# Preface

*Man kann*
*einen jeden Begriff,*
*einen jeden Titel,*
*darunter viele Erkenntnisse gehören,*
*einen logischen Ort nennen.*
Immanuel Kant [258, p. B 324]

This book's title subject, *The Topos of Music*, has been chosen to communicate a double message: First, the Greek word "topos" ($\tau\acute{o}\pi o\varsigma$ = location, site) alludes to the logical and transcendental location of the concept of music in the sense of Aristotle's [20, 592] and Kant's [258, p. B 324] topic. This view deals with the question of *where music is situated* ***as*** *a concept*—and hence with the underlying ontological problem: *What is the type of being and existence of music?* The second message is a more technical understanding insofar as the system of musical signs can be associated with the mathematical theory of *topoi*, which realizes a powerful synthesis of geometric and logical theories. It laid the foundation of a thorough geometrization of logic and has been successful in central issues of algebraic geometry (Grothendieck, Deligne), independence proofs and intuitionistic logic (Cohen, Lawvere, Kripke).

But this second message is intimately entwined with the first since the present concept framework of the musical sign system is technically based on topos theory, so the topos of music receives its topos-theoretic foundation. In this perspective, the double message of the book's title in fact condenses to a unified intention: to unite philosophical insight with mathematical explicitness.

According to Birkhäuser's initial plan in 1996, this book was first conceived as an English translation of my former book *Geometrie der Töne* [340], since the German original had suffered from its restricted access to the international public. However, the scientific progress since 1989, when it was written, has been considerable in theory and technology. We have known new subjects, such as the denotator concept framework, performance theory, and new software platforms for composition, analysis, and performance, such as RUBATO® or OpenMusic. Modeling concepts via the denotator approach in fact results from an intense collaboration of mathematicians and computer scientists in the object-oriented programming paradigm and supported by several international research grants.

Also, the scientific acceptance of mathematical music theory has grown since its beginnings in the late 1970s. As the first acceptance of mathematical music theory was testified to by von Karajan's legendary Ostersymposium "Musik und Mathematik" in 1984 in Salzburg [190], so is the significantly improved present status of acceptance testified to by the Fourth Diderot Forum on Mathematics and Music [365] in Paris, Vienna, and Lisbon 1999, which was organized by the European Mathematical Society. The corresponding extension of collaborative efforts in particular entail the inclusion of works by other research groups in this book, such as the "American Set Theory", the Swedish school of performance research at Stockholm's KTH, or the research on computer-aided composition at the IRCAM in Paris.

Therefore, as a result of these revised conditions, *The Topos of Music* appears as a vastly extended English update of the original work. The extension is visibly traced in the following parts which are new with respect to [340]: Part II exposes the theory of denotators and forms, part V introduces the topological theories of rhythms and motives, part VIII introduces the structure theory of performance, part IX deals with the expressive semantics of performance in the language of performance operators and stemmata (genealogical trees of successively refined performance), part X is devoted to the description of the RUBATO® software platform for representation, analysis, composition, and performance, part XI presents a statistical analysis of musical analysis, part XII concludes the subject of performance with an inverse performance theory, in fact a first formalization of the problem of music criticism.

This does however not mean that the other parts are just translations of the German text. Considerable progress has been made in most fields, except the last part XIV which reproduces the status quo in [340]. In particular, the local and global theories have been thoroughly functorialized and thereby introduce an ontological depth and variability of concepts, techniques, and results, which by far transcend the semiotically naive geometric approach in [340]. The present theory is as different from the traditional geometric conceptualization as is Grothendieck's topos theoretic algebraic geometry from classical algebraic geometry in the spirit of Segre, van der Waerden, or Zariski.

Beyond this topos-theoretic generalization, the denotator language also introduces a fairly exceptional technique of circular concept constructions. This more precisely is rooted in Finsler's pioneering work in foundations of set theory [153], a thread which has been rediscovered in modern theoretical computer sciences [4]. The present state of denotator theory rightly could be termed a Galois theory of concepts in the sense that circular definitions of concepts play the role of conceptual equations (corresponding to algebraic equations in algebraic Galois theory), the solutions of which are concepts instead of algebraic numbers.

Accordingly, the mathematical apparatus has been vastly extended, not only in the field of topos theory and its intuitionistic logic, but also with regard to general and algebraic topology, ordinary and partial differential equations, Pólya theory, statistics, multiaffine algebra and functorial algebraic geometry. It is mandatory that these technicalities had to be placed in a more elaborate semiotic perspective. However, this book does not cover the full range of music semiotics, for which the reader is referred to [361]. Of course, such an extension on the technical level has consequences for the readability of the theory. In view of the present volume of over 1300 pages, we could however not even make the attempt to approach a non-technical presentation. This subject is left to subsequent efforts. The critical reader may put the question whether music is really that complex. The answer is yes, and the reason is straightforward: We cannot pretend that Bach, Haydn, Mozart, or Beethoven, just to name some of the most prominent

composers, are outstanding geniuses and have elaborated masterworks of eternal value, without trying to understand such singular creations with adequate tools, and this means: of adequate depth and power. After all, understanding God's 'composition', the material universe, cannot be approached without the most sophisticated tools as they have been elaborated in physics, chemistry, and molecular biology.

So who is recommended to read this book? A first category of readers is evidently the working scientist in the fields of mathematical music theory, the soft- and hardware engineer in music informatics, but also the mathematician who is interested in new applications from the above fields of pure mathematics. A second category are those theoretical mathematicians or computer scientists interested in the Galois theory of concepts; they may discover interesting unsolved problems. A third category of potential readers are all those who really want to get an idea of what music is about, of how one may conceptualize and turn into language the "ineffable" in music for the common language. Those who insist on the dogma that precision and beauty contradict each other, and that mathematics only produces tautologies and therefore must fail when aiming at substantial knowledge, should not read such a book.

Despite the technical character of *The Topos of Music*, there are at least four different approaches to its reading. To begin with, one may read it as a philosophical text, concentrating on the qualitative passages, surfing over technical portions and leaving those paragraphs to others. One may also take the book as a dictionary for computational musicology, including its concept framework and the lists of musical objects and processes (such as modulation degrees, contrapuntal steps) in the appendices. Observe however, that not all existing important lists have been included. For example, the list of all-interval series and the list of self-addressed chords are omitted, the reader may find these lists in other publications. Thirdly, the working scientist will have to read the full-fledged technicalities. And last, but not least, one may take the book as a source for ideas of how to go on with the whole subject of music. The GPL (General Public License[1]) software sources in the appended CD-ROM may support further development.

The prerequisites to a more in-depth reading of this book are these. Generally speaking, a good acquaintance with formal reasoning as mathematics (including formal logic) preconizes, is a conditio sine qua non. As to musicology and music theory, the familiarity with elementary concepts, like chords, motives, rhythm, and also musical notation, as well as a real interest in understanding music and not simply (ab)using it, are recommended. For the more computer-oriented passages, familiarity with the paradigm of object-oriented programming is profitable. We have not included the appendix on mathematical basics because it should help the reader get familiar with mathematics, but as an orientation in fields where the specialized mathematician possibly needs a specification of concepts and notation. The appendix was also included to expose the spectrum of mathematics which is needed to tackle the formal problems of computational musicology. It is by no means an overkill of mathematization: We have even omitted some non-trivial fields, such as statistics or Lambda calculus, for which we have to apologize.

There are different supporting instances to facilitate orientation in this book. To begin with, the table of contents and an extensive subject and name index may help find one's keywords. Further, following the list of contents, a leitfaden (on page xxix) is included for a generic navigation. Each chapter and section is headed by a summary that offers a first orientation

[1] A legal matter file is contained in the book's CD-ROM, see page xxx.

about specific contents. Finally, the book is also available as a file `ToposOfMusic.pdf` with bookmarks and active cross-references in the appended CD-ROM (see page xxx for its contents). This version is also attractive because the figures' colors are visible only in this version.

In order to obtain a consistent first reading, we recommend chapters 1 to 5, and then appendix A: Common Parameter Spaces (appendix B is not mandatory here, though it gives a good and not so technical overview of auditory physiology). After that, the reader may go on with chapter 6 on denotators and then follow the outline of the leitfaden (see page xxix).

This book could not have been realized without the engaged support of nineteen collaborators and contributors. Above all, my PhD students Stefan Göller and Stefan Müller at the MultiMedia Laboratory of the Department of Information Technology at the University of Zurich have collaborated in the production of this book on the levels of the LaTeX installation, the final production of hundreds of figures, and the contributions sections 20.2 through 20.5 (Göller) and sections 46.3 through 46.3.6.2 (Müller). My special gratitude goes to their truly collaborative spirit.

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Vulpera, June 2002 Guerino Mazzola

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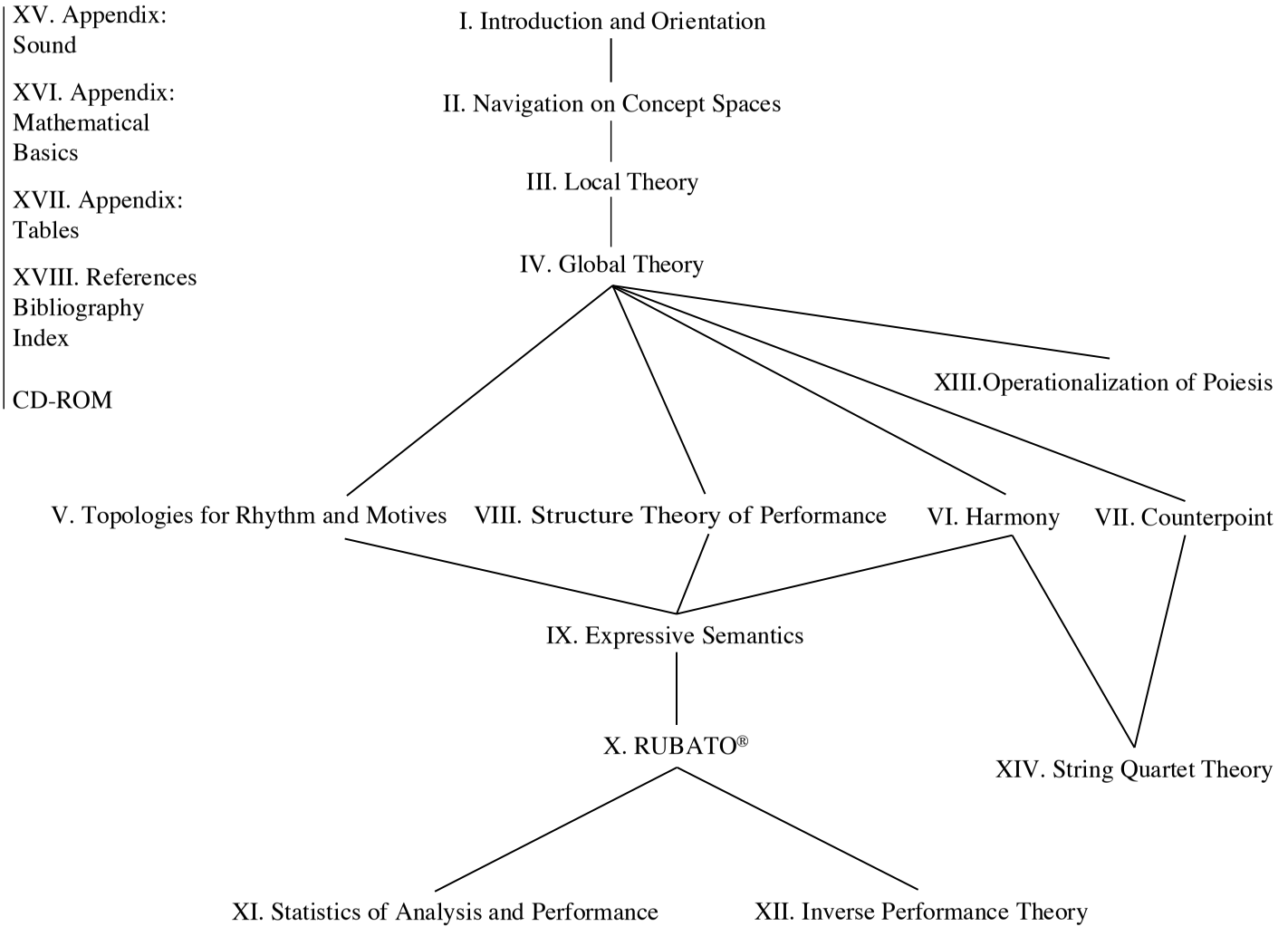
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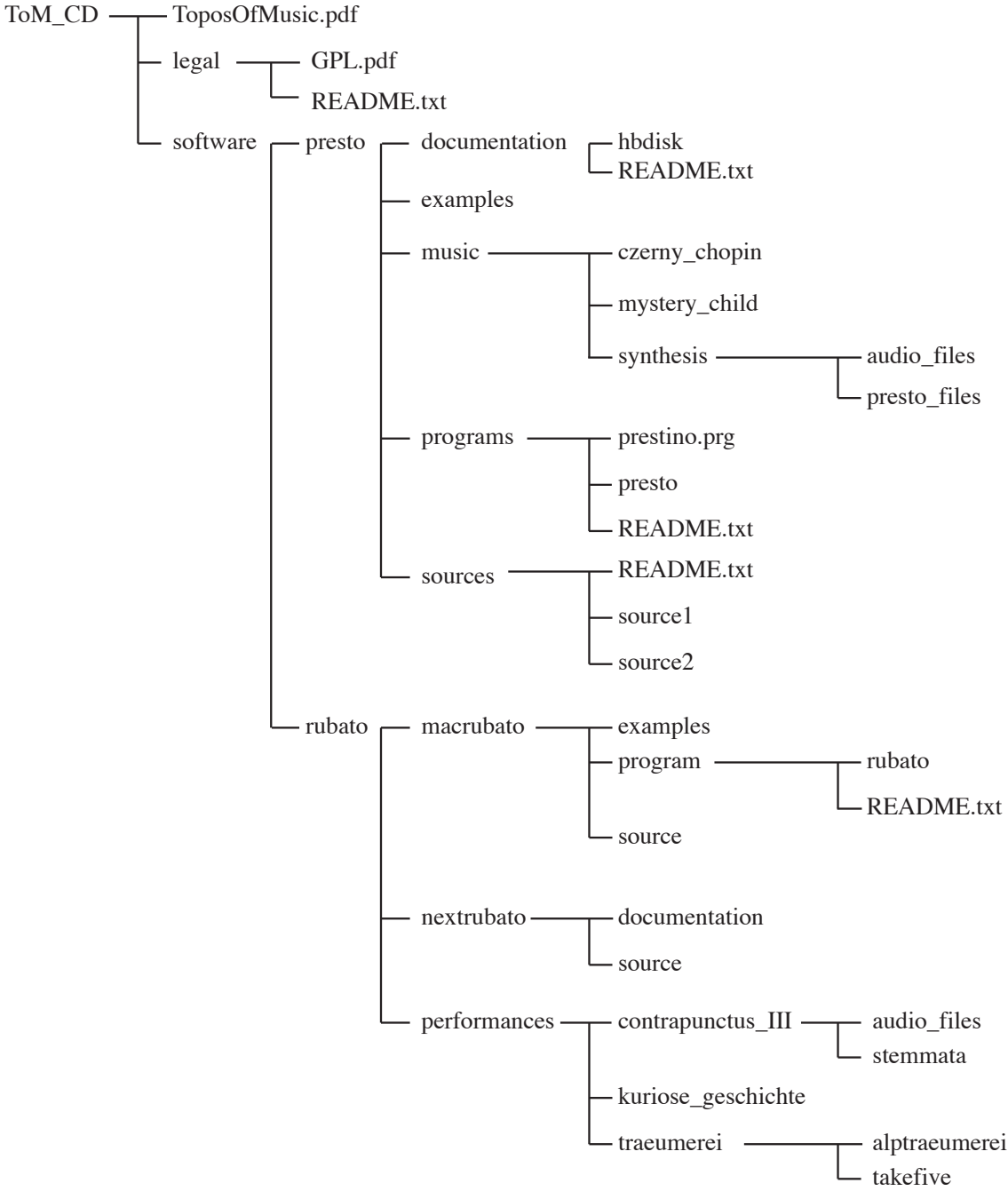
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# Part I

# Introduction and Orientation

# Chapter 1

# What is Music About?

*Musik ist das ganze Leben.*
Rudolf Wille [578]

**Summary.** This chapter describes the overall extension of music-related activities in the space-time of human existence. The basic scope of the systematization as described in this book is declared. Our selection of corresponding fundamental scientific domains is not meant as a qualitative judgment over other scientific or artistic domains; it merely names the scientific pillars of the musical realm.

– Σ –

This book deals with the topos, the very concept of music. However, music appears in a wide range of human realms. It is a universal phenomenon of symbolic and physical, formal and emotional, individual and social, systematic and historic presence. Therefore, developing a concept of music should not get off with a definition *ex machina*, but offer propaedeutic orientation tools in order to make the reader *understand why* certain conceptual mechanisms or definitions are built. The need for such a support reveals a strong distinction from chemistry, physics or other natural sciences. The point is not that these sciences are dispensed from the fundamental question of what they are about. Rather the characteristic difference to musicology—and other humanities—is that natural sciences offer a fast access to effective activities, to the "working scientist" paradigm: Realization of one's scientific status by doing science.

"Doing" musicology is much less easy. The historical overhead in musicology is an index of the effort requested to get off the ground, to navigate within a safe concept framework. Later in this book we shall discuss some of the reasons for this deficiency, but for the time being, we just notice that musicology still lacks a stable concept framework, see [361] for a detailed discussion of the "unscientific status of musicology". It is one of the main goals of this work to develop such a framework. In fact—as a result of a lengthy concept development—an explicit concept framework for musicology will only be achieved in section 19.2. We shall, however, not deal with the full reality of music, as it appears in psychological, physiological, social, religious and political contexts. For a semiotically motivated systematization of music which comprises these aspects, refer to [361].

