g^{-1} \in G_{\mathbf{K}'}$$

Let $\mathrm{B} \in \mathcal{B}$, $\mathrm{C} \in \mathcal{B}'$, then

$$\mathrm{B} \sim_g \mathrm{C} \text{ iff } \mathcal{J} \cong_g \mathcal{J}' \text{ and } \mathrm{B}^g \sim_{\mathbf{K}'} \mathrm{C}.$$

This definition of isomorphism constitutes the only difference with the definition in Chuaqui 1980, Section 5, of compound probability structures. We could use the same definition as there, but I believe that the new one is an improvement of the former; in particular because

$$\mathcal{J} \cong_g \mathcal{J} \text{ (i.e. with } \mathcal{J}' = \mathcal{J}) \text{ iff } g \in G_{\mathbf{K}}.$$

b) We now introduce the notion of isomorphism between sets $\mathbf{H}_T$ and $\mathbf{H}_{T'}$ for T, $T' \in \mathbf{F}$. These isomorphisms are pairs of functions $\langle h,k\rangle$ such that h is an isomorphism between $\langle T, \leqslant_T\rangle$ and $\langle T', \leqslant_{T'}\rangle$, and k is a one to one function from $\mathbf{H}_T$ onto $\mathbf{H}_{T'}$ with the following properties.

(i) If $t \in T$, f, $f' \in \mathbf{H}_T$ and $f \restriction T_t = f' \restriction T_t$ (i.e. what occurs before t is the same for f as for f'), then $k(f) \restriction T'_{h(t)} = k(f') \restriction T'_{h(t)}$ (i.e. what occurs before $h(t) \in T'$ is the same for $k(f)$ as for $k(f')$).

(ii) $\mathbf{H}(f,t)$ and $\mathbf{H}(k(f),h(t))$ are isomorphic (in the sense of (a)) for $f \in \mathbf{H}_T$ and $t \in T$. In fact, the corresponding isomorphism $g_{f,t}$ is such that if $\mathcal{L} = j(t)$ for a certain $j \in \mathbf{H}_T$ with $j \restriction T_t = f \restriction T_t$, then $g^*_{f,t}\mathcal{L}_{st} = (k(j)(h(t)))_{st}$. Notice that $j(t)$ is an outcome in $\mathbf{H}(f,t)$ and $k(j)(h(t))$, an outcome in $\mathbf{H}(k(f),h(t))$.

c) Now let $\mathbf{A} \subseteq \mathbf{H}_T$ and $\mathbf{B} \subseteq \mathbf{H}_{T'}$ for T, $T' \in \mathbf{F}$. We say that $\mathbf{A} \sim \mathbf{B}$ iff there is an isomorphism h of $\langle T, \leqslant_T\rangle$ onto $\langle T', \leqslant_{T'}\rangle$ and there are S, S' and k such that,

(i) $\langle S, \leqslant_S\rangle$ is a subtree of $\langle T, \leqslant_T\rangle$ and $S' = h^*S$.
(ii) $\mathbf{H}_S$ is isomorphic to $\mathbf{H}_{S'}$ by $\langle h \restriction S, k\rangle$.
(iii) For every $t \in S$ and $f \in \mathbf{H}_S$ the corresponding parts of $\mathbf{A}$ and $\mathbf{B}$ by $\langle h,k\rangle$ are equivalent, i.e. if we define $\mathbf{A}(f,t) = \{j(t) : j \in \mathbf{A} \text{ and } j \restriction T_t = f \restriction T_t\}$ then $\mathbf{A}(f,t) \approx_{g_{f,t}} \mathbf{B}(k(f),h(t))$.
(iv) For $t \in T-S$ and $f \in \mathbf{H}_T$, $\mathbf{A}(f,t)$ is equivalent to the sure event at its level, i.e. $\mathbf{A}(f,t) \sim_{f,t} \mathbf{H}(f,t)$.
(v) Similarly, for $t' \in T'-S'$ and $f' \in \mathbf{H}_{T'}$, $\mathbf{B}(f',t') \sim_{f't'} \mathbf{H}(f',t')$.

The measure μ on the compound events should be invariant under $\sim$. In Chuaqui 1980, a procedure for defining such a measure is given. The algebra of compound events consists just of the measurable sets with respect to this measure.

As a first example of a compound probability structure, I shall take the case of independent trials of the same experiment. Assume that the experiment is modeled by a simple probability model $\mathcal{J} = \langle \mathcal{K}, \mathcal{B}, \mu\rangle$ and that there are n independent trials where $n \leqslant \omega$. We assume a symmetry relation is defined on $\mathcal{J}$, say $\sim_{\mathcal{K}}$.

The causal structure for the compound model is constituted by the subtrees of $\langle n, =\rangle$. Since the trials are independent, there is no causal relation between them; thus, we take $=$ for the partial ordering.

The compound set of outcomes is ${}^n\mathbf{K}$, where $\mathcal{K} = \langle \mathbf{K}, \mathcal{O}\!\mathcal{L}\rangle$. For every $f \in {}^n\mathbf{K}$ and $t \in n$, $\mathbf{H}(f,t) = \mathcal{J}$. Since $\langle n\rangle$ is a very simple tree, all $\langle m, =\rangle$ with $m \subseteq n$

are subtrees. Also, the only condition for the isomorphism of two such subtrees is that they have the same cardinality, and any isomorphism can be extended to an automorphism of $<n, =>$, i.e. a permutation of n.

Events are subsets of $\mathbf{H}_m = {}^m\mathbf{K}$ for $m \subseteq n$. For any $\mathbf{A} \subseteq \mathbf{H}_m$, $f \in \mathbf{H}$ and $t \in m$ $\mathbf{A}(f,t) = \{j(t): j \in \mathbf{A}\}$; thus $\mathbf{A}(f,t)$ is independent of f and $\mathbf{H}(f,t) = \mathbf{K}$.

Suppose that $\mathbf{H}_m$ and $\mathbf{H}_{m'}$ are given and that h is a one to one function from m onto m'. Let us analyze which k are such that $<h,k>$ is an isomorphism of $\mathbf{H}_m$ onto $\mathbf{H}_{m'}$. k should be a one to one function from $\mathbf{H}_m$ onto $\mathbf{H}_{m'}$. Since for every $t \in m$, $T_t = \emptyset$, condition b(i) is always satisfied. Condition b(ii) requires that there be an isomorphism g_t of $\mathcal{J}$ onto itself (i.e. $g_t \in G_{\mathbf{K}}$) such that $g_t^*(j(t)_{\text{st}}) = (k(j)(h(t)))_{\text{st}}$ for every $j \in \mathbf{H}_m$. Thus, if $\mathbf{B} \subseteq \mathbf{H}_m$ and $\mathbf{C} \subseteq \mathbf{H}_{m'}$ then $\mathbf{B} \sim \mathbf{C}$ iff $\mathbf{B}(f,t) \sim_{\mathbf{K}} \mathbf{B}(f,h(t))$ for every $t \in m$.

Therefore, the compound measure is given by the product measure, and the compound events are the measurable sets according to this product measure.

As a second example, take $T = \{t_0,t_1\}$, a set of two elements, with $t_0 \leqslant_T t_1$, and its subtrees, as $\mathbf{F}$. For t_0, we have the simple probability model $\mathcal{J}_0 = <<\mathbf{K}_0,\mathcal{O}\!\ell_0>,\mathcal{B}_0,\mu_0>$. For each $\mathfrak{L} \in \mathbf{K}_0$, we have the simple probability model $\mathcal{J}_{\mathfrak{L}} = <<\mathbf{K}_{\mathfrak{L}},\mathcal{O}\!\ell>,\mathcal{B}_{\mathfrak{L}},\mu_{\mathfrak{L}}>$ The compound outcomes are the functions f with $D_0 f = T$ and such that $f(t_0) \in \mathbf{K}_0$ and $f(t_0) \in \mathbf{K}_{f(t_0)}$. Suppose, further, that $\mathcal{K}_{\mathfrak{L}}$ is not isomorphic to $\mathcal{K}_{\mathfrak{L}'}$, for $\mathfrak{L} \neq \mathfrak{L}'$.

The only subtree of $<T,\leqslant_T>$ is $<\{t_0\},\leqslant_T>$. There is only one automorphism of $<T,\leqslant_T>$, namely the identity.

It is clear that $H(f,t_0) = \mathcal{J}_0$ and $H(f,t_1) = \mathcal{J}_{f(t_0)}$ for every outcome f. Let us see which are the k's for which the pair <identity, k> is an isomorphism of $\mathbf{H}_T$ onto $\mathbf{H}_T$. k must be a one to one function from $\mathbf{H}_T$ onto $\mathbf{H}_T$; b(i) can be expressed by:

(1) if $f(t_0) = f'(t_0)$, then $k(f)(t_0) = k(f')(t_0)$,

(b-ii) adds two conditions:

(2) $g_{f,t_0} \in G_{\mathbf{K}}$

(3) $g_{f,t} \in G_{\mathbf{K}_{f(t_0)}}$.

Thus, if $\mathbf{B},\mathbf{C} \subseteq \mathbf{H}_T$, then $\mathbf{B} \sim \mathbf{C}$ if and only if (i) and (ii), or (i) and (iii) are satisfied, where

(i) $\mathbf{B}(f,t_0) \sim_{\mathbf{K}_0} \mathbf{C}(f,t_0)$

(ii) For every $\mathfrak{L} \in \mathbf{K}_0$, we have $\mathbf{B}(\mathfrak{L}) \sim_{\mathbf{K}_{\mathfrak{L}}} \mathbf{C}(\mathfrak{L})$, where $\mathbf{B}(\mathfrak{L}) = \{f(t_1): f \in \mathbf{B}, f(t_0) = \mathfrak{L}\}$.

(iii) For every $\mathfrak{L} \in \mathbf{K}_0$, $\mathbf{B}(\mathfrak{L}) \sim_{\mathbf{K}_{\mathfrak{L}}} \mathbf{K}_{\mathfrak{L}}$.

The compound measure μ is defined by

$$\mu(\mathbf{B}) = \int_{\mathbf{B}(f,t_0)} \mu(\mathbf{B}(\mathfrak{L}))d_{\mu_0}.$$

As an instance of this last example, assume that $\mathbf{K}_0 = \{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$ where $\mathcal{L}_i$ represents the choosing of urn i (i.e. there are three urns 1,2,3). In urn i there are n_i balls with m_i white balls. $\mathbf{K}_{\mathcal{L}_i}$ represents the choosing of a ball from urn i. The event "a white ball is chosen" is represented by the set of outcomes f, where $f(0) \in \mathbf{K}_0$ and $f(1) \in \mathbf{K}_{f(0)}$ is a model where a white ball is selected. Let us call this event W. Suppose that $\mu_0(\mathcal{L}_1) = \mu_0(\mathcal{L}_2) = \mu_0(\mathcal{L}_3)$. Then

$$\mu(W) = \int_{\mathbf{K}_0} \mu_{\mathcal{L}_i}(W(\mathcal{L}_i))\,d\mu_0$$

$$= (\mu_{\mathcal{L}_1}(W(\mathcal{L}_1)) + \mu_{\mathcal{L}_2}(W(\mathcal{L}_2)) + \mu_{\mathcal{L}_3}(W(\mathcal{L}_3)))\cdot\frac{1}{3}\ .$$

4. PROBABILITY AS A LOGICAL RELATION.

The justification of the Connecting Principle is given via the conception of probability as partial truth that was developed in Chuaqui 1977 and 1980. This is a logical conception of probability. We, thus, need a language $\mathcal{L}$.

A probability model $\mathcal{J} = \langle \mathbf{K}, \mathcal{B}, \mu\rangle$ is *appropriate for* $\mathcal{L}$ if the following two conditions are satisfied:

(i) The similarity type of the structures in $\mathbf{K}$ is the same as that of $\mathcal{L}$.

(ii) $\mathcal{B}$ includes the set $\{Mod_{\mathbf{K}}(\phi)$: ϕ a sentence of $\mathcal{L}\}$, where $Mod_{\mathbf{K}}(\phi) = \{\mathcal{L} \in \mathbf{K} : \mathcal{L} \models \phi\}$.

If $\mathcal{J}$ is an appropriate probability model for $\mathcal{L}$, we define, as in the earlier papers,

$$P_{\mathcal{J}}(\phi) = \mu(Mod_{\mathbf{K}}(\phi)),$$

for all sentences ϕ of $\mathcal{L}$.

P provides a logical interpretation of probability as a relation between a sentence and a probability structure $\mathcal{J}$, which is considered as an interpretation of the sentence. $P_{\mathcal{J}}(\phi)$ represents a measure of the "degree of partial truth of ϕ under the interpretation $\mathcal{J}$". Notice that $\mathcal{J}$ has a dual role. On one hand, it is a model of reality. On the other hand, it serves as an interpretation of the language. The usual relational structures (or possible models) of logic can also be seen in this dual role. But in this latter case, reality is completely specified and, hence, every sentence is either true or false.

This logical interpretation of probability can serve to justify the *Connecting Principle*, which is now reformulated as follows:

Let C_X be any reasonable "degree of belief" function of a person who accepts the proposition X that the probability structure $\mathcal{J}$ is an adequate description of the situation involved, and that $P_{\mathcal{J}}(\phi) = r$, for a certain

$r \in [0,1]$. Then

$$C_X(\phi) = r$$

The person p should believe ϕ to the degree $P_J(\phi)$, because he believes ϕ to be true to this degree. Thus, the connection between factual probability and degree of belief is obtained via truth. This is very natural because we believe what we believe to be true. Thus, we should believe ϕ with degree α when we believe ϕ to be true with degree α. This is similar to the relation between usual logic and belief. If somebody believes ϕ and that ϕ logically implies ψ then he should believe ψ.

Degrees of belief should be applied to propositions instead of sentences. Thus, a more accurate description should involve Intensional Logic (such as that of Reinhardt 1980).

Notice that I put $C_X(\phi)$ and not $C(\phi|X)$. This is so, because I believe the acceptance of J does not change C by conditionalization (see Kyburg 1980 and Chuaqui 1980, Section 2).

My main differences with Bayesians (at least with strict Bayesians) are two. In the first place, I do not believe that probabilities (or degrees of belief) can be assigned to all events. Only given a well-defined situation in which the possible outcomes are determined, it may be possible to assign probabilities. In the second place, as it was mentioned before, I do not believe that the only changes in the probability (or degree of belief) function proceed by conditionalization. The discussion of these matters would take us too far, so they will be left for another paper.

5. A NOTE ON THE PRINCIPLE OF INSUFFICIENT REASON.

As an example of the application of the ideas given previously, I shall analyse the *Principle of Insufficient Reason* or *Principle of Indifference*:

"The Principle of Indifference asserts that if there is no *known* reason for predicating of our subject one rather than another of several alternatives, then relatively to such knowledge the assertions of each of these alternatives have an *equal* probability" (version of Keynes 1921, p.42).

This principle may be interpreted in two different ways: cognitive and factual. We can say that the equal probability is established when one *knows* of no reason or when *there are* in fact no reasons. The Principle as stated by Keynes (and also as stated by J. Bernoulli and Laplace) gives the first interpretation. In this form, it is indeed contradictory, as the well-known paradoxes show. However, a factual interpretation is also possible and may not be contradictory. In fact, a careful reading of Laplace 1820, leads me to believe that his intention was factual, although the actual wording is clearly cognitive.

I believe that my symmetry relation for events in probability structures is such a factual interpretation. In order to illustrate these ideas I shall give an analysis of Bertrand's paradox (Bertrand 1889). It is more convenient to introduce first the notion of random variable. Let $\mathbf{H}$ be the set of outcomes of a probability structure (simple or compound). Then a function $X: \mathbf{H} \to \mathbf{R}$ ($\mathbf{R}$ the set of real numbers) is a random variable if and only if $X^{-1*}A \in \mathcal{B}$ (the algebra of events) for every Borel set $A \subseteq \mathbf{R}$.

The experiment for Bertrand's paradox is the choosing of a chord at random in a circle and determining the probability distribution of the lengths of the chords. If the experiment is not further specified, we would have a simple probability structure $(\mathbf{K}, \langle A \rangle)$ where A is the set of chords and $\mathbf{K}$ contains all models of the form $\langle A, S \rangle$ where S consists of one element of A (the chord selected). We define the random variable $X: \mathbf{K} \to \mathbf{R}$ by $X(\langle A, S \rangle)$ = the length of the chord in S. The σ-algebra of events $\mathcal{B}$ should be $\{X^{-1*}A\text{: } A \text{ Borel, } A \subseteq \mathbf{R}\}$. $G_{\mathbf{K}}$ is the set of all permutations of A that transform $\mathcal{B}$ into $\mathcal{B}$. In a sense, they are a counterpart of Borelian functions on $\mathbf{R}$. It is not difficult to prove that there is no measure invariant under $G_{\mathbf{K}}$.

So we should specify the experiment further. In fact, the origin of Bertrand's paradox arises from the fact that we can specify this experiment in several different ways which yield different distributions. Each of these ways can be put into the framework of my probability structures. I shall indicate how this can be done for two of these specifications.

(a) Choose two points on the circle and draw the chord between them. We have a compound probability structure with a causal structure consisting of the subtrees of the tree $\langle \{t_0, t_1\}, = \rangle$, i.e. $T = \{t_0, t_1\}$ consists of two independet elements. Let $\mathcal{K} = \langle \mathbf{K}, \mathfrak{A} \rangle$ where $\mathbf{K}$ consists of all structures $\mathfrak{L}_0 = \langle C, r, O \rangle$ with C the points in the circle, r rotations, and O the selected point; $\mathfrak{A} = \langle C \rangle$. The experiment in question is modeled by $\mathbf{K}$ with the random variable $X: {}^{T}\mathbf{K} \to \mathbf{R}$ where $X(f)$ = the distance between the point selected at $f(t_0)$ and that selected at $f(t_1)$; $\mathcal{B} = \{X^{-1*}(A)\text{: } A \text{ a Borel subset of } \mathbf{R}\}$. The group $G_{\mathbf{K}}$ corresponds to the rotations of the circle. Thus, there is an invariant measure.

(b) Select first a point on the circle and then a point on the radius through the first point. The chord chosen is the perpendicular to the radius through the second point. We are again in front of a compound probability structure consisting of the subtrees of $\langle T, \leq_T \rangle$ where $T = \{t_0, t_1\}$ has two elements and $t_0 \leq_T t_1$. At t_0 we have the same $\mathcal{K}$ as in (a). For each $\mathfrak{L}_0 \in \mathbf{K}$ we have $\mathcal{K} = \langle \mathbf{K}_{\mathfrak{L}_0}, \mathfrak{C} \rangle$ where $\mathbf{K}_{\mathfrak{L}_0}$ is the set of structures of the form $\langle D, t, I \rangle$, D contains the points on the radius passing through the point O, t the translations of the line modulus the radius, and I the point selected on the radius; $\mathfrak{C} = \langle D \rangle$. In order to complete the model we need a random variable Y: $\mathbf{H} \to \mathbf{R}$, where $\mathbf{H} = \{f$: $D_0 f = \{t_0,t_1\}$, $f(t_0) \in \mathbf{K}$ and $f(t_0) \in \mathbf{K}_{f(t_0)}\}$ is the set of outcomes, and $Y(f)$ is the length of the chord perpendicular to the radius selected at $f(t_0)$ and through the point selected at $f(t_1)$. The algebra of events is $\{Y^{-1*}A$: A a Borel subset of $\mathbf{R}\}$.

The group of $\mathbf{K}\mathfrak{L}_0$ is the group of translations; so again there is an invariant measure. By the usual analysis (see, e.g. Parzen 1960) we can obtain the distributions that are expected.

6. CLASSICAL AND BAYESIAN STATISTICAL INFERENCE REVISITED.

The purpose of this section is to improve the analysis of classical and Bayesian statistical inference given in Sections 3 and 4 of Chuaqui 1980, in the light of the modifications introduced so far.

First, I shall analyze classical inference. Let us assume that we have an experiment which can be repeated and we propose a simple probability model $\mathcal{J} = \langle\langle \mathbf{K},\mathfrak{A}\rangle,\mathcal{B},\mu\rangle$ for it. This is now a factual model, so we can assume it as a scientific hypothesis. We then repeat the experiment n times for a large n and obtain a sequence of results. The probability of events consisting of sequences of results is computed by building the compound probability structure $^{\omega}\mathbf{K}$ and proceeding as in Chuaqui 1980. Recall that here we have a compound probability structure, with causal structure $\mathbf{F}$ composed of the subtrees of $\langle\omega, =\rangle$, i.e. all moments in ω are independent. Events are subsets of $^{\omega}\mathbf{K}$, the set of outcomes. If a compound event $\mathbf{A}$ occurs which has a low probability according to $^{\omega}\mathbf{K}$ and high according to another structure $^{\omega}\mathbf{K}'$, then we reject the original hypothesis that $\mathcal{J} = \langle\langle \mathbf{K},\mathfrak{A}\rangle,\mathcal{B},\mu\rangle$ is the adequate simple probability model.

This account is the same as that of Chuaqui 1980, and can be completed as there. What I would like to make precise is the type of simple probability models $\mathcal{J}$ that can be taken as hypothetical models. Suppose, first, that the experiment is that of tossing a coin. In this case, we have a complete physical explanation of the phenomenon and the model $\mathcal{J}$ can be built accordingly. However, there are many cases where the only known facts are frequencies observed in sequences of repeated experiments. Thus, the only natural $\mathcal{J}$ is one that just mimics the choosing of elements of a set. We have to assume that there is an unknown physical explanation for this way of choosing.

For instance, suppose that we observe that the relative frequency of the event is about 1/3. Then we should assume a $\mathbf{K}$ with models of the form $\mathfrak{A}_0 = \langle A,E,O\rangle$ where A contains three elements, E (the event considered) contains one fixed element of A, and O (the chosen element) contains one element of A, different for each model. The intrisic part is $\langle A,E\rangle$ and the structural part $\langle A,O\rangle$.

If the relative frequency tends to $\sqrt{2}/2$, for instance, then our structures in $\mathbf{K}$ should be like that of a symmetrical roulette $<C,r,f,E,0>$, with E a fixed interval of length $\sqrt{2}/2$ and 0 the chosen element.

However, these types of models without any physical explanation are not completely satisfactory, because we do not know the mechanism for choosing 0.

Thus, in a sense, the assumptions are not as well substantiated in the purely statistical case as when we have physical models. A case in point is the observed relation between cigarette smoking and cancer. The evidence is almost purely statistical, since there are no generally accepted physiological models for this phenomenon. I believe this is one of the reasons for the difficulties in accepting this connection as proven. The statistical evidence had to be overwhelming for the general public to accept that cigarrete smoking increases the chances of getting cancer.

Something similar is true for Bayesian inference. Here we have the causal tree $<\{t_0,t_1\}\leqslant_T>$ with $t_0 \leqslant_T t_1$, a simple probability model $\mathcal{J}_0 = <<\mathbf{K},\mathcal{Ol}_0>, B_0,\mu_0>$, and for each $\mathfrak{L} \in \mathbf{K}_0$, another simple probability model $\mathcal{J}_{\mathfrak{L}}$. We may assume as hypothesis any $\mathcal{J}_0$. The trouble here is that it is often the case that we have no evidence for $\mathcal{J}_0$ ($\mathcal{J}_0$ determines what are usually called "a priori" probabilities). Thus, these models might be less justified than the classical ones. Also, we might have evidence for a simple probability structure that admits no invariant measure. In this case, Bayesian methods cannot be used. Only if we have good evidence for a $\mathcal{J}_0$ that admits a probability measure, the method is perfectly adequate.

Given such probability models as the $\mathcal{J}_0$'s for the prior probabilities or the $\mathcal{J}$'s based only on frequencies, it is one of the aims of science to replace them by probability models based on physical laws.

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Facultad de Matemáticas
Universidad Católica de Chile
Santiago, Chile.

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STUDIES IN PARACONSISTENT LOGIC II: QUANTIFIERS AND THE UNITY OF OPPOSITES

Newton C.A. da Costa and Robert G. Wolf

ABSTRACT. In this paper, the propositional logics introduced in a previous work (N.C.A. da Costa and R.G. Wolf, *Studies in paraconsistent logic I: the dialectical principle of the unity of opposites*, Philosophia 9(1980),pp.189-217) are extended to first-order predicate calculi. Our aim is to formalize certain aspects of dialectics, as they are interpreted by McGill and Parry (V.J. McGill and W.T. Parry, *The unity of opposites: a dialectical principle*, Science and Society 12(1948),pp.418-444).

In da Costa and Wolf 1980, we constructed a sentential calculus DL whose purpose was to formalize the dialectical principle of the unity of opposites, as that principle has been interpreted by McGill and Parry. As we insisted, such a sentencial logic is only a first step toward richer, more philosophically useful logics. Here we plan to extend DL (note, not the second system DL^* also formulated in da Costa and Wolf 1980) to a first-order predicate logic DL^Q and show that motivations have not been sacrificed in the move to DL^Q. We shall also indicate how DL^Q can be extended, in a similar way as DL was previously, to DL^{Q*}.

We shall assume that our previous paper is available and will not repeat the motivating remarks we gave there nor some of the more easily adapted technical results. This paper will therefore be more straighforwardly technical, but such technicalities are, we feel, vital to the enterprise. Before moving on to such technical aspects, we would like however to remark that this (and the previous) paper is meant also to show the value of paraconsistent logics -those logics intended to formalize non-trivial inconsistent theories- in treating philosophical problems. Paraconsistent logics are as yet too little known or appreciated within the logical community. Hopefully, successful application of such logics will help change that situation.

1. THE FORMALIZATION OF DL^Q AND SOME METATHEOREMS.

DL^Q has the following primitive symbols. 1- The connectives: $\supset, \wedge, \vee$, and ^{o}. 2- The quantifiers: $\forall$ (for all) and $\exists$ (there exists). 3- Individual variables: an infinitely denumerable set of individual variables which we do not need to specify. 4- Three disjoint sets of individual constants, $\underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{C}$ such that $\underset{\sim}{A} \cup \underset{\sim}{B} \cup \underset{\sim}{C} = \underset{\sim}{D} \neq \emptyset$. 5- Three disjoint non-void sets, $\underset{\sim}{A}'$, $\underset{\sim}{B}'$ and $\underset{\sim}{C}'$, containing constant predicate symbols of any rank n, $0 < n < \omega$. 6- For every n, $0 < n < \omega$, an infinite denumerable set of predicate variables of rank n. 7- Parentheses. The individual variables and constants are called *terms*.

The common syntactical notions, for example those of formula, proof, the symbol $\vdash$, and deduction, are introduced as usual. The letters A, B and C, with or without subscripts, will be employed as metalinguistic variables for formulas; x, y and z, with or without subscripts, will denote individual variables; a, b and c are syntactical variables for individual constants; t will denote any term. The symbol of equivalence, $\equiv$, is introduced in the usual way. The metalinguistic abbreviations of implication and of equivalence are respectively $\Rightarrow$ and $\Leftrightarrow$.

DL^Q is an extension of DL, so we shall assume the axiom schemata given for DL in da Costa and Wolf 1980 (note that we are using schemata). To get DL^Q, we add the following schemata and rules, which are subjected to the standard restrictions:

A18. $C \supset A(x) / C \supset \forall x A(x)$

A19. $\forall x A(x) \supset A(t)$

A20. $A(t) \supset \exists x A(x)$

A21. $A(x) \supset C / \exists x A(x) \supset C$

A22. $\forall x (A(x))^{o} \supset (\forall x A(x))^{o}$

A23. $\forall x (A(x))^{o} \supset (\exists x A(x))^{o}$

A24. If A and B are congruent formulas in the sense of Kleene 1952, p.153, or one is obtained from the other by suppression of vacuous quantifiers, then $A \equiv B$ is an axiom.

As before A22 and A23 insure that the stability operador ^{o} makes well-behaved formulas obey the laws of classical logic.

Theorem 1 of da Costa and Wolf 1980 generalizes to this new context.

THEOREM 1. *All schemata and rules of classical positive predicate logic are valid in* DL^Q.

Proof. Consequence of the postulates of DL^Q.

In the next theorem, some notations of Kleene 1952 are employed:

THEOREM 2. *If* A, B, C_A *and* C_B *are formulas satisfying* ~~*the conditions*~~ *of theorem 14 of Kleene 1952, pp. 151-152, we have:* 1- *If the occurence of* A *in* C_A *is not within the scope of an occurrence of* $\neg$ *or of* ^{o}*, then:* $A \equiv B \vdash^{x_1,\ldots,x_n} C_A \equiv C_B$; 2- *If the prime componentes of* A, B, C *are* $A_1, A_2, \ldots A_k$, *then:* $A_1^o, A_2^o, \ldots, A_k^o, A \equiv B \vdash^{x_1,\ldots,x_n} C_A \equiv C_B$.

Proof. As in Kleene 1952: the postulate of DL^Q are selected partly so that this theorem would hold.

Theorem 3 of da Costa and Wolf 1980 also generalizes to this new context:

THEOREM 3. *Let* $\Gamma \cup \{A\}$ *be a set of formulas of* DL^Q*, in which* ^{o} *does not occur, and whose prime components are* $A_1, A_2, \ldots, A_n$. *Then* $\Gamma \vdash A$ *in the classical predicate calculus iff* $\Gamma, A_1^o, A_2^o, \ldots, A_n^o \vdash A$ *in* DL^Q.

It seems evident that theorem 3 can be generalized to cope with the case in which the formulas of $\Gamma \cup \{A\}$ belong to DL^Q, without any restrictions on the formulas.

THEOREM 4. DL^Q *is undecidable.*

Proof. Consequence of theorem 3 and of Church's result that the classical predicate calculus is undecidable.

We can in an obvious way introduce strong negation ~ into DL^Q just as was done with DL. Then the corollary to theorem 7 of da Costa and Wolf 1980 also generalizes to this new situation:

THEOREM 5. *In* DL^Q*, the symbols* $\supset, \wedge, \vee, \sim, \forall$ *and* $\exists$ *satisfy all the postulates of the classical predicate calculus. In particular, the following are provable:*

(i) $\vdash (A \supset B) \supset ((A \supset \sim B) \supset \sim A)$

(ii) $\vdash A \vee \sim A$

(iii) $\vdash \sim A \supset (A \supset B)$

(iv) $\vdash \sim\sim A \equiv A$

(v) $\vdash \forall x A(x) \equiv \sim\exists x \sim A(x)$

(vi) $\vdash \exists x A(x) \equiv \sim\forall x \sim A(x)$

(vii) $\vdash \forall x \forall y A \equiv \forall y \forall x A$

(viii) $\vdash \forall x \forall y A(x,y) \supset \forall z A(z,z)$

(ix) $\vdash \forall z A(z,z) \equiv \exists x \exists y A(x,y)$

(x) $\vdash \sim\forall x \forall y \forall z A \supset \exists x \exists y \exists z \sim A$.

Proof. Left to the reader. We note that the reasons for having the classical predicate calculus interpretable as a subsystem of DL^Q are the same as having the classical setential calculus interpretable as a subsystem of DL.

The concept of a *k-transform* of a formula (cf. Kleene 1952, p.178) is easily extended to the classical predicate calculus with individual constants and also to DL^Q. Definition of such a notion is useful to prove a motivationally crucial conservative extension result.

THEOREM 6. *If* $\Gamma \vdash A$ *in* DL^Q, *then any* k-*transform of* A *can be deduced in* DL *from the* k-*transforms of the formulas of* Γ.

Proof. Similar to the classical one, but taking into account the fact that the prime propositional components of the formulas of DL will not be propositional symbols (variables or constants), but predicate symbols of rank n, followed by n occurrences of the symbols (numerals) 1,2,3,...,k ($0 < n < \omega$).

The next theorem is crucial.

THEOREM 7. DL^Q *is a conservative extension of* DL, i.e. *schemata not valid in* DL *are not valid in* DL^Q *either*.

Proof. Apply theorem 6.

The import of theorem 7 is that adding quantifiers to DL does not disturb the intuitions underlying DL. If DL^Q did not conservatively extend DL, then either DL^Q would verify a formula scheme, in the vocabulary of DL, which on the intuitions that we are assuming is a false theorem; or DL would have been poorly formulated as it left out a theorem which it could have contained, since it uses only the connectives of DL, and which is on the same dialectical intuitions a true theorem. In one case DL^Q would be branded false, since it would lead from true assumptions to false conclusions; in the other, DL would be at best inadequate, precisely in the area where we have claimed adequacy.

The way in which we have added the quantifiers is not the only possible way; indeed it might be valuable to try other options. The value of the approach taken here is that it makes DL^Q as close as possible to the classical predicate logic (as indicated, the classical predicate calculus is close to DL^Q in another way; it can be interpreted as a subsystem of DL^Q). For our purposes, this is good for two reasons: 1 - it facilitates proving metatheorems and obtaining technical information about DL^Q; and 2 - it isolates the intuitions which separate dialectical logic from classical logic from the intuitions underlying other issues in the philosophy of logic. We do not need here to fight intuitionistic, modal or relevant battles, though we may opt to do so elsewhere.

To pinpoint some of the significance of theorem 7, we note that the following schemata are *not* valid in DL^Q:

(xi) $A \wedge \neg A. \supset B$

(xii) $A \wedge \neg A. \supset \neg B$

(xiii) $\neg A \supset (A \supset B)$

(xiv) $A \supset (\neg A \supset B)$

(xv) $\neg A \supset (A \supset \neg B)$

(xvi) $A \supset (\neg A \supset \neg B)$

(xvii) $(A \supset B) \supset (\neg B \supset \neg A)$

(xviii) $\neg\neg A \equiv A$

(xix) $\neg (A \wedge \neg A)$

(xx) $A \wedge \neg A$

However DL^Q also deviates from classical predicate logic on some properly quantificational theorems.

THEOREM 8. *In* DL^Q, *the following schemata are* not *valid:*

(xxi) $\neg \exists x \neg A(x) \equiv \forall x A(x)$

(xxii) $\neg \forall x \neg A(x) \equiv \exists x A(x)$.

Proof. Consider the k-transforms of the above schemata and apply theorem 6.

It is important to realize that we *don't* want either (xxi) or (xxii) to be valid. If we consider cases where neither $A(x)$ or $\neg A(x)$ is applicable, then (xxi) and (xxii) should fail.

We now move on to the semantics for DL^Q, which, as we shall see, is a generalization of that for DL. For similar semantics for predicate calculi, see Arruda and da Costa 1977 and Alves and Moura 1978.

2. A SEMANTICS FOR DL^Q.

A sentence is a formula without free individual variables. In what follows, Γ and A will always denote respectively a set of sentences and a sentence.

DEFINITION 1. Let D be a nonvoid set. An interpretation of DL^Q in D is a function i which associates to each individual constant of DL^Q an element of D. The diagram language of DL^Q relative to D is denoted by DL^QD. (See Schoenfield 1967.) A valuation of DL^Q in D, having i as its base, is a function v of the set of sentences DL^QD on $\{0,1\}$, such that:

(1) v satisfies the conditions of a valuation of DL;

(2) $v(\forall x A(x)) = 1 \Leftrightarrow$ For every individual constant c of DL^QD, $v(A(c)) = 1$;

(3) $v(\exists xA(x)) = 1 \leftrightarrow$ For some individual constant c of DL^QD, $v(A(c)) = 1$;
(4) $v(\forall x(A(x))^o) = 1 \Rightarrow v((\forall xA(x))^o) = v((\exists xA(x))^o) = 1$;
(5) If A and B are sentences satisfying the conditions of postulate A24, then $v(A) = v(B)$;
(6) For any individual constants of DL^QD, a and b, if $i(a) = i(b)$, then $v(A(a)) = v(A(b))$.

The valuation v *satisfies a sentence* A of DL^QD (and of DL) if $v(A) = 1$.

DEFINITION 2. Suppose that $\Gamma \cup \{A\}$ is a set of sentences of DL^Q and that v is any valuation; v is said to be a *model of* Γ if, for every element B of Γ, $v(B) = 1$. A is called a *semantic consequence of* Γ, if every model v of Γ is such that $v(A) = 1$. In this case, we write $\Gamma \vDash A$. If $\Gamma = \emptyset$, we write $\vDash A$, and A is said to be *valid.*

As in da Costa and Wolf 1980, these definitions are meant to be as unstartling as possible. By paralleling previously studied cases, especially the classical one, our proofs also parallel, and can be presented very quickly.

We now move to prove DL^Q sound and complete relative to the above semantics. As typical, the soundness half is immediate.

THEOREM 9. *In* DL^Q: $\Gamma \vdash A \Rightarrow \Gamma \vDash A$.

Proof. By induction on the length of a deduction of A from Γ.

We now turn to the completeness half, which is very close to the classical case, given the following definitions.

DEFINITION 3. Γ is said to be *trivial* if for every sentence A, $\Gamma \vdash A$; otherwise Γ is called *nontrivial.* Γ is said to be *inconsistent* if there is a sentence A such that $\Gamma \vdash A$ and $\Gamma \vdash \neg A$; otherwise Γ is *consistent.* Γ is said to be *$\neg$-incomplete* if there is a sentence A such that $\Gamma \nvdash A$ and $\Gamma \nvdash \neg A$; otherwise Γ is called *$\neg$-complete.* Γ is *maximally nontrivial* if it is nontrivial and is not properly contained in any nontrivial set.

DEFINITION 4. Γ is called a *Henkin set* if, for every formula A(x) having x as its sole free individual variable, there exists an individual constant c (of DL^Q) such that $\Gamma \vdash \exists xA(x) \supset A(c)$.

We now prove some crucial preparatory theorems.

THEOREM 10. *If Γ is a nontrivial (Henkin) set, it is contained in a max-*

imal nontrivial (Henkin) set.

Proof. Similar to the classical proof.

THEOREM 11. *Let* Γ *denote a nontrivial Henkin set; then, one has:*

(1) Γ *has all the properties of a maximal nontrivial set of* DL*;*

(2) $\forall x A(x) \in \Gamma \Leftrightarrow$ *For every individual constant* c (of DL^Q), $A(c) \in \Gamma$;

(3) $\exists x A(x) \in \Gamma \Leftrightarrow$ *There is an individual constant* c (of DL^Q) *such that* $A(c) \in \Gamma$;

(4) *If* A *and* B *are sentences as in postulate* A24*, then* $A \equiv B \in \Gamma$.

Proof. The classical proof is immediately adaptable to the present situation.

THEOREM 12. *Every (consistent or inconsistent;* ¬*-complete or* ¬*-incomplete) nontrivial Henkin set has a model.*

Proof. Analogous to the proof of the corresponding theorem for DL.

COROLLARY. *Any nontrivial set of sentences of* DL^Q *has a model.*

We have now proven everything necessary for completeness.

THEOREM 13. *In* DL^Q: $\Gamma \models A \Rightarrow \Gamma \vdash A$.

Proof. Again, analogous to the classical case.

COROLLARY. $\Gamma \vdash A \Leftrightarrow \Gamma \models A$.

Proof. See theorems 9 and 13.

As the model theory here is so close to the classical case, some results of the usual model theory can be extended to DL^Q. For example, though we shall not prove it, a Löwenheim-Skolem theorem for any denumerable Γ in DL^Q is: if Γ has a model, then Γ has a infinite denumerable model. Such results are of course of more than technical interest, as they indicate that we have enough control over DL^Q to use it in (later) applications, without worrying overmuch as to DL^Q behaving in a pathological manner, fouling up attempted proofs of interesting results. It is worth while to observe that our semantics is a generalization of classical semantics; in particular, Tarski's scheme T remains valid.

We prove one last, motivationally important theorem:

THEOREM 14. DL^Q *is consistent and nontrivial.*

Proof. Consequence of theorem 6 and of the fact that DL is consistent and nontrivial.

3. THE UNITY OF OPPOSITES, AND CONCLUSION.

We can (as with DL*) formulate the dialectical principle of the unity of opposities explicitly using the resources of DL^Q (and also use some of the symbols introduced way back at the beginning of the paper).

We can formulate two important forms of the principle of the unity of opposities (see da Costa and Wolf 1980) as follows:

First form (#5 of McGill and Parry 1948): If $a \in \underset{\sim}{A}$ and P is a unary constant predicate symbol of A', Then:

A25'. $\neg(P(a) \wedge \neg P(a))$.

Second form (#6 of McGill and Parry 1948): Suppose that $b \in \underset{\sim}{B}$ and that Q is a unary constant predicate symbol belonging to $\underset{\sim}{B}'$; under these conditions, we have:

A26'. $Q(b) \wedge \neg Q(b)$.

Finally in order to insure the existence of well-behaved formulas, we assume:

A27'. $\forall x_1 x_2 \ldots \forall x_n (R(x_1, x_2, \ldots, x_n))^o$,

where R is a constant predicate symbol of rank n, $0 < n < \omega$, belonging to $\underset{\sim}{C}$.

A25' and A26' can obviously be generalized in various ways. DL^{Q*} (DL + A25' + A26' + A27') is both inconsistent and nontrivial and presumably can be interpreted as a logic of vagueness without any real difficulty.

We conclude that DL^Q (and DL^{Q*}) is a well-motivated and technically well-behaved logic which answers to (some of) the intuitions behind a dialectical logic. Its development and investigation promises to be of interest to philosephers interested in dialectical philosophy. Extension of DL^Q to get a tensed dialectical predicate logic seems to offer no real difficulty.

On to tenses!

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University of São Paulo
Institute of Mathematics
São Paulo, SP, Brazil.
and
Department of Philosophical Studies
Southern Illinois University
Edwardsville, Illinois, U.S.A.

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SORTS OF HUGE CARDINALS

C.A. Di Prisco and J. Henle

In this note we consider some large cardinal properties related to huge cardinals. We establish some connections between these notions and the concepts of multihuge cardinals and superhuge cardinals introduced in [B.DP.T].

A cardinal κ is *huge* if there is an elementary embedding $j:V\to M$ of the universe V into a transitive model containing all the ordinals, such that κ is the critical point of j and M is closed under sequences of size $j(\kappa)$. If κ is huge and j is an embedding as described above, we say that $\lambda = j(\kappa)$ is a *target* for κ and denote this by $\kappa \to (\lambda)$. We use the notation $\kappa \underset{j}{\to} (\lambda)$ to make explicit which is the embedding under consideration. A cardinal κ is α *times huge* if there are cardinals $\lambda_0 < \lambda_1 < \ldots < \lambda_\xi < \ldots (\xi < \alpha)$ such that for each $\xi < \alpha$, $\kappa \to (\lambda_\xi)$. A cardinal κ is *superhuge* if it is α times huge for every ordinal α.

In the paper cited above it is shown that if the existence of a 2-huge cardinal is consistent (see for example [S.R.K]), then so is the existence of a superhuge cardinal (moreover, it is consistent that a stationarily superhuge cardinal exists).

The large cardinal properties considered in the present work are in between 2-huge and superhuge (*or stationarily superhuge*) in consistency strength. We also prove that, in this sense, stationarily superhugeness is strictly stronger than mere superhugeness.

1. A TARGET LIMIT OF TARGETS.

The proof of the consistency of a superhuge cardinal from the consistency of 2-huge cardinal [B.DP.T] indicates that it is enough to have a multihuge cardinal with a target which is a limit of targets to obtain the consistency of superhugeness, in fact:

THEOREM 1.1. *Let κ and λ be cardinals such that $\kappa \to (\lambda)$ and $\{\alpha < \lambda \mid \kappa \to (\alpha)\}$ is unbounded in λ. Then $V_\lambda \models$ " κ is superhuge and the limit of superhuge car-*

dinals", and $V_\kappa \models$ " *there are unboundedly many superhuge cardinals".*

Before proceeding to prove this theorem we need a lemma. We will denote by $\kappa \to (<\lambda)$ the fact that the targets of κ are unbounded below λ, $\kappa \to (\leq\lambda)$ means that $\kappa \to (<\lambda)$ and $\kappa \to (\lambda)$.

LEMMA 1.2. *If* $\kappa \to (<\lambda)$ *and* $\lambda \to (\gamma)$ *then* $\kappa \to (<\gamma)$.

Proof. The proof of the lemma is just routine; suppose $\kappa \underset{j}{\to} (<\lambda)$ and $\lambda \underset{k}{\to} (\gamma)$. Then, if $k: V \to M$, by elementarity M satisfies "*there are unboundedly many targets for* κ *below* γ", and by the closure properties of M this is true in V.

Proof of Theorem 1.1. Let λ be as in the statement and let $j: V \to M$ be such that $j(\kappa) = \lambda$ and ${}^\lambda M \subset M$. $\alpha < \lambda$ is a target for κ, i.e. $\kappa \to (\alpha)$, if and only if there is a normal ultrafilter on $[\alpha]^\kappa$. The inaccessibility of λ guarantees that this normal ultrafilter belongs to M, thus $M \models$ " κ *has unboundedly many targets below* λ". From here it follows that the set $A = \{\alpha < \kappa \mid \alpha$ has unboundedly many targets below $\kappa\}$ is in the ultrafilter on κ generated by j; therefore, $V_\kappa \models$ "*There are unboundedly many superhuge cardinals*". By the lemma 1.2, the set of targets of each element of A is unbounded below λ, and thus $V_\lambda \models$ "κ *is superhuge and limit of superhuge cardinals".*

COROLLARY 1.3. Con("*There is a cardinal* κ *with a target which is limit of targets for* κ") *implies* Con("*There are unboundedly many superhuge cardinals*") *and* Con(" *There is a superhuge cardinal limit of superhuge cardinals*").

In fact we have:

COROLLARY 1.4. *If* κ *is superhuge and has a target which is limit of targets of* κ *then there is a normal ultrafilter on* κ *concentrating on superhuge cardinals.*

We have thus seen that having a huge cardinal with a target limit of targets is consistency-wise stronger than superhugeness. Nevertheless, the local character of the former property implies that the first cardinal with this property is not above the first superhuge (suppose κ_1, the first huge with a target limit of targets, is above κ, the first superhuge, then it is enough to consider an elementary embedding associated to $\kappa \to (\lambda)$ for a λ bigger than the target limit of targets of κ_1, and apply the usual kind of argument. Corollary 1.4. above indicates that $\kappa_1 \neq \kappa$, so we have that if both cardinals exist, $\kappa_1 < \kappa$).

Moreover, the existence of a huge cardinal with a target limit of targets does not imply the existence of a superhuge cardinal. Indeed, if κ_1 is the first such cardinal and λ is its first target limit of targets, then let γ be the first inaccessible cardinal above λ, then $V_\gamma \models$ " κ *has a target limit of targets and there is no superhuge cardinal*". Nevertheless, the existence of a cardinal κ with a target limit of targets, does imply multihugeness.

PROPOSITION 1.5. *κ has a target limit of targets strongly implies that κ is many times huge. More precisely, there is a normal ultrafilter on κ concentrating on cardinals with a huge collection of targets.*

Proof. Let $\kappa \to (\leqslant\lambda)$ and let $j:V \to M$ be the embedding associated to $\kappa \to (\lambda)$. Then $M \models$ "$\kappa \to (<\lambda)$". Thus $\{\alpha < \kappa | \alpha \to (<\kappa)\}$ belongs to the normal ultrafilter induced by j on κ.

2. A STATIONARY COLLECTION OF TARGETS.

In this section, we show that the concept of stationarily superhugeness is strictly stronger than that of superhugeness.

PROPOSITION 2.1. *Let κ be a cardinal with a set of targets stationary below a regular cardinal λ. Then κ has a target which is a limit of targets. The converse is not true.*

Proof. Let $A \subseteq \lambda$ be the set of targets of κ below λ. And let $B = (A)$ = the set of limits of elements of A. The set B is closed and unbounded (below λ) so there is $\gamma \in A \cap B$. The cardinal γ is a target limit of targets of κ.

The converse is not true: just take V_γ where γ is the first strongly inaccessible above the first target limit of targets. In this model there is a target limit of targets but no stationary set of targets. Moreover, the existence of a cardinal with a target limit of targets does not imply the consistency of the existence of a cardinal with a stationary set of targets, because the preceding argument shows that Con("$\exists\kappa$ *with a stationary set of targets*") implies Con("$\exists\kappa$ *with a target limit of targets*"). So if a cardinal with a target limit of targets implies the consistency of a cardinal with a stationary set of targets we would have that the theory ZFC + "$\exists\kappa$ *with a target limit of targets*" implies its own consistency.

COROLLARY 2.2. *If κ is stationarily superhuge then there is a normal ultrafilter on κ concentrating on superhuge cardinals; moreover, there is a normal ultrafilter on κ concentrating on superhuge cardinals with a target limit of targets.*

Proof. The proof of Proposition 2.1 shows that κ has a stationary class of targets which are limits of targets. From this and corollary 1.4 we obtain that that there is a normal ultrafilter on κ concentrating on superhuge cardinals.

For the second part, suppose $\lambda_1 < \lambda_2$ are targets of κ which are limits of targets and let $j: V \to M$ be the elementary embedding associated with $\kappa \to (\lambda_2)$. We observe that $M \models$ "$\kappa \to (\lambda_1)$, λ_1 *is a limit of targets of* κ, *and* κ *has unboundedly many targets below* λ_2". i.e. $M \models$ "κ *has a target limit of targets and* $\kappa \to (<\lambda_2)$". Therefore the set $\{\alpha < \kappa \mid \alpha$ has a target limit of targets and $\alpha \to (<\kappa)\}$ belongs to the normal ultrafilter induced on κ by j. But by lemma 1.2, all these cardinals α are superhuge cardinals.

3. MANIFOLD HUGE CARDINALS.

DEFINITION. A sequence $\{\kappa_\alpha\}_{\alpha<\gamma}$ of cardinals is a γ-*fold sequence* if $\kappa_\alpha \to (\kappa_{\alpha+1})$ for all $\alpha, \alpha+1 < \gamma$ and $\kappa_\alpha \to (\kappa_\lambda)$ for all $\alpha < \lambda < \gamma$, λ a limit ordinal.

For cardinals $\kappa < \gamma$, we say that κ is α-*fold huge* (resp. $<\alpha$-*fold huge*) and γ is its α-*fold target* ($<\alpha$-*fold target*) if there is an $\alpha+1$-fold (α-fold) sequence $\{\kappa_\xi\}_{\xi<\alpha+1}$ ($\{\kappa_\xi\}_{\xi<\alpha}$) with $\kappa_0 = \kappa$ and $\kappa_\alpha = \gamma$ ($\bigcup_{\xi<\alpha} \kappa_\xi = \gamma$). We denote this by $\kappa \overset{\alpha}{\to} (\gamma)$ ($\kappa \overset{<\alpha}{\to} (\gamma)$).

The various gradations of κ-fold hugeness form a hierarchy between 2-huge cardinals and those discussed above. The following chart summarizes this ordering. The symbol " $- \Rightarrow \ldots$" indicates that Con(ZFC+ $-$) implies Con(ZFC+...) but not the reverse. The numbers refer to the proofs that follow.

κ *is 2-huge* $\overset{1}{\Rightarrow} \exists\gamma(\kappa \overset{\gamma}{\to} (\gamma)) \overset{2}{\Rightarrow} \exists\gamma(\kappa \overset{<\gamma}{\to} (\gamma))$

$\overset{3}{\Rightarrow}$ κ *is* α-*fold huge for all* α

$\overset{4}{\Leftrightarrow}$ κ *is super* α-*fold huge for all* α

$\overset{10}{\Rightarrow}$ κ *is super* κ-*fold huge* $\overset{6}{\Rightarrow}$ κ *is* κ-*fold huge*

$\overset{5}{\Rightarrow}$ κ *is super* $<\kappa$-*fold huge* $\overset{6}{\Rightarrow}$ κ *is* $<\kappa$-*fold huge*

$\overset{5}{\Rightarrow}$ κ *is super* α-*fold huge for all* $\alpha < \kappa$

$\overset{6}{\Rightarrow}$ κ *is* α-*fold huge for all* $\alpha < \kappa$ $\overset{7}{\Rightarrow}$ ...

$\overset{7}{\Rightarrow}$ κ *is super* β-*fold huge* $\overset{6}{\Rightarrow}$ κ *is* β-*fold huge*... (for $\beta < \alpha$)

and for λ a limit ($\lambda < \beta$):

$\overset{7}{\Rightarrow}$ κ *is super* λ-*fold huge* $\overset{6}{\Rightarrow}$ κ *is* λ-*fold huge*

$\overset{5}{\Rightarrow}$ κ *is super* $<\lambda$-*fold huge* $\overset{6}{\Rightarrow}$ κ *is* $<\lambda$-*fold huge*

$\overset{5}{\Rightarrow}$ κ *is super* α-*fold huge for all* $\alpha < \lambda$

$\overset{6}{\Longrightarrow}$ κ *is α-fold huge for all* $\alpha < \lambda \Rightarrow \ldots$

$\overset{7}{\Longrightarrow}$ κ *is 2-fold huge* $\overset{8}{\Longrightarrow}$ κ *is stationarily superhuge*

$\overset{9}{\Longrightarrow}$ κ *has a target the limit of targets*

$\overset{9}{\Longrightarrow}$ κ *is superhuge, the limit of superhuge cardinals*

$\overset{6}{\Longrightarrow}$ *there are unboundedly many superhuge cardinals*

$\overset{6}{\Longrightarrow}$ κ *is superhuge.*

In general, the prefix "*super*" indicates unboundedly many targets of the relevant sort, e.g. "κ *is super <κ-fold huge*" means that for unboundedly many γ, $\kappa \overset{<\kappa}{\rightarrow} (\gamma)$. First a simple lemma:

LEMMA 3.1. *If* $\{\kappa_\alpha\}_{\alpha<\gamma}$ *is a γ-fold sequence, then* $\kappa_\alpha \rightarrow (\kappa_\beta)$ *for all* $\alpha < \beta < \gamma$

Proof. It is true by definition for $\beta = \lambda$ a limit ordinal and, by induction on n, it is true for $\lambda+n$ since $\kappa \rightarrow (\delta)$ and $\delta \rightarrow (\eta)$ implies $\kappa \rightarrow (\eta)$ (see [B.DP.T], theorem 2C).

Our proofs will now proceed in each case by showing that the first property implies the consistency of the second.

1. Let κ be 2-huge and $j:V \rightarrow M$ witness this fact. $M \models$ "$\kappa \rightarrow (j(\kappa))$", so if μ is the normal ultrafilter induced by j on κ, $X = \{\alpha < \kappa | \alpha \rightarrow (\kappa)\} \in \mu$. Define $G:[\kappa]^2 \rightarrow 2$ by $G(\{\alpha,\beta\}) = 0$ if and only if $\alpha \rightarrow (\beta)$. Let $Y \subseteq X$ be homogeneous for G with $Y \in \mu$. For $\alpha \in X$, since $\alpha \rightarrow (\kappa)$ is true in M, $\alpha \rightarrow (\beta)$ for a set of β's in the ultrafilter μ, so there is a $\beta \in Y$ such that $\alpha \rightarrow (\beta)$. Thus, $G''[Y]^2 = \{0\}$. So every $\alpha \in Y$ has the property $\alpha \overset{\kappa}{\rightarrow} (\kappa)$.

2. We use Lemma 3.1. Let $\kappa \overset{\gamma}{\rightarrow} (\gamma)$ and $j:V \rightarrow M$ be an embedding associated with the fact that $\kappa \rightarrow (\gamma)$. As $M \models$ "$\kappa \overset{<\gamma}{\rightarrow} (\gamma)$", there are unboundedly many $\alpha < \kappa$ such that $\alpha \overset{<\kappa}{\rightarrow} (\kappa)$.

3. Let $\kappa \overset{<\gamma}{\rightarrow} (\gamma)$ and $\{\kappa_\alpha\}_{\alpha<\gamma}$ be a γ-fold sequence with $\kappa_0 = \kappa$ and $\bigcup_{\alpha<\gamma} \kappa_\alpha = \gamma$. For $\alpha < \gamma$, let $\kappa_\alpha \underset{j_\alpha}{\rightarrow} (\kappa_{\alpha+1})$. Then $j_\alpha V \models$ "κ_α *is an α-fold target of* κ", so there are (in V) unboundedly many $\delta < \kappa_\alpha$ which are α-fold targets of κ. If $\kappa_1 \underset{k_\alpha}{\rightarrow} (\kappa_\alpha)$ (such a k_α exists by lemma 3.1), then $k_\alpha V \models$ "*there are unboundedly many α-fold targets of κ below* κ_α", and then (in V) there are unboundedly many $\delta < \kappa_1$ which are α-fold targets of κ. As this holds for all $\alpha < \gamma$, we have that $\forall\alpha < \kappa_1$ $\exists\lambda < \kappa_1(\kappa \overset{\alpha}{\rightarrow} (\lambda))$. So, by the closure properties of j_0V, this statement holds in j_0V. Hence the set $\{\beta < \kappa | \forall\alpha < \kappa \ \exists\lambda < \kappa(\beta \overset{\alpha}{\rightarrow} (\lambda))\}$ is in the ultrafilter induced by j_0 on κ. Therefore, V_κ is the required model of ZFC+"$\exists\beta$, β *is α-fold huge for all* α". (Notice that we only used $\kappa \rightarrow (\kappa_1)$ and $\kappa_1 \overset{\kappa_1}{\rightarrow} (\gamma)$).

4. Obviously the second property implies the first. Suppose that κ is α-fold huge for every α. Given ξ and η we want to show that there is an η-fold target of κ

above ξ. Indeed, if we take $\gamma > \max(\xi^+, \eta^+)$, as κ is γ-fold huge, the first γ-fold target of κ must be above ξ. Moreover, there is a $\gamma+1$-fold sequence for κ above ξ. But then, by Lemma 3.1, there is an $\eta+1$-fold sequence for κ above ξ.

5. Proofs of this sort follow a similar pattern. We prove, for example that κ is κ-fold huge $\Rightarrow$ κ is super $<\kappa$-fold huge. Let $\{\kappa_\alpha\}_{\alpha<\kappa+1}$ be a $\kappa+1$-fold sequence with $\kappa_0 = \kappa$. Claim: $V_{\kappa_1} \models$ "κ *is super* $<\kappa$*-fold huge*". If not, let $\beta < \kappa_1$ be the number of $<\kappa$-fold targets of κ below κ_1. Let $\kappa_1 \underset{j}{\to} (\kappa_\kappa)$; then $jV \models$ "κ *has at least* $\beta+1$ $<\kappa$*-fold targets below* κ_κ" hence (V satisfies) κ has at least $\beta+1$ $<\kappa$-fold targets below κ_1, a contradiction.

6. Routine. For example, if κ is super $<\kappa$-fold huge, let γ be an inaccessible above a $<\kappa$-fold target of κ. Then V_γ is our model.

7. In general, κ is β-fold huge $\Rightarrow$ κ is super α-fold huge for $\alpha < \beta$, by the method of 5 above.

8. We prove that κ is 2-fold huge strongly implies that κ has a stationary set of targets below a regular cardinal. By hypothesis there is a sequence $\{\kappa_0, \kappa_1, \kappa_2\}$ with $\kappa_0 = \kappa$, $\kappa \underset{j_1}{\to} (\kappa_1)$ and $\kappa_1 \underset{j_2}{\to} (\kappa_2)$. We have that $j_2V \models \kappa \to (\kappa_1)$, and therefore the set $S = \{\xi < \kappa_1 | \kappa \to (\xi)\}$ is a stationary subset of κ_1. Moreover, for each $\xi \in S$, $j_1V \models$ "$\kappa \to (\xi)$", and thus $j_1V \models$ "κ *has a stationary set of targets below* κ_1" (since all subsets of κ_1 belong to j_1V and if $j_1V \models$ "$A \subseteq \kappa_1$ *is closed and unbounded*", then A is really closed and unbounded below κ_1). From here we conclude that $\{\alpha < \kappa | \alpha$ has a stationary set of targets below $\kappa\}$ is in the ultrafilter induced by j_1 on κ.

9. Proved previously.

10. Trivial.

This ordering can be expanded further in several ways. First, for all properties not involving unboundedness, an additional property can be placed above by requiring superhugeness as well, e.g., between "κ *is super* $<\kappa$*-fold huge*" and "κ *is* $<\kappa$*-fold huge*" can be placed: "κ *is* $<\kappa$*-fold huge and superhuge*" (If $\{\kappa_\alpha\}_{\alpha<\kappa}$ is a $<\kappa$-fold sequence, let $\gamma, \lambda > \kappa$ be such that $\kappa \underset{j}{\to} (\gamma)$, $\gamma \to (\lambda)$, and $V_\gamma \models$ "κ *is* $<\kappa$*-fold huge*". Use the technique of 8, to show $jV \models$ "κ *has a stationary set of targets below* γ". Conclude that $\{\alpha < \kappa | V_\kappa \models$ "α *is superhuge and* α *is* $<\alpha$*-fold huge*"$\}$ is in the ultrafilter induced by j on κ). Second, for most properties, the existence of unboundedly many, or stationarily many cardinals are two ways of generating statements of greater consistency power. Similarly, one can consider the analogue of stationarily superhuge, for example, a cardinal κ with a stationary set of κ-fold targets.

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Instituto Venezolano de Investigaciones Científicas.
Caracas, Venezuela.

Smith College
Northampton, Massachusetts, U.S.A.

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THE COMPLETENESS AND COMPACTNESS OF A THREE-VALUED FIRST-ORDER LOGIC

Itala M.L. D'Ottaviano

ABSTRACT. The strong completeness and the compactness of a three-valued first order predicate calculus with two distinguished truth-values are obtained. The system was introduced in *Sur un problème de Jaśkowski*, I.M.L. D'Ottaviano and N.C. A. da Costa, C.R. Acad.Sc. Paris 270A (1970),pp.1349-1353, and has several applications, especially in paraconsistent logics.

1. INTRODUCTION.

A theory T is said to be *inconsistent* if it has as theorems a formula and its negation; and it is said to be *trivial* if every formula of its language is a theorem.

A logic is *paraconsistent* if it can be used as the underlying logic for inconsistent but nontrivial theories.

Jaśkowski, motivated by some ideas of Łukasiewicz, was the first logician to construct a system of paraconsistent propositional logic (see [11], [12] and [13]). His principal motivations were the following: the problem of the systematization of theories which contain contradictions, as it occurs in dialectics; the study of theories in which there are contradictions caused by vagueness; the direct study of some empirical theories whose postulates or basic assumptions could be considered, under certain aspects, as contradictory ones (see [2] and [3]).

Jaśkowski proposed the problem of constructing a propositional calculus having the following properties:

i) an inconsistent system based on such a calculus should not be neccessarily trivial;

ii) the calculus should be sufficiently rich as to make posible most of the usual reasonings;

iii) the calculus should have an intuitive meaning.

Jaśkowski himself introduced a propositional calculus which he named "Discussive logic" and which was a solution to the problem. However he did recognize it was not the only solution (or even the best); in [11] he states:

> "Obviously, these conditions do not univocally determine the solution, since they may be satisfied in varying degrees, the satisfaction of condition (iii) being rather difficult to appraise objectively".

In a previous paper (see [10]), we presented a propositional system, denoted by $\mathbf{J}_3$, which is another solution to Jaśkowski's problem. A characteristic of $\mathbf{J}_3$ is that it is a three-valued system with two distinguished truth-values. Furthermore, it reflects some aspects of certain types of modal logics.

In the same paper, we extended $\mathbf{J}_3$ to the first-order predicate calculus with equality $\mathbf{J}_3^*$=.

Some of these results about $\mathbf{J}_3$ were improved by J. Kotas and N.C.A. da Costa (see [15]).

Our aim here is to develop further the calculus $\mathbf{J}_3$.

In Sec. 2 we axiomatize $\mathbf{J}_3$ and establish relations between this calculus and several known logical systems like, for example, intuitionism. We especially emphasize the close analogy between $\mathbf{J}_3$ and Łukasiewicz' three-valued propositional calculus $\mathcal{L}_3$.

Our solution to Jaśkowski's problem is discussed in the latter part of Sec 2.

In Sec. 3 we introduce the $\mathbf{L}_3$-Languages, among whose predicate symbols may appear in addition to identity other equalities. We axiomatize $\mathbf{J}_3$-theories, which are three-valued extensions of $\mathbf{J}_3^*$=, and we introduce a semantics for them.

In Sec. 4, after obtaining some theorems about first-order $\mathbf{J}_3$-theories, we define a strong equivalence which is compatible with the fact that the matrices defining $\mathbf{J}_3$ have more than one distinguished truth-value. This relation allows us to prove the Equivalence Theorems for $\mathbf{J}_3$-theories and the Reduction Theorem for non-Trivialization.

Finally, in Sec. 5, after giving a suitable definition of canonical structure, we present a Henkin-type proof for the Completeness Theorem and the Compactness Theorem.

In this paper, definitions, theorems and proofs, when analogous to the corresponding classical ones, will be omitted.

The Model-theory we developed for $\mathbf{J}_3$ allows us to obtain $\mathbf{J}_3$-versions of the following classical results: Model Extension Theorem, Łoś-Tarski Theorem, Chang-Łoś Susko Theorem, Tarski Cardinality Theorem, Löwenheim-Skolem Theorem, Quantifier Elimination Theorem and many of the usual theorems on categoricity.

Some of the above results about $\mathbf{J}_3$ were also extended to $\mathbf{J}_n$-theories, $3 \leqslant n \leqslant \aleph_0$.

The mentioned results about J_n-theories and Model-theory will appear elsewhere.

2. THE CALCULUS J_3.

The *propositional calculus* J_3 is given by the matrix $M = \langle\{0,\frac{1}{2},1\}, \{\frac{1}{2},1\}, \vee, \nabla, \neg\rangle$, where $\vee$, ∇ and $\neg$ are defined as follows:

$A \vee B$

A \ B	0	½	1
0	0	½	1
½	½	½	1
1	1	1	1

A	∇A
0	0
½	1
1	1

A	¬A
0	1
½	½
1	0

The *set of truth-values* and the *set of distinguished truth-values* are denoted by V and V_d respectively.

The *formulas* of J_3 are constructed as usually from the propositional variables, by means of $\vee$, ∇ and $\neg$, and parentheses. To write the formulas, schemas, etc. we use the conventions and notations of [14], with evident adaptations.

The concept of a *truth-function* is the usual one. The truth-functions defined by the tables above are denoted by $H_\vee$, H_∇, and $H_\neg$.

A *truth-valuation* ν for J_3 and the *truth-value* $\nu(A)$ for a formula A are defined in the standard way; and we observe that A is *valid* in M if, for every evaluation ν, $\nu(A)$ belongs to V_d (see, for example, [22]).

The following *abbreviations* will be used:

$$A \,\&\, B =_{def} \neg(\neg A \vee \neg B)$$
$$\Delta A =_{def} \neg \nabla \neg A$$
$$\neg^* A =_{def} \neg \nabla A$$
$$A \rightarrowtail B =_{def} \nabla \neg A \vee B$$
$$A \twoheadrightarrow B =_{def} (A \rightarrowtail B) \,\&\, (\neg B \rightarrowtail \neg A)$$
$$A \supset B =_{def} \neg \nabla A \vee B$$
$$A \equiv B =_{def} (A \supset B) \,\&\, (B \supset A)$$

$\neg$ is called *weak negation* or simply negation, $\neg^*$ is called *strong negation*, and $\supset$ *basic implication* of J_3.

We present the tables of some of the non-primitive connectives:

A	¬*A
0	1
½	0
1	0

A	ΔA
0	0
½	0
1	1

$A \twoheadrightarrow B$

A \ B	0	½	1
0	1	1	1
½	½	1	1
1	0	½	1

$A \supset B$

$A \backslash B$	0	½	1
0	1	1	1
½	0	½	1
1	0	½	1

$A \equiv B$

$A \backslash B$	0	½	1
0	1	0	0
½	0	½	½
1	0	½	1

In the following theorems, we mention only those results which are useful to the proofs of later theorems.

THEOREM 2.1. *The following schemas of* $\mathbf{J}_3$ *are valid in* M:

$\neg\neg A \equiv A$
$\neg^* A \supset \neg A$
$A \vee \neg A$
$\neg(A \& \neg A)$
$A \& (B \vee \neg B) \equiv A$
$\neg(A \vee B) \supset \neg A \& \neg B$
$A \vee B \equiv \neg(\neg A \& \neg B)$
$\neg(A \& B) \equiv \neg A \vee \neg B$
$\nabla A \equiv \neg\Delta\neg A$
$(A \supset \neg A) \supset \neg A$
$(\neg A \supset A) \supset A$
$\neg(\nabla A \vee \neg\nabla A) \supset B$
$((A \supset B) \supset A) \supset A$
$(A \supset B) \supset (A \rightarrowtail B)$
$(A \rightarrowtail B) \supset (\neg B \rightarrowtail \neg A)$.

$\nabla A \equiv A$
$\nabla A \equiv \nabla\nabla A$
$\neg A \vee \nabla A$
$A \& \neg A \equiv \neg A \& \nabla A$
$A \vee \nabla A \equiv \nabla A$
$\neg\nabla A \supset (\nabla A \supset B)$
$A \supset (\neg\nabla A \supset B)$
$\nabla(A \& B) \equiv \nabla A \& \nabla B$
$\nabla(A \vee B) \equiv \nabla A \vee \nabla B$
$A \rightarrowtail (B \rightarrowtail A)$
$(\neg A \rightarrowtail \neg B) \rightarrowtail (B \rightarrowtail A)$
$(A \rightarrowtail B) \rightarrowtail ((B \rightarrowtail C) \rightarrowtail (A \rightarrowtail C))$
$(A \rightarrowtail \neg B) \rightarrowtail A) \rightarrowtail A$
$\Delta(A \rightarrowtail B) \rightarrowtail \Delta(\Delta A \rightarrowtail \Delta B)$

THEOREM 2.2. *The following schemas are not valid in* $\mathbf{J}_3$:

$\neg A \supset (A \supset B)$
$A \supset (\neg A \supset B)$
$\neg A \supset (A \supset \neg B)$
$A \supset (\neg A \supset \neg B)$
$A \& \neg A \supset B$
$A \& \neg A \supset \neg B$
$(A \equiv \neg A) \supset B$
$(A \equiv \neg A) \supset \neg B$
$(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$.

$(A \supset B) \supset (\neg B \supset \neg A)$
$(\neg A \supset \neg B) \supset (B \supset A)$
$(A \supset \neg B) \supset (B \supset \neg A)$
$(\neg A \supset B) \supset (\neg B \supset A)$
$(A \equiv B) \supset (\neg A \equiv \neg B)$
$A \vee (B \& \neg B) \equiv A$
$A \supset B \equiv \neg(A \& \neg B)$
$A \supset B \equiv \neg A \vee B$

It can be verified that, instead of $\vee$, ∇ and $\neg$ it is possible to use only $\neg$ and $\rightarrowtail$ as primitive connectives of $\mathbf{J}_3$, considering $A \vee B$ and ∇A as abbreviations respectively of $(A \rightarrowtail B) \rightarrowtail B$ and $\neg A \rightarrowtail A$.

So, there is a close analogy between $\mathbf{J}_3$ and Łukasiewicz' three-valued propositional calculus $\mathcal{L}_3$, defined by the matrix $M' = \langle\{0,\frac{1}{2},1\}, \{1\}, \neg, \rightarrowtail\rangle$, in which the Łukasiewicz-Tarski operators $\neg$ and $\rightarrowtail$ are given by the respective tables of $\mathbf{J}_3$ (see [4]).

$\mathbf{J}_3$ can be axiomatized by:

Axiom 1 : $\Delta(A \rightarrowtail (B \rightarrowtail A))$

Axiom 2 : $\Delta((A \rightarrowtail B) \rightarrowtail ((B \rightarrowtail C) \rightarrowtail (A \rightarrowtail C)))$

Axiom 3 : $\Delta((\neg A \rightarrowtail \neg B) \rightarrowtail (B \rightarrowtail A))$

Axiom 4 : $\Delta(((A \rightarrowtail \neg A) \rightarrowtail A) \rightarrowtail A)$

Axiom 5 : $\Delta(\Delta(A \rightarrowtail B) \rightarrowtail \Delta(\Delta A \rightarrowtail \Delta B)$

*Rule R*1 : $\dfrac{A, \Delta(A \rightarrowtail B)}{B}$

*Rule R*2 : $\dfrac{\nabla A}{A}$

The completeness theorem for $\mathbf{J}_3$ is proved from the completeness of $\mathcal{L}_3$, due to Wajsberg (see [4] and [23]), using the following theorem.

THEOREM 2.3. *If* A *is a theorem of* $\mathcal{L}_3$, *then* ΔA *is a theorem of* $\mathbf{J}_3$.

Proof. As the axioms 1 to 4 are the axioms of $\mathcal{L}_3$ preceeded by Δ, if A is an axiom of $\mathcal{L}_3$, then ΔA is a theorem of $\mathbf{J}_3$.

Let A be obtained from B and $B \rightarrowtail A$ by the rule $\frac{B, B \rightarrowtail A}{A}$ of $\mathcal{L}_3$. By induction hypothesis, ΔB and $\Delta(B \rightarrowtail A)$ are theorems of $\mathbf{J}_3$. By axiom 5 and R_1 we obtain $\Delta(\Delta B \rightarrowtail \Delta A)$. Applying R_1, we have that ΔA is a theorem of $\mathbf{J}_3$.

THEOREM 2.4. (Completeness theorem for $\mathbf{J}_3$). *A formula* A *is a theorem of* $\mathbf{J}_3$ *if and only if* A *is valid in* M.

Proof. A straightforward induction shows that if A is a theorem of $\mathbf{J}_3$, then A is valid in M. On the other hand, if A is valid in M, then $\nu(\nabla A) = 1$ for every truth-valuation ν. By the axiomatization and completeness of $\mathcal{L}_3$, both ∇A and $\Delta(\nabla A \rightarrowtail \nabla A)$ are theorems of $\mathcal{L}_3$. By the above theorem and R_1, ∇A is a theorem of $\mathbf{J}_3$. By R_2, A is a theorem of $\mathbf{J}_3$.

COROLLARY (Modus Ponens Rule). *If both* A *and* $A \supset B$ *are theorems of* $\mathbf{J}_3$, *then* B *is a theorem of* $\mathbf{J}_3$.

However, contrary to $\mathcal{L}_3$, the Rule of Modus Ponens is not valid with respect to $\rightarrowtail$.

For some of the theorems that follow it will be convenient to assume that the language of $\mathbf{J}_3$ contains, as primitive symbols, all the connectives introduced so far. In particular we shall often identify $\mathbf{J}_3$ with the set of M-valid formulas in the expanded language.

The following theorems will be used in the proofs of many of the results about $\mathbf{J}_3$.

THEOREM 2.5. $\mathbf{J}_3$ *is a non-conservative extension of the classical positive propositional calculus with connectives* $\vee$, $\&$, $\supset$, $\equiv$.

THEOREM 2.6. $\mathbf{J}_3$ *is a conservative extension of the classical propositional calculus with connectives* $\neg^*$, $\vee$, $\&$, $\supset$ *and* $\equiv$

THEOREM 2.7. $\mathbf{J}_3$ *is a non-conservative extension of Łukaseewicz' three-valued logic* $\mathcal{L}_3$ *with connectives* $\neg, \rightarrowtail$.

THEOREM 2.8. $\mathbf{J}_3$ *is not functionally complete.*

Proof. It is not possible to define a connective, from the primitive conectives of $\mathbf{J}_3$, such that its truth-value is identically ½.

On the other hand, if we add the Słupecki T operador to the primitive connectives of $\mathbf{J}_3$, the calculus becomes functionally complete (see [21]).

By Theorem 2.4, the formulas $\neg A \supset (A \supset B)$, $A \supset (\neg A \supset B)$, $A \supset (\neg A \supset \neg B)$, $(A \& \neg A) \supset B$, $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$, $A \supset (B \& \neg B) \equiv A$, etc., are not theorems of $\mathbf{J}_3$. So, in $\mathbf{J}_3$, in general, it is not possible to deduce any formula whatsoever from a contradiction. Therefore, based on such a calculus we can construct nontrivial inconsistent deductive systems, in the sense of [11]. So, $\mathbf{J}_3$ satisfies condition (i) of Jaśkowski's problem.

By Theorem 2.5 to 2.8, $\mathbf{J}_3$ is quite a strong system, which evidently satisfies Jaśkowski's condition (ii).

$\mathbf{J}_3$ admits intuitive interpretations. For instance, it can be used as the underlying logic of a theory whose preliminary formulation may involve certain contradictions, which should be eliminated in a later reformulation. This can be done as follows; among the truth-values of $\mathbf{J}_3$, 0 can represent falsity, 1 truth, and ½ can represent the provisional value of a proposition A, so that both A and the negation of A are theorems of the theory, in its initial formulation; in a later reformulation, the truth-value ½ should be reduced, at least in principle, to 0 or to 1.

Therefore, $\mathbf{J}_3$ is a solution to Jaśkowski's problem.

$\mathbf{J}_3$ can also be used as a foundation for paraconsistent systems, in the sense of da Costa (see [5], [6], [7] and [8]). In this case, the value 0 represents falsity, 1 truth, and ½ represents the logic value of a formula that is simultaneously true and false.

Finally, as the calculus J_3 was constructed from $\mathcal{L}_3$, it is possible to obtain similar calculi J_n, from Łukasiewicz n-valued calculi $\mathcal{L}_n$, $3 \leqslant n < \aleph_0$.

3. SEMANTICS FOR FIRST-ORDER J_3-THEORIES.

The *symbols* of a first-order L_3-language are the individual variables, the function symbols, the predicate symbols, the primitive connectives $\neg$, $\vee$ and ∇, the quantifies $\exists$ and $\forall$, and the parentheses.

The *identity* = must be among the predicate symbols. Other equalities can be especified among the predicate symbols.

We use x,y,z and w as syntactical variables for individual variables; f and g, for function symbols; p and q, for predicate symbols, and c for constants.

The definitions of *term, atomic formula* and *formula* are the usual ones; a, b,c, etc. are syntactical variables for terms and A,B,C, etc. for formulas.

By an L_3-*language* we understand a first-order language whose logical symbols include the ones mentioned above.

The symbols &, $\rightarrowtail$, $\twoheadrightarrow$, $\supset$, $\equiv$, Δ and $\neg^*$ are defined in the L_3-languages, as in J_3.

Free occurrence of a variable, open formula, closed formula, variable-free term and *closure of a formula* are used as in [22].

The definition of a is *substitutible for* x *in* A is also the usual one.

We let $b_{x_1,\ldots,x_n}[a_1,\ldots,a_n]$ be the term obtained from b by replacing all occurrences of $x_1,\ldots,x_n$ by $a_1,\ldots,a_n$ respectively; and we let $A_{x_1,\ldots,x_n}[a_1,\ldots,a_n]$ be the formula obtained from A by replacing free occurrences of $x_1,\ldots,x_n$ by $a_1,\ldots,a_n$ respectively.

Whenever either of these is used, it will be implicitly assumed that $x_1,\ldots,x_n$ are distinct variables and that, in the case of $A_{x_1,\ldots,x_n}[a_1,\ldots,a_n]$, a_i is substitutible for x_i, $i = 1,\ldots,n$.

In the following definitions, let L be an L_3-language.

