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Variations on da Costa C Systems and Dual-Intuitionistic Logics I. Analyses of C_ω and CC_ω

Abstract. Da Costa's C systems are surveyed and motivated, and significant failings of the systems are indicated. Variations are then made on these systems in an attempt to surmount their defects and limitations. The main system to emerge from this effort, system CC_ω, is investigated in some detail, and "dual-intuitionistic" semantical analyses are developed for it and surrounding systems. These semantics are then adapted for the original C systems, first in a rather unilluminating relational fashion, subsequently in a more illuminating way through the introduction of impossible situations where *and* and *or* change roles. Finally other attempts to break out of impasses for the original and expanded C systems, by going inside them, are looked at, and further research directions suggested.

Newton da Costa will occupy a significant position in the future annals of logic as one of the founders, promoters, and chief early innovators of paraconsistent logic, the new logical paradigm that displaced classical logic. The main – by no means the only – paraconsistent logics proposed by da Costa, the C systems, offer elementary replacements for classical and intuitionistic systems where the data goes "bad", i.e. inconsistent. Unfortunately, for other purposes, the C systems have serious drawbacks, both philosophical and technical.[1] One of the aims of the present contribution is to indicate ways of overcoming some of these deficiencies, of improving the C systems.

Among the technological shortcomings of the C systems are these: First, the systems do not conform to the conditions of adequacy proposed for them; in particular, they do not meet the condition (IV of da Costa [6] p. 498) that they should contain principles of classical [or intuitionist] logic so far as these do not interfere with other conditions of adequacy, specifically requirements of

* I have been much helped through comments and corrections, criticism and proofs, generously offered by Newton da Costa, Chris Mortensen, John Slaney and Igor Urbas. A part of my role is no more than that, not of under-labourer, but of organiser of their results.

The paper, still incomplete at the edges, has been a good while in the making. It was first drafted in around 1980 for a Festschrift for Newton da Costa; hence the introductory remarks, which it has since seemed should stand. More recent literature on the C systems will be taken in account in a sequel.

[1] These are considered in detail in [15] and [16], and additional criticism may be found in [19]. The C systems are presented in several publications of da Costa, e.g. the basic survey paper [6].

paraconsistency. Secondly, the systems do not contain a proper primitive negation connective, the primitive "negation" $\neg$ failing requirements for a negation determinable (so at least Priest and Routley argue in [15]). Thirdly, the systems appear to lack natural and elegant algebraic and semantical formulations, largely because they fail to guarantee intersubstitutivity of equivalents (see Mortensen [13]). In some cases, notably that of system C_ω, these drawbacks could be avoided, or mitigated in the case of the first objection, by strengthening the systems. As a separate issue, there are powerful reasons (again advanced in [15]) for weakening the positive basis – Hilbert's positive logic (positive intuitionism) – on which the dual-intuitionistic C systems are based.

Allowing for both types of variations, weaking of the underlying positive logic and strengthening of the negation superstructures, gives the *general* da Costa logics, some of which are here investigated by semantical means. In due course, some of the original C systems, and direct variations upon them, will also be considered and new and neater analyses provided.

1. The C systems and some variations on C themes

The basic C system C_ω simply adds to positive (intuitionistic) logic, H, with primitive connectives $\{\rightarrow, \&, \vee\}$, and perhaps $\neg$, two axioms concerning negation $\neg$; namely, Excluded Middle, EM, $A \vee \neg A$ and Dialectic Double negation DDN, $\neg\neg A \rightarrow A$. That is, C_ω has the following postulates:

$A \rightarrow B \rightarrow A$	$A \rightarrow B \rightarrow . A \rightarrow (B \rightarrow C) \rightarrow . A \rightarrow C$
$A \& B \rightarrow A$	$A \& B \rightarrow B$
$A \rightarrow . B \rightarrow (A \& B)$	$A \rightarrow A \vee B$
$B \rightarrow A \vee B$	$A \rightarrow C \rightarrow . B \rightarrow C \rightarrow . A \vee B \rightarrow C$
$A \vee \neg A$	$\neg\neg A \rightarrow A$

A, $A \rightarrow B/B$ (i.e. Modus Ponens is the sole rule).

The system C_ω was apparently obtained by the very simple strategy of deleting the (allegedly) paraconsistent trouble-making scheme of Reductio, $A \rightarrow B \rightarrow . A \rightarrow \neg B \rightarrow . \neg A$, from Kleene's list of postulates for propositional calculus S ([11] p. 82). In Brasil there were thought to be two main faults in classical logic from a paraconsistent stance, that it contained Reductio and that it included Non-Contradiction, $\neg(A \& \neg A)$. But, conveniently, the latter was derived from the former in Kleene, so it sufficed to remove just Reductio.[2]

C_ω is, in certain respects the dual of intuitionistic sentential logic I (of Heyting [10]). Both extend positive logic H (i.e. $C_\omega - \{A \vee \neg A, \neg\neg A \rightarrow A\}$). But whereas intuitionism rejects Excluded Middle and asserts Non-Contra-

[2] On this historical note to C_ω see Asenjo [2].

diction, $\neg(A \& \neg A)$, system C_ω and the da Costa C systems generally assert Excluded Middle and reject Non-Contradiction (the latter as a requirement of adequacy); and whereas intuitionism asserts $A \rightarrow \neg\neg A$ and rejects DDN, the C systems do the reverse.[3] This duality also takes semantical shape: whereas intuitionism is essentially focussed on evidentially incomplete situations excluding inconsistent situations, the C systems admit inconsistent situations but remove incomplete situations (hence the semantics of da Costa and Alves [7] for C_n, $1 \leqslant n < \omega$, in terms of maximal non-trivial, but possibly inconsistent, theories). Should not an adequate logical theory allow both, both inconsistent situation *and* evidentially incomplete situations? This indicates the route through weakened positive logic to relevant logics (see [15] Chapter 3).

The basic system C_ω does not enable the classical behaviour of apparently classical wff to be derived. The presumption (based on now questioned consistency assumptions, which are rejected in leading relevant paraconsistent systems) is that wff divide into two classes: nonclassical wff, requiring paraconsistent treatment, which are such that both they and their negations hold, and classical wff that are not like this and so conform to Non-Contradiction. That is, A is *classical* (in *one* obvious sense) if not both A and $\neg A$; in symbols $A^\circ = \neg(A \& \neg A)$.

System C_1 adds to C_ω the theses that "classical wff" do behave classically, in conforming to the omitted Reductio postulate, and that formation conditions are respected classically, that is, that compounds of classical wff are classical. So it is that C_1 results from C_ω by addition of these postulates:

$$B^\circ \rightarrow .A \rightarrow B \rightarrow .A \rightarrow \neg B \rightarrow .\neg A \qquad A^\circ \& B^\circ \rightarrow (A \& B)^\circ$$
$$A^\circ \& B^\circ \rightarrow (A \vee B)^\circ \qquad A^\circ \& B^\circ \rightarrow (A \rightarrow B)^\circ$$

The remaining compounding principle $A^\circ \rightarrow (\neg A)^\circ$, a postulate of an earlier formulation of C_1, proved derivable (see da Costa [6] p. 500, n. 2).