In this vein we want to start the preliminary orientation by presenting the *fundamental activities* related to music, and then their foundation upon *established scientific research fields*.

## 1.1 Fundamental Activities

**Summary.** The musical realm distributes among four types of artistic or scientific action and reflection: production, reception, documentation, and communication. These activities testify to a universally ramified presence of music in culture.

– Σ –

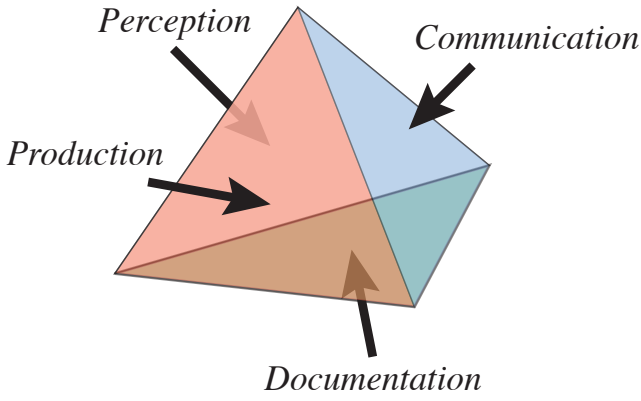


Figure 1.1: The four fundamental activities in music are visualized as sides of a tetrahedron.

The following classification of music-related activities probably applies to many cultural fields. We recognize that developing a consciousness of their presence *in music* sheds a particular light on this subject. This is due to the fact that music traditionally relies on a strong connection between artistic facticity and intellectual reflection. In most sciences, and even arts, such an intertwinement is an exotic aspect, but musicology has to deal with it and this makes the case a very special one. The four activities are as follows: production, reception, documentation, and communication. For the sake of coherent representation, and because of the discussion in section 1.2, we represent these activities as sides of a tetrahedron, see figure 1.1.

These concepts are understood as follows:

→ *Production* refers to whatever is recognized as being on the side of the 'making' of music, and this on any level or reality. For example, we can make a sound on an instrument, or write or let a computer software 'write' a composition in a mental reality, or make an expressive performance. But we cannot 'make' an analysis in this sense, since this kind of activity receives a given musical body.

→ *Reception* refers to whatever is recognized as being on the side of 'taking' something from music on any level or reality. For example, we may hear a sound or let a machine decompose

it into its partials, or make an analysis of a musical score—by hand or supported by a computer. But we may not perform a composition in the sense of a creative restatement. Whatever is on the side of a production of some new musical entity is understood as being on the productive side.

→ *Documentation* deals with whatever is related to bringing musical facts into more or less permanent sign systems. Musical notation, data base systems for musical objects, tradition of musical texts, the question of how and on which media ethnomusical performances should be stored, all this is subject to music documentation.

→ *Communication* is related to whatever happens when the three preceding activities are put into relation with each other. Production leads to documents, documents are retrieved and interpreted by instrumentalists or musicologists, live production of musical sounds by improvising musicians is perceived by other musicians of the band, then answered, and so on. Communication deals with the omnidirectional processes between production, documentation, and reception.

We do not contend that these four fundamental activities represent the only possible classification scheme, but they are broad enough to evidence the immense variety of perspectives when dealing with music. And we insist on the fact that all these activities are of comparable relevance. Let us make this clear. Since music is an art where time plays an omnipresent role, the making of music, its physical production in performance cannot be separated from the product as a preliminary task; it is substantial. The same holds for reception. On the other hand, documentation is not a necessary evil. It evokes the fundamental question of identity of an artistic work: Can we say that Beethoven's Fifth is that determined text? Or is performance intrinsically part of the Fifth's identity? Trying to understand music under permanent abstraction from any one of these aspects cannot succeed. And, above all, they are *activities* and not passive contemplation, a basic fact that we shall discuss in depth in chapter 5.

It is further remarkable that the synopsis of these equally important music activities draws a picture of perfect encyclopedic universality. In the "Encyclopédie" of Denis Diderot and Jean le Rond d'Alembert, we have three basic conceptual coordinates [14, 25] which go back to the global geography of science as designed by Francis Bacon in [34]: Reason/Philosophy, Documentation/History, and Imagination/Arts. If we compare these entities to the music activities, they correspond as follows: *Imagination/Arts* is the counterpart of *production*; the arts express a creative activity of making a work from one's imagination. *Reason/Philosophy* is the counterpart of *reception*; it deals with what in German is better conceived as "Vernunft", which means to perceive, receive, understand, observe something which is presented to your intellect. *Documentation/History* is verbally the same as in our list. What we have not inserted so far is the activity of *communication*. Now, the *genuine* encyclopedic concern is communication of the global orbit of human knowledge. Without this driving motor, reason, imagination, and documentation are worthless abstracta.

Summarizing, we can state that the main concerns in the classical encyclopedia of the Age of Enlightenment are congruent to the four basic music activities. This is not only a comfortable *raison d'être* for music as a cultural phenomenon, it also has deep consequences for the construction of a viable concept framework. Such a body must comprise a variety of

ontological[1] perspectives and cannot refrain from contents in favor of formal virtuosity. The system of predicates to be developed in chapter 18 will account for this specification.

## 1.2 Fundamental Scientific Domains

**Summary.** The fundamental scientific domains relating to the activities described in the preceding section include semiotics, physics, mathematics, and psychology. This is explicated and justified—in particular with regard to the only apparant elimination of the historical perspective which dominates traditional musicology. The relation between the two lists of activities and domains is discussed.

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To begin with, we have to make clear what "fundamental scientific domains" should signify. We may view the collection of sciences as being a hierarchy of disciplines with its particular relations being defined by a substantial dependency. For instance, chemistry is dominated by physics, but not vice versa. And every natural science is dominated by mathematics. Many of the hierarchical relations are quite problematic, blurred or otherwise unclear by the very definition of basics of particular disciplines; we have to accept such general uncertainties and will proceed in full consciousness of the provisional state of the art of epistemological classification. Now, if one has to locate a special research subject, such as music, within the hierarchy of disciplines, one is looking for a minimal set, a kind of *'basis'* of disciplines which are necessary to cover the subject. In this sense, we argue that four sciences: semiotics, physics, mathematics, and psychology, are such a 'basis'.

Let us explain this selection. We have to show that the four disciplines are sufficient to cover the field of music and that no one of these sciences is superfluous in that it either is not involved at all, or that it is only involved via a proper sub-discipline. At this point, we cannot circumvent a provisional description of what music is about, though this is only a first approximation in order to delimitate the scientific overall extension.

**Definition 1** (First provisional version) *Music is a system of signs composed of complex forms which may be represented by physical sounds, and which in this way mediate between mental and psychic contents.*

To deal with such a system, semiotics are naturally evoked. To describe these forms, mathematics is the adequate language. To represent these forms on the physical level, physics are indispensable. And to understand the psychic contents, psychology is the predestined science. Is this selection a "basic" one as claimed above? There are several critical points which we should clarify. First, the historic dimension of music seems to be neglected. However, semiotics has a strong competence in diachronic study of a sign system. Music history is prominently the *history of a system of significant signs* (much like diachronic linguistics), and here, diachronic semiotics is the adequate research field, whereas general history would be too generic in view of the given subject. Musical forms are not only mediated via physical representation. They have a strong properly mental ("symbolic") existence, and this evokes mathematics as a directly

[1] Ontology is the philosophical discipline dealing with the concept of "being".

involved research aspect. Physics has a prominent role in music, and this not only with respect to sound production (classical acoustics and physics of musical instruments), but also in computer science which is concerned with data representation and processing in music—be it on the generic level of software or on the level of advanced digital sound synthesis and analysis. Hence, we do "absorb" computer science and sound engineering within physics, together with the mathematical formalisms in theoretical computer science. Finally, social aspects of music are viewed as objects of social psychology on one, and socio-semiotics on the other hand. Also pedagogical problems (music education) are subsumed under the label of psychology since it is the human psyche which is educated.

It is an extremely important point to recognize the *relative autonomy* of these four research fields in their descriptive force. This means that we do not, and never will hope or hypothesize reductionism whatsoever. There is no interest to view psychological reality as a surface of neurophysiological depth structure. In this sense, our selection is not pragmatic but ontological. We come back to this theme in chapter 2.

To complete the picture, we should try to fix the relative positions of the four basic sciences with respect to the four fundamental activities of production, reception, documentation, and communication. It is evident that activity is more or less related to some of the four scientific disciplines. If we "label" each vertex of the tetrahedron shown in figure 1.1 by a basic science, it turns out that every triangular activity side is "spanned" by three of four vertices. If we pay attention to the obvious semantics of this configuration, we have to place the labels in such a way that each activity is delimited by the three most relevant sciences for its execution. We propose the labeling drawn in figure 1.2 and leave it to the reader to judge our selection.

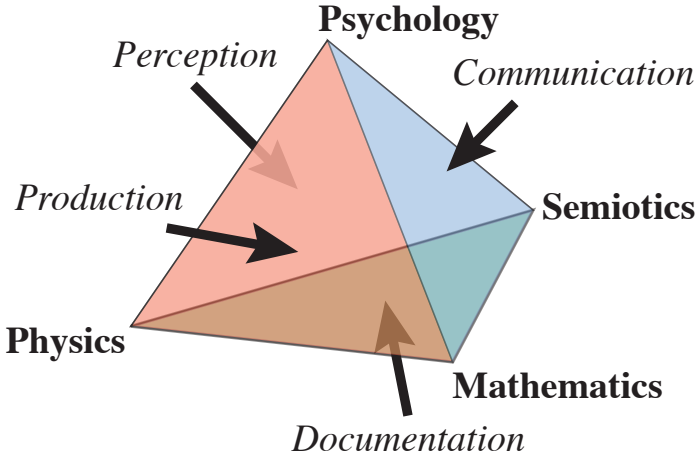


Figure 1.2: The four fundamental activities, resp. their counterparts of basic sciences in music, are visualized as sides, resp. vertices, of a tetrahedron.

Let us simply give one example: Documentation is concerned with signs, their formal representation, and the physical storage techniques. The psychological aspect is less substantial,

whence documentation is "spanned" by the respective vertices. We do, however, not overdress this elegant configuration as a "magic" tetrahedron, the visualization merely serves as a concise and relatively well positioned synopsis of the overall situation, where the question "What is music about?" can be initiated. And it is also a very practical *orientation scheme for designing future music research policy and strategy.*

# Chapter 2

# Topography

*Die Linien des Lebens sind verschieden*
*Wie Wege sind, und wie der Berge Gränzen.*
*Was wir hier sind, kann dort ein Gott ergänzen*
*Mit Harmonien und ewigem Lohn und Frieden.*
Friedrich Hölderlin: Die Linien des Lebens... (1843) [469]

**Summary.** This chapter deals with an ontological orientation in the subject of music. It was already contended in chapter 1 that music is communication, has meaning and mediates on the physical level between its mental and psychological levels. Such an orientation is topographic in nature since it offers a number of ontological "dimensions" and "coordinates" to profile musicological discourse and helps avoiding misplaced or blurred arguments. The topography involves three mutually independent dimensions: communication, reality, and semiosis. The local nature of this orientation scheme is discussed.

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Whoever is dealing with formal, logical, mathematical or computational methods in music does not necessarily have to be an expert in music philosophy. Nonetheless, these methods require a sufficiently refined orientation within the complex ontology of music in order to avoid erroneous interpretations of results obtained by use of these exact methods. It is well known that the precision of mathematical results, together with a poor knowledge about the delicate ontology of music may provoke a dogmatism for which mathematics is unjustly made responsible.

On the other hand, the *extensive* concept framework which we want to set up cannot even be approached without a firm pointer to musical ontology. In the following "topography of music" we shall sketch an ontological model without any dogmatic or unmodifiable character, though constantly available as a powerful coordinate system to locate the problems within the organism of music.

In view of the generic (and first provisional) delimitation of the concept of music (definition 1), it is proposed to set up from the very beginning a "three-dimensional" ontology, i.e. an ontological system which tells rather *where* the concept of music subsists than *what* its being

is like. Let us shortly deviate on this philosophical line to justify the somewhat particular approach. To be clear, we are not concerned with music as such (a Kantian "Ding an sich") but with music that we localize and conceive *within* our knowledge system. Following Immanuel Kant [258, p. B 324], a concept is a local place, a topos in the knowledge space. Ontology then questions the place of the concept, the space-coordination of the music topos.

We do, in other words, stress three ontological coordinates as an intrinsic unity pointing at the concept of music. These coordinates are

- *REALITY*
- *COMMUNICATION*
- *SEMIOSIS*[1]

We shall hereafter (sections 2.1, 2.2, 2.3) introduce them in detail. To understand music as a whole, you have to specify *simultaneously* its levels of reality, its semiotic character, and its communicative extension. Being a fact of music means having these three perspectives or ontological coordinates. Omitting any one of these determinants is an abstraction (though not an aberration) from the full ontology.

It might be argued that this topographical reduction of ontology simply delegates the difficult question to the ontological coordinates: instead of asking for one ontological specification, we have to examine three partial specifications. The argument is correct, but it is also true that these partials are simpler, more elementary. Dealing with semiosis *without* being concerned with its level of reality (and vice versa) is a huge advantage. Giving the ontology a topographical "turnaround" simplifies the problem. Whether the overall ontological question is settled by this procedure is not answered here. Let us instead begin with a closer inspection of the three proposed coordinates—we shall come back to this subject after that in section 2.5.

## 2.1 Layers of Reality

**Summary.** The dimension of musical reality involves *physical*, *psychological*, and *mental* layers. It is not question of reducing one of these realities to the others: Either of them has an autonomous existence which can at most be transformed into others, but not eliminated. The great majority of reality-specific phenomena cannot even be translated into external layers without substantial deficiencies.

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Music takes place on a wide range of realities. They may be grouped into physical, mental and psychological levels. Differentiation of realities is crucial for avoiding widespread misunderstandings about the nature of musical facts.

A representative example of this problem is Fourier's theorem, roughly stating that every periodic function is a unique sum of sinoidal components. Its a priori status is a mental one, a theorem of pure mathematics. In musical acoustics it is often claimed that—according to Fourier's theorem—a sound "is" composed of "pure" sinoidal partials. However, there is no

[1] Semiosis refers to the fact that music deals with a system of signs.

physical law to support this claim. Without a specific link to physics, Fourier's statement is just one of an infinity of mathematically equivalent orthonormal decompositions based on "pure" functions of completely general character, see [127, ch. VI]. To give the claim a physical status, it would be necessary to refer to a concrete dynamical system, such as the cochlea of the inner ear (see appendix, section B.1), which is physically sensitive to the first seven partials in Fourier's sense.

Methodologically, there is no reason nor is it ontologically possible to reduce one reality to others. The problem is rather to describe the *transformation rules* from the manifestation of a phenomenon in one reality to its correspondences within the others. To be clear, a neurophysiological transformation ("explanation") of a psychological phenomenon does not, however, conserve the psychological ontology of the phenomenon. The specific phenomenon within the psychological topos *corresponds* to another phenomenon within the physiological topos. But ontologically, the phenomena do not collapse.

We now give an overview of the three fundamental topoi of reality and their specific characters.

### 2.1.1 Physical Reality

Music is essentially manifested as an acoustical phenomenon, made by means of special instruments and perceived by humans. Nonetheless, its acoustical characteristics are less—if at all—condensed within a unique physical sound quality than in the physical input-output systems for sound processing. The use of physical sounds is a function of very special devices for synthesis and analysis. This is reflected in the fact that to this date, there is no generally accepted classification method for musical sounds, see section 12.3 for a thorough discussion. This is not due to missing synthesis or analysis methods and techniques. Besides classical analog sound synthesis methods as they are realized on musical instruments, there are various digital sound synthesis methods, see appendix A.1.2.

The problem is rather that classification of musical sounds is arbitrary *without* reference to their semantic potential, see section 12.3.3. Physical reality of music is only relevant as an interface between 'expressive' and 'impressive' dynamical systems. All physical sounds are equally natural, be they produced by a live violin performance, a computer driven synthesizer via loudspeakers, or by the tape patchwork of musique concrète.

On the other hand, the central receptive system for music is the human auditory system: Peripherical and inner ear, auditory nerve, its path through multiple relays stations of the brain stem, the neo- and archicortical centers for auditory processing and memory, such as Heschl's gyrus and the hippocampal formation, see appendix B. This extremely complex physiological system is far from being understood. Even though some insights into the dynamics of the cochlear subsystem do exist, it is not known which analysis of the musical sounds takes place on the higher cortical levels. It is not even clear how the elementary pitch property of an ordinary tone is recognized [72]. This means that on the cognitive level, human sound analysis is not yet understood. Therefore, reference to particular sound representation models is good for synthesis options, and for speculative models of cognitive science [292], but *not* as a firm reference to human sound processing.

### 2.1.2 Mental Reality

Just like mathematical, logical and poetical constructions, musical creations germinate as autonomous mental entities. It is a common misunderstanding that musical notation is an awkward form to designate physical sound entities. Being a trace of intrinsically human activity, the phenomenological surface of music is linked to mental schemes which we call scores: oral or written text frames of extra-physical specification. Scores are mental guidelines to an ensemble of musical objects. They reflect the fact that music is composed as well as analyzed on a purely mental level. Obviously, scores do point at physical realization, but only as a *projection* of a mental stratum into physical reality. A fact of harmony or counterpoint is an abstractum much the same as an ideal triangle in geometry. In this sense, playing a chord on a piano corresponds to drawing a triangle on a sheet of paper.

### 2.1.3 Psychological Reality

Besides its physical manifestation and its mental framework, music fundamentally expresses emotional states of its creators and emotionally affects its listeners. This was already known to Pythagoreans [541] and defined as a central issue of music by René Descartes [126], see [120]. Such an emotional reality of music is neither subordinate nor abusive, on the contrary: to music lovers the emotional response is a dominant aspect, individually, and socially. It is not by chance that song titles such as "It don't mean a thing if it ain't got that swing" de facto reduce musical meaning to an emotional category, the feeling of swing. Like other realities of music, the psychological dimension cannot and needn't be reduced to others, it is a manifestation of an autonomous, irreducible ontology.