DEFINITION 3.1. A *structure* $\mathfrak{A}$ for a first-order L_3-language L consists of:

i) a nonempty set $|\mathfrak{A}|$, called universe of $\mathfrak{A}$;

ii) for each n-ary function symbol f of L, a function $f_{\mathfrak{A}}$ from $|\mathfrak{A}|^n$ to $|\mathfrak{A}|$;

iii) for each n-ary predicate symbol p of L, other than =, an n-ary predicate $p_{\mathfrak{A}}$, such that $p_{\mathfrak{A}}$ is a mapping from $|\mathfrak{A}|\times\ldots\times|\mathfrak{A}|$ to $\{0,\frac{1}{2},1\}$.

As in [22], we construct the language $L(\mathfrak{A})$; define $\mathfrak{A}(a)$ for each variable free term of $L(\mathfrak{A})$, and define $\mathfrak{A}$*-instance* of a formula A.

We use i and j as syntactical variable for the names of individuals of $\mathfrak{A}$.

DEFINITION 3.2. The *truth-value* $\mathfrak{A}(A)$ for each closed formula A in $L(\mathfrak{A})$ is given by:

i) if A is a = b, then $\mathfrak{A}(A) = 1$ iff $\mathfrak{A}(a) = \mathfrak{A}(b)$; otherwise, $\mathfrak{A}(A) = 0$;
ii) if A is $p(a_1,\ldots,a_n)$, where p is not =, then $\mathfrak{A}(A) = p_{\mathfrak{A}}^{\cdot}(\mathfrak{A}(a_1),\ldots\mathfrak{A}(a_n))$;
iii) if A is $\neg B$, then $\mathfrak{A}(A)$ is $H_{\neg}(\mathfrak{A}(B))$;
iv) if A is ∇B, then $\mathfrak{A}(A)$ is $H_{\nabla}(\mathfrak{A}(B))$;
v) if A is $B \vee C$, then $\mathfrak{A}(A)$ is $H_{\vee}(\mathfrak{A}(B), \mathfrak{A}(C))$;
vi) if A is a $\exists xB$, then $\mathfrak{A}(A) = \max\{\mathfrak{A}(B_x[i])/i \in L(\mathfrak{A})\}$;
vii) if A is a $\forall xB$, then $\mathfrak{A}(A) = \min\{\mathfrak{A}(B_x[i])/i \in L(\mathfrak{A})\}$.

DEFINITION 3.3. (1) A formula B of $L(\mathfrak{A})$ is *true* in $\mathfrak{A}$ (or $\mathfrak{A}$ is a model of B) iff $\mathfrak{A}(B) \in V_d$.

(2) A formula A of L is *valid* in $\mathfrak{A}$ iff for every $\mathfrak{A}$-instance A' of A, A' is true in $\mathfrak{A}$.

A *first-order predicate calculus* $\mathbf{J}_3^{*}=$ is the formal system whose language is an $\mathbf{L}_3$ plus the following, with the usual restrictions (see [14]):

Axiom 6 : $\forall x(x = x)$
Axiom 7 : $x = y \supset (A[x] \equiv A[y])$
Axiom 8 : $A_x[a] \supset \exists xA$
Axiom 9 : $\forall xA \supset A_x[a]$
Axiom 10: $\exists xA \equiv \neg\forall x\neg A$
Axiom 11: $\forall xA \equiv \neg\exists x\neg A$
Axiom 12: $\neg\exists xA \equiv \forall x\neg A$
Axiom 13: $\neg\forall xA \equiv \exists x\neg A$
Axiom 14: $\nabla\exists xA \equiv \exists x\nabla A$
Axiom 15: $\nabla\forall xA \equiv \forall x\nabla A$
Rule R3 : ($\exists$-*introduction rule*): $\dfrac{A \supset C}{\exists xA \supset C}$
Rule R4 : ($\forall$-*introduction rule*): $\dfrac{C \supset A}{C \supset \forall xA}$

THEOREM 3.1. $\mathbf{J}_3^{*}=$ *is a conservative extension of* $\mathbf{J}_3$.

Proof. We apply the Hilbert-Bernays theorem of k-transforms, that can be extended to this case.

THEOREM 3.2. $\mathbf{J}_3^{*}=$ *is an extension of the classical predicate calculus, with connectives* $\neg^{*}$, $\vee$, $\&$, $\supset$, $\equiv$, $\exists$ *and* $\forall$.

DEFINITION 3.4. A *first-order* J_3-*theory* is a formal system T such that:
i) the language of T, L(T), is an $\mathbf{L}_3$-language;

ii) the axioms of T are the axioms of $J_3^*=$, called the logical axioms of T, and certain further axioms, called the non-logical axioms;
iii) the rules of T are those of $J_3^*=$.

A is a *theorem of* T, in symbols: $\vdash_T A$, and B is a *semantical consequence* of a set Γ of formulas of L(T) are defined in the standard way. If B is a semantical consequence of Γ, then we shall also say that "B is valid in Γ".

THEOREM 3.3. (Validity Theorem): *Every theorem of a* J_3*-theory* T *is valid in* T.

4. SOME THEOREMS IN FIRST-ORDER J_3-THEORIES AND THE EQUIVALENCE THEOREM.

DEFINITION 4.1. A J_3-theory T is *finitely trivializable* if there exists a fixed formula F such that, for any formula A, $F \supset A$ is a theorem of T (see [2]).

THEOREM 4.1. *The* J_3*-theories are finitely trivializable.*

Proof. Any formula $\neg(\neg\nabla A \vee \nabla A)$ trivializes a J_3-theory.

The following results hold in any J_3-theory T:

Generalization Rule: If $\vdash_T A$, then $\vdash_T \forall xA$.
Substitution Rule: Is $\vdash_T A$ and A' is an instance of A, then $\vdash_T A'$.
Substitution Theorem: a) $\vdash_T A_{x_1,\ldots,x_n}[a_1,\ldots,a_n] \supset \exists x_1 \ldots x_n A$;
b) $\vdash_T \forall x_1 \ldots \forall x_n A \supset A_{x_1,\ldots,x_n}[a_1,\ldots,a_n]$
Distribution Rule: If $\vdash_T A \supset B$, then $\vdash_T \exists xA \supset \exists xB$ and $\vdash_T \forall xA \supset \forall xB$.
Closure Theorem: If A' is the closure of A, then $\vdash_T A$ if and only if $\vdash_T A'$.
Theorem on Constants: If T' is a J_3-theory obtained from T by adding new constants (but no new nonlogical axioms), then for every formula A of T and every sequence $e_1,\ldots,e_n$ of new constants, $\vdash_T A$ if and only if $\vdash_{T'} A_{x_1,\ldots,x_n}[e_1,\ldots,e_n]$.

In the case of classical logic, the equivalence ≡ behaves as a congruence relation with respect to the other logical symbols. Unfortunately this is not the case in J_3-theories, for it is possible to have $\vdash_T A \equiv B$ and $\nvdash_T \neg A \equiv \neg B$.
However we can introduce a stronger equivalence, $\equiv^*$, which is a $J_3^=$-congruence relation and thus allow us to prove a J_3-version of the equivalence theorem (see [22]).

DEFINITION 4.2. $A \equiv^* B =_{def} (A \equiv B) \& (\neg A \equiv \neg B)$.

THEOREM 4.2. *If* T *is a* $\mathbf{J}_3$*-theory and* $\vdash_T A \equiv^* B$, *then* $\vdash_T A$ *if and only if* $\vdash_T B$.

THEOREM 4.3. (Equivalence Theorem). *Let* T *be a* $\mathbf{J}_3$*-theory and let* A' *be obtained from* A *by replacing some occurrences of* $B_1, \ldots, B_n$ *by* $B_1', \ldots, B_n'$ *respectively. If* $\vdash_T B_1 \equiv^* B_1', \ldots, \vdash_T B_n \equiv^* B_n'$, *then* $\vdash_T A \equiv^* A'$.

Proof. After considering the special case when there is only one such occurrence and it is all of A, we use induction on the length of A.

For A *atomic*, the result is obvious.

A *is* $\neg C$ *and* A' *is* $\neg C'$, where C' results from C by replacements of the type described in the theorem. By induction hypothesis, $\vdash_T C \equiv^* C'$, that is, $\vdash_T C \equiv C'$ and $\vdash_T \neg C \equiv \neg C'$. As by Theorem 2.4, $\vdash_T C \equiv \neg\neg C$ and $\vdash_T C' \equiv \neg\neg C'$, we have $\neg\neg C \equiv \neg\neg C'$. So $\neg C \equiv^* \neg C'$.

A *is* ∇C *and* A' *is* $\nabla C'$, with $\vdash_T C \equiv^* C'$. From $\vdash_T C \equiv C'$, it follows that $\vdash_T \neg^* C \equiv \neg^* C'$, by Theorem 2.6. Also from $\vdash_T C \equiv C'$ it follows that $\vdash_T \nabla C \equiv \nabla C'$, since $\vdash_T \nabla C \equiv C$ by Theorem 2.4. Therefore, $\vdash_T \nabla C \equiv^* \nabla C'$.

A *is* $C \vee D$ *and* A' *is* $C' \vee D'$, with $\vdash_T C \equiv^* C$ and $\vdash_T D \equiv^* D'$. As by theorem 2.6,

$$\vdash_T ((C \equiv C') \& (D \equiv D')) \supset ((C \vee D) \equiv (C' \vee D'))$$

and

$$\vdash_T ((\neg C \equiv \neg C') \& (\neg D \equiv \neg D')) \supset ((\neg C \& \neg D) \equiv (\neg C' \& \neg D')$$

we have that $\vdash_T C \vee D \equiv C' \vee D'$ and $\vdash_T \neg(C \vee D) \equiv \neg(C' \vee D')$.

A *is* $\exists x C$ *and* A' *is* $\exists x C'$, with $C \equiv^* C'$. By Distribution Rule, $\vdash_T \exists x C \equiv \exists x C'$ and $\vdash_T \forall x \neg C \equiv \forall x \neg C'$. Using Axiom 12 we complete the proof.

If A *is* $\forall x C$ *and* A' *is* $\forall x C'$, with $\vdash_T C \equiv^* C'$, the proof is similar.

In the spirit of the equivalence theorem, we have the following corollaries and remark.

COROLLARY 1. *In a* $\mathbf{J}_3$*-theory* T, *it is possible to replace:*

i) $\neg\neg A$ by A;
ii) $\neg^*\neg^* A$ by $\neg\neg^* A$;
iii) $\neg(A \vee B)$ by $\neg A \& \neg B$;
iv) $\neg^*(A \vee B)$ by $\neg^* A \& \neg^* B$;
v) $\forall x A$ by $\neg\exists x \neg A$;
vi) $\neg\exists x A$ by $\forall x \neg A$;
vii) $\neg\forall x A$ by $\exists x \neg A$;
viii) $\nabla\exists x A$ by $\exists x \nabla A$;
ix) $\nabla\forall x A$ by $\forall x \nabla A$.

Proof. It is enough to verify that $\vdash_T \neg\neg A \equiv^* A$, $\vdash_T \neg^*\neg^* A \equiv^* \neg\neg^* A$, etc.

COROLLARY 2. *In a* $\mathbf{J}_3$*-theory* T, *if* $\vdash_T x = y$, *then, for every formula* A, A(x) *can be replaced by* A(y).

REMARK. Although $\vdash_T \neg^*\neg^* A \equiv A$, it is not possible, in general, to replace $\neg^*\neg^* A$ by A.

DEFINITION 4.3. A formula A' is a *variant of* A just in case A' has been obtained from A by renaming bound variables.

THEOREM 4.4. (Variant Theorem). *If* A' *is a variant of* A, *then* $\vdash_T A \equiv^* A'$.

Proof. In view of Theorem 4.3 and Corollary 1, it is enough to observe that $\vdash_T \exists x B \equiv^* \exists y B_x[y]$.

Let T[Γ] be the $\mathbf{J}_3$-theory whose non-logical axioms are those of T plus the formulas of the set Γ.

THEOREM 4.5. (Reduction Theorem). *Let* Γ *be a set of formulas in the* $\mathbf{J}_3$*-theory* T *and let* A *be a formula of* T. A *is a theorem of* T[Γ] *if, and only if, there is a theorem of* T *of the form* $B_1 \supset \ldots \supset B_n \supset A$, *where each* B_i *is the closure of a formula in* Γ.

Given a non-empty set Γ of formulas we let:

$\Gamma_{\vee\neg\forall\nabla}$ = {B | B is a disjunction of negations of closures of formulas of the type ∇A, with A ∈ Γ}

$\Gamma_{\vee\neg\nabla\forall}$ = {C | C is a disjunction of negations of formulas of the type ∇A', where A' is the closure of a formula of Γ}

THEOREM 4.6. (Reduction Theorem for non-trivialization). *Let* Γ *be a non-empty set of formulas in a* $\mathbf{J}_3$*-theory* T. *Then the extension* T[Γ] *is trivial, if and only if, there is a theorem of* T *which belongs to* $\Gamma_{\vee\neg\forall\nabla}$.

Proof. The corollary to the replacement theorem gives us that every formula of $\Gamma_{\vee\neg\forall\nabla}$ is strongly equivalent to a formula of $\Gamma_{\vee\neg\nabla\forall}$. The proof of the theorem can be completed using the properties of strong negation.

COROLLARY. *If* A' *is the closure of* A, *then the formula* A *is a theorem of* T *if, and only if,* $T[\neg^* A']$ *is trivial.*

5. THE COMPLETENESS AND THE COMPACTNESS THEOREMS FOR J_3-THEORIES

We study certain aspects of the J_3-theories and present a Henkin-type proof of the completeness theorem for this type of many-valued theories.

DEFINITION 5.1. If T is a J_3-theory containing a constant, and if a and b are variable-free terms of T, then:

i) $a \sim b =_{def} \vdash_T a = b$;
ii) $a^o = \{b | a \sim b\}$.

DEFINITION 5.2. A canonical structure for the J_3-theory T is the structure $\mathfrak{A}$:

i) whose universe $|\mathfrak{A}|$ is the set of all equivalence classes under $\sim$;
ii) $f_{\mathfrak{A}}(a_1^o,\ldots,a_n^o) = (f(a_1,\ldots,a_n))^o$;
iii) $p_{\mathfrak{A}}(a_1^o,\ldots,a_n^o)$ is in V_d iff $\vdash_T p(a_1,\ldots,a_n)$.

Observe that (iii) could have been replaced by

$$p_{\mathfrak{A}}(a_1^o,\ldots,a_n^o) = 0 \quad \text{iff} \quad \nvdash_T p(a_1,\ldots,a_n).$$

THEOREM 5.1. *If* $\mathfrak{A}$ *is a canonical structure for* T *and* $p(a_1,\ldots,a_n)$ *is a variable-free atomic formula in* L(T), *then:*

i) $\mathfrak{A}(p(a_1,\ldots,a_n)) = 0$ *iff* $\nvdash_T p(a_1,\ldots,a_n)$
ii) $\mathfrak{A}(p(a_1,\ldots,a_n)) = ½$ *iff* $\vdash_T p(a_1,\ldots,a_n)$ *and* $\vdash_T \neg p(a_1,\ldots,a_n)$;
iii) $\mathfrak{A}(p(a_1,\ldots,a_n)) = 1$ *iff* $\vdash_T p(a_1,\ldots,a_n)$ *and* $\nvdash_T \neg p(a_1,\ldots,a_n)$.

Proof. ii) If $\mathfrak{A}(p(a_1,\ldots,a_n)) = ½$ then $\mathfrak{A}(\neg p(a_1,\ldots,a_n)) = ½$. By the last definition, $\vdash_T p(a_1,\ldots,a_n)$ and $\vdash_T \neg p(a_1,\ldots,a_n)$.

On the other hand, if $\vdash_T p(a_1,\ldots,a_n)$ and $\vdash_T \neg p(a_1,\ldots,a_n)$, also by Definition 5.2, $\mathfrak{A}(p(a_1,\ldots,a_n)$ and $\mathfrak{A}(\neg p(a_1,\ldots,a_n))$ belong to V_d. Then, $\mathfrak{A}(p(a_1, \ldots,a_n)) = ½$.

iii) If $\mathfrak{A}(p(a_1,\ldots,a_n)) = 1$, then $\mathfrak{A}(\neg p(a_1,\ldots,a_n)) = 0$; then, $\vdash_T p(a_p,\ldots,a_n)$ and $\nvdash_T \neg p(a_1,\ldots,a_n)$.

On the other hand, if $\vdash_T p(a_1,\ldots,a_n)$ and $\nvdash_T \neg p(a_1,\ldots,a_n)$, we have that $\mathfrak{A}(p(a_1,\ldots,a_n))$ belongs to V_d and $\mathfrak{A}(\neg p(a_1,\ldots,a_n))$ does not belong to V_d; if $\mathfrak{A}(p(a_1,\ldots,a_n) = ½$ then $\mathfrak{A}(\neg p(a_1,\ldots,a_n)) = ½$ and, so, $\vdash_T \neg p(a_1,\ldots,a_n)$. Then, $\mathfrak{A}(p(a_1,\ldots,a_n)) = 1$.

Now, (i) is immediate.

As a consequence of the theorem we obtain that there is exactly one canonical structure for a J_3-theory. Furthermore, as in the calssical case, in order for a canonical structure to characterize the theorems of a theory, the theory must be in some sense maximal, for there may be a closed formula A such that $\nvdash_T A$, $\nvdash_T \neg A$ and $\nvdash_T \neg^* A$.

DEFINITION 5.3. A formula A of a J_3-theory T is *undecidable* in T if neither A nor $\neg^* A$ is a theorem of T. Otherwise, A is *decidable* in T.

DEFINITION 5.4. A J_3-theory T is *complete* if it is non-trivial and if every closed formula of T is decidable in T.

THEOREM 5.2. *A J_3-theory T is complete if, and only if, T maximal in the class of nontrivial theories.*

DEFINITION 5.5. A J_3-theory T is a *Henkin J_3-theory* if for every closed formula $\exists xA$ of T, there is a constant e such that $\exists xA \supset A_x[e]$ is a theorem of T.

THEOREM 5.3. *If T is a Henkin J_3-theory, then for every closed formula $\forall xA$ in T there is a constant e such that $A_x[e] \supset \forall xA$ is a theorem of T.*

Proof. As T is a Henkin J_3-theory, there is e, such that $\vdash_T \exists x \neg^* A \supset \neg^* A_x[e]$. We obtain the desired result, by successive applications of Theorem 2.6.

THEOREM 5.4. *If T is a complete Henkin J_3-theory and $\mathfrak{A}$ is the canonical structure for T, then for all closed formulas A of L[T]:*

i) $\mathfrak{A}(A) = 0$ *iff* $\nvdash_T A$

ii) $\mathfrak{A}(A) = \frac{1}{2}$ *iff* $\vdash_T A$ *and* $\vdash_T \neg A$

iii) $\mathfrak{A}(A) = 1$ *iff* $\vdash_T A$ *and* $\nvdash_T \neg A$.

Proof. By induction on the height of A. For *atomic* A, the result follows from Theorem 5.1.

Case: A *is* $\neg B$. **i)** If $\mathfrak{A}(A) = 0$, then $\mathfrak{A}(B) = 1$. Thus $\nvdash_T \neg B$, that is $\nvdash_T A$. On the other hand if $\nvdash_T A$, then since T is complete $\vdash_T \neg^* A$, and then $\vdash_T \neg A$, $\vdash_T \neg\neg B$, $\vdash_T B$. Thus we have that $\vdash_T B$ and $\nvdash_T \neg B$, from which if follows that $\mathfrak{A}(B) = 1$ and that $\mathfrak{A}(A) = 0$.

ii) If $\mathfrak{A}(A) = \frac{1}{2}$, then $\mathfrak{A}(B) = \frac{1}{2}$. Thus $\vdash_T B$ and $\vdash_T \neg B$, from which it follows that $\vdash_T \neg A$ and $\vdash_T A$, the converse is analogous.

iii) If $\mathfrak{A}(A) = 1$, then $\mathfrak{A}(B) = 0$ and thus $\nvdash_T B$. Since T is complete, $\vdash_T \neg^* B$ and thus $\vdash_T \neg B$. Since $\nvdash_T B$, we obtain that $\vdash_T \neg\neg B$, in other words, we have that $\vdash_T A$ and $\nvdash_T \neg A$.

Assume next that $\nvdash_T A$ and $\vdash_T A$, that is, $\nvdash_T \neg\neg B$ and $\vdash_T \neg B$. Then $\nvdash_T B$, and so by induction $\mathfrak{A}(B) = 0$, from which it follows that $\mathfrak{A}(A) = 1$.

Case: A *is* $B \vee C$. **i)** If $\mathfrak{A}(A) = 0$ then $\mathfrak{A}(B) = 0$ and $\mathfrak{A}(C) = 0$. Hence $\nvdash_T C$ and $\nvdash_T B$, from which it follows, since T is complete, that $\nvdash_T B \vee C$. The converse is analogous.

ii) If $\mathfrak{A}(A) = \frac{1}{2}$, then either: $\mathfrak{A}(B) = \frac{1}{2}$ and $\mathfrak{A}(C) = \frac{1}{2}$,
or $\mathfrak{A}(B) = \frac{1}{2}$ and $\mathfrak{A}(C) = 0$,
or $\mathfrak{A}(B) = 0$ and $\mathfrak{A}(C) = \frac{1}{2}$.

Let us only consider the situation when $\mathfrak{A}(B) = \frac{1}{2}$ and $\mathfrak{A}(C) = 0$ (the others are analogous). The induction hypothesis gives us that

$$\vdash_T B, \quad \vdash_T \neg B, \quad \nvdash_T C.$$

Since T is complete we obtain that $\vdash_T \neg^* C$ and $\vdash_T \neg C$. From $\vdash_T B$ we get $\vdash_T B \vee C$, and from $\vdash_T \neg B$ and $\vdash_T \neg C$ we may conclude that $\vdash_T \neg(B \vee C)$.

Conversely, suppose that $\vdash_T B \vee C$ and $\vdash_T \neg(B \vee C)$. The latter gives us that $\vdash_T \neg B$ and $\vdash_T \neg C$. From the former, since T is complete, we obtain that either $\vdash_T B$ or $\vdash_T C$. The induction hypothesis allows us then to conclude that $\mathfrak{A}(B \vee C) = \frac{1}{2}$.

iii) If $\mathfrak{A}(A) = 1$, then either:
$\mathfrak{A}(B) = 1$ and $\mathfrak{A}(C) = 0$,
or $\mathfrak{A}(B) = 1$ and $\mathfrak{A}(C) = \frac{1}{2}$,
or $\mathfrak{A}(B) = 1$ and $\mathfrak{A}(C) = 1$,
or $\mathfrak{A}(B) = 0$ and $\mathfrak{A}(C) = 1$,
or $\mathfrak{A}(B) = \frac{1}{2}$ and $\mathfrak{A}(C) = 1$.

We will only consider the case when $\mathfrak{A}(B) = 1$ and $\mathfrak{A}(C) = \frac{1}{2}$. The induction hypothesis gives us that

$$\vdash_T B, \quad \nvdash_T \neg B, \quad \vdash_T C, \quad \vdash_T \neg C.$$

From the first we obtain that $\vdash_T (B \vee C)$. Suppose on the other hand that $\vdash_T \neg(B \vee C)$. Then $\vdash_T (\neg B \wedge \neg C)$, from which it would follow that $\vdash_T \neg B$, contradicting that $\nvdash_T \neg B$. Thus $\nvdash_T \neg(B \vee C)$.

On the other hand, suppose that $\vdash_T (B \vee C)$ and $\nvdash_T \neg(B \vee C)$. Then from the completeness of T we obtain that either

$$\vdash_T B \quad \text{or} \quad \vdash_T C.$$

From $\nvdash_T \neg(B \vee C)$, we obtain that

$$\nvdash_T \neg B \text{ and } \nvdash_T \neg C.$$

The induction hypothesis then gives us that $\mathfrak{A}(B \vee C) = 1$.

Case: A *is* ∇B. **i)** If $\mathfrak{A}(\nabla B) = 0$. Then $\mathfrak{A}(B) = 0$. Thus $\nvdash_T B$; from which it follows that $\nvdash_T \nabla B$. Converse, analogous.

ii) $\mathfrak{A}(\nabla B)$ is never $\frac{1}{2}$.

iii) $\mathfrak{A}(\nabla B) = 1$ then either $\mathfrak{A}(B) = \frac{1}{2}$ or $\mathfrak{A}(B) = 1$.

Subcase: $\mathfrak{A}(B) = \frac{1}{2}$. Then $\vdash_T B$ and $\vdash_T \neg B$, from which we obtain $\vdash_T \nabla B$ and $\vdash_T \nabla \neg B$. Using that T is complete we conclude $\vdash_T \nabla B$, and $\vdash_T \neg \nabla B$.

Subcase: $\mathfrak{A}(B) = 1$. Then $\vdash_T B$ and $\nvdash_T \neg B$. Suppose that $\vdash_T \neg \nabla B$. Then since $\vdash_T B$, we should obtain that T is trivial, which we are assuming it is not. Thus

$\nvdash_T \neg\nabla B$ and $\vdash_T \nabla B$. On the other hand, suppose that $\vdash_T A$ and $\nvdash_T \neg A$. That is suppose that

$$\vdash_T \nabla B \quad\text{and}\quad \nvdash_T \neg\nabla B.$$

Then $\vdash_T B$, and either $\vdash_T \neg B$ or $\nvdash_T B$. In one case the induction hypothesis gives that $\mathfrak{A}(B) = \frac{1}{2}$, and in the other that $\mathfrak{A}(B) = 1$. Thus $\mathfrak{A}(\nabla B) = 1$ in both. That is $\mathfrak{A}(A) = 1$.

Case: A *is* $\exists xB$. **i)** If $\mathfrak{A}(A) = 0$, then for every variable-free term b, $\mathfrak{A}(B_x[b]) = 0$, and by induction hypothesis this is equivalent to $\nvdash_T B_x[b]$. As T is a Henkin theory this gives us that $\nvdash_T \exists xB$. The converse does not need to use that T is a Henkin theory.

ii) If $\mathfrak{A}(A) = \frac{1}{2}$. Then for all b we have that $\mathfrak{A}(B_x[b]) \leqslant \frac{1}{2}$. The induction hypothesis then tells us that

(1) for those b such that $\mathfrak{A}(B_x[b]) = \frac{1}{2}$ (and there is at least one such): $\vdash_T B_x[b]$ and $\vdash_T \neg B_x[b]$.

(2) for the remaining b's: $\nvdash_T B_x[b]$ and (because T is complete) $\vdash_T \neg B_x[b]$. Thus we have that for all constants b: $\vdash_T \neg B_x[b]$; from which it follows that $\vdash_T \forall x \neg B$, i.e. $\vdash_T \exists xB$. From (1) we obtain $\vdash_T \exists xB$.

Conversely, suppose that $\vdash_T A$ and $\vdash_T \neg A$; that is $\vdash_T \exists xB$ and $\vdash_T \neg\exists xB$. Using that T is a Henkin theory and induction, we obtain an e such that $\vdash_T B_x[e]$, $\vdash_T \neg B_x[e]$, and thus $\mathfrak{A}(B_x[e]) = \frac{1}{2}$. A proof by contradiction shows that there is no b such that $\mathfrak{A}(B_x[b]) = 1$. Hence $\mathfrak{A}(\exists xB) = \frac{1}{2}$.

iii) If $\mathfrak{A}(A) = 1$, then there is at least one b such that $\mathfrak{A}(B_x[b]) = 1$. From the induction hypothesis, we obtain that $\vdash_T B_x[b]$ and $\nvdash_T \neg B_x[b]$. From the former, we obtain that $\vdash_T \exists xB$. Suppose next contrary to what we want to show, that $\vdash_T \neg\exists xB$. Then $\vdash_T \exists x \neg B$ and thus $\vdash_T \neg B_x[b]$, a contradiction. Thus $\nvdash_T \neg\exists xB$.

COROLLARY 1. *Let* T *be a complete Henkin* $\mathbf{J}_3$*-theory,* $\mathfrak{A}$ *the canonical structure for* T *and* A *a closed formula of* T*; then,* $\mathfrak{A}(A)$ *belongs to* V_d *if and only if* A *is a theorem of* T.

COROLLARY 2. *If* T *is a complete Henkin* $\mathbf{J}_3$*-theory, then the canonical structure for* T *is a model of* T.

By the above corollary, to prove the completeness of a $\mathbf{J}_3$-theory T, as in the classical case, it is enough to show that it is possible to extend T to a complete Henkin $\mathbf{J}_3$-theory.

Thus, given a nontrivial $\mathbf{J}_3$-theory T, we will first extend it, conservatively, to a Henkin $\mathbf{J}_3$-theory T_c, and then extend it to a complete Henkin $\mathbf{J}_3$-theory T_c'.

Given a $\mathbf{J}_3$-theory T with language $\mathbf{L}$, we proceed as in [22] and define the special constants of level n, the language L_c with the special constants, and introduce the special axioms for the special constants.

DEFINITION 5.6. Let T be a J_3-theory with language L. Then T_c is the Henkin J_3-theory whose language is L_c and whose nonlogical axioms are the nonlogical axioms of T plus the special axioms for the special constants of L_c.

THEOREM 5.5. T_c *is a conservative extension of* T.

Proof. By Theorem 4.4 and by Theorem 5.3, the proof is similar to the classical one.

THEOREM 5.6. (Lindenbaum's Theorem). *If* T *is a nontrivial* J_3*-theory, then* T *admits a complete simple extension.*

Finally, we can obtain the completeness theorem for J_3-theories.

THEOREM 5.7. (Completeness Theorem). *A* J_3*-theory* T *is nontrivial if, and only if, it has a model.*

Proof. If $\mathfrak{A}$ is a model of T and A is a closed formula in T, then $\mathfrak{A}(A \& \neg^* A) = 0$. So, by the validity Theorem, $A \& \neg^* A$ is not a theorem in T. Then T is nontrivial.

If T is nontrivial, then we extend T to T_c, which is a non-trivial Henkin J_3-theory. By Lindenbaum's Theorem, we can extend T_c to a complete Henkin J_3-theory T_c'. By Corollary 2 to Theorem 5.4, T_c' has a model $\mathfrak{A}$. Therefore, $\mathfrak{A} \restriction L(T)$ is a model of T.

THEOREM 5.8. (Gödel's Completeness Theorem). *A formula* A *in the* J_3*-theory* T *is a theorem in* T *if, and only if, it is valid in* T.

Proof. By supposing that the closed formula A is a theorem in T and using the above Completeness Theorem, we shall show that there is no model of T in which A is not valid.

Therefore, suppose that the closed formula A is a theorem in T.

By the corollary to the Reduction Theorem for non-Trivialization, $\vdash_T$ A if and only if $T[\neg \nabla A]$ is trivial; which, by Theorem 5.7, is equivalent to $T[\neg \nabla A]$ not having a model.

On the other hand, a model of $T[\neg \nabla A]$ is a model $\mathfrak{A}$ of T in which $\neg \nabla A$ is valid, that is, a structure $\mathfrak{A}$ such $\mathfrak{A}(\neg \nabla A) = 1$. This is equivalent to $\mathfrak{A}(\nabla A) = 0$, and so $\mathfrak{A}(A) = 0$.

Therefore, $\vdash_T$ A if and only if A is valid in T.

COROLLARY 3. *If* T *and* T' *are* $\mathbf{J}_3$*-theories with the same language, then* T' *is an extension of* T *if, and only if, every model of* T' *is a model of* T.

THEOREM 5.9. (Compactness Theorem). *A formula* A *in a* $\mathbf{J}_3$*-theory is valid in* T *if, and only if, it is valid in some finitely axiomatized part of* T.

COROLLARY 4. *A* J_3*-theory* T *has a model if, and only if, every finitely axiomatized part of* T *has a model.*

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IMECC
Universidade Estadual de Campinas
Campinas, São Paulo, Brazil.

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A THEORY OF VARIABLE TYPES

Solomon Feferman[1]

1. INTRODUCTION.

Several publications in recent years have presented various formal theories T in which considerable portions of mathematical practice (particularly analysis) can be more or less directly formalized and which are proof-theoretically weak; cf. Feferman 1977, Takeuti 1978 and Friedman 1980. Indeed, on the classical side we have such T which are conservative over PA (Peano's Arithmetic)[2].

The paper Feferman 1977 will be taken as the point of reference here (but the reader need not be familiar with it to follow the present paper). It used *functional finite type theories* as the basic framework. One of the theories, denoted Res-$\hat{Z}^{(\omega)}+(\mu)$ is shown there to be conservative over PA, but stronger theories for more substantial portions of mathematics were also dealt with.

For the past several years I have been engaged (off and on) in working up the material of my 1977 paper into a book. One of the first improvements in carrying on that project was to obtain a theory VT of *variable types* which provides a much more natural framework for the direct formalization of mathematics. In this paper the system VT is further improved and presented in print for the first time. VT and its extension and restrictions to be considered are described formally in Sec. 2. Some conservation results are stated in Sec. 3 and their proofs are outlined. The concluding Sec. 4 outlines how one goes about formalizing substantial portions of classical and modern analysis in the VT systems.

2. VARIABLE TYPE SYSTEMS.

In ordinary functional finite type theories one begins by specifying the

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(2) There are comparable results for constructive theories T, e.g., such T in which Bishop's constructive analysis can be formalized and which are conservative over HA (Heyting's Arithmetic); cf. Friedman 1977, Feferman 1979 (esp. pp. 217 ff.), and Beeson 1980.

type symbols $\sigma,\tau,\ldots$. For Z^{ω} these are generated from 0 by closure under $\sigma,\tau \mapsto \sigma\times\tau$, $(\sigma \to \tau)$[(3)]. For each type (symbol) one then has variables $x^{\tau},y^{\tau},z^{\tau},\ldots$ of type τ. The intended interpretation is that these range over M_{τ} where $M_0 = \mathbb{N}$ = the set of natural numbers, $M_{\sigma\times\tau} = M_{\sigma}\times M_{\tau}$ and $M_{(\sigma\to\tau)}$ is a (the) set of (all) functions from M_{σ} to M_{τ}. One advantage of such a setting is that functional existence axioms are simply provided by the *typed λ-calculus*. However, there is no natural way of forming *sub-types* $\{x^{\sigma}|\phi(x^{\sigma},\ldots)\}$ in this framework and then iterating the operations of $\times$ and $\to$ applied to them, etc. Further, equations between individual terms $t_1 = t_2$ are permitted only between terms of the same type. If we were to regard members of $\{x^{\sigma}|\phi(x^{\sigma},\ldots)\}$ as belonging to a new type $\sigma\restriction\phi$, we could not say that an object of type $\sigma\restriction\phi$ is (equal to) an object of type σ. The VT systems to be described here have the following advantages: (i) the types are *variable* , so that statements of generality can be expressed directly, yet (ii) every individual term t is still syntactically of a unique type, and hence (iii) the typed λ-calculus may be extended to this language; but also (iv) equations between terms of arbitrary type are admitted, and (v) we can apply separation to form sub-types from given types.

The basic system to be described is denoted VT_0. To specify its language we generate simultaneously the following syntactic classes, together with the relation, t *is of type* T:

1. **individual terms** s,t,...
   a) With each type term T is associated an infinite list of individual variables $x^T,y^T,z^T,\ldots$ (of type T).
   b) If s is of type S and t of type T, then (s,t) is of type S×T.
   c) If u is of type S×T, then $p_1(u)$ is of type S and $p_2(u)$ is of type T.
   d) If s is of type S and t is of type (S→T), then ts is of type T.
   e) If t is of type T, then $\lambda x^S.t$ is of type S→T.

2. **type terms** S,T,...
   a) Each type variable X,Y,Z,... is a type term.
   b) If S,T are type terms and ϕ is a formula, then S×T, S → T, and $\{x^S|\phi\}$ are type terms.

3. **formulas** $\phi,\psi,\ldots$.
   a) Each equation $t_1 = t_2$ (t_1,t_2 of arbitrary type) is a formula.
   b) If ϕ, ψ are formulas so also are $\neg\phi$ and $\phi \to \psi$
   c) If ϕ is a formula and S is a type term, then $\forall x^S\phi$ is a formula.

NOTE. (i) In extensions of the language we may specify some *individual*

(3) Actually the Z^{ω} described in the 1977 paper only built types by the operation $\sigma,\tau \mapsto (\sigma \to \tau)$. The dispensability of product types is familiar from the combinatory literature.

constants (of certain types) and *type constants*, which are then counted as individual terms and type terms, resp. Other means of constructing individual terms may also be supplied. (ii) Quantifiers are not applied to type variables. This simplifies the conservation arguments below. However, one can also extend those results to VT systems *with* quantified type variables.

Before stating the axioms of VT_0 we make some abbreviations and conventions.
(1) The operators $\wedge, \vee, \leftrightarrow, \exists x^S(\ldots)$ are defined classically.
(2) $\forall x^S \phi(x,\ldots)$ is written for $\forall x^S \phi(x^S,\ldots)$, and similarly for $\exists x^S \phi(x,\ldots)$. That is, once the type of a variable is established, we suppress it in the following context.
(3) *Types* are also called *classes* and *type variables* are called *class variables*, etc. The former terminology figures in our syntactic description, the latter in our mathematical uses of the theory.
(4) $t \in T$ is defined as $\exists x^T(t = x)$, where 'x^T' does not occur in t. $\{x \in T | \phi\}$ is written for $\{x^T | \phi\}$, and $\forall x \in T(\phi)$ for $\forall x^T(\phi)$.
(5) $S \subseteq T$ is defined as $\forall x \in S\ (x \in T)$, i.e., as $\forall x^S\ \exists y^T(x = y)$.
(6) $S = T$ is defined as $S \subseteq T \wedge T \subseteq S$.
(7) We write $t(s,\ldots)$ for $t(s/x)$ when $t(x,\ldots)$ is written for t; similarly for $\phi(x,\ldots)$ and $\phi(s,\ldots) = \phi(s/x)$.

AXIOMS OF VT_0:

I. **Abstraction-Application.**
$\forall y \in X[\lambda x^X.t(x,\ldots)y = t(y,\ldots)]$.

II. **Pairing-Projections.**
i) $\forall x \in X\ \ \forall y \in Y[p_1(x,y) = x \wedge p_2(x,y) = y]$
ii) $\forall z \in X{\times}Y[z = (p_1(z),p_2(z))]$.

III. **Separation.**
$\{x \in X | \phi(x,\ldots)\} \subseteq X \wedge \forall y \in X[y \in \{x \in X | \phi(x,\ldots)\} \leftrightarrow \phi(y,\ldots)]$.

The logic of VT_0 is that of the many-sorted classical predicate calculus. Since the type variables are treated as free, we use the rule of substitution for these: $\phi(X,\ldots)/\phi(T,\ldots)$.

The system VT is an extension of VT_0 with axioms for the *natural numbers*. We adjoin a constant type symbol $\mathbb{N}$, individual constants 0 and sc, and individual terms r_T for each type term T, where 0 is of type $\mathbb{N}$, sc is of type $(\mathbb{N} \to \mathbb{N})$, and r_T is of type $(\mathbb{N}{\times}T \to T){\times}T \to (\mathbb{N} \to T)$. The variables 'n,' 'm,' 'p' with or without subscripts are reserved for variables of type $\mathbb{N}$. We write n' for sc(n). We shall tend to use letters 'f,' 'g,' etc. for members of function types $(S \to T)$.

AXIOMS OF VT ($= \mathrm{VT}_0$ *plus*):

IV. 0, Successor.

i) $(n' \neq 0)$.

ii) $(n' = m' \to n = m)$.

V. Induction.

$0 \in X \wedge \forall n[n \in X \to n' \in X] \to \mathbb{N} \subseteq X.$

VI. Recursion.

$f \in (\mathbb{N}\times X \to X) \wedge a \in X \wedge r_X(f,a) = g \to g0 = a \wedge gn' = f(n,gn).$

REMARK. Officially, Axiom VI would be written

$$\forall f^{(\mathbb{N}\times X\to X)}\forall a^X \forall g^{\mathbb{N}\to X}\forall x^{\mathbb{N}}[r_X(f,a) = g \to g0 = a \wedge gx' = f(x,gx)].$$

We put $1 = 0'$, $2 = 1'$, etc. Then $\{0,1\}$ is defined as $\{n|n = 0 \vee n = 1\}$. By a *characteristic function* on T we mean an element c of $T \to \{0,1\}$. Identify 0 with "true" and 1 with "false"; then write $x \in c$ for $cx = 0$. The elements of $T \to \{0,1\}$ are also called *sets, more precisely subsets of* T, and we also write $S(T)$ for the class of all such, i.e., for $T \to \{0,1\}$. *Set-induction* (on $\mathbb{N}$) or *restricted induction* is the principle

$$c \in S(\mathbb{N}) \wedge 0 \in c \wedge \forall n(n \in c \to n' \in c) \to \forall n(n \in c).$$

This is equivalent to the statement

$$f,g \in (\mathbb{N} \to \mathbb{N}) \wedge f0 = g0 \wedge \forall n(fn = gn \to fn' = gn') \to \forall n(fn = gn),$$

as well as the same with $g = \lambda n\cdot 0$. By *restricted recursion* we mean the principle VI taken only for $X = \mathbb{N}$; this means use only of primitive recursion with values in $\mathbb{N}$. By Res-VT is meant the system VT in which V is replaced by restricted induction and VI by restricted recursion.

Primitive recursive arithmetic and Kleene's extension of it to higher finite types are routinely developed in Res-VT. The following compares the present systems with those of Feferman 1977.

LEMMA. (i) VT *is an extension of* Z^ω.

(ii) Res-VT *is an extension of* Res-$\hat{Z}^\omega$.

Each type symbol τ of Z^ω corresponds to a closed type term T_τ, where $T_0 = \mathbb{N}$, $T_{(\sigma\times\tau)} = T_\sigma \times T_\tau$ and $T_{(\sigma\to\tau)} = (T_\sigma \to T_\tau)$. The T_τ are called *the finite types*.

For classical analysis we need to adjoin various non-constructive functions to VT. The first of these is the *unbounded minimum operator* μ of type $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$. When this is adjoined as a constant symbol, the associated axiom is taken to be:

$$(\mu) \qquad f \in (\mathbb{N} \to \mathbb{N}) \wedge fn = 0 \to f(\mu f) = 0 \wedge \mu f \leqslant n.$$

This allows us to define *quantification over* $\mathbb{N}$ as a functional operator: $\exists^{\mathbb{N}} f = [0$ if $f(\mu f) = 0$, 1 otherwise$]$. Stronger systems are obtained by introducing functionals corresponding to the *Suslin quantifier*, *quantification over* $(\mathbb{N} \to \mathbb{N})$, etc. We shall not detail those here.

The final principle to be considered is the *Axiom of Choice* taken as a s eme:

$$(\mathrm{AC}) \qquad \forall x^X \exists y^Y \phi(x,y) \to \exists z^{X\to Y} \forall x^X \phi(x,zx).$$

We denote by $(\mathrm{AC})_{S,T}$ the result of replacing X by S and Y by T in (AC). By *restricted* $(\mathrm{AC})_{S,T}$ is meant the statement:

$$\mathrm{Res}(\mathrm{AC})_{S,T} \qquad c \in \mathcal{S}(S\times T) \wedge \forall x^S \exists y^T [(x,y) \in c] \to \exists z^{S\to T} \forall x^S [(x,zx) \in c].$$

In other words, this takes A only for matrices ϕ which define sets. The scheme $(\mathrm{AC})_{\mathbb{N},\mathbb{N}}$ is already quite st ng (stronger than full second-order analysis). Writing $\mathbb{N}_0$ for $\mathbb{N}$ and $\mathbb{N}_1$ for $\to \mathbb{N}$, we write $(\mathrm{AC})_{0,1}$ for $(\mathrm{AC})_{\mathbb{N}_0,\mathbb{N}_1}$. In Feferman 1977 the scheme $\mathrm{Res}(\mathrm{AC})_{S,T}$ all finite types was denoted (QF-AC) [QF = quantifier-free]. We shall u the same designation here. Then $(\mathrm{QF\text{-}AC})_{0,1}$ is $\mathrm{Res}(\mathrm{AC})_{\mathbb{N}_0,\mathbb{N}_1}$. By way of com ison with familar systems, Z^{ω} + (μ) + $(\mathrm{QF\text{-}AC})_{0,1}$ contains the second-order sy m (-AC). The same thus holds for VT + (μ) + $(\mathrm{QF\text{-}AC})_{0,1}$

3. CONSERVATION RESULTS.

The *type levels* lev(T) of inite type terms are defined by $\mathrm{lev}(\mathbb{N}) = 0$, $\mathrm{lev}(S\times T) = \max(\mathrm{lev}(S),\mathrm{lev}(T))$ and $\mathrm{lev}(S\to T) = \max(\mathrm{lev}(S)+1,\mathrm{lev}(T))$. For $T = T_\tau$, we put $\mathrm{lev}(\tau) = \mathrm{lev}(T)$. By a *second-order sentence of the language of* Z^{ω} is meant one, all of whose variables are of type-level $\leqslant 1$.

MAIN THEOREM.

(i) VT ± (μ) ± $(\mathrm{QF\text{-}AC})$ *a conservative extension of* Z^{ω} ± (μ) ± $(\mathrm{QF\text{-}AC})_{0,1}$ *for second-order* .

(ii) *The same holds w* Res-VT *in place of* VT *and* Res-$\hat{Z}^{\omega}$ *in place of* Z^{ω}.

COROLLARY.

(i) Res-VT + (μ) + $(\mathrm{QF\text{-}AC})_{0,1}$ *is a conservative extension of* PA.

(ii) VT + (μ) + $(\mathrm{QF\text{-}AC})_{0,1}$ *is a conservative extension of* $(\Sigma^1_1\text{-AC})$ [*and hence of* $(\Pi^0_1\text{-CA})_{<\varepsilon_0}$].

The corollary follows by Feferman 1977, 8.6-8.7 [and Friedman's theorem for Σ^1_1-AC also proved loc. cit.]. Similar conservation results may be obtained with adjunction of stronger functional constants.

The steps in the proof of the main theorem are now outlined. For simplicity we concentrate on the reduction of VT to Z^{ω}. Each of the other stated results follows by a parallel argument.

Step 1. *Reduction of* VT *to a theory* CT *of (semi-) constant types.* CT differs from VT in that it has no type variables, though it has type terms which may vary depending on individual parameters. (For this reason they are called *semi-constant*.) The terms and formulas of CT are generated as in 1.-3. of the preceding section, omitting 2.a) (type variables), but including $\mathbb{N}$, 0, sc, and r_T for each semi-constant type term T. The axioms of CT are obtained from those of VT by substituting semi-constant type terms throughout for the type variables. The logic of CT is the same as for VT except that one can dispense with the substitution rule for types. It is readily seen that VT is conservative over CT.

Step 2. *Reduction of* CT *to a theory* FT *of finite types.* The finite types were defined above. The language of FT is a part of CT with two essential restrictions: (i) there are no sub-type terms $\{x \in S|\phi\}$, and (ii) equations $t_1 = t_2$ are allowed only between terms of the *same* finite type. The axioms of FT consist of appropriate restrictions to its language of: I (Abstraction-Application), II (Pairing-Projections), IV (0, successor), V (Induction), and VI (Recursion), where now V consists of all instances of the *induction scheme* $\phi(0) \wedge \forall n[\phi(n) \rightarrow \phi(n')] \rightarrow \forall n\phi(n)$ for ϕ a formula of FT. (Note the Axiom III is dropped). The proof that CT is conservative over FT is by a model-theoretic argument. With each model $\boldsymbol{M}$ of FT is associated a model $\boldsymbol{M}^*$ of CT which satisfies the same sentences of FT.

Without loss of generality one can assume that the types of $\boldsymbol{M}$ are disjoint. Let L_M^*, L_M be the languages of CT, FT, resp. with constants for all the individuals in $\boldsymbol{M}$. With each term or formula of L_M^* is associated a corresponding term or formula of L_M which will be its interpretarion in $\boldsymbol{M}$, except that type terms S are interpreted as pairs $(A,\phi(x))$ or formal terms $S^* = \{x^A|\phi(x)\}$ with A a finite type (of FT) and ϕ a formula of L_M. Given also $T^* = \{x^B|\phi(x)\}$ of the same kind, we take

$$(S\times T)^* = \{z^{A\times B}|\phi(P_1(z)) \wedge \psi(P_2(z))\},$$
$$(S\rightarrow T)^* = \{z^{A\rightarrow B}|\forall x^A[\phi(x) \rightarrow \psi(z(x))]\},$$
$$\{x^S|\theta(x)\}^* = \{x^A|\phi(x) \wedge \theta^*(x)\}.$$

t^* is then defined in an obvious way for individual terms t. Next, for formulas, if s, t are terms of type S, T, resp. we take

$$(s = t)^* = \begin{cases} s^* = t^* & \text{if } A = B \\ 0 \neq 0 & \text{if } A \neq B. \end{cases}$$

This definition is appropriate since if $A \neq B$, then A and B are disjoint by hypothesis. $(\ )^*$ preserves $\neg$ and $\to$, while $(\forall x^S \theta(x))^* = \forall x^A[\phi(x) \to \theta^*(x)]$ for $S^* = \{x^A | \phi(x)\}$. It is then straightforward to prove that this interpretation of L_M^* in L_M serves to define a model M^* of CT [4]

Step 3. *Reduction of* FT *to* FT[0] *with type* 0 *equations.* The system FT[0] is obtained from FT by use only of those formulas built up from equations $t_1 = t_2$ between objects of type-level 0. Equality at higher types is introduced by definition. This is used in re-expressing Axioms I, II, VI of VT and CT. To verify the laws of equality at higher types we need the axiom Ext of Extensionality. It is then shown that FT[0] + (Ext) is interpretable in FT[0], by the following (formal) model of *hereditarily extensional objects.* With each finite type A is associated a pair of formulas $x \equiv_A y$ and $E_A(x)$ for objects x, y of type A by:

(i) $E_N(x) \leftrightarrow x = x$, $x \equiv_N y \leftrightarrow x = y$,

(ii) $E_{A\times B}(x) \leftrightarrow E_A(P_1(z)) \wedge E_B(P_2(z))$, $z \equiv_{A\times B} w \leftrightarrow P_1(z) \equiv_A P_1(w) \wedge P_2(z) \equiv_B P_2(w)$,

(iii) $E_{(A\to B)}(z) \leftrightarrow \forall x^A[E_A(x) \to E_B(zx)] \wedge \forall x^A \forall y^A[x \equiv_A y \to zx \equiv_B zy]$,

$z \equiv_{(A\to B)} w \leftrightarrow \forall x^A[E_A(x) \to zx \equiv_B wx]$.

Note that when $\mathrm{lev}(A) \leqslant 1$ we have $\forall x^A . E_A(x)$. It follows that FT is conservative over FT[0] for second-order statements. (This is the point where restriction of conservation to second-order statements enters the Main Theorem).

Step 4. The system FT[0] is actually a form of Z^ω. As noted in ftn.3, the system Z^ω of Feferman 1977 is practically the same, but without product types. The latter are eliminable in the presence of extensionality, i.e., FT[0] + (Ext) is conservative over Z^ω + (Ext). Then Z^ω + (Ext) is reduced to Z^ω as in Step 3. The present step is unnecessary if the conservation results of Feferman 1977, 8.6-8.7 are established directly for FT[0] in place of Z^ω. That can be done by the same methods described loc. cit.

Step 5. The conservation results apply to extensions by the Axioms (μ) and/or $(\text{QF-AC})_{0,1}$ since these are second-order statements.

Finally it may be seen that each step can be carried out just as well to reduce Res-VT to Res-$\hat{Z}^\omega$, again with conservation for second-order statements,

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(4) Because of the interpretation of VT (via CT) in FT, I have also called VT a theory of *variable finite types* and denoted it VFT. For formalisms in which one can construct *transfinite types* cf. Feferman 1979.

and thence the same for extensiones by (μ) and/or $(QF\text{-}AC)_{0,1}$.

4. MATHEMATICS IN Res-VT+(μ)

The following is an outline of an informal development which can be formalized directly in Res-VT+(μ). This shows that a considerable portion of mathematical analysis is *predicative* and, indeed, is no stronger than PA. Cf. also Feferman 1977, §3.2, Takeuti 1978, and Friedman 1980[5]. In the approach taken here neither extensionality nor AC is needed, though both can be admitted to a certain extent by the formal results of Sec. 3.

A,B,C,...,X,Y,Z range over classes (which are treated formally as the type variables of VT). All constructions on classes are given explicitly, so all statements about classes are given in universal form $\forall X_1,\ldots,\forall X_n\phi(X_1,\ldots,X_n)$; this is justified in our framework when the formula $\phi(X_1,\ldots,X_n)$ is established in Res-VT+(μ). *Structures* are of the form $\mathbf{A} = \langle A,E,R_1,\ldots,R_m,f_1,\ldots,f_n,a_1,\ldots,a_p\rangle$ where $E \subseteq A^2$, $R_i \subseteq A^{k_i}$, $f_i \in A^{\ell_i} \to A$, and $a_i \in A$, and E is a congruence relation on A. E is called the *equality relation of* $\mathbf{A}$ and is often denoted $=$, even $=_A$. A homomorphism between structures $\mathbf{A} = \langle A,=_A,\ldots\rangle$, and $\mathbf{A}' = \langle A',=_{A'},\ldots\rangle$ of the same signature is a member h of $A \to A'$ such that $\forall x \in A\ \forall y \in A\,[x=_A y \to hx =_{A'} hy]$, and h preserves the operations and relations of $\mathbf{A}$. The appropriate notion of injective and surjective homomorphisms then leads to the notion of *isomorphism* for such structures.

We start with $\mathbf{N} = \langle \mathbb{N},=,<,+,\cdot,0,1\rangle$, where $=_{\mathbb{N}}$ is the identity relation. The *integers* $\mathbb{Z}$ are then defined to be $\mathbb{N}\times\mathbb{N}$ with $(x_1,y_1) =_{\mathbb{Z}} (x_2,y_2) \leftrightarrow x_1+y_2 = x_2+y_1$. An ordered integral domain structure $\mathbf{Z} = \langle \mathbb{Z},=_{\mathbb{Z}},<_{\mathbb{Z}},+_{\mathbb{Z}},\cdot_{\mathbb{Z}},0_{\mathbb{Z}},1_{\mathbb{Z}}\rangle$ is put on $\mathbb{Z}$ in the usual way, so that one has an injective homomorphism h of $\mathbf{N}$ into $\mathbf{Z}$, and $\mathbb{Z}$ is generated from the range of h. Similarly one passes from $\mathbf{Z}$ to the *rationals* $\mathbf{Q} = \langle \mathbb{Q},=_{\mathbb{Q}},<_{\mathbb{Q}},\ldots\rangle$, i.e., the quotient field of (an image of) $\mathbf{Z}$. Finally the *real number system* $\mathbf{R} = \langle \mathbb{R},=_{\mathbb{R}},<_{\mathbb{R}},\ldots\rangle$ is defined by taking $\mathbb{R}$ to consist of all *Cauchy sequences* of rationals, i.e.,

$$\mathbb{R} = \{z \in \mathbb{Q}^{\mathbb{N}} \mid \forall m\ \exists n\ \forall k_1,k_2 [k_1 \geqslant n \wedge k_2 \geqslant n \to |zk_1 - zk_2| < \tfrac{1}{m+1}]\}$$

where the expresion '$|zk_1-zk_2| < \frac{1}{m+1}$' is evaluated in $\mathbf{Q}$. The relations $=_{\mathbb{R}}, <_{\mathbb{R}}$ and the operations on $\mathbb{R}$ are then defined as usual. $\mathbf{R}$ forms an ordered field which is *Cauchy complete* in the sense that every Cauchy sequence of reals has a limit in $\mathbb{R}$. But $\mathbf{R}$ is not (provably) complete in the Dedekind sense that every Dedekind section in $\mathbb{Q}$ determines a real. The *complex number system* $\mathbf{C} = \langle \mathbb{C},=_{\mathbb{C}},\ldots\rangle$ is obtained in the standard way from $\mathbf{R}$.

From the reals we can move to *metric spaces*. All of the topological work

(5) For further original sources on predicative mathematics cf. Feferman 1964 and the references there to the work of Weyl, Lorenzen, and Kreisel.

is done with *separable metric spaces* $\mathbf{A} = (A,\ldots)$ wich carry as part of their structure a dense countable subset $\langle x_n \rangle_{n\in\mathbb{N}} \in A^{\mathbb{N}}$. Among the spaces wich are specially used are the real and complex finite-dimensional spaces $\mathbb{R}^n$ and $\mathbb{C}^n$, Cantor space $2^{\mathbb{N}}$, and Baire space $\mathbb{N}^{\mathbb{N}}$. All of these (and more) are show to be *locally sequentially compact*, i.e., every bounded sequence contains a convergent subsequence. The proof uses *König's Lemma*, which is here applied to trees t which are represented as members of $\mathbf{S}(\mathbb{N})$ (i.e., which have a characteristic function). Here the operator μ and the associated operator $\exists^{\mathbb{N}} \in (\mathbb{N}^{\mathbb{N}} \to \{0,1\})$ make an essential appearance. The definition of an infinite path through t is primitive recursive in $\exists^{\mathbb{N}}$ (and t). Only restricted induction is necessary to verify the required property of the path.

One cannot prove (local) *compactness* of these spaces in the usual sense of reduction of open covers to finite subcovers, but one can give a form of this for *countable open covers*. Some further general theorems which can be established in this setting for (Cauchy) complete separable metric spaces are the *Baire Category Theorem* and the *Contraction Mapping Theorem*.

Turning to classical analysis, the objects one deals with must usually be *presented* with additional information so as to be able to operate with them by the limited functional means provided in Res-VT+(μ). For example, an element of the class C(A,A') of *continuous functions* from A to A' (where A,A' are given metric spaces), is a pair (f,δ) for which f is a mapping in $A \to A'$ and δ is a *modulus-of-continuity function* $\delta(x,\varepsilon)$, i.e., such that

$$d_A(x,y) < \delta(x,\varepsilon) \to d_{A'}(f(x),f(y)) < \varepsilon.^{(6)}$$

Similarly, *uniformly continuous functions* are given as pairs (f,δ) where δ is a *modulus-of-uniform continuity* $\delta(\varepsilon)$ for f. It is shown for countably compact spaces that continuity implies uniform continuity and that maxima and minima are attained. Sequences and series of functions are studied in C(A,$\mathbb{R}$) when A is sequentially compact. This forms a metric space with respect to the sup-norm $\|f-g\| = \sup_{x\in A}|f(x)-g(x)|$; the *Stone-Weierstrass Theorem* can be proved, thus showing C(A,$\mathbb{R}$) to be separable.

Most classical topics in the *differential and integral calculus* (*Riemann integration*) go through quite readily. The extensions to *complex analysis* are

(6) In this respect we follow Bishop's lead in his development of constructive analysis (Bishop 1967); cf. also Feferman 1979, esp.pp.177 ff. The use of (μ) and thence of $\exists^{\mathbb{N}}$ is a way of incorporating mathematically what Bishop calls the *Limited Principle of Omniscience*, LPO. Bishop says that his results are constructive substitutes ϕ' for classical counterparts ϕ, such that ϕ'+LPO implies ϕ. Thus the formalization of Bishop's work in a system conservative over HA (cf.ftn.2 above) implies the formalization (in principle) of the corresponding body of classical mathematics in a system conservative over PA. The point of the approach here is to be able instead to step as directly as possible from current classical mathematics to its formalization in systems of known limited strength.

also straightforward, as are establishment of the properties of the familiar stock of transcendental functions.

New considerations are required when one passes to more modern topics, viz. *measure theory* and *functional analysis*. Standard approaches which start Lebesgue measure theory in $\mathbb{R}^n$ with outer measure $\mu^*(X)$ make essential use of the g.l.b. operation on sets of reals, which in turn requires Dedekind completeness of $\mathbb{R}$; but that is not available in VT+(μ). Instead, one can define *measurable sets* X and their *measure* $\mu(X)$ directly, using sequences of covering approximations to each of X and the complement of X by countable unions of open intervals. Another elegant route is to obtain the theory of *Lebesgue integration* directly using Riesz's approach: every measurable function is represented as a difference of two monotone sequence of step functions which converge *a.e.*, and its integral is defined in terms of integrals of step functions. For this only the concept of set of *measure* 0 is needed. Then the theory of measurable sets is obtained from the integration theory. It turns out all of that can be carried out in Res-VT+(μ). However, when performing operations on measurable functions and sequences of such, one must consistently work with *presentations* of them in terms of sequences of step functions (as described).

Finally, one can obtain the main initial material from functional analysis for *linear operators* on *separable Banach spaces* and *Hilbert spaces*. Usable forms of the Riesz Representation Theorem, Hahn-Banach Theorem, Uniform Boundedness Theorem, and the Open Mapping Theorem are obtained (under heavy use of separability). Finally, I have verified that one can obtain the principal results of the *spectral theory* of compact self-adjoint operators on a Hilbert space. It seems then that all *applicable analysis* can be carried out in this conservative extension of PA.

A theme running throughout this development is that the *l.u.b.* (or *g.l.b.*) property of the reals, which is constantly used in classical analysis, but which is not derivable in VT+(μ), can be avoided by dealing systematically with *sequences of reals* rather than *sets of reals*. For bounded sequences we *do* have l.u.b., g.l.b. (and sequential compactness more generally).

There are of course many results of theoretical analysis which cannot be derived in this setting. Additionally, by the result of Paris-Harrington 1977, there are simple combinatorial Π_2^0 statements which are consequences of RT (infinite Ramsey's Theorem) but which are not provable in Res-VT+(μ). We leave the question of what can be done in various extensions of this theory to another occasion.

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Depts. of Mathematics and Philosophy.
Stanford University
Stanford, California, U.S.A.

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SEPARABLY REAL CLOSED LOCAL RINGS

André Joyal and Gonzalo E. Reyes*

It is well known that several types of "variable reals" arising in topos theory (Dedekind reals, Cauchy reals, smooth reals, etc.) are not real closed, in the sense that a polynomial (with "variable real" coefficients) may change sign in an interval without having a zero in that interval.

This phenomenon stems from the fact roots of polynomials are not, in general continuous functions of the coefficients, as the example of the cubic x^3+px+q shows. Indeed, there is no continuous function $x(p,q)$ such that

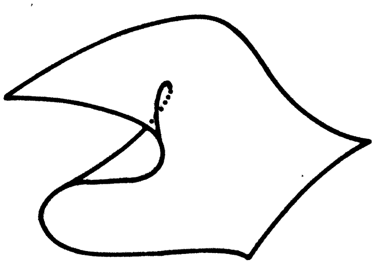
$$x(p,q)^3 + px(p,q) + q = 0$$

in a neighbourhood of $p = 0$, $q = 0$. This is most easily visualized by looking at the catastrophe map χ ("the cusp")

$$\chi\colon \{(p,q,x) \in R^3 | x^3 + px + q = 0\} \to R^2$$

defined by

$$\chi(p,q,x) = (p,q).$$



surface $x^3 + px + q = 0$

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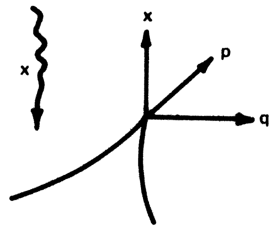


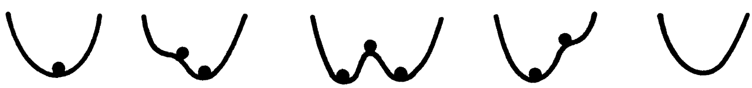
curve $4p^3 + 27q^2 = 0$.

It is obvious that X has no continuous sections in any neighbourhood of $p = 0$, $q = 0$ (just go around a circle with centre (0,0) in the (p,q) plane).

This phenomenon has the following physical interpretation: the positions of equilibrium of a dynamical system given by the potential

$$V = \frac{x^4}{4} + \frac{px^2}{2} + qx$$

do not depend continuously on the parameters p, q. In pictures.



For more information on this subject, the reader may consult Poston-Stewart (1978).

The question then arises as to the appropriate topos theoretic notion of "real closed field". A.Kock (1979) has proposed the notion of "separably real closed local ring", meaning a commutative ring with unit which is local, Henselian and has a real closed residue field. We recall (see e.g. Raynaud (1970)) that a ring A is *local* if it has exactly one proper maximal ideal m_A. A is *Henselian* if any simple root in its residue field $k_A = A/m_A$ of a polynomial $p \in A[t]$ can be lifted to a (necessarily unique) root of p in A.

To show the appropriateness of his notion, Kock (loc.cit.) proves that
(i) it generalizes the notion of a real closed field, which is just a separably real closed field,
(ii) various sheaves of continuous, C^∞, analytic,... real functions in appropriate spatial toposes are examples of this notion,
(iii) it is coherent (see e.g. Johnstone (1977) or Makkai-Reyes (1977) for this notion),
(iv) it is ε-stable (or infinitesimally stable) in the sense of Kock (1979).

A.Kock (loc.cit.) also conjectured that the object of Dedekind reals, in an arbitraty elementary topos with a natural number object, is a separably real closed local ring (object). For the particular case of Grothendieck toposes, this conjecture was verified by P. Johnstone (1979).

In this paper we prove Kock's conjecture, as well as related results about Cauchy reals and smooth reals.

Our main tool is a strengthening of Tarski's theorem on elimination of quantifiers in real closed fields, due to Coste and Coste-Roy, Delzell, Bocknak, and Efroymson (see e.g. Coste and Coste-Roy (1979)). This result provides a new coherent axiomatization of the notion of the title which is the key for the whole proof.

Throughout the paper, we shall use the set theoretical language as described, e.g. in Boileau-Joyal (1981).

1. THE COHERENT AXIOMATIZATION.

We say that a local ring A is *ordered* if it has an order relation, $<$, which is compatible with the ring operations and induces a linear order in its residue field $k_A = A/m_A$. More explicity, $<$ is assumed to satisfy the following axioms:

$1 > 0$

$(x > 0 \wedge y > 0) \rightarrow (x+y > 0 \wedge x \cdot y > 0)$

$(x \text{ invertible}) \leftrightarrow (x > 0 \vee -x > 0)$

We define $x > y \leftrightarrow x-y > 0$.

Our axiomatization will be formulated in the language L of the theory of ordered ring with $+$, $-$, $\cdot$, 0, 1, $>$ as non logical symbols.

If A is any ordered local ring and ϕ any formula of L whose free variables are among $x = (x_1,\ldots,x_n)$, we let

$$A_x(\phi) = \{a \in A^n | A \models \phi[a]\}$$

be the "extension of ϕ in A".

The strengthening of the theorem of Tarski mentioned in the introduction is the following.

THEOREM. *Let* ϕ *be a formula of* L *whose free variables are among* $x = (x_1,\ldots,x_n)$. *Assume that* $R_x(\phi) \subset R^n$ *is open (in the usual topology). Then there is a formula* ϕ_0 *having the same free variables of* ϕ *and of the form*

$$\bigvee_i \bigwedge_j P_{ij}(x) > 0$$

where the P_{ij} *are terms (i.e. polynomials) and such that* $K \models \forall x(\phi \leftrightarrow \phi_0)$, *for any real closed field* K.

The *crucial* property of such a formula ϕ_o is this: it is preserved and reflected by local homomorphisms (i.e. homomorphisms which reflect invertible elements) between ordered local rings.

To state our main result, we shall identify a monic polynomial $p(t) = t^n + p_1t^{n-1}+\dots+p_n$ with the sequence $p = (p_1,\dots,p_n)$ of its coefficients. Consider the following formula of L:

$$\delta(p) \equiv (p(0).\,p(1) < 0) \wedge (p'(0) > 0) \wedge (p'(1) > 0) \wedge \forall x(0<x<1 \to p'(x)>0).$$

Using the previous theorem and the fact that $R_p(\delta) \subset R^n$ is open (in the usual topology), there is another formula δ_o of L of the form $\bigvee_i \bigwedge_j P_{ij}(p) > 0$ such that $K \models \forall p(\delta \leftrightarrow \delta_o)$, for any real closed field K.

PROPOSITION. *If* A *is any ordered local ring, then* $A \models \forall p(\delta_o \to \delta)$.

Proof. Assume that $A \models \delta_o[p]$. From the theorem and the crucial property of δ_o we conclude that $\bar{k}_A \models \delta[(i\circ\tau_A)(p)]$ where $A \xrightarrow{\tau_A} k_A$ is the canonical map and $k_A \xrightarrow{i} \bar{k}_A$ is the inclusion of k_A in its real closure $\bar{k}_A$. A fortiori, $k_A \models \delta[\tau_A(p)]$ and this implies that $A \models \delta[p]$.

THEOREM. *The following is a (coherent) axiomatization of the notion of a separably real closed local ring:*

(i) *commutative ring with unit axioms,*

(ii) *localness:*

$$\begin{cases} \neg(0 = 1) \\ (x \textit{ invertible}) \vee (1-x \textit{ invertible}), \end{cases}$$

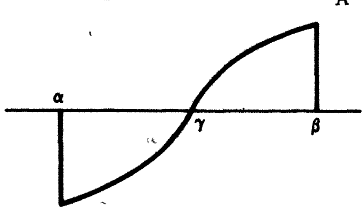
(iii) *order:*

$$\begin{cases} 1 > 0 \\ (x>0 \wedge y>0) \to (x+y > 0 \wedge x\cdot y > 0) \\ (x \textit{ invertible}) \leftrightarrow (x>0 \vee -x>0), \end{cases}$$

(iv) $\delta_o(p) \to \exists x(0<x<1 \wedge p(x)=0 \wedge p'(x)>0)$.

(Notice that, since $p = (p_1,\dots,p_n)$ is a sequence of n elements, (iv) is actually an infinite set of axioms, one for each $n = 1,2,\dots$)

Proof. Assume (i) - (iv). Let $p \in k_A[t]$ be either $p = t^2-\delta$, with $\delta > 0$ or a monic polynomial of odd degree such that $(p,p') = 1$. Going over to the real closure $k_A \xrightarrow{i} \bar{k}_A$, we find a simple root γ of p in $\bar{k}_A$. In pictures:



for some interval (α,β) such that $p' > 0$ throughout that interval (this is always possible, by changing p to -p, if necessary). Using the transformation $u \mapsto \alpha+(\beta-\alpha)u$, we may assume that $\alpha = 0$, $\beta = 1$, $\gamma \in (0,1)$, i.e.

$$k_A \models \delta_o[i(p)].$$

By the crucial property of δ_o (reflection under local homomorphisms between ordered local rings)

$$A \models \delta_o[\tilde{p}],$$

where $\tilde{p} \in A[t]$ is any lifting of p to A.

From (iv), $\tilde{p}$ has a simple root in A whose image in k_A is a (simple) root of $p \in k_A[t]$. This shows that k_A is real closed.

To show that A is Henselian, let $\alpha \in k_A$ be a simple root of $p \in A[t]$. Since k_A is real closed (as just proved), $k_A \models \delta_o[\tau_A(p)]$ (using a transformation of the form $u \mapsto \alpha+(\beta-\alpha)u$ again). Hence $A \models \delta_o[p]$ and so there is a simple root $a \in A$ of p. If $\tau_A(a) = \alpha$, there is nothing else to prove; if not, write $p = (t-a)q$ in $A[t]$ and use induction on the degree of p.

The proof in the other direction is obvious.

2. APPLICATIONS TO VARIABLE REALS.

COROLLARY 1. (Kock's conjeture). *The object R_D of Dedekind reals in any elementary topos with an object of natural numbers is a separably real closed local ring.*

Proof. It is well known that R_D satisfies axioms (i) - (iii) (see, e.g. Johnstone (1977)). Assume that $R_D \models \delta_o[p]$. Since $\delta_o \to \delta$ is (equivalent to) a coherent sequent and is true for all local ordered rings (whose theory is coherent), then it is true in R_D by the Metatheorem of Makkai-Reyes (1977). Therefore $R_D \models \delta[p]$ and the result follows from the following.

LEMMA. *Let $f \in R_D^{R_D}$ be a locally uniformly continuous function such that*

(i) $x < y \to f(x) < f(y)$

(ii) $f(\infty) = \infty$

(iii) $f(-\infty) = -\infty$

in the obvious sense.

Then f *is a homeomorphism.*

Proof. Just check that, for each $r \in R_D$, the pair $\{a \in Q \mid f(a) < r\}$, $\{a \in Q \mid f(a) > r\}$ constitutes a Dedekind cut.

COROLLARY 2. *The (object of) Cauchy reals, R_D, in any elementary topos E with natural number object is a separably real closed local ring.*

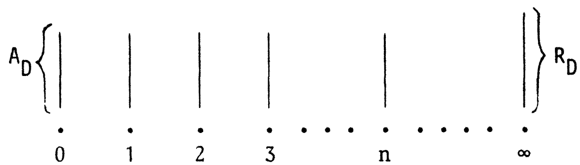
Proof. Let $Sh(N \cup \{\infty\})$ be the E-topos of sheaves over the Alexandroff compactification of $N \in E$. The (object of) Dedekind reals (resp. the algebraic reals) will be denoted, in both toposes, by R_D (resp. A_R). Let $R \in Sh(N \cup \{\infty\})$ be defined by

$$R = \{x \in R_D \mid [\![x \in A_R]\!] \supseteq N\} \ .$$

More precisely

$$R(V) = \{x \in C^0(V, R_D) \mid \forall x \in N \cap V(a(x) \in A_R)\}$$

for all $V \in O(N \cup \{\infty\})$. As an étale space, R has the following representation:



It is immediate that $R_\infty \in E$, the germ at ∞, is isomorphic to the ring of Cauchy sequences of A_R modulo of Frechet filter of cofinite subsets of N. Since every algebraic real has a decimal expansion, a diagonal argument shows that the quotient of R by the Cauchy sequences converging to 0 may be identified with the Cauchy reals:

$$R_\infty \twoheadrightarrow R_C \in E.$$

Furthermore, since separably real closed local rings are closed under non-trivial quotient, it is enough to prove that R_∞ is such a ring. On the other hand, since this notion is coherent and the operations of taking germs preserves coherent logic, we only need to prove that $R \in Sh(N \cup \{\infty\})$ is separably real closed. This is a consequence of the following.

LEMMA. *Let* $R_1 \rightarrowtail R_2$ *be a local monomorphism between separably real closed local rings. Then* $R = \{x \in R_2 \mid [\![x \in R_1]\!] \subseteq U\}$ *is again such a ring, for all* $U \rightarrowtail 1$.

Proof. Only axiom (iv) needs a proof. Let $p \in R[t]$ be a monic polynomial of degree n such that $R \models \delta_0[p]$. Since R_2 is separably real closed, the unique root in $(0,1) \subset R_2$ of any monic polynomial satisfying δ_0 defines a morphism

$$\xi\colon \{p \in R_2^n \mid R_2 \models \delta_0[p]\} \to R_2$$

whose restriction to R_1^n lies in R_1 (given that R_1 is separably real closed).

But $[\![p \in R_1^n]\!] \supset U$ and so $[\![\xi(p) \in R_1]\!] \supseteq [\![p \subset R_1^n]\!] \supseteq U$, i.e. the unique root of p in (0,1) lies in R. □

To formulate our next result, let $C^\infty \in E$ be the (internal) theory $C^\infty = \{R^n\colon n \in N\}$ whose n-ary operations are $C^\infty(R^n, R) \subset R^{R^n}$, the (internal) smooth functions from R^n into R. (We are using R to denote the object of Dedekind reals in E).

A C^∞-*ring* in $\mathcal{E}$ is an (internal) functor $A \in \mathcal{E}^{C^\infty}$ which preserves products. A is *local* if and only if A(R) is a local ring (object) in $\mathcal{E}$.

COROLLARY 3. *Any local* C^∞*-ring in* $\mathcal{E}$ *is separably real closed.*

Proof. There are two sources of difficulties in this proof. First, the theory of separable real closed local rings was formulated by using < as a non-logical relation symbol. We need to relate > with smooth functions, which are all what C^∞-rings know about. This is easily done, by constructing (as usual) a "characteristic" function of the open set $\{x \in R \mid x > 0\}$, i.e. a function $\chi \in C^\infty(R,R)$ such that

$$x > 0 \text{ iff } \chi(x) \text{ invertible.}$$

We now reformulate the axioms for a separably real closed local rings in terms of χ:

(i), (ii) as before,

(iii) $\chi(1)$ *invertible*

$(\chi(x)$ *invertible* $\wedge \chi(y)$ *invertible*$) \rightarrow (\chi(x+y)$ *invertible* $\wedge \chi(x\cdot y)$ *invertible*$)$

$(x$ *invertible*$) \rightarrow (\chi(x)+\chi(-x)$ *invertible*$)$

$(\chi(x)$ *invertible*$) \rightarrow (x$ *invertible*$)$

(iv) replace "$x > 0$" by "$\chi(x)$ *invertible*" throughout.

Now comes the second difficulty: our axioms are not equations and hence do not hold, a priori, in a C^∞-ring. Our solution (lemma 3) is to show that axioms of this type are consequences of equations true in R.

We need some auxiliary results.

LEMMA 1. (Existence of bump functions). *For all* $\varepsilon \in R$, $\varepsilon > 0$ *there is* $r_\varepsilon \in C^\infty(R,R)$ *such that*

$$r_\varepsilon(x) \begin{cases} = 0 \text{ if } |x| > \varepsilon \\ > 0 \text{ if } |x| < \varepsilon \\ = 1 \text{ if } x = 0. \end{cases}$$

In particular, $r_\varepsilon(x)$ *is invertible if and only if* $|x| < \varepsilon$.

Proof. The usual proof which starts from the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leqslant 0. \end{cases}$$

is constructive and valid in $\mathcal{E}$. Furthermore, such an f is clearly defined on Q and is uniformly continuous. Hence, it has a unique extension to R.

LEMMA 2. *Let* $\phi \in C^\infty(R^n)$ *and* $U = \{x \in R^n \mid \phi(x)$ *invertible*$\}$. *Then*

$$C^{\infty}(R^{n+1})/(y\phi(x)-1) \approx C^{\infty}(U).$$

Proof. Let ρ: $C^{\infty}(R^n) \to C^{\infty}(U)$ be the "restriction" map defined by

$$\rho f(x,y) = f(x,\frac{1}{\phi(x)}).$$

We claim that ρ is surjective, i.e. any $h \in C^{\infty}(U)$ may be "extended" to some $f \in C^{\infty}(R^{n+1})$. The argument below was suggested by Ngo van Quê.