C_1 exhibits some anomalies; for example, the intuitionistically invalid (and unjustifiable) Peirce's "law", $((A \rightarrow B) \rightarrow A) \rightarrow A$, is provable, with the result that the positive logic of C_1 turns out to be just positive classical logic, and whatever A, $A \& \neg A$ is classical, i.e. $(A \& \neg A)^\circ$ holds, defeating certain paraconsistent objectives. Da Costa found a sequence of systems which progressively mitigate such anomalies. The point is that the "classical" operation can be iterated without collapse. Define $A^n =_{\mathrm{df}} A^\circ$ "n"$^\circ$ times, for

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[3] There are deeper grounds for adopting the dual forms tied to the 'laws' of dialectic, especially the law of the negation of negation. Given the enriching role of negation, $A \rightarrow \neg\neg A$ appears to fail, but DDN stands: see further [18] and [8].

The duality and possibility of dual-intuitionistic logics, connected with dialetic, was the first noted by Popper [14]; but he developed no logics of the type, and indeed erroneously concluded that only extraordinarily weak logics of the type were possible.

$n \leqslant 1$; and to collect together all iterated forms up to n, define $A^n =_{\mathrm{df}} A^1 \& \ldots \ldots \& A^n$. System C_n, for $1 \leqslant n < \omega$, simply replaces super$^\circ$ in the postulates for C_1 by super$^{(n)}$. That is, C_n results from C_ω by addition of the following:

$$B^{(n)} \to .A \to B \to .A \to \neg B \to .\neg A \qquad A^{(n)} \& B^{(n)} \to (A \& B)^{(n)}$$
$$A^{(n)} \& B^{(n)} \to (A \vee B)^{(n)} \qquad A^{(n)} \& B^{(n)} \to (A \to B)^{(n)}.$$

Illuminating motivation for the higher C_n systems has, however, yet to be produced. Moreover there is nothing sacrosanct about the C_n way of constructing the hierarchy, as Bunder's alternative hierarchies (in [5]) help to show. So there remains a real point in casting about in different directions for motivated enlargements on C_ω and interpretationally satisfactory dual-intuitionistic systems.

A fundamental principle missing from the C systems, so algebraic and semantic investigations soon disclose, is that of (implication-guaranteeing)[4] intersubstitutivity of complication, i.e.

SE. $A \to B,\ B \to A / D(A) \to D(B)$,

where $D(A)$ is some wff containing A and $D(B)$ results from $D(A)$ by replacing one (derivatively, zero or more) occurrence of A by B. It is this omission that renders algebraic analyses, of C systems, while not impossible, so problematic (see further Mortensen [13]). Yet the apparatus for an inductive proof of SE is available in C systems, *except* for negation. One obvious initial addition is accordingly of Rule Contraposition.

RC. $A \to B / \neg B \to \neg A$

Call the resulting systems *CC systems*, i.e. $CC_n = C_n + \mathrm{RC}$, for $1 \leqslant n < \omega$.

Addition of RC, a main strategy in what follows, also makes it a straightforward matter to obtain an elegant semantics, extending that for positive logic, for systems such as CC_ω. In fact, the semantics works by providing conditions for the more basic system CC, i.e. $H + \mathrm{RC}$, and indeed works, with adjustments to the implicational evaluation rule, for extensions by RC of positive logics much weaker that I, including weaker positive strict and positive relevant logics (see also Basic Contraposition Logic, *BCL*, in [20] Chapter 2). This indicates a main semantical route taken in subsequent sections.

However the approach comes to grief with CC_1, which collapses to classical logic S i.e. (C in da Costa's notation). There is the further point

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[4] In modal terms, this is an $S2$ form of substitutivity, which contrasts with the weaker $S1$ form: $A \to B,\ B \to A,\ D(A)/D(B)$. The weaker form has hitherto rendered semantical and algebraic investigation less tractable. Modal analogies (of $\neg$ with $\Box$) in fact provide a main semantical key in what follows.

in casting about for other variations on those C_n systems stronger than C_ω and CC_ω.

PARTIAL SEPARATION THEOREM. *Though* $CC_\omega \neq S$, $CC_n = S$ *for* $1 \leqslant n < \omega$.

PROOF.[5] The following matrices, for C_ω, but with sole designated value 1, satisfy all principles of CC_ω but not further theorems of S such as $\neg A \rightarrow .A \rightarrow B$:

$\rightarrow$	1	2	0	$\neg$
*1	1	2	0	0
2	1	1	0	1
0	1	1	1	1

&	1	2	0
*1	1	2	0
2	2	2	0
0	0	0	0

$\vee$	1	2	3
*1	1	1	1
2	1	2	2
0	1	2	0

Thus valuations of & and $\vee$ are simply given by meet and join respectively on the augmented classical lattice 0-2-1; and $\rightarrow$ and $\neg$ are assessed classically on classical values $\{1,0\}$. To falsify $\neg A \rightarrow .A \rightarrow B$, assign A value 2 and B value 0. To validate RC suppose $\neg B \rightarrow \neg A$ is undesignated and so takes value 0. It cannot take value 2 since then $\neg A$ would have value 2 which is impossible. Then since $\neg B$ never takes value 2, $\neg B$ has value 1 and $\neg A$ value 0; so A has value 1 and B value 2 or 0. But then in either case $A \rightarrow B$ has an undesignated value.

The following sequence of theorems is derivable in CC_ω: $(A \& \neg A) \rightarrow A$ (from C_ω), $\neg A \rightarrow \neg(A \& \neg A)$ (by applying RC), $\neg A \rightarrow A^\circ$ (by definition), $A \& \neg A \rightarrow A \& (\neg A \& A^\circ)$ (applying C_ω), $A \& \neg A \rightarrow A \& \neg^* A$, where $\neg^*$, defined $\neg^* B =_{\mathrm{df}} \neg B \& B^\circ$, is the so-called strong negation of C_1. But in C_1, $\neg^*$ has all properties of classical negation (see [6] p. 500, Theorem 5). Hence in CC_1, $\neg^*(A \& \neg^* A)$, whence by definition of $\neg^*$, $\neg(A \& \neg^* A)$. But applying RC to the end member of the theorem sequence above, $\neg(A \& \neg^* A) \rightarrow \neg(A \& \neg A)$; so detaching $\neg(A \& \neg A)$. But the addition of this principle of Non-Contradiction to C_1 results in S, since it restores Reductio, $A \rightarrow B \rightarrow .A \rightarrow \neg B \rightarrow .\neg A$, and so Kleene's full list of postulates for S (see [6] p. 499). Hence $CC_1 = S$.

A similar result, $CC_n = S$, extends to C_n systems with $1 < n < \omega$. As da Costa observes ([6] p. 501), "in general, the results valid for C_1 can be adapted to apply to C_n, $2 \leqslant n < \omega$". For the adaption here, it is enough to define a strong negation, $\neg^{(n)}$, thus $\neg^{(n)} A =_{\mathrm{df}} \neg A \& A^{(n)}$, and to show $\neg A \rightarrow A^{(n)}$ in CC_n. For $\neg^{(n)}$ has the properties of classical negation ([6] p. 503), so the remainder of the argument is as before for CC_1. Proof of $\neg A \rightarrow A^{(n)}$, i.e. $\neg A \rightarrow A^1 \& A^2 \& \ldots \& A^n$, is by induction. It is shown, given $\neg A \rightarrow A^{m-1}$, that $\neg A \rightarrow A^m$; from which the result can be assembled. Suppose then $\neg A \rightarrow A^{m-1}$.

[5] Leading details of the proof were indicated by Loparic and da Costa.