## 2.2 Molino's Communication Stream

**Summary.** Communication is the second dimension of our topography. Following Jean Molino, it splits into a three-part stream starting from the *poietic* instance of the *creator* of a work of art, traversing the mediating *neutral datum* of the virtual *work*, and ending on the receptive side of the *listener's esthesic* perception.

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According to Jean Molino [377], we describe the tripartite communicative character of music, see figure 2.1. Molino's scheme is an abstraction to the essentials; they may be complex and interlocked in concrete situations, see section 2.5. Its structure is comparable to Oskar Bätschmann's "grosses abstrakt-reales Bezugssystem der Auslegung" which he proposed as a hermeneutical framework for the analytical discourse in fine arts [43]. Like Molino's scheme, Bätschmann's construction issued from the insight that a dispassionate discourse on arts, be it "parler peinture" or "parler musique", cannot succeed without considering their communicative structure.

Molino's scheme partitions the communication process into three instances: creator, work, and listener. To these, three types of analytical discourse correspond: poiesis, neutral level, and esthesis. These concepts are technical terms which have to be explicated in the following. In

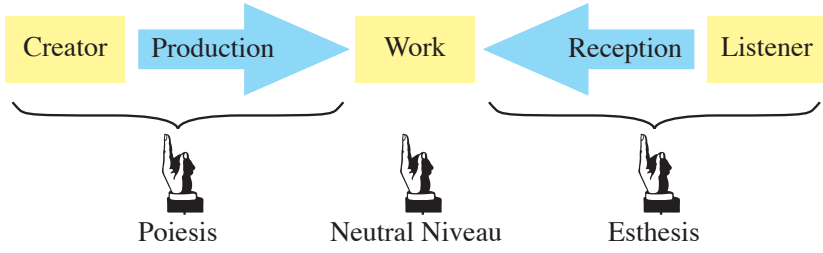


Figure 2.1: The three-part scheme of music communication following Molino.

particular, Molino does not prejudice such a thing as causality between creator and listener by means of the work, or receptive passivity of the listener towards work and creator.[2]

### 2.2.1 Creator and Poietic Level

To begin with, a *creator* instance subsumes all factors which are essential and sufficient for the production of a musical work. This level describes the "sender" instance of the "message", classically realized by the composer. The corresponding analytical discourse of the work's *poiesis* has been introduced by Etienne Gilson [183]. According to the Greek etymology ($\pi o\iota\varepsilon\iota\nu$ = to make), "poiesis" relates to the one who makes the work of art. Poiesis is concerned with the individual condition of the creator as a result of its history of development as well as with the role of the broader socio-cultural frame of the work. In specific cultural contexts or in a more refined discourse about art production, the creator instance may as well be the musician or the performing artist. In jazz, for example, the improvisational aspect is a genuine making of the music, and the performance of a classical piece of music is a creational act.

Poiesis is a sharp analytical instrument. Already the very definition of a sound by means of its defining parameters, such as pitch, color, etc. bears a strong poietical flavor—quite contrary to the naive and widespread belief. For example pitch (logarithm of frequency) is no invariant of the sound, it usually results from the sound construction as a product of a periodic vibration of a determined frequency and of an envelope (see appendix A.1.2.1). Neither frequency of the periodic vibration nor the envelope are uniquely determined by their product. And, worse, not even the frequency of an ideal periodic vibration is uniquely determined from the vibration as such. In fact, Fourier's theorem only states uniqueness of its coefficients *if* the fundamental period is given. But the latter is not automatically inscripted in a vibration. Such parameters are part of poiesis: the way of making the sound, not the "neutral" sound object. This shows that poiesis is not restricted to psychological aspects of "how to fabricate a composition", it can relate to completely "objective" physical or mathematical facts.

[2] We should like to relate these concepts to the activities of production, reception, documentation, and communication, as described in section 1.1. Activities are instances of what here is denoted by "communication". What we call "poietic level" corresponds to "production activity", "reception" is on the side of "esthesic level", "documentation" relates to "neutral level", and the activity of communication corresponds to the transition processes between all the three levels in Molino's system. We have nonetheless maintained these concept frameworks since Molino's technical terminology does not reflect the aspect of overall activity in music.

### 2.2.2 Work and Neutral Level

The *work* is the turning point of musical communication. It subsumes whatever determines directly the production of sound events within a specific context. Its identification depends upon the contract of creator (sender) and listener (receiver) on the common object of consideration. Musicology has difficulties with the work concept since its determinants seem to be of fuzzy nature. This relates to the dramatic difference between music and fine arts. As a matter of fact, a painting is determined down to its humidity and temperature—at least from the moment it has gained 'eternal' appreciation and allocation in a professional museum. In music, the definition of a work is left with ample variants. Decisions about including sound color or tuning prescriptions in Bach's "Wohltemperiertes Klavier" do in any case redefine a new and different work. And above that, such a decision transcends the strict definition and establishes a *valuation* of the work in the light of historical parameters of Bach's creativeness.

In this context, a *score* may be viewed as being the *organizational scheme* of a work. In fact, physical realization of music is always based upon mental schemes which we call scores: oral or written text frames of extra-physical specification. Apart from classical western traditions where this fact is evident, the score concept is also adequate for describing traditions which are more towards ethnomusicological antipodes. First example: In the music of Noh theater [269], there are different score instances, e.g. for vocal utai music denoted in melodic units (fushi) to the right of texts, or for the hayashi notation systems for flutes and drums. Second example: The improvisational culture of jazz—which in its making only marginally relates to traditional western scores—is based on the concept of the interior score ("partition intérieure" [487]). This means that, even for free jazz improvisers, there is an interior reference system of lexical character, together with a selection code which guides performance. The fundamental fact behind the basic role of the score concept for music is that human organization in a complex time-space of acoustical and gestural nature cannot be executed without an interpersonal spiritual orientation common to the responsible participants.

Following Molino, the analytical discourse which—independently of the selection of the tools—is strictly oriented towards the given work is termed *neutral level*. In particular, the historical justification for a work's delimitation is *not* part of the neutral level. Molino's concept has only attracted controversies since it—faultily—seems to imply the preference of a particular analytical tool/method to the exclusion of other possible approaches. But such a thing as the "unique ideal interpretation" is precisely the famous *unicorn of hermeneutics*—a superfluous illusion. A momentous analytical perspective is just one among many possible accounts of the work, and neutral analysis is the variety of *all* these perspectives.

Such a work may be the divine creation of a medieval tone system, a work of computer music which is explicated as far as its acoustical details, or the beat of waves on the open waters of the sea, no matter: Neutral level means propaedeutic analytical precision work preceding any valuation whatsoever. Evidently, exact methods will play an eminent role in realizing such a task.

### 2.2.3 Listener and Esthesic Level

The *listener* is the instance which perceives and interprets a given work. It is subject to the same type of determinants as the creator. Coordinates of a listener will vary from case to case and

coincide only punctually with each other or with the creator's coordinates. The characteristic difference between listener(s) and creator is the time arrow towards the common work. Whereas the creator produces a work, the listener perceives and interprets an already existing work. From the moment of a work's existence, the creator's existence is fixed whereas the number of the listeners' existences grows as a function of those who are engaged in the work.

According to a proposal of Paul Valéry, [537] the analytical discourse on the listener is termed *esthesis*, following the Greek term $\alpha\iota\sigma\theta\eta\sigma\iota\sigma$ (perception), in order to avoid confusion with esthetics, the study of beauty. Esthesis deals with perceptive valuation of a work by the listener. This valuation of the work's attributes is realized as a function of the listener's individually variable position. This interpretative valuation from a determined perspective is not less active than the creator's activity in the creation of the work. And this not only regarding the artistic performance, but quite generally. Like poiesis, esthesis is integral part of the analytical elements constituting the existence of a work.

The distinction between poietical and esthesic levels is fundamental for a sound conceptualization in musicology. Only in very rare cases can we expect the conservation of results from a poietical discourse within the esthesic point of view, and vice versa. Nonetheless, we observe again and again the faulty trial to implant historically and systematically clearcut *poietical* concepts, such as "pitch", into an esthesic discourse. The famous paradigmatic justification of such a malfunction of analysis is known under the dictum that "the ear is the highest musical instance". On one hand, such a dogmatic phantom as "the ear" does not exist, is incompatible with the multiplicity of listening cultures. On the other, the ear fails as a metonymy of cortical music processing: there is a huge transformation process between the auditory cortex and the ear's cochlear system.

Poietics cannot be taken as the ideal access to a music work. There is no thing such as a causal relation between creator and work. The latter is a result which may be generated by very different processes. Within the communication process, the creator above all differs from the listener by his/her position in time. This fact becomes manifest when we imagine the work's production process as a fictional unfolding in a *retrograde* movie. This visualizes a well-known insight from fine arts theories that the artist/creator is the first observer. In this way, retrograde poiesis is embedded in esthesis. And vice versa, the esthesic discourse may be restated as a manifold of variants of imaginary retrograde poieses.

**Example 1** The problem of symmetries in music is a good illustration of the communication-sensitive aspect. A classical conflict concerning the role of retrograde in music arises from the observation that this construction "cannot be heard and thus is a problematic feature". Communication coordinates make this discussion more transparent: The retrograde construction as a poietic technique is a common compositional tool. It fits into the toolkit of contrapuntal constructs for organizing the compositional corpus. On the other hand, the esthesic perspective of the retrograde is concerned with the question whether and how clearly such a construct can be decoded by the listener. This latter question is a completely different topic and cannot be identified with the former. More precisely, the role of the retrograde as an organizational instance is not a function of its perceptibility as an isolated structure. The psychological question of whether a retrograde can be perceived is rather this: "Can a retrograde be distinguished from random?" Finally, retrograde structures may be recognized as objective facts within the neutral level of the score without being either constructed be the composer or consciously perceived by

the listener. Summarizing, the communicative coordinates help localizing and thereby making more precise the musicological discourse.

#### 2.2.3.1 The Problem of Identity

From the preceding communication-theoretic considerations, the *identity of a work* is triply stratified. To begin with, we envisage a work that is given before any analysis is performed. We call this data its *abstract* identity. This first identity is then enriched to yield the *neutral identity* as a result of the multiplicity of neutral, valuation-free analyses. This identity is the basis for a variety of esthesic—in particular: retrograde poietic—valuations. The esthesic identity is built from all these esthesic valuations and their mutual relations.

Accordingly, a work is only identified when the *infinite* process of esthesic valuation, built upon abstract and neutral identification, is completed. This yields at least one justification for an incessant performance practice in classical music, but see section 32.2 for more details on performance and identification.

## 2.3 Semiosis

**Summary.** Independently on what is communicated, and on which level of reality this takes place, music intrinsically involves complex signification processes. The generic setup of semiotics understands a sign as being a tripartite object. It consists of a *significant* expression, inducing the *signification* act of translation, and thereby producing the expression's content, the *significate*. We give a short review of dichotomies of structuralist semiotics. Musical semiosis reveals a complex concatenation of meta- and connotation-layers in the sense of Louis Hjelmslev. Rather than being absent, musical meaning is distributed over a sequence of semiotic subsystems.

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By use of a highly developed textuality of musical notation as well as by the very intention of musical expression, music is structured as a complex system of signs. This is not only a marginal aspect: Music is one of the most developed non-linguistic systems of signs. We shall first present the elementary sign character of musical objects (subsections 2.3.1, 2.3.2, and 2.3.3), and then (2.3.4) review the overall semiotic perspective of music.

### 2.3.1 Expressions

Already the earliest medieval music notation is motivated by the very nature of the graphical neumes: etymologically as well as substantially they are gestural hints pointing at movements in pitch and rhythm. This coincides with the latin etymology of sign: signare = to point at, give a hint. Neumes are *aliquid pro aliquo*, they express something in the sense of semiotics. Besides and beyond musical notation, music is often viewed as an expression of emotions, spiritual contents or gestural units. In any case, music has a phenomenological surface that is organized in a spatio-temporal syntax. Albeit more complex than linguistic syntax, the musical syntax shares some of its characteristics, see section 2.3.4.

### 2.3.2 Content

According to the famous dictum of Eduard Hanslick [206], we learn that: "Der Inhalt der Musik sind tönend bewegte Formen." This evidences that the notated complex of musical graphems is not the content but points at some kind of sounding content: they *mean* something. Hanslick's characterization is a minimal semantic setup but *some kind of content* can be identified. The remarkable aspect of Hanslick's approach is that it associates musical content with mathematical content. As a matter of fact, a triangle is a mathematical object that essentially reduces to form. But mathematicians do associate it with a content, usually with a platonic entity addressed by abstract symbols or by drawings with precise quantifications.

### 2.3.3 The Process of Signification

Signification is the most important instance for the realization of a sign. It is responsible for the transformation from the expressive surface to the hidden meaning. The fact that performance is such a central issue of music gives a strong proof of the qualified presence of signification in music. Without the laborious effort of turning significant expressions into the reality of their meaning, music is not what it is meant to be. For musical signs, this semiotic process bears a highly differentiated structure which is sensitive to Ferdinand de Saussure's dichotomies [41, 471], we review these items in the following section 2.3.4.

### 2.3.4 A Short Overview of Music Semiotics

Semiotics of music is a complex subject which cannot be dealt with in this context. For a full account, we refer to [361]. This does, however, not dispense us from a short overview of this subject since there is no hope to understand music without a minimum of semiotical prerequisites. We shall describe music semiotics from the perspective of structuralist semiology as it has been sketched by Roland Barthes [41] as a generalization of the linguistic theory of Ferdinand de Saussure [471]. We pursue this project according to André Martinet's *principle of relevance* [318]. It states that selection of materials and methods has to be justified by relevant criteria in the sense of points of view towards the actual object—music in our case.

Let us recall that Barthes' position does not mean that music is interpreted as being a type of language. On the contrary, we shall repeatedly stress significant differences between the music system and linguistics.

To say that a system is semiotic means that it is articulate according to the fundamental stratification of signs into significant, signification, and significate layers (Saussure's signifiant/signification/signifié). This stratification articulates the process of production of meaning (the signification), which points from the expressive surface of the mediator (the significant), to the underlying meaning (the significate). Slightly deviating from Saussure's original approach, the existentiality of the significate is that of a concept, i.e. it is part of a "lexical body", of a system of sayable things ($\lambda\varepsilon\kappa\tau o\nu$ in the stoic tradition, in contrast to the "real" things or to the psychological imagination, the latter being Saussure's preferred significates).

Like every existing semiotic system, music is embedded in history. If we investigate the system's state at a determined moment, we deal with *synchronic* analysis. Of course, the concept of a moment really means a short duration wherein the system remains fixed. If, instead, the system's objects are observed in their temporal development, we deal with *diachronic* analysis.

For example, the etymology of the concept of consonance is a diachronic subject, whereas looking at semantics of the augmented triad in Franz Liszt's "Faust" symphony points at a synchronic analysis. Evidently, sychronicity and diachronicity are not strictly separable since—just as with physics—the space-time continuum of music is indecomposable.

The special class of *shifter* or *deictic* signs is very important in music and is characterized as follows: Their significate transcends the lexical reality and penetrates the unsayable existence of the system's user. Whereas, for example, the concept of a sonata form is lexical, the emotional significate of music can neither be "lexicalized" by decree, nor verbalized whatsoever—contrary to numerous flopped attempts. In linguistics, shifters (such as "I", "here", "now") are exceptional signs. In music deixis, such as emotional signification, is the standard situation. But this does not mean that the lexical part of the system is marginal. On the contrary, a rich deixis cannot succeed without a rich lexical support. It is evidently a difficult business to "hit" the unsayable by use of an arsenal of lexical approximation pointers.

As a function of the type of signification process, Saussure distinguishes between *arbitrary* and *motivated* signification. Signification is arbitrary if it is defined by pure convention. For example, the signification "highten a pitch by a semitone!" of the sharp # is pure convention, whereas the signification of a tremolated kettledrum beat towards the significate "thunder" is motivated as onomatopoeia. Notice that the arbitrariness of musical notation is a major source of the elitarian music culture; fortunately, a huge number of semiotically well-motivated music software has started to break this artificial obstacle to the world of music. A standard means for diminishing arbitrariness is the use of system immanent logic or of instances exterior to the system which are at disposal as given evidences. The latter is the case if our system lives in the context of other systems (e.g. software environments) which may help to operationalize it. A famous example of system immanent logic is Roman Jakobson's *poetical function*. It enables production of significates by means of "projection of the paradigmatic axis into the syntagmatic axis" (see below and sections 11.6.1, 48.2.1).

As a whole, the system has sign character but it is composed from single signs. If two signs are adjacent in space or time they are called contiguous. Contiguity is "in praesentia". It is a fact of concrete nature, no further abstraction is needed. A chain or network[3] of signs in successive contiguity is called syntagm, in linguistics also: syntagmatix axis. A progression of degrees or a contrapuntal interval sequence are syntagmata in time; a texture of four parts (soprano, alto, tenor, bass) or a discantus which is set against a cantus firmus voice, these are syntagmata in pitch space. As a consequence, syntagmatic articulation of music looks quite different from linguistics. Whereas linguistic syntagm is one-dimensional (in time), the music syntagm extends simultaneously in several dimensions.

If two signs are similar then they are called in *apposition*. (Martinet uses the term "opposition" in linguistics but in music this binary term is too narrow.) Similarity may regard any of the three layers—significant, signification, or significate. The collection of all signs which are in apposition to a given one define the *paradigm* of that sign. Apposition is a relation "in absentia". It requires a high degree of abstraction, we come back to this important difference to the syntagmatic contiguity in section 49.4. For example, tones which are perceived with loudness equal to that of a reference tone define the paradigm of loudness of this tone. Two such tones do not necessarily yield the same loudness perception (see appendix B.2)!