Let $h \in C^{\infty}(U)$. Define, for $\varepsilon > 0$,

$$f(x,y) = \begin{cases} r_{\varepsilon}(y-\frac{1}{\phi(x)})h(x) & \text{if } \phi(x) \text{ invertible} \\ 0 & \text{if } |y\phi(x)-1| > \varepsilon|\phi(x)|. \end{cases}$$

Using the fact that R is an ordered local ring, one easily checks that $f \in C^{\infty}(R^{n+1})$. Indeed either $|\phi(x)| < \frac{1}{\varepsilon+|y|}$ and hence $|y\phi(x)-1| > \varepsilon|\phi(x)|$ or $|\phi(x)| > \frac{1}{2(\varepsilon+|y|)}$, which implies that $\phi(x)$ invertible. Clearly,

$$h(x) = f(x,\frac{1}{\phi(x)}) \text{ for all } x \in U.$$

Assume now that $f \in \mathrm{Ker}(\rho)$. Using Hadamard's lemma for $f \in C^{\infty}(R^n \times R,R)$:

$$f(x,t) - f(x,s) = (t-s)\int_0^1 \frac{\partial f}{\partial t}(x,s+(t-s)u)du$$

we conclude the existence of some $f_1 \in C^{\infty}(R^{n+2})$ such that

$$f(x,y) - f(x,\frac{1}{\phi(x)}) = (y-\frac{1}{\phi(x)})f_1(x,y,\frac{1}{\phi(x)}).$$

Define

$$\nu(x,y) = \begin{cases} \frac{f(x,y)}{y\phi(x)-1} & \text{if } y\phi(x)-1 \text{ invertible} \\ \frac{f_1(x,y,\frac{1}{\phi(x)})}{\phi(x)} & \text{if } \phi(x) \text{ invertible} \end{cases}$$

Once again, it is easily checked that $\nu \in C^{\infty}(R^{n+1})$. Therefore, $f(x,y) = \nu(x,y)(y\phi(x)-1) \in (y\phi(x)-1)$.

LEMMA 3. *Let* $\phi(x)$, $\theta(x,z)$, $\psi(x,t)$ *be smooth functions. Assume the existence of some* $h \in C^{\infty}(U)$ *such that* $\theta(x,h(x)) = 0$ *and* $\psi(x,h(x))$ *invertible, for all* $x \in U$, *where* $U = \{x \in R^n \mid \phi(x)$ *invertible*$\}$. *Then any* C^{∞}*-ring satisfies the sentence:*

$$\forall x(\phi(x) \textit{ invertible} \to \exists z(\theta(x,z)=0 \wedge \psi(x,z) \textit{ invertible})).$$

Proof. By Lemma 2, we can "extend" $h(x)$ to $f(x,y)$ in such a way that

$$\theta(x,f(x,y)) = 0 \text{ modulo } (y\phi(x)-1)$$

and

$$\psi(x,f(x,y)) \text{ invertible modulo } (y\phi(x)-1).$$

In other words, there are smooth functions ν, ν_1 and ψ_1 such that the following equations are valid in R:

$$\theta(x,f(x,y)) = \nu(x,y)(y\phi(x)-1)$$

$$\psi(x,f(x,y))\psi_1(x,y)-1 = \nu_1(x,y)(y\phi(x)-1).$$

Any C^∞-ring is a model of these equations.

To finish the proof of Corollary 3, we notice that all axioms for separably real closed local rings, as reformulated in terms of X, are of the form stated in Lemma 3. The result thus follows from Corollary 1.

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Université de Montréal
Montréal, Québec
Canada.

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ON INTUITIONISTIC SENTENTIAL CONNECTIVES I

E.G.K. López-Escobar

INTRODUCTION.

Recently there have appeared a series of articles on a non-classical logic, called the *Heyting-Brouwer Logic* (**H-B L**), see [2], [3], [5] and [6]. The **H-B** Logics are obtained by the addition of new sentential connectives to intuitionistic logic so that the resulting Lindenbaum algebras enjoy some duality properties.

In a pseudo-Boolean algebra $A = \langle A, \cap, \cup, \Rightarrow, 0 \rangle$, the element $a \Rightarrow b$ is the *pseudo-complement of* a *relative to* b and has the property that for every $x \in A$:

$$x \leqslant a \Rightarrow b \quad \text{iff} \quad a \cap x \leqslant b.$$

The dual notion to the pseudo-complement is the *pseudo-difference*. The pseudo-difference of b and a is denoted, when it exists, by "$b \dot{-} a$" and it has the property that for every $x \in A$:

$$x \geqslant b \dot{-} a \quad \text{iff} \quad a \cup x \geqslant b.$$

As is well known, a Boolean algebra always has both pseudo-complements and pseudo-differences. On the other hand, pseudo-Boolean algebras (also called Heyting algebras) have pseudo-complements but may fail to have pseudo-differences. The dual of Heyting algebras, the Brouwerian algebras, have pseudo-differences but may fail to have pseudo-complements.

The fusion of Heyting algebras and Brouwerian algebras are called *semi-Boolean algebras*; that is, $A = \langle A, \cap, \cup, \Rightarrow, \dot{-}, 0, 1 \rangle$ is a semi-Boolean algebra iff $\langle A, \cap, \cup, \Rightarrow, 0 \rangle$ is a Heyting algebra and $\langle A, \cap, \cup, \dot{-}, 1 \rangle$ is a Brouwerian algebra.

The *Heyting-Brouwer Sentential Calculus*, **H-B SC**, is the extension of the Intuitionistic Sentential Calculus, **ISC**, obtained by adding a new sentential connective $\dot{-}$ to the language. The axioms and rules of inference were chosen so that the resulting Lindenbaum algebras are semi-Boolean algebras.

An interesting development of the H-**B SC**, obtain by C. Rauszer in [6], is that there is a complete and sound semantics for **H-B SC** in terms of Kripke models. The condition for a formula $(A \dot{-} B)$ to be forced (or satisfied) at a node N of Kripke model $K = \langle K, \leqslant, \ldots \rangle$ is given by:

$$K,N \Vdash (A \dot{-} B) \text{ iff } \exists N'_{N' \leqslant N}[K,N' \Vdash A \;\&\; K,N' \nVdash B].$$

From the Kripke semantics for **H-B SC** it immediately follows that **H-B SC** is a conservative extension of the **ISC**. Thus it would appear that $\dot{-}$ might be considered as a new *intuitionistic* connective.

Unfortunately in a (complete) semi-Boolean algebra we have the following distributive law:

$$\bigcap_i (b \cup a_i) = b \cup \bigcap_i a_i$$

so that it is not surprising that in the **H-B** Predicate Calculus the schema:

$$\forall x(B \vee Ax) . \supset . B \vee \forall x Ax,$$

is provable, and thus **H-B PC** is not a conservative extension of the Intuitionistic Predicate Calculus, **IPC**. Thus we have second thoughts on whether $\dot{-}$ should be considered as an intuitionistic sentential connective.

This leads to the following problem:

PROBLEM. *Suppose that* S *is a schema, essentially involving quantifiers, such that* S *is provable in the Classical Predicate Calculus,* **CPC**, *but not in the* **IPC**. *Then, is there a sentential connective* $\oplus$ *(with associated rules) so that* ISC + $\oplus$ *is a conservative extension of* ISC *and* **IPC** + $\oplus$ $\vdash$ S?

Possible examples for S are: $[P \supset \exists x Qx . \supset . \exists x(P \supset Qx)]$ and $[\forall x \neg\neg Px . \supset . \neg\neg \forall x Px]$. A reason why we believe it may be possible, at least for some schemas is the following observation (suggested to us by the corresponding property for semi-Boolean algebras).

THEOREM. *If* **H** *is an extension of the* **IPC** *such that there is a binary operation* F *on the formulas of* **H** *such that for all formulas* A,B,C *of the first-order language:*

$$\mathbf{H} \vdash F(A,B) \supset C \quad \text{iff} \quad \mathbf{H} \vdash A \supset (B \vee C);$$

then the schema (restricted to first-order formulas):

$$\forall x(P \vee Qx) . \supset . P \vee \forall x Qx$$

is provable in **H**.

Proof. Let P,Qx be formulas of **IPC** and let a be an individual parameter which does not occur in $\forall x(P \vee Qx)$, nor in $F(\forall x(P \vee Qx),P)$. Then:

$$\mathbf{IPC} \vdash \forall x(P \vee Qx) . \supset . P \vee Qa$$
$$\mathbf{H} \vdash \forall x(P \vee Qx) . \supset . P \vee Qa$$
$$\mathbf{H} \vdash F(\forall x(P \vee Qx),P) \supset Qa$$
$$\mathbf{H} \vdash \forall x[F(\forall x(P \vee Qx),P) \supset Qx]$$
$$\mathbf{H} \vdash F(\forall x(P \vee Qx),P) \supset \forall x Qx$$

$H \vdash \forall x(P \vee Qx). \supset .P \vee \forall xQx.$

1. SENTENTIAL CONNECTIVES IN INTUITIONISTIC LOGIC.

Before we can decide what it is meant by an intuitionistic sentential connective, we must have some agreement on what is understood by intuitionistic logic. And to define "intuitionistic logic" one must first define intituitionism and intuitionistic mathematics. Troelstra [8] suggests the following:
"*Intuitionistic mathematics*" is mathematics consistent with L. E. J. Brouwer's reconstruction of mathematics.
"*Intuitionism*" refers to the body of concepts used in the development of intuitionistic mathematics.
"*Intuitionistic logic*" is a formalization of (a part of) intuitionism.

It would thus appear that the place to look for intuitionistic connectives is in "Intuitionism" rather than in Intuitionistic Mathematics or Intuitionistic Logic. Since the principal activity in Intuitionistic Mathematics is obtaining constructions (that prove, or justify, mathematical assertions), we find that the concept "*the construction* c *proves* A" is one of the fundamental concepts of Intuitionism. Or in other words, Intuitionism encompasses some, perhaps informal, theory of constructions T. We shall further assume that in T there are (possibly partial) predicates of the form:

$\pi(c, \ulcorner\theta\urcorner)$ read: *the construction* c *proves* θ

$\pi_A(c)$ read: *the construction* c *proves* A,

and (possibly partial) operations of the form:

c'd = *the result of applying the construction* c *to* d ,
c:d = *the ordered pair of the constructions* c *and* d *(also a construction).*

1.1. Intuitionistic sentential connectives.

The conditional is usually the most problematic of the connectives. However in classical logic, once one accepts the truth tables then the mysticism of the conditional, as well as of the other connectives, disappears.

Similarly for intuitionistic logic. The intuitionistic conditional $(A \supset B)$ is explained by giving the conditions under which a construction proves $(A \supset B)$; i.e.

$$\pi_{(A \supset B)}(c{:}d) \quad \text{iff} \quad \pi(c, \ulcorner\pi_A(x) \rightarrow \pi_B(d'x)\urcorner).$$

And correspondingly for the other connectives $\vee$, $\wedge$, $\perp$. For "new" connectives we can then proceed as follows:

Suppose that P is a sentential parameter and that $C_P(a)$ is a formula of the theory **T** of constructions in which there is at least one occurence of π_P. Then $C_P(a)$ can be used to define a unary sentential connective $\underset{\sim}{C}$ by stipulating that for all formulae A of the extended language:

$$\pi_{\underset{\sim}{C}_A}(c:d) \quad \text{iff} \quad \pi(c, \ulcorner C_A(d) \urcorner),$$

where $C_A(d)$ is the formula obtained from $C_P(a)$ by replacing all occurrences of P by A, and all occurrences of a by d.

Given a language L, then by L + $\underset{\sim}{C}$ we understand the extension of L obtained by adding the (unary) sentential connective $\underset{\sim}{C}$; **SC** is the sentential language with the connectives $\wedge$, $\vee$, $\supset$, and $\perp$.

NISC is the natural deduction axiomatizacion of ISC, in the language **SC** (see Prawitz [1]).

1.1.1. DEFINITION. *A sentential connective* $\underset{\sim}{C}$ *is* axiomatizable w.r.t. **T** *iff there is a finite set* **R** *of rules such that for every formula* A *of* **SC** + $\underset{\sim}{C}$; *if* **NISC** + **R** $\vdash$ A *then there is a construction* c *such that in* **T**, $\pi_A(c)$.

1.1.2. DEFINITION. *Suppose that* $\underset{\sim}{C}$ *is an axiomatizable connective*, **R** *its associated rules and* **H** *an extension of intuitionistic logic not containing the connective* $\underset{\sim}{C}$. *Then the connective* $\underset{\sim}{C}$ *is an* intuitionistic sentential connective w.r.t. H *iff* (**H+R**) *is a conservative extension of* **H**.

1.1.3. DEFINITION. $\underset{\sim}{C}$ *and its associated rules, is an* intuitionistic sentential connective *iff* $\underset{\sim}{C}$ *is an intuitionistic sentential connective* w.r.t. *every intuitionistic logic* **H** *such that* **H** *does not contain* $\underset{\sim}{C}$.

2. THE SENTENTIAL CONNECTIVES $\ulcorner$, $\Box$, $\Diamond$ AND MODELLINGS.

In the **H-B PC** the unary connective $\ulcorner$ can be defined by $\ulcorner A = ((\perp \supset \perp) \dot{-} A)$ and the corresponding condition in the Kripke models is:

$$K,N \Vdash \ulcorner A[\![s]\!] \quad \text{iff} \quad \exists N'_{N' \leqslant N}(K,N' \nVdash A[\![s]\!]),$$

where s is an assignment of the individual parameters of A to the individuals *at the node* N. It is almost immediate from the above that for it to make sense the Kripke model must be one of constant domains, otherwise $K,N' \Vdash A[\![s]\!]$ might fail for the wrong reasons. And it is well known that the formulae valid in all Kripke models with constant domains is *not* a conservative extension of the **IPC**.

Thus any attempt to discover, through the use of Kripke models, if there is an intituitionistic connective corresponding to $\ulcorner$ appears to be doomed from the start. Nevertheless the Kripke models give us a hint of what to look for. The interpretation of Kripke models as stages of positive research (see [6], page 36) give us: *to assert* $\ulcorner A$ *at stage* N *we need to know that there exists an earlier stage* N' *such that our information about* A *is not sufficient to verify* A *at stage* N'.

Since under most interpretations, c and d come before c:d, the above remarks suggest that a possible formula $C_p(a)$ for the connective $\ulcorner$ is $\neg\pi_p(a)$ so that:

$$\pi_{\ulcorner A}(c{:}d) \quad \text{iff} \quad \pi(c, \ulcorner\neg\pi_A(d)\urcorner). \tag{*}$$

Unfortunately, if we wish to use (*) in order to discover an axiomatization for $\ulcorner$ we must first develop part of the theory **T** of constructions and the currently available theories of constructions are quite complicated.

Thus we shall use a more ad hoc method for obtaining an axiomatization. Namely, we take the Beth semantics, which is complete and sound for **IPC** and *which uses constant domains,* and try to accomodate the connective $\ulcorner$. The semantics then suggest a set **R** of rules so that **NISC+R** is sound and complete. Once we have a set **R** of rules we can return to the theory **T** of constructions and verify that **R** is indeed an axiomatization (in the sense of definition 1.1.1).

As a matter of fact, the Beth semantics leads us to another connective (which is inter definable with $\ulcorner$) and which, in certain respects, is much more natural.

2.1. Beth models for SC+$\ulcorner$.

We extend the usual definition of satisfaction (forcing) in Beth structures to formulae of SC+$\ulcorner$ by adding the clause:

$$B,N \Vdash \ulcorner A[\![s]\!] \quad \text{iff} \quad \exists N'_{N' \leqslant N}(B,N' \nVdash A[\![s]\!]).$$

Then we define:

$$\text{VAL} = \{A \mid \text{for all Beth structures } B \text{ and all assignments } s \text{ in } B\text{: } B \Vdash A[\![s]\!]\}.$$

An induction on the complexity of the formula A of SC+$\ulcorner$ give us the following:

2.1.1. LEMMA. *For all Beth structures* B,

(1) *If* $B,N \Vdash A$ *and* $N \leqslant N'$ *then* $B,N' \Vdash A$

(2) $B,N \Vdash A$ *iff* $\forall\beta_{N\in\beta}\exists t(B,\bar\beta t \Vdash A)$,

where β *ranges over paths through* B, "N ∈ β" *expresses that the node* N *belongs to the path* β *and* $\bar\beta t$ *is the node* <β0,β1,...,β(t-1)>.

2.1.2. DEFINITIONS. $\Box A = \neg\ulcorner A$, $\quad\Diamond A = \ulcorner\neg A$.

2.1.3. COROLLARIES.

(1) $B,N \Vdash \ulcorner A$ *iff* $B \nVdash A$ *iff* $B \Vdash \ulcorner A$.

(2) $B,N \Vdash \Box A$ *iff* $\forall N'(B,N' \Vdash A)$ *iff* $B \Vdash A$ *iff* $B \Vdash \Box A$.

(3) $B,N \Vdash \Diamond A$ *iff* $\exists N'(B,N' \Vdash A)$ *iff* $B \Vdash \Diamond A$ *iff* $B \Vdash \Box\Diamond A$.

2.1.4. LEMMA. *The following schema belong to* VAL:

(1) $\Box A \supset A$.

(2) $A \supset \Diamond A$.

(3) $\Box(A \supset B) . \supset . \Box A \supset \Box B$.

(4) $\Box A \supset \Box\Box A$.

(5) $\Diamond A \supset \Box\Diamond A$.

(6) $\ulcorner A \equiv \neg\Box A$.

(7) $A \vee \ulcorner A$.

(8) $\neg\Box\neg A \equiv \Diamond A$.

(9) $\Diamond\Box A \supset \Box A$.

(10) $\Box A \vee \neg\Box A$.

2.1.5. LEMMA. *There are instances of the following schemas which do not belong to* VAL:

(1) $A \supset \Box A$.

(2) $\Diamond A \supset A$.

(3) $\Box(A \vee B) . \supset . \Box A \vee \Box B$.

(4) $\Box \exists x Ax \supset \exists x \Box Ax$.

(5) $(\Box A \supset \Box B) \supset \Box(A \supset B)$.

(6) $\neg\Box A \supset \Box\neg A$.

(7) $\neg\Diamond\neg A \supset \Box A$.

2.1.6. DEFINITION. *A formula* A *of* SC+$\ulcorner$ *is* essentially modal (e.m.) *iff either:*

(i) *for some* B, $A = \Box B$, *or*

(ii) *for some* B, $A = \Diamond B$, *or*

(iii) *for some* e.m. B *and* C, $A = (B \vee C)$ *or* $A = (B \wedge C)$, *or*

(iv) *for some* e.m. B, $A = \exists x B$ *or* $A = \forall x B$, *or*

(v) A *is* $\bot$, *or*

(vi) *for some* e.m. B, $A = \neg B$ $(= B \supset \bot)$.

2.1.7. LEMMA. *If* A *is an e.m. formula then* $(A \supset \Box A)$, $A \vee \neg A$ *and* $A \equiv \Box A$, *all belong to* VAL.

From the above results we see that the sentential combinations $\neg\ulcorner$ and $\ulcorner\neg$ behave as modal operators. Since modal operators are better understood than weak (paraconsistent) negations and since according to lemma 2.1.4 $\ulcorner A$ is (semantically) equivalent to $\neg\Box A$, we shall now change to the language PC($\Box$) in which $\Box$ is a sentential connective and $\ulcorner$, $\Diamond$ are abbreviations for $\neg\Box$, $\neg\Box\neg$, respectively. The definition for the satisfaction of $\Box A$ in Beth models is given by 2.1.

2.2. Rules of inference for $\Box$.

$$(\ulcorner E)\ \dfrac{\begin{matrix} \not{\Box A} \\ \vdots \\ B \end{matrix} \qquad \begin{matrix} \not{\neg\Box A} \\ \vdots \\ B \end{matrix}}{B}$$

$(\Box E)\ \dfrac{\Box A}{A}$ $\qquad\qquad$ $(\Box I)\ \dfrac{A}{\Box A}$

$$(\Diamond E)\ \dfrac{\Diamond A \qquad \begin{matrix} [A] \\ \vdots \\ B \end{matrix}}{B}$$

Restriction on the $\Box$I *rule*: Every undischarged assumption formula on which A depends must be an essentially modal formula.
Restriction on the $\Diamond$E *rule*: B and every undischarged assumption formula on which B depends (except possibly A) must be an essentially modal formula.

2.3. Some theorems of NMPC.

By **NMPC** we understand the system of natural deduction obtained by adjoining the rules 2.2. to the intuitionistic system.

2.3.1. THEOREM. *The following schemas are provable in* **NMPC**:

(1) $A \vee \neg A$.
(2) $\neg\neg\Box A \supset A$.
(3) $\Diamond A \wedge \Box(A \supset B) . \supset . B$ *provided* B *is an e.m.f.*
(4) $\Diamond A \supset \neg\Box\neg A$.
(5) $\Box\neg B \supset \neg\Box B$.
(6) $\Diamond A \equiv \neg\Box\neg A$.
(7) $\Diamond\Box A \supset \Box A$.
(8) $(A \supset \neg\Box A) \supset \neg\Box A$.
(9) $\neg\neg A \supset \Diamond A$
(10) $\Box A \vee \neg\Box A$
(11) $A \vee \neg A$, *if* A *is an e.m.f.*
(12) $A \equiv \Box A$, *if* A *is an e.m.f.*

2.4. Soundness theorem for NMPC.

For every formula A *of* PC($\Box$), *every Beth structure* $\mathcal{L}$ *and every assignment* S *in* $\mathcal{L}$ *:*

$$\textit{if}\ \ \textbf{NMPC} \vdash A \ \ \textit{then}\ \ \mathcal{L} \Vdash A\,[\![S]\!].$$

Proof. By induction on the length of the derivation in **NMPC**.

2.5. Completeness theorem for NMPC.

We prove the (weak) completeness theorem in the form that if a sentence S of PC($\Box$) is not provable in **NMPC** then there is a Beth structure $\mathcal{L}_S$ such that $\mathcal{L}_S \nVdash S$.

2.5.1. DRAMATIS PERSONAE.

(1) S, an unprovable sentence of **NMPC**.

(2) $S^* = \ulcorner S$ (= $\neg\Box S$),

(3) QS = the (finite) set of quasi-subformulae of S^*,

(4) $\mathcal{F} = QS \cup \{\neg G: G \in QS\} \cup \{\Box G: G \in QS\} \cup \{\neg\Box G: G \in QS\} \cdot$
$\cup \{\Box\neg G: G \in QS\} \cup \{\neg\Box\neg G: G \in QS\} \cup \{\bot\}$

(5) L = {A: A is a formula and A is a substitution instance of some $G \in \mathcal{F}$},

(6) Par (θ) = the set of individual parameters ocurring in θ,

(7) L_k = {A: A $\in$ L and Par(A) $\subseteq \{a_o,..,a_{k-1}\}$},

(8) EM = {A: A $\in$ L and A is an e.m. formula} ,

(9) $EM_k = EM \cap L_k$,

(10) NM = L − EM,

(11) $NM_k = L_k - EM_k$,

(12) $A_o, A_1, \ldots$ is an enumeration with infinite repetition of NM, such that for each k, $A_k \in NM_k$,

(13) Γ = {A: A $\in$ EM and $S^* \vdash A$},

(14) $\Gamma_1 \supseteq \Gamma$ is such that $\Gamma_1 \subseteq$ EM is consistent and for each formula $\exists xBx \in \Gamma_1$ there corresponds an individual parameter a_{2i} such that $B(a_{2i}) \in \Gamma_1$. Furthermore if A $\in$ EM and $\Gamma_1 \vdash A$ then $A \in \Gamma_1$,

(15) $\Gamma_2 = \Gamma_1 \cup \{\neg G: G \in EM$ and $G \notin \Gamma_1\}$.

2.5.2. CONSTRUCTION OF THE TREE $\sum$.

Basis: $\sum(0) = \sum(<\ >) = \Gamma_2$.

Recursion step: Suppose $\sum(\vec{u}) = \sum(<u_o,\ldots,u_{k-1}>)$ has already been defined. Consider then the formula A_k ($\in L_k$).

Case 1: A_k is neither a disjunction nor an existential formula then we have 3 subcases to consider.

Subcase 1: $\sum(\vec{u}) \cap L_k \vdash A_k$.
Then we define $\sum(\vec{u}^\frown<1>) = \sum(\vec{u}) \cup \{A_k\}$.

Subcase 2: $\sum(\vec{u}) \cap L_k \vdash \neg A_k$.
Then we define $\sum(u^\frown<0>) = \sum(u)$.

Subcase 3: Neither subcase 1 nor subcase 2.
Then we define $\sum(\vec{u}^\frown<0>) = \sum(\vec{u})$, $\sum(\vec{u}^\frown<1>) = \sum(\vec{u}) \cup \{A_k\}$.

Case 2: $A_k = B_1 \vee B_2$. Then the definition of $\sum$ is changed as follows:

Subcase 1: $\sum(\vec{u}^\frown<1>) = \sum(\vec{u}) \cup \{A_k, B_1\}$, if consistent,
$\sum(\vec{u}^\frown<2>) = \sum(\vec{u}) \cup \{A_k, B_2\}$, if consistent.

Subcase 2: No change.

Subcase 3: $\sum(\vec{u}^\frown<0>) = \sum(\vec{u})$,
$\sum(\vec{u}^\frown<1>) = \sum(\vec{u}) \cup \{A_k, B_1\}$, if consistent,
$\sum(\vec{u}^\frown<2>) = \sum(\vec{u}) \cup \{A_k, B_2\}$, if consistent.

Case 3: $A_k = \exists x B(x)$. The definitions are then:

Subcase 1: $\sum(\vec{u}^\frown\langle 0\rangle) = \sum(\vec{u}) \cup \{A_k, B(a_{2k+1})\}$,

Subcase 2: No change.

Subcase 3: $\sum(\vec{u}^\frown\langle 0\rangle) = \sum(\vec{u})$,

$\sum(\vec{u}^\frown\langle 1\rangle) = \sum(\vec{u}) \cup \{A_k, B(a_{2k+1})\}$.

2.5.3. PROPERTIES OF THE TREE $\sum$.

(1) If $A \in L_k$, $\ell th\ (\vec{u}) = k$ then
$\sum(\vec{u}) \cap L_k \vdash A$ iff $\forall\beta_{\vec{u}\in\beta\in\sum}\ \exists t[\sum(\bar{\beta}t) \cap L_t \vdash A]$

(2) If $A \in L_k$, $\ell th\ (\vec{u}) = k$, then
$\sum(\vec{u}) \cap L_k \vdash \neg A$ iff $\forall\beta_{\vec{u}\in\beta\in\sum}\forall t \geqslant k[\sum(\bar{\beta}t) \cap L_t \nvdash A]$

(3) If $(A \supset B) \in L_k$, $\ell th\ (\bar{u}) = k$, then
$\sum(\vec{u}) \cap L_k \vdash (A \supset B)$ iff $\forall\beta_{\vec{u}\in\beta\in\sum}\forall t \geqslant k[$if $\sum(\bar{\beta}t) \cap L_t \vdash A$ then $\sum(\bar{\beta}t) \cap L_t \vdash B]$

(4) If $A \vee B \in L_k$. $\ell th\ (\bar{u}) = k$ then
$\sum(\vec{u}) \cap L_k \vdash (A \vee B)$ iff $\forall\beta_{\vec{u}\in\beta\in\sum}\exists t \geqslant k[$Either $\sum(\bar{\beta}t) \cap L_t \vdash A$ or $\sum(\bar{\beta}t) \cap L_t \vdash B]$.

2.5.4. DEFINITION OF THE BETH MODEL $\mathcal{L}_S$. $\mathcal{L}_S$ is the Beth model $\langle\sum,\leqslant,V\rangle$, where $\sum$ is the tree just defined and V is the function defined on the nodes of $\sum$ such that $V(\langle u_0,\ldots,u_{k-1}\rangle) =$

$\{A$: A is an atomic formula, $A \in L_k$ and $\sum(\langle u_0,\ldots,u_{k-1}\rangle) \cap L_k \vdash A\}$.

2.5.5. RELATION BETWEEN THE TREE $\sum$ AND $\mathcal{L}_S$. If $k = \ell th(\vec{u})$, then for all formulae $A \in L_k$, $\sum(u) \cap L_k \vdash A$ iff $\mathcal{L}_S$, $\vec{u} \Vdash A\|S\|$, where S is the identity assignment.

2.5.6. COROLLARY. $\mathcal{L}_S \Vdash \neg\Box S$ *and thus* $\mathcal{L}_S \nVdash S$.

2.5.7. THEOREM. **NMPC** *is a conservative extension of* **NPC** *(i.e. of the intuitionistic predicate calculus).*

2.7. The connectives $\ulcorner$, $\Box$ and theories of constructions.

Now that we have a set of rules of inference for the new connectives we must consider if they are an axiomatization, in the sense of 1.1.1, w.r.t. a theory of constructions.

The most problematic rule is ($\ulcorner$E), which in presence of the rules for v is equivalent to the axiom schema A v $\ulcorner$ A. Now the schema A v $\ulcorner$ A is forced upon us because we are considering satisfaction in Beth structures and, at least from a classical viewpoint, A is valid in the Beth structure B or it is not, from which it follows that $B \Vdash A \vee \lceil A$.

One way to avoid such possible unfaithful results would be to use an intuitionistic metatheory on the Beth structures, however we feel that would be counterproductive since one of the pleasant aspects of Beth (and Kripke) semantics is that one can operate on them using classical techniques to obtain results about intuitionistic theories.

So, for the time being, we are stuck with the axiom schema $A \vee \lceil A$. Is there then any reasonable interpretation of $\lceil$ in terms of constructions?

When the predicates π_A, π were first introduced by Kreisel it was stipulated that they be decidable (the argument being that one always knew if one had a proof or not). Unfortunately too liberal use of that principle quickly leads to a contradiction; nevertheless it is a useful heuristic principle, so we shall temporarily adopt it.

Originally we had stated that a construction e proved $\lceil A$ just in case that $c = c{:}d$ and $\pi(c, \ulcorner \neg\pi_A(d)\urcorner)$. Now if π_A is decidable then we need not give explicitly the construction that proves $\neg\pi_A(d)$. In other words, it suffices that $\pi_{\lceil A}$ satisfy the conditions:

$$\pi_{\lceil A}(d) \quad \text{iff} \quad \neg\pi_A(d).$$

Now the decidability of π_A also gives us:

$$\pi_A(d) \vee \neg\pi_A(d).$$

Hence for all proof constructions d we have that:

$$\pi_A(d) \vee \pi_{\lceil A}(d).$$

From which it follows that for all proof constructions d there corresponds a proof construction d^* such that:

$$\pi_{A \vee \lceil A}(d^*).$$

$\Box A$ was originally introduced as $\neg\lceil A$, so $\pi_{\Box A}(c{:}d)$ iff $\pi(c, \ulcorner\neg\pi_{\lceil A}(d'x)\urcorner)$ iff $\pi(c, \ulcorner\neg\neg\pi_A(d'x)\urcorner)$ iff $\pi(c', \ulcorner\pi_A(d'x)\urcorner)$.

It now is routine to verify that relative to the informal theory of constructions considered above, the rules of inference given in 2.2 form an axiomatization for $\Box$ in the sense of definition 1.1.1.

Section 2.5 then give us that the connective $\Box$ (with the associated rules 2.2) is a intuitionistic connective w.r.t. the Intuitionistic Predicate Calculus.

CONJECTURE. *There is an extension* H *of the* IPC *such that the connective* $\Box$ *(with the rules 2.2) is not a intuitionistic connective* w.r.t.H (see §1).

3. CONCRETE MODELS.

A concrete model for a theory of (proof) constructions is an arithmetically definable model, with the natural numbers as the domain of constructions and decidable predicates P, P_A as interpretations of π, π_A respectively.

In Troelstra [8] a concrete model is given for **IIL**, the intuitionistic implicational logic (with the rules ($\supset$I) and ($\supset$E)). In this section we show how to extend the model to the extension **IIL**$_\square$ of **IIL** obtained by the addition of the unary connective $\square$ and the rules ($\square$I) and ($\square$E) of 2.2.

3.1. The simple model for IIL$_\square$.

3.1.1. SOME PRELIMINARY DEFINITIONS.

(1) PR a formal (intuitionistic) number theory including at least primite recursive arithmetic. Prf(x,y) is the canonical, primitive recursive, proof predicate for PR.

(2) 0^n, the n-th numeral (in the language of PR).

(3) "A", the numeral corresponding to the Gödel number of A.

(4) $\langle A\rangle(x)$ the primitive recursive term such that:

$$PR \vdash \langle A\rangle(0^n) = 0^k,$$

where k is the Gödel number of A_0^x.

(5) Der(x,y) is the canonical, primitive recursive, proof predicate for **IIL**$_\square$ such that PR $\vdash$Der(0^n,"A") iff n is the Gödel number of a derivation in **IIL**$_\square$ of A.

(6) μ is the canonical, primitive recursive term such that if n is the Gödel number of the derivation Π_1 of (A $\supset$ B), m is the Gödel number of the derivation Π_2 of A, and k is the Gödel number of the derivation of B obtained by ($\supset$E) from Π_1 and Π_2, then

$$PR \vdash \mu(0^n,0^m) = 0^k.$$

(7) δ is the canonical, primitive recursive term such that if n is the Gödel number of the derivation Π of $\square$A and k is the Gödel number of the derivation of A obtained from Π by ($\square$E), then PR $\vdash \delta(0^n) = 0^k$.

(8) (j,j_1,j_2) are primitive recursive terms forming an onto pairing system.

3.1.2. DEFINITION OF C_p. To each sentential parameter p of **IIL**$_\square$ we assign a primitive recursive predicate C_p such that PR $\vdash C_p(x) \supset$ Der(x,"p").

3.1.3. DEFINITION OF T_A. To each formula A of **IIL**$_\square$ we assign a (primitive recursive) predicate T as follows:

(i) if A is a sentential parameter p, then $T_A(x) \equiv C_p(x)$,

(ii) if A is not a sentential parameter then, then $T_A(x) \equiv$ Der(x,"A").

3.1.4. DEFINITION OF Pr_A AND f_A. To each formula A of **IIL□** we assign a primitive recursive predicate Pr_A and primitive recursive function f_A such that:

$$PR \vdash T_A(x) \supset Pr_A(f_A x)$$

and

$$PR \vdash Pr_A(y) \supset Der(j_2 y, "A").$$

(i) If A is the sentential parameter p, then

$$Pr_p(x) \equiv T_p(j_2 x) \wedge (j_1 x = j_2 x)$$
$$f_p(x) = j(x,x).$$

(ii) If A = (B ⊃ C) and Pr_B, Pr_C, f_B and f_C have already been defined, then we proceed as follows:

Assume that $T_{(B\supset C)}(0^n)$ and $Pr_B(y)$. Then $Der(0^n, "(B \supset C)")$, $Der(j_2 y, "B")$. Hence $Der(\mu(0^n, j_2 y), "C")$ and thus $T_C(\mu(0^n, j_2 y))$. Therefore $Pr_C(f_C(\mu(0^n, j_2 y)))$. In other words, we have shown that:

$$PR \vdash T_{(B\supset C)}(0^n) \supset [Pr_B(y) \supset Pr_C(t(0^n, y))],$$

where $t(x,y) = f_C(\mu(x, j_2 y))$. From the latter we obtain

$$PR \vdash T_{(B\supset C)}(0^n) \supset \forall y[Pr_B(y) \supset Pr_C(t(0^n, y))]. \qquad (*)$$

Furthermore, the Gödel number of the proof of (*) is primitive recursive in n so that there is a term g such that

$$PR \vdash Prf(g(x), \langle T_{(B\supset C)}(x) \supset \forall y[Pr_B(y) \supset Pr_C(t(x,y))]\rangle(x)).$$

Also, using the fact that $T_{(B\supset C)}(x)$ is primitive recursive, we obtain that there is a term h such that

$$PR \vdash T_{(B\supset C)}(x) \supset Prf(h(x), \langle T_{(B\supset C)}(x)\rangle(x)).$$

Combining the last two observations and using the μ function, we obtain a primitive recursive θ such that:

$$PR \vdash T_{(B\supset C)}(x) \supset Prf(\theta(x), \langle \forall y[Pr_B(y) \supset Pr_C(t(x,y))]\rangle(x)).$$

Thus we define:

$$Pr_{(B\supset C)}(x) \equiv T_{(B\supset C)}(j_2 x) \wedge Prf(j_1 x, \langle \forall y[Pr_B(y) \supset Pr_C(t(x,y))]\rangle(x))$$
$$f_{(B\supset C)}(x) = j(\theta(x), x).$$

(iii) If A = □B and we have at hand Pr_B and f_B then we proceed as follows:

Assume that $T_{\Box B}(0^n)$. Then $T_B(\delta(0^n))$ and hence $Pr_B(f_B(\delta(0^n)))$. Let s be the primitive recursive term such that $s(x,y) = f_B(\delta(x))$, then what we have shown is that

$$PR \vdash T_{\Box B}(0^n) \supset \forall y\, Pr_B(s(0^n, y)).$$

Procceding as in case (ii) we then obtain a primitive recursive φ such that

$$PR \vdash T_{\Box B}(x) \supset Prf(\varphi(x), \langle \forall y\, Pr_B(s(x,y)) \rangle (x)) .$$

Thus we define:

$$Pr_{\Box B}(x) \equiv T_{\Box B}(j_2 x) \supset Prf(j_1 x, \langle \forall y\, pr_B\, (s(x,y)) \rangle (x)) .$$

$$f_{\Box B}(x) = j(\varphi(x), x).$$

3.1.5. DEFINITION OF P AND P_A.

$P(x,y) \equiv Der(j_2 x, y) \wedge Prf(j_1 x, y)$

$P_A(x) \equiv Pr_A(x)$.

3.1.6. THEOREM. *For each* B, C *of* $\mathbf{IIL}_\Box$ *and natural number* n*:*

(1) $PR \vdash P_{(B \supset C)}(0^n) \equiv P(j_1 0^n, \text{"}\forall y [P_B(y) \supset P_C(t_n(y))]\text{"})$.

(2) $PR \vdash P_{\Box B}(0^n) \equiv P(j_1 0^n, \text{"}\forall y P_B(s_n(y))\text{"})$.

(3) *If* $\mathbf{IIL}_\Box \vdash B$ *then for some* m*,* $PR \vdash P_B(0^m)$.

3.2. Extension to $\mathbf{IIL}_{\Box\ulcorner}$.

Since Pr_A is a decidable predicate , so is $\neg Pr_A$ and thus we may trivially extend the concrete model to the extension $\mathbf{IIL}_{\Box\ulcorner}$ of $\mathbf{IIL}_\Box$ obtained by adding the unary connective $\ulcorner$ and the following rules of inference (suggested by 2.2):

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array} \qquad \begin{array}{c}[\ulcorner A]\\ \vdots \\ B\end{array}}{B} \qquad\qquad \frac{\begin{array}{c}[A]\\ \vdots \\ \ulcorner A\end{array}}{\ulcorner A}$$

$$\frac{\ulcorner \ulcorner A}{A} .$$

The extension in the concrete model is as follows:

$$T_{\ulcorner A}(x) \equiv Der(x, \text{"}\ulcorner A\text{"}),$$

$$Pr_{\ulcorner A}(x) \equiv T_{\ulcorner A}(j_2 x) \wedge \neg Pr_A(x) .$$

We still obtain that there is a primitive recursive $f_{\ulcorner A}$ such that:

$$PR \vdash T_{\ulcorner A}(x) \supset Pr_{\ulcorner A}(f_{\ulcorner A} x) ;$$

in fact, $f_{\ulcorner A}(x) = j(x,x)$. For suppose that $T_{\ulcorner A}(x)$. Then clearly $T_{\ulcorner A}(j_2 f_{\ulcorner A} x)$. Now suppose, for reductio ad absurdum, that $Pr_A(f_{\ulcorner A} x)$. That is, suppose that $Pr_A(j(x,x))$. Then $Der(x, \text{"}A\text{"})$, but $Der(x, \text{"}\ulcorner A\text{"})$. Thus $\neg Pr_A(f_{\ulcorner A} x)$. In other words, we have shown that $Pr_{\ulcorner A}(f_{\ulcorner A} x)$.

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DERIVATION-BOUNDED GROUPS

K. Madlener* and F. Otto

ABSTRACT. For some problems which are defined by combinatorial properties good complexity bounds cannot be found because the combinatorial point of view restricts the set of solution algorithms. In this paper we present a phenomenon of this type with the classical word problem for finitely presented groups. A presentation of a group is called E_n-*derivation-bounded* (E_n-d.b.), if a function $k \in E_n$ exists which bounds the derivations of the words defining the unit element. For E_n-d.b. presentations a pure combinatorial E_n-algorithm for solving the word problem exists. It is proved that the property of being E_n-d.b. is an invariant of finite presentations, but that the degree of complexity of the pure combinatorial algorithm may be as far as possible from the degree of complexity of the word problem itself.

The complexity of logical theories and of algorithmic problems in algebraic structures has been object of intensive studies during the last years ([Av], [Av-Madl], [Can], [Can-Gat], [Fer-Rac], [Gat], [Madl]). One interesting aspect in the proofs of good lower and upper bounds is the fact that some of these result were achieved not only by using combinatorial methods but also by using algebraic arguments. Even more, for some problems which are defined by combinatorial properties good complexity bounds cannot be found because the combinatorial point of view restricts the set of solution algorithms.

In this paper we want to present a phenomenon of this type within the classical word problem for finitely presented groups ([M-K-S]).

Let $\Sigma = \{s_1,\dots,s_m\}$ be a finite alphabet, $\bar{\Sigma} = \{\bar{s}_1,\dots,\bar{s}_m\}$ a disjoint copy of Σ ($\bar{s}_i$ is the formal inverse of s_i), $\underline{\Sigma} = \Sigma \cup \bar{\Sigma}$, and $\underline{\Sigma}^*$ the set of words over $\underline{\Sigma}$. For $w = a_1\dots a_n \in \underline{\Sigma}^*$, $a_i \in \underline{\Sigma}$, let be $w^{-1} = \bar{a}_n\dots\bar{a}_1$ ($\bar{\bar{s}} = s$), let $n = |w|$ be the length of w, e the empty word, and $L \subset \underline{\Sigma}^*$.

The group G given by the *presentation* $<\Sigma;L>$ can be viewed as the set of equivalence classes of the Thue system

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$$T = (\Sigma;\ \{w = e \mid w \in L \cup L^{-1} \cup (s\bar{s}, \bar{s}s:\ s \in \Sigma\}\}),$$

where u ~ v if there is a derivation from u to v in T. The set of equivalence classes forms a group with the operations $[u]\cdot[v] = [uv]$ and $[u]^{-1} = [u^{-1}]$, $[e]$ being the unit element.

Σ is the set of *generators*, and L is the set of *defining relators* of this presentation. If Σ is finite, $<\Sigma;L>$ is a *finitely generated* (f.g) *presentation* of G, and G is called f.g.. If L is finite, too, then $<\Sigma;L>$ is a *finite presentation* of G, and G is *finitely presented* (f.p.).

The *word problem* for the presentation $<\Sigma;L>$ of G is the problem of deciding for an arbitrary word $w \in \Sigma^*$ whether w defines the unit element of G or not, i.e. the membership to the set $\{w \in \Sigma^* \mid w \underset{G}{=} e\} = \{w \in \Sigma^* \mid$ there is a derivation from w to e in T$\}$. It is well known that the complexity of the word problem for G is independent of the chosen f.g. presentation for G, and we can speak therefore about the complexity of the word problem *for* G.

We call an algorithm solving the word problem for $<\Sigma;L>$ a *natural algorithm* (n.a.) if for $w \underset{G}{=} e$ it produces a derivation $w = w_o \to \ldots \to w_m = e$ in the Thue system T. Of course the length of a produced derivation is a lower bound for the complexity of a n.a..

From each solution of the word problem for $<\Sigma;L>$ we can define a n.a. simply by generating all derivation in T for the words w with $w \underset{G}{=} e$, in some ordering.

Some questions concerning the n.a. arise. Does the complexity of any n.a. give information about the complexity of the word problem? Of course, it gives an upper bound, but does it give a lower bound in any way, too? Starting with an algorithm which solves the word problem can we produce a n.a. of the same complexity? Given two presentations of the same group, what is the relation between the complexities of natural algorithms in both presentation?

We introduce the concept of *derivation bounded presentations* to formulate these questions more precisely and also to give the answers. Let K be any complexity class of word functions. We will restrict ourselves to the *Grzegorczyk classes* E_n which are well known ([Weih]). A finite presentation $<\Sigma;L>$ is called K-*derivation-bounded* (K-d.b.) if there is a function $k \in K$ such that every word $w \in \Sigma^*$ which defines the unit element of $<\Sigma;L>$ can be derived to e in T, within no more than $|k(w)|$ steps.

For a K-d.b. presentation there is always a standard n.a. for solving the word problem. In order to decide for a word $w \in \Sigma^*$ whether $w \underset{G}{=} e$, just produce all possible derivation in T which start with w, of length bounded by $|k(w)|$, and test whether e has been derived. If $K = E_n$ $(n \geq 3)$ this is an E_n-algorithm. In particular the word problem for an E_n-d.b. finite presentation is decidable.

On the other hand if there is a natural E_n-algorithm solving the word problem for <Σ;L> then <Σ;L> is E_n-d.b..

We will prove the following results.

(a) If a f.g. group has an E_n-d.b. finite presentation for some $n \geq 1$ then every finite presentation of this group is E_n-d.b.. So the standard n.a. is an E_n-algorithm for all finite presentations of this group, for $n \geq 3$.

(b) Every f.g. group G with E_n-decidable word problem ($n \geq 3$), and hence any countable group with E_n-decidable word problem ([Ott]), can be embedded into a f.p. group having an E_n-d.b. presentation. This means that a n.a. of the same complexity can effectively be constructed from an algorithm solving the word problem for G, but in general for a larger group only. The restriction of this n.a. solves the word problem for G, but is general it is not a n.a. for G.

These two facts give the hope that at least for f.p. E_n-d.b. groups with $n \geq 3$ an optimal n.a. exists. But this hope is disappointed by the following fact.

(c) For every $n \geq 4$ there is a f.p. E_n-but not E_{n-1}-d.b. group G having an E_3-decidable word problem. So G has no natural E_{n-1}-algorithm for solving the word problem although there is an E_3-algorithm for solving it. Thus the complexity of any n.a. may be as far as possible from the complexity of the word problem. These results show that combinatorial properties of a Thue system are not sufficient to prove good complexity bounds for the word problem. Similar results can be proved for semigroups.

Since there is a f.p. group with E_3-decidable word problem such that none of its finite presentations allows a natural E_3-algorithm, the following question seems to be natural: is there an infinite "easy" presentation of this group for which a natural E_3-algorithm exists?

Of course one could take all relators of the group as defining relators of a presentation, which then trivially is E_0-d.b., since each derivation is of length 1. But such a presentation is not "easy" because the full complexity of the word problem is contained in the defining relators and so in the presentation. Let an *easy* presentation of a group be one for which the set of defining relators is E_1-decidable. Then we have:

(d) Every f.g. group G with E_n-decidable word problem ($n \geq 3$) has a f.g. presentation with an E_1-decidable set of defining relators which allows a natural E_n-algorithm for solving the word problem.

Similar questions may be posed for finitely axiomatized (f.a.) theories. Are natural decision algorithms for f.a. theories optimal, or are there easily decidable theories for which the optimal proofs in any finite axiomatization are too long?

1. E_n-DERIVATION-BOUNDED GROUPS.

1.1. DEFINITION. Let $G = <\Sigma;L>$ be a group, and let $w \in \Sigma^*$ be such that $w \underset{G}{=} e$.

a) A *derivation from* w is a sequence of words $w = w_o, w_1, \ldots, w_k \equiv e$ from Σ^* such that w_{i+1} is formed by insertion of a word u between any consecutive symbols of w_i, or before w_i, or after w_i, or by deletion of a word u if it forms a block of consecutive symbols of w_i. In both cases u must be a member of $L \cup L^{-1} \cup \{s\bar{s}, \bar{s}s : s \in \Sigma\}$. Here L^{-1} is defined as $\{w^{-1} | w \in L\}$, where $e^{-1} \equiv e$, $(ws)^{-1} \equiv \bar{s}w^{-1}$, $(w\bar{s})^{-1} \equiv sw^{-1}$, and $\equiv$ denotes the identity of the free monoid $\underline{\Sigma}^*$. k is the *length* of this derivation.

b) Let be $n \geqslant 1$, $<\Sigma;L>$ is E_n-*derivation-bounded* (E_n-d.b.) if there is a function $k \in E_n(\Sigma)$ satisfying for all $w \in \Sigma^*$: $w \underset{G}{=} e$ implies that there is a derivation from w of length $\leqslant |k(w)|$, where $|\ |$ denotes the length of a word, i.e. the number of letters. Then k is called an E_n-*bound* for $<\Sigma;L>$.

Of course a natural algorithm for solving the word problem exists for a finite E_n-d.b. presentation.

1.2. LEMMA. *Let* $n \geqslant 1$, *and* $k \in E_n(\Sigma)$ *be such that* $k(e) \equiv e$. *Then there is a monotonous function* $k_1 \in E_n(\underline{\Sigma})$ *satisfying:* $|k_1(u)| + |k_1(v)| \leqslant |k_1(uv)|$ *and* $|k(w)| \leqslant |k_1(w)|$ *for all* $u,v,w \in \underline{\Sigma}^*$.

Proof. **n = 1**. Let $k \in E_1(\Sigma)$ with $k(e) \equiv e$. Then $\exists c \geqslant 1\ \forall w \geqslant \underline{\Sigma}^*\ (|k(w)| \leqslant c|w|)$. Define k_1 by $k_1(w) \equiv w^c$, then $k_1 \in E_1(\underline{\Sigma})$, k_1 is monotonous, and $|k(w)| \leqslant |k_1(w)|$ for every $w \in \Sigma^*$. Let $u,v \in \Sigma^*$ then $|k_1(u)| + |k_1(v)| = c|u| + c|v| = c|uv| = |k_1(uv)|$. **n $\geqslant$ 2**. Let $k \in E_n(\underline{\Sigma})$ with $k(e) \equiv e$. Then there is a monotonous function $k' \in E_n(\underline{\Sigma})$ satisfying $|k(w)| \leqslant |k'(w)|$ and $k'(e) \equiv e$. Define $k_1(e) \equiv e$, $k_1(ws) \equiv vk(k_1(w), k'(ws))$, where the function $vk \in E_1(\underline{\Sigma})$ denotes the concatenation of two words. Then

$$k_1(s_{i_1}^{\mu_1} \ldots s_{i_r}^{\mu_r}) \equiv \bigodot_{j=1}^{r} k'(s_{i_1}^{\mu_1} \ldots s_{i_j}^{\mu_j})$$

and therefore $|k_1(w)| \leqslant |w| \cdot |k'(w)|$; since k' is monotonous and $n \geqslant 2$, $k_1 \in E_n(\underline{\Sigma})$ k_1 is also monotonous, and k_1 a bound for k. Now,

$$|k_1(u)| + |k_1(v)| = \sum_{j=1}^{|u|} |k'(s^j)| + \sum_{j=1}^{|v|} |k'(s^j)|$$
$$\leqslant \sum_{j=1}^{|u|} |k'(s^j)| + \sum_{j=1}^{|v|} |k'(us^j)| = \sum_{j=1}^{|uv|} |k'(s^j)| = |k_1(uv)|.$$

This proves Lemma 1.2.

1.3. REMARK. If k is an E_n-bound it may be assumed that $k(e) \equiv e$. But then because of 1.2 it may be assumed that k is monotonous and satisfies $|k(u)| + |k(v)| \leqslant |k(uv)|$.

Now we give an example of an E_o-d.b. presentation.

1.4. LEMMA. $F = <\Sigma;\emptyset>$, *the free group generated by* Σ, *is* E_0-d.b.

Proof. Define $k(w) \equiv w$, then $k \in E_0(\Sigma)$. Now let $w \in \Sigma^*$ such that $w =_F e$. This means $\gamma_f(w) \equiv e$, where γ_f denotes the function calculating the free reduction. But the execution of the free reduction gives a derivation from w of length $\frac{1}{2}|w|$. So k is an E_0-bound for $<\Sigma;\emptyset>$.

The following three propositions give technics to construct E_n-d.b. presentations of groups from given E_n-d.b. presentations, such that the groups defined by the given presentations are embedded into the groups defined by the constructed presentations.

1.5. PROPOSITION. *Let* $H_1 = <\Sigma_1;L_1>$ *and* $H_2 = <\Sigma_2;L_2>$ *be groups such that* $<\Sigma_1;L_1>$ *and* $<\Sigma_2;L_2>$ *are* E_n-d.b. *for some* $n \geq 2$. *Then*
a) *the presentation* $<\Sigma_1 \cup \Sigma_2;L_1,L_2>$ *of* H_1*H_2 *is* E_n-d.b., *and*
b) *the presentation* $<\Sigma_1 \cup \Sigma_2;L_1,L_2,ab\bar{a}\bar{b}: a \in \Sigma_1, b \in \Sigma_2>$ *of* $H_1 \times H_2$ *is* E_n-d.b.

Proof. Without loss of generality it may be assumed that Σ_1 and Σ_2 are disjoint alphabets. Let $k_1 \in E_n(\Sigma_1)$ and $k_2 \in E_n(\Sigma_2)$ be E_n-bounds for $<\Sigma_1;L_1>$ and $<\Sigma_2;L_2>$, respectively, and let $w \equiv u_0v_0u_1v_1\ldots u_lv_l$, $u_i \in \Sigma_1^*$, $v_i \in \Sigma_1^*$, where u_i and v_i are the syllables of w.

a) $w = e$ in H_1*H_2. Then there is an $i \in \{0,\ldots,l\}$ such that $e \neq u_i =_{H_1} e$ or $e \neq v_i =_{H_2} e$. So within no more than $|k_1(u_i)|$, respectively $|k_2(v_i)|$, steps w can be derived to a word w' containing less syllables than w. Hence there is a derivation from w of length

$$\mu \leq |k_i \circ \Pi_{\Sigma_1}(w)| + |k_2 \circ \Pi_{\Sigma_2}(w)|,$$

where $\Pi_{\Sigma_i} \in E_1(\Sigma_1 \cup \Sigma_2)$ denotes the projection onto Σ_i^*.

Define for $s \in \Sigma_1 \cup \Sigma_2$, $U_s(w) = s^{|w|}$, which is an E_1-function. Let $k(w) = vk(k_1 \circ U_{a_1}(w), k_2 \circ U_{b_1}(w))$ for some $a_1 \in \Sigma_1$, $b_1 \in \Sigma_2$. Then $k \in E_n(\Sigma_1 \cup \Sigma_2)$ with

$$|k(w)| = |k_1 \circ U_{a_1}(w)| + |k_2 \circ U_{b_1}(w)| \geq |k_1 \circ \Pi_{\Sigma_1}(w)| + |k_2 \circ \Pi_{\Sigma_2}(w)|$$

since $|\Pi_{\Sigma_1}(w)| \leq |w| = |U_{a_1}(w)|$ and $|\Pi_{\Sigma_2}(w)| \leq |w| = |U_{b_1}(w)|$. Hence k is an E_n-bound for $<\Sigma_1 \cup \Sigma_2;L_1,L_2>$.

b) $w = e$ in $H_1 \times H_2$. Then $w = \Pi_{\Sigma_1}(w)\,\Pi_{\Sigma_2}(w)$ in $H_1 \times H_2$, $\Pi_{\Sigma_1}(w) =_{H_1} e$, and $\Pi_{\Sigma_2}(w) =_{H_2} e$. There is a derivation from $\Pi_{\Sigma_1}(w)$ of length not exceeding $|k_1 \circ \Pi_{\Sigma_1}(w)|$ in $<\Sigma_1;L_1>$, and there is a derivation from $\Pi_{\Sigma_2}(w)$ of length not exceeding $|k_1 \circ \Pi_{\Sigma_2}(w)|$ in $<\Sigma_2;L_2>$. w can be derived to $\Pi_{\Sigma_1}(w)\Pi_{\Sigma_2}(w)$ by sequences of the form $ba \vdash_1 ab\bar{a}\bar{b}ba \vdash_2 ab$. Therefore $\Pi_{\Sigma_1}(w)\Pi_{\Sigma_2}(w)$ can be derived from w within no more than $3|\Pi_{\Sigma_1}(w)| \cdot |\Pi_{\Sigma_2}(w)|$ steps. Define $VK(w,e) \equiv e$, $VK(w,us) \equiv vk(VK(w,u),w)$ Then

$$VK(w,u) \equiv w^{|u|} \text{ and } VK \in E_2(\Sigma_1 \cup \Sigma_2).$$

Now let $k(w) \equiv vk((VK(w,w))^3, vk(k_1 \circ U_{a_1}(w), k_2 \circ U_{b_1}(w)))$. Since $n \geq 2$, $k \in E_n(\Sigma_1 \cup \Sigma_2)$ and

$$|k(w)| \geq 3|w|^2 + |k_1 \circ \Pi_{\Sigma_1}(w)| + |k_2 \circ \Pi_{\Sigma_2}(w)| \geq 3|\Pi_{\Sigma_1}(w)| \cdot |\Pi_{\Sigma_2}(w)| + |k_1 \circ \Pi_{\Sigma_1}(w)| + |k_2 \circ \Pi_{\Sigma_2}(w)|.$$

Hence k is an E_n-bound for $<\Sigma_1 \cup \Sigma_2; L_1L_2, ab\bar{a}\bar{b}: a \in \Sigma_1, b \in \Sigma_2>$.

1.6. PROPOSITION. *Let* $H = <\Sigma;L>$ *be* E_n-d.b. *for some* $n \geq 3$.
a) *If* $H^* = <H,t;\bar{t}u_itv_i^{-1}: i = 1,\ldots\ell>$ *is an* HNN-*extension of* H *with rewriting functions* ω_u *for* $<u_1,\ldots,u_\ell>_H$ *and* ω_v *for* $<v_1,\ldots,v_\ell>_H$ *bounded by polynomials, then the given presentation of* H^* *is* E_n-d.b.
b) *If* $H^* = <H,t_1,\ldots,t_k;\bar{t}_iu_{ij}t_iv_{ij}^{-1}: j = 1,\ldots,\ell_i,\ i = 1,\ldots,k>$ *is an* HNN-*extension of* H *with rewriting functions* ω_{u_i} *for* $<u_{i1},\ldots,u_{i\ell_i}>_H$ *and* ω_{v_i} *for* $<v_{i1},\ldots,v_{i\ell_i}>_H$, $i = 1,\ldots,k$, *bounded by polynomials, then the given presentation of* H^* *is* E_n-d.b. (See [Lyn-Sch] for the definition of HNN-extension).

Proof. As part (b) is nothing else than a finite iteration of part (a) it suffices to prove part (a).

Define $\varphi: U = <u_1,\ldots,u_\ell>_H \to V = <v_1,\ldots,v_\ell>_H$ as follows: If $w \in \Sigma^* \cap U$, then $w =_H \omega_u(w) \equiv \prod_{j=1}^{\lambda} u_{ij}^{\varepsilon_j}$. Let $\varphi(w) \equiv \prod_{j=1}^{\lambda} v_{ij}^{\varepsilon_j}$. Define $\bar{\varphi}: V \to U$ analogously. Now φ and $\bar{\varphi}$ realize the isomorphisms used for constructing the HNN-extension H^* of H. ω_u and ω_v are bounded by polynomials, and so are φ and $\bar{\varphi}$. Therefore $c \geq 1$ and $d \geq 2$ can be chosen in such a way that for all $w \in \Sigma^*$, $|\omega_u(w)|$, $|\omega_v(w)|$, $|\varphi(w)|$, $|\bar{\varphi}(w)| \leq c|w|^d$ are valid.

Define $f(e) \equiv e$, $f(ws) \equiv f(w)s$, $s \in \Sigma$.

$$f(wt) \equiv \begin{cases} u\varphi(v) & \text{if } f(w) \equiv u\bar{t}v,\ v \in \Sigma^* \cap U \\ f(w)t & \text{otherwise} \end{cases}$$

$$f(w\bar{t}) \equiv \begin{cases} u\bar{\varphi}(v) & \text{if } f(w) \equiv utv,\ v \in \Sigma^* \cap V \\ f(w)\bar{t} & \text{otherwise.} \end{cases}$$

According to [Av-Madl] 3.2, p.94, f is a t-reduction function for H^* satisfying

$$\forall w \in (\Sigma \cup \{t\})^* \ |f(w)| \leq 2^{2^{cd|w|}}.$$

Let $k_H \in E_n(\Sigma)$ be an E_n-bound for $<\Sigma;L>$, and let be $w \in (\Sigma \cup \{t\})^*$ such that $w =_{H^*} e$. Then $f(w) \in \Sigma^*$ and $f(w) =_H e$, f(w) results from w by pinching out $\frac{1}{2}|w|_t$ t-pinches, and subsequently f(w) can be derived to e in $<\Sigma;L>$ within no more than $|k_H \circ f(w)|$ steps. Let

$$w \equiv w_0t^{\mu_1}w_1..t^{\mu_k}w_k,\ w_0,\ldots,w_k \in \Sigma^*,\ \ \mu_1,\ldots,\mu_k \in \{\pm 1\}$$

and

$$t^{\mu_i}w_it^{\mu_{i+1}}$$

be the leftmost t-pinch contained in w.

$$\mu_i = -1 : w \equiv w_0t^{\mu_1}..w_{i-1}\bar{t}w_itw_{i+1}..w_k \xrightarrow{(1)}$$

$$w_0..w_{i-1}\bar{t}w_i(\omega_u(w_i))^{-1}\omega_u(w_i)tw_{i+1}..w_k \xrightarrow{(2)}$$

$$w_o..w_{i-1}\bar{t}\omega_u(w_i)tw_{i+1}..w_k \xrightarrow{(3)}$$
$$w_o..t^{\mu_{i-1}}w_{i-1}\;(w_i)w_{i+1}t^{\mu_{i+2}}..w_k \equiv: \; w'$$

ad (1), $|\omega_u(w_i)|$ trivial relators are inserted.

ad (2), $w_i(\omega_u(w_i))^{-1} \underset{H}{=} e$, and so $w_i(\omega_u(w_i))^{-1}$ can be derived to e in $\langle\Sigma;L\rangle$ within at most $|k_H(w_i(\omega_u(w_i))^{-1}|$ steps.

ad (3), $|\omega_u(w_i)|_u$ = the number of generators $u_1,\ldots,u_\ell$ in $\omega_u(w_i)$. Now (3) can be realized by $|\omega_u(w_i)|_u$ steps of the following kind:

(a) Insertion of $t\bar{t}$.

(b) Insertion of $v_j^{-1}v_j$ by using trivial relators.

(c) Deletion of $\bar{t}u_jtv_j^{-1}$.

Hence within at most

$$m_1 = |\omega_u(w_i)|+|k_H(w_i(\omega_u(w_i))^{-1})|+|\omega_u(w_i)|_u\cdot(2+\max_{j=1,..,\ell}|v_j|)$$

steps the first t-pinch of w can be pinched out.

$$m_1 \leqslant |\omega_u(w_i)|\cdot\{3+\max_{j=1,..,\ell}|v_j|\}+k_H(w_i(\;_u(w_i))^{-1})| =: \; m_2$$

since

$$|\omega_u(w_i)|_u \leqslant |\omega_u(w_i)|.$$

$$\mu_i = 1 : w \equiv w_o t^{\mu_1}..w_{i-1}tw_i\bar{t}w_{i+1}..w_k \xrightarrow{(1)}$$
$$w_o..w_{i-1}tw_i(\omega_v(w_i))^{-1}\omega_v(w_i)\bar{t}w_{i+1}..w_k \xrightarrow{(2)}$$
$$w_o..w_{i-1}t\omega_v(w_i)\bar{t}w_{i+1}..w_k \xrightarrow{(3)}$$
$$w_o..t^{\mu_{i-1}}w_{i-1}\,\bar{}(w_i)w_{i+1}t^{\mu_{i+2}}...w_k \equiv: \; w'$$

ad (1), $|\omega_v(w_i)|$ trivial relators are inserted.

ad (2), $w_i(\omega_v(w_i))^{-1} \underset{H}{=} e$, and so $w_i(\omega_v(w_i))^{-1}$ can be derived to e in $\langle\Sigma;L\rangle$ within no more than $|k_H(w_i(\omega_v(w_i))^{-1})|$ steps.

ad (3), by $|\omega_v(w_i)|_v$ steps of the following kind (3) can be realized:

(a) Insertion of $\bar{t}u_j^{-1}t\bar{t}u_jt$ by using trivial relators.

(b) Deletion of $v_j\bar{t}u_j^{-1}t$ $(\equiv(\bar{t}u_jtv_j^{-1})^{-1})$ and of $t\bar{t}$.

In this way u_jt is derived from tv_j. Hence within at most

$$m_1^+ = |\omega_v(w_i)|+|k_H(w_i(\omega_v(w_i))^{-1})|+|\omega_v(w_i)|_v\cdot(4+\max_{j=1,..,\ell}|u_j|)$$

steps the first t-pinch of w can be pinched out.

$$m_1^+ \leqslant |\omega_v(w_i)|\cdot\{5+\max_{j=1,...,\ell}|u_j|\}+|k_H(w_i(\omega_i(w_i))^{-1})| =: \; m_2^+$$

since

$$|\omega_v(w_i)|_v \leqslant |\omega_v(w_i)|.$$

Let $A = \max_{j=1,\dots,\ell}\{|u_j|,|v_j|\}$, and $a \in \Sigma$. Now the first t-pinch of w can be pinched out in at most $c|w|^d\cdot\{5+A\}+|k_H(a^{(c+1)|w|^d})|$ steps. Let w_i' be the word formed by pinching out the first i t-pinches of w.

ASSERTION. *Let* $i \in \{1,2,\dots,\frac{1}{2}|w|_t\}$. *Then* $|w_i'| \leqslant (c+1)^{d^{2i-1}}|w|^{d^i}$, *and* w_i' *can be derived from* w_{i-1}' *within* m_i' *steps where* m_i' *satisfies*

$$m_i' \leqslant (5+A)\cdot(c+1)^{d^{2i-1}}|w|^{d^i}+|k_H(a^{(c+1)^{d^{2i-1}}|w|^{d^i}})|.$$

Proof. By induction on i.

$i = 1$: $w_1' \equiv w'$, then $|w_1'| = |w|-|w_i|-2+|\varphi^{\mu}(w_i)|$

$$\leqslant |w|+c|w_i|^d \leqslant |w|+c|w|^d \leqslant (c+1)|w|^d \leqslant (c+1)^d|w|^d.$$

$$m_1' \leqslant c|w|^d(5+A)+|k_H(a^{(c+1)|w|^d})| \leqslant (5+A)(c+1)^d|w|^d+ k_H(a^{(c+1)^d|w|^d})|.$$

$i \mapsto i+1$: w_{i+1}' is formed from w_i' by pinching out a t-pinch, then

$$\begin{aligned}|w_{i+1}'| &\leqslant |w_i'|+c|w_i'|^d \leqslant (c+1)|w_i'|^d\\ &< (c+1)\cdot\{(c+1)^{d^{2i-1}}|w|^{d^i}\}^d = (c+1)^{d^{2i+1}}|w|^{d^{i+1}}\\ &\leqslant (c+1)^{d^{2i+1}}|w|^{d^{i+1}},\end{aligned}$$

and

$$\begin{aligned}m_{i+1}' &< (5+A)c|w'|^d+|k_H(a^{(c+1)|w_i'|^d})|\\ &< (5+A)c\cdot\{(c+1)^{d^{2i-1}}|w|^{d^i}\}^d+|k_H(a^{(c+1)\cdot\{(c+1)^{d^{2i-1}}|w|^{d^i}\}^d})|\\ &= (5+A)c(c+1)^{d^{2i}}|w|^{d^{i+1}}+|k_H(a^{(c+1)^{d^{2i+1}}|w|^{d^{i+1}}})|\\ &\leqslant (5+A)(c+1)^{d^{2i+1}}|w|^{d^{i+1}}+|k_H(a^{(c+1)^{d^{2i+1}}|w|^{d^{i+1}}})|\end{aligned}$$

Let w^+ be the word formed by pinching out all t-pinches of w. Then $w^+ \equiv w'_{\frac{1}{2}|w|_t}$, and hence

$$|w^+| \leqslant (c+1)^{d^{|w|_t-1}}|w|^{d^{(\frac{1}{2})|w|_t}} \leqslant \{(c+1)|w|\}^{d^{|w|}}$$

The derivation from w to w^+ can be performed within

$$\begin{aligned}m^+ = \sum_{i=1}^{\frac{1}{2}|w|_t} m_i' &\leqslant \sum_{i=1}^{\frac{1}{2}|w|_t}\{(5+A)(c+1)^{d^{2i-1}}|w|^{d^i}+ k_H(a^{(c+1)^{d^{2i-1}}|w|^{d^i}})|\}\\ &\leqslant \tfrac{1}{2}|w|_t\cdot\{5+A)(c+1)^{d^{|w|}}w^{d^{|w|}}+|k_H(a^{((c+1)|w|)^{d^{|w|}}})|\}\end{aligned}$$

steps. At last, w^+ is derived to e in $\langle\Sigma;L\rangle$ within at most

$$|k_H(w^+)| \leqslant |k_H(a^{((c+1)|w|)^{d^{|w|}}})| \text{ steps.}$$

Hence there is a derivation from w in the given presentation of H^* of length not exceeding

$$m_w = m^++|k_H(w^+)| \leqslant |w|\{(5+A)((c+a)|w|)^{d^{|w|}}+|k_H(a^{((c+1)|w|)^{d^{|w|}}})|\}.$$

Define $d_1(w) \equiv a^{d^{|w|}}$, $d_2(w) \equiv VK(w,a^{c+1})$, and

$$d_3(w,e) \equiv a,\ d_3(w,us) \equiv VK(d_3(w,u),w).$$

Then $d_1 \in E_3(\Sigma \cup \{t\})$, $d_2 \in E_2(\Sigma \cup \{t\})$, $d_3 \in E_3(\Sigma \cup \{t\})$, $d_2(w) = w^{c+1}$, $|d_2(w)| \equiv (c+1)|w|$, and $d_3(w,u) \equiv a^{|w||u|}$. $d_4(w) \equiv d_3(d_2(w), d_1(w))$ is a function from $E_3(\Sigma \cup \{t\})$ satisfying

$$d_4(w) \equiv a^{((c+1)|w|)d^{|w|}}$$

and $k(w) \equiv VK(vk(VK(d_4(w), a^{5+A}), k_H \circ d_4(w)), w)$ is from $E_n(\Sigma \cup \{t\})$ satisfying:

$$|k(w)| = |w|\{(5+A)((c+1)|w|)d^{|w|} + k_H(a^{((c+1)|w|)d^{|w|}})|\} .$$

Hence k is an E_n-bound for the given presentation of H^*. Thus this presentation is E_n-d.b.

1.7. PROPOSITION. *The* $H = \langle\Sigma;L\rangle$ *be* E_n-b.d. *for some* $n \geq 2$. *If* $H^* = \langle H,t;tu_it u_i^{-1} : i = 1, \ldots, \ell\rangle$ *is an* HNN-*extension of* H *with the identity as isomorphism and with a rewriting function* $\omega \in E_n(\Sigma)$ *for* $\langle u_1,\ldots,u_\ell\rangle_H$, *then the given presentation of* H^* *is* E_n-d.b.

Proof. Define $f(e) \equiv e$, $f(ws) \equiv f(w)s$, $s \in \Sigma$,

$$f(wt^{\mu}) \equiv \begin{cases} uv & \text{if } f(w) \equiv ut^{-\mu}v, \quad v \in \Sigma^* \cap \langle u_1,\ldots,u_\ell\rangle_H \\ f(w)t^{\mu} & \text{otherwise} \end{cases}$$

f is a t-reduction fuction for H^* satisfying $|f(w)| \leq |w|$. Let $w \in (\Sigma \cup \{t\})^*$ with $w =_{H^*} e$. Then $f(w) \in \Sigma^*$ and $f(w) =_H e$. Therefore w can be derived to e by first pinching out all the t-pinches of w and thereafter deriving the resulting word to e in $\langle\Sigma;L\rangle$. $\omega(e) \equiv e$ may be assumed. Then according to Lemma 1.2 there is a monotonous function $\omega_2 \in E_n(\Sigma)$ satisfying $|\omega(w)| \leq |\omega_2(w)|$ and $|\omega_2(u)| + |\omega_2(v)| \leq |\omega_2(uv)|$ for every $w,u,v \in \Sigma^*$.

Let $k_H \in E_n(\Sigma)$ be an E_n-bound for $\langle\Sigma;L\rangle$, and let $w \in w_o t^{\mu_1} .. t^{\mu_r} w_r$, $w_o,..,w_r \in \Sigma^*$, $\mu_1,\ldots,\mu_r \in \{\pm 1\}$, with $w =_{H^*} e$ contain the t-pinch $t^{\mu_i} w_i t^{\mu_{i+1}}$. This t-pinch can be pinched out by the following sequence of operations:

$$\begin{aligned} w &\equiv w_o t^{\mu_1} w_1 .. w_{i-1} t^{\mu_i} w_i t^{\mu_{i+1}} w_{i+1} .. w_r \xrightarrow{(1)} \\ & w_o .. w_{i-1} t^{\mu_i} w_i (\omega(w_i))^{-1} \omega(w_i) t^{\mu_{i+1}} w_{i+1} .. w_r \xrightarrow{(2)} \\ & w_o .. w_{i-1} t^{\mu_i} \omega(w_i) t^{\mu_{i+1}} w_{i+1} .. w_r \xrightarrow{(3)} \\ & w_o .. t^{\mu_{i-1}} w_{i-1} \omega(w_i) w_{i+1} t^{\mu_{i+2}} .. w_r \xrightarrow{(4)} \\ & w_o .. t^{\mu_{i-1}} w_{i-1} w_i (\omega(w_i))^{-1} \omega(w_i) w_{i+1} t^{\mu_{i+2}} .. w_r \xrightarrow{(5)} \\ & w_o .. t^{\mu_{i-1}} w_{i-1} w_i w_{i+1} t^{\mu_{i+2}} .. w_r \equiv : w' . \end{aligned}$$

ad (1), $|\omega(w_i)|$ trivial relators are inserted.

ad (2), $w_i(\omega(w_i))^{-1} \underset{H}{=} e$, and hence $w_i(\omega(w_i))^{-1}$ can be derived to e in $\langle\Sigma;L\rangle$ within at most $|k_H(w_i(\omega(w_i))^{-1})|$ steps.

ad (3), $|\omega(w_i)|_u$ steps of the following form:

$\mu_i = -1$: (a) Insertion of $tu_j^{-1}u_j\bar{t}$ by using trivial relators

(b) Deletion of $\bar{t}u_jtu_j^{-1}$

In this way $u_j\bar{t}$ is derived from $\bar{t}u_j$.

$\mu_i = 1$: (a) Insertion of $\bar{t}u_j^{-1}t\bar{t}u_jt$ by using trivial relators

(b) Deletion of $u_j\bar{t}u_j^{-1}t$ $(\equiv(\bar{t}u_jtu_j^{-1})^{-1})$ and of $t\bar{t}$:

$$tu_j \to tu_j\bar{t}u_j^{-1}t\bar{t}u_jt \to t\bar{t}u_jt \to u_jt.$$

ad (4), $w_i(\omega(w_i))^{-1}$ can be derived from e by inverting the derivation of (2).

ad (5), $|\omega(w_i)|$ trivial relators are deleted.

Hence the t-pinch of w can be pinched out within

$$m' \leqslant |\omega(w_i)|+|k_H(w_i(\omega(w_i))^{-1})|+|\omega(w_i)|_u\cdot(4+\max_{j=1,\ldots,\ell}|u_j|)$$
$$+\ |k_H(w_i(\omega(w_i))^{-1}|+|\omega(w_i)| \leqslant |\omega(w_i)|\cdot(6+\max_{j=1,\ldots,\ell}|u_j|)+2|k_H(w_i(\omega(w_i))^{-1})|$$

steps, since $|\omega(w_i)|_u \leqslant |\omega(w_i)|$. Let $A = \max_{j=1,\ldots,\ell}|u_j|$. Then

$$m' \leqslant |\omega(w_i)|(6+A)+2|k_H(w_i(\ (w_i))^{-1})| \leqslant 2|k_H(w_i\omega_2(w_i))|+(6+A)|\omega_2(w_i)|,$$

since $|\omega(w_i)| = |\omega(w_i)^{-1}| \leqslant |\omega_2(w_i)|$, and k_H being monotonous

$$\leqslant 2|k_H(a^{|w|+|\omega_2(a^{|w|})|})+(6+A)|\omega_2(a^{|w|})\ ,$$

since $|w_i| \leqslant |w|$, and k_H and ω_2 being monotonous.

$\frac{1}{2}|w|_t$ t-pinches must be pinched out. Of course $|w'| \leqslant |w|$. Hence w can be derived to f(w) in the given presentation of H^* within m^* steps where m^* satisfies:

$$m^* \leqslant \tfrac{1}{2}|w|_t\cdot(2|k_H(a^{|w|+|\omega_2(a^{|w|})|})|+\ (6+A)|\omega_2(a^{|w|})|\}$$
$$\leqslant |w|\{|k_H(a^{|w|+|\omega_2(a^{|w|})|})|+(3+A)|\omega_2(a^{|w|})|\}.$$

f(w) is derived to e in $\langle\Sigma;L\rangle$ within at most $\tilde{m} \leqslant |k_H\circ f(w)| \leqslant |k_H(a^{|w|})|$ steps, as $|f(w)| \leqslant |w|$ and k_H being monotonous. Hence w can be derived to e in the given presentation of H^* within m steps where m satisfies:

$$m = m^*+\tilde{m} \leqslant |w|\{|k_H(a^{|w|+|\omega_2(a^{|w|})|})|+(3+A)|\omega_2(a^{|w|})|\}+|k_H(a^{|w|})|.$$

Define

$$k(w) \equiv vk(VK(vk(k_H\circ vk(U_a(w),U_a\circ\omega_2\circ U_a(w)),\ VK(\omega_2\circ U_a(w),a^{3+A})),w),k_H\circ U_a(w)).$$

Then $k \in E_n(\Sigma \cup \{\underline{t}\})$ and k satisfies:

$$|k(w)| = |w|\{|k_H(a^{|w|+|\omega_2(a^{|w|})|})|+(3+A)|\omega_2(a^{|w|})|\}+\ k_H(a^{|w|})|.$$

Therefore k is an E_n-bound for the given presentation of H^*, which is E_n-d.b. herewith.

2. AN EMBEDDING INTO DERIVATION-BOUNDED GROUPS.

The proposition in Sec. 1 give examples of embeddings of d.b. groups into d.b. groups. But now the question arises whether a group possessing no E_n-d.b. presentation can be cambedded into a E_n-d.b. group. The answer to this question is given by the next theorem and its corollary.

2.1. THEOREM. *Let* $G = \langle\Sigma;L\rangle$ *be f.g. with* $WP_G \in E_n(\Sigma)$, *i.e. the word problem for the given presentation* $\langle\Sigma;L\rangle$ *of* G *is* E_n*-decidable, for some* $n \geq 3$. *Then there is a finite* E_n-d.b. *presentation* $\langle\Delta;M\rangle$ *of a group* H *such that* G *can be embedded in* H.

Proof. Starting with $\langle\Sigma;L\rangle$ we construct $\langle\Delta;M\rangle$ in a few number of steps. Let $\hat{L} = \{w \in \Sigma^* | w \underset{G}{=} e\}$, and $\hat{G} = \langle\Sigma;\hat{L}\rangle$. Then $\hat{G}$ is f.g., $WP_{\hat{G}} \in E_n(\Sigma)$, $\hat{G} \cong G$, via the identity mapping, and for each word $w \in \Sigma^*$ with $w \underset{\hat{G}}{=} e$ there is a derivation of length 1 in $\langle\Sigma;L\rangle$, because $w = e$ in $\hat{G}$ implies $w \in \hat{L}$.