By RC, $\neg A^{m-1} \to \neg\neg A$. But $\neg\neg A \to A$; so $\neg A^{m-1} \to A$. Hence A^{m-1} & $\neg A^{m-1} \to A$, whence by RC again, $\neg A \to \neg(A^{m-1} \& \neg^{m-1} A)$, i.e. $\neg A \to A^m$.

Despite its initial appeal, CC_ω fails to meet one of da Costa's, requirements of adequacy. As Urbas's investigations have shown, Non-Contradiction $\neg(A \& \neg A)$ is a theorem of CC_ω, and so of all its extensions. Proof is simply as follows: $\neg A \to \neg(A \& \neg A)$, upon contraposing $A \& \neg A \to A$. Similarly, $\neg\neg A \to \neg(A \& \neg A)$, whence by disjunction composition, $\neg A \vee \neg\neg A \to \neg(A \& \neg A)$. Hence, by Excluded Middle (or by the weaker K principle $\neg A \vee \neg\neg A$); $\neg(A \& \neg A)$.

NC Lemma. *$\neg(A \& \neg A)$ is a theorem of CC_ω.*

Hence again, $CC_1 = S$. More generally, in the presence of RC and basic principles for & and $\vee$, the two traditional "laws of thought", Non-Contradiction and Excluded Middle hang out together. For example, given $CC + \{\neg\neg A \to A\}$, if follows in dual fashion, $\neg(A \& \neg A) . \to A \vee \neg A$. For $A \to (A \vee \neg A)$, whence $\neg(A \vee \neg A) \to \neg A$; analogously $\neg(A \vee \neg A) \to \neg\neg A$, whence by DDN $\neg(A \vee \neg A) \to A$; so composing $\neg(A \vee \neg A) \to A \& \neg A$. Then contraposing and DDN yield the result.

The NC lemma also indicates that other rule-weakening routes with appeal in varying C systems are blocked. Observe that the strengthening of RC to theorem-status, viz $A \to B \to . \neg B \to \neg A$ (equivalently in the context its adoption as a strong rule, $A \to B \vdash \neg B \to \neg A$), would collapse C_ω to S again; that is, $C_\omega + A \to B \to . \neg B \to \neg A = S$. For firstly then, $A \to B$, $A \to \neg B \vdash A \to \neg A$, since from $A \to B$, $A \to \neg B$ it follows $\neg B \to \neg A$, $A \to \neg B$, whence $A \to \neg A$. But $(A \to \neg A) \to \neg A$ in C_n for $1 \leqslant n \leqslant \omega$. For $(A \to \neg A) \to . (\neg A \to \neg A) \to . A \vee \neg A \to . \neg A$; than permute out theorems of C_n. So $A \to B$, $A \to \neg B \vdash \neg A$, whence results by the Deduction Theorem full Reductio, and so Kleene's axiomatisation of S again. Such a collapse is avoided in CC_ω because Contraposition applies only in *rule* form.

A similar rule amendment may appear to be what is required in properly formulating the theme that classical wff behave classically. It is clear that the postulates of C_1 concerning super-zeroed wff (i.e. the postulates that distinguish C_1 from C_ω), especially the first, could be pulled back to rule connections. And certainly it suffices, more drastically, to excise the first postulate, to remove Reductio. For it is this postulate, $B^\circ \to . A \to B \to . A \to \neg B \to . \neg A$, that can be singled out as producing the collapses of CC_1 to S.

Lemma. $CC_1 - (B^\circ \to . A \to B \to . A \to \neg B \to . \neg A) \subset S$.

Proof. It is certainly contained in S since the addition of full Reductio takes it to S. To show containment is proper extend the first part of the previous theorem. The matrices introduced there satisfy all the classical compounding principles, since A° always takes the designated value 1. But they falsify $B^\circ \to .A \to B \to .A \to \neg B \to .\neg A$, for the following assignments: $A = 1$, $B = 2$. It is appealing then to consider the systems resulting from CC_ω by addition of the same postulates as C_n adds to C_ω except that in place of the first, qualified Reductio, appears the rule.

QR. $B^{(n)}/A \to B \to .A \to \neg B \to \neg A$.

But the argument to the NC lemma shows in effect that this route goes nowhere of paraconsistent interest. Because the premisses of the rule are delivered, the systems collapse again to S.

There are three ways to go, given that we are not interfering with fundamental & and $\vee$ principles: abandonment of Excluded Middle, abandonment of da Costa's requirement of adequacy, or abandonment of RC and therewith the duality of inconsistency with incompleteness. We have elsewhere argued in favour of the removal of da Costa's requirement of adequacy, a main reason being that the investigation of inconsistency does not require the removal of Non-Contradiction (see [15] and [18]). We turn to semantical investigations of the first two options.

2. Semantics for CC and for extensions such as CC_ω

The duality with respect to intuitionism may be suggestively extended to the semantical analysis. Whereas negation in intuitionism is assessed from a modal perspective in terms of impossibility, as $\sim\Diamond$, a dual role would analyse it through nonnecessity, as $\sim\Box$, i.e. equivalently $\Diamond\sim$. In terms of a more fashionable analogy, whereas intuitionistic negation is failure, more precisely what leads to or implies failure (i.e. $\neg A$ iff $A \to F$), da Costa negation is instead what success doesn't lead to (i.e. $\neg A$ iff $T \nrightarrow A$; as the characterisation already involves a negation, elimination in favour of a constant T is not preshadowed). The modal analogy suggests an evaluation rule for $\neg$ of the form:

$$I(\neg A, a) = 1 \text{ iff, for some } b \in K, Sab \text{ and } I(A, b) = 0,$$

with S a two-place relation on world set K satisfying conditions as yet to be determined. In fact such a negation rule emerges on an alternative translation-based semantics for intuitionistic logic, so there is little doubt but that is represents a negation (determinable).

There is a ready-made semantics to hand for positive logic, H, namely, in the form given by Beth [13], or more conveniently Kripke [12], through world structures of the form $M = (G, K, R)$ where $G \in K$ and R is an ordering relation on K. For the basic system CC (i.e. $H + RC$), this semantics can simply be built upon (at the same time, anticipating subsequent developments, the notation is adapted to that of [20]).

A *CC model structure* (*m.s.*) M is a structure $(T, K, \leqslant, S)$, where $T \in K$ and $\leqslant$ and S are two-place relations on K such that the ordering relation $\leqslant$ is reflexive and transitive (and optionally antisymmetric), and the further relation S satisfies universally the following condition:

si) where $a \leqslant c$ and Sab then Scb.

It is because of this latter condition, required for hereditariness below, that S must be distinguished from the ordering relation $\leqslant$. Only under restrictive paraconsistency-upsetting conditions can S be equated with $\leqslant$. For negation-free extensions of H, S simply drops out.

A *valuation* v *in* M is a function v on initial wff and worlds with values in $\{1, 0\}$, such that

vi) where $a \leqslant b$ and $v(p, a) = 1$ then $v(p, b) = 1$, for every $a, b \in K$ and every initial wff p.

Interpretations effectively extend valuations to all wff, inductively as follows:

$I(p, a) = v(p, a)$;
I&) $I(A \& B, a) = 1$ iff $I(A, a) = 1 = I(B, a)$;
Iv) $I(A \vee B, a) = 1$ iff $I(A, a) = 1$ or $I(B, a) = 1$;
I→) $I(A \to B, a) = 1$ iff for every b in K such that $a \leqslant b$ and $I(A, b) = 1$ then $I(B, b) = 1$;
I¬) $I(\neg A, a) = 1$ iff for some b in K such that $Sab\, I(A, b) = 0$

Remaining definitions, of truth, validity and so forth, are standard (see e.g. [20] p. 302). Preliminaries to the soundness theorem for CC are a simplification of those for relevant logics (as in [20]).