[3]This extension transcends the usual definition, we allow a multiplicity of simultaneous concatenations, not only one chain such as the case in ordinary language.

According to Louis Hjelmslev [226], a given semiotic system can appear as a sign-theoretic instance (significant, signification, or significate) of another such system. If it appears as a significant layer, we speak about *connotation*. The significant system is the termed *denotative level*, where as the superior system is termed *connotative level*. If, conversely, a system occupies the significate layer of another system, the superior system is termed *metalevel* or *metasystem*, whereas the inferior system represents the *objectlevel* or *objectsystem*. Generalizing Hjelmslev's approach, one should also envisage the case where a system appears as the *signification layer* of its supersystem. In general, semiotic systems are complex interlockings of connotative, metasystemic, and signification layers. For example, the double articulation in language: grapheme ⇒ acoustical significate ⇒ conceptual significate can also be observed in music. Here, in Hanslick's terminology, we have the double articulation Note ⇒ Ton ⇒ tönend bewegte Form. On the other hand, grammar of language and harmony condivide their metasystemic character [70, **?**]. For an overview of the entire music system articulation in the sense of Hjelmslev, see [361].

The last of the famous Saussurean dichotomies—it is also decisive for distinguishing music from language—concerns *language* against *speech* (Saussure: langage/parole, Barthes: structure/procès). This means the contrast between general reglementation and concrete syntagms and paradigms as far as dialects and revolutionary poetical invention. In order to avoid confusion with the linguistic context, we shall adopt the dichotomy of (music) *norm* /(music) *process* (corresponding to language/speech). This dichotomy shows a strong difference to the linguistic dichotomy in so far as such a thing as a music norm is neither a synchronic nor a diachronic constant. And emotional significates of music by definition forbid any normative fixation, they are music processes, "sound speech"("Klangrede") as they used to call it. On the other hand, the musical denotation level in classical score notation is extremely normative, and as such a source for elite formation whose control is often confused with musicality. The analphabet differs from the score illiterate in so far as production of musical meaning has very little to do with score notation whereas competence in linguistic articulation is intimately related to lexical competence.

The Hjelmslev articulation of music [361] shows that musical meaning is not concentrated on a semiotic spot, it is on the contrary distributed among several system layers of denotation and connotation. In other words, musical meaning is the result of a complex development of signs, it grows from simple categories of signification to more and more all-embracing and even transcendental categories as they appear in highly spiritual musical expression.

## 2.4 The Cube of Local Topography

**Summary.** Putting together the three dimensions of musical topography, the topographic cube is defined. While each of the described dimensions splits into three "coordinate" values, the cube involves $3^3 = 27$ distinguished topographic locations. Their role in guiding a sophisticated discourse on music is discussed and exemplified. However, this cube's orientation is of local nature, it does not provide a global point of view.

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After presenting the ontological dimensions of reality, communication, and semiosis, we may view these contributions as coordinate axes of a more complete musical ontology. This

means that an ontological localization can be interpreted as a point in a three-dimensional cube spanned by the axes of reality, communication, and semiosis, each of them being articulated in three "values":

- reality: physical–psychological–mental.
- communication: creator–work–listener
- semiosis: significant–signification–significate

This suggests a geometric representation of the ontological variety as a "topographic cube" of musical ontology, see figure 2.2. To describe the ontological position of a musical object, we have to specify its three coordinates in reality, communication, and semiosis. Each coordinate may take one of three values as listed above. This produces a set of $3^3 = 27$ possible topographic locations. Of course, it is not necessary that a general object be localized at a single location, it may as well occupy any subset of the cube. So, the 27 points are just the elementary positions from which more complex ontological situations are composed.

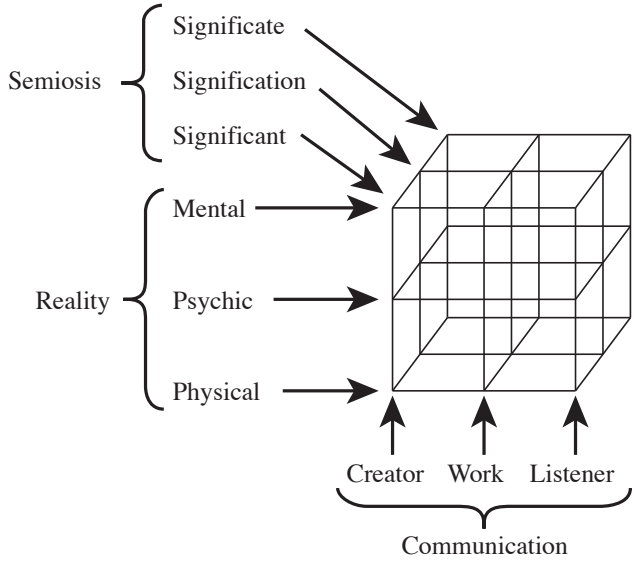


Figure 2.2: The cube of musical ontology.

To illustrate this topography, we present a classical text of music critique. It is the review of a concert by Franz Liszt written by Ludwig Rellstab in Berlin's ≪Vossischen Zeitung≫, cited following Knapp [265, p.86]:

> *Unter dieser Erweckung der vorteilhaftesten Eindrücke setzte er sich an das Instrument. Jetzt wird ein neuer Geist in ihm lebendig. Er lebt die Musikstücke in sich, die er vorträgt. Während er mit erstaunenswürdigster Gewalt der Mechanik eigentlich alles leistet, was bisher von irgend jemand einzelnem bezwungen worden*

*ist, und außerdem noch ein ganzes Füllhorn neuer Erfindungen, völlig ungekannter Effekte und mechanischer Kombinationen vor uns ausschüttet, so daß die aufs höchste gespannte Erwartung und Forderung sich weit überflügelt sieht: bleibt doch der eigentümliche Geist, den er diesen Formen einhaucht, das bei weitem anziehendere, anregendere und fesselndere Element. Diese geistige Bedeutsamkeit seines Kunstwerkes prägt sich aber auf das lebendigste in seiner Persönlichkeit aus. Die Affekte seines Spiels werden zu Affekten seiner leidenschaftlich aufgetürmten Seele und finden in seiner Physiognomie und Haltung den treuesten Spiegel. Seine künstlerische Leistung wird zugleich eine Tatsche des Inneren, sie bleibt nicht getrennt von ihm, sondern wirkt in dem mächtigen Bündnis mit dem Geist, der sie erzeugt.*

Rellstab addresses all three levels of reality. Words such as "Instrument", "Gewalt der Mechanik", "mechanische Kombination", "Haltung", and "Physiognomie" refer to physical reality.—Expressions such as "Affekt seines Spiels", "Affekte der leidenschaftlich aufgetürmten Seele", "anregend", "fesselnd", and "lebt die Musik in sich" refer to psychological reality.—The mental level is addressed in expressions such as "neuer Geist wird lebendig", "Geist wird den Formen eingehaucht", "geistige Bedeutsamkeit", "Bündnis mit dem Geist", and "Tatsache des Inneren".

The communicative dimension specifies in its poietical coordinate with expressions such as "trägt vor", "*haucht* Geist *ein*", Geist "erzeugt" künstlerische Leistung", "Erfindung (Füllhorn)".—The neutral level is addressed in passages such as "Musikstücke vortragen", "Geist den Formen einhauchen", and "Kunstwerk". The esthesic coordinate is addressed by allusions like "wecket Eindrücke", "vor uns ausschüttet", "Erwartung (der Hörer)", "Forderung", "anregend", and "fesselnd".

The semiotic system is instantiated when Rellstab talks about the significant coordinate in expressions such as "Musikstück", "Kunstwerk", "Form", "Persönlichkeit als Gefäß für die Bedeutung", "Physiognomie", and "Haltung". (Observe that "Haltung", for instance, simultaneously addresses physical reality and the semiotic significant.)—Signification is traced in expressions such as "lebt die Musikstücke", "geistige Bedeutsamkeit", "einhauchen", and "in dem mächtigen Bündnis mit dem Geist".—Finally, significates are addressed in "Leidenschaft", "Geist", "Affekt", and "Überflügelung der Erwartung und Forderung".

## 2.5 Topographical Navigation

**Summary.** The topographic cube offers a local and recursive orientation. Hence the corresponding discursive navigation receives a ramified path structure: A priori, each topographic location may open or participate in another topographic variety. The local and recursive nature of such a ramified navigation is described and exemplified.

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Observe that the topographic specification of a fact of music is a local resp. recursive one in the following sense which one may visualize as a *conceptual zoom-in effect*:

- Parts of a sign may be entire signs of their own right. For example, the significate of a sign in a metasystem—by definition—is an entire sign of the objectsystem.

- Also, in the communication chain, the performing artist is a creator for the auditory, but he/she includes an entire communication process, starting from the composer, and being communicated through the score.

- Third, regarding levels of reality, an acoustic sound is essentially a physical entity, but its description refers to mental instances, such as real numbers for parameter values.

In other words, the topographic cube yields a local conceptual orientation, and by regression, the topographical description of a fact of music may open a complex tree of ramifications, each knot being loaded by a localization within the cube. Supposedly, there is no consistent ontology without such a self-referential regression. In particular, it is not necessary to introduce such a thing as a "topographic metacube" for the description of metamusical facts (e.g. harmony syntactics) since a metalanguage precisely *means* recursiveness on the level of the significate.

# Chapter 3

# Musical Ontology

*Musik ist ohne Begriffe.*
Hans Heinrich Eggebrecht [101, p.192]

**Summary.** This chapter introduces the difficult subject of musical ontology. Such a discussion is substantial for a reconciliation of traditional musicology and innovative perspectives in cognitive and computational musicology.

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It is a common argument of traditional musicologists against cognitive and computational methods that the full extent of musical being cannot be grasped by these methods. It is claimed that the very depth and transcendence of music are beyond any analytical and quantitative reasonment.

We do not condivide such a credo for the following reasons. Above all, cognitive and computational methods are neither restricted to quantification, nor to simple analysis. Rather modern mathematics and logic is an issue of conceptual explicitness; simple numerical quantification is a very special subject of old-style mathematics that has been overridden by structure theories since the development of set theory, modern algebra, topology, and geometric logic in the first half of the twentieth century. Further, semiotics of general sign systems has proven that formal explicitness on the levels of significants and signification processes does not prevent significates from important *added value* of depth and transcendence.

In other words, depth and transcendence are not missed by computational and cognitive approaches. Rather we can observe an increasing distribution of aspects of depth and transcendence among the conceptual topography. This provokes a crucial question of ontology: Can the problem of whatness be tackled by a differentiated discussion of the problem of whereness? At this point, we should make more precise the ontological problem.

In fact, it is not a question of specifying the music's whatness as a Kantian "Ding an sich", but of how we *conceive* music: "What is the concept of music?" But a concept is—again in Kant's words [258]—a topos, a logical place. That is, asking for the concept of music involves a topos in a particular 'concept space'. Recall [417] that platonic ideas live in an ontological fundus, the hyperouranios topos, which is an explicit topographic site: Relative position and hierarchical location of its instances serve as basements for the whatness of ideas.

It is therefore a legitimate procedure to open the ontological discourse by a view of the conceptual topography of music.

## 3.1 Where is Music?

**Summary.** The difficult localization of musical existence is due to three factors: (1) the interweaved usage of topographically distinct locations in the classical ontological discourse; (2) the tendency of classical musicologists to override topographical complexity by all-embracing breviloquent approaches in order to uphold the discourse; (3) the genuinely related activities of thinking and playing—music is one of the best operationalized fields of human cognition. We discuss and exemplify these factors.

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A closer look at musicological terminology reveals that many concepts are topographically blurred since their breviloquent function inhibits differentiation. For example, the concept pairing of consonance and dissonance is of complex reality (see Appendix B.3). It relates to psychic categories, to mental compositional principles, and to physical reality of hearing and cortical sound processing. Questioning this concept pairing *without* topographical differentiation (of reality coordinates in our case) yields a confusion of meanings and creates bunches of pseudo-problems (see loc. cit.). Historically, one can understand such a difficulty as a capitulation in view of exorbitant complexity, but systematically, it is necessary to distribute a concept over its topographical specializations. This procedure at first sight destroys comfortable unity and simplicity, but science is not about comfort, it is about truth. One may then, after establishing a differentiated and *adequate* terminology, return to unity by conceptual constructions which include complexity rather than repress it.

A most dramatic breviloquent repression strategy within traditional musicology was proposed by Hans Heinrich Eggebrecht on various occasions [134, 135]. Centered around the fundamental question, where to localize music ontologically, he concludes that music resides in the "I", a kind of super-breviloquent subject icon. This conclusion is deduced from a discussion of where music could be found elsewhere: in the work, in the composer, etc. Eggebrecht rightly observes that music is not exclusively in the work, nor is it exclusively in the composer, etc. Whence he draws the erroneous conclusion that only a kind of abstract ego may comprise the totality of music. It cannot be discussed in detail, how radical a declaration of scientific bankruptcy such a consequence does represent. But it is evident that the foundation of musical ontology in a whatever abstract subject *opens any issue of subjectivity* and prevents scientific discourse whenever it suits corresponding needs. A musicological discourse which is founded in the subject can go on forever, but it is for nothing.

A third difficulty for locating musical ontology is a fundamental, but difficult relation between thinking and making music. From the very beginning of European music theory in the Pythagorean tradition, playing an instrument (the monochord in those early days) was a method to gain evidence of the transcendent truth of music. It was contended that a metaphysical tetractys symbol was the core of musical ontology, and by playing the monochord, we could access that transcendent being [330, 541]. This approach was not just a comfortable way to understand music, it was the unique solution to gain understanding. In our days, this is still a

strong belief among musicologists, as the following proposition by Helga de la Motte [121, p.232] shows: "Musikalisches Denken ist grundsätzlich Denken in Musik." Not only is it admitted that 'thinking music by examples' is useful, no, it is fundamentally so! This is, of course, in the line of Eggebrecht's belief that music is without concepts. Thinking music is thinking *in* music, beyond words, *ineffable*, to rephrase it with the words of Diana Raffman [432].

*On the one hand*, this problematic stems from a poor localization on the semiotic axis. It is not recognized that signs of non-verbal type—mathematical formalisms, for example—could effectively resolve simple *verbal* ineffability. Sticking to common language, the dominant part of traditional musicological language, is the wrong strategy. No non-trivial insight in modern physics could be obtained via common language—not to speak of modern mathematics. We shall see in part VIII on performance that performance nuances, one of Raffman's advanced arguments for ineffability of music, can be perfectly formulated in the mathematical language of vector fields. In other words, thinking music by its direct invocation is first of all the proof of a defective language. The claim that something is *definitely ineffable* presupposes knowledge of any possible language, an absurdity.

And it is wrong to believe that music is a special issue of science in that it deals with objects that point at non-mental strata of reality. Psychology, for example, deals with emotions, physics deals with elementary particles. All these objects share aspects that transcend human conceptualization. But we may very well conceive them in a system of knowledge, model their behavior with remarkable success for our cognition. Music is not less and not more accessible than physics. But we have to establish a sophisticated system of signs in order to grasp the significates, common language is not the tool for a musical concept space.

*On the other hand*, music is substantially tied to its communicative dimension: Unlike physics, music was always pronouncedly *distributed* over poiesis, neutral niveau, and esthesis. However, this aspect is completely disjoint from the question of ineffability. It simply stresses the need for a communicative differentiation when dealing with music. Forgetting about the making of music, boiling it down to an object independently of its making and perceiving is a basic error, just as it is an error to make physical experiments without quantum mechanical uncertainty and complementarity. A famous defective argument which stems from the confusion of communicative coordinates is the judgment of contrapuntal symmetries in music. For instance, the retrograde movement has a completely different existentiality as a compositional (poietic) tool, than when discussed as a (esthesic) fact of perception, we have already discussed this issue in example 1 of section 2.2.3. Summarizing, we can state that most of the musical objects cannot be treated properly outside their topographically specified ontological modality.

## 3.2 Depth and Complexity

**Summary.** The ramified discourse based on orientation via local topographic cubes is described as a tool for generating controlled complexity in understanding music. The fundamental pointer character of this tree structure is compared to pretended depth in traditional approaches. The latter is identified as a rhetoric chain of external reference pointers.

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Both, traditional musicology and mathematics, use the word "depth", however in significantly different ways. To begin with, mathematicians speak of a "deep theorem" or a "deep

theory" to indicate that it marks a result of a complex development, and that this result gives a compact, pregnant insight into the addressed complexity.

For example, Fermat's last theorem[1] is a deep result since it involves an incredible complexity of mathematical structures, but it is a very compact, in fact elementary, statement. In this sense depth involves complexity, but rather in an *encapsulated* form; its surface shows a simple appearance.

Semiotically speaking, mathematical depth deals with simple significants that are built from complex, viz ramified and recursive signification to connotation and/or metasystem levels.

What, then, is the difference to depth in musicology? Strictly speaking, it is *pretended* encapsulation without the license to access contents. Let us make this clear in the following example. Eggebrecht, an authority of traditionalist German musicology, has stated that "music is without concepts" (see the catchword heading this chapter [101, p.192]). This is a prototypical deep statement since the reader is supposed to know that Eggebrecht is a scientific authority who has written a large number of books and papers on music. In fact, the citation is one of seven final theses in a book entitled "Was ist Musik?" which Eggebrecht and the other German pope of traditionalist musicology, Carl Dahlhaus, have written to discuss the 'ultimate and deepest' questions about music. At first sight, Eggebrecht's statement provokes a contradiction: How can somebody write hundreds of pages about music and simultaneously claim that conceptualization must fail? The solution is *depth*: We have to recognize that Eggebrecht has gone an incredibly long and audacious way in search of the 'real thing', but alas! has to report that—ultimately—the real thing is beyond conceptual comprehension. We, his humble pupils and admirers have to accept that music is an unreachable secret and He, The Master, does rightly encapsulate a deep insight. This capsule can—however—not be opened, the encapsulation remains locked, and this is mandatory for two reasons.