Let $\tilde{\Sigma} = \{\tilde{s} | s \in \Sigma\}$ be a copy of Σ satisfying $\tilde{\Sigma} \cap \Sigma = \emptyset$, $\Sigma_o = \Sigma \cup \tilde{\Sigma}$, and let $\varphi: \Sigma^* \to \Sigma_o^*$ be defined by $\varphi(s) \equiv s$, $\varphi(\bar{s}) \equiv \tilde{s}$. Let $L_o = \{w \in \Sigma_o^* | \exists u \in \hat{L}: \varphi(u) \equiv w\}$ and $G_o = \langle\Sigma_o;L_o\rangle$, then G_o is f.g., $WP_{G_o} \in E_n(\Sigma_o)$, $G_o \cong G$ via φ, the defining relators of G_o do contain only positive letters, and for every word $w \in \Sigma_o^*$ with $w \underset{G_o}{=} e$ there is a derivation of length $\leq 2|w|+1$ in $\langle\Sigma_o;L_o\rangle$, because at first all letters of the form $\bar{s}$ $(s \in \Sigma)$ contained in w must be substituted by $\tilde{s}$ by means of the derivation $\bar{s} \to \bar{s}s\tilde{s} \to \tilde{s}$, then all the letters of the form $\bar{\tilde{s}}$ $(\tilde{s} \in \tilde{\Sigma})$ contained in w must be substituted by s by means of the derivation $\bar{\tilde{s}} \to \bar{\tilde{s}}\tilde{s}s \to s$, as $s\tilde{s}, \tilde{s}s \in L_o$, and at last the produced word $w' \in \Sigma_o^*$ can be deleted in one step.

L_o is an E_n-decidable subset of Σ_o^*. Hence there is a Turing Machine $T = (\Sigma_o, Q_T, q_o, \beta)$, where Q_T is a finite set of states, $q_o \in Q_T$ is the initial state of T, and β is the transition function of T, and a function $g \in E_n(\Sigma_o)$ such that T computes the characteristic function of the set L_o and g is a time bound for T.

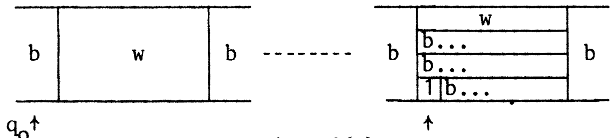
Now T can be modified to get a Turing Machine $\tilde{T} = (\tilde{\Sigma}_o, Q_{\tilde{T}}, q_o, \tilde{\beta})$, where $\tilde{\Sigma}_o$ is a finite alphabet including Σ_o, satisfying the following two conditions:

(1) There is a special state $q_a \in Q_{\tilde{T}}$ called the *accepting state* such that starting at $q_o w$, $\tilde{T}$ eventually reaches the state q_a if and only if $w \in L_o$.

(2) There is a function $k_T \in E_n \in (\tilde{\Sigma}_o \cup Q_{\tilde{T}})$ satisfying for all $u,v \in \Sigma_o^*$, $q_j \in Q_{\tilde{T}}$: starting at the configuration uq_jv, $\tilde{T}$ halts within $|k_T(ug_jv)|$ steps if $\tilde{T}$ reaches the accepting states q_a afeter all.

Especially it is E_n-decidable whether starting at uq_jv, $\tilde{T}$ eventually reaches the state q_a. For that $\tilde{T}$ works as follows:

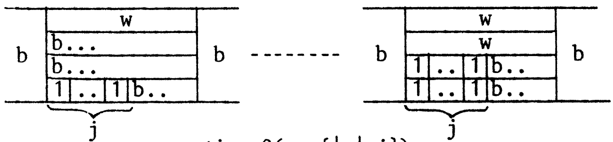
Start:



time: O(n)

The tape is divided into four tracks. The input is copied onto track N° 1. Below the leftmost letter of the copied input a "1" is printed onto track N° 4.

Loop:



time: $O(\max\{|w|, j\})$

Track N° 1 is copied onto track N° 2, and track N° 4 is copied onto track N° 3. If a letter $a \in \tilde{\Sigma}_o - \Sigma_o$ is contained in w, or if a letter $a \neq 1$ is contained in the inscription of track N° 4, $\tilde{T}$ halts at the state q_-, a nonaccepting state. Otherwise $\tilde{T}$ simulates T starting at $q_o w$ on its track N° 2. Ahead of each step of this simulation a "1" is erased from track N° 3. If T halts and accepts, then $\tilde{T}$ halts at state q_a. If T halts without accepting, then $\tilde{T}$ halts at state q_-. If the whole inscription of track N° 3 is erased before reaching the end of the computation of T, then $\tilde{T}$ breaks off the simulation of T, cleans track N° 2, adds a "1" to the inscription of track N° 4, and starts the loop again. For carrying out this computation, $\tilde{T}$ needs two additional tracks as scratch paper to note the direction of the beginning of the inscription of track N° 1, and with it the beginning of the inscriptions of tracks N° 3 and N° 4, and the direction in which the actual cell of track N° 2 is situated in relation to the position of the head of $\tilde{T}$.

Now the following is satisfied: for $w \in L_o$ starting at $q_o w$, T halts and accepts. Hence starting at q_o, w $\tilde{T}$ reaches the state q_a. On the other hand if starting at $q_o w$, $\tilde{T}$ reaches the state q_a, then $w \in \Sigma_o^*$, and T halts and accepts starting at $q_o w$, i.e. $w \in L_o$.

With the input $w \in \Sigma_o^*$, T does not carry out more than $|g(w)|$ steps, $\tilde{T}$ simulates T step by step. Each step of this simulation takes $\tilde{T}$ at most $O(|g(w)|)$ steps, for $\tilde{T}$ must erase a "1" from track N° 3. Altogether, $\tilde{T}$ simulates $\sum_{i=1}^{|g(w)|} i = O(|g(w)|^2)$ steps of T.

Hence $\tilde{T}$ needs $O(|g(w)|^3)$ steps to carry through the simulation of T with input w. If $\tilde{T}$ is started at an arbitrary configuration $uq_j v$, it simulates T starting at a configuration depending on $uq_j v$ on track N° 2 for as many steps as the inscription of track N° 3 tells. This takes no more than $O(|uv|^2)$ steps. Afterwards $\tilde{T}$ simulates T, starting at a well defined initial configuration $q_o w$, where w is the inscription of track № 1 of the configuration uq_jv, if $w \in \Sigma_o^*$. Otherwise $\tilde{T}$ halts at state q_-. As $n \geq 3$ there is a function $k_{\tilde{T}} \in E_n(\tilde{\Sigma}_o \cup Q_{\tilde{T}})$ satisfying condition (2).

According to [Av-Madl], p.89, a semigroup $\Delta_{\tilde{T}} = (S \cup Q;\pi)$ where $S = \tilde{\Sigma}_o \cup \{h\}$, $Q = Q_{\tilde{T}} \cup \{q\}$, and

$$\pi = \{F_i q_{i_1} G_i = H_i q_{i_2} K_i \mid q_{i_1}, q_{i_2} \in Q,\ F_i, G_i, H_i, K_i \in S^*, \quad i = 1,\dots,N\}$$

can be constructed from $\tilde{T}$, satisfying:

(3) $\forall w \in \tilde{\Sigma}_o^* \ (hg_o wh \underset{\Delta_{\tilde{T}}}{=} q \iff w \in L_o)$.

(4) If $uq_jv = q$, then there is a derivation from uq_jv to q in $\Delta_{\tilde{T}}$ of length not exceeding $2|k_T(uq_jv)|+|uq_jv|$, because it may be assumed that $k_{\tilde{T}}$ is non-decreasing ([Weih]).

Let $u,v \in S^*$ with $uq_jv \underset{\Delta_T}{=} q$. Then $uq_jv \equiv q$, or $u \equiv hu'$, $v \equiv v'h$, $q_j \not\equiv q$, and starting at $u'q_jv'$, $\tilde{T}$ reaches the accepting state q_a. But for doing so, $\tilde{T}$ does not need more than $|k_T(u'q_jv')|$ steps. Hence $uq_jv \equiv hu'q_jv'h$ can be derived to $h\tilde{u}q_a\tilde{v}h$ in $\Delta_{\tilde{T}}$ within at most $|k_{\tilde{T}}(u'q_jv')|$ steps. Of course $|\tilde{u}\tilde{v}| < |u'v'|+$ $|k_{\tilde{T}}(u'q_jv')|$, since $\tilde{T}$ can increase the length of its tape inscription by at most one per step. It takes $\Delta_{\tilde{T}}$ $|\tilde{u}\tilde{v}|$ steps to derive hq_ah from $h\tilde{u}q_a\tilde{v}h$ by erasing $\tilde{u}\tilde{v}$; hq_ah can be derived to q within one step. Hence $\Delta_{\tilde{T}}$ can derive q from uq_jv within at most

$$2|k_{\tilde{T}}(uq_jv)|+|u'v'|+1 \leq 2|k_{\tilde{T}}(uq_jv)|+|uq_jv| \quad \text{steps.}$$

Define $k_\Delta(w) \equiv vk(vk(k_{\tilde{T}}(w),k_{\tilde{T}}(w)),w)$. Then $k_\Delta \in E_n(S \cup Q)$, and k_Δ bounds the derivation of words $w \in (S \cup Q)^*$ with $w \underset{\Delta_{\tilde{T}}}{=} q$ to q in $\Delta_{\tilde{T}}$.

Now a Britton tower of groups is constructed:

$D_o = \langle x;\emptyset\rangle$

$D_1 = \langle x,S;\bar{s}xs = x^2 (s \in S)\rangle$

$D_2 = \langle D_1,Q;\emptyset\rangle$

$D_3 = \langle D_2,r;\bar{r}_i\bar{F}_i q_{i_1} G_i r_i = \bar{H}_i q_{i_2} k_i, \bar{r}_i s x r_i = s\bar{x}(s \in S,\ i = 1,\dots,N)\rangle$ where $\overline{(s_{i_1}..s_{i_m})} \equiv \bar{s}_{i_1}..\bar{s}_{i_m}$

$D_4 = \langle D_3,t;\bar{t}xt = x,\bar{t}rt = r(r \in R)\rangle$

$D_5 = \langle D_4,k;\bar{k}xk = x,\bar{k}rk = r(r \in R),\bar{k}\bar{q}tqk = \bar{q}tq\rangle$

$\cong \langle D_5,t_o,k_o;t_o = (\bar{h}q_o)^{-1}t(\bar{h}g_o),k_o = hk\bar{h}\rangle$

$\cong \langle D_3,t_o,k_o;(\bar{h}q_o\bar{t}_o\bar{q}_oh)a(\bar{h}q_ot_o\bar{q}_oh) = a,\ (\bar{h}\bar{k}_oh)a(\bar{h}k_oh) = a(a \ \ \{x\} \cup R),$
$(\bar{h}\bar{k}_oh)\bar{g}\bar{h}q_ot_o\bar{q}_o\bar{h}g(\bar{h}k_oh) = \bar{q}\bar{h}q_ot_o\bar{q}_ohq\rangle =: \langle S_6;M_6\rangle = D_6$

$\Sigma_o' = \{s' \mid s \in \Sigma_o\}$, $': \Sigma_o^* \to \Sigma_o'^*$ is defined by $(s^\mu)' \equiv s'^\mu$, $(\bar{s}^\mu)' \equiv \bar{s}'^\mu$, $L_o' = \{w \in \Sigma_o'^* \mid \exists u \in L_o: u' \equiv w\}$, and $G' = \langle \Sigma_o';L_o'\rangle$.

Then $G' = G_o$ via $'$.

$H_o \cong D_6 \times G' = \langle S_6 \cup \Sigma_o';M_6,L_o',as' = s'a(a \in S_6,s' \in \Sigma_o')\rangle$

$H_1 = \langle H_o,d;\bar{d}ss'd = s,\bar{d}\bar{k}_osk_od = \bar{k}_osk_o(s \in \Sigma_o),\bar{d}t_od = t_o,\bar{d}\bar{k}_ot_ok_od = \bar{k}_ot_ok_o\rangle$

$H_2 = \langle H_1, z; \bar{z}sz = s, \bar{z}\bar{k}_o s k_o z = \bar{k}_o s k_o (s \in \Sigma_o), \bar{z}t_o z = t_o d, \bar{z}\bar{k}_o t_o k_o z = \bar{k}_o t_o k_o d\rangle$

$\Delta = S_6 \cup \Sigma_o' \cup \{d,z\}$.

Let M be the set of defining relators of the given presentation of H_2 and $M = \tilde{M} - (L_o' - \{s'\tilde{s}', \tilde{s}'s' \mid s' \in \Sigma'\})$ where $\Sigma' = (\Sigma)' \subseteq \Sigma_o'$

REMARK. $\forall s \in \Sigma(s\bar{s}, \bar{s}s \in \hat{L}) \Rightarrow \forall s \in \Sigma(s\tilde{s}, \tilde{s}s \in L_o) \Rightarrow \forall s' \in \Sigma'(s'\tilde{s}', \tilde{s}'s' \in L_o')$

Let $H = \langle \Delta; M\rangle$. In [Av-Madl] Satz 1.1, p.184, Avenhaus and Madlener prove: H is f.p., $WP_H \in E_n(\underline{\Delta})$, and G embeds in H. It remains to show that $\langle \Sigma; M\rangle$ is E_n-d.b.

According to [Ott]§15,pp.156-173, the following assertions are valid:

D_o is E_o-d.b.

D_1, D_2, D_3, and D_4 are E_3-d.b.

D_5 is E_n-d.b.

For proving these assertions propositions 1.4 until 1.7 are used. At the last part one has to construct a rewriting function $\omega \in E_n(\{\underline{x,t}\} \cup \underline{S} \cup \underline{Q} \cup \underline{R})$, for $\langle x, \bar{q}tq, R\rangle_{D_4}$. After that, proposition 1.7 can be applied. Analogously there is an E_n-rewriting function for $\langle hx\bar{h}, hr\bar{h}, h\bar{q}\bar{h}q_o t_o \bar{q}_o hq\bar{h}\rangle$ in $D_4' = \langle D_3, t_o; (\bar{h}q_o\bar{t}_o\bar{q}_o h)\cdot a(\bar{h}q_o t_o \bar{q}_o h) = a \ (a \in \{x\} \cup R)\rangle$ where D_4' is E_3-d.b. just like D_4. Hence D_6 is E_n-d.b., too.

(5) $\langle \Sigma; M\rangle$ is E_n-d.b.

Proof. a) Let $w' \in \tilde{L}_o' = L_o' - \{s'\tilde{s}', \tilde{s}'s' \mid s' \in \Sigma'\}$, and $w \equiv (w')^{('-1)} \in L_o \subseteq \Sigma_o^*$. Then $hq_o wh \underset{\Delta_{\tilde{T}}}{=} q$, and hence $k_o(w^{-1}t_o w) \underset{D_6}{=} (w^{-1}t_o w)k_o$, due to [Av-Madl], p.185. $w' \in \tilde{L}_o' \subseteq L_o' \Rightarrow w' \underset{H_2}{=} e \Rightarrow w' \underset{H}{=} e$, since $H \cong H_2$ via the identity. Now w' can be derived to e in $\langle \Delta; M\rangle$ as follows:

$$w' \xrightarrow{(1)} w^{-1}\bar{t}_o ww^{-1}t_o ww'd\bar{d} \xrightarrow{(2)} w^{-1}\bar{t}_o ww^{-1}t_o dw\bar{d} \xrightarrow{(3)}$$
$$w^{-1}\bar{t}_o w\bar{z}w^{-1}t_o wz\bar{d} \xrightarrow{(4)} w^{-1}\bar{t}_o w\bar{z}\bar{k}_o w^{-1}t_o wk_o z\bar{d} \xrightarrow{(5)}$$
$$w^{-1}\bar{t}_o w\bar{k}_o w^{-1}t_o wk_o d\bar{d} \xrightarrow{(6)} w^{-1}\bar{t}_o ww^{-1}t_o wd\bar{d} \xrightarrow{(7)} e.$$

ad (1), $2|w|+2$ trivial relators are inserted.

ad (2), by using the commutation relators of H_o w and w' can be mixed within at most $3|w|^2$ steps:

$$ww' \to s_{i_1}s_{i_1}' s_{i_2}s_{i_2}' .. s_{i_\lambda}s_{i_\lambda}' .$$

After that:

$$s_{i_1}s_{i_1}'..s_{i_\lambda}s_{i_\lambda}'d \to d\bar{d}s_{i_1}s_{i_1}'d\bar{d}s_{i_2}s_{i_2}'d\bar{d}..s_{i_\lambda}s_{i_\lambda}'d.$$

(Insertion of $\lambda = |w|$ trivial relators)

$$\to ds_{i_1}s_{i_2}..s_{i_\lambda} = dw$$

(Insertion of $\bar{s}_{i_j}s_{i_j}$ and deletion of $\bar{d}s_{i_j}s_{i_j}'d\bar{s}_{i_j}$).

Taken altogether this derivation doesn't need more than $3|w|^2+3|w|$ steps.

ad (3), $w^{-1}t_o dw \equiv \bar{s}_{i_\lambda}..\bar{s}_{i_1}t_o ds_{i_1}..s_{i_\lambda}$

$$\to (\bar{z}\bar{s}_{i_\lambda}z)(\bar{z}s_{i_\lambda}z\bar{s}_{i_\lambda})..(\bar{z}\bar{s}_{i_1}z)(\bar{z}s_{i_1}z\bar{s}_{i_1})(t_o d\bar{z}\bar{t}_o z)(\bar{z}t_o z)$$

$$(s_{i_1}\bar{z}\bar{s}_{i_1}z)(\bar{z}s_{i_1}z)..(s_{i_\lambda}\bar{z}\bar{s}_{i_\lambda}z)(\bar{z}s_{i_\lambda}z)$$

(Insertion of $6|w|+3$ trivial relators),

$$\to \bar{z}\bar{s}_{i_\lambda}z..\bar{z}\bar{s}_{i_1}z\bar{z}t_o z\bar{z}s_{i_1}z..\bar{z}s_{i_\lambda}z$$

(Deletion of $2|w|+1$ relators of the form $\bar{z}s_{i_j}z\bar{s}_{i_j}, s_{i_j}\bar{z}\bar{s}_{i_j}z, t_o d\bar{z}\bar{t}_o z$),

$$\to zw^{-1}t_o wz$$

(Deletion of $2|w|$ trivial relators).

Altogether (3) needs at most $|10|w +4$ steps.

ad (4), $w^{-1}t_o w \to (w^{-1}t_o w\bar{k}_o w^{-1}\bar{t}_o wk_o)(\bar{k}_o w^{-1}t_o wk_o)$

(Insertion of $2|w|+3$ trivial relators).

Let $k_6 \in E_n(S_6)$ be an E_n-bound for $<S_6;M_6>$. Then $w^{-1}t_o w\bar{k}_o w^{-1}\bar{t}_o wk_o$ can be derived to e in $<S_6;M_6>$ within at most $|k_6(w^{-1}t_o w\bar{k}_o w^{-1}\bar{t}_o wk_o)| = |k_6(x^{4|w|+4})|$ steps. Hence;

$$(w^{-1}t_o w\bar{k}_o w^{-1}\bar{t}_o wk_o)(\bar{k}_o w^{-1}t_o wk_o) \to \bar{k}_o w^{-1}t_o wk_o \text{ in } <\Delta;M>$$

within at most $|k_6(x^{4|w|+4})|$ steps, and (4) can be carried out within not more than $|k_6(x^{4|w|+4})|+2|w|+3$ steps.

ad (5), $\bar{z}\bar{k}_o w^{-1}t_o wk_o z \equiv \bar{z}\bar{k}_o\bar{s}_{i_\lambda}..\bar{s}_{i_1}t_o s_{i_1}..s_{i_\lambda}k_o z$

$$\to (\bar{k}_o\bar{s}_{i_\lambda}k_o)(\bar{k}_o s_{i_\lambda}k_o\bar{z}\bar{k}_o\bar{s}_{i_\lambda}k_o z)(\bar{k}_o\bar{s}_{i_{\lambda-1}}k_o)(\bar{k}_o s_{i_{\lambda-1}}k_o\bar{z}\bar{k}_o\bar{s}_{i_{\lambda-1}}k_o z)..$$

$$(\bar{z}\bar{k}_o t_o k_o z d\bar{k}_o\bar{t}_o k_o)(\bar{k}_o t_o k_o d)(\bar{z}\bar{k}_o s_{i_1}k_o z\bar{k}_o\bar{s}_{i_1}k_o)$$

$$(\bar{k}_o s_{i_1}k_o)(\bar{z}\bar{k}_o s_{i_2}k_o z\bar{k}_o\bar{s}_{i_2}k_o)..(\bar{z}\bar{k}_o s_{i_\lambda}k_o z\bar{k}_o\bar{s}_{i_\lambda}k_o)(\bar{k}_o s_{i_\lambda}k_o)\bar{z}\bar{k}_o k_o z$$

(Insertion of $10|w|+6$ trivial relators),

$$\to \bar{k}_o\bar{s}_{i_\lambda}k_o\bar{k}_o\bar{s}_{i_{\lambda-1}}k_o..\bar{k}_o\bar{s}_{i_1}k_o\bar{k}_o t_o k_o d\bar{k}_o s_{i_1}k_o..\bar{k}_o s_{i_\lambda}k_o\bar{z}\bar{k}_o k_o z$$

(Deletion of $2|w|+1$ relators of the form $\bar{z}\bar{k}_o s_{i_j}k_o z\bar{k}_o\bar{s}_{i_j}k_o, \bar{k}_o s_{i_j}k_o\bar{z}\bar{k}_o\bar{s}_{i_j}k_o z$ $(s_{i_j} \in \Sigma_o)$, $\bar{z}\bar{k}_o t_o k_o z d\bar{k}_o\bar{t}_o k_o$),

$$\to \bar{k}_o w^{-1}t_o k_o d\bar{k}_o s_{i_1}k_o..\bar{k}_o s_{i_\lambda}k_o.$$

(Deletion of $|w|+2$ trivial relators),

$$\to \bar{k}_o w^{-1}t_o k_o d(\bar{k}_o s_{i_1}k_o d\bar{k}_o\bar{s}_{i_1}k_o d)(d\bar{k}_o s_{i_1}k_o d)..(\bar{k}_o s_{i_\lambda}k_o d\bar{k}_o\bar{s}_{i_\lambda}k_o d)(d\bar{k}_o s_{i_\lambda}k_o d).$$

(Insertion of $5|w|$ trivial relators),

$$\to \bar{k}_o w^{-1}t_o k_o dd\bar{k}_o s_{i_1}k_o d..d\bar{k}_o s_{i_\lambda}k_o d$$

(Deletion of $|w|$ relators of the form $\bar{k}_o s_{i_j}k_o d\bar{k}_o\bar{s}_{i_j}k_o d$ $(s_{i_j} \in \Sigma_o)$).

$$\to \bar{k}_o w^{-1}t_o wk_o d.$$

(Deletion of $2|w|$ trivial relators).

Hence (5) can be carried out within $21|w|+9$ steps.

ad (6), $\bar{k}_o w^{-1} t_o w k_o \rightarrow (\bar{k}_o w^{-1} t_o w k_o w^{-1} \bar{t}_o w)(w^{-1} t_o w)$

(Insertion of $2|w|+1$ trivial relators).

Hence $\bar{k}_o w^{-1} t_o w k_o w^{-1} \bar{t}_o w$ can be derived to e in $<S_6;M_6>$ within $|k_6(x^{4|w|+4})|$, steps, and so $(\bar{k}_o w^{-1} t_o w k_o w^{-1} \bar{t}_o w)(w^{-1} t_o w) \rightarrow w^{-1} t_o w$ in $<\Delta;M>$ within at most $|k_6(x^{4|w|+4})|$ steps. Hence (6) doesn't need more than $|k_6(x^{4|w|+4})|+2|w|+1$ steps altogether.

ad (7), $2|w|+2$ trivial relators are deleted.

Taken altogether, there is a derivation from w' in $<\Delta;M>$ of length not exceeding $2|k_6(x^{4|w|+4})|+3|w|^2+42|w|+21$. Define

$$k'(w) \equiv vk(vk(k_6 U_x(w^8), k_6 U_x(w^8)), VK(VK(w,w), x^{66})).$$

Then $k' \in E_n(\Delta)$, k' is nondecreasing, for all $u,v \in \underline{\Delta}^*(|k'(u)|+|k'(v)| \leq |k'(uv)|)$ and for every $w' \in L_o'$ there is a derivation from w' in $<\Delta;M>$ of length bounded by $|k'(w')|$.

b) Let $w \in \underline{\Delta}^*$ with $|w|_z = 0$ and $w \underset{\bar{H}}{=} e$, and so $w \underset{H}{=} e$. According to the proof of Proposition 1.6 (a), w can be derived to e in H_1 in the following way:

$$w \xrightarrow{(1)} w' \xrightarrow[\text{in } H_o]{(2)} \pi_{\underline{S}_6}(w')\pi_{\underline{\Sigma}_o'}(w') \xrightarrow[\text{in } D_6]{(3)} \pi_{\underline{\Sigma}_o'}(w') \xrightarrow[\text{in } G']{(4)} e$$

(d-pinches are pinched out in H_1, in step (1))

This derivation can be simulated in $<\Delta;M>$:

ad (1), d-pinches are pinched out in the following way:

$$d^{-\mu}ud^{\mu} \rightarrow \bar{d}^{\mu}u(\omega_\mu(u))^{-1}\omega_\mu(u)d^{\mu}$$

(Insertion of $|\omega_\mu(u)|$ trivial relators),

$$\rightarrow \bar{d}^{\mu}u_1u_2\omega_\mu(u)d^{\mu}$$

(Within $3(|u|+|\omega_\mu(u)|)^2$ steps $u(\omega_\mu(u))^{-1}$ can be transformed into u_1u_2 where $u_1 \in \underline{S}_6^*$ and $u_2 \in \underline{\Sigma}_o'^*$),

$$\rightarrow \bar{d}^{\mu}u_2\omega_\mu(u)d^{\mu}$$

$(u(\omega_\mu(u))^{-1} \underset{\bar{H}}{=} e$, and so $u_1 \underset{D_6}{=} e$ and $u_2 \underset{G'}{=} e$. But then u_1 can be derived to e in $<S_6;M_6>$ within at most $|k_6(u_1)|$ steps),

$$\rightarrow \bar{d}^{\mu}\omega_\mu(u)d^{\mu}$$

(In u_2, $\bar{s}'$ is substituted by s', and $\bar{\bar{s}}'$ is substituted by s': $\bar{s}' \rightarrow \bar{s}'s'\tilde{s}' \rightarrow \tilde{s}'$, and $\bar{\bar{s}} \rightarrow s'\tilde{s}'\bar{\bar{s}}' \rightarrow s'$. Let $\tilde{u}_2$ be the result of these substitutions. Then $\tilde{u}_2$ can be derived from u_2 within at most $2|u_2|$ steps. Since $e \underset{G'}{=} u_2 \underset{G'}{=} \tilde{u}_2$, $\tilde{u}_2 \in L_o'$, and because of (a), $\tilde{u}_2$ can be derived to e in $<\Delta;M>$ within no more than $|k'(\tilde{u}_2)|$ steps),

$$\rightarrow \varphi^{\mu}(u)$$

($7|\omega_\mu(u)|$ steps of the form: Insertion of trivial relators, deletion of trivial relators, and deletion of a d-relator).

Let $A_1 = \langle ss', \bar{k}_o s k_o (s \in \Sigma_o), t_o, \bar{k}_o t_o k_o \rangle_{H_o}$, $B_1 = \langle s, \bar{k}_o s k_o (s \in \Sigma_o), t_o, k_o t_o k_o \rangle_{H_o}$, and φ and $\bar{\varphi}$ denote function realizing the isomorphisms $A_1 \to B_1$ and $B_1 \to A_1$, respectively. According to [Av-Madl] Lemma 1.4, p.187, there are constants $c \geq 1$ and $d \geq 2$ sastisfying $|\omega_{A_1}(w)|$, $|\omega_{B_1}(w)|$, $|\varphi(w)|$, $|\bar{\varphi}(w)| \leq c|w|^d$. Hence for pinching out the d-pinch $\bar{d}^\mu u d^\mu$ one doesn't need more than

$$8c|u|^d+3(c+1)^2|u|^{2d}+|k_6(x^{(c+1)|u|^d})|+2(c+1)|u|^d+|k'(x^{(c+1)|u|^d})|$$
$$\leq 13(c+1)^2|u|^{2d}+ k_6(x^{(c+1)|u|^d})|+|k'(x^{(c+1)|u|^d})|$$

steps in $\langle\Delta;M\rangle$. Let w_i' be the word formed from w by pinching out i d-pinches. Then by the proof of Prop. 1.6 (a),

$$|w_i'| \leq (c+1)^{d^{2i-1}}|w|^{d^i}.$$

Therefore every d-pinch $\bar{d}^\mu u d^\mu$ pinched out at (1) is bounded by

$$|u| \leq (c+1)^{d^{|w|}}|w|^{d^{|w|}}$$

Hence there is a function $k_1' \in E_n(N)$ bounding the number of steps needed for carrying out (1), since $n \geq 3$. Of course w' satisfies $|w'| \leq ((c+1)|w|)^{d^{|w|}}$.

ad (2), by using the commutation relators of H_o and some trivial relators, w' can be transformed into $\pi_{S_6}(w')\pi_{\Sigma_o}'(w')$, within at most $3|w'|^2$ steps. So this transformation can be bounded by a function $k_2' \in E_n(N)$.

ad (3), there is a derivation from $\pi_{S_6}(w')$ in $\langle S_6;M_6\rangle$ consisting of no more than $|k_6 \circ \pi_{S_6}(w')| \leq |k_6(x^{|w'|})|$ steps, and so there is a function $k_3' \in E_n(N)$ bounding this derivation.

ad (4), within at most $2|\pi_{\Sigma_o}'(w')|$ steps each $\bar{s}'$ and each $\bar{\bar{s}}'$ contained in $\pi_{\Sigma_o}'(w')$ can be substituted by $\bar{\bar{s}}'$ or s', respectively. In this way $\pi_{\Sigma_o}'(w')$ is transformed into a word $\tilde{w} \in L_o'$ which can be derived to e in $\langle\Delta;M\rangle$ within at most $|k'(x^{|\tilde{w}|})| \leq |k'(x^{|w'|})|$ steps because of (a). Hence (4) is bounded by a function $k_4' \in E_n(N)$, too.

So there is a function $\tilde{k} \in E_n(N)$ bounding the derivations from w to e in $\langle\Delta;M\rangle$ for all $w \in \Delta^*$ satisfying $|w|_z = 0$ and $w \underset{H}{=} e$.

c) Let $w \in \Delta^*$ with $|w|_z > 0$ and $w \underset{H}{=} e$, and so $w \underset{H_2}{=} e$. According to the proof of Prop. 1.6 (a), w can be derived to e in H_2 as follows:

$$w \xrightarrow{(1)} w' \xrightarrow[\text{in } H_1]{(2)} e$$

(z-pinches are pinched out in H_2, in steps (1)).
This derivation can be simulated in $\langle\Delta;M\rangle$:

ad (1), z-pinches are pinched out in the following way

$$\bar{z}^\mu u z^\mu \to \bar{z}^\mu u(\omega_\mu(u))^{-1}\omega_\mu(u)z^\mu$$

(Insertation of $|\omega_\mu(u)|$ trivial relators),

$$\to \quad \bar{z}^{\mu}\omega_{\mu}(u)z^{\mu}$$

$(u(\omega_{\mu}(u))^{-1} \underset{H_1}{=} e$ and $|u(\omega_{\mu}(u))^{-1}|_z = 0$. Hence $u(\omega_{\mu}(u))^{-1}$ can be derived to e in <Δ;M> within at most $\tilde{k}(|u|+|\omega_{\mu}(u)|)$ steps because of (b)),

$$\to \quad \varphi^{\mu}(u)$$

($8|\omega_{\mu}(u)|$ steps of the form: insertion of trivial relators, deletion of a z-relator, and deletion of trivial relators). Let

$$A_2 = \langle s,\bar{k}_o s k_o (s \in \Sigma_o), t_o, \bar{k}_o t_o k_o \rangle_{H_1},\quad B_2 = \langle s,\bar{k}_o s k_o (s \in \Sigma_o), t_o d, \bar{k}_o t_o k_o d\rangle_{H_1},$$

and φ and $\bar{\varphi}$ denote functions realizing the isomorphisms $A_2 \to B_2$ and $B_2 \to A_2$, respectively. Because of [Av-Madl] Lemma 1.5, p.187, there are constants $\alpha,\beta \geq 2$ satisfying:

$$|\omega_{A_2}(w)|,\ |\omega_{B_2}(w)|,\ |\varphi(w)|,\ |\bar{\varphi}(w)| \leq \alpha|w|^{\beta}.$$

Hence for pinching out the z-pinch $z^{\mu}uz^{\mu}$ one only needs $\alpha|u|^{\beta}+\tilde{k}((\alpha+1)|u|^{\beta}) + 8\alpha|u|^{\beta} = 9\alpha|u|^{\beta}+\tilde{k}((\alpha+1)|u|^{\beta})$ steps.

Let w_i' denote the word formed from w by pinching out i z-pinches. By the proof of Prop. 1.6 (1), $|w_i'| \leq (\alpha+1)^{\beta^{2i-1}}|w|^{\beta^i}$. Hence any z-pinch $\bar{z}^{\mu}uz^{\mu}$ pinched out at (1) satisfies $|u| \leq ((\alpha+1)|w|)^{\beta^{|w|}}$, and therefore the number of steps necessary to realize (1) can be bounded by a function $k_1'' \in E_n(N)$. Furthermore $|w'| \leq ((\alpha+1)|w|)^{\beta^{|w|}}$.

ad (2), $|w'|_z = 0$ and $e \underset{H}{=} w \underset{H}{=} w'$. Hence, because of (b), w' can be derived to e in <Δ;M> within at most $\tilde{k}(|w'|)$ steps and so, there is a function $k_H' \in E_n(\Delta)$ bounding the derivations of all words $w \in \underline{\Delta}^*$ with $w \underset{H}{=} e$ in <Δ;M>.
Therefore <Δ;M> is E_n-d.b.

2.2. COROLLARY. *Every countable group* G *having an* E_n*-decidable word problem for some* $n \geq 3$ *can be embedded into a f.p. group* H *possessing a finite* E_n-d.b. *presentation.*

Proof. Every countable group G having an E_n-decidable word problem for some $n \geq 2$ can be embedded into a f.g. group G_1 having an E_n-decidable word problem too ([Ott] Thm. 12.1, p.117).

3. F.P. E_n-DERIVATION BOUNDED GROUPS AND THE WORD PROBLEM.

For finite E_n-d.b. presentations of groups there is a standard natural algorithm for solving the word problem. But of what degree of complexity is this algorithm, and how is this degree of complexity related to the selected finite presentation?

3.1. THEOREM. *Let* H = <Σ;L> *be f.p. and* E_n-d.b. *for some* $n \geq 3$. *Then the standard natural algorithm for <Σ;L>, as it is described in the introduction,* *is an* E_n*-algorithm. In particular the word problem for* $<\Sigma;L>$ *is* E_n*-decidable.*

Proof. Let $\Sigma = \{s_1,..,s_m\}$, $L = \{w_1,..,w_\ell\} \subseteq \Sigma^*$, and $k \in E_n(\Sigma)$ be an E_n-bound for $<\Sigma;L>$. Without loss of generality $m \geqslant 3$ may be assumed, for otherwise auxiliary generators and defining relators can be added.

If $w \in \Sigma^*$ with $w \overset{=}{_H} e$, then there is a derivation from w in $<\Sigma;L>$ of length not exceeding $|k(w)|$. During each step of this derivation a word $u \in Rel = L \cup L^{-1} \cup \{s\bar{s},\bar{s}s | s \in \Sigma\}$ is inserted or deleted. L contains ℓ, and Σ contains m elements only. Hence there are only $2(\ell+m)$ possible choices for u. Define λ as the length of the longest possible word u. Then every word v found in that bounded derivation from w satisfies $|v| \leqslant |w|+\lceil\frac{\lambda}{2}\rceil\cdot k(w)|$, where $\lceil\mu\rceil$ denotes the least natural number greater than or equal to μ, because in order to derive a word of greater length from w more than $\frac{1}{2}|k(w)|$ steps are necessary, but then in order to derive this word to e more than $\frac{1}{2}|k(w)|$ steps are needed, again contradicting the fact that the derivation from w is bounded by $|k(w)|$. Define

$$\mu_w = |w|+\lceil\tfrac{\lambda}{2}\rceil\cdot|k(w)|.$$

A step of a derivation can be encoded as a triple (i_1,i_2,i_3) of natural numbers such that $i_1 \in \{0,1\}$, $i_2 \in \{1,2,...,2(\ell+m)\}$, and $i_3 \in \{0,1,2,..,\mu_w\}$. Here $i_1 = 0$ stands for "insertion", $i_1 = 1$ for "deletion" of the relator with the number i_2 at the position described by i_3. Hence there are $\nu_w = 2\cdot 2\cdot(\ell+m)\cdot(\mu_w+1)$ different steps which can be chosen in a derivation of w. Therefore there are not more than $(\nu_m)^{|k(w)|}$ possible derivations from w of length $|k(w)|$. In order to decide $w \overset{=}{_H} e$, it is sufficient to apply these derivations one after another to w, and to test whether one of these derivations produces e. Define $f_1(e) \equiv e$, $f_2(e) \equiv s_1$, $f_1(ws) \equiv f_2(w)$, $f_2(ws) \equiv vk(f_1(w),s_1)$ then $f_1,f_2 \in E_1(\Sigma)$, satisfying

$$f_1(w) \equiv s_1^{\lceil\frac{|w|}{2}\rceil}, \quad f_2(w) \equiv s_1^{\lceil\frac{|w|+1}{2}\rceil}$$

Let $ML(w) \equiv vk(U_{s_1}(w),VK(U_{s_1}\circ k(w),f_1(s_1^{\lambda})))$ where $\lambda = \max_{u\in Rel}|u|$. Then $ML \in E_n(\Sigma)$ and

$$ML(w) \equiv s_1^{|w|+\lceil\frac{\lambda}{2}\rceil\cdot|k(w)|} \equiv s_1^{\mu_w}$$

Each step in a derivation is described by a triple

$(i_1,i_2,i_3) \in \{0,1\}\times\{1,2,..,2(\ell+m)\}\times\{0,1,..,\mu_w\}$, and so it can be encoded as a word over Σ, namely as

$$s_1^{i_1+1}\, s_2^{i_2}\, s_3^{i_3+1}$$

which is a word of length not exceeding $2+2(\ell+m)+\mu_w+1 = 2(\ell+m)+3+|w|+\lceil\frac{\lambda}{2}\rceil\cdot|k(w)|$. Hence a derivation of w can be described by a word of length at most

$$(2(\ell+m)+3+|w|+\lceil\tfrac{\lambda}{2}\rceil\cdot|k(w)|)\cdot|k(w)|.$$

Let

$$LDA(w) \equiv VK(vk(s_1^{2(\ell+m)+3},ML(w)),k(w))$$

then LDA $\in E_n(\Sigma)$ satisfying

$$\mathrm{LDA}(w) \equiv s_1^{(2(\ell+m)+3+|w|+\lceil\frac{\lambda}{2}\rceil\cdot|k(w)|)\cdot|k(w)|}.$$

In order to decide whether $w \underset{H}{=} e$ is valid or not one only has to check whether there is a word u of length at most |LDA(w)| describing a derivation from w to e in $<\Sigma;L>$. Now a Turing Machine M will be defined to test for a pair $(w,u) \in (\Sigma^*)^2$ whether u is the description of a derivation from w, by trying yo apply u to w. In an initial part of u is the description of a derivation from w to e, then M will halt with its output tape being empty, but if u doesn't meet this condition, then M will print the letter "s_1" and halt.

Let M have two input tapes, one output tape, and four auxiliary tapes.

1) w is the inscription of the first input tape, and u is the inscription of the second one.

2) w is copied onto the first auxiliary tape, while u is copied onto the second one. This can be done within $2|w|+2|u|+3$ steps. i.e. *amount of time* (*A.t.*) = $2|w|+2|u|+3$.

3) The elements of the set Rel are printed onto the third auxiliary tape separated by a "b", respectively.

A.t. $\leqslant 2\ (\lambda+1)\cdot 2\cdot(\ell+m) \leqslant 8\lambda(\ell+m)$.

I1:

..b	w	b..
↑		

I2:

..b	u	b..
↑		

A1:

..b	w	b..
↑		

A2:

..b	u	b..
↑		

A3:

..b	w_1	b	w_1^{-1}	b	w_2	b..b	w_1^{-1}	b	$s_1\bar{s}_1$	b..b	$\bar{s}_m s_m$	b..
↑												

4) If u starts with a letter $s \neq s_1$, then outputs s_1 and halts. *A.t.* = 3.
If u starts with s_1, then mind "insertion". *A.t.* = 3.
If u starts with s_1^2, then mind "deletion". *A.t.* = 4.
If u starts with s_1^i for an $i > 2$, then outputs s_1 and halts. *A.t.* = 4.

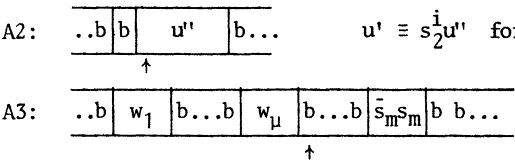
A2:

..b	b	u'	b..
		↑	

$u \equiv s_1^i u'$ for some $i \in \{1,2\}$.

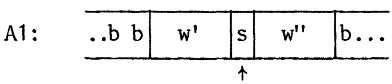
If u' starts with a letter $s \neq s_2$, then outputs s_1 and halts. *A.t.* = 2.

If u' starts with s_2^i, then for i-1 times M puts the head of its third auxiliary tape onto the next symbol "b" to the right of the actual position of the head. After that this head performs one step to the right. *A. t.* $\leqslant i(\lambda+1)+1$.
If M reads a "b" on its third auxiliary tape, then output s_1 and halts. *A. t.* = 2.
Otherwise, the head of A3 is pointing to the relator which shall be inserted or deleted from w.



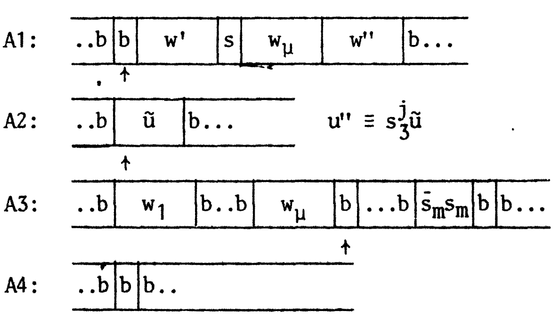
$u' \equiv s_2^i u''$ for some $i \in \{1,\dots 2(\ell+m)\}$

If u'' starts with a letter $s \not\equiv s_3$, then output s_1 and halts *A.t.* = 2.
If u'' starts with s_3^j, then the operation R (i.e. make a step to the right) is executed on A1, j-1 times. *A.t.* = j.
If the head of A1 is now pointing at a cell containing "b", and if M has to delete the relator marked on A3, then M prints "s_1" and halts. *A.t.* = 2.
If the head of A1 is pointing at a cell containing "b", if $j \geqslant 2$, and if M has to insert the relator marked on A3, then M prints "s_1" and halts. *A.t.* = 2.
Otherwise, the head of A1 is pointing at the first letter of w which shall be erased or behind which the indicated relator shall be inserted.



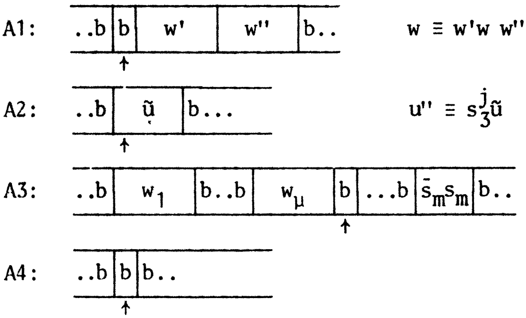
$w \equiv w'sw''$

5) ***Insertion:*** The indicated relator is copied from A3 onto A4, subsequently w'' is appended at the rigth end of this copy, and at last w'' is erased from A1. *A.t.* $\leqslant \lambda+|w|+1$.
If j = 1, then the inscription of A4 is copied onto A1, in the course of which it is erased from A4. Otherwise the inscription of A4 is appended to the inscription of A1 (w's), at which it is erased from A4. The head of A1 is put onto the first "b" to the left of the inscription of A1.
A.t. $\leqslant |w|+2(|w|+\lambda+1)+|w|+\lambda+1 = 4|w|+3\lambda+3$.



$u'' \equiv s_3^j \tilde{u}$

Deletion. The indicated relator is compared to the subword of w, beginning at the position the head of A1 is pointing at. By doing so, the subword of w is erased. If this subword of w and the indicated relator do nos coincide, then M prints "s_1" and halts. Otherwise an initial part or an internal segment of w has been erased. In the first case the head of A1 performs one step to the left, in the second case M appends the remained end of w to the remained initial part by using the tape A4 as scratch paper. At last M puts the head of A1 onto the first "b" to the left of the inscription of A1. *A.t.* $\leq \lambda+2|w|+\lambda+2+2|w|+1 = 4|w|+2\lambda+3$.



6) The head of tape A3 returns to the left.
A.t. $\leq (\lambda+1)\cdot 2\cdot(1+m)+2 \leq 4\lambda(1+m)+2$.
If the inscription of tape A1 is e, then M halts because e has been derived from w. Otherwise M continues with step (4).
A.t. $= 2$.
Of course M eventually halts satisfying $f_M(w,u) \equiv e$ iff an initial part of u is describing a derivation from w. Altogether M has the following amount of time.

$$T_M(w,u) \leq 2|w|+2|u|+3+8\lambda(\ell+m)+|u|\cdot\{4+|u|(\lambda+1)+1+2+|u|+2+ 5(|w|+\lambda|u|)+4\lambda+4+4\lambda(\ell+m)+2+2\}.$$

(In the course of the computation w may grow, but it cannot become larger than $|w| + |u|$)

$$= 2|w|+2|u|+3+8\lambda(\ell+m)+|u|\cdot\{5|w|+(6\lambda+2)|u|+4\lambda(\ell+m+1)+17\}.$$

But λ,ℓ,m are constants, and so $f_M \in E_2(\underline{\Sigma})$ because of [Weih] Kap. 4.3, Satz 2. Now we have:

$$w \underset{H}{=} e \quad \text{iff} \quad \exists u \in \underline{\Sigma}^* \; (|u| \leq \text{LDA}(w)| \text{ and } f_M(w,u) \equiv e)$$
$$\text{iff} \quad \exists u \leq vk(\text{LDA}(w),s_1) \; (f_M(w,u) \equiv e).$$

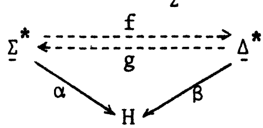
But as $n \geq 3$, $E_n(\underline{\Sigma})$ is closed under bounded quantification and therefore $w \underset{H}{=} e$ is E_n-decidable by the standard n. a. implemented above. Hence, $WP_H \in E_n(\underline{\Sigma})$.

Next we prove that E_n-derivation boundedness is an invariant of finite presentations.

3.2. THEOREM. *Let* $H = \langle\Sigma;L\rangle$ *be* f.p. *and* E_n-d.b. *for some* $n \geq 1$. *Then every finite presentation for* H *is* E_n-d.b., *too.*

Proof. Let be Σ, L, and k as in the proof of Theorem 3.1, and let $\langle\Delta;M\rangle$, $\Delta = \{t_1,..,t_r\}$, $M = \{u_1,..,u_s\} \subseteq \underline{\Delta}^*$, be another finite presentation for H. Then, for all $s_i \in \Sigma$ there is $v_i \in \underline{\Delta}^+$ such that s_i and v_i define the same element of the group H. Define $f(e) \equiv e$, $f(ws_i^\mu) \equiv vk(f(w),v_i^\mu)$. Then for all $w \in \underline{\Sigma}^*$, w and f(w) define the same element of the group H, and there is a constant $c_1 > 0$ such that $|f(w)| \leq c_1 \cdot |w|$.

$\forall t_j \in \Delta_1 \exists x_j \in \underline{\Sigma}^+$, t_j and x_j define the same element of H. Moreover, $g(e) \equiv e$, $g(wt_j^\mu) \equiv vk(g(w),x_j^\mu)$. Then for all $w \in \underline{\Delta}^*$, w and g(w) define the same element of H, and there is a constant $c_2 > 0$ such that $|g(w)| \leq c_2 \cdot |w|$.



Then $\beta(w) \underset{H}{=} \alpha\circ g(w) \underset{H}{=} \beta\circ f\circ g(w)$, and so $w \underset{H}{=} f\circ g(w)$. Also $|f\circ g(w)| \leq c_1|g(w)| \leq c_1 \cdot c_2 \cdot |w|$. Especially $t_j^\mu(f\circ g(t_j^\mu))^{-1} \underset{H}{=} e$. Hence for each $t_j^\mu \in \underline{\Delta}$ there is a derivation from $t_j^\mu(f\circ g(t_j))^{-1}$ to e in $\langle\Delta;M\rangle$ of length $\ell_{j,\mu}$. If $c_3 = \max\{\ell_{j,\mu} \mid j = 1,\dots,r, \mu \in \{\pm 1\}\}$, then $f\circ g(t_j)$ can be derived from t_j^μ in $\langle\Delta;M\rangle$ within at most $c_4 = c_3+1$ steps by the following sequence:

$$t_j^\mu \xrightarrow[c_3]{} t_j^\mu \bar{t}_j^\mu (f\circ g(\bar{t}_j^\mu))^{-1} \xrightarrow[1]{} (f\circ g(\bar{t}_j^\mu))^{-1} \equiv f\circ g(t_j^\mu).$$

Hence every word $w \in \underline{\Delta}^*$ can be derived to $f\circ g(w)$ within $c_4|w|$ steps.

For every $u \in$ Rel, $f(u) \underset{H}{=} e$, and therefore there is a derivation from f(u) to e in $\langle\Delta;M\rangle$ of length ℓ'_u. If $c_5 = \max\{\ell'_u \mid u \in \text{Rel}\}$, then f(u) can be derived to e in $\langle\Delta;M\rangle$ within no more than c_5 steps. Let $w \in \underline{\Delta}^*$ with $w \underset{H}{=} e$, then $g(w) \underset{H}{=} e$, too. Hence there is a derivation from g(w) to e in $\langle\Sigma;L\rangle$ of length not exceeding $|k\circ g(w)| \leq |k(s_1^{c_2|w|})|$:

$$g(w) \equiv u_0 \to u_1 \to .. \to e.$$

But then

$$f\circ g(w) \equiv g(u_0) \xrightarrow[c_5]{} f(u_1) \xrightarrow[c_5]{} \dots \xrightarrow[c_5]{} f(e) \equiv e$$

in $\langle\Delta;M\rangle$, i.e. there is a derivation from $f\circ g(w)$ to e in $\langle\Delta;M\rangle$ of length not exceeding $c_5|k(s_1^{c_2|w|})|$. Now w can be derived to e in $\langle\Delta;M\rangle$ in the following manner:

$$w \xrightarrow[c_4|w|]{} f\circ g(w) \xrightarrow[c_5|k(s_1^{c_2|w|})|]{} e.$$

Of course there is an E_n-function bounding this derivation. Hence $<\Sigma;M>$ is E_n-d.b.

The last theorem shows that the property of being E_n-d.b. does not depend on the chosen finite presentation. It merely depends on the group. Hence a f.p. group is called E_n-d.b. if one, and therewith each, of its finite presentations is E_n-d.b. A conclusion of the proof of the last theorem is the fact that even every f.g. presentation of a f.p. E_n-d.b. group is E_n-d.b. But of course each f.p. E_n-d.b. group has a f.g. E_0-d.b. presentation, i.e. $<\Sigma;\{w \in \Sigma^* | w \underset{G}{=} e\}>$ for example. Therefore the property of being E_n-d.b. does depend on the chosen f.g. presentation of a group.

It remains to answer the question wheter for f.p. E_n-d.b. groups with $n \geq 3$ an optimal n.a. exists. The following theorem gives an answer in the negative sense.

3.3. THEOREM. *For every* $n \geq 4$ *there is a* f.p. *group* $G_5 = <S_5;L_5>$ *such that the word problem for* G_5 *is* E_3-*decidable, but* $<S_5;L_5>$ *is only* E_n-, *but not* E_{n-1}-d.b. *Especially there is no finite* E_3-d.b. *presentation for* G_5.

Proof. Let $n \geq 4$. The f.p. group G_5 will now be constructed in the same manner as the group D_5 has been constructed in the proof of Theorem 2.1. Only the underlying Turing Machine will be modified. Let $S' = \{s_1,s_2,s_3\}$ and $L = S'^*$, and let $T = (S',Q_T,q_0,\beta)$ be a single tape machine acting as follows. For every $w \in S'^*$, starting at q_0w, T computes $A_n(w,w)$ where $A_n \in E_n(S')$ denotes the n-th Ackermann function over S' ([Weih]). After that T enters the accepting state q_a and halts. For carrying out this computation T has to execute more than $|A_n(w,w)|$ steps. On the other hand, T can be chosen in such a way that there exists a function $g \in E_n(S')$ which bounds the time, i.e. the number of steps T needs for its computation ([Weih] Kap.4.4, Satz 1).

Now T can be modified to get $\tilde{T} = (\tilde{S},Q_{\tilde{T}},q_0,\tilde{\beta})$, where $\tilde{S}$ is a finite alphabet containing S' such that there is a function $k_{\tilde{T}} \in E_n(\tilde{S} \cup Q_{\tilde{T}})$ satisfying.

$\forall u,v \in S^* \forall q_j \in Q_{\tilde{T}}$, starting at the configuration uq_jv, $\tilde{T}$ halts in the accepting state q_a within at most $|k_{\tilde{T}}(uq_jv)|$ steps.

This modification is done in the same way as the one used in the proof of theorem 2.1, with the only exception that the non-accepting state q_- is omitted, i.e. instead of entering q_-, $\tilde{T}$ enters the accepting state q_a. Since for every $w \in S'^*$, starting at q_0w, T halts in the state q_a, $\tilde{T}$ also halts in the state q_a, starting at any configuration uq_jv. The execution time of T is bounded by the function $g \in E_n(S')$. Hence there is a function $k_{\tilde{T}} \in E_n(\tilde{S} \cup Q_{\tilde{T}})$ satisfying the condition formulated above. Of course, starting at q_0w, $\tilde{T}$ has to carry out more than $|A_n(w,w)|$ steps for avery $w \in S'^*$, too.

CLAIM. *Let* $S = \tilde{S} \cup \{h\}$, $Q = Q_{\tilde{T}} \cup \{q\}$ *and* $\Delta = (S \cup Q; \pi)$, *where* $\pi = \{F_i q_{i_1} G_1 = H_i q_{i_2} K_i \mid q_{i_1}, q_{i_2} \in Q, F_i, G_i, H_i, K_i \in S^*,\ i = 1,\dots,N\}$ *is the semigroup constructed from* T *according to* [Av-Madl],*p.89. Then the following three conditions are satisfied:*

(1) $\forall u,v \in S^* \forall q_j \in Q$ $(uq_jv \underset{\Delta}{=} q \iff uq_jv \equiv q$ or $u \equiv hu'$, $v \equiv v'h$, with $u',v' \in \tilde{S}^*$ and $q_j \not\equiv q)$.

(2) $\forall w \in S^* \forall q_j \in Q$ $(uq_jv \underset{\Delta}{=} q \to \exists$ derivation from uq_jv to q in Δ, of length not exceeding $2|k_{\tilde{T}}(uq_jv)|+|uq_jv|)$.

(3) $\forall w \in S'^*$ $(hq_owh \underset{\Delta}{=} q$, but there is no derivation from hq_owh to g in Δ of length $\leqslant |A_n(w,w)|)$.

Proof.

ad (1) "$\Rightarrow$". Let $uq_jv \underset{\Delta}{=} q$, but $uq_jv \not\equiv q$. Then $q_j \not\equiv q$, $u \equiv hu'$ and $v \equiv v'h$ for some $u',v' \in \tilde{S}^*$.

"$\Leftarrow$". Let $u',v' \in \tilde{S}^*$, $q_j \in Q_{\tilde{T}}$. Then $u'q_jv' \overset{*}{\underset{\tilde{T}}{\Rightarrow}} q_a$, and so $hu'q_jv'h \underset{\Delta}{=} hq_ah \underset{\Delta}{=} q$.

ad (2) This can be proved in exactly the same way as the corresponding statement in the proof of Theorem 2.1 was proved. Hence there is a function $k_\Delta \in E_n(S \cup Q)$ which bounds the derivations from $w \in (S \cup Q)^*$ to q in Δ if $w \underset{\Delta}{=} q$.

ad (3) Δ simulates $\tilde{T}$, step by step. But starting at q_ow, $\tilde{T}$ has to execute more than $|A_n(w,w)|$ steps before reaching q_a. Therefore Δ has a carry out more than $|A_n(w,w)|$ steps to reach q, too, when started at hq_owh.

Now a Britton tower of groups is constructed:

$G_o = \langle x, \emptyset \rangle$, $S_o = \{x\}$,

$G_1 = \langle G_o, S; \bar{s}xs = x^2 (s \in S)\rangle$, $S_1 = S_o \cup S$,

$G_2 = \langle G_1, Q; \emptyset\rangle$, $S_2 = S_1 \cup Q$,

$G_3 = \langle G_2, R; \bar{r}_i \bar{F}_i q_{i_1} G_i r_i = \bar{H}_i q_{i_2} k_i, \bar{r}_i s x r_i = s\bar{x} (s \in S, 1 \leqslant i \leqslant N)\rangle$, $S_3 = S_2 \cup R$,

$G_4 = \langle G_3, t; \bar{t}xt = x,\ \bar{t}rt = r (r \in R^2)\rangle$, $S_4 = S_3 \cup \{t\}$,

$G_5 = \langle G_4, k; \bar{k}ak = a (a \in \{x, \bar{q}tq\} \cup R)\rangle$, $S_5 = S_4 \cup \{k\}$, $R_x = R \cup \{x\}$.

Of course $G_o, G_1, \dots, G_5$ are f.p. Furthermore they satisfy ([Av-Madl]):

(α) For $i = 1,\dots,4$, G_i is an HNN-extension of G_{i-1}, there is a reduction function $f_i \in E_3(\underline{S}_i)$ for G_i, and the word problem for G_i is E_3-decidable.

(β) There is a function $g \in E_3(\underline{S}_3)$ satisfying:

- $\forall w \in \underline{S}_3^*$ $(g(w) \underset{G_3}{=} uw$ for some $u \subset \underline{R}_x^*)$.
- If $w \in \underline{S}_3^*$ is R-reduced, there is no $u \in \underline{R}_x^*$ such that there is a R-pinch in $ug(w)$ just on the border $u - g(w)$.
- If $w \in \underline{S}_3^*$ is R-reduced, and if $g(w) \equiv ur_i^\mu v$ where $u \in \underline{S}_2^*$, $v \in \underline{S}_3^*$, then w has the form $u'r_i^\mu v$ for some $u' \in \underline{S}_3^*$.

(γ) Over $\underline{S}_3^*$ define the predicate: $\tilde{P}(u) \iff \exists w_1, w_2 \in \underline{R}_x^*$ $(w_1uw_2 \underset{G_3}{=} q)$.

- If $u \in \underline{S}_3^*$ is reduced and $v \equiv g((g(u))^{-1})^{-1}$, then: $\tilde{P}(u)$ iff $v \in \underline{S}_2^*$ and $\tilde{P}(v)$.
- If $v \in \underline{S}_2^*$ is reduced and v' is the result of deleting all x and $\bar{x}$ symbols of v, then:

$\tilde{P}(v)$ iff $\exists X,Y \in S^*$, $q_j \in Q$ ($v' \equiv \bar{X}q_jY$ and $\tilde{P}(Xq_jY)$).
- $\forall X,Y \in S^*$, $q_j \in Q$ ($\tilde{P}(\bar{X}q_jT)$ iff $Xq_jY \underset{\Delta}{=} q$).

Assertion. $\tilde{P} \in E_3(\underline{S}_3)$.

Proof. Let $u' \in \underline{S}_3^*$. Then $u \equiv f_3(u')$ satisfies $u' \underset{G_3}{=} u$, and so $\tilde{P}(u')$ iff $\tilde{P}(u)$. Let $v \equiv g((g(u))^{-1})^{-1}$. Then because of (γ), $\tilde{P}(u)$ iff $v \in \underline{S}_2^*$ and $\tilde{P}(v)$, since u is reduced. Let $\tilde{v} \equiv f_2(v)$, and $v' \equiv \pi_{\underline{S} \cup Q}(\tilde{v})$. If $v \in \underline{S}_2^*$, then the following is true because of (γ):

$$\tilde{P}(v) \text{ iff } \exists X,Y \in S^*,\ q_j \in Q \quad (v' \equiv \bar{X}q_jY \text{ and } \tilde{P}(\bar{X}q_jY)).$$

Altogether we have thus:

$\tilde{P}(u')$ iff $\tilde{P}(u)$ iff $v \in \underline{S}_2^*$ and $\tilde{P}(v)$
iff $v \in \underline{S}_2^*$ and $\exists X,Y \in S^*, q_j \in Q$ ($v' \equiv \bar{X}q_jY$ and $\tilde{P}(\bar{X}q_jY)$)
iff $v \in \underline{S}_2^*$ and $\exists X,Y \in S^*, q_j \in Q$ ($v' \equiv \bar{X}q_jY$ and $Xq_jY \underset{\Delta}{=} q$).

But $u, v, \tilde{v}$, and v', and therewith also Xq_jY, are E_3-computable from u'. $Xq_jY \underset{\Delta}{=} q$ is E_1-decidable because of (1). Hence $\tilde{P} \in E_3(\underline{S}_3)$.

Now let $u \in \underline{S}_4^*$ be such that $f_4(u) \equiv u_o t^{\mu_1} u_1 .. t^{\mu_m} u_m$, $u_i \in \underline{S}_3^*$, $\mu_i \in \{\pm 1\}$. According to the proof of [Av-Mad] Lemma 4.9, p.102, the following assertion is satisfied:

$$u \in \langle x, \bar{q}tq, R\rangle_{G_4} \text{ iff } u_o u_1 .. u_m \in \langle x,R\rangle_{G_3} \text{ and } \bigwedge_{i=0}^{m-1} \tilde{P}((u_o u_1 .. u_i)^{-1}).$$

But $\langle x,R\rangle_{G_3}$ is E_3-decidable because of the proof of [Av-Madl] Lemma 4.6, p.100. Hence $\langle x,\bar{q}tq,R\rangle_{G_4}$ is E_3-decidable and so G_5 is an E_3-admissible HNN-extension of G_4. Hence $WP_{G_5} \in E_3(\underline{S}_5)$.

According to [Ott] §15, pp.156-173, the presentation $\langle S_5;L_5\rangle$ of G_5 is E_n-d.b.

Now let $w \in S'^*$, then $q_o w \overset{*}{\underset{T}{\vdash}} ..q_a..$; so $q_o w \overset{*}{\underset{T}{\vdash}} ..q_a..$, and therefore $hq_o wh \underset{\Delta}{=} q$. $\bar{k}\bar{h}w^{-1}\bar{q}_o ht\bar{h}q_o whk \underset{G_5}{=} \bar{h}w^{-1}\bar{q}_o ht\bar{h}q_o wh$ according to [Rot] Lemma 12.13, p.229. Therefore, there is a derivation from $\bar{k}\bar{h}w^{-1}\bar{q}_o ht\bar{h}q_o whk\bar{h}w^{-1}\bar{q}_o ht\bar{h}q_o wh$ to e in $\langle S_5;L_5\rangle$. During this derivation $\bar{k}$ and k must be eliminated by using relators of the form $\bar{k}aka^{-1}$ ($a \in \{x,\bar{q}tq\} \cup R$). But for that, $\bar{h}w^{-1}\bar{q}_o ht\bar{h}q_o wh$ must be rewritten into a word $u \in (\underline{\{x,\bar{q}tq\} \cup R})^*$. Let $u \equiv u_o \bar{q}t^{\mu_1}qu_1..\bar{q}t^{\mu_1}qu_1$, $u \in \underline{R}_x$, $\mu_i \in \{\pm 1\}$ be such that

$$\bar{h}w^{-1}\bar{g}_o ht\bar{h}q_o wh \underset{G_4}{=} u \equiv u_o \bar{q}t^{\mu_1}qu_1..\bar{q}t^{\mu_1}qu_1.$$

$\bar{h}w^{-1}\bar{q}_o ht\bar{h}q_o wh$ is t-reduced in G_4. Hence there is an $i \in \{1,\dots,\ell\}$ such that $u \equiv u_o\bar{q}t^{\mu_1}qu_1..\bar{q}t^{\mu_\ell}u_\ell \underset{G_4}{=} u_o..u_{i-1}\bar{q}tqu_iu_{i+1}..u_\ell \underset{G_4}{=} \gamma_f(u_o..u_{i-1})\bar{q}tq\gamma_f(u_i..u_\ell)$, where γ_f denotes the free reduction. Then $\bar{h}w^{-1}\bar{q}_o ht\bar{h}q_o wh \underset{G_4}{=} u \underset{G_4}{=} v_1\bar{q}tqv_2$ with $v_1 \equiv \gamma_f(u_o..u_{i-1})$ and $v_2 \equiv \gamma_f(u_i..u_\ell)$. So, $\bar{h}w^{-1}\bar{q}_o ht\bar{h}q_o whv_2^{-1}\bar{q}\bar{t}qv_1^{-1} \underset{G_4}{=} e$. Hence there is a $v_3 \in \underline{R}_x^*$ freely reduced with $\bar{h}q_o whv_2^{-1}\bar{q} \underset{G_3}{=} v_3$. But $v_3^{-1}\bar{h}q_o whv_2^{-1} \underset{G_3}{=} q$ with v_3^{-1}, $v_2^{-1} \in \underline{R}_x^*$ freely reduced. So $|v_3^{-1}|_R = |v_2^{-1}|_R$. According to the proof of [Rot] Lemma 12.18, p.304, $\pi_R(v_2^{-1})$ describes a derivation from hq_owh to q in Δ. Because of (3) such a derivation contains more than $|A_n(w,w)|$ steps. This means $|v_2^{-1}|_R > |A_n(w,w)|$, and therefore $|A_n(w,w)| \leq |v_2^{-1}|_R \leq |v_2^{-1}| \leq |u_1..u_1| \leq |u| - 3$. Therefore, a word of length $2|w|+7$, namely $\bar{h}w^{-1}\bar{q}_oht\bar{h}q_owh$, is substituted by a word of length $> |A_n(w,w)|+3$, namely u.

Let $\alpha = \max\{|y|: y \in L_5 \cup L_5^{-1} \cup \{s\bar{s},\bar{s}s | s \in S_5\}\}$. Then in order to construct a word of length $> |A_n(w,w)|+4$ from a word of length $2|w|+7$, at least $\lceil\frac{1}{\alpha}(|A_n(w,w)|-2|w|-3)\rceil$ steps are necessary. Hence a derivation from $\bar{h}w^{-1}\bar{q}_oht\bar{h}q_owh$ to a word $u \in (\underline{\{x,\bar{q}tq\} \cup R})^*$ needs at leas $\lceil\frac{1}{\alpha}(|A_n(w,w)|-2|w|-3)\rceil$ steps. Therefore every derivation from $\bar{k}\bar{h}w^{-1}\bar{g}_oht\bar{h}q_owhk\bar{h}w^{-1}\bar{q}_o\bar{h}t\bar{h}q_owh$ to e in $<S_5;L_5>$ needs at least $\lceil\frac{1}{\alpha}(|A_n(w,w)|-2|w|-3)\rceil$ steps, i.e. in order to derive a word of length $3|w|+16$ to e in $<S_5;L_5>$ at least $\lceil\frac{1}{\alpha}(|A_n(w,w)|-2|w|-3)\rceil$ steps are necessary.

Hence $<S_5;L_5>$ is not E_{n-1}-d.b., which proves Theorem 3.3.

3.4. COROLLARY. *For every* $n \geq 4$ *there is a f.p. group having an* E_3*-decidable word problem such that each finite presentation of this group is* E_n*-, but not* E_{n-1}-d. b.

Proof. Theorem 3.3 and Theorem 3.2.

3.5. COROLLARY. *For every* $4 \leq m < n$ *there is a f.p. group such that the word problem for this group is* E_m*-, but not* E_{m-1}*-decidable, and each finite presentation of this group is* E_n*-, but not* E_{n-1}-d.b.

Proof. Let $G_1 = <\Sigma_1;L_1>$ be f.p. having an E_3-decidable word problem and being E_n-, but not E_{n-1}-d.b. (3.3). Let $H = <\Delta;M>$ be f.g. having an E_m-, but not E_{m-1}-decidable word problem. Then there is a group $G_2 = <\Sigma_2;L_2>$ which is f.p. and E_m-d.b. s.t. $H \hookrightarrow G_2$ (2.1). According to 3.1, G_2 has an E_m-decidable word problem. The word problem of G_2 is not E_{m-1}-decidable since the word problem of H is not either. Hence G_2 is not E_{m-1}-d.b. because of 3.1. Let $G = G_1 * G_2 = <\Sigma_1 \cup \Sigma_2;L_1,L_2>$. Then G is f.p., the word problem for G is E_m-, but not E_{m-1}-decidable, and the given presentation of G, and therewith each finite presentation of G, is E_n-, but not E_{n-1}-d.b. (1.5 a)).

This last corollary shows that even for f.p. groups the complexity of a n.a. for solving the word problem can be of an arbitrarily higher degree than the complexity of the word problem itself.

3.6. REMARK. According to a remark in [Av-Madl], p.93, the word problem of the group G_5 constructed in the proof ot Theorem 3.3 is even E_2-decidable, since the special word problem of the underlying semigroup Δ is E_1-decidable because of (1), p.155. Hence for every $n \geq 3$ there is a f.p. group having an E_2-decidable word problem and being E_n-, but not E_{n-1}-d.b.

4. NATURAL E_n-ALGORITHMS FOR E_n-DECIDABLE GROUPS.

For f.p. groups the property of E_n-derivation-boundedness leads to a natural E_n-algorithm for solving the word problem of the group. If a presentation has infinitely many relators we have infinitely many possibilities of inserting a relator in each step of a derivation, but only a finite number of deletions of a defining relator are possible, since only subwords are deleted. For non-f.p. groups a stronger concept of derivation-boundedness is therefore needed which guarantees the existence of a natural algorithm of the same complexity. There are several different possible definitions of d.b. group presentations for non-f.p. groups. We choose the following one, in which the allowed derivations are restricted.

4.1. DEFINITION. Let $G = \langle\Sigma;L\rangle$ f.g. The presentation $\langle\Sigma;L\rangle$ is *strongly* E_n-*derivation bounded* (s. E_n-d.b.) if there is a function $k \in E_n(\Sigma)$ such that for any $w =_G e$ in Σ^*, there is a derivation $w \equiv w_0 \to w_1 \to \ldots \to w_\ell \equiv e$ in $\langle\Sigma;L\rangle$ such that (i) $\ell \leq |k(w)|$, (ii) only trivial relators are inserted. Such a derivation is called a *strongly* E_n-*bounded derivation*.

4.2. OBSERVATION. a) Let $G = \langle\Sigma;L\rangle$ f.p. Then for all $n \geq 1$, $\langle\Sigma;L\rangle$ is s.-E_n-d.b. iff $\langle\Sigma;L\rangle$ is E_n-d.b. (The insertion of a relator u can be simulated by the insertion of uu^{-1} by using trivial relators and the delection of u^{-1}. So the length of the derivation is at most increased by the factor $\mu = (\max\{|u| : u \in L\}+1)$.

b) Let $n,p \geq 0$, and $q := \max\{n,p,3\}$. If $G = \langle\Sigma;L\rangle$ is s.E_n-d.b. with $L \subseteq \Sigma^+$, E_p-decidable, then there is a natural algorithm $x \in E_q(\Sigma)$ for the word problem of $\langle\Sigma;L\rangle$, i.e.

$$x(w) \equiv \begin{cases} (w_0,w_1,\ldots,w_\ell) & \text{if } w =_G e, \text{ and } w \equiv w_0 \to w_1 \to \ldots \to w_\ell \equiv e \text{ is a strongly} \\ & E_n\text{-bounded derivation from } w \text{ to } e \text{ in } \langle\Sigma;L\rangle. \\ \# & \text{if } w \neq_G e. \end{cases}$$

The proof of this fact is similar to the proof of Theorem 3.1. The only difference is that only strongly E_n-bounded derivations are considered.

c) The property of being strongly E_n-d.b. is dependent on the chosen presentation of the group. Let $n \geq 2$, $\Delta = \{a_i \mid i \geq 1\}$, and

$$G = \langle\Delta; a_1^i (i \geq 1), a_i a_1^{A_{n+1}(i,i)} (i \geq 2)\rangle,$$

where A_{n+1} is the n+1st Ackermann-function ([Rit]Def.1.1, p.1028).

For all $w \in \underline{\Delta}^*$, $w \underset{G}{=} e$, i.e. $G \cong \langle e \rangle$, and so $WP_G \in E_1(\underline{\Delta})$. Let $F = \langle b,c;\emptyset \rangle$ and $K = F*G \cong F$. Then $WP_K \in E_1(\underline{\Delta} \cup \underline{\{b,c\}})$. Finally let

$$\begin{aligned} H &= \langle K,t;\bar{t}b^n cb^n cb^n cb^n t = b^n cb^n a_n cb^n cb^n : n \geq 1 \rangle \\ &\cong \langle b,c,t;\ (\bar{b}\bar{c}\bar{b}\bar{t}bcbcbcbt\bar{b}\bar{c}\bar{b}\bar{c})^i (i \geq 1), \bar{b}^i\bar{c}\bar{b}^i\bar{t}b^i cb^i cb^i cb^i t\bar{b}^i\bar{c}\bar{b}^i\bar{c}(\bar{b}\bar{c}\bar{b}\bar{t}bcbcbcb \\ &\qquad t\bar{b}\bar{c}\bar{b}\bar{c})^{A_{n+1}(i,i)} : i \geq 2 \rangle \\ &=: \langle \Sigma;L_{n+1} \rangle \\ &\cong \langle \Sigma;\bar{t}b^i cb^i cb^i cb^i t\bar{b}^i\bar{c}\bar{b}^i\bar{c}\bar{b}^i\bar{c}\bar{b}^i : i \geq 1 \rangle =: \langle \Sigma;L' \rangle. \end{aligned}$$

Then $\langle \Sigma;L' \rangle$ is s.E_2-d.b. and $\langle \Sigma;L_{n+1} \rangle$ is s.E_{n+1}-d.b. but not s.E_n-d.b.

Since there are f.p. groups with E_3-decidable word problem for which no finite presentation allows a natural E_3-algorithm (the group $G_5 = \langle S_5;L_5 \rangle$ in 3.3 has this property), we ask whether there is an infinite strongly E_3-d.b. presentation for this group, and further on whether this is the case for all E_n-decidable f.g. groups.

For the group G_5 we get that the presentation

$$\langle S_5;L_5,\bar{k}\bar{t}Y^{-1}\bar{q}_j\bar{X}^{-1}ht^{\varepsilon}\bar{h}\bar{X}q_jYhk\bar{h}Y^{-1}\bar{q}_j\bar{X}^{-1}h\bar{t}^{\varepsilon}\bar{h}\bar{X}q_jYh : \varepsilon \in \{\pm 1\}, X,Y \in \tilde{S}^*,\ q_j \in Q-\{q\} \rangle.$$

has an E_1-decidable set of defining relators, and that it is in fact s.E_3-d.b. So a natural E_3-algorithm exists for this special presentation. We want to prove that such easy presentation can be constructed for all E_n-decidable f.g. groups ($n \geq 3$). Therefore we need the following technical lemma, which is proved by standard methods.

4.3. LEMMA. *Let Σ with $|\Sigma| > 1$, $t \in \Sigma$, be a finite alphabet, and $\emptyset \neq L \subseteq \Sigma^*$ be E_n-decidable for some $n \geq 3$. Then there is a function $g \in E_1(\Sigma)$ such that*

(a) $g(\{t^i | i \geq 0\}) = L$.

(b) *There exists a function $k \in E_n(\Sigma)$ satisfying:*
$\forall w \in \Sigma^*(w \in L \rightarrow \exists i \leq |k(w)| : g(t^i) \equiv w)$,

i.e. L is enumerated by an E_1-function g such that for each word w an index can be calculated by an E_n-function.

4.4. THEOREM. *Let $G = \langle \Sigma;L \rangle$ be f.g. with E_n-decidable word problem for some $n \geq 3$, and let $t \notin \Sigma$. Then G has a non-finite presentation $\langle \Sigma,t;L_g \rangle$ such that*

(1) $L_g \subseteq (\underline{\Sigma} \cup \{\underline{t}\})^*$ *is E_1-decidable.*

(2) $\langle \Sigma,t;L_g \rangle$ *is strongly E_n-d.b.*

Proof. Let $\tilde{L} := \{w \in \underline{\Sigma}^+ | w \underset{G}{=} e\}$. $\tilde{L}$ is E_n-decidable in $\underline{\Sigma}^*$, and so $\tilde{L}$ is E_n-decidable in $(\Sigma \cup \{\underline{t}\})^*$. Because of Lemma 4.3 there is a function $g \in E_1(\underline{\Sigma} \cup \{\underline{t}\})$ such that $g(\{t^i | i \geq 0\}) = \tilde{L}$ and there exists a function $k \in E_n(\underline{\Sigma} \cup \{\underline{t}\})$ satisfying:

$$\forall w \in (\underline{\Sigma} \cup \{\underline{t}\})^*(w \in \tilde{L} \rightarrow \exists i \leq |k(w)|(g(t^i) \equiv w)).$$

Let $L_g = \{t, t^i g(t^i) : i \geqslant 0\}$. Then

$$\langle\Sigma,t;L_g\rangle = \langle\Sigma,t;t,t^i g(t^i): i \geqslant 0\rangle \cong \langle\Sigma;g(t^i): i \geqslant 0\rangle = \langle\Sigma;\tilde{L}\rangle$$
$$\cong \langle\Sigma;L\rangle = G,$$

and so $\langle\Sigma,t;L_g\rangle$ is a f.g. presentation of G.

a) ***Claim.*** L_g *is* E_1*-decidable in* $(\underline{\Sigma} \cup \{\underline{t}\}))^*$. We have $w \in L_g$ iff $w \equiv t$ or $w \equiv t^i v$ with $v \in \underline{\Sigma}^+$ and $v \equiv g(t^i)$.

b) ***Claim.*** $\langle\Sigma,t;L_g\rangle$ *is strongly* E_n-d.b. Let $w =_G e$. Then we have the following derivation, where $w' \in \underline{\Sigma}^*$: $w \xrightarrow{1} w' \xrightarrow{2} t^i w' \xrightarrow{3} e$.

ad 1, all t^ε which appear in w are deleted. This takes $|w|_t \leqslant |w|$ steps, and $w' \equiv \pi_\Sigma(w)$ satisfies $|w'| \leqslant |w|$ and $w' =_G e$.

ad 2, if $w' \equiv e$ then we are ready. Let $w' \not\equiv e$. Then $w' \in \tilde{L}$ and because of (b) there is an $i \leqslant |k(w')|$ with $g(t^i) \equiv w'$. Insertion of i trivial relators $\bar{t}t$ and deletion of i relators $\bar{t}$ result in $t^i w'$. Here $2i \leqslant 2|k(w')|$ steps are sufficient.

ad 3, $t^i w' \equiv t^i g(t^i) \in L_g$, and so $t^i w'$ can be deleted within one step. Thus we have a derivation of w to e in $\langle\Sigma,t;L_g\rangle$ of length $m \leqslant |w|+2|k(w')|+1$ in which only trivial relators are inserted. Hence the presentation $\langle\Sigma,t;L_g\rangle$ is s.E_n-d.b.