HEREDITARINESS LEMMA. *For every* $a, b \in K$ *and every wff* A, *where* $a \leqslant b$ *and* $I(A, a) = 1$ *then* $I(A, b) = 1$.

PROOF. The induction steps for connectives & and $\vee$ are immediate and use no conditions on $\leqslant$; that for $\to$ uses however transitivity of $\leqslant$.

ad ¬. Suppose $a \leqslant b$ and $I(\neg A, a) = 1$; to show $I(\neg A, b) = 1$, i.e. for some c, Sbc and $I(A, c) = 0$. But as $I(\neg A, a) = 1$, for some c, Sac and $I(A, c) = 0$, and by si) Sbc.

SOUNDNESS THEOREM. *Where* A *is a theorem of* L *then* A *is* L-*valid, for* L *either* CC *or* H.

PROOF is by the usual induction. A couple of illustrative cases are given to indicate the machinery at work.

ad $A \to .B \to A$. To show $I(A \to .B \to A, T) = 1$ for arbitrary interpretation I in arbitrary m.s. M, suppose for arbitrary a in K, $I(A, a) = 1$ with $T \leqslant a$. Then to show, as required, $I(B \to A, a) = 1$, suppose further $a \leqslant b$ and $I(B, b) = 1$. By the hereditariness lemma, $I(A, b) = 1$ establishing $I(B \to A, a) = 1$. (Note, by contrast with relevant logic, the extent of information not used.)

ad $A \to B / \neg B \to \neg A$. Suppose $\neg B \to \neg A$ is not CC valid. That is, on some interpretation I in some m.s. $M = (T, K, \leqslant, S) I(\neg B \to \neg A, T) \neq 1$, i.e.

for some a with $T \leqslant a$, $I(\neg B, a) = 1$ and $I(\neg A, a) \neq 1$. Then for some b, Sab and $I(B, b) \neq 1$ and also $I(A, b) = 1$. Consider now the m.s. $M' = (b, K, \leqslant, S)$ with base b in place of T. Since $b \leqslant b$, $I(A, b) = 1$ and $I(B, b) \neq 1$, $I(A \to B, b) \neq 1$. Thus $A \to B$ is not CC-valid. (Note verification depends on reflexivity of $\leqslant$.)

For completeness, much labour can be saved by borrowing extensively from proofs of completeness for other negation extensions of H (such as those of Johansson and others, treated by Segerberg [21], or those of Fitch and Nelson, treated in [17])[6]. In particular, notice that the basic result in Segerberg (the optimistic Theorem 2.4) makes no essential use of negation, but holds equally well for H, and also for different negation extensions of system H than those extending J (*where* suitable model structures are defined). We outline the main elements of the argument for any such suitable logic L extending H, filling out requisite details for system CC.

A set S of wff is *L-full* iff S is regular (i.e. all theorems of L are in S), nontrivial (i.e. not all wff are in S), prime (i.e. whenever $A \vee B$ is in S either A is in S or B is in S), and closed under modus ponens (i.e. whenever A and $A \to B$ are in S so also is B). L-full sets (or, in RLR terms, full L-theories) provide the worlds of the canonical model structure, now to be defined. A *canonical* L m.s. M_L *at* L-full set T_L is the structure $(T_L, K_L, \subseteq, S_L)$, where K_L is set of L-full sets, $\subseteq$ is set inclusion (defined on L-full sets), and S_L is defined thus: For a, b in K_L, $S_L\, ab$ iff for every wff A if $\neg A$ is not in a then A is in b, i.e. iff $(A)(\neg A \notin a \supset A \in b)$. In place of $\subseteq$, $\leqslant_L$ may be defined as follows: for $a, b \in K_L$, $a \leqslant_L b$ iff whenever $A \to B \in a$ and $A \in b$ then $B \in b$. For negation free systems such as negation unextended H, S_L again simply drops out.

It is immediate that a canonical H m.s is an H m.s. Since $a \subseteq c$ and $S_L\, ab$ guarantee $S_L cb$, a canonical CC m.s. is a CC m.s. A canonical valuation is defined: $v_L(A, a) = 1$ iff $A \in a$. Immediately, by properties of inclusion again, it is a valuation. The membership definition yields the inductive basis for the main lemma used in establishing completeness.

CANONICAL LEMMA. *For every $a \in K_L$ and every wff A, $I(A, a) = 1$ iff $A \in a$, where I is the canonical valuation extending v_L.*

PROOF is by induction on the structure of wff. The induction steps apply, and those for wff of the forms $A \& B$ and $A \vee B$ follow directly from, the following properties of L-full sets. For any full L-theory S, $A \& B \in S$ iff $A \in S$ and $B \in S$, $A \vee B \in S$ iff $A \in S$ or $B \in S$, and further S is closed under

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[6] Strong completeness can also be proved by elaboration of these methods, or by the adaptation of [20] Chapter 4, to be applied subsequently to establish completeness of systems with relevant positive logics.

L-derivability, i.e. if $S \vdash_L A$ then $A \in S$. Quite generally, set U is *L-derivable* from set S, written $S \vdash_L U$ iff, for some $A_1, \ldots, A_n \in S$ and $B_1, \ldots, B_m$ in U, $\vdash_L A_1 \& \ldots \& A_n \rightarrow B_1 \vee \ldots \vee B_m$. By virtue of the portation principles of H, $S \vdash_L B$ iff for some $A_1, \ldots, A_n$ in S, $A_1 \rightarrow \ldots \rightarrow . A_n \rightarrow B \in L$.

To accomplish the induction steps for implicational and negated wff, the following extension lemma (Lemma 2.2 in [21]) is put to work:

L-FULL EXTENSION LEMMA. *For any wff A, A is L-derivable from nontrivial set S iff A is in all L-full extensions of S, i.e. for every $a \in K_L$, if $S \subseteq a$ then $A \in a$.*[7]

ad $\rightarrow$. A is of the form $B \rightarrow C$. What is to be shown reduces, applying the induction hypothesis, to the biconditional:

$$B \rightarrow C \in a \text{ iff } (b \in K_L)(a \subseteq b \,\&\, B \in b \supset C \in b).$$

One half follows by induction and modus ponens closure. For the converse, suppose $B \rightarrow C \notin a$, and consider $b_1 = a \cup \{B\}$. Then $B \in b_1$, $a \subseteq b_1$, but C is not L-derivable from b_1. For suppose otherwise $a \cup \{B\} \vdash_L C$. Then a $\vdash_L B \rightarrow C$, whence $B \rightarrow C \in a$, contradicting the initial supposition. By L-full extension then, there is some $b \in K_L$ such that $b_1 \subseteq b$, whence $A \in b$ and $a \subseteq b$, but $C \notin b$. That completes details of the interpretation lemma for H.

ad $\neg$. A is of the form $\neg B$. What is to be shown reduces to the biconditional:

$$\neg B \in a \text{ iff, for some } b \text{ in } K_L,\ S_L ab \text{ and } B \notin b,$$

i.e. iff $(Pb \in K_L)$. $(C)(\neg C \notin a \supset C \in b) \,\&\, B \notin b$. Suppose the latter holds. Then for some b, $B \notin b$ and $\neg B \notin a \supset B \in b$, whence $\neg B \in a$. Conversely suppose $\neg B \in a$, and define $b_1 = \{A\colon \neg A \notin b_1\}$, so B is not L-derivable from b_1. By L-full extension there is a b in K_L such that $b_1 \subseteq b$ but $B \notin b$. Since $b_1 \subseteq b$, $S_L ab$.