First, by the pronounced authority, He, The Master has experienced the horrific failure of concepts and we, the pupils, should just accept and admire this report from hell. Second, the capsule's key has been lost, no trace down to the very proof of the *pretended* conceptual failure is at hand. The text [101, p.192] does not give any hint, it simply states the verdict. Such a type of "depth" is completely standard in musicology. You give an encapsulated statement and prevent any access to the (presumably) hidden complexity. Whereas in mathematics, the access to complexity is possible and its realization eventually yields insight, the musicological encapsulation sticks to a pointer to insight, however, the pointer points into the void. You cannot obtain a follow-up, information flow breaks down, and verification of the claim is not offered on the base of explicit complexity. Rather is the dark path to the hidden (pretended) rationale ornamented by metaphors. In our case, Eggebrecht terminates by stigmatizing music with mysterious weirdness, thus turning depth into mystery—and science into fabulation. As a basic rhetoric device, this kind of void pointer has been used in one of the most ambitious handbooks of traditional musicology [103]. The discourse of important parts of that handbook relies on void pointers to never ending chains of implicit (!) external references. Fortunately, musicology is not doomed to void pointer depth. If applied with all its ramifications, the topographic cube (see section 2.4 and figure **??**) offers an effective tool of complexity which can help hammer out depth in the mathematical sense, i.e. without void encapsulation. In section 2.4, we have already

[1] This theorem is one of the most famous and hardest results in the history of mathematics. It was proven by Andrew Wiles in 1994 and states that any non-trivial integer solution of the equation $x^n + y^n = z^n$ implies $n = 1$ or $n = 2$.

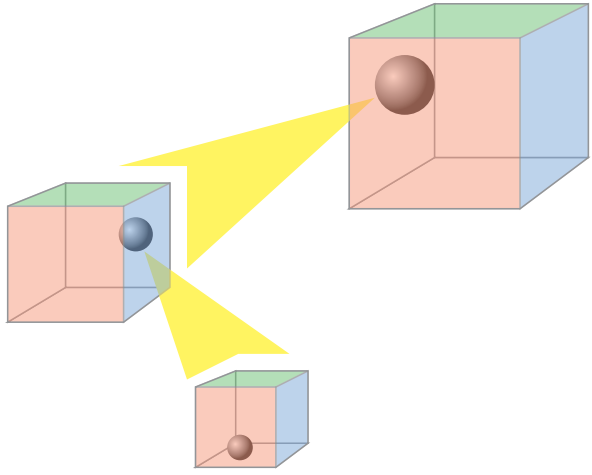

Figure 3.1: The topographic cube of musical ontology has local character. A concept of music (represented by balls in this figure) may open different localizations with successively refined concept analysis.

introduced the local character of ontological topography. It means that the decision to attribute to a given situation specific topographic coordinates is no obstruction to other coordinates in a successive step of refined analysis, see figure 3.1. But it does not mean that this recursion from one local perspective to the next tends to more elementary and eventually "atomic" ontology. Ontological atomism is neither desired nor even possible as a foundation of knowledge: Being has no elements, only perspectives. The cube of topography is a local coordinate system for such perspectives, it offers orientation, but no roots.

**Example 2** On the string of communication, the performing artist is on the esthesic coordinate with respect to the composer. He or she receives the work and analyzes it as a mental entity. This complex identity of the performer is then encapsulated and taken as a new communication unity of the concrete performance. In this role, the performer stays on the poietic side and produces a work of physical reality, namely the sounding performance. This latter is an expression of the artist's reflection and will now transmit the message to the listeners of a concert hall, say. In this description, the listener is still encapsulated as an esthesic instance, but we should open it with respect to what is going on when understanding a performance of a work of art. In this perspective—the inner dialog of the listener with his or her consciousness—the acoustical performance is again transformed into a mental structure and interpreted along the personal lines of musical insight. The result is a new type of work of mental and/or psychic reality which the listener will communicate internally or externally in a further processing of the musical experience. Etc.–

We have just sketched some en- and decapsulation processes on the communicative dimension, but the reader can easily extend this flip-flop of perspectives on any other topographic

dimension. It becomes clear now that opening ontological perspectives is by no means an approach to ontological roots, but it *is* a clarification and gives ample orientation which void pointers definitely *do* refuse.

It is one of the main tasks of part II to develop a profound and detailed navigation formalism through music, a formalism which provides encapsulation and reliable pointers to hidden complexity at the same time. But a word of caution is imperative: Our task cannot be attained without a considerable amount of technical machinery—nihil ex nihilo.

# Chapter 4

# Models and Experiments in Musicology

*Plato's pessimistic picture of*
*empirical observation caused him to*
*deny the validity of physical models and was*
*largely responsible for the*
*eclipse of empiricism for 2,000 years.*
Peter Wegner [563]

**Summary.** This chapter introduces the paradigm of experimental humanities. Such a perspective is basic to all computational, resp. effective, methods in musicology. We discuss the parallel to the epochal Galilean change from speculative to experimental natural sciences.

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In the final chapter of this first introductory part, it is necessary to review the overall epistemology of musicology since the preceding ontological topography has questioned traditional standpoints in a measure that does not allow of uncritical takeover of epistemological fundamentals. Above all, it is the requirement of free access to (encapsuled) complexity that creates a boundless analytical attitude: It is no longer possible to 'cultivate' private, inaccessible regions of knowledge.

This is not only due to the fact that music belongs to humans and no longer appears as a revelation of divinities of whatever flavor. It is also due to the massively improved arsenal of information and communication technology where the knowledge space becomes a concretely accessible site, and any lack of precise or explicit information is immediately blamed.

This situation enforces a new, fundamentally interactive understanding of human knowledge. The paradigm of contemplative science which was essentially promoted by religious constraints [363, 477] has to be abandoned, knowledge is constantly updated and extended. The global knowledge space has become an ocean which is in permanent metamorphosis, on which we navigate and experiment in a spirit of dynamical space-time [362].

We shall make these points more explicit in the following sections. It is useful for the understanding of new paradigms in musicology, however, to have a closer look at the development of modern experimental science which prepared the Galilean revolution. In this preliminary discussion, we refer to Isabelle Stengers' article [485, p.395]. The case of Galileo Galilei is important for musicology since it is intimately related to a common problem: definition of musical tempo and physical velocity. Galileo was concerned with the definition of instantaneous velocity. Recall that, for historical reasons, he could not yet refer to calculus and define velocity as derivative $ds/dt$ of space $s(t)$ as a function of time $t$. He had to develop the very *concept of instantaneous velocity* since the traditional concept of uniformly accelerated motion, as it was understood by medieval scientists such as Nicholas Oresme, could only view it as a succession of portions of uniform motion, changing discontinuously their constant velocity after a short duration. Galileo's concern was not just another speculative concept but resulted from intense observation of the movement of physical bodies on an inclined plane. His proposals (dated around 1604) rather aimed at measuring (by impact) local velocity in a determined moment of time.

Figure 4.1: Galileo's method concentrated on observation and experimental interaction with nature. This detail of a fresco by Giuseppe Bezzuoli (Museo di Storia della Scienza, Florence) shows the famous experiment of a ball running down an inclined plane (right) and the traditional consultation of Aristotelian works (left).

In other words, Galileo's approach was essentially built upon observation and measurement, it did not rely on abstract speculative reflections. The turning point here is precisely the passage from speculative encapsulation to explicit accessibility by 'doing' science. Measuring the body's mechanical impact invokes a momentous property of that body. Though Galileo could

not make this property explicit on mathematical terms, he succeeded in discovering a physical one-to-one correspondence to instantaneous velocity. The interesting point in this procedure is a fundamentally *operational* method: thinking by doing. In fact, if one views Galileo's two-hundred-page notices written around 1604, it becomes clear that he was constantly dialoguing with his experiments, tables, number lists, and diagrams—and not with Aristotle's authoritative writings—in order to construct a homogeneous concept space of physical movement, see figure 4.1.

We have insisted on this episode since the musicological analog of velocity, musical tempo, shows an astonishingly parallel concept history—though nearly five hundred years delayed from physics.

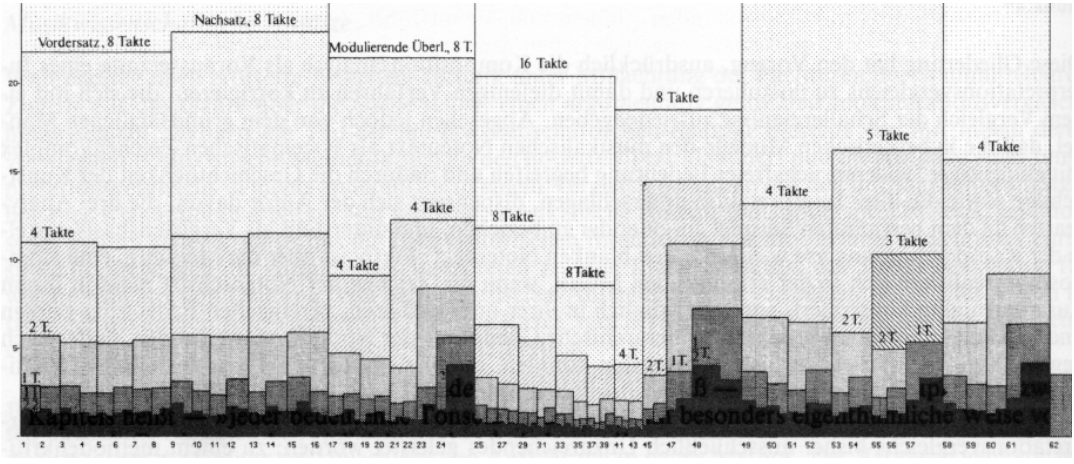


Figure 4.2: The figure shows a typical musicological 'digitized' version of the tempo concept, as used by Hermann Gottschewski [110]. The rectangular regions define regions of constant tempo (visualized here by reciprocal duration values). Since no instantaneous tempo concept is given, the author obtains different averaged tempo regions, according to the respective time window.

In traditional European musicology [110, p.317], we encounter a concept of tempo which corresponds to Oresme's setting: Tempo is the quotient of musical duration (measured in quarters) and physical duration (measured in minutes), this yields the classical Mälzel metronome m.m. in quarters per minute (see section 33.1 for a detailed discussion of tempo). It is essential that this concept does *not* include instantaneous tempo in the sense of a derivative, traditional musicologists deny the musical relevance of such a concept, see figure 4.2.

This refutation stems from the completely speculative handling of tempo in traditional musicology. Many of the tempo-related concepts, such as fermatas, ritardandi, and accelerandi, are encapsulated locked objects, and theorists would not know how to give a workable description of such tempo phenomena. The opening and explication of these void pointers are left to musicians and programmers of music software. Instead of learning from Galileo's experimental observations theorists insist on a discrete tempo fantasm a la Oresme—despite the rich language of modern mathematics.

This parallels physics to musicology in a basic issue. Precisely as Galileo, modern *working*

musicians and engineers are at the cutting edge of explicitness and leave irrelevant speculations to feuilletonism where they are well-placed. We learn from this double experimental revolution that it is not banausic artisanship that enforces opening of concept capsules, but the mere desire to redeem conceptual pointers, and to drive out at last void rhetorics.

Galileo's revolution was the answer to pretended depth of rhetoric discourse. Just as medieval medicine was traditionally split into speculative academics (without license to practice!) and artisanal surgery (practised by charlatans), physics was split into mechanical engineering and speculative philosophy. Scientists such as physicist Galileo and the French surgean Ambroise Paré had to 'dissect' the concepts and—hitherto tabooed—human bodies to explicate complexity [463]. We shall now take a closer look at the epistemological implications of this revolution.

## 4.1 Interior and Exterior Nature

**Summary.** Reasons for the categorical distinction between interior and exterior nature. Can human thought be an experimental field?

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Experimental sciences are classically related to empirical exterior nature. According to a traditional opinion condivided by Immanuel Kant [258], experiments are understood as interrogations of a passive witness. Nature is passive and will answer to the hearing in an objective way, i.e. independently of the interrogator's subject. These two characteristics, passivity and objective response, have since been recognized as erroneous idealizations. Physical experiments are substantially interactive processes, this was evidenced by uncertainty and complementarity principles of quantum mechanics.

But physical nature is also responsive on a more conceptual level. Each response to an experiment may alter the theoretical position and the concepts which drive the experiments. This is exactly Galileo's method, there are no platonic ideas which we have to retrace from empirical reality, the concepts are in incessant metamorphosis and mutate as a result of experimental cognition. As with genetics, only the fittest concepts can survive the experimental struggle.

In what respect is then inner nature so fundamentally different to exterior nature? It is said that inner human nature is subjective and that there is, consequently, no such a thing as objective facts. But the argument of subjectivity is misleading. It is definitely not true that we have a more indepth control of inner nature because of subjectivity. The inner human nature is a vast field of instances which we do not any better access than exterior objects. Subjectivity is only a minor part of the inner nature. Let us give two examples:

In *mathematics*, it is admitted that the Peano axioms for number theory[1] are trivial creations of the human mind. But the impression vanishes immediately when abording such a simple question as the famous and still unsolved Goldbach conjecture[2]: Can every even natural number be written as the sum of two prime numbers? This makes evident that the complexity of

[1] (1) The number 0 (zero) is a natural number; (2) every natural number $a$ has a unique successor $a^+$ among the natural numbers; (3) if $a^+ = b^+$, then $a = b$; (4) 0 is not a successor; (5) if a property $\phi$ for natural numbers is such that 1) $\phi(0)$ and 2) $\phi(a)$ implies $\phi(a^+)$, for every $a$, then the property holds for all natural numbers.

[2] Conjecture established by German mathematician Christian Goldbach, 1690-1764.

our mental creations is not more under control than physical objects. We believe that the Peano axioms are under control. But in their immediate conceptual vicinity, there are statements which do escape our mental power.

In *musical composition*, we typically deal with the creation of a score. But it is an illusion to believe that the richness of such a creation can be controlled by the composer. His or her contribution is but a *germ* of an incredible complexity that implies combinatorial processes, interpretation strategies, semiotic stratification and so on. In other words, a composition is quite the same as a piece of exterior nature, the part of subjectivity vanishes in front of an exorbitant structural variety.

So, what is the difference between exterior and interior nature? It relies on the more apparant, though not fundamental, autonomy of exterior 'natural' phenomena. But a second view reveals an inner nature—especially on the level of man-made universes such as mathematics and music—which is not less complex and uncontrollable than exterior nature. *As a piece of uncontrolled nature, Bach's "Art of the Fugue" is not fundamentally different from a supernova out in interstellar space.*

## 4.2 What Is a Musicological Experiment?

**Summary.** The Pythagorean tradition and the general concept of a music instrument. Candidates for experimental layers. Testing the congruence of a scientific model with corresponding material, such as a composition, performance, or analysis.

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We know from section 3.1 that Pythagorean tradition offers a type of "thinking by doing". Can we view this paradigm as an archaic version of Galileo's experimental approach? We could if it provided us with a conceptual laboratory where concepts could be fabricated while making music. But this fails, playing the monochord was nothing more and nothing less than a passive verification of already fixed ideas, in fact the metaphysical tetractys. It was a kind of doing on the ground of prefabricated thoughts.

Nonetheless, the insistence on the aspect of activity in thinking music has survived all speculative assaults from tetractys to Keplerian harmony. The shape in which we still encounter this method has of course changed. No longer can it be found under the auspices of divinities, it has been saved by composers, the experimental scientists of music. This is somewhat exaggerated since composers often condivide the status of medieval charlatans. But in their best performance, such as with Pierre Boulez, Giorgy Ligeti, or Iannis Xenakis, they really appear as prophets of a future type of experimental musicologists.

What happens is that these forerunners are left alone by a speculative void-pointed musicology. Already did Arnold Schönberg quit the theoretical background in view of the speculative overhead that could not cope with requirements of effective conceptualization in harmony[3]. The so-called "emancipation of dissonance" is among others the emancipation of experimental musicology from speculative rhetorics. It is not by chance that Boulez demanded to blow up all opera houses.

[3]This does not contradict his book on harmony [478] which is much less a warmed-up catechism than a critical review of failures of harmony. However, Schönberg's treatise on harmony did rather stress than solve those problems.

So what is a musicological experiment? It must clearly include both, thinking and playing, reflection and action. But in the form of mutual inspection, i.e. it should neither be a passive sonification of given speculations nor should it reduce to speculation without musical realization. So the decisive criterion is a bidirectional dialog between a flexible conceptualization and an intelligent sonic realization. Such a dialog can only succeed if the transformation channels between thinking and playing are explicit, operational and highly efficient.

These requirements are necessary, but not sufficient to conceive such a thing as a musicological experiment. We are still missing the adaptation of the crucial concept of experience to the context of humanities. It is by no means clear that such a category can subsist within inner nature. This will be dealt with in the next section.

## 4.3 Questions—Experiments of the Mind

**Summary.** Experience in the humanities. Separating speculative questions from musicological experiments. Necessary and sufficient conditions for questions to be accessible on the level of the experimental setup. Improvement of scientific conceptualization by constraints from experimental options.

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The naive view of experience in natural sciences evokes a bunch of facts in the space-time of exterior nature. But this is not what an experimental setup is dealing with; already in its very data processing are the 'naked facts' codified in numbers, figures, and all kinds of symbols. What is accessible to scientific reflection is already a body of information in a well-structured concept space. Measurement of physical events includes a transformation into *data streams within a mental knowledge space*. Experience then boils down to a complex navigation process along this data stream, and under use of statistical orientation tools and theoretical hypotheses to be verified or falsified.

When put together with the foregoing approach to the experimental setup, this yields a concept of a dialog between a set of data that are embedded in a concept space, and a set of analytical tools living in the same concept or knowledge space. In this perspective, the question underlying the experiment appears as a search mission towards a statement which fits the theoretical orientation defined by hypotheses and evaluation tools.