4.5. COROLLARY. *Let* $G = \langle\Sigma;L\rangle$ *be f.g. with* E_n*-decidable word problem for some* $n \geqslant 3$*. Then there exists a f.g. presentation for* G *with an* E_1*-decidable set of defining relators such that the word problem for this presentation can be solved by a natural* E_n*-algorithm.*

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Universität Kaiserslautern
Fachbereich Informatik
Kaiserslautern, West Germany.

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LANGUAGES EXTENDING L(Q)

Jerome Malitz

ABSTRACT. We present a survey of the model theory of the quantifiers Q^n and $Q^{m,n}$, where $M \models_\kappa Q^n_\kappa \bar{x}\phi\bar{x}$ means that there is a κ-powered subset X of M such that $M \models_\kappa \phi\bar{a}$ whenever $a_1, \ldots a_n \in X$, and $M \models_\kappa Q^{m,n}\bar{x}\bar{y}\phi\bar{x}\bar{y}$ means that there is a κ powered subset X of M such that $M \models \phi\bar{a}\bar{b}$ whenever $a_1,\ldots,a_m \in X$ and $b_1,\ldots,b_n \notin X$. Some recent results are announced and several open problems are given.

1. INTRODUCTION.

Over the past several years, there has been considerable work done in the model theory of languages more expressive than the first order predicate calculus L. Our interests have centered about the languages L^n and $L^{m,n}$ introduced in $[MM]_1$. The first adds the quantifier Q^n to L, where $M \models_\kappa Q^n\bar{x}\phi$ means that there is κ powered subset X of M such that $M \models \phi\bar{a}$ whenever $a_1,\ldots,a_n \in X$. The second adds $Q^{m,n}$ to L, with $M \models_\kappa Q^{m,n}\bar{x}\bar{y}\phi\bar{x}\bar{y}$ meaning that some κ powered subset X of M exists such that $M \models_\kappa \phi\bar{a}\bar{b}$ for all $a_1,\ldots,a_m \in X$ and all $b_1,\ldots,b_n \notin X$. While considerable progress has been made in the study of these languages over the past few years, many fundamental questions remain open. Our intention here is to present a survey of known results, some recent unpublished results, and some of the open problems.

Section 2 is devoted to preliminaries, notation and definitions that will be used throughout the paper.

Section 3 is concerned with compactness questions for the L^n languages.

Section 4 considers the relative expressive power of these languages.

Section 5 is concerned with decidability questions arising in the context of the L^n languages.

Some recent results for $L^{m,n}$ are presented in section 6.

Some open problems are described in section 7.

This survey is in no way comprehensive, either in the results stated or the problems, mentioned. Rather, the material represents the personal interests of the author.

2. PRELIMINARIES.

We use i,j,k,ℓ,m,n to denote natural numbers; $\alpha,\beta,\gamma,\delta$ to denote ordinals; κ, λ,μ,ν to denote uncountable cardinals; κ^+ is the cardinal successor of κ, cX is the cardinality of X, and ${}^nX = \{(x_1,\ldots,x_n): x_i \in X \text{ for } i = 1,\ldots,n\}$.

n termed sequences $(x_1,\ldots,x_n)$ will be denoted by $\bar{x}$.

In [J], Jensen introduced the combinatorial principle $\Diamond_\kappa$: *there is set of subsets of* κ, $\{S_\alpha: \alpha < \kappa\}$ *such that for all* $X \subseteq \kappa$, $\{\alpha \in \kappa: X \cap \alpha = S_\alpha\}$ *is stationary* (i.e. meets every closed bounded subset of κ). He proved that if $V = L$ then $\Diamond_\kappa$ holds for every regular κ. $\Diamond_\kappa$ will appear in the hypotheses of several of the theorems we shall mention.

M and N will be used to denote structures. tM is the type of M. $|M|$ is the universe of M. If s is a type then $M\restriction s$ is the reduct of M to s. If $A \in tM$, A a unary relation symbol, then $M\restriction A$ is the relativization of M to A.

Let $\mathcal{Q}$ be a set of quantifiers, L the first order predicate calculus. $L(\mathcal{Q})$ is the language obtained by adjoining the quantifiers in $\mathcal{Q}$ to L, i.e, to the definition of fm for L we add the clause:

if $Q \in \mathcal{Q}$ and Q binds n variables and $\bar{v}$ is a sequence of n variables and $L(\mathcal{Q})$ then $Q\bar{v}\phi \in L(\mathcal{Q})$.

For $\mathcal{Q} = \{Q\}$ we write $L(Q)$.

The language L^n is $L(Q^n)$ where Q^n binds n variables. For each κ, $M \models Q^n\bar{v}\phi$ is given a κ interpretation: there is a κ powered subset X of $|M|$ such that $M \models \phi\bar{a}$ for all $\bar{a} \in {}^nX$.

$L^{m,n}$ is $L(Q^{m,n})$ where $Q^{m,n}$ binds m+n variables. The κ interpretation of $M \models Q^{m,n}\bar{u}\bar{v}$ is that for some κ powered proper subset X of $|M|$ and all $\bar{a} \in {}^mX$, $\bar{b} \in {}^n\tilde{X}$ we have $M \models \phi\bar{a},\bar{b}$. (The restriction that X be proper is necessary to avoid vacuous satisfaction of $Q^{m,n}\bar{u}\bar{v}\phi$).

$L^{<\omega} = L(\mathcal{Q})$ where $\mathcal{Q} = \{Q^1,Q^2,\ldots\}$.

We may write L^n_κ, Q^n_κ, $\models_\kappa$, etc. when the κ interpretarion is intended.

If Σ is a set of sentences then Mod Σ is the set of models of Σ. $Th_\Sigma M = \{\sigma \in \Sigma: M \models \sigma\}$, $Th_\Sigma K = \{\sigma \in \Sigma: M \models \sigma$ for all $M \in K\}$. $M \equiv_\Sigma N$ means that $Th_\Sigma M = Th_\Sigma N$.

Given two languages L_1 and L_2 we write $L_1 \leqslant L_2$ if for all $\Sigma_1 \subseteq L_1$ there is some $\Sigma_2 \subseteq L_2$ such that Mod $\Sigma_1 = ((\text{Mod}\Sigma_2)\restriction A)\restriction t\Sigma_1$. $L_1 < L_2$ means $L_1 \leqslant L_2$ but $L_2 \nleqslant L_1$. If the Σ_2 can always be chosen to be of type $t\Sigma_1$ we write $L_1 \leqslant\!\cdot L_2$.

$Val(L_1)$ is the set of valid L_1 sentences. L_1 is axiomatizable if $Val(L_1)$ is recursively enumerable.

A language L_1 is κ-compact if wherever $\Sigma \subseteq L_1$, $c\Sigma < \kappa$, and Mod $\Sigma = \emptyset$ then there is some finite subset $\Delta \subseteq \Sigma$ such that Mod$\Delta = \emptyset$. We say L_1 is countably compact if it is ω_1-compact.

3. COMPACTNESS, AXIOMATIZABILITY, AND LOWENHEIM-SKOLEM RESULTS FOR $L^{<\omega}$.

Clearly, L^1_κ is not fully compact. Indeed if $\Sigma = \{\neg Q_\kappa v(v = v)\} \cup \{c_\alpha \neq c_\beta : \alpha < \beta < \kappa\}$ then every subset of Σ of power $< \kappa$ has a model but Σ does not.

In [K], Keisler proves that L^1_κ is κ compact for all uncountable κ and axiomatizable for regular uncountable κ. His proof for $\kappa = \omega_1$ provided a starting point for our proof of compactness of $L^{<\omega}_{\omega_1}$. Recently, considerable progress has been made in the study of compactness for the L^n languages but many fundamental questions are still open.

THEOREM 3.1.1. $(\Diamond_{\omega_1})$ $L^{<\omega}_{\omega_1}$ *is countable compact and axiomatizable.*
3.1.2. $(\Diamond_{\kappa^+}, \Diamond_{\kappa^{++}})$ $L^{<\omega}_{\kappa^{++}}$ *is* κ^{++} *compact and axiomatizable.*

The first result was proved in $[MM]_1$. The κ^{++} compactness of $L^{<\omega}_{\kappa^{++}}$ is asserted in [S]. The axiomatizability of $L^{<\omega}_{\kappa^{++}}$ is not found in the literature but can be obtained as in Theorem 9.5 of $[MM]_1$.

The assumption $\Diamond_{\omega_1}$ in 3.1 is not necessary as was shown in $[MM]_1$ p.257, and similar arguments show that it is not necessary for 3.1.2 either.

THEOREM 3.2.1. $(\Diamond_{\omega_1})$ *If* $\sigma \in \mathrm{Val}(L^{<\omega}_{\omega_1})$ *then* $\sigma \in \mathrm{Val}(L^{<\omega}_\kappa)$ *for every regular* κ.

3.2.2. $(\Diamond_{\kappa^+}, \Diamond_{\kappa^{++}})$ *If* $\sigma \in \mathrm{Val}(L^{<\omega}_\lambda)$ *then* $\sigma \in \mathrm{Val}(L^{<\omega}_{\kappa^{++}})$ *for all regular* λ.

This first clause is found in $[MM]_1$ the second is a consequence of [S] but is not found there.

THEOREM 3.3. *If* κ *is weakly compact then* $L^{<\omega}_\kappa$ *is* κ *compact and axiomatizable.*

In fact if $\lambda < \kappa_\alpha$ for all $\alpha \in \lambda$ and each κ_α is weakly compact then $L(\{Q^n_{\kappa_\alpha} : n \in \omega, \alpha \in \lambda\})$ is λ compact. This is a straightforward generalization of 3.3 which appears in $[MM]_1$.

THEOREM 3.4. *If* κ *is weakly compact then* $\mathrm{Val}(L^{<\omega}_\kappa) \supseteq \mathrm{Val}(L^{<\omega}_\lambda)$ *for all* λ.

This is found in $[MM]_1$. Notice that the sentence

$$\forall uv[Ruv \to Rvu] \to [Q^2 uvRuv \vee Q^2 uv \neg Ruv]$$

is in $\mathrm{Val}(L^{<2}_\kappa)$ just in case κ is weakly compact.

When κ is a limit cardinal there is a natural alternative interpretation for Q_n: $M \models_\kappa Q^n \bar{v}\phi$ means that for all $\lambda < \kappa$ there is a λ powered subset $X \subseteq |M|$

such that $M \models_\kappa \phi\bar{a}$ for all $\bar{a} \in {}^nX$. In [MM]$_1$ we prove

THEOREM 3.5. *If κ is a strong limit cardinal and if $\lambda < \kappa$ then L^n_κ in the limit interpretation is λ compact.*

The problem of compactness for languages of the form $L(Q^m_\kappa, Q^n_\lambda)$ has for the most part been intractable to date. However, the following result appears in [Mo]$_1$.

THEOREM 3.6. *Let $\lambda < \kappa$ with κ weakly compact and $L^{<\omega}_\lambda$ λ-compact. Let $\mathcal{Q} = \{Q^n_\kappa : n = 1,2,\ldots\} \cup \{Q^n_\lambda : n = 1,2,..\}$. Then $L(\mathcal{Q})$ is λ-compact and axiomatizable.*

In [MR] the Q^n quantifiers are generalized to higher order suggested by writing $\exists X \forall v_1,\ldots,v_n \in X$ instead of $Q^n v_1,\ldots,v_n$. We let:

$$P^1_\kappa(R) = R$$
$$P^2_\kappa(R) = \{S : S \subseteq R \text{ and } cS \geqslant \kappa\}$$
$$P^{2+n}_\kappa(R) = P^2_\kappa(P^{n+1}_\kappa(R)) \quad n = 1,2,\ldots .$$

Let X^n_i, $i = 0,1,2,\ldots$ be variables ranging over $P^n_\kappa(|M|)$ in the κ interpretation for $n > 1$. An n-*order properly descending* quantifier is one of the form

$$B_n B_{n-1} \ldots B_2 B_1$$

where

$$B_n \text{ is } \exists X^n_j \text{ for some } j$$

and for $m < n$

$$B_m \text{ is a sequence } B_{m,1},\ldots,B_{m,k_m}$$

where

$$B_{m,i} \text{ is } \exists X^m_{i_1}, X^m_{i_2},\ldots,X^m_{i_{n(i)}} \in X^{m+1}_{\ell_i}$$

for some $X^{m+1}_{\ell_i}$ occurring in B_{m+1}. We identify X^1_i with the first order variable v_i. As an example,

$$\exists X^3_0 \, \forall X^2_0, X^2_1 \in X^3_0 \, \forall v_0, v_1 \in X^2_0 \, \forall v_2 \in X^2_1 [Rv_0v_1 \wedge \neg Rv_0v_2]$$

asserts the existence of a subset of the universe partitioned into κ many κ powered equivalence classes by R.

Let $\mathcal{Q}^*$ be the collection of all n-th order properly descending quantifiers for all n. Let $L^* = L(\mathcal{Q}^*)$. In [MR] it is shown that

THEOREM 3.7. ($\Diamond_{\omega_1}$) *L^* is countably compact and axiomatizable in the ω_1 interpretation.*

The results in [S] can be used to generalize this to interpretations κ^{++} when $\Diamond_{\kappa^+}$ and $\Diamond_{\kappa^{++}}$ both hold, to give: L^* is κ^{++} compact and axiomatizable in the κ^{++} interpretation. Moreover, the analogs of 3.2.1, 3.2.2, 3.3, 3.4. all hold for L^*.

Fix a similarity type t with a unary relation symbol B. Vaught defined the two cardinal type of a structure M to be $(c|M|, cB^M)$. In [Mor] and [V] it is proved that if for all $n > 0$ there is a κ such that Σ has a model M of two cardinal type $(2^\kappa_n, \kappa)$, then Σ has a model of any two cardinal type (λ, μ) where $\lambda \geqslant \mu \geqslant c\Sigma + \omega$ (Here $2^\kappa_1 = \kappa$, $2^\kappa_{n+1} = 2^\lambda$ where $\lambda = 2^\kappa_n$). The following theorem from $[MM]_1$ generalized this.

THEOREM 3.8. *Let* $\Sigma \subseteq L$ *and let* R *be an* n-*ary relation symbol in* $\tau\Sigma$. *Suppose for each* n *there is a* κ *and a model* M *of* Σ *such that* $c|M| = 2^\kappa_n$ *and* $M \models \neg Q^n_\chi \bar{v} R\bar{v}$. *Then for every* $\kappa > \lambda \geqslant c\Sigma + \omega$ *there is a model* M *of* Σ *such that* $c|M| = \kappa$ *and* $M \models \neg Q^n_\lambda \bar{v} R\bar{v}$.

4. RELATIVE EXPRESSIVE POWER OF THE L^n LANGUAGES.

In $[MM]_1$ we showed that $L^1_\kappa \leqslant L^2_\kappa$ for all regular κ. In an unpublished paper, S. Garavaglia proved that $L^n_\kappa <\cdot L^{n+1}_\kappa$. Recently, using a forcing argument, it is shown in [RS] that

THEOREM 4.1.1. *Assuming* $\Diamond_{\omega_1}$, $L^n_{\omega_1} < L^{n+1}_{\omega_1}$ *for all* n.

Combining this result with the techniques in [S] one easily obtains

THEOREM 4.1.2. *Assuming* $\Diamond_{\kappa^+}$ *and* $\Diamond_{\kappa^{++}}$, $L^n_{\kappa^{++}} < L^{n+1}_{\kappa^{++}}$.

P. Rothmaler and P. Tuschik [RT] give sufficient conditions for the elimination of the L^n quantifiers for a countable first order theory. So elementary classes whose theories satisfy the conditions can not be split by means of L^n sentences.

5. DECIDABLE QUESTIONS.

Here we mention a few results about the decidability of models, decidability of theories, and the decidability of sentences with respect to theories. In several of these instances one can view the results as showing the expressive strength of L^n over L^1.

It is easy to find structures whose L theories are decidable but whose L^1_κ

theories are not, for example, take $M = \langle A, B_n \rangle_{n \in \omega}$ where for some nonrecursive set X, B_n is countably infinite iff $n \in X$. On the other hand, for $n \geq 1$ we do not know of such structures whose L^n_κ theory is decidable but whose L^{n+1}_κ theory is not.

In [R] one finds an example of a "natural" class of structures whose $L^1_{\omega_1}$ theory is decidable but whose $L^2_{\omega_1}$ theory is not.

THEOREM 5.1. (Rubin). *The* L^1 *theory of boolean algebras is decidable but the* L^2 *theory is not.*

The decidability of the L^1 theory of boolean algebras was discovered independently by M. Weese [W].

A number of other decidability results of this nature are mentioned and an extensive bibliography is given in D. Seese [Se]. Many of the decidability results can be found in [BSTW]. In particular, the reader should see H. Tuschik [T] for results on the decidability of L^n theories of linear orderings.

In another direction Macintyre [Ma], Morgenstern $[Mo]_2$, and Schmerl and Simson [SS] turn their attention to L^2 extensions of Peano's arithmetic. The axiomatization given in $[MM]_1$ (correct and, with $\diamondsuit_{\omega_1}$, complete for validities in the ω_1 interpretation) is correct for the ω interpretation. When the usual first order version of the Peano arithmetic is enriched by adding all instances of the induction schema involving L^2 formulas we get the theory P^2 (Morgenstern observes that the Q^1 quantifier can be defined in arithmetic using L and that the quantifiers Q^n for $n > 2$ can be defined in arithmetic using L^2). In $[Mo]_2$ and [Ma] it is shown that truth for first order formulas in arithmetic can be defined in P^2, which leads to

THEOREM 5.2. *The Harrington Paris combinatorial principle is provable in* P^2.

Simson and Schmerl broaden this to show that even stronger combinatorial principles considered by Friedman, McAloon and Gunison are also provable in P^2. This leads naturally to the problem of finding a "meaningful" statement of P^2 or Peano's arithmetic that is undecidable in P^2 (of course by Gödel's 2nd theorem there are undecidable L statements in P^2). Morgenstern has noticed that Kruskals theorem [K] is statable in P^2 and this is a candidate.

6. THE $L^{m,n}$ LANGUAGES.

The languages $L^{m,n}$ were introduced in $[MM]_1$, being called $L^{\#}$ there. It was shown there that even $L^{1,1}$ is *not* countably compact in any infinite power. The purpose of presenting this language there was to show that the L^n languages could not be generalized in this direction without losing compactness properties. However, in [Ma] we began to investigate the model theory of $L^{m,n}$. Regarding the relative expressive power of these languages we have

THEOREM 6.1. $L^m_\kappa \leqslant\cdot L^{m,n}_\kappa \leqslant L^{2,2}_\kappa$ *for all* m,n. *When* L^m_κ *is countably compact then* $L^m_\kappa < L^{m,n}_\kappa$.

All questions about relative expressive power not answered by 6.1 are open. The only other bit of information on these languages is

THEOREM 6.2 *Let* $\sigma \in L^{1,1}$ *and suppose there is a model of* σ *in the* κ *interpretation where* κ *is regular and* $\kappa > \omega$. *Then there is a model of* σ *in the* ω *interpretation.*

The expressive strength of $L^{m,n}$ makes a generalization of this theorem desirable. For example, the sentence $\neg Q^{m,1}\bar{u}v[f\bar{u} \neq v]$ asserts that f is not closed on a κ powered subset of the universe. It follows that in a finite functional type one can express the property of a Jónsson algebra. A strengthening of the theorem above would yield results such as: if there is a Jónsson algebra in $\mathrm{Mod}_\kappa\Sigma$ then there is one in $\mathrm{Mod}_\lambda\Sigma$.

7. OPEN PROBLEMS.

This list of problems is by no means comprehensive, instead it represents the author's particular interests. In many of these problems only relative consistency results can be hoped for.

Is $L^{<\omega}$ $<\kappa$-compact in the κ interpretation when the cofinality of $\kappa \geqslant \omega_1$?

At the moment we do not know if L^2 is countably compact in the $\beth_{\omega_1}$ interpretation or in the first strongly inaccessible interpretation.

In the cases where compactness is known, completeness is also, at least in the sense that the validities are recursively enumerable. Positive answers to any of the above should yield completeness results also.

Let Val_κ be the set of validities of $L^{<\omega}$ in the κ interpretation. Let κ and κ' be successor cardinals and let λ and λ' be of cofinality strictly between ω and κ. Let μ and μ' be inaccessible but not weakly compact, ν and ν' weakly compact. We suspect that $\mathrm{Val}_\kappa = \mathrm{Val}_{\kappa'} \subset \mathrm{Val}_\lambda = \mathrm{Val}_{\lambda'} \subset \mathrm{Val}_\mu = \mathrm{Val}_{\mu'} \subset \mathrm{Val}_\nu = \mathrm{Val}_{\nu'}$ (It is easy to see that $\mathrm{Val}_\kappa \not\supseteq \mathrm{Val}_\lambda \not\supseteq \mathrm{Val}_\mu \not\supseteq \mathrm{Val}_\nu$).

A purely set theoretic combinatorial statement equivalent to the countable compactness of $L^{<\omega}$ might be an interesting new axiom for set theory.

We have mentioned that L^{n+1} is more expressive than L^n (even up to relativised reducts). Can this be sharpened in the following way? Let $M = \langle A, R^M, \ldots \rangle$ where R^M is a symmetric n+1-ary relation and the cardinality of A is $\kappa > \omega$. Is there some N equivalent to M with respect to the language L^n such that $N \models Q^{n+1}\bar{x}R\bar{x} \vee Q^{n+1}\bar{x} \neg R\bar{x}$?

Regarding the $L^{m,n}$ languages, there are two obvious questions. In view of Theorem 6.1 it is natural to investigate the relative expressive power of $L^{1,1}$, $L^{1,2}$, $L^{2,1}$, and $L^{2,2}$.

Theorem 6.2 raises the following questions. For what $m,n \in \omega, \kappa, \lambda$ will satisfiability in the κ interpretation of $\sigma \in L^{m,n}$ imply satisfiability of σ in the λ interpretation? In particular, we do not know if satisfiability of $\sigma \in L^{1,1}$ in the κ interpretation, κ uncountable, regular and $> \lambda$ implies satisfiability of $\sigma \in L^{1,1}$ in the ω_2 interpretation. Nor do we know if satisfiability of $\sigma \in L^{1,2}$ or $L^{2,1}$, or $L^{2,2}$ in the κ interpretation, κ uncountable and regular implies the satisfiability of σ in the ω_1 interpretation.

Theorem 5.2 presents an R.E. extention P^2 of Peano's arithmetic in which one can prove the combinatorial principles of Harrington and Paris which are independent of Peano's arithmetic. At the moment there is no 'natural' sentence independent of L^2 that is known. In particular, it is not known if Kruskal's theorem [K] is decidable in P^2.

For each $n \geqslant 1$ is there a (natural) structure whose L^n theory is decidable but whose L^{n+1} theory is not?

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Department of Mathematics
University of Colorado
Boulder, Colorado, U.S.A.

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WHAT IS A MATHEMATICAL THEORY?

Jan Mycielski

Since Hilbert's and Skolem's work in foundations of mathematics we got used to mathematizing the concept of a theory as a theory formalized in first order logic. This view was very fruitful since it generated model theory and proof theory, but it may have obscured the fact that there are possibilities of other more abstract mathematizations of the concept of a theory which raise other deep and interesting problems. It is the purpose of this lecture to point out two such mathematizations, and the way in which one of them leads to a mathematical concept of finitistic theory.

1. $\tau\pi$-THEORIES.

By a *normal theory* we mean a theory T which is formalized in first order logic with equality and axiomatized by a finite set of axioms or axiom schemata (see [10]) such that $\forall xy\,[x = y]$ is not a theorem of T. By a proof in such a theory we mean a Hilbert style proof from the axioms.

Let Σ be a finite alphabet and Σ^* the set of all words, i.e. finite sequences of elements of Σ. For any $\xi \in \Sigma^*$, $|\xi|$ denotes the length of ξ.

A $\tau\pi$-*theory* is a set of pairs $T \in \Sigma^* \times \Sigma^*$ such that there exists a polynomials $p(x,y)$ and a Turing machine M such that, for any $(\tau,\pi) \in \Sigma^* \times \Sigma^*$. M can decide in time $\leqslant p(|\tau|,|\pi|)$ if $(\tau,\pi) \in T$.

If $(\tau,\pi) \in T$ then τ is called a *theorem* of T and π is called a *proof* of in T.

Every normal theory defines a $\tau\pi$-theory since the time necessary to check the correctness of a Hilbert style proof in a normal theory can be estimated from above by a polynomial of the length of that proof.

Now, a $\tau\pi$-theory T will be called *amenable* (to *automatization*) iff there exists another polynomial $P_0(x,y)$ and another Turing machine M_0 such that, given any word $\tau \in \Sigma^*$ and any positive integer n, M_0 can decide in time $\leqslant p_0(|\tau|,n)$ if there exists a $\pi \in \Sigma^*$ with $|\pi| \leqslant n$ such that $(\tau,\pi) \in T$. (Notice that if we replaced the condition $\leqslant p_0(|\tau|,n)$ by the condition $\leqslant p_0(|\tau|,c^n)$ where $c = \operatorname{card}\Sigma$, then the concept would trivialize since every $\tau\pi$-theory would be amenable).

It is clear that after Gödel's discovery that all sufficiently strong theories are undecidable, the next question which should have presented itself is the question if the $\tau\pi$-theories (corresponding to normal theories) are amenable. But we had to wait until 1971 (the paper of Cook [1]) for a clear statement of that question. To end this part of my talk let me formulated the following proposition (which is implicit in [1]).

PROPOSITION. *The following three statements are equivalent to each other:*

i) $P \neq NP$;

ii) *There exists a $\tau\pi$-theory which is not amenable;*

iii) *Every $\tau\pi$-theory which is defined by a normal theory is not amenable.*

I think that this proposition constitutes the best way of explaining the great importance of Cook's conjecture $P \neq NP$ for the foundations of mathematics. (Its importance in computer science is also well known [3]).

2. INTERPRETABILITY.

Now we want to introduce you to another abstraction which we call the *locall interpretability type*, or, *chapter* of a first order theory.

First a sentence σ without functions symbols nor equality is *interpretable* in a theory T if one can substitute the variable of σ by n-tuples of variables (for some integer n), and the relation symbols of σ by formulas which may have additional free variables (called parameters of the interpretation) such that the existential closure of the resulting formula is a theorem of T. of course, if σ and σ' are sentences of the same shape, i.e., if they differ only by the names of their relation symbols, then σ is interpretable in T iff σ' is interpretable in T.

For any first order theory T the *chapter* of T, in symbols $|T|$, is the set of all shapes of sentences interpretable in T. Let $\mathcal{J}$ be the set of all chapters of theories. Thus $\mathcal{J}$ is a family of sets.

It is easy to check that the partial order $<\mathcal{J}, \subseteq>$ constitutes a complete lattice, since the intersection of any set of chapters is again the chapter of some theory.

From the point of view of ordinary informal mathematics the chapter $|T|$ of a theory T is no less interesting than T itself. E.g., $|T|$ does not depend on the choice of the primitive symbols of T, in fact $|T|$ is immune to extensions of T by means of defined symbols, and, $|T|$ reflects very well the mathematical strength of T. Thus a study of the lattice $<\mathcal{J}, \subseteq>$ seems very important. In [5] we have published a preliminary study of this lattice. E.g, $<\mathcal{J}, \subseteq>$ is Brouwerian, its zero has one succesor, etc. Now we want to point out some open problems:

(A) Does $<\mathfrak{J}, \subseteq>$ have any automorphisms? If it does, are the types of some important theories like PA or ZF fixed points of all automorphisms? (Similar problems for some lattices of equational theories were recently solved by Kezek [4]).

(B) We say that a theory T is *connected* iff for all, a,b $\in J$ if $a \vee b = |T|$ then $a = |T|$ or $b = |T|$. P. Pudlâk has shown [9] that many interesting theories are connected. Are the theories of real closed fields or of algebraic closed fields connected?

3. FINITISM.

A first order theory T will be called *finitistic* iff every finite part of T has finite models. The following proposition follows from Proposition 3(i) of [5].

PROPOSITION. *A theory is finistic iff its type is either zero or the successor of zero in the lattice* $<\mathfrak{J}, \subseteq>$.

It is surprising that there exists finitistic theories (whose type is the successor of zero) with a considerable mathematical content. In fact we have constructed a finitistic recursively axiomatized theory FIN which appears to be as powerful as analysis [6,7,8].

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Department of Mathematics
University of Colorado
Boulder, Colorado, U.S.A.

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THE CONSISTENCY OF A VARIANT OF CHURCH'S THESIS WITH AN AXIOMATIC THEORY OF AN EPISTEMIC NOTION

W.N. Reinhardt

ABSTRACT. In this paper we prove the consistency of a variant of Church's Thesis than can be formulated as a schema in a first order language with a modal operator for intuitive provability. We also conjeture the consistency of a stronger variant.

1. INTRODUCTION.

We consider the language of arithmetic augmented by a new symbol B and the formation rule: If σ is a sentence (or formula) so is $B\sigma$. The informal meaning intended for $B\sigma$ is that σ is intuitively provable, so that for example $\neg B\sigma \wedge \neg B\neg\sigma$ expresses the (absolute) undecidability of σ. This interpretation suggest notions of intuitive decidability, for example

$$\forall x(B\theta(x) \vee B\neg\theta(x))$$

express the intuitive decidability of $\theta(x)$, and these motivate the formulation of our variant of Church's Thesis. Since Turing advocated the view that any intelectual activity of humanscan be carried out by a properly programmed computer, and in particular that theorem proving by an idealized human mathematician is essentially mechanical, the thesis we formulate migth appropriately be called Turing's thesis. I believe that B expresses an important epistemic notion and that the axiomatic theory given here can be used to illuminate for example some controversies regarding the philosophical significance of Gödel's incompleteness theorems. In this paper we leave these issues aside, and simply formulate the theory and prove it consistent with one variant of Church's Thesis. In a later paper we shall discuss these issues and the relation of B to earlier authors. (Gödel 1933, 1951, Löb 1955, Kalmar 1959, Myhill 1960, Lucas 1961, Montague 1963, Benacerraf 1967, Tharp 1973, Wang 1974, Boolos 1979, Shapiro 1980). I would like to thank Andrej Šc̆edrov for pointing out an error in the first version of this paper. In the earlier version a proof was claimed for conjecture-3-1a. of this paper. The problem remains open.

2. ARITHMETIC WITH B.

We split the axioms into four groups: those which may be regarded as applicable to any subject matter (the logical axioms), those peculiar to arithmetic, those involving the truth (or satisfaction) predicate for arithmetic, and those stating Church's Thesis. In the first and third groups some of the axioms are essentially classical (e.g. instances of classical schemas which may however involve B) and others are new, peculiar to languages with B. The arithmetic axioms are essentially classical.

2.1. Logical axioms.

We suppose our languages has variables $v_0, v_1, \ldots$, a one place sentential connective $\neg$ (for negation), a two place sentential connective $\rightarrow$ (for truth functional implication), the universal quantifier $\forall$, a one place sentential connective B (for provability), and equality $\doteq$. We allow relation symbols and certain function symbols, but logic with function symbols in the general case requires restrictions not familiar from classical logic. (In effect, we may allow function symbols for recursive functions with no restrictions, or arbitrary function symbols with certain caveats which will be mentioned). We shall use $\neg$, $\rightarrow$, $\forall$ not as names for symbols but as names for operations. Thus if θ, ϕ are formulas, $\neg\theta$, $(\theta \rightarrow \phi)$ are to be formulas. We treat defined connectives such as $\vee$, $\wedge$, $\leftrightarrow$ similarly.

We have the usual formation rules for first order languages, plus the rule: if θ is a formula, $B\theta$ is a formula with the same free variables as θ. A sentence is a formula with no free variables.

In order to state the axioms for the truth predicate (in §2.3), it will be convenient to suppose that all syntactic objects have been identified with their Gödel numbers, in one of the usual ways. Thus the syntactic operations $\neg$, $\rightarrow$, etc. are all primitive recursive. It will not much matter how this is done, but for convenience in describing substitution operations, one may think of formulas as strings of symbols. What is important is that the various syntactic operations $\neg$, $\rightarrow$, substitution, etc. will be primitive recursive.

DEFINITION 2.1.

a) By a B-*closure* of a formula θ we mean a sentence obtained from θ by iterated applications of universal quantification and B. Thus $\forall x \forall y\, x \doteq y$, $B\forall x B \forall y B\, x \doteq y$, and $B\forall x \forall y\, x \doteq y$ are all B-closures of $x \doteq y$. If σ is a sentence it is a B-closure of itself.

b) $\theta(x/y)$ is the expression obtained from θ by replacing all free occurrences of x in θ by y.

c) By the *logical axioms* we understand the B-closures of the following schemas (where u,x,y,z are variables):

L1. truth functional tautologies

L2. $\forall x(\theta \to \phi) \to (\forall x\theta \to \forall x\phi)$

L3. $\forall y(\forall x\theta \to \theta(x/y))$, where x,y are variables and x is free for y in θ,

L4. $\theta \to \forall x\theta$, where x is not free in θ,

L5. $x \doteq x$

L6. $x \doteq y \to [\theta(u/x) \to \theta(u/y)]$, where u is free for x,y in θ,

L7. $B(\theta \to \phi) \to (B\theta \to B\phi)$

L8. $B\theta \to \theta$

L9. $B\theta \to BB\theta$

L10. $B\forall x\theta \to \forall xB(\exists y(x = y) \to \theta)$, where y is a variable distinct from x, and $\exists$ means as usual $\neg\forall\neg$,

L11. $B\exists zB(t = z) \to [\forall x\theta \to \theta(x/t)]$, where x,z are variables, t is a term and
i) z does not occur free in t
ii) x is free for t in θ,

L12. $\exists y(t = y) \to [\forall x\theta \to \theta(x/t)]$, provided that i), ii) above hold, and in addition x does not occur free within the scope fo B in θ.

d) A *theory* in the language with B is a set of sentences containing the logical axioms and closed under modus ponens. We write $A \vdash \sigma$ to mean σ is in every theory including A. We write as usual $\vdash\sigma$ for $\emptyset \vdash \sigma$.

We note:

PROPOSITION 2.2.

a) *Suppose that* A *is a set of sentences such that whenever* $\sigma \in A$, $B\sigma \in A$. *Then* $A \vdash \sigma$ *implies* $A \vdash B\sigma$.

b) *If the sentence* σ *is a classical validity, in a language with no function symbols, then* $\vdash\sigma$.

c) *If the sentence* σ *is a classical validity in a language with function symbols* f_i, *and* A *is the set of sentences* $\forall x\exists y\ f(\vec{x}) = \vec{y}$, *then* $A \vdash \sigma$.

Proof. a) The only rule is modus ponens, so apply L7, L2.
b) Since L1-L6 are the usual classical schemas, this is obvious.
c) By L12, it is sufficient to see that $A \vdash \exists y(t = y)$ for all terms t built from the f's. This is easily seen by induction on t; e.g. if $t = f(t_1,t_2)$, and $A \vdash \exists y_1(t_1 = y_1) \wedge \exists y_2(t_2 = y_2) \wedge \forall y_1 y_2 \exists z(f(y_1,y_2) = z)$, then by L12, $A \vdash \exists z(f(t_1,t_2) = z)$.

We note also the following:

L3' $\qquad \forall x\theta \to \theta(x/y)$, where x is free for t in θ, provided that we restrict the introduction of function symbols to (for example) primitive recursive functions. This is because the antecedent of L11 says that t is a term which may be effectively evaluated (this is the caveat referred to earlier) and it will follow from our arithmetic axioms that this is so for primitive recursive terms. The antecedent in L12 may of course be omitted altogether if we follow the usual practice of classical logic of assuming all functions everywhere defined.

We remark without proof that the axioms L11-L12 are chosen so that the following is true.

THEOREM 2.3. *If* T *is a theory with* $T\vdash B\forall\vec{x}\exists!y$, *no* x_i *occurs in the scope of* B *in* θ, f *is new function symbol, and* T' *has as axioms* T *together with all* T *closures of*

$$\theta(\vec{x},y) \leftrightarrow f(\vec{x}) = y,$$

Then T' *is a conservative extension of* T.

In particular, if T has no function symbols, then we may add them with impunity, but only by observing the caveats regarding the principle of universal instantiation with complex terms (the case of functions $fx = y$ corresponding to $\theta(x,y)$ with x in the scope of B would require a further restriction in L12).

2.2. Arithmetic axioms.

By the language of (first order) Peano arithmetic we understand the language with an individual constant for 0, a unary function symbol for the succesor operation S, and a function symbol d for each primitive recursive definition of a function f_d (we could of course do with only function symbols for plus and times, but it will be convenient to have terms for certain primitive recursive functions). Just as we used $\bar{\forall}$, $\bar{\neg}$, $\bar{=}$ etc. for syntactic operations on formulas and variables, we use $\bar{0}$, $\bar{S}$, $\bar{f}_d$ for syntactic operations on terms. Thus $\bar{0}$ is a term, and if t is a term, so are $\bar{S}(t)$, $\bar{f}_d(t)$. If f is primitive recursive, we shall often write $\bar{f}$, leaving the reader to find d.

REMARK 2.4. It does not much matter how one thinks of the definitions d, except that "d is such a definition" should be primitive recursive, and "x is the denotation of t" should be definable for terms t built up with the $\bar{f}_d$'s. We observe however that a nice way is the following simultaneous definition of function symbols and terms. We write V(e) for the free variable of the expression e.

1. $\bar{0}$ is a term with $V(p) = \emptyset$.

2. If v is a variable, v is a term with $V(v) = \{v\}$.
3. $\bar{S}$ is a function symbol with $V(\bar{S}) = \emptyset$.
4. If f is a function symbol and t is a term then f(t) is a term with $V(f(t)) = V(f) \cup V(t)$.
5. If t,s are terms and v is a variable, $v \notin V(t)$, then Pvst is a function symbol with $V(Pvst) = V(t) \cup V(s) - \{v\}$.

Modulo an assignment of functions f_d to the function symbols d, an assignment a to the variables now determines (in the usual way) a value $t[a]$ for each term. We explain the assignment f_d as follows. If we single out a variable, say u, to stand as argument, there is a (one place) function f_t for each term, namely $f_t(n)$ is just the value of t under the assignment which is like a except for assigning to u the value n. We intend that for a function symbol d, and term du (obtained by concatenating d with u), f_d will be the same as f_{du} (with u singled out). In particular, for the functional symbol d = Pvst, the value at n of the one-place function f_d is obtained by interating f_s (determined by selecting v as the argument) n times, starting with f_t (which, since $v \notin V(t)$, is constant when v is selected to mark the argument):

$$f_d(n) = f_s f_s \ldots f_s f_t = f_s^{(n)} f_t \quad ,$$

i.e.

$$f_d(0) = f_t$$
$$f_d(n+1) = f_s f_d(n).$$

Thus the axioms for d are (the closures of)

$$d(0) \doteq t$$
$$d(\bar{S}v) \doteq s(v/dv).$$

Note that the syntactic operation $\bar{+}$ takes variables x,y and produces the term $(x \bar{+} y)$ with free variables x,y, whereas the function symbol for "adding x to" or "iterate the succesor operation starting with x" is $\dot{P}y\bar{S}yx$, with one free variable x. Thus $x \bar{+} y$ is $Py\bar{S}yxy$. Similarly, using Polish notation and dropping the bars for legibility, $\cdot xy = +x +x \ldots +x0 = (+x)^{(y)}0 = (Py + xy0)y$.

DEFINITION 2.5. Let L be any language which includes the language of Peano arithmetic. The *Peano axioms for* L are the B-closures (or the ordinary closures, if L is classical) of the following:

A1. The usual Peano axioms for $\bar{0}$ and $\bar{S}$, which assert that S is 1-1 and onto all but 0. We takes this to include $\exists x(\bar{0} \doteq x)$ and $\exists y(\bar{S}x \doteq y)$.

A2. The usual Peano axioms stating that +, $\cdot$ satisfy their recursive definitions, and in general similar axioms for each primitive recursive function symbol $\bar{f}$, including $\forall\vec{x}\exists y(f(\vec{x}) = y)$.

A3. The usual induction schema, allowing formulas from L:

$$\theta(x/\bar{0}) \wedge \forall x(\theta \to \theta(x/\bar{S}x)) \to \forall x\theta.$$

THEOREM 2.6.

a) $B\forall xB\exists y(x = y)$. *This yields the* B-*closures of* $B\forall x\theta \to \forall xB\theta$ *and* $\exists xB\theta \to B\exists x\theta$.

b) *From the* B-*closure of* $\theta \to B\theta$, *we obtain those of* $\exists x\theta \to B\exists x\theta$ *and* $\exists x\theta \to \exists xB\theta$.

c) *For each (primitive recursive) term* t *and variable* z, *we have the* B-*closure of* $t = z \to B(t = z)$. *In particular*

(i) $Sx = z \to B(Sx = z)$

(ii) $x+y = z \to B(x+y = z)$

(iii) $x\cdot y = z \to B(x\cdot y = z)$

(iv) $\exists zB(t = z)$

(v) $x < y \to B(x < y)$

(vi) $t \neq z \to B(t \neq z)$

(vii) $t_1 = t_2 \to B(t_1 = t_2)$.

d) *We have the* B-*closures of* $\forall x < y\,(B\theta) \to B(\forall x < y\,\theta)$.

REMARK. We have suppressed the bars and dots in the interests of readability.

Proof. a) We prove the B-clousures of $\exists y(x = y)$ by induction on x. $\exists y(0 = y)$ is a classical validity, so $B\exists y(0 = y)$ by Prop. 2.2. For the induction note that $\forall x\exists y(Sx = y)$ is a classical validity, so again by Prop. 2.2, $B\forall x\exists y(Sx = y)$. By L10 then

$$\forall xB(\exists y(x = y) \to \exists y(Sx = y)),$$

and by L7, L2

$$\forall x(B\exists y(x = y) \to B\exists y(Sx = y)),$$

which is the induction step.

We note that L10 now simplifies to $B\forall x\theta \to \forall xB\theta$,

a) con't. Furthermore by Prop. 2.2, $B\forall x(\theta \to \exists x\theta)$, so by L10 $\forall xB(\theta \to \exists x\theta)$, so by L7 $\forall x(B\theta \to B\exists x\theta)$, whence $\exists xB\theta \to B\exists x\theta$.

b) From $\forall x(\theta \to B\theta)$ we get $\forall x(\sim B\theta \to \sim\theta)$, $\forall x \sim B\theta \to \forall x \sim \theta$, $\sim\forall x\sim\theta \to \sim\forall x\sim B\theta$, i. e. $\exists x\theta \to \exists xB\theta$. Also, we have $\exists xB\theta \to B\exists x\theta$, so $\exists x\theta \to B\exists x\theta$.

c) We show that $\forall x(Sx = z \to B(Sx = z))$ by induction on z. The case z = 0 is vacuously true, as $\forall x \neg(Sx = 0)$. For the induction step we need the

LEMMA 2.7. *We have the* B-*closure of* $x = y \to B(x = y)$.

Proof. We give two proofs, first using the equality axioms, then using only induction. Now L6 gives

$$x = y \to [B(x = u)(u/x) \to B(x = y)(u/y)]$$

i.e.

$$x = y \to [B(x = x) \to B(x = y)].$$

But L5 gives $B(x = x)$, so a tautology yields

$$x = y \to B(x = y).$$

We now prove $\forall y(x = y \to B(x = y))$ by induction on x. First, $\forall y(0 = y \to B(0 = y))$, this is done by induction on y. If $y = 0$, it is $B(0 = 0)$. Since $0 = 0$ is classicaly valid, Prop. 2.2 gives $B(0 = 0)$. The conclusion of the induction step is $\forall y(0 = Sy \to B(0 = Sy))$, which is vacuolusly satisfied. We now return to the induction on x. We assume $\forall y(x = y \to B(x = y))$, and are to prove $\forall y(Sx = y \to B(Sx = y))$. Again we proceed by induction on y. The case $y = 0$ is vacuously satisfied. So we must see $\forall y(Sx = y \to B(Sx = y)) \to \forall y(Sx = Sy \to B(Sx = Sy))$. Now $\forall y \forall x(x = y \to Sx = Sy)$ is a classical theorem of Peano arithemtic, so by Prop. 2.2 and A10, $\forall y \forall x B(x = y \to Sx = Sy)$, and thus

$$B(x = y) \to B(Sx = Sy).$$

Consequently
$$\begin{aligned} Sx = Sy &\to x = y \\ &\to B(x = y) \quad \text{induction on } x \\ &\to B(Sx = Sy). \end{aligned}$$

This completes the induction on y, hence that on x, and so the lemma.

We return to the induction on z for the theorem. We show directly that the conclusion of the induction step holds, namely

$$\forall x(Sx = Sz \to B(Sx = Sz)).$$

This is because
$$\begin{aligned} Sx = Sz &\to x = z \\ &\to B(x = z) \text{ by the Lemma.} \end{aligned}$$

As in the second proof of the Lemma,

$$B\forall x(x = y \to Sx = Sy)$$
$$\forall x B(x = y \to Sx = Sy)$$
$$\forall x(B(x = y) \to B(Sx = Sy)),$$

so

$$Sx = Sz \to B(Sx = Sz),$$

as desired.

For the iterations of S, note first that the above argument shows $0 = z \to B(0 = z)$. Now suppose that we have the B-closures of

$$f(x,0) = k(x)$$
$$f(x,n+1) = g(x,fn),$$

and those of $kx = z \to B(kx = z)$, $g(x,y) = z \to B(g(x,y) = z)$.

[W]e show,

$$f(x,n) = z \to B(f(x,n) = z)$$

by induction on n. For n = 0, this is

$$\begin{aligned} f(x,0) = z &\to k(x) = z \\ &\to B(k(x) = z) \\ &\to B(f(x,0) = z). \end{aligned}$$

For Sn it is

$$\begin{aligned} f(x,Sn) = z &\to g(x,f(x,n)) = z \\ &\to \exists u(f(x,n) = u \wedge g(x,u) = z) \\ &\to B\exists u(f(x,n) = u \wedge g(x,u) = z) \quad \text{(by b)} \\ &\to B(f(x,Sn) = z). \end{aligned}$$

This takes care of c) and in particular (i)-(iii); (iv) follows from b). Writing $x < y$ as $\exists z(x+Sz = y)$, so does (v). We get (vi), (viii) the same way, viewing $t \neq z$ as $\exists u(t = u \wedge (t < u \text{ or } u < t))$, and $t_1 = t_2$ as $\exists z(t_1 = z = t_2)$.

d) This is a straightforward induction on y.

2.3. Axioms for the truth predicate.

The axioms are the usual ones for arithmetical truth, with the addition of a clause for sentences $B\sigma$. Satisfaction is definable from truth, since for example if θ has one free variable x, then $\models \theta[n]$ iff $\models \theta(x/\bar{n})$, where $\bar{n}$ denotes n. We must, however, state more than the truth schema $Tr(\bar{\sigma}) \leftrightarrow \sigma$ to get the corresponding satisfaction schemas such as

$$Sat(\bar{\theta},y) \leftrightarrow \theta(y).$$

We shall need several syntactic operations, and formulas and terms arithmetizing syntactic notions, which we summarize in

NOTATION 1.8.

a) For each $n \in w$, $\bar{n}$ is the term $\bar{S}\ldots\bar{S}\bar{0}$ (n iterations).
The corresponding function is b: $n \mapsto \bar{n}$.

b) $Sb(\theta,x,n)$ is $\theta(x/\bar{n})$, the result of substituting $\bar{n}$ at the free occurrences of x in θ.

c) $Vb(x)$, $Tm(x)$, $Fm_L(x,y)$, $Sent_L(x)$, are formulas (of Peano arithmetic) expresing "x is a variable", "x is a primitive recursive term", "x is a formula of L with one free variable y", and "x is a sentence of L". L may be classical or allow B, but does not admit Tr_L; Tr_L is always a unary predicate not in L.

d) $den(x,y)$ is a formula expressing "x is a primitive recursive term and y is the denotation of x".

e) Recall that if f is a primitive recursive function of n arguments, then $\bar{f}$ is a syntactic operation taking n terms to a term. If for example f has 2 arguments and u, v are variables $\bar{f}(u,v)$ will have two free variables u, v, and for all m, n, $\bar{f}(\bar{m},\bar{n})$ will denote f(m,n). For the axioms we are interested specifically in $\bar{\neg}$, $\bar{\rightarrow}$, $\bar{\forall}$, $\bar{=}$, $\bar{B}$, and $\overline{Sb}$. Note that for example $(\bar{\theta} \bar{\rightarrow} \bar{\phi})$ denotes $\theta \rightarrow \phi$, $\bar{B}\bar{\theta}$ denotes $B\theta$, and $\overline{Sb}(\bar{\theta},\bar{x},\bar{n})$ denotes $Sb(\theta,x,n)$.

DEFINITION 2.9. The *satisfaction axioms for* L are the B-closures of the following. Here u,v,w,x,y,z are variables.

S1. $Tm(u) \wedge Tm(v) \rightarrow$
$$Tr(u \bar{=} v) \leftrightarrow \exists z(den(u,z) \wedge den(v,z))$$

S2. $Sent(x) \rightarrow$
$$Tr(\bar{\neg}\ x) \leftrightarrow \neg Tr(x)$$

S3. $Sent(x) \wedge Sent_L(y) \rightarrow$
$$Tr(x \bar{\rightarrow} y) \leftrightarrow (Tr(x) \rightarrow Tr(y))$$

S4. $Fm(x,w) \rightarrow$
$$Tr(\bar{\forall}wx) \leftrightarrow \forall z Tr(\overline{Sb}(x,w,z))$$

S5. $Sent(x) \rightarrow$
$$Tr(\bar{B}x) \leftrightarrow BTr(x)$$

Note that for appropriate L, S1-S5 are the usual satisfaction axioms over Peano arithmetic; S5 is the obvious addition when the language includes B. For readability we omitted the subscript L from Sent, Fm, and Tr.

DEFINITION 2.10.

a) **P** is classical first order Peano arithmetic, i.e. the classical theory of axioms A1-A3. **BP** is Peano arithmetic in the language with B adjoined, i.e. the (Def. 2-1d) theory of A1-A3.

b) If A is a theory in a language L with finitely many function and relation symbols, then A^+ has in addition to the axioms of A the (B)-closures of the satisfaction axioms for Tr_L.

2.4. Church's Thesis.

Let U(e,n) be an r.e. formula universal for r.e. sets (provably in P). Now we may state the version of Church's Thesis which concern us. They are

DEFINITION 2.11.

CT. $\forall n(\theta n \rightarrow B\theta n) \rightarrow \exists n \forall n(\theta n \leftrightarrow Uen)$

BCT. $B[\forall n(\theta n \rightarrow B\theta n) \rightarrow \exists e \forall n(\theta n \leftrightarrow Uen)]$.

REMARK 2.12.

a) Note that CT implies that every intuitively decidable set (i.e. $\forall x[B\theta \vee B\neg\theta]$) is recursive, as then both $\forall x(\theta \rightarrow B\theta)$ and $\forall x(\neg\theta \rightarrow B\neg\theta)$, so both θ and $\neg\theta$ define r.e. sets, hence θ is recursive.

b) We remark without proof that the effectivized version of CT

ECT. $B\forall n(\theta n \to B\theta n) \to \exists e B\forall n(\theta n \leftrightarrow Uen)$

is refutable (this is essentially the content of Godel's first incompleteness theorem).

3. ON THE CONSISTENCY OF BCT.

We can now state the main theorem.

THEOREM 3.1. *The theory* $(BP)^+$ *is consistent with* CT. *That is, Peano arithmetic in the language with* B *and the truth (or satisfaction) predicate is consistent with the weak form of Church's Thesis. Specifically,* $(BP)^+$ *includes the axiom groups* L1-L12, A1-A4, *and* S1-S5.

The following corollary can be stated without the fuss of arithmetization required for 3·1.

COROLLARY 3.2. *The theory* BP *is consistent with* CT.

These results are much weaker than the corresponding conjectures for BCT:

C- **jecture 3-1a.** $(BP)^+$ *is consistent with* BCT.
Cc -**jecture 3-2a.** BP *is consistent with* BCT.

We begin by indicating the proof in outline. We shall first prove 3.2, then observe that the same method works to obtain 3.1. The proof proceeds by an interpretation **I**. **I** interprets $(BP)^+$ into the languages of P^+ so that an important instance of CT holds. Namely, that where θ is the formula Bw(x) expressing "x is an intuitively provable Tr-free sentence", i.e. the formula $Tr(\bar{B}(x))$. CT(Bw) is equivalent to the assertion that Bw is r.e.

Since Bw is r.e., all instances $CT(\theta)$ for θ not involving Tr follow from this. This proves 3.2. The proof of 3.1 proceeds the same way, beginning with an interpretation I of $(BP^+)^+$ into the language of P^{++} and using $CT(Bw^+)$ to complete the proof.

With each formula θ and assignment a to the free variables θ we can associate a sentence σ which asserts that θ holds of the assigned numbers. For example, to $\theta(u)$ (with one free variable u) and assignment a which sets u to n, we associate $\theta(\bar{n})$; to $\theta(u,v)$ and $a = (u/m, v/n)$ we associate $\theta(\bar{n},\bar{m})$, etc. In the sequel it will be convenient to associate with each θ a term $[\theta]$ with the same free variables as θ such that, under the assignment a, $[\theta]$ will denote the above mentioned sentence. For example, in the case $\theta(u)$, $\overline{Sb}(\bar{\theta},\dot{u},\bar{u})$

is such a term.

DEFINITION 3.3.

a) Let $u_1,\dots,u_n$ be the free variables of θ. Then

$$[\theta] = \overline{Sb}(\dots\overline{Sb}(\overline{Sb}(\bar\theta,\bar u_1,u_1),\bar u_2,u_2)\dots,\bar u_n,u_n)$$

(if $n = 0$, $[\theta] = \bar\theta$). Note that $\theta \mapsto [\theta]$ is primitive recursive.

b) $Bw(x)$ is the formula $Sent_L(x) \wedge Tr(\bar B(x))$ of $(BP)^+$ (which expresses "x is an intuitively provable sentence of L").

c) Let π be any formula of classical arithmetic (not involving Tr). We define the interpretation $\mathbf{I}$ ($\mathbf{I}_\pi$ if we need to make π explicit) in two stages. First, for formulas not involving Tr, then for those which do.

(i) $\alpha^{\mathbf{I}} = \alpha$ if α is atomic (without Tr).

$$(\neg\theta)^{\mathbf{I}} = \neg\theta^{\mathbf{I}}$$

$$(\theta \to \phi)^{\mathbf{I}} = \theta^{\mathbf{I}} \to \phi^{\mathbf{I}}$$

$$(\exists u\theta)^{\mathbf{I}} = \exists u\theta^{\mathbf{I}}$$

$$(B\theta)^{\mathbf{I}} = \theta^{\mathbf{I}} \wedge \pi[\theta^{\mathbf{I}}]$$

(this interpretation of B was suggested by M.H. Löb). It will be convenient to introduce the notation

$$B_\pi\theta = \theta \wedge \pi[\theta].$$

(ii) Let I_0 be the (primitive recursive) mapping (from formulas of arithmetic with B but without Tr to formulas of arithmetic with neither B nor Tr) defined by (i) (and the condition $I_0(x) = 0$ if x is not such a formula). We put

$$(Tr(x))^{\mathbf{I}} = Tr(\bar{I}_0(x)),$$

where $\bar{I}_0(x)$ is a primitive recursive term representing I_0.

PROPOSITION 3.4. CT(Bw), *the instance of* CT *obtained by taking* $\theta(x) = Bw(x)$, *is equivalent to*

$$\exists e\forall x(Bw(x) \leftrightarrow U(e,x)),$$

BCT(Bw) *to*

$$B\exists e\forall x(Bw(x) \leftrightarrow U(e,x)),$$

and ECT(Bw) *to*

$$\exists eB\forall x(Bw(x) \leftrightarrow U(e,x)).$$

Proof. It suffices to prove the antecedent

$$B\forall x(Bw(x) \to B\,Bw(x)).$$

Now $Bw(x)$ is $Tr(\bar B(x))$, and S5 is the B-closure of

$$Tr\bar B(x) \to B\,Tr(x).$$

Thus

$$\begin{aligned} Bw(x) &\to B\ Tr(x) && \text{by S5} \\ &\to B\ BTrx && \text{by L9} \\ &\to B\ Tr(\bar{B}(x)) && \text{by S5} \\ &\to B\ Bw(x). \end{aligned}$$

THEOREM 3.5. *There is a r.e. formula* π *such that under the interpretation* I_π, $(BP)^+$ *is true, as well as* CT(Bw). *In particular*

(a) $\sigma \in BP$ *implies* $P \vdash \sigma^I$

(b) $\sigma \in (BP)^+$ *implies* $P^+ \vdash \sigma^I$

(c) $\sigma =$ CT(Bw) *implies* $P^{++} \vdash \sigma^I$.

Proof. We break the proof into a series of lemmas and propositions. The first lemma asserts the existence of a π satisfying the hypotheses of the others.

LEMMA 3.6. *There is an r.e. formula* π *such that*

(i) $P^+ \vdash \sigma$ *implies* $P \vdash \pi[\sigma]$

(ii) $P \vdash \pi[\theta \to \phi] \to \pi[\theta] \to \pi[\phi]$

(iii) $P \vdash \pi[\theta] \to \pi[\pi[\theta]]$

(iv) $P \vdash \theta \to \pi[\theta]$, *for* θ *any primitive recursive equation.*

(v) $P \vdash Sent_L(y) \to \pi([Tr(y)] \leftrightarrow y)$

(vi) $P \vdash Sent_L(x) \wedge Sent_L(y) \wedge \pi(x \to y) \wedge \pi(x) \to \pi(y)$

(vii) $P^+ \vdash Sent_L(x) \wedge \pi_0(x) \to Tr(x)$,

where $\pi_0(x)$ *expresses "*x *is provable in* P*".*

(viii) $P^{++} \vdash Sent_L(x) \wedge \pi(x) \to Tr(x)$

In (v)-(viii), L *is the language of* P.

Proof. Take π to be a standard formula expressing provability in P^+. Then (i)-(iii) are the schemas traditionaly used to prove Gödel's second incompleteness theorem. The proof may be found for example in the second volume of Hilbert and Bernays. (iv) is the main lemma used in the proof of (iii). (v) is the formalization of $Tr(\bar{\sigma}) \leftrightarrow \sigma$, and will be discussed at lemma 3.17. (vi) is the main lemma used in proving (ii); we list it separately for ease of reference. (vii) is a standard fact about P^+; it contains enough to prove the correctness of P. The specific theorem of P we will need to apply this to is

(vii') $\quad Fm(x,w) \to \pi_0(\bar{I}\bar{Sb}(x,w,z) \leftrightarrow \bar{Sb}(\bar{I}x,w,z))$

which formalizes Claim 3.12.

PROPOSITION 3.7. *Let* A *be a theory such that*

(i) $A \vdash \sigma$ *implies* $A \vdash \pi[\sigma]$

Suppose furthermore that

(a) *The universal closures of the* **I** *images of the schema*

$$B\forall x\theta \to \forall x B\theta$$

are provable in A.

(b) *The universal closure of the* **I** *image of* ϕ *is provable in* A.

Then the **I** *image of any* B-*closure of* ϕ *is provable in* A.

(Thus it will suffice to show in P^+ the universal closures of the **I** images of the schemas we wish to interpret, including that of (a)).

Proof. Let σ be the universal closure of $\phi^{\mathbf{I}}$. From (i) we get $A \vdash \sigma$ implies $A \vdash B$. Using (a), the prefixed B may be moved in the quantifier prefix anywhere before $\phi^{\mathbf{I}}$. The process may be repeated with a new prefixed B to obtain all B-closures of $\phi^{\mathbf{I}}$.

We now examine the various groups of axioms.

PROPOSITION 3.8. *The* **I** *images of the classical schemas* L1, L2, L4, L5 *are logically valid, as well as* L6 *for* θ *atomic, and* L8.

Prop. 3.7 then shows that the **I** images of their B-closures are provable in A.

The rest of the classical axioms (L3, L6) will be taken care of in Lemma 3.10.

Proof. L1, L8. The schema $B_\pi\theta \to \theta$ is tautologous $(\theta \wedge \pi[\theta] \to \theta)$, and $(B\theta \to \theta)^{\mathbf{I}} = B_\pi\theta^{\mathbf{I}} \to \theta^{\mathbf{I}}$ is an instance of this. Evidently **I** takes tautologies to tautologies in general.

L2, L4, L5; L6 for θ atomic. The **I** images of these are all instances of the schema, since **I** preserves the classical logical operations. The only point to check arises in the case of L6, where we must check that $\theta(x/u)^{\mathbf{I}} = \theta^{\mathbf{I}}(x/u)$. For later reference we prove this where u is any term. For α atomic in classical arithmetic, $\alpha^{\mathbf{I}} = \alpha$; since $\alpha(x/u)$ is also atomic,

$$[\alpha(x/u)]^{\mathbf{I}} = \alpha(x/u) = \alpha^{\mathbf{I}}(x/u).$$

(Here the brackets are used only as parentheses). For α of the form Tr(t),

$$[Tr(t)(x/u)]^{\mathbf{I}} = [Tr(t(x/u))]^{\mathbf{I}} = Tr\bar{\mathbf{I}}_0(t(x/u)),$$

whereas

$$[Tr(t)]^{\mathbf{I}}(x/n) = [Tr(\bar{\mathbf{I}}_0(t))](x/u) = Tr(\bar{\mathbf{I}}_0 t(x/u)),$$

which establishes the desired equality.

PROPOSITION 3.9. a) *Suppose that in* A *we have the universal closures of*

(i) $\pi[\theta]$, *for* θ *logically valid*,

(ii) $\pi[\theta \to \phi] \to (\pi[\theta] \to \pi[\phi])$

Then the universal closures of the **I** *images of the schemas* L7, L8, L10 *(propositional and quantifiers axioms for* B) *are provable in* A.

b) *Moreover, the hypothesis* (a) *of prop. 3.7 is also satisfied, that is* $B_\pi \forall x\theta \to \forall x B_\pi \theta$.

c) *If in addition*

(iii) $\pi[\theta] \to \pi[\pi[\theta]]$

then L9 *is also provable in* A.

Proof. For L7, we show the schema

$$B_\pi(\theta \to \phi) \to (B_\pi\theta \to B_\pi\phi).$$

This is just $(\theta \to \phi) \wedge \pi[\theta \to \phi] \to (\theta \wedge \pi[\theta] \to \phi \wedge \pi[\phi])$, and follows from (i), $\pi[\theta \to \phi] \to (\pi[\theta] \to \pi[\phi])$. The **I** images of L7 are instances of the former.

L8 was proved already.

For L9, we show the schema $B_\pi\theta \to B_\pi B_\pi \theta$, of which the **I** images of L9 are instances. This is just $\theta \wedge \pi[\theta] \to \theta \wedge \pi[\theta] \wedge \pi[\theta \wedge \pi[\theta]]$, for which it will suffice to see $\pi[\theta] \to \pi[\theta \wedge \pi[\theta]]$. Now $\theta \to (\pi[\theta] \to \theta \wedge \pi[\theta])$ is a tautology, so by (i)

$$\pi[\theta \to (\pi[\theta] \to (\theta \wedge \pi[\theta]))].$$

Thus using (iii) $\pi[\theta] \to \pi[\pi[\theta]]$ and several instances of (ii), we obtain the desired result.

For L10 we show the schema

(a) $$B_\pi \forall x\theta \to \forall x B_\pi \theta.$$

(This will satisfy the hypotheses (a) of 3.7). The schema $B_\pi \forall x\theta \to \forall x B_\pi(\exists y (x = y) \to \theta)$ follows from this using a tautology and the distributivity of B_π over $\to$ (from the proof of L7). Now (a) is just

$$\forall x\theta \wedge \pi[\forall x\theta] \to \forall x(\theta \wedge \pi[\theta]),$$

for which it will suffice to see

$$\pi[\forall x\theta] \to \pi[\theta].$$

Since $\forall x\theta \to \theta$ is a logical validity, by (i) we have $\pi[\forall x\theta \to \theta]$, so (ii) gives the desired result.

LEMMA 3.10. a) *Let* A *include* P *and satisfy assumption* (ii) *of Prop. 3.9a, as well as*

(i) A $\vdash \sigma$ *implies* A $\vdash \pi[\sigma]$.

Then the **I***-images of* L3 *and* L6 *are provable in* A.

Suppose that in addition, A *satisfies*

(iv) A $\vdash (\theta \to \pi[\theta])$, *for all primitive recursive equations* θ.

Then the universal closures of the **I** *images of* L11, L12 *(the universal instantiation schemas for terms), are provable in* A.

Before giving the proof, we make the following remark regarding the significance of the antecedents of L11, L12,

REMARK 3.11.

a) Because of assumption (i), the antecedent of L12 is always satisfied. This is because $\exists y(t = y)$ is logically valid (classically), and (i) assures that $A \vdash B_\pi \exists y(t = y)$.

b) Since the schema $B\forall x\theta \to \forall xB\theta$ yield $\exists xB\theta \to B\exists x\theta$ (see the proof of Theorem 2.6 a), the antecedent of L11 becomes $\exists xB(t = z)$. This is

$$\exists z(t = z \wedge \pi[t = z]).$$

In view of (iv), it is always satisfied for primitive recursive t.

Proof. We prove the I-images of the conclusion of L11, L12,

$$\forall x\theta \to \theta(x/t).$$

(In case t is a variable, (iv) will not be needed, which will prove part a)). Now $(\forall x\theta \to \theta(x/t))^I = \forall x\theta^I \to (\theta(x/t))^I$. Since $\forall x\theta^I \to \theta^I(x/t)$ is an instance of the classical instantiation schema, will thus suffice to prove the

CLAIM 3.12. *In* A *we have the closure of* $(\theta(x/t))^I \leftrightarrow \theta^I(x/t)$.

Proof. The proof proceeds by induction on the formula θ.

1. $\theta = \alpha$ atomic. The proof is the same as in the case $t = u$, u a variable, which was given in prop. 3.8 for L6; we get $(\theta(x/t))^I = \theta^I(x/t)$.

2. $\theta \to \phi$, $\neg\theta$, $\exists y\theta$ are all easily checked since substitution is a homomorphism on these logical operations.

3. $B\theta$. Assume for θ, that $A \vdash \theta^I(x/t) \leftrightarrow (\theta(x/t))^I$. We are to see that $A \vdash (B\theta)^I(x/t) \leftrightarrow (B\theta(x/t))^I$. The l.h.s. of this is $(\theta^I \wedge \pi[\theta^I])(x/t)$, i.e.

$$\theta^I(x/t) \wedge (\pi[\theta^I])(x/t),$$

whereas the r.h.s. is

$$(\theta(x/t))^I \wedge \pi[(\theta(x/t))^I].$$

Thus it will suffice to see that

$$\pi[\theta(x/t)^I] \leftrightarrow (\pi[\theta^I])(x/t).$$

Now by induction, $A \vdash \theta^I(x/t) \leftrightarrow (\theta(x/t))^I$. Using (i)-(iv) we can apply Prop. 3.9 b), and Prop. 3.7 to obtain any B_π closure of this in A, in particular

$$A \vdash \pi[\theta^I(x/t) \leftrightarrow (\theta(x/t))^I].$$

Hence by (ii) we may distribute π to obtain

$$A \vdash \pi[\theta^I(x/t)] \leftrightarrow \pi[\theta(x/t)^I].$$

Thus it will suffice to see in A that

$$\pi[\theta^{\mathbf{I}}(x/t)] \rightarrow (\pi[\theta^{\mathbf{I}}])(x/t).$$

We shall prove now in A the general

SCHEMA 3.13: $\pi[\phi(x/t)] \leftrightarrow (\pi[\phi])(x/t)$.

To do this we need the syntactic

FACT 3.14. *If* x,w *are variables, then* $P \vdash [\theta](x/w) \doteq [\theta(x/w)]$.

We leave this to the reader to check, but note that in the case of θ with one free variable x, it follows from the formalization of

$$(\theta(x/w))(x/\bar{n}) = \theta(x/\bar{n}).$$

Letting

$$\mathrm{Sub}(\theta,x,w) = \theta(x/w),$$

this is

$$\mathrm{Sb}(\mathrm{Sub}(\theta,x,w,n) = \mathrm{Sb}(\theta,x,n),$$

so the formalization is

$$\overline{\mathrm{Sb}}(\overline{\mathrm{Sub}}(v_0,v_1,v_2),v_2,v_3) \doteq \overline{\mathrm{Sb}}(v_0,v_1,v_3)$$

which yields on taking v_0 to $\bar{\theta}$, v_1 to $\bar{x}$, v_2 to $\bar{w}$, and v_3 to w

$$\overline{\mathrm{Sb}}(\overline{\mathrm{Sub}}(\bar{\theta},\bar{w},\bar{w}),\bar{w},\bar{w}) \doteq \overline{\mathrm{Sb}}(\bar{\theta},\bar{x},\bar{w})$$

i.e.

$$\overline{\mathrm{Sb}}(\overline{\theta(x/w)},\bar{w},w) \doteq \overline{\mathrm{Sb}}(\bar{\theta},\bar{x},x)(x/w)$$

i.e.

$$[\theta(x/w)] \doteq [\theta](x/w).$$

We proceed to check the schema 3.13. Now using the above fact,

$$\begin{aligned}(\pi[\phi])(x/w) &= \pi([\phi](x/w))\\ &\leftrightarrow \pi[\phi(x/w)].\end{aligned}$$

In case t is a variable, this is schema 3.13, so the proof of a) is now complete. In general we have, for w a variable not in ϕ or t,

$$\phi(x/t) \leftrightarrow \forall w(t = w \rightarrow \phi(x/w)),$$

$$\begin{aligned}(\pi[\phi])(x/t) &\leftrightarrow \forall w(t = w \rightarrow (\pi[\phi])(x/w))\\ &\leftrightarrow \forall x(t = \dot{w} \rightarrow \pi[\phi(x/w)]),\end{aligned} \quad (1)$$

whereas (using (i), (ii))

$$\pi[\phi(x/t)] \leftrightarrow \pi[\forall w(t = w \rightarrow \phi(x/w))];$$

again using (i) etc. (as in Prop. 3.9b)

$$\to \forall w\pi[t = w \to \phi(x/w)],$$

sp by (ii)

$$\to \forall w(\pi[t = w] \to \pi[\phi(x/w)]).$$

Thus by (iv)

$$t = w \to \pi[t = w],$$

and it follows that

$$\pi[\phi(x/t)] \to \forall w(t = w \to \pi[\phi(x/w)]).$$

Thus by (1)

$$\pi[\phi(x/t)] \to (\pi[\phi])(x/t).$$

This completes one half of the proof of the schema. Note that we have used the computability of t.

For the other half, we use

$$\chi(x/t) \leftrightarrow \exists w(t = w \wedge \chi(x/w)):$$

$$\pi[\phi](x/t) \leftrightarrow \exists w(t = w \wedge \pi[\phi](x/w)) \qquad (2)$$

$$\leftrightarrow \exists w(t = w \wedge \pi[\phi(x,w)])$$

so that by (iv)

$$\to \exists w(\pi[t = w] \wedge \pi[\phi(x/w)]);$$

using (i) with a suitable tautology, and (ii), gives

$$\to \exists w(\pi[t = w \wedge \phi(x/w)])$$

so that using tha validity $\chi \to \exists w\chi$ under π, we have by (ii)

$$\to \pi[\exists w(t = w \wedge \phi(x/w))].$$

Thus applying (i) to (2) with $\chi = \phi$

$$\to \pi[\phi(x/t)]$$

which completes the proof of schema 3.13, and Lemma 3.10, and hence the checking of the logical axioms under I.

We turn now the arithmetical axioms.

PROPOSITION 3.15. *Let* A *be a theory in a classical language* L *including arithmetic, and suppose that* A *satisfies the Peano axioms. Let* π *satisfy the conditions* (i)-(iii) *of Lemma 3.10a, as well as*

(iv) $$A \vdash \bar{S}(x) = w \to \pi[\bar{S}(x) = w],$$
$$A \vdash \bar{0} = w \to \pi[\bar{0} = w].$$

Then the I_π *images of the universal closures of the schemas* A1-A3 *are provable in* A.

Proof. I_π takes A1, A2 to themelves, so no assumption is needed on π. The claim 3.12 for $t = \bar{0}$, $\bar{S}(x)$ obtains and I_π preserves the classical logical symbols, so A3 goes to a formula equivalent to an instance of A3.

This completes the proof of Theorem 3.5a.

Next the satisfaction axioms.

PROPOSITION 3.16. a) *For any formula* π, *the universal closures of the* $\mathbf{I}$ *images of the satisfaction axioms* S1-S3 (*for the language* L^B *of arithmetic with* B) *are provable in* P^+.

b) *If* π *satisfies the conditions*

(vi) $\mathrm{Sent}_L(x) \wedge \mathrm{Sent}_L(y) \wedge \pi(x \rightarrow y) \wedge \pi(x) \rightarrow \pi(y)$,

(vii) $\mathrm{Sent}_L(x) \wedge \underset{0}{\pi}(x) \rightarrow \mathrm{Tr}(x)$,

then the $\mathbf{I}$ *image of* S4 *is provable in* P^+.

c) *If in addition to* (vi) π *satisfies*

(v) $\mathrm{Sent}_L(y) \rightarrow \pi([\mathrm{Tr}(y)] \leftrightarrow y)$,

then S5 *is also obtained.*

Proof. For atomic formulas, $\mathbf{I}_0\alpha = \alpha$, so in P

$$\mathrm{Tm}(u) \wedge \mathrm{Tm}(v) \rightarrow \bar{\mathbf{I}}(u \mathrel{\dot{\bar{=}}} v) \mathrel{\dot{=}} (u \mathrel{\dot{\bar{=}}} v).$$

Thus

$$\begin{aligned}\mathrm{Tr}^{\mathbf{I}}(u \mathrel{\dot{\bar{=}}} v) &\leftrightarrow \mathrm{Tr}\bar{\mathbf{I}}(u \mathrel{\dot{\bar{=}}} v)\\ &\leftrightarrow \mathrm{Tr}(u \mathrel{\dot{\bar{=}}} v),\end{aligned}$$

and S1 follows from S1 in P^+.

S2 becomes

$$\mathrm{Sent}_{(L^B)}(x) \rightarrow (\mathrm{Tr}(\bar{\mathbf{I}}_0 \,\bar{\neg} x) \leftrightarrow \neg \mathrm{Tr}\bar{\mathbf{I}}_0 x). \tag{1}$$

Since $\mathbf{I}_0 \neg\sigma = \neg\mathbf{I}_0\sigma$, in P we have

$$(\bar{\mathbf{I}}_0 \,\bar{\neg} x) \mathrel{\dot{=}} (\bar{\neg}\bar{\mathbf{I}}_0 x)$$

Using this and $P \vdash \mathrm{Sent}_{(L^B)}(x) \rightarrow \mathrm{Sent}_L(\mathbf{I}_0 x)$, (1) follows from S2 in P^+. S3 is proved the same way, formalizing $\mathbf{I}_0(\theta \rightarrow \phi) = \mathbf{I}_0\theta \rightarrow \mathbf{I}_0\phi$.

b) For the proof of S4 we need the formalization of Claim 3.12. Taking $\Lambda = P$ in that Claim, and confining our attention to the case where θ has one free variable w, and t is a term $\bar{n}$, this yields

$$\left\{\begin{array}{l}\text{For all n, and all formulas } \theta \text{ with one free variable w,}\\ \text{(3)}\quad P \vdash (\theta(w/\bar{n}))^{\mathbf{I}} \leftrightarrow \theta^{\mathbf{I}}(w/\bar{n}).\end{array}\right. \tag{2}$$

The proof of (2) may be carried out in P; since (3) may be written as

$$P \vdash \mathbf{I}\mathrm{Sb}(\theta,w,n) \leftrightarrow \mathrm{Sb}(\mathbf{I}\theta,w,n).$$

and π_o expresses provability in P, the formalization of (2) is

(vii') $$P \vdash Fm(x,w) \rightarrow \pi_o(\bar{I}\overline{Sb}(x,w,z) \bar{\leftrightarrow} \overline{Sb}(\bar{I}x,w,z))$$

as indicated after Lemma 3.6. Here Fm(x,w) means for L with B,Tr. At this point we only need it for L^B without Tr; for such we get $Fm_{(L^B)}(x) \rightarrow \bar{I}(x) = \bar{I}_o(x)$, of course. Now S4 is

$$Fm(x,w) \rightarrow (Tr(\bar{\forall}wx) \leftrightarrow \forall z Tr(\overline{Sb}(x,w,z))),$$

so we must see that $Fm_{(L^B)}(x,w) \rightarrow$

$$Tr(\bar{I}_o\bar{\forall}wx) \leftrightarrow \forall z Tr \bar{I}_o\overline{Sb}(x,w,z).$$

Now, $I_o\forall u\theta = \forall u I_o\theta$, so in P, $\bar{I}_o\bar{\forall}wx \doteq \bar{\forall}x\bar{I}_o x$. Thus

$$Tr(\bar{I}_o\bar{\forall}wx) \leftrightarrow Tr(\bar{\forall}w\bar{I}_o x),$$

and using S4 in P^+,

$$\leftrightarrow \forall z Tr\overline{Sb}(I_o x,w,z).$$

Using (vii'), (vii), and S2,3 in P^+ gives the result.

c) S5. We shall need the following

LEMMA 3.17. a) *If* θ *is any formula of the language* L *of Peano arithmetic, then in* P^+ *we have the closure of* $Tr[\theta] \leftrightarrow \theta$.

b) *In particular, restricting* θ *to sentences gives: for all sentences* σ *of* L, $P^+ \vdash (Tr(\bar{\sigma}) \leftrightarrow \sigma)$.

c) *The proof of* (b) *can be carried out in* P, *so that if* π *expresses probability in* P^+ *(or something stronger)*

$$P \vdash \forall y(Sent_L(y) \rightarrow \pi([Tr(y)] \bar{\leftrightarrow} y))$$

Proof. These are standard results so we indicate the proof only briefly.

a) Proved by induction on θ. The notation $[\theta]$ extends to terms t. For atomic formulas use $[t \doteq s] \doteq ([t] \bar{\doteq} [s])$ and den([t],t). For the other cases use $[\neg\theta] = \bar{\neg}[\theta]$, etc., and the corresponding clause of the satisfaction axioms.

b) is immediate from a).

c) Note that

$$Tr(\bar{\sigma}) = (Tr(y))(y/\bar{\sigma}) = Sb(Tr(y),y,\sigma),$$

so its formalization is $\overline{Sb}(\overline{Tr(y)},\bar{y},y) = [Tr(y)]$.