COMPLETENESS THEOREM. *Where A is L-valid then A is a theorem of L, for L either CC or H.*

PROOF. Suppose A is not a theorem of L. Then A is not L-derivable from L, so there is an L-full set T_L to which A does not belong. Form the canonical m.s. at T_L and take the canonical valuation at it. By the canonical lemma, $I(A, T_L) \neq 1$, whence A is not L-valid.

Given semantics for CC, modelling changes can be wrung in the usual fashion. In particular, the further axiom schemes of CC_ω have the following,

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[7] The lemma can be strengthened, with set of wff U replacing A. This is one route to stronger completeness results.

independent, modelling conditions:

DDN	$\neg\neg A \rightarrow A$	mDDN.	whenever Sab there is some c such that Sbc and $c \leqslant a$; or better, where Sab then Sba.
EM[X]	$A \vee \neg A$.	mEM.	STT, or Saa.

Proof of adequacy of the semantics reduces to these steps:

ad DDN. Suppose, on the contrary, DDN is invalid. Then for some world a in some model, $I(\neg\neg A, a) = 1$ but $I(A, a) \neq 1$. By the first, for some b, Sab and $I(\neg A, b) \neq 1$. Hence for every c such that $Sbc\, I(A, c) = 1$. By the modelling condition, since Sab for some c_1, Sbc_1 and $c_1 \leqslant a$. Hence, as $I(A, c_1) = 1$ by hereditariness $I(A, a) = 1$, which is impossible.

ad EM. Suppose otherwise $I(A \vee \neg A, T) \neq 1$. Then $I(A, T) \neq 1 \neq I(\neg A, T)$. So for every b such that $STb\ I(A, b) = 1$. Since STT, $1 \neq I(A, T) = 1$ which is impossible.

ad mEM. It suffices to show for arbitrary A such that $\neg A \notin T$, $A \in T$. Since $\neg A \vee A \in T$, $\neg A \in T$ or $A \in T$, but $\neg A \notin T$, whence the result.

ad mDDN. Suppose $S_T ab$, and further that $\neg A \notin b$. As $S_T ab$, whatever C, if $\neg C \notin b$ then $\neg\neg C \in a$. Hence $\neg\neg A \in a$, whence by DDN, $A \in a$. Reassembling, $S_T ba$.

In summary, a CC_ω m.s. is a CC m.s. where S is reflexive and symmetric. CC_ω has an unexpectedly elegant semantics.

It would no doubt simplify these modellings if S could be removed. An obvious way to do this would be to define Sab as $b \leqslant a$, which guarantees soundness (as also transitivity) at once since condition si) is immediate. Completeness would then require both EM and the negative paradox principle (ex falso quodlibet)

EFQ. $\neg B \rightarrow . B \rightarrow A$.

For consider a direct argument to establish the inductive step for $\neg$ in the interpretation lemma (indirection using $b \subseteq a$ iff $S_L ab$ yields the same result). It has to be shown that

$$\neg B \in a \text{ iff } (Pb \in K_L)(b \subseteq a \,\&\, B \notin b).$$

The argument from right to left uses only LEM and so succeeds in CCX, i.e. CC+EM. However the argument from left to right appears to require exclusion in worlds – a step from $\neg B \in a$ to $B \notin a$ and so EFQ.

EFQ collapses CCX into classical logic S. By EFQ, $B \,\&\, \neg B \rightarrow A \,\&\, \neg A$ and conversely, so $\perp$ can be defined in a formula independent way, as $B \,\&\, \neg B$. Hence, by EFQ again, $\perp \rightarrow A$ (Segerberg's postulate I for intuitionistic logic I). It remains to establish the inclusion of minimal logic J, for which the implications $\neg A \rightarrow . A \rightarrow \perp$ (immediate by EFQ) and $A \rightarrow \perp \rightarrow . \neg A$, defining $\perp$ in J, jointly suffice (cf. Segerberg's formulation of J). Proof of $A \to (A \,\&\, \neg A) \to.\ \neg A$ results from Reductio, $A \to \neg A \to.\ \neg A$, which follows in turn from Summation, $A \to \neg A \to. A \vee \neg A \to \neg A$, and EM.

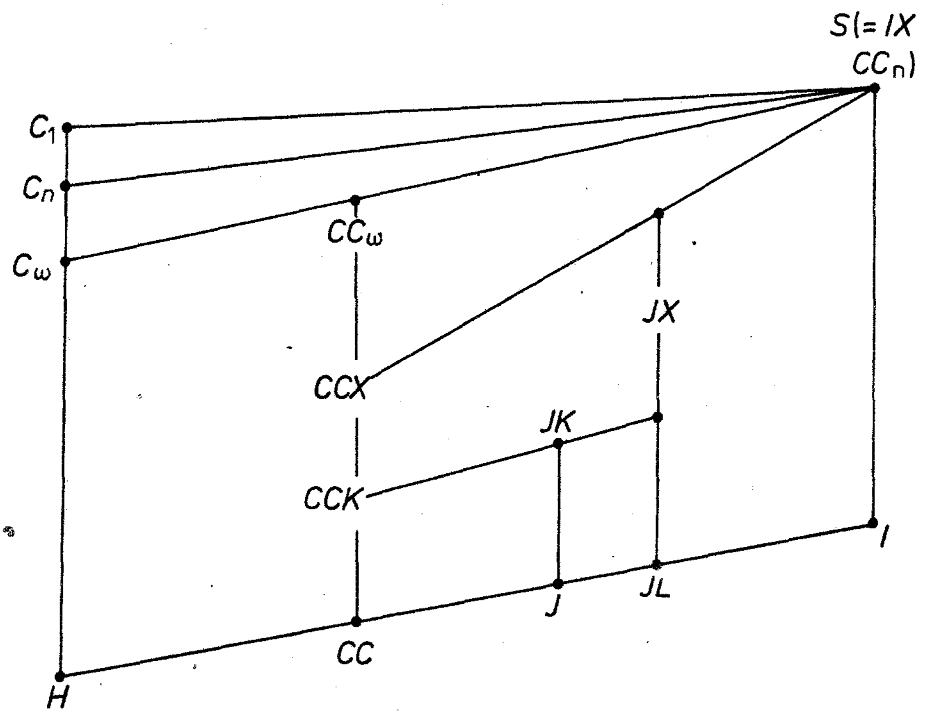

Figure 1. A map of systems, combining them with some main extensions of J studied by Segerberg [21].

Both the semantics so far provided for CC_ω and other systems and those to come for C_ω and the C system, lend themselves to algebraic adaptation. The methods of analysis are those already displayed for modal and relevant logics (for which see [4]). As with these other logics, the algebraic analyses yield in turn matrices for the systems included, and thereby give an interpretation to the matrices generated. Alternatively, the matrices can be generated directly from *finite* world modellings of the semantics. In this fashion, together with value truncation, main matrices deployed in the study of C systems can be semantically and algebraically explained.

3. No-fail semantics for C_ω and extensions, and much improved semantics

In the absence of appropriate equivalence substitution rules, more elegant first order semantics may appear to fail for C_ω. But workable, if postulate-mirroring, neighbourhood-style semantics are not far to seek.