In other words, *the experiment turns out to be a search mission while navigating on a concept space*, the navigation being driven by two 'dialog partners': the data stream and the hypotheses and evaluation tools. But such a generic restatement of the concept of an experiment has *erased any specific reference to exterior nature*. We just deal with navigation on a data stream issued from a specific research object and a bunch of analytical orientation devices. This entire process is located on a (mental) knowledge space.

Under this perspective, an experiment may naturally occur in the context of humanities. Navigation may thus be restated as interactive interpretation of a given human work, be it music, painting, literature or whatever can be subsumed under the general structure of a verbal or non-verbal semiotic 'text'. Navigation on such a text crystallizes in form of a search for structural coherence between the variety of the text's syntax and semantics on one side, and corresponding hypotheses and (text) theories as a background of orientation on the other. Observe that the

concept of empirical experience has been absorbed by the concept of navigation on a stream in a concept or knowledge space. The parameter of external physical time is no longer relevant, navigation unfolds on an 'abstract' space, and time is replaced by an abstract 'curve parameter' of the navigation trajectory.

We should stress that the experimental dialog is strongly interactive in that there is no priority of theories over 'data streams'. A theory may completely mutate under the influence of new directions in artistic creativity. And conversely, the inner nature is no "nature morte", it is in constant renewal.

This being the case, it is important to review the role of speculative void pointer concepts under the experimental paradigm. In the situation of an explicit dialog between given data and theoretical instances, void pointer concepts are immediately ruled out. For example, an encapsulated *locked* concept of dominant, tonic and subdominant in function harmony cannot resolve any task of harmonic analysis for a concrete score since it asks for harmonic function values of *any possible chord* that may theoretically appear in the given score. In cases where the function value(s) cannot be determined, the data stream cannot be navigated and the experiment breaks down. Galilei could not work with the medieval velocity chimaeras since they evidently provided no navigation through the attentive observation of balls running down the inclined plane. *Experiments are excellent exterminators of empty pointers: Their dialog immediately lays bare deficiencies hidden in locked concept pointers.*

The experiment, if viewed as a question of the mind, not as an autistic reflection on locked concepts, rather as a dialoguing navigation, can help us understand what happens when we build concepts and theories.

## 4.4 New Scientific Paradigms and Collaboratories

**Summary.** Mutual influence of communicative networks and conceptual precision. Doing musicological experiments on a distributed laboratory. Redefining scientific competence from the communicative point of view.

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From the preceding discussion it is straightforward that musicology (and humanities in general) in its experimental profile cannot subsist in the private ambience of classical humanities. The experimental navigation dialog is demanding for all partners, and the very nature of dialog enforces data exchange on powerful channels. Socially restated, such a type of science has to be realized in a strongly collaborative style. On the level of institutions, this demands the structure of collaboratories[4], as introduced by the US Department of Energy. Just as physicists could not survive in isolated research units, musicologists have to initiate intense communication within research in order to succeed. We insist on communication *while doing research*, not merely in noble and non-binding conference small talk. Collaboration is meant as a working style, not as a title for informal politeness.

Moreover, the communicative network of collaborative science enforces a radical conceptual precision. This is not only about the formal constraints when dealing with computer programs, it

[4]The term "collaboratory" was coined by Bill Wulf in 1993 and is merged from "collaboration" and "laboratory", see [363] for further information.

is really about communicating as a working method. Learning by *doing* science can only succeed on the basis of unlimited access to encapsulated complexity. Evidently, scientific competence must then be rebuilt on the fundament of conceptual communicativeness while private regions of knowledge will lose their historically grown honorability.

We should keep in mind that the entire navigation and orientation metaphor in the preceding discussion of the knowledge or concept space is not merely general knowledge science slang, but more concretely an adequate expression of the topographical (in fact: topological) setting which was developed in chapter 2. Navigation happens to move along (local) coordinates of topographic cubes, and we shall learn in part II that *the general metaphor of a concept space can be realized in form of a veritable geometric space.*

# Part II

# Navigation on Concept Spaces

# Chapter 5

# Navigation

*Verlassen wir das strenge Labyrinth der zementierten Begriffe,*
*und ergehen wir uns ungezwungen in improvisierten Architekturen,*
*und dies selbst dann, wenn wir reden.*
Pierre Boulez [60, II, p.78]

**Summary.** Well-conceived information produces knowledge. But conceiving means building concepts, and this amounts to having well-structured access modes. This enforces the development of concept-oriented access modalities to given information, including associated concept spaces. Music is an excellent field to exemplify such knowledge spaces. It has become virtually impossible to navigate through music information without developing powerful concept formats.

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We initiate this part by a reflection on conceptual navigation since the concept spaces which will be defined in the subsequent chapters are strongly motivated by universal orientation paradigms while surfing on encyclopedias of inhomogeneous music-related information and knowledge.

More precisely, observe that knowledge involves two components: information and its mental organization. Whereas boiling down knowledge to mere information would mean getting drowned in a sea of amorphous substance, abstraction from information towards pure mental organization would mean getting stuck on Georg Wilhelm Friedrich Hegel's germs of logic, i.e. the zero state of philosophy as exposed in the first pages of [214]. Both reductions cannot cope with what knowledge deals with, viz building *concepts of something*. Building concepts then amounts to conceiving this data, organizing the access to information and doing this by use of well-defined access modes. Knowledge has an object, and it reaches that object via its concept. Conceiving means being able to navigate in a conceptual coordinate system to attend that something. In other words, pure information is substantial, but it is uncontrollable without a conceptual form, a coordinate system where we may place and retrieve substance.

This is precisely why so-called digital information has nothing to do with knowledge whatsoever, and why therefore the digital paradigm is completely irrelevant to the yoga of electronic age: Digital substance, reduced to the $BIT = \{0, 1\}$ of two substance values, 0 and 1, or OFF

and ON, the name does not matter, is just a minimal substance set, but it cannot *per se* be responsible for any kind of knowledge. In fact, a digital record is worthless without being organized in a concept form, i.e. as encoded knowledge. 'Understanding', i.e., decoding information is needed in order to control and manage its knowledge potential. The digital age is *not* centered around "Bits and Bytes" but around their accessibility and handling, in short: *data navigation.*

From the above we learn that a powerful concept system for any field of knowledge must provide us with a thorough navigation method that works on an extensive concept space. And from the preceding discussion regarding depth and complexity (section 3.2) we deduce that navigation must have access to any encapsulated concept, and permission to navigate on any possible "concept path".

It is essential for a successful concept navigation methodology in music to realize a highly interactive and non-authoritative, autonomous orientation and decision strategy. Rather than ending in obscure concept mist conceptual navigation should lay bare semantical lacunae, and this means developing a language for the open-ended navigation if ever this is the state of the art. In no case is it any longer acceptable to be navigated by dogmatic magisterial guides. This is what the Boulez citation heading this chapter suggests: To create a conceptual environment that allows quasi-improvisational and free-will driven discourse on music. Boulez must have felt that "cemented concept labyrinths" no longer cope with the requirements of advanced musicology, be it on the creative, or be it on the reflexive side.

Now, the term "improvisational" in Boulez' statement is somewhat misleading. It suggests an antagonist to the score-dominated way of making music. Like improvisation it should be interactive—this is to be retained. But it is much more and much more important than a way of making music. The key word is " cemented". Refraining from cemented concept architectures means that the concept architecture has become dynamic, ever-changing and soft. Concrete is a rigid material, hard and insensitive to whatever. Boulez' vision is that of a flexible concept environment, rather than improvisational it should be termed dynamic, i.e. interactive and adapting to changing demands.

The problem is how to realize such a dynamic concept architecture without loosing rigor and reliability. We shall propose such an architecture in chapter 6. For the time being, let us concentrate on the generic navigation paradigm as suggested by Boulez and stressed in the ongoing transformation from energy to information society. In particular we should reconsider navigation as an interactive interpenetration of knowledge agents and data bases. This latter idea turns out to be one of the deepest concerns of music in the making of our own world.

## 5.1 Music in the EncycloSpace

**Summary.** We discuss the navigation problem on music data in relation to the general concept of an encyclopedic knowledge space, the *EncycloSpace*. The latter is introduced as an upgrading of the classical concept of an encyclopedia, as conceived by the French encyclopedists Denis Diderot and Jean le Rond d'Alembert. The upgrading is characterized by three changes: (1) the static cosmos is replaced by a dynamically developing data organism in space-time; (2) the passive "speculum mundi", i.e. the purely receptive view, is replaced by an interactive instrumental relation to the data organism; (3) the textually oriented alphabetic ordering is generalized to a universal navigational orientation. Consequently, encyclopedic navigation splits

into a receptive and a productive variant.

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In [363], a general definition of an encyclopedic knowledge space, called EncycloSpace, was given. Let us shortly recall that definition and then explain the concept's genealogy and ingredients:

**Definition 2** EncycloSpace *is the topological corpus of global human knowledge which evolves dynamically in a virtual space-time, is coupled to human knowledge production in an interactive and ontological way, and allows of unrestricted navigation according to universal orientation within a hypermedially represented concept space.*

According to Sylvain Auroux' analysis of Denis Diderot's Encyclopédie [25], the encyclopedic principles of the Age of Enlightenment can be viewed as a combination of the *completeness* of a dictionary, the *unity* of philosophical synopsis, and the *discoursivity* of a mathematically inspired ordered representation (the alphabetic ordering of words), see figure 5.1.

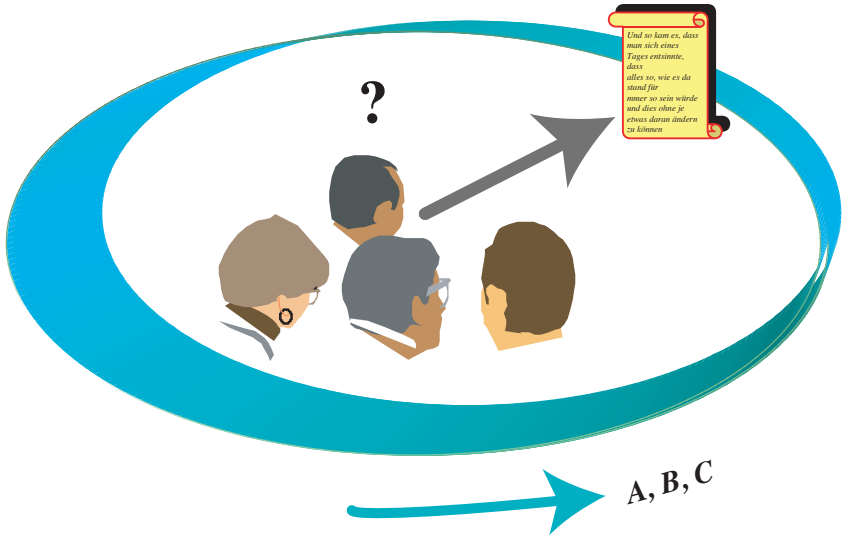


Figure 5.1: The principles of classical encyclopedia: completeness (complete circumference of the circle), unity (the circle's perfect shape), and discoursivity (alphabetical discourse along the circle's line).

In a critical review of these encyclopedic characteristics, the Encyclopédie is recognized as being a faithful realization of the medieval idea of a "speculum mundi": "L'encyclopédie assume par sa fixité relative le rôle d'un miroir du monde naturel, tel qu'il peu s'offrir aux sujets connaissants." This insight specifies the attribute of completeness, symbolized by the

orbit, as a fixed cosmos. This is a traditional static world view, similar to the photography of a stationary system. However, such a snapshot cannot claim to represent a world where knowledge is rapidly and incessantly growing. We have to accept that the purely spatial coordinates of a virtual cosmos (in Diderot's words: "connaissances éparses sur la surface de la terre") must be completed by the time coordinate. We have to deal with a dynamic universe of knowledge embedded in virtual space-time, and subjected to laws of coupled synchronic and diachronic nature (see section 2.3.4 for these semiotic concepts). The development of knowledge is not an accumulation of essentially immutable and isolated time-slices. See figure 5.2 for the following discussion.

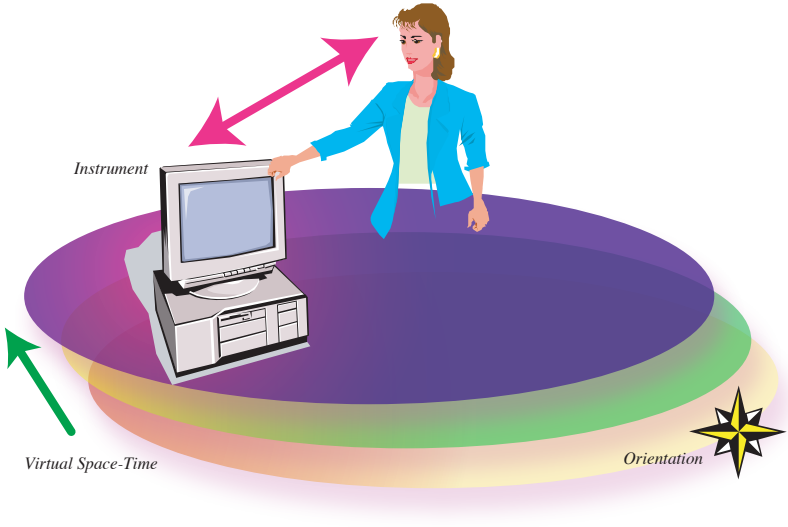


Figure 5.2: The EncycloSpace characteristics: Universal orientation in a dynamic knowledge space-time which is interactively coupled to human knowledge production.

Further, the change of this dynamic system evidently does not happen by an autonomous activity: It is a result of a constant and substantial interaction of humanity with the corpus of knowledge. Much like political and civilization dynamics or brute continental shift, the encyclopedic body is an open system which is incessantly reshaped in synergy with human knowledge production. Consequently, the metaphor of a "speculum", i.e. a passive vision of given things, is not adequate. We do not only look at prefixed things, we do also define and redefine them, add new knowledge and relativize old-fashioned approaches. One could then replace the metaphor of a "speculum" by that of an "instrumentum". This latter is perfectly adequate to the computer as a bidirectionally active interface between humans and knowledge bases. This aspect is substantial since it also questions the very nature of knowledge. Concepts are no eternal entities who live out in a platonic sky (hyperouranios topos). They rather represent operational units with an ontology that is rightly defined by the very accessibility of conceptual components.

(In a more radical perspective, it could be argued that concepts are operators.) In this vein, understanding the concept of "house", for example, boils down to the way you activate its components (roof, walls, windows, etc.) when asked to prove your understanding (to yourself or to others, there is no difference).

Finally, the order of representation of knowledge within the Encyclopédie is twofold: On the very surface, it is the alphabetic ordering dictated by the traditional textual representation. More in-depth, it is the geographic orientation given by the Encyclopédie's cross-reference system [25, p.321 f]. Whereas the former is strictly linear, the latter is rather a partial relation of "pointers", far from being an ordering in the technical sense. This is also due to the fact that the concepts are not cast in a formal language and hence, beyond the alphabetic ordering no intrinsic organization is visible on the representational level. This point is not just a marginal note concerning formal aspects of knowledge. Rather do we envisage the in-depth question of the structure of the space where concepts live. We know from Plato's allegory of the cave [417], from Aristotle's "Topic" [20] and from Immanuel Kant's comment on Aristotle's "Topic" in "Kritik der reinen Vernunft" [258], that the spatial metaphor for the ontology of concepts is crucial for any effective discussion of the system of concepts. In this sense, ontology of concepts is essentially topology: a study of the topoi where concepts subsist. In this theoretical sense, alphabetic navigation is not sufficient, conceptual navigation must provide tools and paradigms which transcend the textual alphabetism and include generic principles of answering to primordial discursive questions: Where do I come from? Where am I? In what direction do I proceed?

From a more operative and practical point of view, the alphabetic ordering inherent in the linguistic text paradigm must be completed by orderings which are genuinely related to non-textual data formats, such as geometric spaces, or spaces of set collections (see chapter 6 for precise definitions). Such more general data types emerge in a natural way in hypermedia documents including visual and acoustic instances, such as pictures, diagrams, sounds, music or dance scores, for example. Navigation orderings have to cope with hypermedia orientation paradigms for practical and for theoretical reasons: There is no reason why the completely arbitrary textual reduction should and could grasp the intrinsic ordering of concepts and thoughts. To be clear, the brute reduction to the definitely textual binary code encompasses virtually every possible information record. But it is a different business to understand information—this is definitely not in the reach of the binary code. "Worüber man nicht sprechen kann, darüber muss man schweigen." This killer sentence which terminates Ludwig Wittgenstein's "tractatus logico-philosophicus" [580] collapses in the age of hypermedia discourse. It should be replaced by a recommendation of visual/geometric, acoustic/musical, or haptic/gestural alternatives to textual dead ends.

Summarizing, we have reviewed the concept of Diderot's Encyclopédie and updated its three attributes to meet the needs of a dynamic, active, and universally oriented knowledge society. Thus, we may baptize this update by the name of EncycloSpace, and determine the concept as explained in the above definition 2.

Though "navigation" stems from Latin " navigare", and this one from "navis" plus "agere", which means giving motion to a ship, the present meaning of "navigation" is restricted to the passive steering of a vehicle which is already moved by some motor. However, navigation on the EncycloSpace must regain the original activity, since EncycloSpace has been determined as an interactively handled body of knowledge, a body which is not built, altered and developed by mysterious forces but by our genuine *agitation* upon the object. Therefore, the original

etymology of "navigation" is restituted: We shall distinguish between *receptive* and *productive* navigation. Receptive navigation means moving around within the body of knowledge without changing it. In contrast, productive navigation does change the body of knowledge. We will discuss these two modalities in the following sections.

## 5.2 Receptive Navigation

**Summary.** Receptive navigation leaves the EncycloSpace unaltered. It deals with universal orderings to find and represent knowledge for optimized understanding. Since musical data are of very inhomogeneous and general types, universal navigation concepts are essential in innovative musicology.