Proof of 3.16. The new satisfaction axiom is

$$Sent(x) \rightarrow (Tr(\bar{B}x) \leftrightarrow BTr(x)),$$

so under **I** it becomes

$$Sent(x) \rightarrow (Tr(\bar{I}_o\bar{B}x) \leftrightarrow B_\pi Tr(I_o x)). \quad (1)$$

Here Sent(x) expresses "is a sentence in the language L^B of arithmetic with

B", but of course without Tr. Now

$$I_0 B\sigma = \mathbf{I}_0\sigma \wedge \pi[\mathbf{I}_0\sigma] = \mathbf{I}_0\sigma \wedge \pi(\overline{\mathbf{I}_0\sigma})$$
$$= \mathbf{I}_0\sigma \wedge \mathrm{Sb}(\pi, y, \mathbf{I}_0\sigma).$$

Where y is the free variables of π. The formalization of this is

$$\mathrm{Sent}(x) \rightarrow \bar{\mathbf{I}}_0\bar{B}x \doteq \bar{\mathbf{I}}_0 x \bar{\wedge}\ \overline{\mathrm{Sb}}(\bar{\pi}, \bar{y}, \bar{\mathbf{I}}_0 x) \ ,$$

which is provable in P. Thus we have, suppressing the antecedent Sent(x),

$$\mathrm{Tr}(\bar{\mathbf{I}}_0\bar{B}x) \leftrightarrow \mathrm{Tr}(\bar{\mathbf{I}}_0 x \bar{\wedge}\, \overline{\mathrm{Sb}}(\bar{\pi}, \bar{y}, \bar{\mathbf{I}}_0 x)) \tag{2}$$

so that by S2, S3 in P^+,

$$\leftrightarrow \mathrm{Tr}(\bar{\mathbf{I}}_0 x) \wedge \mathrm{Tr}(\overline{\mathrm{Sb}}(\bar{\pi}, \bar{y}, \bar{\mathbf{I}}_0 x)).$$

Now

$$B_\pi \mathrm{Tr}(\bar{\mathbf{I}}_0 x) \leftrightarrow \mathrm{Tr}(\bar{\mathbf{I}}_0 x) \wedge \pi[\mathrm{Tr}\bar{\mathbf{I}}_0 x], \tag{3}$$

so to check (1) it will by (2), (3) be enough to check

$$\mathrm{Tr}(\overline{\mathrm{Sb}}(\bar{\pi}, \bar{y}, \bar{\mathbf{I}}_0 x)) \leftrightarrow \pi[\mathrm{Tr}\bar{\mathbf{I}}_0 x].$$

Now in P

$$\mathrm{Sent}_{(LB)}(x) \rightarrow \mathrm{Set}_L(\bar{\mathbf{I}}x),$$

so it will suffice (by instantiating $\bar{\mathbf{I}}_0 x$ for y) to see

$$\mathrm{Sent}_L(y) \rightarrow (\mathrm{Tr}(\overline{\mathrm{Sb}}(\bar{\pi}, \bar{y}, y)) \leftrightarrow \pi[\mathrm{Tr}(y)]),$$

i.e., suppressing the antecedent, and noting $[\pi] = \overline{\mathrm{Sb}}(\bar{\pi}, \bar{y}, y)$

$$\mathrm{Tr}[\pi(y)] \leftrightarrow \pi[\mathrm{Tr}(y)]. \tag{4}$$

The result now follows easily from Lemma 3.17. By 3.17a,

$$\mathrm{Tr}[\pi(y)] \leftrightarrow \pi(y).$$

While by 3.17c,

$$\pi([\mathrm{Tr}(y)] \mathbin{\bar{\leftrightarrow}} y). \tag{5}$$

Thus by (vi) (modus ponens for π),

$$\pi[\mathrm{Tr}(y)] \leftrightarrow \pi(y). \tag{6}$$

Combining (5), (6) gives (4) and hence completes the proof of 3.16.

This completes the proof of Theorem 3.5b.

We now turn to CT(Bw).

PROPOSITION 3.18. *If* π *satisfies the condition* (vi) (vii) *of Prop. 3.16 b, as well as*

(viii) $P^{++} \vdash \mathrm{Sent}_L(x) \wedge \pi(x) \rightarrow \mathrm{Tr}(x)$, *then the* **I** *image of* CT(Bw) *is provable in* P^+.

Proof. The **I** image of CT(Bw) is

$$\exists e \forall x(Bw^{\mathbf{I}}(x) \leftrightarrow Uex),$$

i.e.

$$\exists e \forall x(Sent(x) \wedge Tr(\bar{I}_o \bar{B} x) \leftrightarrow Uex).$$

As before (in 3.16c), this is

$$Bw^{\mathbf{I}}(x) \leftrightarrow Tr(\bar{I}_o x) \wedge Tr(\overline{Sb}(\bar{\pi}, \bar{y}, \bar{I}_o x))$$
$$\leftrightarrow Tr(\bar{I}_o x) \wedge \pi(\bar{I}_o x).$$

But since π is a correct notion of proof (viii),

$$\leftrightarrow \pi(\bar{I}_o x).$$

Finally, since π is an r.e. formula, there is a Turing machine e which enumerates the arithmetical sentences σ such that $P^+ \vdash \sigma$; since I_o is recursive, it is evident that $\{\sigma | P^+ \vdash \sigma^{I_o}\}$ is also. I.e. that

$$\exists e \forall x(\pi(\bar{I}_o x) \wedge Sent_{(L^B)}(x) \leftrightarrow Uex),$$

so that

$$\exists e \forall x(Bw^{\mathbf{I}}(x) \leftrightarrow Uex)$$

as desired.

This completes the proof of Theorem 3.5.

It is now easy to prove Corollary 3.2. We must see the schema BCT for formulas θ of the language of BP (i.e. without Tr).

Proof of 3.2. Suppose that $\forall x(\theta \rightarrow B\theta)$. Then

$$\theta \leftrightarrow B\theta$$
$$\leftrightarrow BTr[\theta]$$
$$\leftrightarrow Bw[\theta]$$
$$\leftrightarrow U(e, [\theta]).$$

Now this just says that the inverse image of a certain r.e. set (Bw) under the recursive function $f(n) = \theta(\bar{n})$ is also r.e. Indeed, there is a primitive recursive function g such that if e is the Turing machine for a set E, g(e) will be the Turing machine for $f^{-1}E$ (and moreover, one can get g effectively from f). Thus we have in P,

$$\forall e(\forall x(U(e, \bar{f}x) \leftrightarrow U(\bar{g}e, x)), \quad (1)$$

so that

$$\forall e[\forall x(Bw(x) \leftrightarrow Uex) \rightarrow \forall x(Bw(\bar{f}x) \leftrightarrow U(\bar{g}e, x))].$$

Since $\bar{f}(x) = [\theta(x)]$, this completes the proof of Corollary 3.2.

To complete the proof of Theorema 3.1, we extend the definition of **I** to the language of $(BP)^{++}$ using for $\pi(x)$ a formula expressing "$P^{++} \vdash x$": Let $\mathbf{I}_1$ be the previous definition and put

$$Tr_+^{\mathbf{I}}(x) = Tr_+(\bar{\mathbf{I}}_1(x)).$$

We show that **I** interpretes $(BP)^{++} + CT(Bw_+)$.

Prop. 3.8 requires the addition clauses (obtained by replacing Tr by Tr_+ and $\mathbf{I}_0$ by $\mathbf{I}_1$):

$$[Tr_+(\tau)(x/u)]^{\mathbf{I}} = [Tr_+(t(x/u))]^{\mathbf{I}} = Tr_+\bar{\mathbf{I}}_1(t(x/u)),$$

$$[Tr_+(\tau)]^{\mathbf{I}}(x/u) = [Tr_+(\bar{\mathbf{I}}_1(\tau)](x/u) = Tr_+(\bar{\mathbf{I}}_1 t(x/u)).$$

In Prop. 3.16, the first new satisfaction axiom is $S1^+$:

$$Tm(u) \to (Tr_+\overline{Tr}u \leftrightarrow Tr(u)),$$

which goes to $Tm(u) \to$

$$Tr_+(\bar{\mathbf{I}}_1\overline{Tr}u) \leftrightarrow Tr(\bar{\mathbf{I}}_0(u)).$$

Now

$$Tr^{\mathbf{I}}(x) = Tr\bar{\mathbf{I}}_0(x),$$

so in P

$$(\bar{\mathbf{I}}_1\overline{Tr}\,x) \doteq (\overline{Tr}\,\bar{\mathbf{I}}_0 x).$$

Thus

$$\begin{aligned} Tr_+(\mathbf{I}_1\overline{Tr}u) &\leftrightarrow Tr_+(\overline{Tr}\bar{\bar{\mathbf{I}}}_0 u) \\ &\leftrightarrow Tr_+|Tr\bar{\mathbf{I}}_0 u] \\ &\leftrightarrow Tr(\bar{\mathbf{I}}_0 u). \end{aligned}$$

The new axioms $S2^+$, $S3^+$, $S4^+$ are the same as the old with Tr replaced by Tr_+, and the antecedent syntactic conditions (Fm,Sent) chaged accordingly. Thus the proofs are the same with I_0 replaced by $\mathbf{I}_1$. The proof of $S5^+$ requires use of

$$P^{++} \vdash Tr_+[\theta] \leftrightarrow \theta\ ,$$

and the various formalizations of Lemma 3.17. The proof is again the same, replacing Tr by Tr_+, I_0 by $\mathbf{I}_1$. The proof of $CT(Bw_+)$ requires the correctness of π, which is now provable in P^{+++}, but is otherwise the same. Theorem 3.1 now follows by the same argument as in the proof of 3.2.

We would like to make a few remarks to indicate the nature of the difficulty in extending these results to obtain conjectures 3.1a, 3.2a.

The situation now is that we have T consistent (where T is $(BP)^{\dagger}$), $T \vdash \alpha$, and we would like to show the consistency of $T \cup \{B\alpha\}$. However, it is certainly not true in general that $T \cup \{B\alpha\}$ is consistent. For example, if α is $(\gamma \wedge \neg B\gamma)$, then $B\alpha$ (and hence $T \cup \{B\alpha\}$) is inconsistent:

$$B(\gamma \wedge \neg B\gamma) \rightarrow B\gamma \wedge B \neg B\gamma$$
$$\rightarrow B\gamma \wedge \neg B\gamma.$$

Now CT(Bw) certainly implies that there are sentences such as γ; in fact BCT implies for some arithmetical θ

$$B\exists e(\theta(e) \wedge \neg B\theta(e))$$

so that

$$\exists e(\theta(e) \wedge \neg B\theta(e))$$

is provable in T.

We remark that one obvious way to try to interpret BCT(Bw) is to use the same method used for CT(Bw) but to replace $\pi[\theta]$ by $\pi[\alpha \rightarrow \theta] = \pi_\alpha[\theta]$. It turns out that no such interpretation will have BCT(Bw). In fact, if π is any reasonable proof predicate (i.e. one satisfying Gödel's second theorem), then de-Jongh and the author have observed that BCT(Bw) fails in the Löb interpretation using π.

To see this, consider the following schema, which is a consequence of BCT:

1) $B\exists e\forall x(B\theta(x) \leftrightarrow U(e,[\theta(e)]))$,

where $\theta(x)$ is a formula with one free variable x. This is equivalent to BCT(Bθ), since the correspondence $x \mapsto [\theta(x)]$ may be assumed to be 1-1 recursive. Apply the fixed point theorem to obtain $\theta(x)$ an arithmetical formula (no B's, no Tr's, etc.) so that

2) $B\forall x(\theta(x) \leftrightarrow \neg U(x,[\theta(x)]))$.

Now we also have in general

3) $B\forall x(B\theta(x) \rightarrow \theta(x))$.

Thus taking x to be e, we have $B\exists e\phi(e)$, where $\phi(e)$ is the conjunction of

1') $B\ (e) \leftrightarrow U(e,[\theta(e)])$
2') $\theta(e) \leftrightarrow \neg U(e,[\theta(e)])$
3') $B\theta(e) \rightarrow \theta(e)$.

Since $\theta(e) \wedge \neg B\theta(e)$ is a tautological consequence of $\theta(e)$, we thus have

4) $B\exists e(\theta(e) \wedge \neg B\theta(e))$.

Consequently, any interpretation which makes (1) come out true will also have (4). In particular, if $B\theta$ is interpreted as $\theta \wedge \pi[\theta]$ (which, since θ is arithmetical, will be the case for any Löb interpretation), the truth of 4) gives

$$\pi[\exists e(\theta(e) \wedge \neg(\theta(e) \wedge \pi[\theta(e)])]$$
$$\vdash_\pi \exists e(\theta(e) \wedge \neg\pi[\theta(e)])$$
$$\vdash_\pi \exists e(\neg\pi[\theta(e)])$$

$$\vdash_{\pi} \exists x \neg \pi(x)$$
$$\vdash_{\pi} Con_{\pi},$$

in other words, Gödel's second theorem fails for π.

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Department of Mathematics
University of Colorado
Boulder, Colorado, U.S.A.

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NEGATION AND CONTRADICTION

R. Routley and V. Routley

The problems of the meaning and function of negation are disentangled from ontological issues with which they have been long entangled. The question of the function of negation is the crucial issue separating relevant and paraconsistent logics from classical theories. The function is illuminated by considering the inferential role of contradictions, contradiction being parasitic on negation. Three basic modellings emerge: a cancellation model, which leads towards connexivism, an explosion model, appropriate to classical and intuitionistic theories, and a constraint model, which includes relevant theories. These three modellings have been seriously confused in the modern literature: untangling them helps motivate the main themes advanced concerning traditional negation and natural negation. Firstly, the dominant traditional view, except around scholastic times when the explosion view was in ascendency, has been the cancellation view, so that the mainstream negation of much of traditional logic is distinctively nonclassical. Secondly, the primary negation determinable of natural negation is relevant negation. In order to picture relevant negation the traditional idea of negation as otherthanness is progressively refined, to nonexclusive restricted otherthanness. Several pictures result, a reversal picture, a debate model, a record cabinet (or files of the universe) model which help explain relevant negation. Two appendices are attached, one on negation in Hegel and the Marxist tradition, the other on Wittgenstein's treatment of negation and contradiction.

1. THE PHILOSOPHICAL CENTRALITY OF NEGATION, AND THE HISTORICAL ENTANGLEMENT OF NEGATION WITH ONTOLOGICAL ISSUES.

Negation is a fundamental, but ill-understood, ill-explained and much disputed notion across a wide philosophical spectrum. It is not only the central notion in recent and momentum-gathering disputes between classically-inclined logicians and alternative people (called by rougher classical types 'deviants'); it is also, for instance, a crucial notion in much Buddhist philosophy, such as theories of meaning, relation and cognition (see, e.g., Shaw, Matilal, and references therein). But in order 'to make sense of the use of negation in Buddhist philosophy in general, one needs to venture outside the perspective of the standard (i.e. classical) notion of negation' (Matilal, p.2). As well, negation, along with its derivate nothingness, is a key notion in modern European philosophy, for instance, in the modern tradition Sartre considers himself to belong to, from Hegel through Husserl and Heidegger (see Sartre, p.6 ff); again the negations involved are, almost invariably, nonclassical.

However very many of the problems, accounted problems of negation in the literature, are not really problems of negation simpliciter at all but are rather problems of nonexistence, which arise from the alleged riddle of not-being or nonexistence; of how it can be truly said, as it sometimes is, that A does not exist (or that A is not) when the truth of the statement implies that A does exist (that A is). Such are central issues in Greek philosophy, from Parmenides through Plato's *Sophist*; such are the main worries of late nineteenth and early twentieth century traditional logicians over negative judgements and negative terms; such are many of the problems in existentialism over not-being and nothingness. As to the second consider, for example, the main logical difficulties Joseph finds with negative judgements:

> Judgement... refers to the existent, whose manner of being is as we conceive. But the real is positive: it only exists by being something, not by being nothing. A negative judgement declares what is not, and how can this express it as it is?(1)

As to the third, consider for instance Sartre's (somewhat devious but similar) argument to the objective existence of non-being and nothing (p.5)(2). All these moves and their difficulties, are based in one way or another upon the Ontological Assumption, according to which what is a subject of true discourse must exist, a thoroughly fallacious assumption whose manifold defects have already been exposed (in Routley, [22], Chapter 1).

These perennial "problems" persist in contemporary logical theory. Much of Russell's problem with negation, for example, is again an ontological one:

> if the sun is not shining there is not a fact *sun-not-shining* which is affirmed by the true statement "the sun is not shining" (p.520).

For if there were, it would exist, yet how can such negativity exist? It cannot according to Russell, in sharp contrast to existentialists, who grasp the other horn of the dilemma the Ontological Assumption generates from negative facts. Fortunately, Russell (erroneously) thinks, negation is eliminable: '...

"not" is unnecessary for a complete description of the world' (p.520). He makes two gestures towards showing this, one psychological and quite unsatisfactory -'what is happening [in the belief that "the sun is not shining"] is that I am inhibiting the impulses generated by the belief that the sun is shining -and one logical -'"not" is eliminated from our fundamental apparatus' through 'the positive predicates "true" and "false"'. But this is no elimination without cheating, namely reclassifying the negative predicate 'false' as a positive predicate (which makes negation itself positive, since it can then be positively defined!). For, otherwise, as falsity involves negation, 'false' being defined commonly (for a very wide range of logical theories) in terms of 'true' and 'not', the account Russell presupposes, that a statement not-A is true iff A is false, is circular, and begs the question (cf. Quine, p.84). Similarly the psychological account is circular at bottom, since 'inhibit' is a negative verb (derived using the negative prefix 'in'). In fact the indefinability of negation in various positive logics is readily demonstrated (see, e.g., Goddard and Routley, chapter 5). Negation cannot be eliminated without cheating: nor can it be dispensed with without very serious impoverishment of discourse, as Griss's attempt to construct a negationless mathematics and von Dantzig's an affirmative mathematics have revealed (see, e.g., Fraenkel and Bar-Hillel, p.239 ff).

2. IT IS AS REGARDS NEGATION THAT RELEVANT AND PARACONSISTENT LOGICS DIVERGE FUNDAMENTALLY FROM CLASSICAL LOGIC.

Even with the ontological problems duly disposed of, many problems remain, especially as to the meaning and function of negation. Some of these problems are grappled with in what follows, especially the problems of characterising, picturing and modelling negations of relevant (and also paraconsistent) logics, and, what overlaps these, negations of natural language. A *relevant* logic can be characterized, approximately for present purposes, as a logic whose pure sentential part conforms to Belnap's weak relevance requirement, namely that there is no thesis of the form A → B, (i.e. that A implies or entails that B) where A and B do not share a variable. (A *standard* relevant logic is one that conservatively extends the first degree system FD, the first degree of system E and R and very many other relevant logics. Much of the discussion of relevant logic that follows is conceived in terms of standard systems, though some points will apply more widely).

The fundamental divergence of relevant (and paraconsistent) logics from classical logic is as to negation, its logical and inferential behaviour. Indeed at the first degree stage (where no nested implications occur), *relevant and classical logic differ just over negation* (see Routley, Meyer, Plumwood and Brady, *Relevant Logics and Their Rivals* [23], hereafter RLR, Chapter 2).

Negation is accordingly, really the crucial notion for the choise of logical theory, as also for comparisons of logical theories, for appreciation of the varieties and character of competing theories of entailment, and so on.

Parasitic on negation is contradiction. A contradictory situation is one where both B and ~B (it is not the case that B) hold for some B. An explicit contradiction is a statement of the form B and ~B. A statement C is contradictory, it is often said, if it entails both B and also ~B for some B, etc. Contradiction is always characterized in terms of negation and the logical behaviour of contradictions is dependent on that of negation. Different accounts of negation result not merely in different conceptions of contradiction and of incompatibility, they likewise correspond to different accounts of what constitutes a describable world, what constitutes a logically assessible world. Classical negation restricts such worlds to possible worlds, excluding contradictory and incomplete worlds.

Contradictory situations play a prominent role in world semantics for relevant logics. Most conspicuously, nontrivial contradictory situations are deployed in counterexamples to the harder Lewis paradox of implication, the spread principle, ex falso quodlibet, $A \wedge {\sim}A \rightarrow B$ (or in rule form $A, {\sim}A \Rightarrow B$), which spreads contradictions everywhere and trivializes all contradictory situations. For suppose c is a nontrivial situation, i.e. not everything holds at c, but c is contradictory. Then for some A and B, A and ~A both hold in c but B does not. Therefore A and ~A does not entail B, for this would require that in whatever situation A and ~A hold B does also.

It is at the same time evident that classical logic and classically-based logics rule out nontrivial inconsistent situations, and so exclude an important class of theories, of *much* philosophical and other interest. More generally, the excluded class is that of paraconsistent theories. The core idea is that a paraconsistent theory is one that contains true contradictions without triviality. It is immediate that paraconsistent logics, logics that can serve as the basis for paraconsistent theories are, rather radically, nonclassical.

Many relevant logics are paraconsistent logics, but not all are. For example, Ackermann's logic Π' (which has the same theorems as Anderson and Belnap's system E) is not a paraconsistent logic; and similarly for almost any relevant logic that has Material Detachment, $A, {\sim}A \vee B \rightarrow B$, as a primitive (or immediate) rule. Systems E and R and all their subsystems and many of their extensions are paraconsistent logics. Relevant logics do not however exhaust paraconsistent logics. There are many irrelevant paraconsistent logics, e.g. the main systems of da Costa, the earlier systems of Priest, etc. Relevant and paraconsistent logics thus properly overlap. Standard systems in the overlap are of primary interest in what follows. For these systems a theory of negation, of relevant negation, is especially important.

3. BASIC MODELLINGS OF NEGATION IN TERMS OF DIFFERENT RELATIONS OF ~A TO A.

Theories of negation differ, very obviously, in the roles they allow, or assign to, contradictions. Contradictions may be allowed no inferential role (they imply nothing, except perhaps themselves), a total inferential role (they imply everything), or some limited inferential role (they imply some things, such as their contradictory components, but not others). There are, correspondingly three initial ways to classify theories of negation, in terms of the relation of ~A to A.

1. ~A deletes, neutralizes, erases, cancels A (and similarly, since the relation is symmetrical, A erases ~A), so that ~A together with A leaves nothing, no content. The conjunction of A and ~A says nothing, so nothing more specific follows. In particular, A ∧ ~A does not entail A and does not entail ~A. Accordingly, the *cancellation* (erasure, or neutralization) *model* leads towards connexivism, a position (much discussed in RLR) distinguished by the following two theses- First, that already cited, that explicit contradictions do not entail their components, and secondly, that A does not entail ~A. The second thesis emerges naturally under the neutralization view, for instance, as follows. Entailment is inclusion of logical content. So, if A were to entail ~A, it would include as part of its content, what neutralizes it, ~A, in which event it would entail nothing, having no content. So it is not the case that A entails ~A, that is Aristotle's thesis, ~(A → ~A) holds.(3)

There is reasonable, but not conclusive, evidence that Aristotle did adhere to Aristotle's thesis. And assuming that he did certainly has great explicative advantages, for example the full theory of the syllogism translates into connexive quantificational logic without loss or qualification (as Angell, and also McCall, has pointed out); the theory of immediate inference also emerges intact (for inferential but not implicational form). Whether or not Aristotle was operating with connexive assumptions, there is a long historical line of logicians and philosophers who have assumed a cancellation picture, from Boethius in medieval times through to Strawson, Körner, and many others in modern times (see RLR). One striking intermediate example is Berkeley, who advances the following claims in his attack on the calculus (*The Analyst*, p. 73):

> Nothing is plainer than that no just conclusion can be directly drawn from two inconsistent premises. You may indeed suppose any thing possible: But afterwards you may not suppose anything that destroys what you first supposed: or, if you do, you must begin *de nove*... [When] you... destroy one supposition by another... you may not retain the consequences, or any part of the consequences, of your first supposition so destroyed.

Cancellation views are prevalent in one place where they are particularly damaging, in so-called expositions of Hegel (but there is a basis for this ascription in Hegel himself, as will appear). Given this phenomenon it is not surprising that Hegel's logic has appeared so intractable to commentators. Here, for instance, is what the Marxist logician Havas has to say as regards Hegel's theory:

> ...the Aristotelian principle of non-contradiction is a general principle of metalogic, which can be said to bring out a necessary condition to be satisfied by all human thought and all of the systems of logic; namely, the condition that it is a logical contradiction, and therefore, a logical mistake to assert both something and its opposite. This is one of the elementary but necessary conditions of sound reasoning, because if one asserts something to be true and, insisting on this assertion, one also asserts that this very assertion is not true, then his assertions will neutralize each other and, in consequence of this, no knowledge will be acquired (p.7).

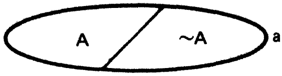
Apart from being unfaithful to Hegel, who (correctly) says that there is nothing unthinkable about contradictions, thereby repudiating the laws-of-thought myth, and who accepted no such simple neutralization view, the Aristotelian principle is not a metalogical principle concerning the logic of assertion.

The second model for negation is that embodied in contemporary classical and intuitionist logics:

2. ~A explodes, or fully implodes, A (and similarly A explodes ~A) in such a way that ~A together with A yields everything, total content. The conjunction A and ~A says everything, so everything follows. $A \wedge {\sim}A$ entails B, for arbitrary and irrelevant B, so the *explosion* (or destruction) *model* is inevitably paradoxical. The paradoxical character of classical logic for example, can accordingly be obtained with very few further assumptions from the character of its negation.

Under weak, and relatively noncontroversial, conditions on other connectives $({\sim}, \wedge, \vee)$, the explosion model delivers classical negation, according to which negation ~, is evaluated according to the classical semantical rule $I({\sim}A,a) = 1$ iff $I(A,a) \neq 1$, i.e. A holds at world a iff A does not hold at a, for every world a (deployed in the semantic evaluation of entailment). (Under alternative conditions the model yields intuitionistic negation.) Conversely, classical negation, i.e. negation conforming to the classical rule, yields the explosion view, since there is no world where both A and ~A hold but B does not, whence, on the semantical theory,[4] $A \wedge {\sim}A \rightarrow B$. Thus under weak conditions the explosion view is that of classical negation.

Classical negation offers a complete exclusion model of negation, more precisely, an exclusion and exhaustion view: for each world a and each statement A, ~A excludes A from holding in a, and ~A united with A exhausts a, one or other must hold in a. The picture is that commonly offered for the real world (as e.g. in Hospers, p.212) simply relativised to world a, namely



where the ellipse represents the whole of a, all statements of a. '"Not A" will cover all territory (of a) other than what is covered by "A"' (p.223).

Quine and many others (e.g. D. Lewis, Copeland) think that classical negation is "our ordinary" negation and that there is no alternative to it, for any alternative would 'change the subject' from negation. Of course they never *argue* that it is our ordinary negation; they simply *assume* that it is. So it is in Quine's main defence of classical negation, which occurs in a famous passage in 'Deviant Logics' (p.81) where he considers two parties, α and β say, who proceed as follows: α, adopting a 'popular extravagance', rejects the law of non-contradiction and accepts A and ~A occasionally. β objects that this 'would vitiate all science' and uses the paradoxes to show that everything would follow so 'forfeiting all distinctions between true and false'. Party α tries to 'stave' this off by 'compensatory adjustments', by rigging the logic so as to isolate contradictions (in good paraconsistent fashion). In Quine's view,

> neither party knows what he is talking about. They think that they are talking about negation, '~', 'not'; but surely the notation ceased to be recognisable as negation when they took to regarding some conjunctions of the form 'p. ~p' as true, and stopped regarding such sentences as implying all others.

Quine's case is however vitiated by being described in a thoroughly incoherent (indeed inconsistent) fashion; for example, party β is described as objecting that 'everything would follow' and as adopting what appears to be the classical view, yet Quine asserts that '*neither* party knows what he is talking about' because *neither* adopts the classical view, having just described one of his disputants as doing so. Nor are Quine's conclusions independently warranted. The paradoxes of strict implication are not built into the ordinary notion of negation, into the particle 'not'. The English negation determinable 'not' is not so determined (as distinct from the classical negation determinate). Quine has failed to observe the distinction, and has done something which entirely begs the question at issue: equated, without any trace of argument, the natural language 'not' with classical negation. Thus what he goes on to claim has no secure basis:

> Here evidently lies the deviant logician's predicament: when he tries to deny the doctrine he only changes the subject (p.81).

There is no predicament: the "deviant" may be trying, with more success than the classicist, to explicate the core notion 'not'. On Quine's viewpoint, no distinct systems can explicate the same (preanalytic) connectives - which is a reduction to absurdity of the position. Moreover, were Quine right no "deviant" could reject the classical doctrine, he would only be changing the subject. Yet elsewhere Quine admits (and has to admit on his theory of unrestricted revisability [4.1]) the possibility of rejecting the doctrine (e.g. on p.84, three pages later):

> .It is hard to face up to the rejection of anything so basic [as classical negation, etc.] If anyone questions the meaningfulness of classical negation, we are tempted to say in defense that the negation of any given closed sentence is *explained* thus: it is true if and only if the given sentence is not true... However our defense here begs the question... [since] we use the same classical 'not'.

Nor is it the meaningfullness of classicall negation that is at issue: it is its correctness, and its uniqueness. The semantical recipe given in explanation does not separate classical negation from various other negations, e.g. the relevant negation of Π', which [can] satisfy the same recipe. Accordingly, the recipe does not explain classical negation (without further assumptions, such as a one-world assumption), nor does it show its uniqueness.

Although classical negation is not, unlike connexive negation, a subtraction operation, a taking *away* of something already given, it involves certain subtraction features. By contrast relevant negation does not involve subtraction features; ~A does not imply the taking away or elimination of A, but adds a further condition (although one related to A by certain constraints); ~A does not have entirely dependent status in the way it does classically. These differences are already reflected in the structure of the complete possible worlds of classical logic, as distinct from the worlds of relevant logic. In the classical case when ~A is added to a world, quite a bit may have to be taken out of the world, e.g. A (and what implies it) if it is there, in order to consistencize the world; whereas in the relevant case ~A can simply be added without any consistencizing subtractions. More generally, worlds can be simply combined and statements added to worlds without the need to delete anything, *because* what is being added are further conditions, not the taking away of conditions already given. This is the route to a straightforward, and relevant, theory of counterfactuals (in sharp contrast to the irrelevant classically-based theories which presently dominate the literature[5]).

3. On the third part of the trichotomous classfication, ~A neither cancels nor explodes A, rather ~A *constrains* but does not totally control A. This allows for different positions, including one which will be of especial concern in what follows, namely relevant negation. Equally as "natural" as the cancellation model, and much more natural than the explosion view of contradiction is the relevant model, according to which contradictions have exactly the same sort of inferential status as other types of propositions, that is, they imply some propositions and fail to imply other propositions and are subject to the same laws.

The normal semantical rule for evaluating relevant negation which again is derivable under modest conditions (see RLR, 2.9), is as follows:

$$I(\sim A,a) = 1 \text{ iff } I(A,a^*) \neq 1,$$

i.e. ~A holds in world a iff A does not hold in world a^*, the opposite or reverse of a. The normal rule, which *generalises* the classical rule, differs from the classical rule in the occurrence of function *, a function which has generated much discussion. A major objective in what follows is further explanation of the * function. It is not difficult to show that negation so evaluated has the leading properties sought, e.g. A and ~A are suitably independent though nonetheless related; A and ~A may both fail together and differently both may hold together; A and ~A neither cancel nor implode one another.

It is also not too difficult to indicate how requisite allowance for incomplete and inconsistent worlds, both sorts of which are called for in the semantical evaluation of inference, leads to the normal rule for negation. Such was the historical route: given that the paradoxes of strict implication, (1) $A \wedge \sim A \Rightarrow B$; and (2) $C \Rightarrow D \vee \sim D$, are indeed paradoxes and false of entailment, and that entailment (at the first degree) amounts to truth (or holding) preservation over worlds, then their semantical evaluation must allow for worlds where A and ~A (strictly $A \wedge \sim A$) hold but B does not, i.e. for non-trivial inconsistent worlds, and for worlds where C holds but neither D nor ~D do, i.e. for nonnull incomplete worlds.(6) The classical rule has to be rejected. With only very weak (De Morgan) conditions on negation, e.g. $\sim(A \wedge B) \leftrightarrow \sim A \vee \sim B$, the normal (star) rule is inevitable.

To both sum-up and anticipate: the star rule may be variously seen as a generalization of the classical negation rule, as a generalization that is inevitable if inconsistent and incomplete worlds are to be symmetrically allowed for, as deriving from a general analysis of negation as a certain type of one-place connective, as a way of reducing a 4-valued picture to a two-valued one (the American plan to an Australian one), as a natural reversal operation in semantic tableaux and in worlds modellings (all these explications are featured in RLR).

4. HOW THE MAIN THEORIES HAVE BEEN CONFUSED.

There is much confusion of these three different theories in the modern literature, much of it engendered by classical logicians' identification of their negation with "the" real and natural negation. This is responsible for many gratuitous problems. For example, Strawson thinks that he is giving an account of negation which explains its behaviour in modern logical theory, but he offers the cancellation account, and then (correctly) arrives at principles implying Aristotle's thesis which would trivialize the modern theory.

The first and second views (i.e. 1 and 2) are spectacularly confused in Findlay's exposition of Hegel, where neither view is particularly appropriate. For Findlay both gives a self-nullifying account of contradictions and also uncritically assumes without any pause a classical explosion view:

> All these doctrines [of Hegel's Dialectic] are extremely hard to stomach, since a contradiction is, for the majority of logical thinkers, a self-nullifying utterance, one that puts forward an assertion and then takes it back in the same breath, and so really says nothing.(7) And it can readily be shown that a language system which admits even *one* contradiction among its sentences, is also a system in which *anything whatever* can be proved... (p.76)(8).

The last claim is false, since there are many language systems containing only isolated contradictions; in particular, systems where contradiction really is self-nullifying are commonly of this type. Findlay cannot, on his own grounds, have it both ways.

It is accordingly not surprising that Findlay is bound to say that Hegel, whatever he might *say*, did not mean by 'contradiction' contradiction.

> ... whatever Hegel may *say* in regard to the presence of contradictions in thought and reality, the sense in which he admits such contradictions is determined by his *use* of the concept and not what he says about it... it is plain that he cannot be using it in the self-cancelling manner that might at first seem plausible. By the presence of 'contradictions' in thought and reality, Hegel plainly means the presence of opposed, antithetical *tendencies*... (p. 77, similarly, p.193).

Such is the myth, which is recounted with minor variations, by the majority of commentators upon Hegel. That was not Hegel's position, as Hegel emphasized. As Findlay himself elsewhere remarks,

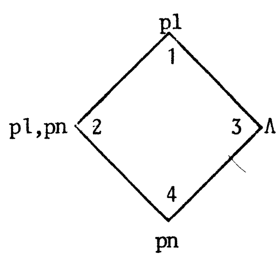
> Hegel makes it as plain as possible, that it is not some watered-down equivocal brand of contradiction, but straight-forward head-on contradiction, that he believes to exist in thought and the world... (p.77),

not antithetical or opposed tendencies, etc. To the question of how negation and contradiction did function on Hegel's logic, the dialectic will return.

The confusion is not confined to nonlogicians, but appears to originate in Aristotle. Russell, one of the main architects of classical logic, affords yet another, and striking example. For in *Human Knowledge: Its Scope and Limits*, Russell tries to explain negation: but he outlines (on p.519) a way of introducing negation which leads *not* to classical negation but rather to a relevant negation (which indicates that even some of those who thought they were arguing for classical negation have gone astray). Russell links negation ('not') with 'No'; correspondingly affirmation is linked with 'Yes'. But "Yes" means "Pleasure this way", and "No" "pain that way" according to Russell, whence the correspondences:

Affirmation	Negation
Yes	No
Pleasure, that way (pl)	Pain, that way (pn).

The first thing to observe about such an explanation is that the explaining terms are *neither exclusive nor exhaustive*. For one phenomenon may yield neither pleasure nor pain, another can produce both pleasure and pain. So what Russell's analogy leads to is not a two-valued picture but a four-valued lattice, with the following Hasse diagram:



where Λ indicates neither pl nor pn.

Since the negation operation, N say, defined on the lattice (in terms of pl yielding pn and vice versa) plainly takes us just from top and bottom and vice versa, the sides being fixed points, upon representing *and* and *or* in the usual way (in the way Russell invariably took them) as lattice join and meet, what results is a model of relevant logic, specifically of tautological entailment (as presented e.g. in Anderson and Belnap).

For the operations yield at once the following ∧ ∨ ~ matrices

∧	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

∨	1	2	3	4	~
1	1	1	1	1	4
2	1	2	1	2	2
3	1	1	3	3	3
4	1	2	3	4	1

→	1	2	3	4
1	1	4	4	4
2	1	1	4	4
3	1	4	1	4
4	1	1	1	1

Upon taking 1 as the only designated value (a natural choice since it is the only case of unmixed pleasure, the only clearly desirable element), and upon *defining* an implication → by the → matrix above, tautological entailment automatically results.[9] For the above, Smiley matrices, are characteristic (see Anderson and Belnap, p.161). Nothing Russell goes on to say alters the initial picture, which he rightly says is 'remote from what the lôgician means by 'not'.' But his attempt to fill the intervening steps, to get to what Russell the logician at least means is simply this:

> "not" means something like: "you do right to reject the belief that ...". And "rejection" means, primarily, a movement of aversion. A belief is an impulse towards some action, and the word "not" inhibits this impulse (pp.519-20).

The picture remains four-valued, since the one thing may both impel and repel or may, alternatively, do neither. The four-valued picture is hardly new, going back to the Megarian logicians. Some of the Stoics tried to reduce the truth value picture to a three-valued one, discarding the value, both (true and false).

The valued (matrix) picture may however, like the three-valued picture be reduced to a two-valued worlds picture. The 4-valued matrices can be derived from a semantical model with just two worlds: T, the real world or in this case the pleasure world, and T^*, its reverse, here the pain world. What is of importance for present purposes in this semantical analysis of the matrices is the fact that the negation rule required by the analysis is the star rule, in the form

$$I(\sim A,T) = 1 \text{ iff } I(A,T^*) \neq 1 \text{ and}$$
$$I(A,T^*) = 1 \text{ iff } I(A,T) \neq 1, \text{ where of course } T^{**} = T.$$

5. MAIN THEMES CONCERNING TRADITIONAL NEGATION, ORDINARY AND NATURAL NEGATION, AND THEIR MODELS.

Neither the explosion nor the cancellation view is satisfactory. The explosion view is strongly paradoxical,[10] the cancellation view is weakly paradoxical (at least as it stands). The cancellation view does not have each contradiction entailing everything, and all inconsistent theories trivialized in the way that the explosion theory does; but it does have each contradiction entailing each other, $A \wedge \sim A \leftrightarrow B \wedge \sim B$, for arbitrary A and B. For A and ~A, and B and ~B, say exactly the same, namely nothing.[11] The explosion view is wrong because contradictions are not so destructive: there are various different non-trivial inconsistent theories. The simple cancellation view is also defective, since not all contradictions carry the same information: they differ in what they entail, some of them entailing some things, others other things.

The negation of Hegel's logic, like that of any paraconsistent logic, does not, and cannot, conform to the classical view 2, nor does it conform to view 1.[12] For not only did Hegel reject the idea that contradictions could not be separately thought ('Contradiction is the very moving principle of the world: and it is ridiculous to say that contradiction is unthinkable', *Logic* p. 174; cf. too Findlay, p.75 where several references are cited: 'it is absurd to say that contradictions are unthinkable'); he also certainly held that in thinking contradictions, one was not thinking nothing, or *merely* a self-cancelling thought; for, quite the contrary, in thinking that Being is identical with Nothingness and is also not identical therewith, one is thinking an explicitly contradictory thought of fundamental importance. While modern paraconsistent theories are usually not as extravagant as to the range, type, or centrality of the contradictions asserted, the intention is much the same: accounts of negation of type 3 are required.

Before considering in more detail what such negations are like, it is worth inquiring, and important to inquire -since classical logic is wont to claim that history (as well as God and Truth and Language) is on its side - what traditional negation, the negation of traditional logic (if there was such a single creature) was like. What was the tradition, especially as regards negation? There wasn't a single unified tradition, there were various competing traditions in particular as to negation and implication. These competing positions are especially evident in the debate as to implication in ancient Alexandria, and in the controversies of scholastic writings. Despite the competition, there seems, at least from post-scholastic times, to have been a dominant view, namely the cancellation view.(13) It should be stressed that this is very much a working hypothesis. There is a great deal of difficult assemblage of historical evidence still to be accomplished (both for modern and for earlier periods).(14) A weaker theme, on somewhat firmer ground, is that *the mainstream or dominant negation of traditional logic is distinctly nonclassical.* Some of the evidence supporting this first working hypothesis will emerge below.

It is also important to inquire what natural negation, negation of natural language, is like, because part at least of the logical enterprise concerning negation is to reflect key features of that negation. Again it has been assumed, with precious little evidence, that classical negation fulfils this role. Many considerations tell against this assumption (see RLR 2). It is important to see through classical negation's pretensions to be the ordinary normal intuitive notion of natural language and logical thought - compared with which alternative negations such as relevant negation must be seen as 'deviant', 'peculiar','queer', abnormal, contrived, or purely formalistic. For seeing through its pretensions is an essential part of seeing through classical (implication) theory and seeing why relevant (implication) theory should replace it.

In fact the situation is pretty much the reverse of the conventional picture. Relevant negation has a better claim to be the (primary) negation determinate of natural language than classical (if indeed there is a unique natural language negation, which is to be doubted). A second working hypothesis is, then, that *relevant negation is a natural negation.* (A, because the negation determinable is probably the most commonly occurring natural language negation; see further RLR, 2.9).

6. NEGATION AS OTHERTHANNESS, AND PROGRESSIVE MODIFICATION OF THE TRADITIONAL PICTURE.

In order to discuss the traditional idea that negation is otherthanness, and to consider negation in its historical setting, especially that of the nineteenth and earlier twentieth century work (when logic began its modern revival), it is helpful to introduce some of the ideas of Boole and Venn in an exact fashion. Consider, in particular a Boole-Venn interpretation of sentential logic S: such an interpretation can be extended to other logics, e.g. modal logics. Such an interpretation j is a mapping from (initial) wff of S to V which consists of a composite with (at least two) components, e.g. a geometrical area, a set, a mereological class, such that the following conditions are met:

$j(\sim A) = V - j(A)$;
$j(A \& B) = j(A) \cap j(b)$ i.e. the common part
$j(A \vee B) = j(A) \cup j(B)$ i.e. the union (of areas).

A wff C of S is said to be BV-*valid* iff, for every mapping j, $j(C) = V$, i.e. the interpretation is always the whole of V. Then no doubt soundness and completeness may be established: a wff C is a theorem of S iff it is BV-valid. Further, assignments under j may be reduced to assignments for initial wff only, and the conditions given used to definitionally extend the assignments to all wff. What is of especial interest is however not the familiar results but the rendering, or intended interpretation, of function j. There are at least three overlapping groups of readings:

1. Geometrical reading with j(A), or $|A|$ as it will sometimes be written, as what A covers (cf. Hospers above), or the area (or territory) of A.

2. Set-theoretical readings, with j(A) some set, e.g. the set of cases where A in true (i.e. the range of A). Set readings are central in the nineteenth century theories of complex terms - in the context of which negation was characteristically discussed. For this the wff of S are reconstrued as complex terms, ~A as non-A (e.g. non-animal), A v B as A or B (e.g. animal or plant), etc. Then j(A) is the extension of A, e.g. j(horse) comprises (all) horses. While the operations of *and* and *or* are relatively straighforward in forming complex terms, several logicians were distinctly worried about *not*, and in fact opted, as we shall see, for a non-Boolean interpretation of *not*.

3. Propositional readings, where j(A) is some proposition or sum of propositions. In particular, an *obvious* j function, exploited below, is that which maps each initial wff to the proposition it expresses.

The late nineteenth century view was that negation, which applied to terms and also to judgements, is otherthanness, and on the prevailing view restricted otherthanness. Thus according to Baldwin's *Encyclopedia* (p. 147) 'Not-A = other than A - a second thing to A'. But *it was not anything other than* A. Joseph, for one, considers the view that 'whatever it (the positive term) be, the negative term covers everything else', and rejects it. His conclusion is that

> A positive term and its corresponding negative (e.g. blue and not-blue) may then be said to divide between them not indeed the whole universe, but the limited universe of things, to which they belong (p. 44fn).[15]

Venn himself acknowledges such limits:

> It is quite true that we always do recognize a limit, sometimes expressed but more often tacit, as to the extent over which not-X is to be allowed to range; and also we not infrequently do so in respect of X itself; so long as these expressions are set before us in words, and not in symbols only.

Though he continues, 'Between them X and not-X must fill up the whole field of our logical enquiry', he admits restricted fields, and allows that we can choose the universe (of discourse) - admissions that run him into serious trouble where different negative terms call for different restricted fields. Venn's procedure and the Boolean picture captured in the j-functions then break down.

It is by now more or less clear how to repair matters. The picture on the left gives way to the picture on the right:

V (perhaps restricted) V (unrestricted)

In functional terms it is a little more complex, but again now evident enough. A further operation * is added to the modelling and the rule for negation amended to:

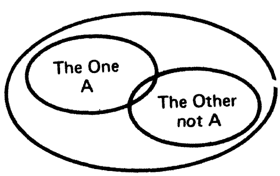
$j(\sim A) = j^{*}(A)$, i.e. roughly

$= V_A - j(A)$, where V_A is the universe as restricted by A.

Some of the intended properties of * are evident enough, e.g. $j^{**} = j$, i.e. * is involutary, and on British perceptions $j \cap j^* = \Lambda$, i.e. j and j* are always exclusive.

To the resulting picture Hegel added, in effect, a significant liberalizing element - one that is forced if contraposition principles (etc.) are to be duly respected, especially at the statement stage - namely that exclusiveness is not always guaranteed. The reason is, according to Hegel, that for certain A (of categorial type) A = not-A. It is not however necessary to accept, for anything that follows, this difficult doctrine: it would be nearer the mark to say that what is supposed is that there are situations where $A \leftrightarrow \sim A$, and better to say, still less demandingly, where both A and ~A hold. It is enough to say, with Simone de Beauvoir (p.18) that presence and absence are not mutually exclusive, or that what A covers does not fully exclude what ~A covers. The liberalized picture which emerges is important:

V (unrestricted)

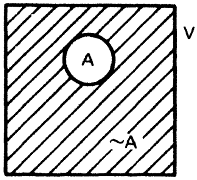


Given that j* need not exclude j, the rule $j(\sim A) = j^*(A)$ again suffices (equivalently, and revealing more structure, $j(\sim A) = V - \bar{j}(A)$, where $\bar{j}$ is the inverse of j: see RLR, 13.5). To glance ahead, we shall simply put The One and The Other back to back, as in a phonograph record, and we will have, more or less, the sought picture of negation.

7. TRANSPOSING THE HEGELIAN PICTURE: RESTRICTED OTHERTHANNESS, REVERSAL AND OPPOSITES.

The next task is to transpose the whole business (as preclassical thinkers like Joseph also tried to do) from the term to the statement level. The Hegelian picture goes over intact, and what results interpretationally are functions extended not according to Boolean but according to De Morgan lattice logic (for details see Anderson and Belnap, or RLR). The negation is no longer classical, but *relevant*.

In terms of relevant negation we can see classical negation as a depauperate one-dimensional notion, which forces us to consider *otherness* with respect to a single universe consisting of everything. In classical logic negation, ~A, is interpreted as the universe without |A|, everything in the universe other than what A covers, as reflected in the Venn diagram:



The square V comprises the universe

The universe can be interpreted as the sum of propositions. Thus where atomic wff *p* is interpreted, naturally enough, as the proposition it expresses, ~*p* amounts to every proposition in the universe other than the proposition that *p*.

Relevance problems come straight out of this; for irrelevance is written in at the bottom. All contradictions have the same interpretation, namely V: hence each entails all others and indeed everything. Paradoxes are inevitable.

It is corollary that ~*p* cannot be independently identified, it is entirely dependent on *p*. This relates, more than coincidentally, to alienation (compare what Simone de Beauvoir has to say to alienation of women where 'woman' is identified as 'other than man'; and is not positively identified, only introduced as alien to the primary notion, 'man'). The negation ~A of A is (so to say) alien to A.

Relevant negation can, however, preserve much of the *otherness* notion of traditional negation (without the counterproductive alienation features). But relevant and classical negation differ firstly as regards what the otherness is considered in relation to. In the case of classical negation it is otherness with respect to the universe. In the case of relevant negation it is otherness with respect to a much more restricted state, such that *p* and its negation do not (interpretationally) exhaust the universe between them.

Such a restricted otherness notion is provided by *reversal*, which gives the other side of something. The *lead* side and the *other*, or opposite, side do not yield everything, the universe, by any means,[16] any more than *p* and ~*p* yield everything with relevant negation. Reversal is in fact a restricted *other than* notion - on the other side is not all territory other than *p*, representing everything other than *p*. With reversal otherthanness operates in a relevantly-restricted universe. The reverse direction (or sense) is not *any* direction other than the forward or given one.

The reversal picture can be filled out in several apposite (and of course connected) ways, both more superficially syntactically, since in one sense the reverse of *p* is ~*p*, and less superficially semantically. Consider first the *debate*, or *dialectical*,[17] *model* which reveals the type of restricted situation with respect to which otherness (the rest of the situation) is assessed. A debate can be represented as the p-issue, or the p-question, when the issue is as to whether p or $\sim p$. One side asserts, argues, or defends p, the other side $\sim p$. Or, as we say, p and $\sim p$ are each sides of the issue as to whether p, one side being the opposite (X or reverse) of the other. The sides are clearly *issue-restricted*, and so accordingly is the complementation. To present the case for one side, e.g. the positive or affirmative, and to present the case for the other side, the negative, is *not* to present the case for everything, to exhaust what can be said, etc.

The debate model indicates that classical negation itself carries the seeds of irrelevance. Thus if one is debating an issue, whether p or $\sim p$, classical negation would allow anything at all that wasn't p as relevant to truth of one half. Thus in debating say, uranium mining one could introduce say, child care centres as relevant to one side of case. The notion of relevance is similarly destroyed, since anything confirming anything which is not p is relevant to the debate. Notions of aboutness, of case, issue, relevance, confirmation and evidence, are all seriously distorted, in a systematic way, by classical negation (as independently shown in much detail in RLR and [22]). The systematic distortion is a result of the restriction to (complete) possible (consistently describable) worlds, a restriction forced by retention of classical negation. There is a similar, and similarly forced, distortion of other intensional functors, e.g. of deontic functors such as obligation (with respect to moral conflicts), of psychological functors such as belief (with respect to inconsistent beliefs), etc. etc.

Classical negation is a depauperate one-dimensional concept which distorts the functions of natural language and limits the usefulness of the logic it yields. Classical negation may *seem* natural, firstly because we (or rather some, the brainwashed among us) have become accustomed to it and perhaps impressed by its computer applications and arithmetical analogues, and secondly because (like material implication itself) it captures *one* dimension of negation, but it has rejected the other dimensions (e.g. restrictedness). Classical negation gives a simple account which is a limiting case, but one which, like that of frictionless surface or perfectly elastic body, does not occur in experience.

8. SEMANTICAL MODELS: WORLDS ON RECORD AND TAPE:

The debate model can be given a more semantical turn. In the p-issue, $\sim p$ is asserted, or presented as true, on one side, a say (i.e. $a \models \sim p$ in obvious notation), while the reverse, namely p, is asserted, or presented as true, on the opposite side a* (i.e. symbolically $a^* \models p$). Now one side succeeds in a debate, or establishes its case, iff the opposite side does not; therefore a $\models \sim p$ iff a* $\not\models$ p. That is, a version of the star rule naturally emerges from the debate model more semantically considered. Statement $\sim p$ is made, or presented as, true at side or situation a iff p is made, or presented as, true at its opposite a*.

The debate model leads directly to the *record cabinet model.* The cabinet, which can represent the files of the universe, is full of records, each record is an *issue,* or question, with p on one side and $\sim p$ on the other side, for every atomic p (at least). From this point of view classical negation takes p as one side of one record, and $\sim p$ as everything else in the cabinet (classical theory fails to duly separate issues). Relevant negation takes p as one side of the record and $\sim p$ as the other side of the *same* record, there being many many records in the cabinet. Note well that intensional functions select a program from the cabinet. Such a program may include *both* sides of a record, and may include neither side of various records - in contrast to the published classical picture (the classical picture can be suped-up to avoid the latter defect but not the former).

The cabinet model may be differently oriented. Each record, or tape, represents, e.g. it may just *describe,* a world, a two-sided world. Then where a is one side of a world record, or a world, the opposite side is again a*, where * is the *reversal,* or *flip*, function which gives, whichever side one is in on, the other side. Obviously a** = a, since turning the record over twice takes one back to the initial position. The semantical rule for evaluating negated statements is, as for the debate model, the star rule, $\sim p$ holds at a iff p does not hold at a*. By contrast, the classical rule quite erroneously identifies a side with its opposite.

The records may be ordered or arranged in a way that reflects the relational structure of (two sided) worlds. The *structured record* model corresponds exactly to a natural elaboration of Kripke's valuable sheets-of-paper model of semantic tableaux for normal modal logics. In explaining *alternative sets* Kripke says (63, p.73): 'Informally speaking, if the original ordered set is diagrammed structurally on a sheet of paper, we copy over the entire diagram twice, in one case putting in addition A in the right column of tableau t and in the other case putting B; the two new sheets correspond to the two new alternative sets'. Thus a full construction which consists of a system of alternative sets corresponds to an arrangement of sheets (a sheaf of sheets). For relevant semantic tableaux there are only two innovations. First, whereas with strict implication new related tableaux are introduced one at a time, with relevant implication new related tableaux are introduced two at a time, i.e. in pairs. This reflects the replacement of the two-place alternativeness relation of modal logics, by the three-place alternatives relation of relevant logics. The first innovation is not particularly germane to the present issues (and quasi-relevant systems such as the I systems which require only two-place relations could be adopted for exposition). Second, and more important, then, both sides of the sheets are used. (Relevant logics are conservation-oriented in that even if rather a lot of sheets are introduced, both sides are used; the reverses are not wasted as with modal semantical tableaux). The reversal function * accordingly reverses the page, giving back for front and front for back.

In sum, reversal and opposition have the right properties in leading respects for (the semantics of) relevant negation. Thus the opposite side of something is not the removal of the first side or, for example, everything other than the first side; it is *another* and *further* side, which is relatively independent of its reverse but which is related to it in a certain way. Both sides can co-occur (occur simultaneously) in a framework (e.g. controversy) and one can perfectly well consider both of them. The important point, to say it yet again, is that one side does not somehow *obliterate* or wipe out or entirely exclude or exhaust its opposite. Nor is the reverse, or opposite, just defined negatively as *the other* - it has an independent and equal role on its own behalf.

There is no mystery then about relevant negation. It is an otherthanness notion; it has natural and easy reversal models. There is some mystery however about classical negation, except as an extrapolation, and much mystery as to why some logicians are tempted to apply it everywhere, especially where, as so often, it mucks things up. Indeed, given the naturalness of relevant negation as issue-controlled complementation, versus the unnaturalness of classical; the naturalness of the reversal notion; and the improved ability of relevant negation to account for actual intensional functions in natural languages, relevant negation has a far better claim to be considered the core negation relation of natural language than classical. So much for the classical claim to have the only real natural negation and that relevant negation is queer.

APPENDIX 1. HISTORICAL SIDELIGHTS; NEGATION AND CONTRADICTION IN HEGEL AND HEGELIAN TRADITIONS.

There is not in Hegel a complete and well worked out theory of negation. There is however much that is suggestive, many models, and a clear nonclassical paraconsistent view. According to Hegel, contradiction occurs both in thought and in the world. There are true contradictions in nature, as an analysis of motion shows.

> Something moves, not because it is here at one point of time and there at another, but because at one and the same point of time it is here and not here, and in this here both is and is not ([12], II, p.67).

For details Hegel refers to Zeno's paradoxes of motion. Another important class of true contradictions concerns the categories, which can pass into and be identical with their opposites. Representing propositional identity as a coentailment, there are truths of the form $A \leftrightarrow \sim A$. Hegel nicely contrasts his view with the ordinary view:

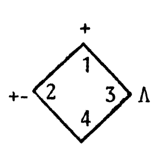
> Ordinarily... contradiction, both in actuality and in thinking reflection, is considered an accident, a kind of abnormality or paroxysm of sickness that will soon pass away ([12], II, p.67).

Ordinarily too, often enough, contradictions are considered unthinkable. According to Hegel however,

> The only correct thing in that statement (that contradiction is unthinkable) is that contradiction does not end the matter, but cancels itself. But contradiction, when cancelled, does not leave abstract identity; for that is only one side of the contrariety ([13], p.174).

The other side is presumably difference. Although there are elements of a cancellation picture both here and elsewhere in Hegel (e.g. [13], p.172 where he compares positive and negative with + and -, which cancel to zero), he rejects a cancellation view. He specifically notes ([12], p.59) that *the ordinary view of contradiction is that it reduces to nothing,* like a vacuum (in itself a revealing piece of historical data). But he says ([12], p.70) that we must pass beyond this one-sided resolution and 'perceive its positive side, when it becomes absolute activity and absolute Ground'.

As well as a severely qualified concellation picture, Hegel offers us a polarity picture of negation, drawn from physics, with Positive and Negative as polar opposites ([13], p.174). This polarity picture rapidly leads to a four-valued model. For some things are both positive and negative, and others are neither. In short we are back with the lattice Russell's theory leads to:



Havas also claims to find a many-valued logic in Hegel, though what his evidence is is unclear;

> ... in Hegel's view, in addition to the values "true" and "false", there is another value, namely, "true and false" and this is the designated value. So, in this case, the value "not-true" is not identical with the value "false",

> since "not-true" means "false, or true and false". If a proposition does not have the value "true", it will have either the value "false" or the value "true and false". Propositions having the value "true and false" are expressions of the actual being of the things, that is, their existence in the dynamical states of coming into being and passing away, and not the mere subsistence of the things.

Perhaps the most disconcerting things about Hegel's logic are firstly, that there appears to be no distinction between acceptable and unacceptable contradictions, all being in a way unacceptable, and generating by themselves motion towards a higher stage in which they are partially resolved, and secondly the sheer extent of contradiction: '*All things are contradictory in themselves*' ([12], p.66). In later idealists such as Bradley it was insisted that contradictions were manifest in appearance, but not in reality; the Absolute was claimed to be self-consistent. Hegel's view seems to have been different; the Absolute was inconsistent: 'the sum-total of all realities ends as absolute contradiction' ([12], p.69).

It is perhaps because of the difficulty of admitting contradictions on such a grand scale as Hegel does that the Marxist tradition, while retaining the thesis of contradictions as pervasive and the source of all movement, watered down the notion of contradiction.

In Marxist theory, the notion of contradiction degenerates - exceedingly low redefinitions of 'contradiction' are invoked. This degenerating use of 'contradiction' which is already beginning in Marx's work has become highly advanced in modern Marxism, where 'contradiction' comes to mean simply 'problem' or '(apparent) conflict' (as often in Mao) or even 'difficulty' (in an Australian radio broadcast). Just one example from *The Trojan Horse:*

> So far no major breakthrough has appeared that is capable of resolving the contradiction of uneven regional development (p. 183, similarly p. 182).

Here the 'contradiction' involves no inconsistency; what there is rather is a problem which has not been satisfactorily resolved.

APPENDIX 2. AN ACCOUNT OF NEGATION AND CONTRADICTION IN WITTGENSTEIN'S WORK.

In the earlier work, the *Notebooks* and especially the *Tractatus*, Wittgenstein runs together, in a way that is ultimately incoherent, exclusion and cancellation models of negation. On the one hand, a classical explosion model of negation and classical truth tables for negation are adopted; negation is represented as total exclusion. There has, of course, to be more to the account of negation than this. In particular, logical constants such as negation, since they would otherwise raise serious difficulties for the picture theory of meaning, call for special treatment, which they obtain through the theory of truth-functions. Negation is simply such a classical function; nothing in reality corresponds to it.

But, on the other hand, significant elements of a cancellation picture are superimposed on the classical view. Although the simple parts of contradictions *and* tautologies have sense, 'the connexions between these paralyse or destroy one another' ([35],p.117). 'Tautology and contradiction are the limiting cases -indeed the disintegration- of the combination of signs' ([34], 4.466). Wittgenstein even says explicitly that 'in a tautology the conditions... cancel one another...' ([34], 4.462), *but* 'cancel' is applied in a different context from that where the cancellation view is explained. (What are cancelled, according to Wittgenstein, are 'the conditions of agreement with the world'). The expected corollaries of a cancellation picture follow, but for tautologies as well as contradictions: they 'say nothing' (e.g. [34], 6.11)(18).

In later work this unstable combination of a hard classical view with a cancellation picture is modified and softened in several respects:

1. Negation is not one thing, or one function. Wittegenstein rejects 'the idea [of earlier work] that there is something common to all negation..., that negation always has "the same meaning" ' ([9], p.540, where these claims are referenced). Wittgenstein now wants to insist that the meaning of negation is not an object, and not an essence. Rather, what the meaning of negation is is shown by 'the way it *works*-the way it is used in the game' ([31], p.55). The exclusion model is not abandoned: rather it is assimilated as one among many (partial) models of negation. So it is also with contradiction (which, along with negation, receives considerable attention in Wittgenstein's later work):

> ... contradiction isn't the *unique* thing people think it is. It isn't the *only* logically inadmissible form and it is, under certain circumstances, admissible (Letters, p.177) .

The no-one-thing theme concerning negation (and derivatively, contradiction) can evidently be assimilated under the determinable theory of negation (of [8], 4.3 and RLR 2.9). There is not a single negation determinate - and in this sense, no essence - but *many*, with classical and connexive negations as (depauperate, and hardly ideal) limiting cases. The determinable theory handles well Wittgenstein's comparison of $\sim(\sim p) = p$ with $(\sim\sim)p = \sim p$, where he says both that the meaning of negation is not different and that there is some truth in our inclination to say that '~' must mean something different in the two cases ([36], p.81).

2. Much of the remainder of Wittgenstein's apparently diffuse, and sometimes incoherent, material on negation and contradiction can similarly be coherently organized within a wide relevant theory of negation, a theory which has a classical core but allows for a wide variety of nonclassical language-games or situations. *The key correspondence* in so reorganizing *is that between language-games* on the one side *and situations or worlds or* - a bit differently - *theories* on the other. Language-games can play a quite analogous semantical role to that played by situations in world semantics (and pragmatics), and indeed, semantical analyses in terms of worlds can be recast in terms of games(19).

Among the normal worlds, comprising class K, of relevant logic, there is a distinguished world T, the factual world, which (on more orthodox accounts) is exclusive and exhaustive, i.e. the classical negation rule is satisfied at T, as it is at the subclass P of K, the complete possible worlds (of modal theory). P does not exhaust K, since K also contains incomplete worlds, which may or may not be possible, and inconsistent worlds. Normal worlds do not of course exhaust worlds (i.e., class W), they only exhaust the worlds required for the assessment of such notions as deducibility and entailment; but for semantical assessment of more highly intensional functors *and* of connexive logic, abnormal worlds are also needed. A precisely analogous picture holds good, in principle(20), for language-games; and we shall simply indicate corresponding languages-games and classes of such games by bar superscripting of corresponding worlds. Thus, for instance, corresponding to T is the true-false language-game $\bar{T}$. Also, most important, corresponding to inconsistent and very incomplete (and so nontrivial) theories in K are fragmentary language-games of $\bar{K}$ in the form of inconsistent calculi (Wittgenstein equates calculi with language-games). Through the difference between $\bar{T}$ and other elements of $\bar{K}$ we can account for such facts as that Wittgenstein does not (in the *Lectures*, for instance) really get beyond a classical truth-table account of negation (good for elements of $\bar{P}$), yet says enough to make it clear that that is inadequate, because we can allow contradictions in systems (cf. [36], p.138) and not admit that everything follows (cf. p.243); so contradictions which occur need not trivialise a calculus in the way that they do classically.

The distinction between $\bar{T}$ and inconsistent calculi in $\bar{K}$ appears in (unnecessarily) accentuated form in Wittgenstein's transitional work (especially [20]). There Wittgenstein separates pure calculi where discourse is not really propositional from the true-false game, which does involve semantical matters such as truth and falsity, and where discourse is propositional(21). He even seems inclined to suggest that, contrary to appearances, a single notion of contradiction does not bridge these distinct areas. Contradiction proper is propositional; 'The idea of contradiction... is that of logical contradiction, and this can occur only in the *true-false game,* that is where we make statements' ([20],p.126). What is *forbidden* in calculi, certain configurations such as Hilbert's $0 \neq 0$, sometimes called 'contradictions', should be represented by 'entirely new sign(s)... the sign Z, say' ([20], pp.175-6). This doubtful distinction is in danger of disappearing even where it occurs (as Waissman's puzzlement about Wittgenstein's position in [20] helps reveal). For Wittgenstein also emphasizes that 'a contradiction proper and a tautology do not *say* anything' ([20], p.531), i.e. even in the true-false game contradictions have no content, a claim Wittgenstein often equates with the claims he also makes that they have no sense, and do not express propositions, but are only sentences (cf. [36], pp.185-6). In this way a propositional role for contradictions which distinguishes them from symbols like $0 \neq 0$ in perhaps inconsistent calculi is undermined. Of course a residual distinction could be retained by saying that in one area the signs are associated with propositions (e.g. their parts express in them), in the other area not. But it would not be worth much. And in any case the effect of such a nonpropositional line on logic, and elsewhere, is devastating (unless a *great* deal of implausible reinterpretation of propositions and proofs in terms of rules is undertaken). For example, things that figure in proofs, arguments (such as from p and ~*p*), beliefs, etc., can no longer figure because they are not duly propositional. In particular, proofs using *reductio* methods are fouled up.

In later work, Wittgenstein abandoned the distinction between types of "contradiction" (but by no means entirely the theme that contradictions do not make sense), and speaks of configurations in calculi as contradictions. It is calculi with such configurations (members of $\bar{W}$-$\bar{P}$), and their mathematical investigation that afford the *limited* emancipation from the requirement of consistency predicted by Wittgenstein ([32], p.332). There is said to be point in developing systems in which contradictions can be generated ([36],pp.188-9), and situations, language-games, where there is such point are outlined.

There are then different (sorts of) language-games with respect to contradictions, some such as $\bar{T}$ where they would be very damaging if they occurred, and some such as calculi games where they do occur but without damage. By way of different language-games, different procedures in the face of contradictions can be allowed for. Thus around the theme of different language-games, many of the pictures and suggestions tried out, often rather inconclusively, in Wittgenstein's later work, can be organized. The reversal pictures of negation (cf. [36], p.180), for example, fit immediately into the wider relevant framework.

Among the pictures tried out are elements that echo Hegel, for instance the sickness presentation of contradictions, the rejection (late on, e.g. [32], p. 130) of the unthinkability of contradictions. Wittgenstein was also aware of, and not entirely unsympathetic to, the Hegel-Parmenidean thesis that an adequate description of motion involves contradictions ([32], V). But Hegel's progressive transcendence of contradictions (which are supposed to appear almost everywhere) is not repeated in Wittgenstein, who would, for the most part, have us stop at contradictions or carefully skirt around them. The sickness presentation is filled out in the *germ* (but not a disease) *image* of a contradiction ([36], pp.138, 211). A contradiction is like a germ in a system, but it does not show that the whole system is diseased. In short, a contradiction in a system is generally a bad thing and to be avoided, but does not reveal triviality. In this connection we are offered something like Kriesel's story regarding Wittgenstein on contradiction: - according to Kriesel, Wittgenstein's rule is: On encountering a contradiction, Stop! There are manifold troubles with this connexive-style rule (brought out in [22], pp.179-80), and it should be rejected. Nor it is so clear that Wittgenstein would have accepted it. What Wittgenstein does say concerning finding a contradiction in a system is that 'the contradiction does not even falsify anything. Let it lie. Do not go there' ([36],p.138). This approach is unsatisfactory. For - contrary to Wittgenstein, who assumes that usefulness implies no contradictions ([32],p.104), that where a calculus has a use contradiction has to be forestalled (cf. [9],p.272)- we may want to *use* the contradiction to show significant things about a system. And we *won't* want to persist with a system which proves trivial (as Wittgenstein occasionally admits, [36], p.243).

3. A cancellation view, strictly incompatible with the classical theory of the *Tractatus*, is increasingly infiltrated in subsequent work. A cancellation picture is already much deployed in transitional work, e.g. 'the rules of Euclidean geometry don't contradict one another, i.e. no rule occurs which cancels out an earlier one (*p* and ~*p*)...' ([31],p.345). It is such a cancellation picture, where (as in the *Tractatus*) contradictions have no content and say nothing, and so are useless, that lies behind Wittgenstein's assumptions that one should not draw *any* conclusions from a contradiction ([36],p.220), or better, that a way should be found of not proceeding from a contradiction ([36],p.223). But both assumptions are inadequate, because often one needs, or wants, to proceed from a contradiction: some contradictions are very useful.

Very many of the pictures and images of negation Wittgenstein later considers are of a cancellation type or can be adjusted to fit a cancellation model. Although Wittgenstein repeatedly alludes to such images, at the same time he depreciates them (e.g. all attempts to explain why a contradiction "won't work" are spurious, [36],p.xviii): they are all said to convert to no more than substitution of one symbolism for another. Even so, such things can have an explanatory and modelling role. Wittgenstein suggests not, because all that is offered is symbolism and figure, so the question of 'how one is going to use it?' ([36],p.181) remains, since any picture can be used in several ways. He goes on to advance the even more dubious description theme that 'anything which we give and conceive to be an explanation of *why* a contradiction does not work is always just another way of saying that we do not want it to work' ([36], p.187).

The assumption that contradictions don't or won't, work and associated themes, e.g. that contradictions are useless, and associated images, etc. the jamming picture ([36],pp.187-9, ascribed to Moore, p.190), are all connexivist in cast. With a contradiction, as when the cogs jam, nothing emerges, 'we cannot do anything with it' ([36],p.191). It is from the same cancellation model that the no-content thesis, which jamming depicts, derives that contradictions do not say anything, a thesis also equivalently (but misleadingly) expressed in 'contradictions don't make sense'.

The cancellation view can be included in the relevant synthesis by appeal to abnormal worlds or language-games, games where contradictions do stop proceedings, and where $A \wedge \sim A$ may have no content. But in assuming, as he often appears to, that games are restricted to those that are classical (effectively, in $\bar{P}$) or those that are of a cancellation type (in a subclass of $\bar{W}$-$\bar{K}$), Wittgenstein much too drastically delimits the games, or worlds, needed in giving a full account of negation. And in assuming that abnormal cancellation-type games are characteristic - '*that* we exclude the contradiction and don't normally give it a meaning is characteristic of our whole use of language' ([36],p.179) - Wittgenstein goes curiously astray. Commonly we do not treat contradictions in this way. We reason on the basis of them (e.g. in reductio arguments), we act on the basis of inconsistent information (cf. the general who acts, and succeeds, on the basis of contradictory reports [36],p.105), we exploit paradoxes when we can, etc.

*

NOTES.

(1) Joseph fails to escape the difficulty though he makes two attempts, contending (falsely): 1 - that 'there is always a positive character as ground of a negation', 2 - 'that A is not B means that it is different from B and not that it is non-existent'. Both routes have since been followed through, and found wanting.

(2) Sartre is firmly entrenched (like the early Russell) in a levels-of-existence doctrine (see, e.g., p.7, middle).

(3) This assumes - what is not unreasonable, but strictly calls for futher argument - that the inner and outer negations are the same.

(4) Assumed, for the time being, at least, is a metatheory for the semantics that can be interpreted classically: cf. RLR, 3.2.

(4.1) Quine's position on "deviant" logics is inconsistent. As Gochet points out (in chapter 7, Section 7), Quine has vacillated between a liberal position, according to which every statement, including any logical law, is open to revision, and an incompatible conservative position (reflected in the previous quotation from p.81) where revision becomes impossible, any change in logical principles being ascribed to changes in the meanings of constants. Quine's later attempt to resolve the matter (in *Roots of Reference*) and admit *limited* revisability, not only leads beyond the confines of truth-functional logic (to three-valued and indeed intensional logic), but is, so Gochet argues, untenable.

(5) Where the classical account is not the crude (but persistent) view that the material-conditional represents 'if ... then', it is a classical-based modal theory in the fashion of Stalnaker and others (see, especially, Harper, Pearce and Stalnaker). Both types are criticised in RLR, where the rudiments of an alternative relevant account are also presented. The philosophical basis of the relevant account is explained in Routley (unpublished).

(6) There are independent arguments, presented in RLR, 2, that ∧ and ∨ semantical rules do not change from orthodoxy.

(7) Similarly below, 'contradictions... condemn (us to) wholesale dumbness', 'it is the mark of a self-contradictory utterance that it describes nothing whatever', etc. Compare also p.25 where self-nullifying is explained in terms of taking back what has been put forward so as to leave nothing standing.

(8) 'So', Findlay continues, p.76,

> "the *whole* of such a system becomes self-nullifying, and infected with contradiction".

The attempt to connect the different cases by the use of 'self-nullifying' fails: for a system is self-nullifying, or rather self-defeating, in saying too much, in being trivial, whereas an utterance is self-nullifying in saying too little, zero.

(9) Alternatively, on different grounds both 1 and 2 may be designated, and a different implication matrix adopted with the same result: see RLR, 2.

(10) That is only symptomatic of the range of things that is wrong with it, on which see RLR, 1.

(11) Strictly there are different positions here depending on whether contradictions are said to imply themselves or not, i.e. whether A → A holds quite generally or not. If not, as with peripatetic logics, weak paradox can be avoided. But then many other problems arise: see RLR, 11. An alternative, sometimes attributed to Wittgenstein, is to say that contradictions lead nowhere, that all argument stops when a contradiction is encountered. As to how unsatisfactory this view is, see [22] pp. 179-80. Moreover, on Wittgenstein's view, contradictions may stand in some language games. They are not always destructive or self-cancelling.

(12) The theme that Hegel's logical theory is a paraconsistent one will be argued elsewhere, as will the theme that contradictions in Hegel's theory are genuine contradictions.

(13) Where does Hegel fit in? Hegel seems to have realized that he was doing something different from traditional logic, that he was in a sense outside of (and extending)the tradition.

(14) We should be grateful to anyone who supplies historical leads to pursue. The situation is much complicated in the case of scholastic logicians by the selection of work that has so far been made available -which typically tries to see these people as anticipating modern established doctrine, the conventional classical wisdom, rather than as investigators of various alternative logic options. The bias of history impedes research of alternatives, so to say. Fortunately that situation may be beginning to change especially with new research into the *obligationes*-literature.

(15) Joseph elaborates his view through examples such as 'intemperate', 'uneven' and 'not-blue', e.g. the latter is equated with 'coloured in some way *other than* blue' (italics added). More generally, not-A will signify what a subject, *which might be* A, will be if it is not A' (p.43).

(16) Otherwise there would be room for only one record company, and only one record from it.

(17) In one of the historical senses of 'dialectical'. A debate can also be 'dialectical' in the other historical sense; for one side may defend both q and ~q. A related model is the *evidence model*, where one side is the evidence for p, the other the evidence for ~p.

(18) It is possible to define a (highly artificial) notion of content which makes *some* elements of such combinations of classical and cancellation views work, for instance thus: where A is analytic ct(A) = 0, and where A is not analytic ct(A) is defined in a standard way, e.g. in terms of consequences of A, or through the class situations where A does not hold. But then content loses its usual (normic) connections, e.g. the ties with consequence are severed, and the logical behaviour of content becomes highly irregular.
L. Goldstein persuaded us that some of Wittgenstein's early work involved a cancellation view.

(19) As explained in detail in [8],7.2. In effect the correspondence is also applied, in different ways, by Lorentzen and Hintikka, where the analogies with game theory are also exploited.

(20) There are some (hardly insuperable) problems in describing maximally consistent language-games.

(21) This sharp contrast, and double role for contradictions, is repeated in Hallett (p.221), again based partly on transitional work. Where propositions *say* something, describe something ([20],p.106), a contradiction is alarming. 'For there can be no contradiction in reality (i.e. T) our description must be wrong'. By contrast, contradictions need not be alarming in mathematics (in K-P). 'But mathematics is always a machine, a calculus. The calculus describes nothing. It can be applied to that to which it can be applied'.

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SOBRE LAS SOLUCIONES PERIODICAS DEL PROBLEMA DE DIRICHLET PARA ECUACIONES DE TIPO PARABOLICO

por

Luis A. ORTEGA

ABSTRACT. It is shown that for a Dirichlet's periodic problema studied by Lazer:

$$L(u) = g(x,t,u)+f_1(x,t)+s\emptyset(x,t) \quad \text{in } \Omega\times\mathbb{R}$$
$$u(x,t+T) \equiv u(x,t), \qquad u\restriction\partial\Omega\times\mathbb{R} = 0$$

where Ω is a bounded domain in $\mathbb{R}^N$, and certain restrictions are assumed for g, f_1, and $\emptyset$, there exists a number $\alpha(f_1)$ such that the problem has at least two solution if $s < \alpha(f_1)$, and at least one if $s = \alpha(f_1)$.

1. Introducción. Ambrosetti y G. Prodi demostraron en [1] un resultado muy importante referente al siguiente problema de Dirichlet.

$$-\Delta u = g(u) + f(x) \text{ en } \Omega, \qquad u \equiv 0 \text{ en } \partial\Omega, \tag{1}$$

en donde Ω es un dominio acotado en $\mathbb{R}^N$, con una frontera $\partial\Omega$ de clase $C^{2+\alpha}$, $g:\mathbb{R}\to\mathbb{R}$ es una función C^2, de valor real, tal que $g''(s) > 0$ para todo $s\in\mathbb{R}$, y

$$0 < \lim_{s\to-\infty} g'(s) < \lambda_1 < \lim_{s\to\infty} g'(s) < \lambda_2, \qquad (2)$$

donde λ_1,λ_2 son los dos primeros valores propios del siguiente problema de valor propio:

$$-\Delta u = \lambda u \text{ en } \Omega, \quad u \equiv 0 \text{ en } \partial\Omega. \qquad (3)$$

Ellos probaron, bajo estas hipótesis, la existencia de una C^1-variedad M en $C^\infty(\bar{\Omega})$, que divide a este espacio en dos conjuntos abiertos G_1 y G_2 con las siguientes propiedades:
(i) Si $f \in G_1$, el problema (1) no tiene solución.
(ii) Si $f \in M$, el problema (1) tiene exactamente una solución.
(iii) Si $f \in G_2$, el problema (1) tiene exactamente dos soluciones.