Relation S on worlds is "superseded" for the present by relations S on worlds and sets of worlds, i.e. on $KXP(K)$, with S subject to the condition

Sii) where $a \leqslant b$ and $Sa\alpha$ than $Sb\alpha$, for every a, b in K and α in $P(K)$.

Otherwise model structures are as before. Only the evaluation rule for negation is varied, to the following:

$$I(\neg A, a) = 1 \text{ iff } Sa[A],$$

$$\text{where } [A] = \{b\colon I(A,b) = 1\} \text{ is the } \textit{range} \text{ of } A.$$

Then system H, now formulated with connective set $\{\rightarrow, \&, \vee, \neg\}$ but no negative postulates, is sound and complete with respect to the semantics outlined. For soundness, only hereditariness, immediate from Sii), requires attention, as all else is as before. For completeness, it suffices to define S and establish the induction step for negation in the canonical lemma and the modelling conditions. For the first, define $Sa\alpha$ iff $(PB)(\neg B \in a \,\&\, |B| = \alpha)$, where $|B| = \{c \in K\colon B \in c\}$. Then the induction step, transforms to $\neg A \in a$ iff $(PB)(\neg B \in a \,\&\, |B| = |A|)$, which is immediate. Condition Sii) follows from the inclusion $a \subseteq b$,

Adequate modelling conditions for the negation postulates of C_ω are then as follows:

DDN.	$\neg\neg A \rightarrow A$	$_n$DDN.	when $Sa\{b\colon Sb\alpha\}$ then $a \in \alpha$
EM.	$A \vee \neg A$	$_n$EM.	$T \in \alpha$ or $ST\alpha$ (or, $a \in \alpha$ or $Sa\alpha$).

Adequacy Theorem. *C_ω is sound and complete w.r.t. the neighbourhood semantics given.*

Proof. Given the details for H, it remains to verify specific modelling conditions for C_ω.

ad DDN and EM. The modelling conditions simply mirror the axioms, e.g. $_n$DDN is what is required to ensure that when $I(\neg\neg A, a) = 1$ then $I(A, a) = 1$.

ad $_n$DDN. Suppose $Sa\{b\colon Sb\alpha\}$ and let $\alpha = \{a\colon A \in a\}$. then $Sa\{b\colon (PB)(\neg B \in a \,\&\, |B| = |A|)\}$, i.e. $Sa\{b\colon \neg A \in a\}$. That is, $(PC)(\neg C \in a \,\&\, \{b\colon \neg A \in a\} = |C|)$, whence $\neg\neg A \in a$. Then by DDN, $A \in a$, i.e. $a \in \alpha$.

ad EM. As $A \vee \neg A \in a$, $A \in a$ or $\neg A \in a$, i.e. $a \in \alpha$ or $Sa\alpha$, where $\alpha = \{A\colon A \in a\}$.

Similar neighbourhood semantics can be supplied for all C_n systems $(1 \leqslant n < \omega)$. It is a matter of mirroring the further postulates in corresponding modelling conditions. The results are hardly perspicuous; indeed the conditions are for the most part less informative than the postulates they correctly mirror. So there is a real point in beginning again with C_ω.

The way to improved semantics is shown by picking up the modal analogy again. The Becker rule for $\Box$-distribution which appears guaranteed by modal semantics, such as that for $S2$, is neatly removed in semantics for systems like $SO.5$ by introducing nonnormal worlds. Why not remove RC, which is exactly the same form as a Becker rule for impossibility, in the same sort of way? A mark-1 $H(\neg)$ m.s M is a structure $(T, K, N, \leqslant, S)$ with $T \in N \subseteq K$, and otherwise like a CC m.s. (though si) could be restricted to N, normal worlds). The important difference from CC models comes with the rule for $\neg$, namely

I$\neg'$) For $a \in N$, $I(\neg A, a) = 1$ iff for some b in K such that Sab $I(A, b) = 0$; otherwise $I(\neg A, a)$ is assigned arbitrarily so long as hereditariness is respected.

That is, for (negation) normal worlds the rule is as before; for abnormal worlds c, $I(\neg A, c) = 1$ or $I(\neg A, c) = 0$, as for v, with the analogue of vi) respected. (Then validation of RC fails because there is no guarantee that world b is normal.) In fact it is easiest to let v do the work of assigning for negated wff at abnormal worlds; and this we shall suppose done. A first deficiency in these semantics, deriving from from hereditariness connections, is that $v(\neg A, c)$, for $c \notin N$, cannot be determined at the outset; its specification may depend upon the evaluation for $I(\neg A, b)$ for some $b \in N$. However effectiveness is not lost; it simply means that recursive specification of I is intertwined with v.

A typical example will illustrate the point and effect of introducing N. The contrapositives of lattice requirements are theorems of CC and CC_ω, but not of $H(\neg)$ and C_ω, a conspicuous defect of these latter systems. Consider, for instance, $\neg A \rightarrow \neg(A \& B)$; and try to fashion a countermodel. Then for some a, $I(\neg A, a) = 1 \neq I(\neg(A \& B), a)$. If $a \in N$, as it must in CC, then for some b, $Sab \& I(A, b) \neq 1$ and further $I(A \& B, b) = 1$ whence $I(A, b) = 1$, which is impossible. But if $a \notin N$, then a model can be designed where the countering assignments are satisfied.

Although, following recent tradition, the worlds of N and its complement are called respectively normal and abnormal, it is worth noticing that the normal worlds are *non*-classical, and in some respects Hegelian, while the abnormal worlds could, in further elaboration of strong C systems, be considered potentially classical.

The appeal of the mark-1 improved semantics begin to fade quickly when we come to C_ω (our present goal: who needs further semantics for $H(\neg)$!). For if we are to validate DDN and EM then special stipulation corresponding just to them is required in abnormal worlds, namely for $a \notin N$.

$\neg$) when $I(A, a) \neq 1$ then $I(\neg A, a) = 1$ and $I(\neg\neg A, a) \neq 1$.

But that is not all: there are fresh problems in validating DDN in normal worlds, and problems in keeping EM in while keeping LNC (i.e. A°) out. Thus, one might almost as well abandon the negation S rule, and rely upon pure stipulation for negation. The procedure will indeed provide a *stipulation semantics* for C_ω, readily extended to C_n systems; it resembles Brasilian semantics for these systems (as e.g. in [7]), except that it is based on pleasant positive logic semantics. For such a stipulative semantics, N is dropped again and S is dropped: $\neg 1$) is required for all a in K; and it is also required that $\neg 2$) where $a \leqslant b$ and $I(\neg A, a) = 1$ then $I(\neg B, a) = 1$. However, we can do better than such stipulative postulate-copying semantics.

The mark-2 approach borrows another old idea from modal semantics (from those proposed by Routley for certain interesting strengthenings of $S0.5$, namely $S0.9$ and $S1$). The idea is that instead of negation-arbitrary worlds, certain better controlled impossible worlds are included in K, specifically worlds where & behaves like $\vee$ and $\vee$ like &. Then stipulation can be avoided; negation behaves according to the S rule everywhere.