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Receptive navigation is the common situation in classical encyclopedias. One is looking for a bunch of information in a more or less well-determined, and *immutable* field of knowledge. The main requirement in this situation is an *optimal orientation environment*. In case of a simple name search, the alphabetic ordering is adequate: We just leaf through the linear alphabetic ordering of words. But this does not cover less trivial search problems. For instance, if we are looking for all chords in the score of Schumann's "Träumerei", say, the alphabetic orientation crashes. Surfing through the set of chords in a determined score is a new situation, it requires ordering principles for chords as special types of note sets. The EncycloSpace features should comprise a representation of all chords built from the 463 notes of the "Träumerei". They should then be able to order these chords according to *universal* ordering principles since we cannot (1) redefine from scratch orderings among EncycloSpace objects and (2) build individual orientation frames for each particular configuration. Given such a universal ordering, every chord could be retrieved according to corresponding intuitive orientation paradigms.

We recognize that inspecting such an ordered chord list is only the last link of an extended chain of encylopedic objects which finally leads to the required chords. We would typically start on the level of composer names, then—once "Robert Schumann" has been found—descend to piano works, then to "Kinderszenen", and finally to "Träumerei". At this point, we will have to surf through different types of musical score signs. Once arrived on the level of notes, representing ordered lists of chords requires non-alphabetic ordering principles. In fact, piano notes are points in a space of four dimensions, pitch, onset, duration, and loudness. Such a space requires different approaches to ordering—and chords do so *a fortiori*. Universal orderings among EncycloSpace objects is a central feature for receptive navigation because it yields orientation and thusly prepares the field for recruitment of added knowledge which is to be created from productive navigation.

Whatever are the searched for objects, it turns out to be crucial for valid receptive navigation to be provided with universal ordering principles: *orientation must be generic, and generically representable on the visual interface.*—This will be an important feature of the denotator system to be introduced in chapter 6.

## 5.3 Productive Navigation

**Summary.** Productive navigation interacts with the EncycloSpace such that the latter is enriched by a surplus value of added knowledge. The EncycloSpace is a repertoire as well as a laboratory. Productive navigation is a central issue in extending musicological and musical knowledge.

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In the above example of an ordered list of chords, we supposed the existence of such a list on the EncycloSpace. However, it is not probable that all such lists of chords drawn from scores of the piano music repertoire are incorporated in the EncycloSpace. More probably most of what is required by an interested user will not be allocated. This can be the case on very different levels of knowledge. In the simplest level, it can be a numerical information we want to obtain, e.g.: "How many quarter notes are contained in Thelonious Monk's original tune "Blue Monk"?" This information usually is not part of the knowledge base, but it can easily be calculated. Searching for the answer is a trivial extension of knowledge, but it differs from receptive navigation: Navigation which is defined by the question and terminates upon its answer produces a (tiny) extension of the EncycloSpace.

On a less trivial level, the calculation of complex data, such as the number of (1) chords in the "Träumerei" which (2) contain more than three notes, which (3) are dominant in F major, and which (4) contain a diminished third, can require too much calculation time to allow forgetting about the result. In this case, we try to save the result and access it as soon as new navigation trips ask for this information.

On a still more invasive level, it may turn out that we have built new concepts, such as a special type of musical motives which meet non-standard conditions. Suppose that such a concept is named "Non-Standard-Motif". We do not want to be excluded from navigation because this concept is not yet part of the EncycloSpace's vocabulary! In other words, Boulez' requirement of an "improvisational" discourse/navigation must include free vocabulary extension and then the operation upon the extended vocabulary.

Summarizing, we see that productive navigation responds to the demand for dynamic navigation, interacting with the body of knowledge, and incessantly producing new data to be added to the given EncycloSpace. Evidently, such an extended functionality adds to the original repertory character of the EncycloSpace the character of a laboratory where questions are transformed into mental experiments as introduced in chapter 4. Besides the technical problem of how to realize such a functionality without losing control, we are also confronted with the level of accessibility of extended EncycloSpace contents by third parties. But this is not a question of fundamental ordering, rather technology should be concerned with the social aspect of navigation-induced knowledge extension.

Boulez' idea of an improvisational concept navigation provokes fundamental questions of communication which are well known from music when improvisation started to penetrate the music market and musicology as a science. In fact, most jazz productions essentially rely on sound tracks which override pure score or songbook data. The introduction of sound files as a major trace of and reference to the musical creativity is responsible for the very existence of the present jazz culture. Though technologically low level, the mere possibility to access ad infinitum an improvised piece of music initiated a discourse about music which was never

possible before. And it enforced the relativization of classical scores as an identification of the musical work.

On the level of conceptual navigation, we experience the same effect: Classical 'scores of knowledge', i.e. hard-coded knowledge data bases in books and encyclopedias, are being confronted with dynamic records of ever moving knowledge streams. 'Improvisation' within knowledge can now be traced and integrated in the global stream of knowledge production. No need to stress the fundamental transformation of knowledge sociology under such perspectives.

*We would like to insist on this background throughout the present book since it exerts an incredible impact on how we recognize important from irrelevant aspects of formal conceptualization in future musicology.*

# Chapter 6

# Denotators

*C'est peut-être dans la façon de se représenter l'esprit qu'on pourrait déchiffrer l'esprit.*
Paul Valéry [538, III]

**Summary.** This chapter introduces the universal data format of denotators to describe musical objects. Denotators generalize the structures of local compositions in mathematical music theory as well as the data model used in the RUBATO® software. They do, however, *not* include deeper semantic layers which are related to physical, psychic, social or religious meaning. Their semiotic structure represents an elaborate denotative baseline and resides in a purely mental level usually attributed to mathematical objects.

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This chapter introduces the formal framework of the entire book. We believe that the preceding chapters have given a huge motivation to elaborate a formally rigorous concept framework. Therefore, we shall no longer comment on the general purpose of this issue. Let us instead describe the genealogy of the denotator concept. After the publication of the German precursor [340] of this volume, and in connection with the development of the composition software *presto*® [338], it was believed that the concept framework of mathematical music theory was sufficiently generic to handle formal issues in computational systematic musicology. This framework was built upon so-called *local and global compositions*. However, the theoretical and program design groundwork for the analysis and performance platform RUBATO® taught the developers that these concepts had several severe drawbacks:

1. They were centered around a special type of mathematical structures (essentially finite point sets in particular parameter spaces) which could not comprise less theoretical objects, such as to include attributes with character strings for names, or other non-numerical data.

2. Also were they 'hard-coded' in the sense that introduction of new concept types was not possible. In particular it was not possible to define concepts by recursion to already given concepts.

3. The mathematical flavor was not compatible with programming requirements for data bases.

4. The naming policy did not allow clear distinction between objects and names (in a strictly semiotic sense).

5. The so-called functorial point of view was not taken seriously enough. Intuitively, this means that we had to stick to one fixed 'ontological' point of view.

6. The objects were not well distinguished from the spaces they live in. This hindered recombination of object selections—which now is possible, and meets the requirements of productive navigation.

The denotator concept framework will eliminate these defects. But it will also require a rather complex concept scheme. We shall develop the details in several steps, starting with a more down-to-earth version of the denotator framework (section 6.1), and then extending to the universally valid generic setup.

Denotators are conceived according to the general criteria of EncycloSpace navigation, so observe that we do not follow mathematical tradition but navigation methodology, including 'mathematical navigation' as a special case.

Also we should take care of *abstraction from any deeper semantic loading with denotators.* The reader should postpone semantic issues to chapter 18, the present discussion is uniquely concerned with the universal denotation formalism. Notwithstanding their formal power, denotators are radically pre-semantic, at least regarding to 'real' meaning of physical, psychic, social or religious character.

## 6.1 Universal Concept Formats

**Summary.** Discussion of three principles of universal data format typology: unity, completeness, and discursivity. These are realized by a typology which is recursive, of extensive ramification, and open to unrestricted modular data recombination. The denotator concept is modeled following the Aristotelian pairing of substance and form or of the geometric paradigm of point and space, respectively. Each of these components: substance and form are arranged according to the structure of signs. We discuss the fundamental pointer nature of points as basic concepts of axiomatized semiotics.

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From the discussion of encyclopedic characteristics: unity, completeness, and discoursivity, we deduce three denotator attributes which cope with these characteristics.

| *encyclopedic characteristic* | $\Longrightarrow$ | *denotator attribute* |
|---|---|---|
| unity | $\Longrightarrow$ | **recursive** typology |
| completeness | $\Longrightarrow$ | extensive **ramification** modes |
| discoursivity | $\Longrightarrow$ | **order** and **recombination** |

The idea is that unity in creating concepts is achieved by means of recursive constructions. One builds new denotators from old ones by systematic reuse. Completeness must be guaranteed by extensive ramification modalities when stepping down along recursive paths. Each ramification mode ensures a particular way of stepping downwards in recursivity. Finally, discoursivity is met by means of free construction and recombination of denotators and by universal orderings among these objects. Unlike with usual data base management systems [551] there is no fixed ensemble of possible denotator constructs. Figure 6.1 illustrates the denotator 'flow chart'.

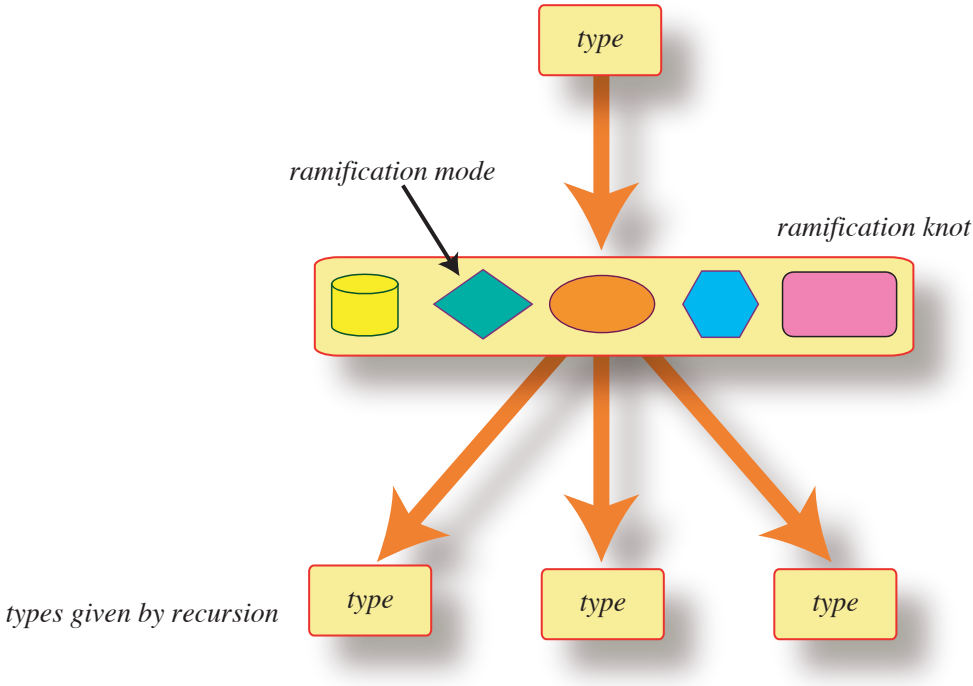


Figure 6.1: Denotators are designed to cope with universal encyclopedic characteristics, such as unity, completeness, and discoursivity. This flow chart visualizes the corresponding denotator attributes. A type refers to other types in a recursive way through different ramification modes as indicated in the ramification knot. Construction of new and recombination of given denotators is non-restrictive. The system of denotators can be ordered in a universal way.

### 6.1.1 First Naive Approach To Denotators

**Summary.** A first approach to denotators is given without requiring further mathematical background.

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The universality of our claim implies that we cannot just set up a generic space and take the objects as being "points" in such a space. Rather is it necessary that each point carries its space with it like a snail carries its shell. Referring to Aristotle's description of real things as a combination of substance and form, we view a denotator as being a substance-point within its form-space, see Figure 6.2.

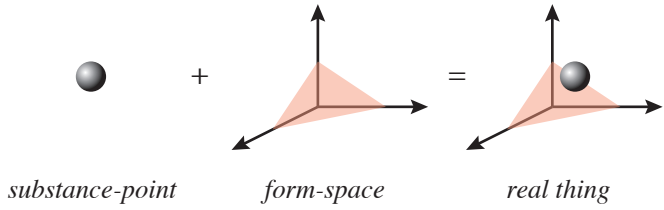


Figure 6.2: Like a real thing, a denotator is a substance-point, together with a proper form-space.

The naive definition[1] of a denotator is split into a recursive definition of the form-space, and then, based upon this concept, the substance-points are defined. The recursive structure of a form

is a triple, consisting of Form-Name, Type, and Coordinator, denoted in this way:

$$Form\text{-}Name \rightarrow Type(Coordinator). \tag{6.1}$$

The Form-Name is any character string, such as 'Note'[2]. By definition, the type is either compound or simple. Compound types are ramification modes, simple types are recursive start points. There are four[3] compound types: Product $\prod$, coproduct $\coprod$, powerset $\{\}$, and synonymy **Syn**. For products and coproducts, the coordinator is a finite sequence $F_1$,..., $F_n$ of forms. For powerset and synonymy[4], the coordinator is one form $F$. By definition, four[5] simple types are possible: STRING, BOOLE, INTEGER, FLOAT. Their coordinators look as follows:

[1] In the mathematically rigorous framework of form semiotics as described in appendix G.5.3, the naive approach is related to the topos **Sets** of sets.

[2] The special '...'quotation marks reduce the word to a character string, i.e. the sequence of characters without any further meaning.

[3] These are just the ones which can be easily understood within this naive introduction. Later, we shall generalize the types, but no fundamentally different ideas will be added.

[4] In the rigorous framework, synonymy is superfluous and may be mimicked by a product with one single factor.

[5] These simple domains will be vastly extended, the present selection is made with regard to practical and programming-oriented usage.

- STRING: The set **CHR** of character strings[6].
- BOOLE: The set $BIT = \{0, 1\}$.
- INTEGER: The set $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ of integers.
- FLOAT: The set $\mathbb{R}$ of (floating point) decimal numbers[7], such as 234.0157, −25.19022879, etc.

Since coordinators are uniquely determined for simple forms, we shall omit the coordinator bracket in these cases, i.e. INTEGER($\mathbb{Z}$) simplifies to INTEGER, etc.

The general naming policy for forms is that two different forms should bear different names. This enables us to identify a form with its name (the name is a "key" in the terminology of database theory [551]). For example, if we have the product form

$$'\text{Piano-Note}' \to \prod(\text{Onset}, \text{Pitch}, \text{Loudness}, \text{Duration}), \tag{6.2}$$

with coordinators Onset, Pitch, Loudness, and Duration, then we want to give it the shorthand name Piano-Note. In this sense, we often adopt the bracketless denotation

$$\text{Piano-Note} \to \prod(\text{Onset}, \text{Pitch}, \text{Loudness}, \text{Duration}) \tag{6.3}$$

if no confusion is likely[8].

With this naming convention in mind we introduce the following simple forms as a first set of ad hoc examples for this 'naive' introduction:

- Onset→FLOAT,
- Pitch→INTEGER,
- Loudness→STRING,
- Duration→FLOAT.

A rigorous expression for such a form would read $'\text{Onset}' \to$ SIMPLE($\mathbb{R}$), for example, and the form itself would be called Onset.

If we deal with complex compound forms and wish to represent them in a more graphical way, the symbolism can be shown in a vertical arrangement, as illustrated for the Piano-Note form in Figure 6.3. Observe that presently, products and coproducts or powersets and synonymies, respectively, are not further distinguished. The former couple has a sequence of forms, the latter one a single form as coordinator. However, the specific difference will emerge when we introduce the substance-points on the respective form-spaces.

[6] We may take the standard set of ASCII characters, but from an informal point of view, just think of usual symbols, such as letters a, b,..., figures 0, 1, 2,..., and other characters, such as brackets, etc.

[7] We do not consider infinite decimal digits, such as $\pi = 3.1415926\ldots$, in this naive setup.

[8] In the general approach to denotator theory, names are also denotators, and the present distinction is obsolete, see appendix G.5.3.

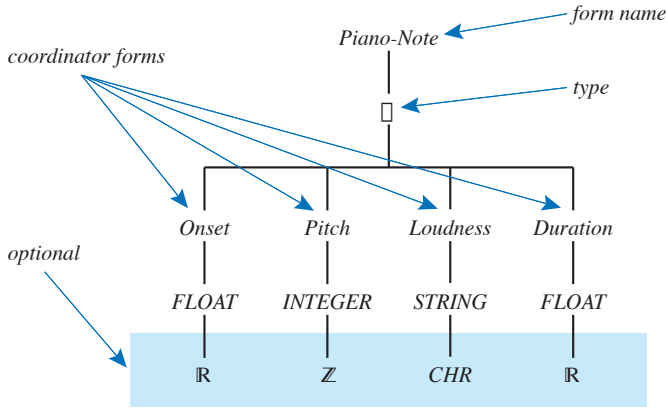


Figure 6.3: The form Piano-Note in graphical representation, including the recursive explication of its coordinator forms Onset, Pitch, Loudness, and Duration.

In our naive setting, a denotator is a triple, consisting of the denotator name Denotator-Name, the form Form, and the coordinates Coordinates; it is denoted by

$$Denotator\text{-}Name \rightsquigarrow Form(Coordinates). \tag{6.4}$$

Again, *Denotator-Name* is a character string, *Form* is a form, and *Coordinates* is an object which is determined according to the form's type. If the type is simple then, by definition, Coordinates is an element of the coordinator of the form. For example, if Type = INTEGER then Coordinates is an integer number, i.e. an element of $\mathbb{Z}$. For example, a denotator for pitch middle C on a keyboard could look like $'C' \rightsquigarrow$Pitch(60), see Figure 6.4 for concrete examples of simple denotators. Again, if there is no risk of confusion, we shall identify a denotator by its denotator name, writing C or C$\rightsquigarrow$Pitch(60) instead of $'C' \rightsquigarrow$Pitch(60), for example. For denotators we cannot however stick to the naming policy of forms since in general, denotator names are of secondary importance. This is so because denotators usually occur in large numbers (such as notes in a score), and we are forced to distinguish them by their coordinates rather than by names. Very often it will even be reasonable to reduce the denotator name to the empty string $''$[9].