Por su parte, M.S. Berger y E. Podolak han demostrado en [9] el siguiente resultado: Para cada $f_1 \in N^1$ existe un número real $\alpha(f_1)$ para el cual el problema de Dirichlet:

$$-\Delta u = g(u) + t\phi + f_1 \text{ en } \Omega, \quad u \equiv 0 \text{ en } \partial\Omega, \qquad (4)$$

no tiene solución si $t > \alpha(f_1)$, tiene exactamente una solución si $t = \alpha(f_1)$ y tiene exactamente dos soluciones si $t < \alpha(f_1)$, donde Φ es una función propia del problema (3) que satisface $\int\Phi^2 = 1$ y $N^* = \{f \in C^\alpha(\bar{\Omega}) \mid \int f\Phi = 0\}$.

J.L. Kazdan y F.W. Warner [5], H. Amann y P. Hess [2] y E.N. Dancer [4] probaron resulrados similares referentes al problema (4). En [6], Alan C. Lazer demostró que dada $f_1 \in N^*$, existe un número $\alpha(f_1)$ tal que el siguiente *problema periódico de Dirichlet*:

$$\{L[u] = g(x,t,u) + f_1(x,t) + s\phi(x,t) \text{ en } \Omega\times\mathbb{R},$$

$$u(x,t+T) \equiv u(x,t)\}, \qquad (5)$$

con $u \mid \partial\Omega\times\mathbb{R} = 0$, tiene solución si $s < \alpha(f_1)$, y no tiene solución si $s > \alpha(f_1)$, en donde ϕ es solución del problema de valor en la frontera:

$$\{L[\phi] = \lambda_1\phi,\ \phi > 0 \text{ en } \Omega\times\mathbb{R},\ \phi(x,t+T) = \phi(x,t),$$

$$\phi \restriction \partial\Omega\times\mathbb{R} \equiv 0\} \qquad (6)$$

cuando λ_1 es el valor propio principal , ϕ^* es la solución del problema de valor en la frontera:

$$\{L^*(\phi^*) = \lambda_1\phi^*, \phi^* > 0 \text{ en } \Omega\times\mathbb{R},$$

$$\phi^*(x,t+T) \equiv \phi^*(x,t), \phi^* \restriction \partial\Omega\times\mathbb{R} \equiv 0\}, \qquad (7)$$

y en donde, finalmente:

$$N^* = \{f \in C^{\alpha}(\bar{\Omega}\times\mathbb{R}) \mid f(x,t+T) \equiv f(x,t), (f,\phi^*)_o = 0\}$$

$$L[u] = \frac{\partial u}{\partial t} - \left(\sum_{j=1}^{N}\sum_{i=1}^{N} a_{ij}(\cdot)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(\cdot)\frac{\partial u}{\partial x_i} + Cu\right),$$

$$(C \leqslant 0), \qquad (8)$$

L^* es el operador adjunto de L, y los coeficientes de L son periódicos con período T.

En el mismo artículo Lazer demostró que si $\hat{\lambda} < \lambda_1$, entonces para todo $f \in C^{\alpha}(\bar{\Omega}\times\mathbb{R})$, $f(x,t+T) \equiv f(x,t)$, existe $u \in C^{2+\alpha}(\Omega\times\mathbb{R})$ tal que

$$L[u] \equiv \hat{\lambda}u + f,\ u(x,t+T) \equiv u(x,t),\ u \restriction \partial\Omega\times\mathbb{R} \equiv 0 \qquad (9)$$

Si $f \geq 0$, $f \neq 0$, entonces $u > 0$ en $\Omega\times\mathbb{R}$. En este artículo demostramos que el problema (5) tiene por lo menos dos soluciones si $s < \alpha(f_1)$ y por lo menos una solución si $s = \alpha(f_1)$.

2. Definiciones y notaciones.

Denotaremos con $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) a un dominio acotado con frontera de clase $C^{2+\alpha}$, para algún $\alpha \in (0,1)$, con $[a,b]$ a un intervalo compacto y con u, a una función de valor real, definida en $\bar{\Omega}\times[a,b] = Q$. Escribimos $u \in C^{\alpha}(Q)$ si el número

$$\bar{H}_{\alpha}^{Q}(u) = \sup_{\substack{(x_k,t_k)\in Q \\ k=1,2 \\ (x_1,t_1)\neq(x_2,t_2)}} \frac{|u(x_1,t_1)-u(x_2,t_2)|}{[|x_1-x_2|^2+|t_1-t_2|]^{\alpha/2}} < \infty$$

El conjunto de todas estas funciones es un espacio de Banach, con la norma $\|u\|_{C^{\alpha}(Q)} = \max\limits_{(x,t)\in Q} \|u(x,t)\| + \bar{H}_{\alpha}^{Q}(u)$.

Se escribe también $u \in C^{2+\alpha}(Q)$, si $u_t, u_{x_i}, u_{x_i x_j}, u$ pertenecen a $C^{\alpha}(Q)$, para $1 \leq i, j \leq N$ y la norma $\|u\|_{C^{2+\alpha}(Q)}$ se define como la suma de las $C^{\alpha}(Q)$-normas de todas estas funciones. Similarmente, $C^{1+\alpha}(Q)$ se define como el conjunto de todas las funciones u definidas en Q, tales que u y u_{x_j} $1 \leq i \leq N$, pertenecen a $C^{\alpha}(Q)$. Se escribe $u \in C^{\alpha}(\bar{\Omega}\times\mathbb{R})$, ($C^{1+\alpha}(\bar{\Omega}\times\mathbb{R})$, $C^{2+\alpha}(\bar{\Omega}\times\mathbb{R})$ respectivamente), si $u \in C^{\alpha}(\bar{\Omega}\times[a,b])$, ($C^{1+\alpha}(\bar{\Omega}\times[a,b])$, $C^{2+\alpha}(\bar{\Omega}\times[a,b])$ respectivamente) para cualquier $a,b \in \mathbb{R}$, $a < b$.

Denotamos con F al espacio de Banach conformado por las funciones $f \in C^{\alpha}(\bar{\Omega}\times\mathbb{R})$, tales que $f(x,t+T) \equiv f(x,t)$, con la norma $|\cdot|_F$ definida por $\|f\|_{C^{\alpha}(\bar{\Omega}\times[0,T])} = |f|_F$. Con

E denotamos al espacio de Banach de todas las funciones $u \in C^{2+\alpha}(\bar{\Omega}\times\mathbb{R})$ que satisfacen $u(x,t+T) \equiv u(x,t)$ en $\bar{\Omega}\times\mathbb{R}$ y $u(x,t) = 0$, para todo $(x,t) \in \partial\Omega\times\mathbb{R}$, con la norma $\|\cdot\|_E$ definida por $\|u\|_E = \|u\|_{C^{\alpha+2}(\Omega\times[0,T])}$.

En lo que sigue, L denotará al operador definido en (8) con las siguientes condiciones:

A_1) Los coeficientes de L son periódicos en t con período T.

A_2) $a_{ij}(\cdot) \in C^{2+\alpha}(\bar{\Omega}\times\mathbb{R})$, $1 \leqslant i, j \leqslant N$, $a_{ij} = a_{ji}$,

$b_i(\cdot) \in C^{1+\alpha}(\bar{\Omega}\times\mathbb{R})$, $1 \leqslant i \leqslant N$,

$C \in C^{\alpha}(\bar{\Omega}\times\mathbb{R})$, $C \leqslant 0$.

A_3) L es uniformemente parabólico.

El operador L^* adjunto de L, tiene la forma siguiente:

$$L^*[u] = -\frac{\partial u}{\partial t} - \left(\sum_{i,j=1}^{N} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i^*(x,t)\frac{\partial u}{\partial x_i} + C^*(x,t)u \right),$$

donde,

$$b_i^* = -b_i + 2\sum_{j=1}^{N} \frac{\partial^2 a_{ij}}{\partial x_i \partial x_j}, C^*(x,t) = C(x,t) - \sum_{i=1}^{N}\frac{\partial b_i}{\partial x_i} + \sum_{i,j=1}^{N}\frac{\partial^2 a_{ij}}{\partial x_i \partial x_j}$$

Con $g:\bar{\Omega}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ denotaremos a una función que cumple lo siguiente:

a) $g(x,t+T,u) = g(x,t,u)$ en $\bar{\Omega}\times\mathbb{R}\times\mathbb{R}$,

b) $\frac{\partial g}{\partial u}$ es continua, $\frac{\partial g}{\partial u}(\cdot,\cdot,v) \in F$ y $g(\cdot,\cdot,v) \in F$ para cualquier $v \in F$.

c) Existen constantes $k_1 > 0$, $k_2 > 0$ y $k_3 > 0$ tales que $g(x,t,u)-\lambda_1 u \geqslant k_1|u|-k_2$, para todo $(x,t,u) \in \bar{\Omega}\times\mathbb{R}\times\mathbb{R}$ y existe un número a tal que: $\left|\frac{\partial g}{\partial u}(x,t,u)\right| < k_3$, para cualquier $u \in [a,\infty)$ y para todo $(x,t) \in \bar{\Omega}\times\mathbb{R}$ (se puede suponer que $\lambda_1 > k_1$, pues $\lambda_1 > 0$).

3. Resultado principal. El principal propósito de esta última sección es probar el siguiente resultado, en donde N^* es el conjunto definido en la introducción.

3.1 TEOREMA. *Para cada* $f_1 \in N^*$, *existe un número* $\alpha(f_1)$ *tal que el problema de Dirichlet:*

$$\{u \in E \text{ y } L[u] = g(\cdot,\cdot,u) + f_1 + s\phi \text{ en } \Omega\times\mathbb{R}\}, \qquad \text{(I)}$$

tiene por lo menos dos soluciones si $s < \alpha(f_1)$ *y por lo menos una si* $s = \alpha(f_1)$.

En la demostración de este teorema usaremos los siguientes lemas.

3.2 LEMA. *Para cada* $f \in F$ *existe una única función* $u \in E$ *que cumple* $L[u] = f$. *Además existen constantes* $\bar{k}_1 > 0$ *(la cual depende de un número* $\beta \in (0,1)$, *fijo) y* $\bar{k}_2$ *tales que:*

a) $\|u\|_E \leq \bar{k}_1 \|L[u]\|_F$,

b) $\|u\|_{1+\beta} \leq \bar{k}_2 \|L[u]\|_\infty$.

3.3 LEMA. *Para cada* $r_o > 0$, *existe una constante* $\bar{R}(r_o) > 0$ *tal que si* $f \in F$, $u \in E$ *y* $L[u] = g(\cdot,\cdot,u)+f$ *en* $\Omega\times\mathbb{R}$ *y* $\|f\|_\infty < r_o$, *entonces* $\|u\|_{1+\beta} \leq R(r_o)$ *(en toda esta sección* $\beta \in (0,1)$ *y se mantiene fijo).*

3.4 LEMA. *Sean*

$$\hat{E} = \{f \in C^{1+\beta}(\bar{\Omega}\times\mathbb{R}) \mid f \equiv 0 \text{ en } \partial\Omega\times\mathbb{R},\ f(x,t+T) \equiv f(x,t) \text{ en } \bar{\Omega}\times\mathbb{R}\}$$

y $i: E \to \hat{E}$ *la inyección de* E *en* $\hat{E}$. *Si* $f_1 \in N^*$, $d \geq 0$ *y*

$M[u] = L[u]+du$ *para todo* $u \in E$, *entonces para cada* $t \in \mathbb{R}$, *la aplicación definida de* $\hat{E}$ *en* $\hat{E}$ *por* $H_t(v) = iM^{-1}(g(\cdot,\cdot,v) + dv + f_1 + t\phi)$, *para cada* $v \in \hat{E}$, *esta bien definida y es continua y compacta* (en toda esta sección $\hat{E}$ tendrá el mismo significado).

Este lema nos permite hablar del grado de Leray-Schauder: $dg(I-H_t, G,p)$ donde G es un conjunto abierto y acotado en $\hat{E}$ y $p \in \hat{E}$.

3.5 **LEMA.** *Si* $f_1 \in N^*$ *y* $t_o < \alpha(f_1)$, *entonces existen dos funciones* $\bar{u} \in E$, $\underline{u} \in E$, *tales que*

$$L[\bar{u}] > g(\cdot,\cdot,\bar{u}) + t_o\phi + f_1 \text{ en } \Omega\times\mathbb{R},$$

$$L[\underline{u}] < g(\cdot,\cdot,\underline{u}) + t_o\phi + f_1 \text{ en } \Omega\times\mathbb{R}, \text{ y } \underline{u} < \bar{u} \text{ en } \Omega\times\mathbb{R}.$$

3.6 **LEMA.** *Sea*

$$\tilde{g}(x,t,s) = \begin{cases} g(x,t,\underline{u}(x,t))+d_1\underline{u}(x,t), & si\ s \leqslant \underline{u}(x,t), \\ g(x,t,s)+d_1 s, & si\ \underline{u}(x,t) < s < \bar{u}(x,t), \\ g(x,t,\bar{u}(x,t))+d_1\bar{u}(x,t) & si\ s \geqslant \bar{u}(x,t), \end{cases}$$

donde $\left|\frac{\partial g}{\partial u}(x,t,s)\right| \leqslant d_1$ *en* $\Omega\times\mathbb{R}\times[\bar{\bar{r}},\infty]$ *y* $\bar{\bar{r}} = \min\limits_{\bar{\Omega}\times\mathbb{R}} \underline{u}$. *Entonces* $\tilde{g}$ *es una función acotada y no decreciente en la variable* s.

El lema anterior se usa en la demostración del siguiente lema:

3.7 **LEMA.** *Si* $t_o < \alpha(f_1)$, *y* $\tilde{H}_{t_o}$ *es la función de* $\hat{E}$ *en* $\hat{E}$ *definida por* $\tilde{H}_{t_o}(v) = iM^{-1}(\tilde{g}(\cdot,\cdot,v)+t_o\phi+f_1)$ *para ca-*

da $v \in \hat{E}$, *donde* $M[u] = L[u]+d_1u$, *y la función* $i:E \to \hat{E}$ *es la inyección, entonces existe un número real* $\bar{R} > 0$ *tal que* $\|\tilde{H}_{t_o}(v)\|_{1+\beta} < \bar{R}$, *para todo* $v \in \hat{E}$.

3.8 **LEMA.** ($\bar{R}$ *mencionado en el Lema 3.7*). *Si*

$$G = \begin{cases} u \in \hat{E} \mid \underline{u} < u < \bar{u} \text{ en } \Omega\times\mathbb{R}, \\ \dfrac{\partial\bar{u}}{\partial n} < \dfrac{\partial u}{\partial n} < \dfrac{\partial\underline{u}}{\partial n} \text{ en } \partial\Omega\times\mathbb{R} \\ \|u\|_{1+\beta} < \bar{R}, \end{cases}$$

entonces:

i) G *es un conjunto no vacío, abierto y convexo de* $\hat{E}$ *y* $\tilde{H}_{t_o} \mid G \equiv H_{t_o} \mid G$.

ii) $dg(I-H_{t_o},G,0) = 1$.

iii) *Existe un número* $R_o > \bar{R}$ *tal que* $dg(I-H_{t_o},B_{R_o},0) = 0$ *donde* $B_{R_o} = \{u \in \hat{E} \mid \|u\|_{1+\beta} < R_o\}$, *y* $dg(I-H_{t_o},D,P)$, *denota el grado de* $(I-H_{t_o})$ *en* P, *relativo a* D.

Demostración del Teorema 3.1. Supongamos que $t_o < \alpha(f_1)$. Por el Lema 3.8, $dg(I-H_{t_o},G,0) = 1$, $dg(I-H_{t_o},B_{R_o},0) = 0$ y $B_{R_o} \supset G$. Por un resultado estándar de la teoría del grado (ver [7], Teorema 4.3.9), lo anterior implica que $dg(I-H_{t_o},(B_{R_o}\setminus\bar{G}),0) = -1$; por tanto, existen dos funciones, $u_1 \in G$, $u_2 \in (B_{R_o}\setminus\bar{G})$, tales que $H_{t_o}(u_1) = u_1$, y $H_{t_o}(u_2) = u_2$. Esto quiere decir que:

$$u_1 = H_{t_o}(u_1) = M^{-1}(g(\cdot,\cdot,u_1)+d_1u_1+f_1+t_o\phi)$$

$$u_2 = H_{t_o}(u_2) = M^{-1}(g(\cdot,\cdot,u_2)+d_1u_2+f_1+t_o\phi).$$

Estas dos identidades son equivalentes a:

$$M[u_1] = L[u_1]+d_1u_1 = g(\cdot,\cdot,u_1)+d_1u_1+t_o\phi+f_1,$$

$$M[u_2] = L[u_2]+d_1u_2 = g(\cdot,\cdot,u_2)+d_1u_2+t_o\phi+f_1,$$

$$L[u_1] = g(\cdot,\cdot,u_1)+t_o\phi+f_1 \quad \text{en } \Omega\times\mathbb{R},$$

$$L[u_2] = g(\cdot,\cdot,u_2)+t_o\phi+f_1 \quad \text{en } \Omega\times\mathbb{R}, \quad u_1 \neq u_2.$$

Para demostrar (ii), denotemos con $\{t_n\}_{n=1}^{\infty}$ a una sucesión de números reales que cumple las condiciones $t_n < \alpha(f_1)$ para cualquier entero $n \geqslant 1$, y $\lim_{n\to\infty} t_n = \alpha(f_1) = t^*$. Lo anterior implica que existe una sucesión de funciones $\{u_n\}_{n=1}^{\infty}$ en E tal que $L[u_n] = g(\cdot,\cdot,u_n)+t_n\phi+f_1$ en $\Omega\times\mathbb{R}$, para todo $n \geqslant 1$. Puesto que $\{t_n\}_{n=1}^{\infty}$ es una sucesión convergente, existe $r > 0$ tal que $\|t_n\phi+f_1\|_\infty \leqslant |t_n|\|\phi\|_\infty+\|f_1\|_\infty < r$ para todo $n \geqslant 1$. Por el Lema 3.3, existe $R(r) > 0$ que satisface $\|u_n\|_{1+\beta} < R(r)$ para todo $n \geqslant 1$. Ahora bien, como la inyección $i:\hat{E} \to F$ es compacta, existe una subsucesión de $\{u_n\}_{n=1}^{\infty}$, denotada también con $\{u_n\}_{n=1}^{\infty}$, que converge a una función $u \in F$, en la norma de F. Por tanto: $\|u_n-u\|_\infty \to 0$ cuando $n \to \infty$. Lo anterior implica que existe $\bar{\bar{r}} > 0$ tal que $u_n \geqslant \bar{\bar{r}}$ y $u \geqslant \bar{\bar{r}}$ en $\bar{\Omega}\times\mathbb{R}$, para todo $n \geqslant 1$. Se puede suponer, por otra parte, que $|\frac{\partial g}{\partial u}(x,t,s)| \leqslant k_3$, para todo $(x,t) \in \bar{\Omega}\times\mathbb{R}$ y $s \geqslant \bar{\bar{r}}$ Por tanto,

$$\begin{aligned}\|g(\cdot\cdot,u_n)-g(\cdot\cdot,u)\|_\infty &= |\frac{\partial g}{\partial u}(\cdot,\cdot,\xi(\cdot)(u_n-u)\|_\infty \\ &\leqslant k_3\|u_n-u\|_\infty \to 0,\end{aligned}$$

cuando $n \to \infty$. Por el Lema 3.2, existe entonces una función $v \in E$ tal que:

$$L[v] = g(\cdot,\cdot,u) + t^*\phi+f_1 \text{ en } \Omega\times\mathbb{R} \tag{10}$$

y

$$\|u_n - v\|_{1+\beta} \leqslant \bar{k}_1 \|L(u_n - v\|_\infty$$

$$= \bar{k}_1 \|g(\cdot,\cdot,u_n) + t_n\phi - g(\cdot,\cdot,u) - t^*\phi\|_\infty$$

$$\leqslant \bar{k}_1 \|g(\cdot,\cdot,u_n) - g(\cdot,\cdot,u)\|_\infty + \bar{k}_1 \|\phi\|_\infty |t_n - t^*|$$

$$\leqslant \bar{k}_1 k_3 \|u_n - u\|_\infty + \bar{k}_1 \|\phi\|_\infty |t^* - t_n| \to 0$$

si $n \to \infty$. Por lo anterior, tenemos que $u_n \to v$, cuando $n \to \infty$, en la norma de F. Como $\|u_n - u\|_\infty \to 0$ cuando $n \to \infty$, tenemos $u \equiv v$. Por (10), podemos concluir finalmente que

$$L[u] = g(\cdot\cdot,u) + t^*\phi + f_1 \text{ en } \Omega\times\mathbb{R}, \text{ donde } t^* = \alpha(f_1). \quad \Delta$$

Demostración del Lema 3.2. Denotemos con $x = (x_1, \ldots, x_n)$ a un punto variable en $\mathbb{R}^n$ y sean $s_o = \min_{\bar{\Omega}} x_1$ y $s_1 = \max_{\bar{\Omega}} x_1$ Si $\psi(x) = e^{\gamma(s_1 - s_o)} - e^{\gamma(x_1 - s_o)}$ donde $\gamma > 0$, entonces $\psi(x) \geqslant 0$ en $\bar{\Omega}$ y $L(\psi)(x,t) \geqslant m\gamma^2 - k\gamma$, donde m es la constante asociada a L por ser uniformemente parabólico, y k es la cota de $|b_1(x,t)|$ en $\bar{\Omega}\times\mathbb{R}$. Seleccionamos $\gamma > 0$ lo suficientemente grande para que $L(\psi)(x,t) \geqslant 1$. Haremos la demostración para $f \in F$, $f(x,t) \geqslant 0$ en $\bar{\Omega}\times\mathbb{R}$, y $f(x,0) \equiv 0$. La demostración en el caso general es fácil, usando este caso particular. Denotemos con $v(x,t) = \psi(x)\|f\|_\infty$ y con $z(x,t)$ a la solución del problema inicial con valor en la frontera:

$$L[z](x,t) = f(x,t) \text{ en } \bar{\Omega}\times[0,\infty), \; z(x,0) \equiv 0,$$

$$z\restriction\partial\Omega\times[0,\infty) \equiv 0.$$

Puesto que $L[z] \geqslant 0$ en $\bar{\Omega}\times[0,\infty)$ y $z \equiv 0$ en $\partial\Omega\times[0,\infty) \cup \Omega\times\{0\}$, síguese, por el principio máximo, que $z(x,t) \geq 0$ en $\bar{\Omega}\times[0,\infty)$. Hagamos $z_1(x,t) \equiv z(x,t+T)$, para $t \geq 0$. Ya que $f \in F$ y los coeficientes de L son T-periódicos en t, entonces $L[z_1] \equiv f$. Como

$$L[z_1-z] \equiv 0, \quad (z_1-z)(x,0) = z_1(x,0) = z(x,T) \geq 0$$

y $(z_1-z)|\partial\Omega\times[0,\infty) \equiv 0$, tenemos que $z(x,t) \leq z_1(x,t) = z(x,t+T)$ en $\Omega\times[0,\infty)$. Por tanto, si $z_m(x,t) \equiv z(x,t+mT)$, resulta que $z_m(x,t) \leq z_{m+1}(x,t)$ para $m \geq 1$, y $(x,t)\bar{\Omega}\times[0,\infty)$. Puesto que $L[v](x,t) = \|f\|_\infty L(\psi)(x,t) \geq \|f\|_\infty \geq f$, y $v(x,0) \geq 0$, se deduce del principio máximo, que $z(x,t) \leq v(x,t) \leq d_1\|f\|_\infty$, donde $d_1 = e^{\gamma(s_1-s_o)}$. Por consiguiente, $0 \leq z_m(x,t) \leq z_{m+1}(x,t) \leq d_1\|f\|_\infty$ en $\Omega\times[0,\infty)$ para todo $m > 1$; esto es, el $\lim_{m\to\infty} z_m(x,t) \equiv u(x,t)$ existe en $\bar{\Omega}\times[0,T]$. Sea $\theta(t)$ una función suave, definida en $[0,\infty)$, tal que $\theta(0) = 0$ y $\theta(t) = 1$ para $t \geq T/2$. Si para cada $m \geq 1$, escribimos $w_m = \theta z_m$, entonces $L[w_m] = \theta\, z_m + \theta f = h_m$, en $\bar{\Omega}\times[0,\infty)$. Ahora bien, por estimaciones previas tenemos que $\|h_m\|_\infty \leq d_2\|f\|_\infty$, en $\bar{\Omega}\times[0,\infty)$, donde d_2 es independiente de f. Dado que $w_m = 0$ en $\partial\Omega\times[0,\infty) \cup \Omega\times\{0\}$, por un resultado conocido (ver [8], págs. 342-343), para cualquier $r > 0$ existe entonces $d_3 = d_3(r)$ tal que, $\|w_m\|_{1+\beta}^{\Omega\times[0,r]} \leq d_3\|h_m\|_\infty^{\Omega\times[0,r]}$. Si $r > T/2$, vemos que

$$\|z_m\|_{1+\beta}^{\Omega\times[T/2,r]} \leq d_3\|h_m\|_\infty^{\Omega\times[0,r]} \leq d_2 d_3\|f\|_\infty, \tag{11}$$

para todo $m \geq 1$.

Como la inyección de $C^{1+\beta}(\bar{\Omega}\times[T/2,r])$ en $C^{\alpha}(\Omega\times[T/2,r])$ es compacta, existe una subsuseción de $\{z_m\}_{m=1}^{\infty}$ que converge en $C^{\alpha}(\bar{\Omega}\times[T/2,r])$ y puesto que cualquiera de tales subsucesiones converge a $u(x,t)$ en $\bar{\Omega}\times[T/2,r]$, tenemos que $u \in C^{\alpha}(\bar{\Omega}\times[T/2,r])$ y $\|z_m-u\|_{\alpha}^{\Omega\times[T/2,r]} \to 0$, cuando $m \to \infty$, para todo $r > T/2$. Denotemos con Φ a una función suave tal que $\Phi(T/2) = 0$, $\Phi(t) \equiv 1$ para $t \geqslant T$. Sea $n > m \geqslant 1$; ya que $\Phi(z_n-z_m) = 0$ en $\partial\Omega\times[T/2,\infty) \cup \bar{\Omega}\times\{T/2\}$, por un resultado conocido (ver [3] pág. 65, Teoremas 6 y 7), para todo $r > T/2$ existe una constante $d_4 = d_4(r)$ independiente de m y n que cumple

$$\|\phi(z_n-z_m)\|_{2+\alpha}^{\Omega\times[T/2,r]} \leqslant d_4\|L(\Phi(z_n-z_m))\|_{\alpha}^{\Omega\times[T/2,r]}$$

Ya que

$$\|L(\Phi(z_n-z_m))\|_{\alpha}^{\Omega\times[T/2,r]} = \|\Phi(z_n-z_m)\|_{\alpha}^{\Omega\times[T/2,r]}$$

$$\leqslant d_5\|z_n-z_m\|_{\alpha}^{\Omega\times[T/2,r]}$$

se tiene que:

$$\|z_n-z_m\|_{\alpha+2}^{\bar{\Omega}\times[T,r]} \leqslant \|\Phi(z_n-z_m)\|_{2+\alpha}^{\bar{\Omega}\times[T/2,r]} \leqslant d_4d_5\|z_n-z_m\|_{\alpha}^{\bar{\Omega}\times[T/2,r]}$$

Lo anterior implica que $\{z_n\}^{\infty}$ converge a u en $C^{2+\alpha}(\bar{\Omega}\times[T,r])$, para cualquier $r > T$. Esto quiere decir que u es de clase $C^{2+\alpha}$ en $\Omega\times[T,r]$, para $r > T$, y para $x \in \bar{\Omega}$ y $t > T$, se tiene entonces $u(x,t+T) = \lim_{m\to\infty} z_m(x,t+T) = \lim_{m\to\infty} z_{m+1}(x,t) = u(x,t)$; además para $x \in \partial\Omega$ y $t \geqslant T$, $u(x,t) = \lim_{m\to\infty} z_m(x,t) = 0$; y existe una única extensión de u a $\bar{\Omega}\times\mathbb{R}$, que es T-periódica en t. Si denotamos esta extensión otra vez con u, entonces $u \in E$ y $L[u] = f$. Retornando a la desigualdad (11) tenemos que

$$\|u\|_{1+\beta} = \lim_{m\to\infty}\|z_m\|_{1+\beta}^{\Omega\times[T,2T]} \leqslant \lim_{m\to\infty}\|z_m\|_{1+\beta}^{\bar{\Omega}\times[T/2,2T]}$$

$$\leqslant d_2d_3\|f\|_{\infty}, \quad \bar{k}_2 = d_2d_3.$$

La unicidad es consecuencia inmediata del principio máximo. Por lo tanto, el operador $L: E \to F$ es uno a uno y sobre. Por el teorema de la función abierta, L^{-1} es continua; en consecuencia, existe una constante $\bar{k}_1$ que satisface la desigualdad (a). ▲

Demostración del Lema 3.3. Usando las hipótesis sobre la función g y las relaciones comunes existentes entre el operador L y su adjunto L^*, es fácil demostrar que para r_o dado, existe $r_1 > 0$ tal que si $u \in E$, $f \in F$, $\|f\|_\infty < r_o$ y $L[u] = g(\cdot\cdot,u)+f$ entonces $\|u\|_\infty \leqslant r_1$. Si $f \in F$, $u \in E$, $\|f\|_\infty < r_o$ y $L[u] = g(\cdot,\cdot,u)+f$, entonces

$$\|u\|_{1+\beta} < \bar{k}_2 \|L[u]\|_\infty \leqslant \bar{k}_2(\|g(\cdot,\cdot,u)\|_\infty + \|f\|_\infty)$$

$$\leqslant \bar{k}_2\left[k_3\|u\|_\infty + \sup_{\bar{\Omega}\times\mathbb{R}}|g(x,t,0)| + \|f\|_\infty\right]$$

$$\leqslant \bar{k}_2\left|k_3 r_1 + r_o + \sup_{\bar{\Omega}\times\mathbb{R}}|g(x,t,0)|\right] \equiv R(r_o). \quad ▲$$

Las demostraciones de los lemas 3.4, 3.6 y 3.7 son inmediatas.

Demostración del Lema 3.5. Tomemos $t_o < t_1 < \alpha(f_1)$, (t_1 fijo). Por (5), existe una función $\bar{u} \in E$ tal que $L[\bar{u}] = g(\cdot,\cdot,\bar{u}) + t_1\phi + f_1$ en $\Omega\times\mathbb{R}$. Como $\phi > 0$ en $\Omega\times\mathbb{R}$, se tiene

$$L[\bar{u}] = g(\cdot,\cdot,\bar{u}) + t_1\phi + f_1 > g(\cdot,\cdot,\bar{u}) + t_o\phi + f_1 \text{ en } \Omega\times\mathbb{R}.$$

Sea $\bar{r}_1 = \min_{\Omega\times\mathbb{R}}(f_1 + t_o\phi)$. Para $f = -k_2 + (\bar{r}_1 - 1)$, por (9), existe una función $\underline{u} \in E$ tal que

$$L[\underline{u}] = (\lambda_1-k_1)\underline{u}-k_2+(\bar{r}_1-1) \leq g(\cdot,\cdot,\underline{u})+(\bar{r}_1-1) < g(\cdot,\cdot,\underline{u})+t_o\phi+f_1$$

en $\Omega\times\mathbb{R}$. Puesto que

$$L[\bar{u}] > g(\cdot\cdot,\bar{u})+t_o\phi+f_1 > (\lambda_1-k_1)\bar{u}+(\bar{r}_1-1)-k_2 \text{ en } \Omega\times\mathbb{R},$$

tenemos:

$$L[\bar{u}-\underline{u}] > (\lambda_1-k_1)(\bar{u}-\underline{u}) \text{ en } \Omega\times\mathbb{R} \tag{12}$$

Hagamos $z = \bar{u}-\underline{u}$ y $h = L[z]-(\lambda_1-k_1)z$ en $\Omega\times\mathbb{R}$. Como $L[u] = (\lambda_1-k_1)z+h$ y $h > 0$ en $\Omega\times\mathbb{R}$, por (9) tenemos que $0 < z = \bar{u}-\underline{u}$ en $\Omega\times\mathbb{R}$. ▲

Demostración del Lema 3.8. La demostración de (i) es consecuencia directa de las definiciones de G y $\tilde{H}_{t_o}$. Para demostrar (ii), supongamos que $v \in \hat{E}$, $u = \tilde{H}_{t_o}(v)$ y $u \in E \subset \hat{E}$. Por el Lema 3.5 tenemos que:

$$M[\bar{u}-u] > d_1\bar{u}+g(\cdot,\cdot,\bar{u})+t_o\phi+f_1-(\tilde{g}(\cdot,\cdot,v)+t_o\phi+f_1)$$

$$= \tilde{g}(\cdot,\cdot,\bar{u})-\tilde{g}(\cdot,\cdot,v) \geq 0 \quad \text{en } \Omega\times\mathbb{R}.$$

De esta desigualdad y del principio del máximo para las ecuaciones parabólicas, se deduce que $\bar{u}-u > 0$ en $\Omega\times\mathbb{R}$ y $\frac{\partial}{\partial\vec{n}}(\bar{u}-u) < 0$ en $\partial\Omega\times\mathbb{R}$. Usando este mismo argumento podemos probar que $\partial u/\partial\vec{n} < \partial\underline{u}/\partial\vec{n}$ en $\partial\Omega\times\mathbb{R}$. Ya que $\|u\|_{1+\beta} = \|\tilde{H}_{t_o}(v)\|_{1+\beta} < \bar{R}$ (ver Lema 3.7), tenemos que $u = \tilde{H}_{t_o}(v) \in G$, para todo $v \in \hat{E}$. Tomemos $\bar{\phi} \in G$, ($\bar{\phi}$ fijo) y consideremos la homotopía compacta $\tilde{H}_\lambda(v) = \lambda\tilde{H}_{t_o}(v)+(1-\lambda)\bar{\phi}$, $0 \leq \lambda \leq 1$, para todo $v \in \hat{E}$. Ya que $\tilde{H}_{t_o}(v) \in G$ y G es un conjunto convexo, se sigue que $\tilde{H}_\lambda(v) \in G$, para cada $v \in \hat{E}$ y $0 \leq \lambda \leq 1$. Por la propiedad de invariabilidad de las homotopías en la teoría del grado, obtenemos: $dg(I-\tilde{H}_o, G, 0) = dg(I-\tilde{H}_1, G, 0)$. Como $\tilde{H}_{t_o} = \tilde{H}_1$, $\tilde{H}_o$ es constante en $\hat{E}$ $(\tilde{H}_o = \bar{\phi})$, se tiene que $1 = dg(I-\tilde{H}_o, G, 0) = dg(I-\tilde{H}_1, G, 0)$. Puesto que $\tilde{H}_{t_o}|_G \equiv H_{t_o}|_G$, se sigue que

$$1 = dg(I-\tilde{H}_o, G, 0) = dg(I-\tilde{H}_1, G, 0) = dg(I-\tilde{H}_{t_o}, G, 0)$$
$$= dg(I-H_{t_o}, G, 0).$$

Para demostrar (iii) seleccionemos un número real t_1 tal que $t_1 > \alpha(f_1)$ y $t_1 > |t_o|$. Para $r_3 > \|f_1\|_\infty + t_1\|\phi\|_\infty \geq \|f_1 + t_1\phi\|_\infty$, por el Lema 3.3 existe $R(r_3) > 0$ tal que si $f \in F$, $\|f\|_\infty < r_3$ y $L|u| = g(\cdot,\cdot,u)+f$ en $\Omega\times\mathbb{R}$, entonces

$$\|u\|_{1+\beta} < R(r_3). \qquad (13)$$

Si $t \in [t_o, t_1]$ y $H_t(u) = u$, $u \in \hat{E}$, entonces $L[u] = g(\cdot,\cdot,u)+f_1+t\phi$ en $\Omega\times\mathbb{R}$ y

$$\|f_1+t\phi\| \leq \|f_1\|_\infty + |t|\,\|\phi\|_\infty \leq \|f_1\|_\infty + t_1\|\phi\|_\infty < r_3.$$

Por esta desigualdad y por (3) tenemos que $\|u\|_{1+\beta} < R(r_3) \leq \max\{R(r_3), \bar{R}+1\} = R_o$. Por tanto, si $\|u\|_{1+\beta} = R_o$, $u \in \hat{E}$ y $t \in [t_o, t_1]$ entonces $u \neq H_t(u)$. La invariabilidad de las homotopías en la teoría del grado nos dice ahora que

$$dg(I-H_{t_o}, B_{R_o}, 0) = dg(I-H_{t_1}, B_{R_o}, 0). \qquad (14)$$

Como $t_1 > \alpha(f_1)$, el problema periódico de Dirichlet $\{u \in E$ y $L[u] = g(\cdot,\cdot,u)+f_1+t_1\phi$ en $\Omega\times\mathbb{R}\}$ no tiene solución en virtud de (5), por consiguiente $(I-H_{t_1})(v) \neq 0$ para cualquier $v \in \hat{E}$, lo cual es equivalente a decir que:

$$dg(I-H_{t_1}, B_{R_o}, 0) = 0. \qquad (15)$$

De (14) y (15) deducimos finalmente que $0 = dg(I-H_{t_1}, B_{R_o}, 0) = dg(I-H_{t_o}, B_{R_o}, 0)$. ▲

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Departamento de Matemáticas
Universidad del Valle
CALI, Valle, Colombia

Departamento de Matemáticas y Estadística
Universidad Nacional de Colombia
BOGOTA. D.E. Colombia.

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COHOMOLOGIA LOCAL DE LAS EXTENSIONES CICLICAS DE GRADO PRIMO

por

Marco Fidel SUAREZ R.

RESUMEN. Se utiliza un teorema de localización de cohomología para demostrar un teorema de Hideo Yokoi sobre cohomología de extensiones cíclicas de grado primo de cuerpos numéricos. Para ello, se reduce el teorema al caso local y se calculan explícitamente los grupos de cohomología de dimensión 0 y -1, mostrando que son isomorfos.

ABSTRACT. In this paper we use a theorem of localization of cohomology to prove a theorem of Hideo Yokoi on the cohomology of a cyclic extension of number fields of primer degree. We reduce this theorem to the local case and compute explicity the cohomology groups of dimensions 0 and -1, observing that they are isomorphic.

§1. **Introducción**. Sean K y L cuerpos numéricos, L extensión finita de K con grupo de Galois G, A y B los respectivos anillos de enteros de K y L. En [6] H. Yokoi consideró a B como $\mathbf{Z}[G]$ módulo y en particular demostró

TEOREMA 1. *Si* K *y* L *son extensiones normales de* $\mathbb{Q}$ *y* L/K *es cíclica de grado primo* p, *entonces todos los grupos de cohomología de* B *con respecto a* L/K *son isomorfos entre sí; en otras palabras* $H^i(G,B) \cong H^j(G,B)$, *para cualquier par de enteros* i *y* j *(ver* [4]).

En el presente artículo demostraremos este resultado utilizando el siguiente teorema de localización ([3]).

TEOREMA 2. *Si* Φ *es un conjunto de ideales primos de* L *que contienen exactamente un divisor* q *de cada ideal primo* $\mathfrak{P}$ *de* K, *entonces para cualquier entero* i

$$H^i(G,B) \simeq \oplus H^i(G_q,\hat{B}_q) \quad (q \in \Phi)$$

donde $\hat{B}_q$ *es el anillo de enteros de la completación* q-*ádica* $\hat{L}_q$ *de* L *y* G_q *es el grupo de descomposición de* q *en* L/K.

En la situación particular del Teorema 1, tenemos el siguiente corolario del Teorema 2.

COROLARIO. *Si* K *y* L *son extensiones normales de* $\mathbb{Q}$ *y* L/K *es cíclica de grado primo* p, *entonces* $H^i(G,B) \simeq H^i(G_q,\hat{B}_q)$ *para todo entero* i *donde* q *es un ideal primo en* L *que divide a* p.

Demostración. Si $\mathfrak{P}$ es cualquier ideal primo en K y $\mathfrak{P} \cap \mathbb{Z} \neq p$, sea q un ideal primo de L que divide a $\mathfrak{P}$; la característica del cuerpo residual $\bar{K}_{\mathfrak{P}}$, siendo distinta de p, no divide el índice de ramificación de q sobre $\mathfrak{P}$; por lo tanto, la extensión local $L_q/K_{\mathfrak{P}}$ es suavemente ramificada (ver [1] I,§5 pág.21) y $H^i(G_q,\hat{B}_q) = 0$ para todo entero i

([3],Cor.1). El resultado del Corolario es entonces consecuencia del Teorema 2. ▲

OBSERVACIONES. Consideramos pertinentes las siguientes observaciones:

(i) El corolario anterior reduce el Teorema 1 al caso de extensiones locales.

(ii) Como G es un grupo cíclico, sabemos ([4], 3-2-1) que para todo entero i y todo G-módulo B

$$H^i(G,B) \simeq H^{i+2}(G,B).$$

Según estas observaciones podemos simplificar el Teorema 1 en la forma siguiente:

TEOREMA 3. *Si* K *es una extensión normal de* $\mathbb{Q}_p$, L/K *es una extensión cíclica de grado* p, A *(resp.*B) *es el anillo de enteros de* K *(resp.*L) *y* G *el grupo de Galois de* L/K, *entonces* $H^0(G,B) \simeq H^{-1}(G,B)$.

Observemos que en caso de que L/K sea suavemente ramificada el resultado es trivial pues todo grupo de cohomología es 0. Por lo tanto podemos suponer que L/K es *totalmente ramificada y fuertemente ramificada.*

§2. Cálculo de los grupos de cohomología $H^0(G,B)$ y $H^{-1}(G,B)$.

A continuación, procederemos a calcular los dos grupos de cohomología anteriores mediante algunos lemas auxiliares.

LEMA 1. *En las mismas condiciones del Teorema 3, el conjunto* SB *de las trazas de los elementos de* B, *constituye un ideal entero de* K, *el cual divide a* $\mathcal{D}$, *"el diferente" de la extensión* ([1], 1,§4,pág.16).

Demostración. Sea $\mathfrak{A}$ = SB; afirmamos que $\mathfrak{A}^{-1} = \mathcal{D}^{-1} \cap K$. En efecto, $\alpha \in \mathfrak{A}^{-1}$ si y sólo si $\alpha \in K$ y para todo $b \in B$, $\alpha S(b) \in A$, si y sólo si $\alpha \in K$ y $S(\alpha b) \in A$ para todo $b \in B$, si y sólo si $\alpha \in K$ y $\alpha \in \mathcal{D}^{-1}$. De lo anterior concluímos que $\mathfrak{A}^{-1} B \subset \mathcal{D}^{-1}$ y $\mathcal{D} \subset \mathfrak{A}B$, es decir $\mathfrak{A} | \mathcal{D}$. ▲

LEMA 2. *Si* $r = v_K(\mathfrak{A})$, $v = v_L(\mathcal{D})$ *donde* v_K *y* v_L *son las valuaciones de* K *y* L *respectivamente, entonces* $v = re+t$, $0 \leqslant t < e$ (e *es el índice de ramificación de la extensión* L/K), *es decir,* $[v/e] = r$ ([ ] *es la función parte entera*).

Demostración. Según el Lema 1, $v \geqslant re$. Si fuera $(r+1)e \leqslant v$ entonces tendríamos $\mathcal{D} \subset P^{r+1}B$ (P el ideal maximal de K) de lo cual $\mathcal{D} \subset q^{(r+1)e}$ (q el idel maximal de L); pero entonces $q^{-(r+1)e} \subset \mathcal{D}^{-1}$, lo cual implicaría

$$q^{-(r+1)e} \cap K \subset \mathcal{D}^{-1} \cap K = \mathfrak{A}^{-1}.$$

Por otra parte, $P^{-(r+1)} \subset q^{-(r+1)e} \cap K$ y en consecuencia $P^{-(r+1)} \subset \mathfrak{A}^{-1} = P^{-r}$, lo cual es absurdo. ▲

En las mismas condiciones del Teorema 3, si σ es un generador del grupo G, definimos el *número de ramificación* de L/K por $N(\sigma) = v_L \frac{(\sigma-1)\pi}{\pi} = i(\sigma)-1$ ($i(\sigma)$ definido en [2], IV,§1) donde π es un elemento primo en L. Es fácil ver que $N(\sigma)$ no depende del primo π.

PROPOSICION. *En las condiciones del Teorema 3,*

$i(\sigma^{\nu}) = i(\sigma)$ *para todo* ν *tal que* $1 < \nu \leqslant p-1$.

Demostración. Siendo L/K fuertemente ramificada, entonces $i(\sigma) > 1$ ([5], 3-6-8) e inductivamente puede demostrarse que, para $1 < \nu \leqslant p-1$,

$$v_L(\sigma^{\nu}-1)\pi = v_L[(\sigma-1)(\sigma^{\nu-1}+\sigma^{\nu-2}+\ldots+\sigma+1)]\pi$$

$$= v_L(\sigma-1)\pi. \quad \blacktriangle$$

LEMA 3. *Si* $s = N(\sigma) - [N(\sigma)/p]$ *y* $n = [K:\mathbb{Q}_p]$, *entonces*

$$H^{o}(G,B) \simeq \mathbb{Z}/p\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/p\mathbb{Z} \text{ (ns/ek veces)}$$

donde e_K *es el índice de ramificación de* p *en* K.

Demostración. Tenemos $v = v_L(\mathcal{D}) = \sum_{\nu=1}^{p-1} i(\sigma^{\nu}) = (p-1)(N(\sigma)+1)$ ([2],Prop.4, IV,§1). Sea $N(\sigma) = pm+t$, $0 \leqslant t < p$; si tomamos $r = p-(t+1)$, entonces también $0 \leqslant r < p$, y sustituyendo en la fórmula para v tendremos

$$v = (p-1)(pm+t+1) = p((p-1)m+t)+r$$

$$= p(N(\sigma)-m)+r = p(N(\sigma)-[N(\sigma)/p])+r = ps+r.$$

De acuerdo con el Lema 2 y la suposición de que L/K es una extensión totalmente ramificada (e = p) concluímos que $SB = P^{s}$. Por otra parte, $H^{o}(G,B) \simeq A/SB$ ([4], 1-5-6); luego $H^{o}(G,B) \simeq A/P^{s}$, de donde

$$\#H^{o}(G,B) = \text{card}(A/P^{s}) = N_{K/\mathbb{Q}p}(P^{s}) = (N_{K/\mathbb{Q}p}(P))^{s} = p^{ns/e_k}$$

pues $n/e_k = [\bar{K}:\bar{\mathbb{Q}}_p]$ (grado residual); puesto que $p = \#G$ anula $H^{o}(G,B)$ ([4], 3-1-6), se tiene el resultado del Lema 3. $\blacktriangle$

Ahora construiremos una base para B sobre A.

LEMA 4. *Para cada* j, $1 \leq j \leq p-1$ *existe* $x_j \in B$ *tal que* $v_L(x_j) = j$ *y* $v_L(\sigma-1)x_j = j+N(\sigma)$.

Demostración. Si π es un elemento primo en L, sea $x_j = \prod_{i=0}^{j-1} \sigma^i(\pi)$ para $1 \leq j \leq p-1$, es claro que $v_L(x_j) = j$. Además, $\sigma x_j/x_j = \sigma^j\pi/\pi$ implica que $(\sigma-1)x_j/x_j = (\sigma^j-1)\pi/\pi$. Ahora bien, $v_L(\sigma^j-1)\pi/\pi = N(\sigma)$ (según la proposición). Por lo tanto $v_L \frac{(\sigma-1)x_j}{x_j} = N(\sigma)$, y entonces

$$v_L(\sigma-1)x_j = v_L(x_j)+N(\sigma) = j+N(\sigma). \quad \blacktriangle$$

LEMA 5. *El conjunto* $\{1 = x_o, x_1, \ldots, x_{p-1}\}$, *constituye una base para* B *sobre* A.

Demostración. Los elementos del anterior conjunto son representantes de todas las distintas coclases de $v_L(K^*)$ en $v_L(L^*)$ ([5], cap.I, 6); además siendo L/K totalmente ramificada $[\bar{L}:\bar{K}] \doteq 1$, entonces dicho conjunto es linealmente independiente sobre A ([5], 1-6-3).

Si π es un primo fijo en B, afirmamos que $B = A+\pi B$. En efecto, si $\alpha \in B$ entonces $\bar{\alpha} = \bar{a}_1 \cdot \bar{1}$, donde $\bar{a}_1 \in \bar{K}$ es la coclase de cierto $a_1 \in A$, y $\bar{\alpha}$ y $\bar{1}$ son las coclases respectivas de α y 1 en $\bar{L}$. Entonces

$$\alpha = a_1 + \pi\alpha_o, \text{ con } \alpha_o \in B.$$

Puesto que $\pi B = \sigma(\pi)B = \sigma^2(\pi)B = \ldots = \sigma^{p-1}(\pi)B$ tendremos

$$\begin{aligned} B &= A+\pi(A+\pi B) = A+\pi(A+\sigma(\pi)B) \\ &= A+\pi A+\pi\sigma(\pi)B = x_oA+x_1A+x_2B = \ldots \\ &= x_oA+x_1A+x_2A+\ldots+x_{p-1}A+\pi^pB. \end{aligned}$$

En otras palabras, para todo $\alpha \in B$.

$$\alpha = \sum_{j=0}^{p-1} c_j x_j + \pi^p b, \text{ para algunos } b \in B,\ c_j \in A;$$

pero $\pi^p B = \pi_o B$, siendo π_o un primo en A; entonces

$$\alpha = \sum_{j=0}^{p-1} a_j^{(o)} x_j + \pi_o \alpha_1; \quad \alpha_1 \in B,\ a_j^{(o)},s \in A;$$

así mismo

$$\alpha_1 = \sum_{j=0}^{p-1} a_j^{(1)} x_j + \pi_o \alpha_2.$$

Obtenemos entonces una sucesión $\alpha_1, \alpha_2, \ldots$, en B tal que $\alpha_n = \sum_{j=0}^{p-1} a_j^{(n)} x_j + \pi_o \alpha_{n+1}$; además, para j fijo, $\{a_j^{(n)}\}$ es una sucesión en A.

Consideremos la serie $\sum_{n=0}^{\infty} a_j^{(n)} \pi_o^n$; puesto que $a_j^{(n)} \in A$, esta serie converge ([5],1-7) hacia un $a_j \in A$. Por la construcción de la sucesión $\alpha_1, \alpha_2, \ldots$, tenemos

$$\alpha = \sum_{k=0}^{n-1} \Big(\sum_{j=0}^{p-1} a_j^{(k)} x_j \Big) \pi_o^k + \pi_o^n \alpha_n, \quad \text{para todo } n \in \mathbb{N},$$

y haciendo tender n a infinito obtenemos $\alpha = \sum_{j=0}^{p-1} a_j x_j$, tal como queríamos demostrar.

LEMA 6. *El conjunto* $\{y_1, y_2, \ldots, y_{p-1}\}$ *donde para todo* j, $1 \leq j \leq p-1$, $y_j = (\sigma-1)x_j$, *es una base de* $(\sigma-1)B$ *sobre* A.

Demostración. Según el lema anterior basta demostrar su independencia lineal. Supongamos que tenemos $\sum_{j=1}^{p-1} a_j y_j = 0$, donde los $a_j \in A$ no son todos cero. Podemos suponer que por lo menos un a_j es una unidad de A. Sea j_o

el menor subíndice tal que a_{j_o} es una unidad, entonces $v_L(a_{j_o}y_{j_o}) = j_o+N(\sigma)$; pero

$$a_{j_o}y_{j_o} = -\sum_{j\neq j_o} a_j y_j$$

y para $j < j_o$,

$$v_L(a_j y_j) = j+N(\sigma)+v_L(a_j) = j+pv_K(a_j)+N(\sigma) \geqslant j_o+N(\sigma);$$

también para $j > j_o$, $v_L(a_j y_j) \geqslant j+N(\sigma) > j_o+N(\sigma)$; por lo tanto

$$v_L(a_{j_o}y_{j_o}) \geqslant \min_{j\neq j_o} \{v_L(a_j y_j)\} > j_o+N(\sigma),$$

lo cual es una contradicción. ▲

Observemos que $H^{-1}(G,B) \simeq B_S/(\sigma-1)B$ ([4],3-2-1); en consecuencia, a fin de calcular este grupo de cohomología debemos estudiar ahora B_S, el cual consiste de los elementos de B cuya traza es cero. Debemos tener en cuenta además que $B_S = B \cap (\sigma-1)L$. En efecto, $B_S \subset B \cap L_S$, pero $L_S = (\sigma-1)L$, porque $H^{-1}(G,L^+) = 0$ ([4],3-1-4).

LEMA 7. *El conjunto* $\{y_j/\pi_o^{(j)}\}_{j=1}^{p-1}$ *donde* π_o *es un primo en* A *y* $(j) = [j+N(\sigma)/p]$ ([ ] *función parte entera*) *es una base de* B_S *sobre* A.

Demostración. Observemos en primer lugar que para $1 \leqslant j \leqslant p-1$,

$$v_L(y_j/\pi_o^{(j)}) = j+N(\sigma)-p\left[\frac{j+N(\sigma)}{p}\right] > 0.$$

Dado $\alpha \in B_S$, de acuerdo con observaciones anteriores

$$\alpha = \sum_{j=1}^{p-1} q_j y_j, \text{ donde los } q_j \in K.$$

Los números enteros $v_L(q_j y_j) = pv_K(q_j)+j+N(\sigma)$ son distintos entre sí, y puesto que $v_L(\sigma) \geq 0$, para todo $1 \leq j \leq p-1$, entonces $v_L(q_j y_j) \geq 0$; es decir,

$$pv_K(q_j) \geq -j-N(\sigma),$$

de donde

$$v_K(q_j) \geq -\frac{j+N(\sigma)}{p}.$$

Si para cada $1 \leq j \leq p$, escribimos

$$q_j = b_j/\pi_o^{(j)} \quad \text{con } b_j \in K$$

entonces

$$v_K(b_j) = v_K(q_j)+(j) \geq \left[\frac{j+N(\sigma)}{p}\right] - \frac{j+N(\sigma)}{p};$$

de aquí se concluye que para $1 \leq j \leq p-1$, $v_K(b_j) \geq 0$; en otras palabras,

$$\alpha = \sum_{j=1}^{p-1} b_j \frac{y_j}{\pi_o^{(j)}}, \text{ con los } b_j \in A. \quad \blacktriangle$$

PROPOSICION. $\sum_{j=1}^{p-1}\left[\frac{j+N(\sigma)}{p}\right] = N(\sigma) - \left[\frac{N(\sigma)}{p}\right] = s$ (*ver Lema 3*).

Demostración. Supongamos en primer lugar que $p|N(\sigma)$, entonces

$$\sum_{j=1}^{p-1}\left[\frac{j+N(\sigma)}{p}\right] = (p-1)\frac{N(\sigma)}{p} = s.$$

Si por el contrario, $p \nmid N(\sigma)$, sea $N(\sigma) = pm+t$, $0 < t < p$; entonces

$$\sum_{j=1}^{p-1}\left[\frac{j+N(\sigma)}{p}\right] = \sum_{j=1}^{p-(t+1)}\left[\frac{N(\sigma)}{p}\right] + \sum_{j=p-t}^{p-1}\left[\frac{j+N(\sigma)}{p}\right] =$$

$$= (p-t-1)m + t(m+1) = N(\sigma)-m = N(\sigma) - \frac{N(\sigma)}{p} = s.$$

LEMA 8. $H^{-1}(G,B) \simeq \mathbb{Z}/p\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/p\mathbb{Z}$ (ns/e_k *veces*).

Demostración.

$$H^{-1}(G,B) \simeq \frac{B_s}{(\sigma-1)B} \simeq \left(\bigoplus_{p-1}^{p-1} Ay_j/\pi_o^{(j)}\right)/\left(\bigoplus_{j-1}^{p-1} Ay_j\right)$$

$$\simeq \bigoplus_{j=1}^{p-1} \frac{Ay_j/\pi_o^{(j)}}{Ay_j} \simeq \bigoplus_{j=1}^{p-1} \frac{A}{\pi_o^{(j)}A}.$$

Por otra parte,

$$\mathrm{Ord}(H^{-1}(G,B)) = \prod_{j=1}^{p-1} \mathrm{Ord}\left(\frac{A}{\pi_o^{(j)}A}\right) = \prod_{j=1}^{p-1} N_{K/\mathbb{Q}_p}(\pi_o^{(j)})$$

$$= \prod_{j=1}^{p-1} (N_{K/\mathbb{Q}_p}(\pi_o))^{(j)} = \prod_{j=1}^{p-1} p^{n(j)/e_k}$$

$$= p^{\frac{n}{e_k}\sum_{j=1}^{p-1}(j)} = p^{ns/e_k}$$

y el hecho de que p anula $H^{-1}(G,B)$ completa la demostración del Lema 8. ▲

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Departamento de Matemáticas
Universidad del Valle
Apartado áereo 2188
CALI, Colombia.

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WEAK APPROXIMATION OF MINIMAL NORM SOLUTIONS OF FIRST KIND EQUATIONS BY TIKHONOV'S METHOD

by

Julio E. GUACANEME

ABSTRACT. Tikhonov's regularization method is considered to find conditions that guarantee orders of weak convergence of approximate solutions of linear ill-posed problems to the true solution. We establish orders of convergence by requiring smoothness conditions on the functional and the true solution, and we establish a converse result to the main theorem.

RESUMEN. Se establecen órdenes de convergencia débil para las soluciones aproximadas obtenidas por el método de regularización de Tikhonov en el caso de problemas lineales "ill-posed" (es decir, aquellos para los cuales las soluciones exactas pueden depender discontinuamente de los parámetros). Para ello se exigen condiciones de suavidad tanto al funcional como a la solución exacta. Esto se hace para la versión clásica infinito-dimensional del método de Tikhonov y también para la versión con elementos finitos. Además, se obtiene un converso al teorema principal, en el cual la suavidad resulta del orden de convergencia.

1. Introducción. In this article we shall be concerned with the integral equation of the first kind

$$Kx(s) = \int_0^1 k(s,t)x(t)dt = g(s) \qquad (1)$$

where $k(s,t) \in L^2([0,1]\times[0,1])$ and $g(s) \in L^2[0,1]$. It is well known that K is a compact linear operator from the Hilbert space $H_1 = L^2[0,1]$ into the Hilbert space $H_2 = L_2[0,1]$. Solving (1) then means finding an x satisfying (1), given $g \in H_2$.

This problem can have solutions x (we do not assume uniqueness; by solution we mean "minimal norm solution") which depend discontinuously on the "data" g, i.e., this is not a well-posed problem (see e.g. [6]). This lack of continuity can have serious numerical consequences since the data g is usually the result of measurements and hence is only imprecisely known.

Regarding the discontinuous dependence upon data, instead of solving (1) we solve a new equation close to (1) which is well posed. This approach is called "regularization". In particular Tikhonov [6] suggests the minimizer x_α of the functional

$$F_\alpha(w,g) = \|Kw-g\|^2 + \alpha\|w\|^2 \qquad (2)$$

as a regularized solution of (1); we use $\|\cdot\|$ to indicate the norm in each of the space H_1 and H_2 and $<\cdot,\cdot>$ to denote the corresponding inner product. The minimizer x_α of this functional is the solution of the equation

$$(\alpha I + \tilde{K})x_\alpha = K^* g,$$

where K^* denotes the adjoint of K and $\tilde{K} = K^*K$, see e.g. [4].

When the approximation x_α is defined, we would like to estimate how far it is from the minimum norm solution x. This can be done in the strong sense by looking at the norm of the difference $\|x-x_\alpha\|$ between the approximate and the true solutions, or in the weak sense by considering the functional $\langle x-x_\alpha, z\rangle$, where $z \in H_1$.

In many applications all we want to know about solutions is the value of some functional $\langle x,z\rangle$ (see e.g. [2]). In such cases we will be concerned with $\langle x,z\rangle$ and $\langle x_\alpha,z\rangle$ rather than x and x_α respectively. That is, our interest is in weak rather than strong approximation. In this paper we will derive estimates for $\langle x-x_\alpha,z\rangle$ under various assumptions on x and z for both the classical infinite dimensional version of Tikhonov regularization and for a finite element version. We consider both the cases of exact data and inaccurate data.

2. Infinite Dimensional Tikhonov Approximation.

For fixed $z \in H_1$ we consider whether

$$\langle x-x_\alpha,z\rangle \to 0 \quad \text{as } \alpha \to 0,$$

i.e., the weak convergence of the Tikhonov approximations to the true solution.

Sufficient conditions for convergence in the weak topology have already been studied by Tikhonov [7], and for more general methods of regularization, by H.W. Engl [1].

Our goal is to establish orders of convergence by imposing conditions on x and the functional z. Since x as well as x_α are members of $\overline{R(K^*)}$, the closure of the range of K^*, and every z in H has an orthogonal decomposition $z = z_1+z_2 \in \overline{R(K^*)} + R(K^*)^\perp$, we have

$$<x-x_\alpha,z> = <x-x_\alpha,z_1>.$$

We therefore may restrict our attention to funcionals $z \in \overline{R(K^*)}$; however to obtain convergence rates we need to impose stronger conditions on z.

THEOREM 2.1. *If* (a) $x \in R(\tilde{K})$ *or* (b) $z \in R(\tilde{K})$ *or* (c) x *and* $z \in R(K^*)$, *then* $<x-x_\alpha,z> = O(\alpha)$.

Proof. (a) If $x \in R(\tilde{K})$, then $|<x-x_\alpha,z>| \leq \|x-x_\alpha\|\|z\| = O(\alpha)$ by [4,Corollary 3.1.1]. (b) For this case let $z = Ku$, then

$$|<x-x_\alpha,z>| = |<\tilde{K}(x-x_\alpha), u>| = |<[I-\tilde{K}(\alpha I+\tilde{K})^{-1}]K^*g, u>|$$
$$= \alpha|<(\alpha I+\tilde{K})^{-1}\tilde{K}x,u>| \leq \alpha\|x\|\|u\|,$$

since $\|(\alpha I+\tilde{K})^{-1}\tilde{K}\| \leq 1$. (c) Let $z = K^*v$, $x = K^*w$, then, setting $\hat{K}:= KK^*$, we have

$$|<x-x_\alpha, z>| = |<K(x-x_\alpha), v> = |<\hat{K}w-\hat{K}(\alpha I+K)^{-1}\hat{K}w, v>|$$
$$=\alpha|<\hat{K}(\alpha I+\hat{K})^{-1}w, v>| \leq \alpha\|w\|\|v\|,$$

which completes the proof. ▲

The best order of weak convergence is $O(\alpha)$ as the following theorem establishes.

THEOREM 2.2. *If* $<x-x_\alpha,z> = O(\alpha)$ *for every* $z \in H_1$, *then* $x = 0$.

Proof. Since $<x-x_\alpha, z> = O(\alpha)$, we have $<(x,x_\alpha)/\alpha, z> \to 0$ as $\alpha \to 0$ $\forall z \in H_1$. In particular the sequence $\{(x-x_\alpha)/\alpha\}$ is weakly convergent and hence bounded, i.e., $\|x-x_\alpha\| = O(\alpha)$. By [4, Theorem 3.2.2] we have $x \in R(K^*K)$, say $x = K^*Kv$. Now using the fact that $x-x_\alpha = \alpha(\alpha I+\tilde{K})^{-1}\tilde{K}v$ (clearly we may take $v \in N(\tilde{K})^\perp = N(K)^\perp$), we then have

$$0 = \lim_{\alpha\to 0} <(x-x_\alpha)/\alpha, z> = \lim_{\alpha\to 0} (\alpha I+\tilde{K})^{-1}\tilde{K}v, z> = <K^+Kv, z>$$

for every $z \in H_1$, so $K^+Kv = 0$, where K^+ is the Moore-Penrose inverse of K (see e.g. [4]). But

$$K^+Kv = P_{N(K)^\perp}v = v$$

and hence $v = 0$, i.e., $x = \tilde{K}v = 0$. ▲

For the converse of Theorem 2.1 we will have, using the same notation as above, the following

THEOREM 2.3. *If* $<x-x_\alpha, z> = O(\alpha)$ *for all* $z \in H_1$, *then* $x \in R(K^*K)$.

Proof. Define $T_\alpha z = <(x-x_\alpha)/\alpha, z>$. This family of linear functionals on H_1 has the properties required by the uniform boundedness theorem, that is,

$$\|T_\alpha z\| = |<\frac{x-x_\alpha}{\alpha}, z>| \leq M(z) \text{ for all } \alpha > 0.$$

M(z) is a bound depending of the element z, and thus there exists a bound M independent of α and independent of z such that $\|T_\alpha\| \leq M$, i.e.,

$$\|T_\alpha z\| = |<\frac{x-x_\alpha}{\alpha}, z>| \leq M\|z\|$$

for all $\alpha > 0$, and for all $z \in H_1$. In particular, for $z = (x-\alpha)/\alpha$,

$$\left\|T\frac{x-x_\alpha}{\alpha}\right\| = \left\|\frac{x-x_\alpha}{\alpha}\right\|^2 \leqslant M\left\|\frac{x-x_\alpha}{\alpha}\right\|,$$

and hence $\|x-x_\alpha\| = O(\alpha)$. By [4 Theorem 3.2.2] we conclude that $x \in R(K^*K)$. ▲

Now that we have analyzed the error-free case we turn our attention to the more realistic case of inexact data g^δ, with a prescribed error bound $\delta : \|g-g^\delta\| \leqslant \delta$.

The Tikhonov regularized solution x_α^δ is the minimizer of

$$F_\alpha(z,g^\delta) = \|Kz-g^\delta\|^2 + \alpha\|z\|^2,$$

or, equivalently, $x_\alpha^\delta = (\alpha I+\tilde{K})^{-1}K^*g^\delta$.

LEMMA 2.4. *If* $z \in R(K^*)$ *then* $\langle x_\alpha-x_\alpha^\delta, z\rangle = O(\delta)$ *for any* $\alpha > 0$.

Proof. Let $z = K^*u$, then

$$\langle x_\alpha-x_\alpha^\delta, z\rangle = \langle K(x_\alpha-x_\alpha^\delta), u\rangle = \langle\hat{K}(\alpha I+\hat{K})^{-1}(g-g^\delta), u\rangle$$

and hence

$$|\langle x_\alpha-x_\alpha^\delta, z\rangle| \leqslant \|g-g^\delta\|\|u\| \leqslant \delta\|u\|,$$

since $\|\hat{K}(\alpha I+\hat{K})^{-1}\| \leqslant 1$. ▲

THEOREM 2.5. *Let* $\alpha = O(\delta)$. *If* (a) $z \in R(\tilde{K})$, *or* (b) z *and* $x \in R(K^*)$, *then* $\langle x-x_\alpha^\delta, z\rangle = O(\delta)$.

Proof. (a) By Theorem 2.1 and the preceding lemma

$$|\langle x-x_\alpha^\delta, z\rangle| \leq |\langle x-x_\alpha, z\rangle| + |\langle x_\alpha-x_\alpha^\delta, z\rangle| = O(\alpha)+O(\delta) = O(\delta)$$

(b) Let $z = K^*w$ and $x = K^*v$, then

$$\langle x-x_\alpha, z\rangle = \langle K(x-x_\alpha), w\rangle = \alpha\langle(\alpha I+\hat{K})^{-1}\hat{K}v, w\rangle$$

and hence

$$|\langle x-x_\alpha, z\rangle| \leq \alpha\|v\|\|w\| = O(\alpha).$$

Using the Lemma again we find that $\langle x-x_\alpha^\delta, z\rangle = O(\alpha)+O(\delta)$. ▲

In the final theorem of this section, we make no assumption on the true solution x.

THEOREM 2.6. *If* $z \in R(K^*)$, *then* $\langle x-x_\alpha, z\rangle = O(\sqrt{\alpha})$.

Proof. Suppose $z = K^*u$, then

$$\langle x-x_\alpha, z\rangle = \langle K(x-x_\alpha), u\rangle.$$

But

$$\|K(x-x_\alpha)\|^2 = \langle K(x-x_\alpha), K(x-x_\alpha)\rangle = \langle\tilde{K}(x-x_\alpha), x-x_\alpha\rangle.$$

But $\tilde{K}(x-x_\alpha) = K^*g-\tilde{K}(\tilde{K}+\alpha I)^{-1}K^*g = \alpha\tilde{K}(\tilde{K}+ I)^{-1}K^*g = \alpha\tilde{K}x_\alpha$. Therefore,

$$\|K(x-x_\alpha)\|^2 = \alpha\langle\tilde{K}x_\alpha, x-x_\alpha\rangle = O(\alpha),$$

since $x_\alpha \to x$ (see |4|). Therefore,

$$\langle x-x_\alpha, z\rangle = O(\|K(x-x_\alpha)\|) = O(\sqrt{\alpha}),$$

completing the proof. ▲

Combining this with Lemma 2.4 we obtain:

COROLLARY 2.7. *If* $z \in R(K^*)$ *and* $\alpha = O(\delta^2)$, *then* $\langle x - x_\alpha^\delta, z\rangle = O(\delta)$. ▲

3. Finite Element Approximations. Groetsch and Guacaneme [3] have proved weak convergence of certain finite element Tikhonov approximation to the minimal norm solution of (1). These approximations are formed by using a sequence of finite dimensional subspaces V_m that increase and are eventually dense in H_1, i.e.,

$$V_1 \subseteq V_2 \subseteq \ldots \text{ and } \overline{\bigcup_m V_m} = H_1 .$$

The finite element Tikhonov approximations $x_{m,\alpha}$ and $x_{m,\alpha}^\delta$ are the minimizers of $F_\alpha(\cdot;g)$ and $F_\alpha(\cdot;g^\delta)$ (see [2]), respectivley, over the finite dimensional space V_m, or equivalently $x_{m,\alpha}, x_{m,\alpha}^\delta \in V_m$ and

$$\langle Kx_{m,\alpha} - g, Ky\rangle + \alpha\langle x_{m,\alpha}, y\rangle = 0$$

$$\langle Kx_{m,\alpha} - g^\delta, Ky\rangle + \langle \alpha x_{m,\alpha}^\delta, y\rangle = 0$$

respectively, for all $y \in V_m$. These conditions are in turn equivalent to

$$x_{m,\alpha} = (K_m^* K_m + \alpha I)^{-1} K_m^* g$$

and

$$x_{m,\alpha}^\delta = (K_m^* K_m + \alpha I)^{-1} K_m^* g^\delta$$

where K_m is the restriction of K to V_m. The number

$$\gamma_m = \|K(I-P_m)\| = \|(I-P_m)K^*\|$$

where P_m is the orthogonal projector of H_1 onto V_m, plays a prominent role; it tell us how well the spaces V_m support the operator K. Note that $\gamma_m \to 0$ as $m \to \infty$ (see e.g. [4]).

To study the approximations $x_{m,\alpha}$ and $x^{\delta}_{m,\alpha}$ we assume the regularization parameter is a function of m, say $\alpha = \alpha(m) \to 0$ as $m \to \infty$.

Before going into the order of convergence we define the inner product $[u,v] = \langle Ku,Kv\rangle + \alpha\langle u,v\rangle$ and the norm $|u|^2 = [u,u] = \|Ku\|^2 + \alpha\|u\|^2$ in the Hilbert space H_1. Under this inner product $x_{m,\alpha}$ is orthogonal projection of x_α onto V_m, (see [5]), i.e.

$$[x_\alpha - x_{m,\alpha}, v] = \langle\tilde{K}(x_\alpha - x_{m,\alpha}), v\rangle + \alpha\langle x_\alpha - x_{m,\alpha}, v\rangle = 0$$

for all $v \in V_m$.

We now give a weak order of convergence result for $x_{m,\alpha}$. For ease of notation below we will replace α_m by α and γ_m by γ, respectively.

THEOREM 3.1. *Assume that* $\gamma = O(\alpha^{\frac{1}{2}})$. (a) *If* x *and* $z \in R(K^*)$ *then* $\langle x - x_{m,\alpha}, z\rangle = O(\alpha)$. (b) *If* $z \in R(K^*)$, *then* $\langle x - x_{m,\alpha}, z\rangle = O(\sqrt{\alpha})$.

Proof. (a) Let $z = K^*w$ and let $x = K^*v$. Then,

$$\langle x - x_{m,\alpha}, z\rangle = \langle x - x_\alpha, z\rangle + \langle x_\alpha - x_{m,\alpha}, z\rangle$$

$$= \langle K(x - x_\alpha), w\rangle + \langle K(x_\alpha - x_{m,\alpha}), w\rangle.$$

Now,

$$K(x - x_\alpha) = \hat{K}v - \hat{K}(\alpha I + \hat{K})^{-1}\hat{K}v = \alpha(\alpha I + \hat{K})^{-1}\hat{K}v;$$

therefore

$$|\langle K(x-x_\alpha),\ w\rangle| \leq \alpha\|v\|\|w\| = O(\alpha).$$

Also

$$\|K(x_\alpha - x_{m,\alpha})\|^2 \leq \|K(x_\alpha - x_{m,\alpha})\|^2 + \alpha\|x_\alpha - x_{m,\alpha}\|^2$$

and remembering that $x_{m,\alpha}$ is the projection of x_α onto V_m under the inner product $[w,v] = \langle Kw,Kv\rangle + \alpha\langle w,v\rangle$, we have

$$\begin{aligned}\|K(x_\alpha - x_{m,\alpha})\|^2 + \alpha\|x_\alpha - x_{m,\alpha}\|^2 &\leq \|K(x_\alpha - P_m x_\alpha)\|^2 + \alpha\|x_\alpha - P_m x_\alpha\|^2 \\ &\leq \|K(I-P_m)^2 x_\alpha\|^2 + \alpha\|(I-P_m)x_\alpha\|^2 \\ &\leq (\gamma^2+\alpha)\|(I-P_m)x_\alpha\|^2 \\ &= (\gamma^2+\alpha)\|(I-P_m)K^*(\alpha I+\hat{K})^{-1}\hat{K}v\|^2 \\ &\leq (\gamma^2+\alpha)\gamma^2\|v\|^2.\end{aligned}$$

Therefore

$$\begin{aligned}|\langle x_\alpha - x_{m,\alpha},\ z\rangle| &\leq \|K(x_\alpha - x_{m,\alpha})\|\|w\| \\ &\leq (\gamma^2+\alpha)^{\frac{1}{2}}\gamma\|v\|\|w\| = O(\alpha).\end{aligned}$$

(b) Let $z = Ku$, using the same decomposition as in part (a) we have

$$\langle x-x_{m,\alpha},\ z\rangle = \langle x-x_\alpha,\ z\rangle + \langle x_\alpha - x_{m,\alpha},\ z\rangle$$

and $\langle x-x_\alpha,\ z\rangle = O(\sqrt{\alpha})$, by Theorem 2.6. Now $\langle x_\alpha - x_{m,\alpha},\ z\rangle =$ $\langle K(x_\alpha - x_m,\alpha),\ u\rangle$, and

$$\|K(x_\alpha - x_{m,\alpha})\|^2 \leq \|K(x_\alpha - x_{m,\alpha})\|^2 + \alpha\|x_\alpha - x_{m,\alpha}\|^2.$$

Using the characterization of $x_{m,\alpha}$ as the projection of x_α

onto V_m, we have as above

$$\|K(x_\alpha - x_{m,\alpha})\|^2 \leq (\gamma^2+\alpha)\|(I-P_m)x_\alpha\|^2 = O(\alpha)\|(I-P_m)x_\alpha\|^2 = O(\alpha),$$

thus

$$<x_\alpha - x_{m,\alpha}, z> = O(\sqrt{\alpha}).$$

We therefore find that

$$<x - x_{m,\alpha}, z> = (\sqrt{\alpha})$$

for case (b). ▲

We see that under suitable conditions the finite element approximations attain the order $O(\alpha)$ of weak convergence, which we now show does not allow improvement.

THEOREM 3.2. *If* $<x - x_{m,\alpha}, z> = O(\alpha)$ *for all* $z \in H_1$, *then* $x = 0$.

Proof. In particular for $z = \tilde{K}(u+w)$ with $u \in V_N$ and $w \in V_N^\perp$, we have

$$<x - x_{m,\alpha}, z> = O(\alpha).$$

By the definition of $x_{m,\alpha}$ we have

$$<\tilde{K}(x - x_{m,\alpha}), u> + \alpha <x_{m,\alpha}, u> = 0 \text{ for } m > N.$$

Therefore

$$\begin{aligned} <x - x_{m,\alpha}, \tilde{K}(u+w)> &= \tilde{K}(x - x_{m,\alpha}), u+w> \\ &= -\alpha <x_{m,\alpha}, u> + <\tilde{K}(x - x_{m,\alpha}), w> \end{aligned}$$

and hence

$$0 = \lim_m \frac{\langle x-x_{m,\alpha}, \tilde{K}(u+w)\rangle}{\alpha} = -\lim_m \langle x_{m,\alpha}, u\rangle + \lim_m \langle \frac{\tilde{K}(x-x_{m,\alpha})}{\alpha}, w\rangle$$

$$= -\lim_m \langle x_{m,\alpha}, u\rangle,$$

for $u \in V_N$. To see this note that by hypothesis $\frac{x-x_{m,\alpha}}{\alpha}$ converges weakly to zero and K is a compact operator, therefore $\tilde{K}(\frac{x-x_{m,\alpha}}{\alpha})$ converges to zero. Using the fact that $x_{m,\alpha} \overset{w}{\to} x$, we then have $0 = \langle x,u\rangle$ for all $u \in V_N$. Since $\bigcup_{n=1}^{\infty} V_n$ is dense in H_1 we have $x = 0$. ▲

Finally, we consider the weak convergence of the approximation $x^{\delta}_{m,\alpha}$ obtained using imprecise data g^{δ}, where $\|g-g^{\delta}\| < \delta$.

LEMMA 3.3. $\|K(x_{m,\alpha} - x^{\delta}_{m,\alpha})\| < \delta$.

Proof. Since

$$K(x_{m,\alpha}-x^{\delta}_{m,\alpha}) = K_m(\alpha I+\tilde{K}_m)^{-1}K^*_m(g-g^{\delta})$$

$$= \hat{K}_m(\alpha I+\hat{K}_m)^{-1}(g-g^{\delta})$$

and

$$\|\hat{K}_m(\alpha I+\hat{K}_m)^{-1}\| \leq 1$$

we get that

$$\|K(x_{m,\alpha}-x^{\delta}_{m,\alpha})\| \leq \|g-g^{\delta}\| \leq \delta. \quad ▲$$

We now show that under appropiate conditions an optimal orde of weak convergence obtains.

THEOREM 3.4. *Assume that* $\gamma = O(\alpha^{½})$. *If either* (a) x *and* $z \in R(K^*)$ *and* $\alpha = O(\delta)$ *or* (b) $z \in R(K^*)$ *and*

$\alpha = O(\delta^2)$, *then* $\langle x - x_{m,\alpha}, z \rangle = O(\delta)$.