An *improved* C_ω m.s. is a structure $(T, K, N, \leqslant, S)$ with $T \in N \subseteq K$, $\leqslant$ a reflexive and transitive relation on K such that ri) where $b \in N$ and $a \leqslant b$ then $a \in N$, and where $b \in J$ (with $J = K - N$) and $a \leqslant b$ then $b \in J$, and S a reflexive and symmetric relation on N such that si). Thus C_ω m.s. extend CC_ω m.s. by adding further worlds, worlds possibly accessible through S relations. Valuation and interpretations are characterised exactly as for CC_ω, except for evaluation rules for & and $\vee$. These rules are as normal for every $a \in N$, but for $a \notin N$, i.e. for $a \in J$, they dualise as follows: $I(A \& B, a) = 1$ iff $I(A, a) = 1$ or $I(B, a) = 1$; $I(A \vee B, a) = 1$ iff $I(A, a) = 1 \Rightarrow I(b, a)$.

ADEQUACY THEOREM FOR C_ω. *C_ω is sound and complete w.r.t. to the improved semantics.*

PROOF varies that for CC_ω. For soundness, the hereditariness lemma has to be elaborated. The further steps for & and $\vee$ in J worlds use ri) also; for if $\leqslant$ leads from T in N to a J world, & and $\vee$ postulates would be swiftly invalidated. But in virtue of ri), verification is as for CC. Both DDN and EM are as for CC_ω.

Completeness requires much new detail. In particular, much use is made of transforms of wff and of sets of wff. The *transform* A^t of A is the wff obtained from A by uniformly replacing (in left to right order) each occurrence of '&' in A by '$\vee$' and each occurrence of '$\vee$' by '&'. The *transform* ∇^t of set ∇ is the set of transforms of ∇.

TRANSFORM LEMMA.

(i) $A \in \nabla$ iff $A^t \in \nabla^t$

(ii) $(A^t)^t = A$

(iii) $(\nabla^t)^t = \nabla$

(iv) $(\neg A)^t = \neg (A^t)$

(v) $(A \to B)^t = (A^t \to B^t)$; $A \to B \in \nabla$ iff $A^t \to B^t \in \nabla^t$

(vi) $(B \& C)^t = B^t \vee C^t$

(vii) $(B \vee C)^t = B^t \& C^t$

(viii) $\nabla \subseteq \Gamma$ iff $\nabla^t \subseteq \Gamma^t$.

A canonical C_ω m.s. varies and elaborates the canonical CC_ω m.s. as follows: N is the class of C_ω-full sets, J is the class of their transforms, and K is $N \cup J$. The canonical C_ω m.s. is a C_ω m.s. Most details are as for CC_ω. For ri) which is new, observe that theorems, such as $A \vee \neg A$, or their trnasforms, such as $A \& \neg A$, belong to every $d \in K$. Such wff serve to label sets in K, and given its definition guarantee ri).

In the canonical lemma there are several new cases, involving worlds in J. The induction steps for $\rightarrow$ and $\neg$ depend in one direction only on logical principles, and go through as before. The converse cases involve transforming shifts. Let $a \in J$, whence $a^t \in N$, and suppose $I(B \rightarrow C, a) = 1$. Then for every b, where $a \subseteq b$ and $B \in b$ then $C \in b$, whence by the lemma, wherever $a^t \subseteq b^t$ and $B^t \in b^t$ then $C^t \in b^t$. Hence by the argument for normal worlds $B^t \rightarrow C^t \in b^t$, i.e. $(B \rightarrow C)^t \in b^t$. Hence $B \rightarrow C \in b$. Similarly, where $a \in J$, suppose $\neg B \in a$. Then $\neg(B^t) \notin a^t$, with a^t in N. As before, there is some c such that $Sa^t c$ and $B^t \notin c$. Since $Sa^t c$, for every C, $\neg C \notin a^t \supset C \in c$, i.e. $\neg C^t \notin a \supset C^t \in c^t$, i.e. $Sa\ c^t$. But as $B \notin c$, $B^t \notin c^t$. So for some d in K, $S_L ad$ and $B \notin d$, as required.

ad &. $I(B \& C, a) = 1$ iff $I(B, a) = 1 = I(C, a)$, i.e. iff $B \in a \& C \in a$, by induction hypothesis, and hence, where $a \notin J$, $B \& C \in a$.
In contrast where $a \in J$, $a^t \notin J$, and by lemmata etc.,

$$\begin{aligned} I(B \& C, a) = \text{ iff } & I(B, a) = 1 \vee I(C, a) = 1 \\ \text{iff } & B \in a \vee C \in a, \text{ by induction hypothesis} \\ \text{iff } & B^t \in a^t \vee C^t \in a^t \\ \text{iff } & B^t \in C^t \in a^t, \text{ since } a^t \notin J \text{ is prime} \\ \text{iff } & (B \& C)^t \in a^t \\ \text{iff } & B \& C \in a \end{aligned}$$

ad $\vee$: Similar to the previous case.
The remainder of the completeness argument is like that for CC_ω.

Before considering extensions of C_ω such as C_1, it is worth inquiring as to where verification of A°, a theorem of CC_ω, fails in C_ω. Since all postulates of C_1 have B° as hypothesis this is decidedly relevant. Verification of A°, i.e. $\neg(A \& \neg A)$, in CC_ω is as follows: Suppose $I(A^\circ, a) \neq 1$, for $a \in N$ or $a = T$. Then $(b)(Sab \supset I(A \& \neg A, b) = 1)$. Since Saa, $I(A, a) = 1 = I(\neg A, a)$. Then for some b, Sab and $I(A, b) \neq 1$. As Sab, $I(A \& \neg A, b) = 1$, so $(\dashv) I(A, b) = 1$, which is impossible. In C_ω the argument fails at the last step, marked $(\dashv)$. For this requires that b is a normal world, which it may not be.

The semantics proposed for C_ω appear to admit of extension of other C_n systems; but even for C_1 (though the positive theory simplifies to classical banality) the details of a properly recursive semantics become discouragingly messy. Let us return instead to the main issue, rectification through extension, or other variation, of the C systems so they accomplish desired work.

4. Other directions: final extensions through EC systems, and going within

Although the results assembled reveal CC_ω as a neat base on which to try to build da Costa style paraconsistent theories with pleasant logical and semantical properties, they do not resolve two problems in particular: Firstly, how is the theme that classical wff behave classically to be represented in CC_ω, given that it cannot, without classical collapse, be done in the fashion of C_1? Secondly, how is a suitable form of SE to be grafted onto C_n systems for n distinct from ω?

The first question is investigated in detail in Urbas, though with largely negative results so far, since the da Costa C hierarchy typically collapses (sometimes in bizarre ways: see Urbas). In any case, an improved resolution of the second question would, it seems, enable the first question to be skirted. To get in a position to try to answer the second question definitively, observe that RC really gives more than is necessary to prove SE by induction. For the proof it is enough that the following weaker equivalential rule hold:

EC. $A \to B,\ B \to A / \neg B \to \neg A.$

Indeed the EC rule is both necessary and sufficient for full substitutivity, SE. Call the systems resulting from C_n systems by the addition of EC, EC_n systems, for $1 \leqslant n \leqslant \omega$. EC_n systems do not enjoy the apparent initial advantages of CC_n systems; for example, EC does not enable quite as pretty a semantical analysis as RC affords. (Even so semantics for $E\omega$ may be obtained by analogy with relational semantics for modal system $S1$, with $\neg$ behaving like $S1$ modal functor $\Box$) More important, it was from the beginning of investigations very doubtful that the relevant EC_n systems, for $n \neq \omega$, differed from classical logic S. Certainly, as soon emerged, there are no finite matrices which distinguish EC_1 from S.