Proof. By Theorem 3.1, in both cases (a) and (b):

$$\langle x - x_{m,\alpha}, z \rangle = O(\delta).$$

Now set $z = K^*w$, then by Lemma 3.3 we have

$$|\langle x_{m,\alpha} - x^{\delta}_{m,\alpha}, z \rangle| = |\langle K(x_{m,\alpha} - x^{\delta}_{m,\alpha}), w \rangle| \leq \delta \|w\|,$$

i.e., $\langle x_{m,\alpha} - x_{m,\alpha}, z \rangle = O(\delta)$, and hence $\langle x - x^{\delta}_{m,\alpha}, z \rangle = O(\delta)$, completing the proof. ▲

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Department of Mathematics
University of Puerto Rico
Río Piedras, Puerto Rico, 00931

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EXISTENCIA DE SOLUCIONES DE ECUACIONES DIFERENCIALES ESTOCASTICAS

Myriam Muñoz de Özak

ABSTRACT. In classic theorems, when we have a stochastic differential equation of the form $dX_t = f(t,X_t)dt + G(t,X_t)dW_t$, $X_{t_0} = \xi$, $t_0 \leq t \leq T < \infty$, where W_t is a Wiener Process and ξ is a random variable independent of $W_t - W_{t_0}$ for $t \geq t_0$, in order to have existence and uniqueness of solutions it is supposed the existence of a constant K such that: (Lipschitz condition) for all $t \in [t_0,T]$, $x,y \in \mathbb{R}^d$,

$$|f(t,x)-f(t,y)| + |G(t,x)-G(t,y)| \leq K|x-y|.$$

And for all $t \in |t_0,T|$ and $x \in \mathbb{R}^d$,

$$|f(t,x)|^2+|G(t,x)|^2 \leq K^2(1+|x|^2).$$

In this article we prove an existence theorem under weaker hypothesis: we require only that f and G be continuous in the second variable and the existence of a function $m \in L^2[t_0,T]$ such that

$$|f(t,x)|+|G(t,x)| \leq m(t)$$

for all $t \in [t_0,T]$ and $x \in \mathbb{R}^d$.

Introducción. Clásicamente cuando se tiene una ecuación diferencial estocástica de la forma

$$dX_t = f(t,X_t)dt + G(t,X_t)dW_t, \quad X_{t_o} = \xi, \ t_o \leq t \leq T < \infty,$$

donde W_t es un proceso de Wiener y ξ una variable aleatoria independiente de $W_t - W_{t_o}$ para $t \geq t_o$, en orden a demostrar la existencia y unicidad de soluciones se supone la existencia de una constante K tal que: a) (Condición de Lipschitz) para toda $t \in [t_o,T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$,

$$|f(t,x)-f(t,y)| + |G(t,x)-G(t,y)| \leq K|x-y|.$$

b) Para toda $t \in [t_o,T]$ y $x \in \mathbb{R}$,

$$|f(t,x)|^2 + |G(t,x)|^2 \leq K^2(1+|x|^2).$$

Véase, por ejemplo Arnold [1]. En este artículo probamos un teorema de existencia debilitando las hipótesis: sólo necesitamos que f y G sean continuas en la segunda variable y exista una función $m \in L^2[t_o,T]$ tal que

$$|f(t,x)| + |G(t,x)| \leq m(t), \quad \forall t \in [t_o,T], \ \forall x \in \mathbb{R}^d.$$

Aunque estas hipótesis son más débiles en el sentido de que no se pone límite al crecimiento de f y G respecto a la variable t, son más fuertes en el sentido de que limitan el crecimiento de las funciones respecto a la variable x, excluyendo por ejemplo las funciones lineales en x.

En la literatura existen teoremas similares, por ejemplo Teorema 5.2 en [H.J. Keisler, *An infinitesimal approach to stochastic analysis*, Memoirs of the A.M.S., 1984], demostrado por métodos no estándar para un espacio $\underline{\Omega} = (\Omega,P,A_t)_{t\in[0,1]}$, en donde Ω es un espacio de muestreo hiperfinito, P es la probabilidad de Loeb y A_t una filtración arbitraria.

La existencia de solución en nuestro artículo, en cambio, depende fuertemente de la filtración específica que se toma en el teorema, es decir la generada por el proceso de Wiener con respecto al cual se integra.

Aparentemente, de los resultados de Barlow en [*One diemsional stochastic differential equations with no strong solution*, Proc. London Math. Society, 1981], si se toma una filtración arbitraria, puede fallar el teorema de existencia.

El esquema de aproximación que usaremos en la demostración del teorema principal es distinto del esquema usual de aproximaciones sucesivas, en el que $X_t^{(n+1)}$ se define en términos de $X_t^{(n)}$, debido a que no se tiene la condición de Lipschitz sino la continuidad, y es posible que estas aproximaciones no converjan a una solución. Además, en el teorema principal sólo se obtiene la existencia, pero no la unicidad, nuevamente porque no se supone una condición de Lipschitz. Como las ecuaciones diferenciales ordinarias son un caso particular de las estocásticas, podemos considerar el siguiente ejemplo, en el cual no se tiene unicidad:

$$dx = f(t,x)dt = x^{1/3}dt, \quad x(0) = 0.$$

Considerando las aproximaciones como en nuestro teorema, obtenemos que $x^{(j)}(t) = 0$ para cualquier $j = 1,2,\ldots$, o sea que obtenemos una solución $x(t) = 0$, pero existe otra solución, $x(t) = (2t/3)^{3/2}$.

§1. **Notación y conceptos básicos.** Tomaremos las definiciones de los conceptos básicos y la nomenclatura del libro de Arnold [1].

Por $(\Omega,\mathcal{U},P)$ denotaremos un espacio de probabilidad. Se dice que una proposición D se cumple con *probabilidad 1*, si D(w) se cumple para toda $w \in \Omega$ excepto en un conjunto de probabilidad cero. D se cumple *estocásticamente* si para todo $\varepsilon > 0$ existe $A \in \mathcal{U}$ tal que $P(A) < \varepsilon$ y D(w) se cumple para todo $w \in \Omega$ tal que $w \notin A$. A lo largo de este trabajo, las letras mayúsculas X, X_n, Y denotarán variables aleatorias. X_A denotará la función característica de A. $L^p(\Omega,\mathcal{U},P) = L^p(P)$ al espacio vectorial normado formado por las clases de equivalencia de variables aleatorias X que coinciden con probabilidad 1 y tales que $E|X|^p < \infty$, $p \geqslant 1$. $L^2[t_o,T]$ será la colección de todas las funciones medibles f tales que $\int_{t_o}^{T} |f(s,x)|^2 ds < \infty$. Las letras manuscritas, $\mathcal{U}$, $\mathcal{F}$, $\mathcal{W}$ denotarán σ-álgebras. $\mathcal{U}(X)$ denotará a la σ-álgebra generada por X. $\mathcal{U}(X_u;t_o \leqslant u \leqslant t)$ será la σ-álgebra generada por las variables aleatorias X_u, con $t_o \leqslant u \leqslant t$. $\{X_t\}_{t\in[t_o,T]}$, o en ocasiones simplemente X_t, denotará un proceso estocástico definido en $(\Omega,\mathcal{U},P)$ con valores en $\mathbb{R}^d$ $(d \geqslant 1)$ y $\{X_t,\mathcal{F}_t\}_{t\in[t_o,T]}$ denotará a una martingala. $\{W_t\}_{t\in[t_o,T]}$, o simplemente W_t, denotará a un proceso de Wiener. $\text{ac-}\lim_{n\to\infty} X_n = X$ denotará a la convergencia con probabilidad 1 de $\{X_n\}_{n\in\mathbb{N}}$ hacia X. $\text{st-}\lim_{n\to\infty} X_n = X$ denotará a la convergencia estocástica o en probabilidad $\{X_n\}_{n\in\mathbb{N}}$ hacia X.

§2. Integral estocástica. Para simplificar la lectura del trabajo daremos la definición detallada de integral estocástica, siguiendo Arnold [1], y citaremos algunos resultados de [1], y de Doob [4], para posterior uso.

Se considera un proceso de Wiener m-dimensional W_t, definido en $(\Omega,\mathcal{U},P)$. Sean t_o un real fijo y no negativo, $\mathcal{W}_t = \mathcal{U}(W_u;t_o \leqslant u \leqslant t)$ y $\mathcal{W}_t^+ = \mathcal{U}(W_s-W_t,\ t \leqslant s < \infty)$. Como W_t tiene incrementos W_s-W_t independientes, entonces $\mathcal{W}_t$ y $\mathcal{W}_t^+$ son independientes.

DEFINICION 2.1. (Arnold, Def.4.3.2, pág.63).
Sea t_o no negativo y fijo; una familia $\{\mathcal{F}_t\}_{t>t_o}$ de subsigma-álgebras de $\mathcal{U}$ se dice *no anticipante* respecto del proceso de Wiener m-dimensional W_t si:

a) $\mathcal{F}_s \subseteq \mathcal{F}_t$ $(t_o \leqslant s \leqslant t)$

b) $\mathcal{F}_t \supseteq \mathcal{W}_t$ $(t \geqslant t_o)$

c) $\mathcal{F}_t$ es independiente de $\mathcal{W}_t^+$ $(t \geqslant t_o)$.

DEFINICION 2.2. (Arnold, Def.4.3.4, pág.63). Una función $G(s,w)$ definida en $[t_o,T]\times\Omega$, con valores matriciales (dxm), medible en ambas variables, se dice *no anticipante* (respecto de la familia no anticipante $\{\mathcal{F}_s\}$) si para todo s fijo, $s \in [t_o,T]$, $G(s,w)$ es $\mathcal{F}_s$ medible. Si además las trayectorias $G(s,w) \in L^2[t_o,T]$, w fijo, con probabilidad 1, entonces a este tipo de funciones no anticipantes las agrupamos en el conjunto denotado con $M_2[t_o,T]$.

Para facilitar la escritura, cuando no haya lugar a confusiones, en algunos casos se omitirá la segunda variable, ya que los conceptos que se estudiarán ahora sólo dependerán de la primera variable.

DEFINICION 2.3. (Arnold 4.4, pág.64). Una función $G \in M_2[t_o,T]$ se dice *escalonada*, si existe una partición $t_o < t_1 < t_2 < \ldots < t_n = T$ del intervalo $[t_o,T]$ tal que $G(s) = G(t_{i-1})$, para todo $s \in [t_{i-1},t_i)$.

DEFINICION 2.4. (Arnold 4.4, pág.64). La *integral estocástica* para funciones escalonadas G respecto al proceso de Wiener W_t la definimos como

$$\int_{t_o}^{t} GdW = \int_{t_o}^{t} G(s)dWs = \sum_{i=1}^{n} G(t_{i-1})(W_{t_i}-W_{t_{i-1}}) \cdot$$

Para definir la integral estocástica de cualquier función en $M_2[t_o,T]$ se utilizará el hecho de que el conjunto de todas las funciones escalonadas en $M_2[t_o,t]$ es denso en $M_2[t_o,t]$ en el siguiente sentido: si $G \in M_2[t_o,t]$, existe una sucesión de funciones escalonadas $G_n \in M_2[t_o,t]$ tal que

$$\underset{n\to\infty}{\text{ac-lím}} \int_{t_o}^{t} |G(s)-G_n(s)|^2 ds = 0 \cdot$$

Como la convergencia con probabilidad 1 implica la convergencia estocástica, se tiene, en resumen, que si $G \in M_2[t_o,t]$, existe entonces una sucesión de funciones escalonadas $G_n \in [t_o,t]$ tal que

$$\underset{n\to\infty}{\text{st-lím}} \int_{t_o}^{t} |G(s)-G_n(s)|^2 ds = 0.$$

Como ya se ha definido la integral para funciones escalonadas, se tiene $\int_{t_o}^{t} G_n(s)dW_s$ para todo n = 1,2,... .

Además, es posible ver que el límite

$$\underset{n\to\infty}{\text{st-lím}} \int_{t_o}^{t} G_n(s)dW_s$$

existe y define una variable aleatoria, llamémosla I(G), la cual no depende de la escogencia de la sucesión $\{G_n\}_{n\in\mathbb{N}}$, de manera que la integral estocástica de G se define como

$$\int_{t_o}^{t} G(s)dW_s = \text{st-lím}_{n\to\infty} \int_{t_o}^{t} G_n(s)dW_s,$$

donde G_n es una sucesión de funciones en $M_2[t_o,t]$ tal que

$$\text{st-lím}_{n} \int_{t_o}^{t} |G(s)-G_n(s)|^2 ds = 0 .$$

TEOREMA 2.5. (Arnold, Theorem 4.4.14, pág.73). *Sean* G, G_1, G_2, G_n, $n = 1,2,\ldots$ *funciones con valores matriciales* $(d\times m)$ *en* $M_2[t_o,t]$. *Si* W_t *denota a un proceso de Wiener* m-*dimensional, entonces si*

$$\text{st-lím}_{n\to\infty} \int_{t_o}^{t} |G(s)-G_n(s)|^2 ds = 0,$$

para una sucesión $\{G_n\}$ *de funciones no necesariamente escalonadas se tiene que*

$$\text{st-lím}_{n\to\infty} \int_{t_o}^{t} G_n dW = \int_{t_o}^{t} G dW.$$

COROLARIO 2.6. (Arnold, Corollary 4.5.5, pág.77). *Si* $\int_{t_o}^{t} E|G(s)|^2 ds < \infty$, *donde* $G \in M_2[t_o,t]$, *entonces para todo* $c > 0$

$$P\left[\left|\int_{t_o}^{t} GdW\right| > c\right] \leq \int_{t_o}^{t} E|G(s)|^2 ds/c^2.$$

Si $G \in M_2[t_o,T]$ y $A \in \mathcal{B}$ (σ-álgebra generada por los conjuntos de Borel de $\mathbb{R}$), entonces $A \subseteq [t_o,T]$ y $G\chi_A \in M[t_o,T]$. Si se ha definido $\int_{t_o}^{T} GdW$, entonces $\int_A GdW = \int_{t_o}^{T} G\chi_A dW$. Si

$t \in [t_o,T]$, se define

$$X_t(w) = \int_{t_o}^{t} G(s,w)dW_s(w) = \int_{t_o}^{T} G(s)X_{[0,t]}dWs.$$

$\{X_t\}_{t \in [t_o,T]}$ es un proceso estocástico con valores en $\mathbb{R}^d$, definido en forma única, salvo equivalencia estocástica $X_{t_o} = 0$ con probabilidad 1

$$X_t - X_s = \int_s^t G(u)dW_u, \quad t_o \leq s \leq t \leq T.$$

TEOREMA 2.7. *Si* X_t *y* Y_t *son martingalas respecto a la misma familia de subsigma-álgebras, entonces* $AX_t + BY_t$ *(donde* A *y* B *son matrices fijas,* pxd, $p \geq 1$*) es una martingala. En particular,* $X_t - X_a$ *es una martingala si* a *es un real fijo,* $t \geq a$.

TEOREMA 2.8. (Doob [4]). *Si* X_t *es una martingala y* $X_t \in L^p(P)$, *entonces* $|X_t|^p$ *es una submartingala y se cumplen las siguientes desigualdades:*

a) $P[\sup_{t\in[a,b]} |X_t| \geq c] \leq E|X_b|^p/c^p, \ \forall c > 0, \ p \geq 1$ (2.2.3)

b) $E[\sup_{t\in[a,b]} |X_t|^p] \leq (p/(p-1))^p E|X_b|^p, \quad p > 1.$ (2.2.4)

TEOREMA 2.9. (Arnold, Theorem 5.11 (a)-(c), pag. 81) *Sean* $G \in M_2[t_o,T]$ *y* $\{F_t\}_{t\in[t_o,T]}$, *una familia no anticipante de subsigma-álgebras. Si*

$$X_t = \int_{t_o}^{t} G(s)dW_s, \quad t_o \leq t \leq T$$

entonces se cumplen las siguientes afirmaciones:

a) X_t *es* F_t*-medible, o sea, no anticipante.*

b) *Si* $\int_{t_o}^{t} E|G(s)|^2 ds < \infty$, $\forall t \leq T$, *entonces* $\{X_t, F_t\}_{[t_o,T]}$ *es una martingala con valores en* $\mathbb{R}^d$*; además para* $t,s \in [t_o,T]$ *se tiene:*

i) $EX_t = 0$

ii) $EX_t X_s' = \int_{t_o}^{\min(t,s)} EG(u)G(u)'du$;

en particular,

$$E|X_t|^2 = \int_{t_o}^{t} E|G(u)|^2 du \tag{2.2.6}$$

iii) $\forall c > 0$, $t_o \leq a \leq b \leq T$,

$$P[\sup_{a \leq t \leq b} |X_t - X_a| \geq c] \leq \int_a^b E|G(s)|^2 ds/c^2 \tag{2.2.7}$$

c) X_t *tiene trayectorias continuas con probabilidad 1.*

§3. Ecuación diferencial estocástica y existencia de soluciones.

Se consideran dos funciones, f con valores en $\mathbb{R}^d$ y G con valores matriciales (dxm), ambas definidas en $[t_o,T] \times \mathbb{R}^d$; $f(t,x)$ y $G(t,x)$ son independientes de $w \in \Omega$, este valor aparece solo cuando se considera $f(t,X_t(w))$ y $G(t,X_t(w))$; ambas funciones se suponen medibles en las dos variables. Sea W_t un proceso de Wiener m-dimensional. Consideramos una ecuación diferencial estocástica de la forma

$$dW_t = f(t,X_t)dt + G(t,X_t)dW_t, \quad Xt_o = \xi, \quad t_o \leq t \leq T < \infty \tag{3.1}$$

o su forma integral

$$W_t = \xi + \int_{t_o}^{t} f(s,X_s)ds + \int_{t_o}^{t} G(s,X_s)dW_s, \quad t_o \leq t \leq T < \infty \tag{3.2}$$

Sabemos por el Teorema 2.9 (a) que, en caso de existir, X_t es un proceso estocástico F_t-medible; $X_{t_o} = \xi$ es una variable aleatoria que debe ser independiente de los procesos $W_t - W_{t_o}$, $t \geq t_o$; por tanto, a partir de este momento se toma $F_t = U(\xi, W_s, s \leq t)$ que debe ser independiente de $W_t^+ = U(W_s - W_t, s \geq t)$.

DEFINICION 3.3. (Arnold, Def.6.1.3,pág.101). Una ecuación de la forma (3.1) se llama *Ecuación Diferencial Estocástica* (de Ito). La variable aleatoria ξ se llama el *valor inicial* en el momento t_o. El proceso estocástico X_t se llama una *solución de la ecuación* (3.1) o (3.2) en el intervalo $[t_o,T]$ si satisface las propiedades siguientes:

a) X_t es F_t-medible o sea no anticipante para $t \in [t_o,T]$.

b) Las funciones $\bar{f}(t,w) = f(t,X_t(w))$ y $\bar{G}(t,w) = G(t,X_t(w))$ (no anticipantes) cumplen con probabilidad 1:

$$\int_{t_o}^{T} |\bar{f}(s,w)| ds < \infty \quad y \quad \int_{t_o}^{T} |\bar{G}(s,w)|^2 ds < \infty$$

c) La ecuación (3.2) se cumple para todo $t \in [t_o,T]$ con probabilidad 1.

Si $G = 0$ la influencia estocástica de la ecuación está dada por ξ.

TEOREMA 3.4. *Dada la ecuación diferencial estocástica*

$$dW_t = f(t,X_t) + G(t,X_t)dW_t, \quad X_{t_o} = \xi, \quad t_o \leq t \leq T < \infty,$$

donde W_t *es un proceso de Wiener m-dimensional y* ξ *una variable aleatoria independiente de* $W_t - W_{t_o}$ *para* $t \geq t_o$, *si las funciones* f, *con valores en* $\mathbb{R}^d$, *y* G *con valores matriciales* (d×m), *ambas definidas y medibles en* $[t_o,T]\times\mathbb{R}^d$ *son tales que* $f(t,\cdot)$ *y* $G(t,\cdot)$ *son continuas para cada* $t \in [t_o,T]$, *y satisfacen:*

$$|f(t,x)|+|G(t,x)| \leq m(t),\ \forall t \in [t_o,T],\ \forall x \in \mathbb{R}^d,\ m \in L^2[t_o,T] \tag{4.1.1}$$

entonces la ecuación tiene en $[t_o,T]$ *una solución* X_t, *con valores en* $\mathbb{R}^d$ *y definida en* Ω, *continua para* $t \in [t_o,T]$ *con probabilidad 1, que satisface la condición inicial* $X_{t_o} = \xi$.

Demostración. La demostración se hará en dos etapas (c.f. Arnold [1]). Supongamos primero que $E|\xi|^2 < K_1 < \infty$. Como $m \in L^2[t_o,T]$, definimos una función auxiliar M(t) así:

$$M(t) = 0, \quad \text{si} \quad t = t_o$$

$$M(t) = \int_{t_o}^{t} m^2(s)ds, \quad t_o < t \leq T.$$

Por su definición, M es una función continua, no decreciente y $M(t_o) = 0$. Definimos ahora las aproximaciones $X_t^{(j)}$ de X_t, $j = 1,2,\ldots$; para facilitar la escritura, escribimos $h = T-t_o$ y para cada $w \in \Omega$ definimos:

$$X_t^{(j)}(w) = \xi(w), \quad \text{si } t_o \leq t < t_o+h/j$$

$$X_t^{(j)}(w) = \xi(w)+\int_{t_o}^{t-h/j} f(s,X_s^{(j)}(w))ds + \int_{t_o}^{t-h/j} G(s,X_s^{(j)}(w))dW_s(w) \ \text{si } t \in [t_o+h/j,T]$$

Es claro que $X_t^{(j)}$ está bien definida, pues si $t_o+h/j \leqslant t \leqslant t_o+2h/j$ entonces

$$X_t^{(j)} = \xi + \int_{t_o}^{t-h/j} f(s,\xi)ds + \int_{t_o}^{t-h/j} G(s,\xi)dW_s \ ,$$

en segundo lugar, si $t_o+2h/j \leqslant t \leqslant t_o+3h/j$, al considerar las integrales, en $f(s,\cdot)$ y $G(s,\cdot)$ sólo se substituyen valores ya definidos de $X_s^{(j)}$, para s en el intervalo $[t_o,t_o+2h/j]$; ésta consideración se puede hacer j veces hasta que t recorra todo el intervalo $[t_o,T]$. Como

$$|f(s,x)| + |G(s,x)| \leqslant m(s) \quad y \quad \int_{t_o}^{T} m^2(s)ds < \infty,$$

entonces

$$\int_{t_o}^{t} |f(s,w)|ds \leqslant \left[(T-t_o)(\int_{t_o}^{t} |f(s,w)|^2 ds)\right]^{\frac{1}{2}} < \infty$$

y

$$\int_{t_o}^{t} |G(s,w)|^2 ds \leqslant \int_{t_o}^{t} m^2(s)ds < \infty$$

De modo que $f \in L^1[t_o,T]$ y $G \in M_2[t_o,T]$ y por el Teorema 2.9 (c) los $X_t^{(j)}$ son procesos estocásticos con trayectorias continuas con probabilidad 1, para $j = 1,2,\ldots$ y $t \in [t_o,T]$. Por la parte (a) del mismo teorema los $X_t^{(j)}$ son además no anticipantes y por la parte (b) son martingalas respecto de la familia $\{F_t\}_{t\in[t_o,T]}$, $F_t = \mathcal{U}(\xi,W_s,s \leqslant t)$. Además en $[t_o,T]$ se obtienen las siguientes acotaciones:

$$E|X_t^{(j)}|^2 = E|\xi|^2 < K_1 < \infty, \quad \text{si } t_o \leqslant t < t_o+h/j$$

Si $t_o+h/j \leqslant t \leqslant T$, como $|x+y+z|^2 \leqslant 3|x|^2 + 3|y|^2 + 3|z|^2$, entonces

$$E|X_t^{(j)}|^2 = E|\xi + \int_{t_o}^{t-h/j} f(s,X_s^{(j)})ds + \int_{t_o}^{t-h/j} G(s,X_s^{(j)})dW_s|^2$$

$$\leqslant 3E|\xi|^2 + 3E|\int_{t_o}^{t-h/j} f(s,X_s^{(j)})ds|^2 + 3E|\int_{t_o}^{t-h/j} G(s,X_s^{(j)})dW_s|^2$$

$$\leqslant 3K_1 + 3(t-h/j-t_o)\int_{t_o}^{t-h/j} E|f(s,X_s^{(j)})|^2 ds + 3\int_{t_o}^{t-h/j} E|G(s,X_s^{(j)})|^2 ds$$

$$\leqslant 3K_1 + 3(T-t_o)\int_{t_o}^{t-h/j} Em^2(s)ds + 3\int_{t_o}^{t-h/j} Em^2(s)ds$$

$$= 3(K_1+(T-t_o+1)M(t-h/j)),$$

usando la desigualdad de Cauchy-Schwarz y el Teorema 2.9 (b,ii). Como M es una función continua en $[t_o,T]$, es uniformemente continua y acotada; luego existe una constante $K_2 > 0$ tal que $M(s) \leqslant K_2$ para todo $s \in [t_o,T]$. Sea $K_3 = 3(K_1+(T-t_o+1)K_2)$, entonces

$$E|X_t^{(j)}|^2 \leqslant K_3, \quad j = 1,2,\ldots, \quad t \in [t_o,T] \tag{3.4.1}$$

Por el Teorema 2.8, para p = 2 se tiene la desigualdad

$$E[\sup_{t\in[t_o,T]} |X_t^{(j)}|^2] \leqslant 4E|X_T^{(j)}|^2 \leqslant 4K_3, \quad j = 1,2,\ldots \tag{3.4.2}$$

Sea ahora $Z_j = \sup_{t\in[t_o,T]} |X_t^{(j)}|^2$, la cual es una variable aleatoria con esperanza finita acotada por $4K_3$, $j = 1,2,\ldots,$ y consideramos la variable aleatoria $U_n = \max_{j\leqslant n} Z_j$, entonces $E(U_n) \leqslant 4K_3$, $n = 1,2,\ldots$; además $E(U_n)$ es una sucesión creciente y acotada de números reales, por lo tanto es convergente. Observamos que U_n es a su vez una sucesión creciente de variables aleatorias que converge a $\sup_j Z_j$; por tanto,

$$E(\sup_j Z_j) = E(\lim_{n\to\infty} U_n) = \lim_{n\to\infty} E(U_n) \leq 4K_3,$$

y por la desigualdad de Chebyshev, para todo $c > 0$ obtenemos

$$P\left[\sup_j Z_j \geq c\right] \leq E(\sup_j Z_j)/c \leq 4K_3/c;$$

en resumen,

$$P\left[\sup_j \sup_{t\in[t_o,T]} |X_t^{(j)}|^2 \geq c\right] \leq 4K_3/c.$$

Sea

$$A_n = \{w : \sup_j \sup_{t\in[t_o,T]} |X_t^{(j)}|^2(w) \geq n^2\},$$

entonces $P(A_n) \leq 4K_3/n^2$. Como $\sum_{n=1}^{\infty} P(A_n) < \infty$, tenemos $P(\lim\sup A_n) = 0$ por el lema de Borel-Cantelli. Sea $A = \lim\sup_n A_n$. Si $w \notin A$, entonces $w \in A_n$ para un número finito de valores de n solamente digamos $n_1, n_2, \ldots, n_k$, sea $K(w) = \max\{n_1, n_2, \ldots, n_k\} + 1$, de modo que $w \notin A_n$ para todo $n \geq K(w)$: entonces para este w

$$\sup_j \sup_{t\in[t_o,T]} |X_t^{(j)}|^2(w) \leq K(w)^2,$$

es decir,

$$|X_t^{(j)}(w)| \leq K(w), \quad j = 1,2,\ldots, \quad t \in [t_o,T]; \qquad (3.4.3)$$

por consiguiente, para todo $w \notin A$, $\{X_t^{(j)}(w)\}$ es uniformemente acotada.

Ahora demostraremos que el proceso $\{X_t^{(j)}\}_j$ es equicontinuo, con probabilidad 1; esto es, existe un conjunto B con $P(B) = 0$ tal que si $w \notin B$, entonces dado $\varepsilon > 0$, existe $\delta > 0$, que depende de ε y de w, tal que si $|t_1-t_2| < \delta$, $t_1,t_2 \in [t_o,T]$, entonces

$$|X_{t_1}^{(j)}(w) - X_{t_2}^{(j)}(w)| < \varepsilon, \quad j = 1,2,\ldots$$

Consideraremos tres casos:

a) Si $t_1,t_2 \in [t_o,t_o+h/j)$, entonces $|X_{t_1}^{(j)}(w) - X_{t_2}^{(j)}(w)| = 0$, $j = 1,2,\ldots$

b) Si $t_1,t_2 \in [t_o+h/j,T]$ entonces, suponiendo $t_1 < t_2$ se tiene

$$E|X_{t_1}^{(j)}-X_{t_2}^{(j)}|^2 = E|\int_{t_1-h/j}^{t_2-h/j} f(s,X_s^{(j)})ds + \int_{t_1-h/j}^{t_2-h/j} G(s,X_s^{(j)})dW_s|^2$$

$$\leqslant 2(t_2-t_1)\int_{t_1-h/j}^{t_2-h/j} E|f(s,X^{(j)})|^2 ds + 2\int_{t_1-h/j}^{t_2-h/j} E|G(s,X_s^{(j)})|^2 ds$$

$$\leqslant 2(T-t_o)\int_{t_1-h/j}^{t_2-h/j} Em^2(s)ds + 2\int_{t_1-h/j}^{t_2-h/j} Em^2(s)ds$$

$$= 2(T-t_o+1)(M(t_2-h/j) - M(t_1-h/j)).$$

c) Si $t_1 \in [t_o,t_o+h/j)$ y $t_2 \in [t_o+h/j,T]$, se reduce simplemente al caso (b) ya que la función $X_t^{(j)}$ es constante respecto a t en $[t_o,t_o+h/j]$, o sea que $X_{t_1}^{(j)}$ se comporta como $X_{t_o+h/j}^{(j)}$.

Como M es uniformemente continua, entonces dado $\varepsilon > 0$ existe $\delta > 0$, que depende sólo de ε, tal que si $[t_1-t_2] < \varepsilon$ con $t_1,t_2 \in [t_o,T]$:

$$|M(t_2-h/j) - M(t_1-h/j)| < \varepsilon, \quad j = 1,2,\ldots$$

Así que resumiento los tres casos, se tiene:

$\forall\varepsilon > 0$ existe δ, que depende de ε, tal que si $t_1,t_2 \in [t_o,T]$ y $[t_1-t_2] \leqslant \delta$ entonces

$$E|X^{(j)}_{t_1}-X^{(j)}_{t_2}|^2 \leqslant 2(T-t_o+1)(M(t_2-h/j)-M(t_1-h/j))$$

$$< 2(T-t_o+1)\varepsilon \qquad (3.4.4)$$

y esto para todo $j = 1,2,\ldots$

Sea $t \in [t_o,T]$ fijo, sea $\delta > 0$ y $t \in [t_o,T-\delta]$; por el Teorema 2.7, $\{X^{(j)}_{t+u} - X^{(j)}_t, \mathcal{F}_{t+u}\}_{u\in[0,\delta]}$ es una martingala y por el teorema 2.8 (b), para $p = 2$ se tiene entonces:

$$E|\sup_{o\leqslant u\leqslant\delta} |X^{(j)}_{t+u} - X^{(j)}_t|^2 \leqslant 4E|X^{(j)}_{t+\delta} - X^{(j)}_t|^2$$

$$< 8(T-t_o+1)\varepsilon \qquad (3.4.5)$$

cuando δ es el de (3.4.4). Sea

$$Y^{(j)}_t = \sup_{0\leqslant u\leqslant\delta} |X^{(j)}_{t+u} - X^{(j)}_t|^2, \quad t \in [t_o,T-\delta], \quad j = 1,2,\ldots$$

Como los racionales de $[t_o,T-\delta]$ son densos en $[t_o,T-\delta]$, por el Teorema 2.2, capítulo II, en [4], para cada $n \in \mathbb{N}$ existe una partición de $[t_o,T-\delta]$, de números racionales $t_o \leqslant s^{(n)}_o \leqslant s^{(n)}_1 \leqslant\ldots\leqslant s^{(n)}_{a_n} \leqslant T-\delta$, tal que

$$\sup_{t_o\leqslant t\leqslant T-\delta} T^{(j)}_t \leqslant \liminf_n \max_{k\leqslant a_n} Y^{(j)}_{s^{(n)}_k}, \quad j = 1,2,\ldots.$$

Por (3.4.5)

$$\lim_n \inf E(\max_{k\leqslant a_n} Y^{(j)}_{s_k^{(n)}}) < 8(T-t_o+1)\varepsilon, \quad j = 1,2,\ldots$$

Entonces

$$E(\sup_{t_o\leqslant t\leqslant T-\delta} Y_t^{(j)}) < E(\lim_n \inf \max_{k\leqslant a_n} Y^{(j)}_{s_k^{(n)}})$$

$$\leqslant \lim_n \inf E(\max_{k\leqslant a_n} Y^{(j)}_{s_k^{(n)}}) < 8(T-t_o+1)\varepsilon, \quad j=1,2,\ldots \tag{3.4.6}$$

Sea ahora $Z_j = \sup_{t_o\leqslant t\leqslant T-\delta} Y_t^{(j)}$, $j = 1,2,\ldots$ Definimos $U_n = \max_{j\leqslant n} Z_j$; como hemos visto, ésta es una sucesión creciente de variables aleatorias que converge a $\sup_j Z_j$, además $E(\sup_j Z_j) < 8(T-t_o+1)$. Nuevamente por la desigualdad de Chebyshev, tenemos para todo $c > 0$:

$$P[\sup_j Z_j \geqslant c] \leqslant E(\sup_j Z_j)/c < 8(T-t_o+1)/c,$$

obteniendo para todo $\varepsilon > 0$ y $c > 0$:

$$P[\sup_j \sup_{t_o\leqslant t\leqslant T-\delta} \sup_{0\leqslant u\leqslant\delta} |X^{(j)}_{t+u}-X^{(j)}_t|^2 \geqslant c] < 8(T-t_o+1)\varepsilon/c, \tag{3.4.7}$$

con δ dependiente de ε como en (3.4.4). Escogemos ahora valores $\varepsilon_n = 1/n^4$ y $c_n = 1/n^2$ para ε y c respectivamente, y consideramos los conjuntos:

$$B_n = \{w:\sup_j \sup_{t_o\leqslant t\leqslant T-\delta} \sup_{0\leqslant u\leqslant\delta} |X^{(j)}_{t+u}-X^{(j)}_t|^2(w) \geqslant 1/n^2\},$$

en donde $P(B_n) < 8(T-t_o+1)1/n^2$ por (3.4.7); aplicando nuevamente el lema de Borel-Cantelli, si $B = \lim_n \sup B_n$, vemos que P(B) = 0. Así mismo si $w \notin B$, entonces $w \in B_n$ para un número finito de valores de n solamente, digamos $n_1, n_2, \ldots, n_r$. Si $N = \max\{n_1, n_2, \ldots, n_r\}+1$, entonces $w \notin B_n$, $n \geq N$; como $\varepsilon_n = 1/n^4$, existe un δ_n, el cual depende de ε_n y de w, pero no de j ni de t, tal que se cumple:

$$\sup_{j} \sup_{t_o \leq t \leq T-\delta_n} \sup_{0 \leq u \leq \delta_n} |X_{t+u}^{(j)} - X_t^{(j)}|^2 < 1/n^2$$

o, lo que el lo mismo, si $|t_1 - t_2| \leq \delta_n$ con $t_1, t_2 \in [t_o, T]$ entonces

$$|X_{t_2}^{(j)}(w) - X_{t_1}^{(j)}(w)| < 1/n, \qquad j = 1,2,\ldots$$

De manera que para cada $w \notin B$, $\{X_t^{(j)}(w)\}_j$, es equicontinua en $[t_o, T]$. Considerando el conjunto $C = A \cup B$, para cada $w \notin C$, $\{X_t^{(j)}(w)\}_j$ es una familia de funciones de t uniformemente acotada (3.4.3) y equicontinua. Por el teorema de Arzèla-Ascoli existe una subsuseción $\{X_t^{(j_k)}(w)\}$ que converge uniformemente en $[t_o, T]$ hacia una función $X_t(w)$, obteniendose así un proceso estocástico $\{X_t\}$, que es el límite uniforme de una subsucesión $\{X_t^{(j_k)}\}_k$ con probabilidad 1; por consiguiente, X_t es no anticipante y de trayectoria continua con probabilidad 1, ya que los $X_t^{(j_k)}$ lo son.

Para que X_t sea solución de la ecuación diferencial estocástica falta ver únicamente que X_t cumple la propiedad (c) de la Definición 3.3. Como f y G son continuas respecto a la segunda variable $t \in [t_o, T]$, entonces

$$\lim_{k\to\infty} f(t, X_t^{(j_k)}) = f(t, X_t) \quad y \quad \lim_{k\to\infty} G(t, X_t^{(j_k)}) = G(t, X_t)$$

con probabilidad 1. Como $|f(s,x)| < m(s)$ y $m \in L^2[t_o, T]$, f es integrable y por el teorema de la convergencia acotada se tiene:

$$\text{ac-lîm}_{n\to\infty} \int_{t_o}^{t} f(s, X_s^{(j_k)})ds = \int_{t_o}^{t} f(s, X_s)ds.$$

Debido a que $G \in X_2[t_o, T]$, se tiene $G \in L^2[t_o, T]$ y como la convergencia de $X_t^{(j_k)}$ hacia X_t es uniforme, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ tal que $\forall n \geq N$; $\forall s \in [t_o, T]$ se tiene que $|G(s, X_s^{(j_n)}) - G(s, X_s)|^2 < \varepsilon$ con probabilidad 1; por consiguiente $\int_{t_o}^{t} |G(s, X_s^{(j_n)}) - G(s, X_s)| ds < (T - t_o)\varepsilon$ con probabilidad 1; es decir,

$$\text{ac-lîm}_{n\to\infty} \int_{t_o}^{t} |G(s, X_s^{(j_n)}) - G(s, X_s)|^2 ds = 0,$$

y, por consiguiente,

$$\text{st-lîm}_{n\to\infty} \int_{t_o}^{t} |G(s, X_s^{(j_n)}) - G(s, X_s)|^2 ds = 0;$$

finalmente, por el Teorema 2.5,

$$\text{st-lîm}_{n\to\infty} \int_{t_o}^{t} G(s, X^{(j_n)})dW_s = \int_{t_o}^{t} G(s, X_s)dW_s,$$

obteniéndose así que X_t es una solución de la ecuación diferencial estocástica planteada en el teorema.

Se demostrará ahora el caso general; es decir, el caso en que ξ es una variable aleatoria cualquiera. Definimos

$$\xi_n = \begin{cases} \xi, \text{ si } |\xi| \leq n \\ 0 \text{ en todo lo demás.} \end{cases}$$

Para cada $n \in \mathbb{N}$ existe entonces una solución $X_t^{(n)}$ a la ecuación

$$X_t = \xi_n + \int_{t_o}^{t} f(s,X_s)ds + \int_{t_o}^{t} G(s,X_s)dWs, \quad t_o \leq t \leq T,$$

fuera de un conjunto C_n tal que $P(C_n) = 0$. Además como $\lim_{n\to\infty} \xi_n = \xi$, se tiene:

$$P[|X_t^{(n)} - X_t^{(m)}| \neq 0] = P[|\xi_n - \xi_m| \neq 0] \to 0$$

cuando $m,n \to \infty$, sin depender de t. Existe entonces un proceso estocástico X_t, tal que $X_t^{(n)} \to X_t$ $(n \to \infty)$ uniformemente con probabilidad 1, ya que $P(\bigcup_n C_n) = 0$.

Resulta, pues, que X_t es una solución de la ecuación diferencial estocástica, completando así la demostración del teorema.

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Departamento de Matemáticas y Estadística
Universidad Nacional de Colombia
Bogotá, D.E. Colombia

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THE NUMERICAL SOLUTION OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS BY THE METHOD OF LINES*

by

Bruce H. EDWARDS

ABSTRACT. The Method of Lines is shown to be a practical and convenient technique for the numerical solution of elliptic partial differential equations. The method produces a system of coupled two-point boundary value problems which are solved using state-of-the-art software.

Section 1. The Method of Lines has long been a popular and convenient technique for the numerical solution of parabolic partial differential equations. The idea is to discretize all but one of the independent variables, which results in a system of coupled ordinary differential equations. In the parabolic case the system of initial value problems can then be solved using state-of-the-art software.

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Because of the recent availability of high quality software for two-point boundary value problems, the Method of Lines has become an attractive technique for elliptic partial differential equations as well ([4], [6]). In this paper we analyze certain aspects of this numerical method applied to Poisson's equation with Dirichlet boundary conditions on a rectangular domain. Extensions to equations with Neumann conditions are immediate.

As observed by the referee, comparisons with other numerical techniques must ultimately be made. We do not pretend that the Method of Lines competes favorably with finite elements or finite differences on most problems. However, it has been successful on certain specific elliptic equations ([4], [6]). Current work on the Method of Lines applied to Fracture Mechanics (to appear) has produced good results compared to such classical techniques as finite difference and boundary elements.

In Section 2 we outline the basic discretization schemes and in Section 3 solve the five-point scheme. Section 4 presents a matrix technique for explicitly solving the system of two-point boundary value problems. The inherent instability for this system is briefly discussed in Section 5. A specific example from [4] is presented in Section 6, which was solved using the two-point boundary value problem code SUPORT [8]. A nonlinear problem [5] having nine distinct solutions is completely solved in Section 7. Here we use the nonlinear two-point boundary value problem code PASVA3 [7], now available through the International Mathematics and Statistics Library IMSL.

Section 2. Following Berezin and Zhidkov [2], we consider Poisson's equation with Dirichlet boundary conditions on a rectangle:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \tag{1}$$

$$u(x,c) = \gamma(x),\ u(x,d) = \delta(x), \quad a \leqslant x \leqslant b,$$

$$u(a,y) = \alpha(y),\ u(b,y) = \beta(y), \quad c \leqslant y \leqslant d.$$

For $n > 3$, let $h = (d-c)/(n+1)$ and $y_k = c+kh$, $k = 0,1,2,\ldots,n+1$ (see Figure 1). The second partial derivative with respect to y is approximated by a three-point central difference scheme

$$\frac{\partial^2 u}{\partial y^2}(x,y_k) = \frac{1}{h^2}(u(x,y_{k+1}) - 2u(x,y_k) + u(x,y_{k-1})) + O(h^2), \tag{2}$$

$k = 1,2,\ldots,n.$

Putting $U_k(x) = u(x,y_k)$ and $f_k(x) = f(x,y_k)$, we have the $O(h^2)$ accurate system of coupled two-point boundary value problems:

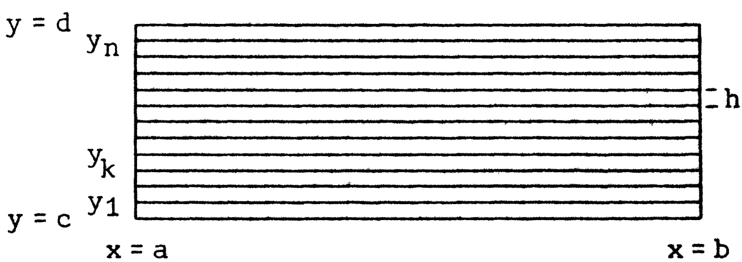

Figure 1

$$U_k'' + \frac{1}{h^2}(U_{k+1} - 2U_k + U_{k-1}) = f_k(x), \qquad k = 1,2,..,n, \qquad (3)$$

$$U_o(x) = \gamma(x), \quad U_{n+1}(x) = \delta(x).$$

$$U_k(a) = \alpha(y_k), \quad U_k(b) = \beta(y_k).$$

Similarly, if we approximate the second partial derivative by a five point scheme,

$$\frac{\partial^2 u}{\partial y^2}(x,y_k) = \frac{1}{h^2}\left(\frac{-1}{12}u(x,y_{k+2}) + \frac{4}{3}u(x,y_{k+1}) - \frac{5}{2}u(x,y_k)\right.$$

$$\left. + \frac{4}{3}u(x,y_{k-1}) - \frac{1}{12}u(x,y_{k-2})\right) + O(h^4),$$

we arrive at the $O(h^4)$ system:

$$U_k'' + \frac{1}{h^2}\left(-\frac{1}{12}U_{k+2} + \frac{4}{3}U_{k+1} - \frac{5}{2}U_k + \frac{4}{3}U_{k-1} - \frac{1}{12}U_{k-2}\right) = f_k, \qquad (4)$$

$k = 1,2,...,n,$

$$U_o(x) = \gamma(x), \quad U_{n+1}(x) = \delta(x),$$

$$U_k(a) = \alpha(y_k), \quad U_k(b) = \beta(y_k).$$

Of course, there is now the need to impose further boundary conditions. One idea [3] is to require that the five-point scheme be compatible with the three-point scheme. This is the approach we will take in Section 3. However, in actual computations, nonsymmetric higher order approximations have been quite successful. For example, one could impose the $O(h^4)$ scheme (see [4]) at the n^{th} line:

$$U_n'' + \frac{1}{h^2}\left(\frac{5}{6}U_{n+1} - \frac{5}{4}U_n - \frac{1}{3}U_{n-1} + \frac{7}{6}U_{n-2} - \frac{1}{2}U_{n-3} + \frac{1}{12}U_{n-4}\right) = f_n(x).$$

$$(5)$$

Berezin [2] also derives the $O(h^4)$ system

$$\frac{5}{6}U_k'' + \frac{1}{12}(U_{k+1}'' + U_{k-1}'') + \frac{1}{h^2}(U_{k+1} - 2U_k + U_{k-1})$$

$$= \frac{5}{6}f_k + \frac{1}{12}(f_{k+1} + f_{k-1}), \quad k = 1,2,\ldots,n, \qquad (6)$$

$$U_o(x) = \gamma(x), \quad U_{n+1}(x) = \delta(x),$$

$$U_k(a) = \alpha(y_k), \; U_k(b) = \beta(y_k).$$

Section 3. Berezin [2] and Jones [4] both solve the three-point scheme (3). Berezin further states the solution to (6). Hence, we will illustrate the techniques involved by solving the five-point scheme (4). Consider first the homogeneous system of two-point boundary value problems:

$$U_k'' + \frac{1}{h^2}(-\frac{1}{12}U_{k+2} + \frac{4}{3}U_{k+1} - \frac{5}{2}U_k + \frac{4}{3}U_{k-1} - \frac{1}{12}U_{k-2}) = 0$$

$$U_o(x) = 0, \quad U_{n+1}(x) = 0; \quad k = 1,2,\ldots,n.$$

This is solved by separation of variables: $U_k(x) = r(k)v(x)$. Substituting into (7) and simplifying, we obtain

$$\frac{v''(x)}{v(x)} = \frac{r(k+2)-16r(k+1)+30r(k)-16r(k-1)+r(k-2)}{12r(k)h^2} = \delta^2 \qquad (8)$$

where δ is a constant. Note that the equation corresponding to (8) for the three-point scheme (3) is

$$\frac{v''(x)}{v(x)} = \frac{r(k+1)-2r(k)+r(k-1)}{-r(k)h^2} = \bar{\delta}^2, \qquad (9)$$

while that for the scheme of equation (6) is

$$\frac{v''(x)}{v(x)} = \frac{r(k+1)-2r(k)+r(k-1)}{-(r(k+1)+10r(k)+r(k-1))h^2/12} = \hat{\delta}^2. \tag{10}$$

The boundary conditions for (7) give $r(0)v(x) = 0$, and $r(n+1)v(x) = 0$, which imply $r(0) = 0$ and $r(n+1) = 0$. If we further require that the five-point scheme be compatible with the three-point shceme, we can establish two more boundary conditions. From (9),

$$r(k+1) - (2-h^2\bar{\delta}^2)r(k) + r(k-1) = 0.$$

Since $r(0) = 0$, we get $r(-1) = -r(1)$. And since $r(n+1) = 0$, $r(n+2) = -r(n)$.

We see, therefore, from (8) that the problem is to solve the difference equation:

$$r(k+2)-16r(k+1)+(30-12\delta^2h^2)r(k)-16r(k-1)+r(k-2) = 0, \tag{11}$$

$k = 1,2,\ldots,n$, $r(0) = r(n+1) = 0$, $r(-1) = -r(1)$; $r(n+2) = -r(n)$.

The general solution is of the form

$$r(k) = c_1\lambda_1^k + c_2\lambda_2^k + c_3\lambda_3^k + c_4\lambda_4^k,$$

where $\lambda_1,\lambda_2,\lambda_3$ and λ_4 are the roots of the polynomial

$$\lambda^4 - 16\lambda^3 + (30-12\delta^2h^2)\lambda^2 - 16\lambda+1.$$

Since this a reciprocal polynomial,

$$\lambda_2 = \lambda_1^{-1} \quad \text{and} \quad \lambda_4 = \lambda_3^{-1}$$

Putting $Z = 30 - 12\delta^2h^2$ and $\omega_i = \lambda_i + \lambda_i^{-1}$ $(i = 1,2,)$, an easy calculation gives

$$\begin{aligned} \omega_1 &= 8 + \sqrt{66-Z} \\ \omega_2 &= 8 - \sqrt{66-Z}\ . \end{aligned} \tag{12}$$

Solving for the roots λ_i $(i = 1,2,3,4)$, we obtain

$$\lambda_1 = \frac{\omega_1 + \sqrt{\omega_1^2 - 4}}{2}, \qquad \lambda_2 = \frac{\omega_1 - \sqrt{\omega_1^2 - 4}}{2}$$

$$\lambda_3 = \frac{\omega_3 + \sqrt{\omega_3^2 - 4}}{2}, \qquad \lambda_4 = \frac{\omega_3 - \sqrt{\omega_3^2 - 4}}{2}.$$

We now apply the boundary conditions given in (11) in order to find the constants C_i $(i = 1,2,3,4,)$:

$$r(0) = 0:\ C_1+C_2+C_3+C_4 = 0$$

$$r(n+1) = 0:\ C_1\lambda_1^{n+1} + C_2\lambda_2^{n+1} + C_3\lambda_3^{n+1} + C_4\lambda_4^{n+1} = 0$$

$$r(-1) = -r(1):\ C_1\lambda_1^{-1}+C_2\lambda_2^{-1}+C_3\lambda_3^{-1}+C_4\lambda_4^{-1} = -(C_1\lambda_1+C_2\lambda_2+C_3\lambda_3+C_4\lambda_4)$$

$$r(n+2) = -r(n):\ C_1\lambda_1^{n+2} + C_2\lambda_2^{n+2} + C_3\lambda_3^{n+2} + C_4\lambda_4^{n+2} = -(C_1\lambda_1^n+C_2\lambda_2^n+C_3\lambda_3^n+C_4\lambda_4^n).$$

However, $\lambda_2 = \lambda_1^{-1}$ and $\lambda_4 = \lambda_3^{-1}$, and we can simplify these four equations to:

$$C_1 + C_2 + C_3 + C_4 = 0$$

$$C_1\lambda_1^{n+1} + C_2\lambda_1^{-(n+1)} + C_3\lambda_3^{n+1} + C_4\lambda_3^{-(n+1)} = 0$$

$$C_1(\lambda_1+\lambda_1^{-1})+C_2(\lambda_1+\lambda_1^{-1})+C_3(\lambda_3+\lambda_3^{-1})+C_4(\lambda_3+\lambda_3^{-1}) = 0$$

$$C_1\lambda_1^{n+1}(\lambda_1+\lambda_1^{-1})+C_2\lambda_1^{-(n+1)}(\lambda_1+\lambda_1^{-1})+C_3\lambda_3^{n+1}(\lambda_3+\lambda_3^{-1}) + C_4\lambda_3^{-(n+1)}(\lambda_3+\lambda_3^{-1}) = 0.$$

These four equations can be solved for C_1,C_2,C_3,C_4 to give $C_1 = -C_2$; $C_3 = -C_4$ and

$$\lambda_1^{n+1}C_1 + \lambda_1^{-(n+1)}C_2 = 0$$

$$\lambda_3^{n+1}C_3 + \lambda_3^{-(n+1)}C_4 = 0.$$

Hence,

$$\lambda_1^{n+1} = \lambda_1^{-(n+1)} \quad \text{and} \quad \lambda_3^{n+1} = \lambda_3^{-(n+1)}.$$

That is,

$$\lambda_1^{2(n+1)} = 1 \quad \text{and} \quad \lambda_3^{2(n+1)} = 1.$$

This implies that

$$\lambda_1 = \exp(\pi is/(n+1)), \quad \lambda_3 = \exp(\pi is/(n+1)), \; s = 0,1,..,n.$$

Then

$$\begin{aligned} r(k) &= C_1\lambda_1^k + C_2\lambda_2^k + C_3\lambda_3^k + C_4\lambda_4^k \\ &= C_1(\lambda_1^k - \lambda_1^{-k}) + C_3(\lambda_3^k - \lambda_3^{-k}) \\ &= C_1[\exp(\pi isk/(n+1)) - \exp(\pi isk/(n+1))] \\ &\quad + C_3[\exp(\pi isk/(n+1)) - \exp(-\pi isk/(n+1))] \end{aligned}$$

$$= C_s \sin(\frac{\pi sk}{n+1}), \quad s = 0,1,\ldots,n.$$

Of course, the value s = 0 is the trivial solution, so the solution to (11) is:

$$r(k) = \sum_{s=1}^{n} C_s \sin(\frac{\pi sk}{n+1}) \ , \quad k = 1,2,\ldots,n.$$

The values of δ can now be calculated from (12):

$$8 + \sqrt{66-Z} = w_1 = \lambda_1 + \lambda_1^{-1} = \exp(\pi is/(n+1)) + \exp(-\pi is/(n+1))$$

$$= 2\cos(\frac{\pi s}{n+1}) \ , \qquad s = 1,2,\ldots,n.$$

Hence

$$8 + \sqrt{66-(30-12\delta^2h^2)} = 2\cos(\frac{\pi s}{n+1})$$

$$\sqrt{9+3\delta^2h^2} = \cos(\frac{\pi s}{n+1}) - 4$$

$$\delta_s^2 = \frac{\cos^2(\frac{\pi s}{n+1}) - 8\cos(\frac{\pi s}{n+1}) + 7}{3h^2}, \quad s = 1,2,\ldots,n.$$

To complete the solution to equation (8) we have

$$\frac{v''(x)}{v(x)} = \delta_s^2 \qquad s = 1,2,\ldots,n,$$

whose solution is

$$v_s(x) = A_s \exp(\delta_s x) + B_s \exp(-\delta_s x)$$

Finally, the solution to (7) is

$$U_k(x) = \sum_{s=1}^{n} \sin(\frac{\pi ks}{n+1})(A_s \exp(\delta_s x) + \exp(-\delta_s x)) \qquad (13)$$

$(k = 1,2,\ldots,n)$,

where

$$\delta_s^2 = \frac{\cos^2(\frac{\pi s}{n+1})-8\cos(\frac{\pi s}{n+1})+7}{3h^2}$$

Problem (4) can now be resolved by variation of parameters.

Note that the solution to (3) is derived in Berezin [2]:

$$U_k(x) = \sum_{s=1}^{n} \sin(\frac{\pi ks}{n+1})(A_s\exp(\bar{\delta}_s x) + B_s\exp(-\bar{\delta}_s x)) \qquad (14)$$

where

$$\bar{\delta}_s^2 = \frac{4\sin^2(\frac{\pi s}{2(n+1)})}{h^2}$$

The solution to (6) is also stated in Brezin:

$$U_k(x) = \sum_{n=1}^{n} \sin(\frac{\pi sk}{n+1})(A_s\exp(\hat{\delta}_s x) + B_s\exp(-\hat{\delta}_s x)) \qquad (15)$$

where

$$\hat{\delta}_s^2 = \frac{24\sin^2(\frac{\pi s}{2(n+1)})}{h^2(5+\cos(\frac{\pi s}{n+1}))} .$$

Section 4. There is a matrix technique that can also be used to solve the system of two-point boundary value problems. This will be illustrated by solving the homogeneous system corresponding to the three-point scheme (3):

$$U_k'' + \frac{1}{h^2}(U_{k+1}-2U_k+U_{k-1}) = 0 \qquad (16)$$

$$U_o(x) = 0, \quad U_{n+1}(x) = 0, \quad k = 1,2,\ldots,n.$$

In matrix form, we have: $U'' = AU$, where $U = (U_1, \dots, U_n)^T$ and A is the n×n matrix:

$$A = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & \dots & & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ & & & & & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

Now put $V_i = U_i'$ and $V_i' = U_i''$. We then have (16) written as a system of (2n) first order differential equations:

$$\begin{bmatrix} U_1' \\ \vdots \\ U_n' \\ V_1' \\ \vdots \\ V_n' \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ & \\ A & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ \vdots \\ U_n \\ V_1 \\ \vdots \\ V_n \end{bmatrix}$$

where 0 is the n×n zero matrix and I_n is the identity matrix.

The eigenvalues λ of the (2n)×(2n) matrix above are easy to determine. For if $U = (U_1, \dots, U_n)^T$ and $V = (V_1, \dots, V_n)^T$ then

$$\begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \lambda \begin{bmatrix} U \\ V \end{bmatrix}$$

implies $V = \lambda U$ and $AU = \lambda V$. That is, $AU = \lambda^2 U$.

The eigenvalues of A are known to be

$$\frac{2 - 2\cos(\frac{\pi s}{n+1})}{h^2}, \qquad s = 1, 2, \dots, n,$$

with corresponding eigenvectors

$$E_s = \begin{bmatrix} \sin(\frac{s\pi}{n+1}) \\ \sin(\frac{2s\pi}{n+1}) \\ \vdots \\ \sin(\frac{ns\pi}{n+1}) \end{bmatrix}, \qquad s = 1,2,\ldots,n.$$

Hence the λ must satisfy

$$\lambda^2 = \frac{2-2\cos(\frac{\pi s}{n+1})}{h^2} = \frac{4\sin^2(\frac{\pi s}{2(n+1)})}{h^2},$$

which implies

$$\lambda_s = \pm\frac{2}{h}\sin(\frac{\pi s}{2(n+1)}); \qquad s = 1,2,\ldots,n.$$

This leads to the same solution as (14).

Section 5. Is is apparent from the previous work that the method of lines leads to a system of unstable two-point boundary value problems. An exponential of the form $\exp(\delta x)$ occurs in each of the solutions (13), (14), (15). Let us analyze (13) in more detail. The worst case would occur if

$$\delta_s^2 = \frac{\cos^2(\frac{\pi s}{n+1})-8\cos(\frac{\pi s}{n+1})+7}{3h^2}$$

were as large as possible. This happens if $s = n$ and we would have

$$\delta_n^2 = \frac{\cos^2(\pi) - 8\cos(\pi) + 7}{3h^2}$$

$$\delta_n \cong 4/\sqrt{3h}.$$

We summarize this result with those of the other two schemes (3) and (6):

Scheme	
(4)	$4/\sqrt{3h} = 2.31/h$,
(3)	$2/h$,
(6)	$\sqrt{6/h} = 2.45/h$.

Section 6. Jones [4] considers the example

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ , \tag{17}$$

$$u(0,y) = u(1,y) = 0,\ u(x,b) = u(x,-b) = \sin(\pi x),$$

where b = .475. The exact solution is known to be

$$u(x,y) = \frac{\cosh(\pi y)\ \sin(\pi x)}{\cosh(\pi b)} \tag{18}$$

By symmetry considerations he is able to simplify the problem to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{19}$$

$$u(0,y) = u_x(.5,y) = 0,\ u(x,b) = \sin(\pi x),\ u(x,y) = u(x,-y).$$

He then solves the three-point method of lines scheme for (19) using the technique of section 2 together with Chebyshev polynomials for the particular solution. The solution to the three-point scheme is

$$U_k(x) = \frac{\cosh(k\theta)\sin(\pi x)}{\cosh((n+1)\theta)} \ , \tag{20}$$

where $\cosh(\theta) = 1 + \pi^2 h^2/2$.

The solution to the five-point scheme is much more complicated and of little interest. Details are available upon request.

It is important to prove that the Method of Lines solution (20) converges to the exact solution (18) as the number of lines n tends to infinity. To this end, consider

$$\cosh(\theta) = 1 + \pi^2 h^2/2 = 1 + \theta^2/2 + \theta^4/4! + \ldots$$

So for small h, $\theta \approx \pi h = \pi b/(n+1)$. Furthermore, $y_k = kb/(n+1) = kh$, which implies that $\pi y \approx k\theta$. Hence, as $n \to \infty$,

$$U_k(x) = \frac{\cosh(k\theta)\sin(\pi x)}{\cosh((n+1)\theta)} \to \frac{\cosh(\pi y)\sin(\pi x)}{\cosh(\pi b)}$$

The convergence question for arbitrary elliptic partial differential questions is much more complicated. Thompson [9] proves that under suitable hypotheses convergence is assured (see Theorem 5.1, page 36).

SUPORT [8] was used successfully to solve this example with both the three and five-point schemes. The results agree with those of Jones [4]. Other codes for two-point boundary value problems could be used as well. This author has had good results with COLSYS [1] and the IMSL routines DTPTB and DVCPR. Details are available from the author upon request.

Section 7. The Method of Lines can be an effective technique for solving nonlinear equations as well. We illustrate this by solving

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 20u - u^3 = 0$$

with boundary conditions u = 0 on the square $[0,\pi]\times[0,\pi]$. The system of boundary value problems is now non-linear:

$$U_k'' + \frac{1}{h^2}(U_{k+1} - 2U_k + U_{k-1}) + 20U_k - U_k^3 = 0.$$

It can be shown theoretically that this elliptic partial differential equation has nine distinct solution [5], page 213 . We used the IMSL routine DVCPR which was originally called PASVA3 [7]. By suitably adjusting the initial profiles, all nine solutions were successfully obtained. For a theoretical analysis of partial differential equations of this type reader is referred to [5].

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Department of Mathematics
University of Florida
Gainesville, FL. 32611
U. S. A.

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BEST APPROXIMATION IN VECTOR VALUED FUNCTION SPACES

by

ROSHDI KHALIL

ABSTRACT. Let T be the unit circle, and m be the normalized Lebesgue measure on T. If H is a separable Hilbert space, we let $L^\infty(T,H)$ be the space of essentially bounded functions on T with values in H. Continuous functions with values in H are denoted by C(T,H), and $H^\infty(T,H)$ is the space of bounded holomorphic functions in the unit disk with values in H. The object of this paper is to prove that $(H^\infty+C)(T,H)$ is proximinal in $L^\infty(T,H)$. This generalizes the scalar valued case done by Axler, S. et al. We also prove that $(H^\infty+C)(T,\ell^\infty)|H^\infty(T,\ell^\infty)$ is an M-ideal of $L^\infty(T,\ell^\infty)|H^\infty(T,\ell^\infty)$, and $V(T,\ell^\infty)$ is an M-ideal of $L^\infty(T,\ell^\infty)$ whenever V is an M-ideal of L^∞, where $V(T,\ell^\infty) = \{g \in L^\infty(T,\ell^\infty): \langle g(t),\delta_n\rangle \in V$ for all $n\}$.

§0. **Introduction.** Let T be the unit circle and m be the normalized Lebesgue measure on T. The space of p-Bochner integrable functions on T with values in a Banach space X is denoted by $L^p(T,X)$, $1 \leqslant p < \infty$. If $p = \infty$, then $L^\infty(T,X)$ is the space of essentially bounded functions on T with values in X. $L^p(T,X)$ are Banach spaces with the usual norms:

$$\|f\|_p = \left(\int_T \|f(t)\|^p dm\right)^{1/p}, \quad 1 \leq p < \infty$$

$$\|f\|_\infty = \text{ess.}\sup_t \|f(t)\|, \quad p = \infty.$$

We refer to, [3], for the basic structure of these spaces.

The problem of best approximation in L^1 and L^∞ is of much interest. The existence of best approximation in $C(\Omega)$ to functions in $L^\infty(\Omega)$, was proved in [9], where Ω is paracompact, and later, [13, 16], was generalized to the vector valued case, where X is assumed to be uniformly convex.

The existence of best approximation in $H^\infty+C$ to functions in $L^\infty(T)$ was proved in [2], where H^∞ is the space of bounded analytic functions in the unit disk and C is C(T), the space of continuous functions on T.

The object of this paper is to try to generalize the result in [2] to the vector valued case, where X is assumed to be a Hilbert space. Further, if V is a closed subspace of $L^\infty(T)$ and

$$V(T,\ell^\infty) = \{g \in L^\infty(T,\ell^\infty) : \langle g(t),\delta_n\rangle \in V \text{ for all } n\}$$

we prove that $V(T,\ell^\infty)$ is proximinal in $L^\infty(T,\ell^\infty)$ whenever V is proximinal in $L^\infty(T)$, noting that ℓ^∞ is not uniformly convex.

In Section 1, we prove some results on $(H^\infty+C)(T,H)$, similar to the scalar valued case. A representation of the dual of $L^\infty(T,H)$ is included. In Section 2, we prove that $(H^\infty+C)(T,H)$ is proximinal in $L^\infty(T,H)$. In Section 3 we give an example, ℓ^∞, where C(T,X) is proximinal in $L^\infty(T,X)$, without X being uniformly convex.

Throughout the paper, if X and Y are Banach spaces, then $X\hat{\otimes}Y$, $X\check{\otimes}Y$ denote the projective and the injective tensor products of X and Y, respectively. $L(X,Y)$ is the space of bounded linear operators from X into Y. The dual of a Banach space X is X^*. The complex numbers are denoted by $\mathbb{C}$.

§1. Vector valued function spaces. We let $H^p(T)$ be the classical Hardy spaces, $1 \leq p < \infty$, and $H^\infty(T)$ is the space of bounded analytic functions in the unit disk. One can consider $H^p(T)$ as closed subspaces of $L^p(T)$, $1 \leq p \leq \infty$, for which $\int_T f(t)e^{int}dm(t) = 0$ for all $n < 0$; $f \in L^p(T)$, [8].

If X is a Banach space, we define

$$H^p(T,X) = \{f \in L^p(T,X) \mid x^* \circ f \in H^p(T),\ x^* \in X^*\}.$$

This definition saves us the trouble of proving the existence of the radial limit if we were to consider $H^p(T,X)$ as functions on $D = \{z \in \mathbb{C}: |z| < 1\}$.

As in the scalar valued case one can prove:

THEOREM 1.1. $H^p(T,X)$ *is a closed subspace of* $L^p(T,X)$, $1 \leq p \leq \infty$.

Now, we take our Banach space to be a separable Hilbert space H. Consequently every element $f \in H^p(T,H)$ has a representation:

$$f = \sum_{n=1}^{\infty} f_n e_n, \quad f_n \in H^p(T),$$

(e_n) is some orthonormal basis of H. For p = 2, one has

$\|f\| = (\sum^{\infty}\|f_n\|^2)^{\frac{1}{2}}$, [7].

Since H is reflexive, $L^p(T,H)$, $1 < p < \infty$ are reflexive Banach spaces, and $L^p(T,H) = [L^{p'}(T,H)]^*$, $L^1(T,H)^* = L^\infty(T,H)$, [3]. From the definition of $H^p(T,H)$ and the fact that the $H^p(T)$, $1 \leqslant p \leqslant \infty$ are w^*-closed in $L^p(T,H)$ the following result follows:

LEMMA 1.2. *$H^p(T,H)$ is w^*-closed in $L^p(T,H)$, $1 < p \leqslant \infty$.*

Set $A(T,H) = H^\infty(T,H) \cap C(T,H)$. If $f \in A(T,H)$, then $f = \sum_{n=1}^{\infty} f_n e_n$, $f_n \in A(T)$, the disk algebra.

If Y is a subspace of the Banach space X, then for $x \in X$, $d(x,Y) = \inf\{\|x-y\| : y \in Y\}$. The proof of the following result is the same as for the scalar valued case and will be omitted, [6].

THEOREM 1.2. *Let $f \in C(T,H)$. Then $d(f,H^\infty(T,H)) = d(f,A(T,H))$.*

The subspace $H^\infty + C \subset L^\infty$ was proved to be a closed subspace, [6]. This result is true for the vector valued case:

THEOREM 1.3. *$H^\infty(T,H) + C(T,H)$ is a closed subspace of $L^\infty(T,H)$.*

THEOREM 1.4. *$[L^\infty(T,H)]^*$ is isometrically isomorphic to the space of finitely additive vector measures which vanish on m-null sets, equipped with the total variation norm.*

Proof. Since one can integrate a vector valued function against finitely additive vector measures, [4], the proof is similar to the scalar valued case, [5], and will be omitted. Q.E.D.

§2. **Best approximation in $L^\infty(T,H)$**.

Let X be a Banach space and Y be a closed subspace of X. For $x \in X$, an element $y \in Y$ is called a *best approximant of* x *in* Y if

$$\|x-y\| = d(x,Y) = \inf\{\|x-z\|, z \in Y\}.$$

If every element $x \in X$ has a best approximant in Y, then Y is called *proximinal* subspace of X. It is an interesting problem to determine whether a given closed subspace of Banach space is proximinal or not. In [2], Axler et al. proved that $H^\infty + C$ is proximinal in $L^\infty(T)$. Luecking, [12], gave a different proof for the same result, using the idea of M-ideals. We refer to the references in [2] for further results on best approximation. The object of this section is to prove that $(H^\infty+C)(T,H)$ is proximinal in $L^\infty(T,H)$, for every separable Hilbert space H.

A closed subspace Y of X is called an L-*summand* of X if there exists a subspace Y' of X such that $X = Y \oplus Y'$ and if $x = y + y' \in Y \oplus Y'$ then $\|x\| = \|y\| + \|y'\|$. Y is called an M-*ideal* of X if $Y^\perp$ is an L-summand of X^*, where $Y^\perp = \{\phi \in X^*: \phi(Y) = 0\}$.

THEOREM 2.1. *$(H^\infty+C)(T,H)|H^\infty(T,H)$ is an M-ideal of $L^\infty(T,H)|H^\infty(T,H)$.*

Proof. We identify $(L^\infty(T,H)|H^\infty(T,H))^*$ with $H^\infty(T,H)^\perp \subseteq L^\infty(T,H)^*$, and $((H^\infty+C)(T,H)|H^\infty(T,H))^\perp$ with $(H^\infty+C)(T,H)^\perp$

$\subseteq L^\infty(T,H)^*$, [6]. Let $F \in H^\infty(T,H)^\perp$. Being an element of $L^\infty(T,H)^*$, it follows from Theorem 1.8 that $F = \sum_{n=1}^{\infty} \mu_n e_n$ for some sequence (μ_n), $\mu_n \in L^\infty(T)^*$, and a (fixed) orthonormal basis (e_n) of H, where

$$<F(E),x> = \sum_{n=1}^{\infty} \mu_n(E)<e_n,x>$$

for every Lebesgue measurable set $E \subset T$, and $x \in H$. If X is the maximal ideal space of $L^\infty(T)$, then $L^\infty(T) \simeq C(X)$, the space of continuous function on X. Hence $L^\infty(T)^* \simeq M(X)$, the space of regular Borel bounded measures on X, [6]. Consequently, every element μ_n in the representation of F can be considered as a countably additive measure on X. Set

$$F_N = \sum_{n=1}^{N} \mu_n e_n.$$

Clearly $<F_N(E),x> \to <F(E),x>$ for each $\hat{m}$-measurable set E in X and each $x \in H$, where $\hat{m}$ is the lifting of the Lebesgue measure m on T to X, [6]. It follows from Grothendieck's theorem, [10], that

$$<F_N(g),x> \to <F(g),x>$$

for ecery $g \in L^\infty(T) \simeq C(X)$ and $x \in H$. Thus we can consider F as a countably additive vector measure on X. Let $F_a+F_s = F$ be the Lebesgue decomposition of F with respect to $\hat{m}$, [3], where F_a is $\hat{m}$-continuous and xF_s is singular with respect to m for all $x \in H$. One can easily show that if $F_a = \sum_1^\infty {}_n e_n$ and $F_s = \sum_1^\infty w_n e_n$, then ν_n is $\hat{m}$-continuous and w_n is orthogonal to $\hat{m}$ for all n. It follows from Pettis theorem, [3], that each ν_n is absolutely continuous with respecto to n. Since (e_n) is an orthonormal basis for H, it follows that $\mu_n = \nu_n + w_n$

is the Lebesgue decomposition of μ_n with respect to $\hat{m}$. Since $F \in H^\infty(T,H)^\perp$, it follows that $\mu_n \in H^{\infty\perp}$ for all n. Consequentely by the abstract Riesz theorem, [6], $w_n \in H^{\infty\perp}$ and $\nu_n \in H^{\infty\perp}$. Define the following map:

$$P: H^\infty(T,H)^\perp \to (H^\infty+C)(T,H)^\perp, \quad P(F) = F_s.$$

It follows from the above argument and theorem 2.4 of [12], that $F_s \in (H^\infty+C)(T,H)^\perp$. Further, since $\|F\| = \|F_a\|+\|F_s\|$, [3], one has P bounded, where $\|F\|$ is the total variation of F. To complete the proof of the theorem, we have to show that P is onto. Let $\phi \in (H^\infty+C)(T,H)^\perp$, and $\phi = \sum_{n=1}^{\infty} \mu_n e_n$ for some $\mu_n \in L^\infty(T)^*$. Then each $\mu_n \in (H^\infty+C)^\perp$, and by Theorem 2.4 of [12], μ_n is orthogonal to $\hat{m}$. Consequentely ϕ is singular to $\hat{m}$. Hence $P(\phi) = \phi$, and P is onto. Q.E.D.

THEOREM 2.2. $(H^\infty+C)(T,H)$ *is proximinal in* $L^\infty(T,H)$.

Proof. Since an M-ideal of a Banach space X is proximinal in X, [1], it follows that $(H^\infty+C)(T,H)/H^\infty(T,H)$ is proximinal in $L^\infty(T,H)/H^\infty(T,H)$. Lemma 1.2. together with a compactness argument imply that $H^\infty(T,H)$ is proximinal in $L^\infty(T,H)$. Thus for $f \in L^\infty(T,H)$, there exists $g \in (H^\infty+C)(T,H)$ such that

$$d(f,(H^\infty+C)(T,H)) = d(f-g,H^\infty(T,H)) = \|f-g-g_o\|$$

for some $g_o \in H^\infty(T,H)$. Thus $g+g_o \in (H^\infty+C)(T,H)$ is a best approximant of f. Q.E.D.

PROBLEM. Is Theorem 2.2 true if H is replaced by arbitrary Banach space? If not, what are those Banach spaces for which the result is true?

§3. Proximinality in $L^{\infty}(T,\ell^{\infty})$. Let ℓ^{∞} be the Banach space of bounded sequences, so that if $f \in \ell^{\infty}$, then $\|f\|_{\infty} = \sup_n |f(n)| < \infty$.

THEOREM 3.1. *Let* V *be a proximinal subspace of* $L^{\infty}(T)$. *Then* $V(T,\ell^{\infty})$ *is proximinal in* $L^{\infty}(T,\ell^{\infty})$.

Proof. Let $f \in L^{\infty}(T,\ell^{\infty})$. Since $L^{\infty}(T,\ell^{\infty}) = (L^1(T,\ell^1))^* = (\ell^1(N,L^1))^* = \ell^{\infty}(N,L^{\infty})$, it follows that

$$\|f\| = \|\sum_{n=1}^{\infty} f_n\delta_n\| = \sup_t \sup_n |f_n(t)| = \sup_n \sup_t |f_n(t)|$$
$$= \sup_n \|f_n\|_{\infty}.$$

Here, $\delta_n(j) = 1$ if $n = j$, $\delta_n(j) = 0$ otherwise. Consider the function $\tilde{f} = \sum_{n=1}^{\infty} \tilde{f}_n\delta_n$, where $\|f_n - \tilde{f}_n\| = d(f_n,V)$. If $g = \sum_{n=1}^{\infty} g_n\delta_n \in V(T,\ell^{\infty})$ then

$$\|f-\tilde{f}\| = \sup_n \|f_n - \tilde{f}_n\| \leqslant \sup_n \|f_n - g_n\| = \|f-g\|.$$

Hence $\tilde{f}$ is a best approximant of f in $V(T,\ell^{\infty})$. Q.E.D.

COROLLARY 3.2. *The spaces* $C(T,\ell^{\infty})$ *and* $(H^{\infty}+C)(T,\ell^{\infty})$ *are proximinal in* $L^{\infty}(T,\ell^{\infty})$.

Proof. Follows from Theorem 3.1 and the fact that $H^{\infty}+C$ and $C(T)$ are proximinal in $L^{\infty}(T)$. Q.E.D.

THEOREM 3.3. *If* V *is an* M*-ideal of* $L^{\infty}(T)$, *then* $V(T,\ell^{\infty})$ *is an* M*-ideal of* $L^{\infty}(T,\ell^{\infty})$.

Proof. Let $B(f^i,r_i)$ be any three open balls in $L^{\infty}(T,\ell^{\infty})$ with centers f^i and radii r_i such that $\bigcap_{i=1}^{3} B(f^i,r_i) \neq \phi$ and $V(T,\ell^{\infty}) \cap B(f^i,r_i) \neq \phi$ for $i = 1,2,3$. Let

$g^i \in V(T,\ell^\infty)$ such that $g^i \in B(f^i,r_i)$. If $g^i = \sum_{n=1}^{\infty} g_n^i \delta_n$ and $f^i = \sum_{n=1}^{\infty} f_n^i \delta_n$, then $\sup \| f_n^i - g_n^i \| < r_i$. Hence $g_n^i \in B(f_n^i, r_i) \subset L^\infty(T)$. By the same argument, we have $\bigcap_{i=1}^{3} B(f_n^i, r_i) \neq \phi$ for all n. It follows that $V \cap (\bigcap_{i=1}^{3} B(f_n^i, r_i) \neq \phi$, for all n, [1]. Let $g_n \in V \cap (\bigcap_{i=1}^{3} B(f_n^i, r_i))$. The function $g = \sum_{n=1}^{\infty} g_n \delta_n \in V(T,\ell^\infty)$. Further

$$\| g - f^i \| = \sup_n \| g_n - f_n^i \| < r_i .$$

Hence $g \in V(T,\ell^\infty) \cap (\bigcap_{i=1}^{3} B(f^i, r_i))$. It follows, [1], that $V(T,\ell^\infty)$ is an M-ideal of $L^\infty(T,\ell^\infty)$. Q.E.D.

Using the same argument of Theorem 3.3 one can prove:

THEOREM 3.4. $(H^\infty + C)(T,\ell^\infty) | H^\infty(T,\ell^\infty)$ *is an* M-*ideal of* $L^\infty(T,\ell^\infty) | H^\infty(T,\ell^\infty)$.

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Current Address:

Department of Mathematics
The University of Michigan
Ann Arbor, Michigan 48109
U. S. A.

Kuwait University
Dept. of Mathematics
P.O. Box 5969
Kuwait

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