SLANEY LEMMA. *There are no finite strong models of EC_1 other than models of classical logic.*[8]

PROOF. EC suffices in context for a derivable rule of replacement:

$$A \to B,\ B \to A,\ C(A)/C(B).$$

Therefore, in any strong matrix for EC_1 the relation on matrix values, λxy ($x \leftrightarrow y$ is designated), is a congruence w.r.t. the connectives. So the quotient algebra, obtained by collapsing the matrix *modulo* this relation, is likewise a model structure for EC_1 and falsifies all formulas falsified by the original. In it there is only one designated value, T, and $a \leftrightarrow b = T$ iff $a = b$.

In order to validate all C_1 theorems this structure must be a Brouwerian lattice (with operations $\wedge$, $\vee$, etc.). For any element a of it,

$$\neg[a \wedge \neg a) \wedge \neg(a \wedge \neg a)] = T, \text{ since } (A \,\&\, \neg A^\circ) \text{ holds.}$$

$$\neg T = \neg\neg[(a \wedge \neg a) \wedge \neg(a \wedge \neg a)]$$

$$\neg T \leqslant a \wedge \neg a \wedge \neg(a \wedge \neg a) \text{ (by DDN)}$$

$$\neg T \leqslant a.$$

[8] In Harrop's sense of finite strong model, i.e. finite, validating the theorems, and closed under all the rules of inference. This result was supplied, in its entirety, by Slaney.

So $\neg T$ is lattice 0. Hence the structure is a Heyting lattice with intuitionist negation $-a$ defined as $a \to \neg T$. But it satisfies Peirce's law, so it is a Boolean algebra, and $-a$ is the (Boolean) complement of a.

Thus if it is *finite* then every element of it is a finite join of atoms. But if $a_1 \ldots a_n$ are all classical elements (i.e. for $1 \leqslant i \leqslant n$, $\neg a_i = -a_i$) then the join $a_1 \vee \ldots \vee a_n$ is classical by applications of the C_1 postulate, $A^\circ \& B^\circ \to (A \vee B)^\circ$. Contrapositively, if there is any non-classical element in such a structure than there is a non-classical atom. But there cannot be a non-classical atom; for since $a \vee \neg a = T$, where a is an atom, $\neg a$ can only be $-a$ or T. If it is $-a$ then a is classical, while if it is T then $a \leqslant \neg a$, whence $a \wedge \neg a = a$ and a is (absurdly) again classical (since $A \& \neg A$ is classical).

Everyone's immediate conjecture, that $EC_1 = S$, did not maintain its conjectural status for long.

COLLAPSE THEOREM. *$EC_n = S$, for $1 \leqslant n < \omega$.*

PROOF, by syntactic means, is given in Urbas, to whom this result is due (see [22]). Leading ideas for the proof were suggested by the strategy of the Slaney lemma.

COROLLARY. *There is no extension of a C_n system, for $1 \leqslant n < \omega$, conforming to* SE *weaker than classical logic S.*
Thus a whole line of investigation for rectifying stronger C systems is closed.[9]

We are left with a small group of systems like CC_ω extending EC_ω satisfying SE and distinct from S, but apparently lacking a decent internal way of representing classicalness. (But no doubt we can add a classicalness functor conservatively, after the fashion of significance logics). For stronger systems there is, as the theorems show, no way of rectifying C systems through extension. It is necessary to move *inside* these systems to achieve desired ends. Such a move is important for other reasons also, for instance in accommodating nontrivially within the one system in a symmetrical way, both paraconsistency *and* paracompleteness (i.e. incompleteness). We are thus led towards a sure way, of independent merit, out of all these problems, namely weakening the positive base H. That approach (already advocated in [15]) will be pursued in a sequel.

References

[1] A.I. ARRUDA and N. C. A. DA COSTA, *Une sémantique pour le calcul $C_1^=$*, ***C.R. Acad Sc.* Paris**, t. 284 (31 Janvier 1977) Series A, pp. 282–97.

[2] F. G. ASENJO, *A calculus of antinomies*, ***Notre Dame Journal of Formal Logic*** 7 (1966), pp. 103–5.

[3] E. W. BETH, ***The Foundations of Mathematics***, North Holland, Amsterdam, 1959.

[9] The conclusion should be put together with Mortensen's analogous conclusion, (in [13]), upon which Urbas's result improves. The way now appears clear to close several open questions arising from Mortensen's investigations.

[4] R. Brady, R. K. Meyer, C. Mortensen and R. Routley, *Algebraic analysis of relevant affixing logics, and other Polish connections*, Research Paper No. 16, Logic Group, Research School of Social Sciences, The Australian National University, 1983.
[5] M. Bunder, *A new hierarchy of paraconsistent logics*, in ***Proceedings of the Third Brazilian Conference on Mathematical Logic*** (edited A. I. Arruda and others), Sociedade Brasiliera de Logica, Sao Paulo, 1980.
[6] N. C. A. da Costa, *On the theory of inconsistent formal systems*, ***Notre Dame Journal of Formal Logic*** 15 (1974), pp. 497–510.
[7] N. C. A. da Costa and E. H. Alves, *A semantical analysis of the calculi C_n*, ***Notre Dame Journal of Formal Logic*** 18 (1977), pp. 621–680.
[8] N. C. A. da Costa and R. Wolf, *Studies in paraconsistent logic I: the dialectical principle of unity of opposites*, ***Philosophia*** 9 (1980), pp. 189–217.
[9] H. B. Curry, ***Foundations of Mathematical Logic***, McGraw, New York 1963; reprinted by Dover, New York, 1976.
[10] A. Heyting, ***Intuitionism***, North Holland, Amsterdam, 1977.
[11] S. C. Kleene, ***Introduction to Metamathematics***, North Holland, Amsterdam, 1952.
[12] S. A. Kripke, *Semantical analysis of intuitionistic logic I*, in ***Formal Systems and Recursive Functions*** (ed. J. Crossley and M. Dummett), North Holland, Amsterdam, 1965.
[13] C. Mortensen, *Paraconsistency and C_1*, Research Paper No. 10, Logic Group, Research School of Social Sciences, The Austrialian National University, 1981; included in [16].
[14] K. Popper, *What is dialectic? **Mind*** 49 (1940), pp. 403–426; reprinted in ***Conjectures and Refutations***, Routledge & Kegan Paul (1976), pp. 313–335.
[15] G. Priest and R. Routley, ***On Paraconsistency***, Research Monograph No. 1, Research School of Social Sciences, The Australian National University; now absorbed in [16].
[16] G. Priest, R. Routley and J. Norman (eds), ***Paraconsistent Logic***, Philosophia Verlag, Munich, 1989.
[17] R. Routley, *Semantical analysis of propositional systems of Fitch and Nelson*, ***Studia Logica*** 30 (1974), pp. 283–298.
[18] R. Routley and R. K. Meyer, *Dialectical logic, classical logic, and the consistency of the world*, ***Studies in Soviet Thought*** 16 (1976), pp. 1–25.
[19] R. Routley, *Dialectical logic, semantics and metamathematics*, ***Erkenntnis*** 14 (1979), pp. 301–331.
[20] R. Routley and others, ***Relevant Logics and Their Rivals***, Ridgeview, California, 1982.
[21] K. Segerberg, *Propositional logics related to Heyting's and Johansson's*, ***Theoria*** 34 (1968), pp. 26–61.
[22] I. Urbas, ***On Brazilian Paraconsistent Logics***, Doctoral dissertation, Australian National University, Canberra, 1987.

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