Let Type be compound[10]. For *product*, and if[11] Coordinator $= F_1, \ldots F_n$, we define Coordinates as being any sequence $f_1, \ldots f_n$ of denotators with forms $F_1, \ldots F_n$, respectively, i.e. $f_1$

[9]Be careful in distinguishing the empty string $''$ from the string $' '$ consisting of one empty space, sometimes also denoted by $\sqcup$!

[10]In this naive setup, coordinates are also denotators. This is however different from the formal setup, but there, we need conceptual circularity on the level of forms and denotator namings. We have preferred this naive approach to ease understanding in a first approach.

[11]In this book, we make a logical, though not very common usage of the ellipsis symbol "...": it means that one has started with a sequence of symbol combinations which follows an evident law, such as $1, 2, \ldots n$, or $a_1 + a_2 + \ldots a_n$. The evidence is built upon the starting unit, such as "1," or "$a_1+$" in our examples, and then

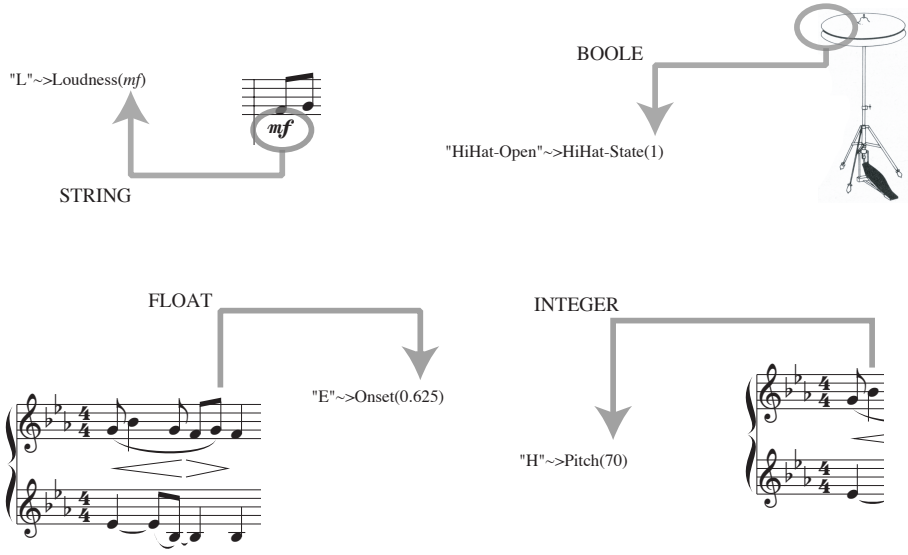


Figure 6.4: Examples of simple denotators.

$=$ $f_1$-Name $\rightsquigarrow F_1(f_1\text{-Coordinates})$, ..., $f_n = f_n$-Name $\rightsquigarrow F_n(f_n\text{-Coordinates})$. For example, if we are given a denotator named ′myNote′[12], and with form Piano-Note, the coordinates are four denotators $E$, $H$, $L$, $D$ with forms Onset, Pitch, Loudness, Duration and Coordinates $e, h, l, d$ in the coordinators $\mathbb{R}, \mathbb{Z}, \mathbf{CHR}, \mathbb{R}$ respectively:

$$\begin{aligned} \text{myNote} \quad \rightsquigarrow \quad & \text{Piano-Note}(E, H, L, D), \\ & E \rightsquigarrow \text{Onset}(e), H \rightsquigarrow \text{Pitch}(h), L \rightsquigarrow \text{Loudness}(l), \\ & D \rightsquigarrow \text{Duration}(d). \end{aligned} \tag{6.5}$$

Figure 6.5 shows the graphic representation of the Piano-Note form (see figure 6.3) with the above denotator myNote inserted.

If Type is *coproduct*, and if Coordinator $= F_1, \ldots F_n$, we define Coordinates as being any one denotator $f_i$ of form $F_i$, for an index $i = 1, ..., n$, i.e.

$$f_i = f_i\text{-}Name \rightsquigarrow F_i(f_i\text{-}Coordinates). \tag{6.6}$$

the following unit, such as "2," or "$a_2+$", and then inducing the following units to be denoted, such as "3," or "$a_3+$", "4," or "$a_4+$", etc., until the sequence is terminated by the last symbol, such as "$n$" or "$a_n$" in our examples. The ellipsis means that the building law is repeated, and as such, it is a meta-sign referring to the inductive offset. Therefore the more common notation $1, 2, \ldots, n$, or $a_1 + a_2 + \ldots + a_n$ is not correct. In the limit, for $n = 3$, it would imply a notation such as $1, 2, , 3$ or $a_1 + a_2 + +a_3$, which is nonsense. Moreover, in complicated indexing situation, the common notation would be overloaded.

[12]If possible, we name denotators with small initials, but exceptions may occur.

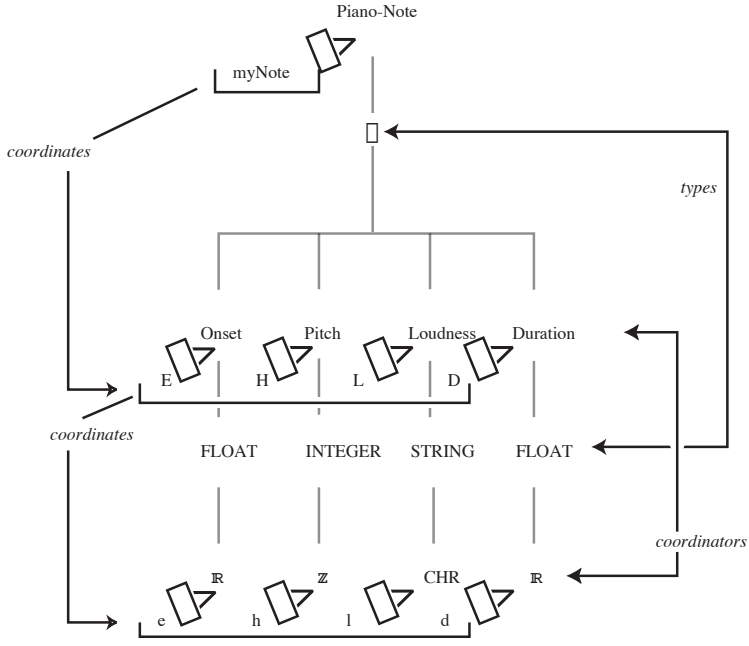


Figure 6.5: A product denotator named ′myNote′ with its coordinates in a recursively complete graphic representation.

In the above example with $F_1, F_2, F_3, F_4$ = Onset, Pitch, Loudness, Duration, let

$$PianoSelector \rightsquigarrow \coprod(Onset, Pitch, Loudness, Duration)$$

be the corresponding coproduct form. Then, with this form, a denotator named ′mySelection′, given by ′mySelection′$\rightsquigarrow$PianoSelector($f$), is defined by the selection of either $f\rightsquigarrow$Onset($\phi$) or $f\rightsquigarrow$Pitch($\phi$) or $f\rightsquigarrow$Loudness($\phi$) or $f\rightsquigarrow$Duration($\phi$), $\phi$ being an element of $\mathbb{R}, \mathbb{Z}, \mathbf{CHR}, \mathbb{R}$, respectively.

If Type is *powerset*, and if Coordinator = $F$, we define Coordinates as being a finite set $S$ of denotators, all having one and the same form $F$. For example, if we consider the form Chord$\rightarrow${}(Pitch), then a denotator myChord$\rightsquigarrow$Chord($S$) is defined by a set $S$ of pitches, i.e. a chord in the classical terminology. For a finite set $S = \{f_1, ..., f_k\}$, we usually omit the curled set brackets in this notation and simply write {}($f_1, ..., f_k$) instead of {}({$f_1, ..., f_k$}). For example, a second degree chord[13] $\text{II}_\text{C}$ in C major consisting of pitch denotators D$\rightsquigarrow$Pitch(62), F$\rightsquigarrow$Pitch(65), and A$\rightsquigarrow$Pitch(69) would be denoted by $\text{II}_\text{C}\rightsquigarrow${}(D,F,A).

If Type is *synonymy*, and if Coordinator = $F$, we define Coordinates as being a denotator $f$ with form $F$. Hence, if the denotator's form is New$\rightarrow$**Syn**($F$), then a denotator with denotator name ′myNewName′ is myNewName$\rightsquigarrow$New($f$). Essentially, this means renaming $f$

[13]We shall, of course, give rigorous definitions of all musicological terms later in this book.

by ′myNewName′ and doing this within the form which is essentially $F$, but with a new name ′New′. Besides a renaming of the space and the point, nothing really happens—unfortunately, this is a very common situation in the humanities. Notice, however, that synonymy is not symmetric, i.e. the new denotator is recursively later than the old one. This is essential for understanding the construction history of concepts.

If a denotator of form *AForm* and coordinates *myCoordinates* is given the empty name ″, we shall denote it by

$$AForm(myCoordinates) \tag{6.7}$$

instead of $''\rightsquigarrow AForm(myCoordinates)$. But we shall not introduce analogous notation for empty-named forms since form names are essential as an identification instance.

### 6.1.2 Interpretations and Comments

**Summary.** Interpretations of the above concepts and comments on non-trivial implications are given.

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**Pointer Character.** To begin with, let us stress the pointer character of the denotator concept. The underlying form concept is self-referential: In general, a form is defined by another (bunch of) form(s), only the reference type is explicit. Recursion *is supposed* to stop at a simple form, and there, we are left with a selection of substance from a common repertoire, viz mathematics and computer programming data types (numbers, digits, strings). Similarly with denotators. We contend that this scheme of recursive pointers is fundamental in human concept construction. Understanding such a concept means navigating on its recursive ramification tree.

This is the way we usually execute our understanding of a concept, e.g. "house": We say that a house consists of a roof, a set of windows and doors, walls, etc. We point at components of the concept. If necessary, we penetrate such components, and prosecute their ramification trees, etc. But we do *not* point automatically at any possible knot of the entire concept tree! If the concept "house" appears in a discourse, we only delve into a minimal level of explicitness. In our setup, this means that we just take the name, or, if empty, the form of a denotator if this specification suffices. Only in case some request is made for more details do we unveil deeper concept levels and follow the path down to coordinators and coordinates.

Such a pointer scheme realizes a very economic handling of concepts. You only "unpack" what is really needed, the rest remains either referred by name or even completely hidden. This approach is similar to the paradigm of object-oriented programming, more precisely, to encapsulation of object data. They have an identity[14], but the contents are not unveiled until really needed.

**Circularity.** Denotators have the outstanding property that they are open to circular definitions, and this is a fundamental extension of concept building rules with respect to the

[14]Usually a pointer to some memory address.

apparently founded recursivity of denotator structures. Let us give a common example to make the point explicit. We want to define the concept of a book with chapters, subchapters, subsubchapters etc. Evidently, a book can have an arbitrary number of chapters, and the depth of sectioning and subsectioning a book is not limited. But we can say that a book always consists of title, text, and a set of chapters.

In turn, each chapter again consists of the same data type, except that we have to enumerate the chapters. We thereof deduce a natural ordering of appearance of the particular chapters. Let us therefore complete the above list by a number:

1. number
2. title
3. text
4. a set of chapters

The problem with this definition is that we have to declare the "set of chapters". In the context of denotators, this problem is solved in the following way: The basic form is termed ′book′, and it has this structure:

$$\begin{aligned} \prime book\prime \quad \to \quad & \prod(No., title, text, chapters), \\ & \prime No.\prime \to INTEGER, \\ & \prime title\prime \to STRING, \\ & \prime text\prime \to STRING, \\ & \prime chapters\prime \to \{\}(book). \end{aligned} \tag{6.8}$$

We have made self-reference to the form *book* within the powerset factor *chapters* of the four-fold product. However, this circularity of the form *book* is not necessarily inherited by substance-points within the form-space. Since a *book*-formed denotator

$$myBook \rightsquigarrow book(..., myChapters)$$

has any set of chapters, we may give it the empty set $\emptyset$ of coordinates:

$$myChapters \rightsquigarrow chapters().$$

In this case, no further specification is needed, and the denotator is completely determined. This may happen anywhere in the recursive regression of such a denotator and therefore, any finite book tree can be captured in the sense we expect from practice.

Nonetheless, circularity of forms and denotators may occur in a much less inoffensive way which takes the very basis of concept construction into task. We come back to this issue in section 6.7.

**Meeting Encyclopedism.** Let us recapitulate the encyclopedic characteristics and their realization on the denotator level. Recursive typology meets *unity* since we are given a unified construction principle of concepts; this is the yoga of pointers. It is, of course, not

possible to demonstrate in a formal sense that our ramification modalities are *complete*. But in view of the rigorous formal setup to be exposed in section 6.2, we can say that the most general mathematical construction principles of products, coproducts[15] and powersets, together with the epistemological factotum of synonymy, gather everything which is known to be relevant for formation of mathematical and database structures[16]. And *discoursivity* is guaranteed by two measures: First, complete freedom of form and denotator construction guarantees recombination. Even operations of *semantic completion* are feasible in the following sense. Suppose, for example, that an ethnological pitch-related form has not yet been made fully explicit and is merely 'sketched' in a simple, STRING-typed form $UnknownPitchType \to$ STRING within its recursion tree. The denotators of form *UnknownPitchType* have only a string of characters, no further contents are available. It may happen in a more in-depth research development that the pitch structure can be specified beyond strings. Then the form *UnknownPitchType* may be replaced by

$$KnownPitchType \to \prod(UnknownPitchType, \text{Pitch}), \tag{6.9}$$

or even by form

$$KnownPitchType \to \mathbf{Syn}(\text{Pitch}), \tag{6.10}$$

where Pitch is the integer-valued form discussed in formula (6.2). The first form relates every denotator pitch string to its numerical value, whereas the second form is a restatement of pitch data via a synonym of the already given Pitch form. Whatever procedure is selected, the old form *UnknownPitchType* can be replaced by the new *KnownPitchType* form in all forms which make use of it—just as in genetic engineering. And *UnknownPitchType*-formed denotators can keep their names, we only need to replace them by synonymous denotators which point at numerical pitch values.

For example, it may happen that a pitch is just named 'F♯', but no special value in terms of frequency is specified. This is a common situation in music theory. Then, in a more explicit situation—of instrumentation, say—this 'symbolic' data should be realized in a specific tuning. This is precisely the need to replace or enrich the string 'F♯' by more involved information, e.g. *KnownPitchType*('F♯', Pitch(66)) as related to keyboard middle C Pitch(60).

The second measure to meet discoursivity is a universally defined ordering principle on denotators. This provides us with an 'omnipresent' orientation for the discourse on denotators. Observe that this ordering system does not restrict to lexical alphabetic ordering but extends to more natural geometric constructions. We shall introduce (still naively, see section 6.8 for the rigorous version) this topic in the following section.

[15] Products will then generalize to so-called limits, coproducts generalize to colimits.

[16] It is astonishing that database theorists have not yet learned to make systematic use of mathematical category theory, although the latter is being used in theoretical computer science.

### 6.1.3 Ordering Denotators and 'Concept Leafing'

**Summary.** We give a naive approach to the universally defined linear orderings among denotators and illustrate the results for musically meaningful denotators.

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The question of what d'Alembert called the "encyclopedic ordering" is fundamental to any encyclopedic enterprise. We contend that the 'EncycloSpace of denotators' should be provided with a linear ordering $<$ (see section C.2 for its definition). Here are three points why such a requirement is indeed fundamental:

- One should be able to have a universal orientation, something that could be termed 'leafing' on the denotator EncycloSpace.
- Orderings on denotators should be a germ for their representation as 'points in coordinator spaces'.
- A denotator search algorithm has to be defined in a universal way, and preceding visualization, since search engines must a priori be able to invoke this algorithm.

It was an interesting conflict within Diderot's Encyclopédie that the alphabetic ordering was so low level with respect to the order of ideas. This is above all due to the informal presentation of ideas—no database concepts were available in 1750.

We introduce the linear ordering on denotators and their forms by recursion, i.e. by inheritance from already defined orderings on the coordinates and their coordinators. To simplify the discussion, we shall stick to non-circular forms. To begin with, if we look at *simple coordinators*, we are provided with well-known linear orderings: The coordinator **CHR** of STRING has the *lexicographic ordering* realized in every dictionary (see example 64 in appendix C.2 for this concept). The coordinator $BIT$ of BOOLE is ordered by $0 < 1$, integers and decimal numbers are given their standard orderings.

To define a linear ordering among denotators $D$ with

$$D\text{-}Name \rightsquigarrow D\text{-}Form(D\text{-}Coordinates), \tag{6.11}$$

we proceed as follows: Suppose that we have defined linear order relation $<^*$ for couples of denotators regardless of their names: $Form1(Coordinates1) <^* Form2(Coordinates2)$. Then we can easily extend the ordering lexicographically with priority on $<^*$:

$$'D1\text{-}Name' \rightsquigarrow Form1(Coordinates1) < 'D2\text{-}Name' \rightsquigarrow Form2(Coordinates2)$$

if and only if either

$$Form1(Coordinates1) <^* Form2(Coordinates2)$$

or

$$Form1(Coordinates1) = Form2(Coordinates2),$$

and in the latter case,

$$'D1\text{-}Name' < 'D2\text{-}Name'$$

in the lexicographic ordering on **CHR**. Therefore, we can concentrate on couples of denotators regardless of their names. The same holds for forms: If we have settled forms without name, the lexical naming ordering just refines the nameless ordering. So we may forget about names and concentrate on the structure $Type(F_1, ...F_n)$ of a form. Again, we proceed lexicographically with priority to the type. First, let us order the types as follows:

$$\text{BOOLE} < \text{INTEGER} < \text{STRING} < \text{FLOAT} < \mathbf{Syn} < \prod < \coprod < \{\} \tag{6.12}$$

Then, if the type is fixed, we set[17]

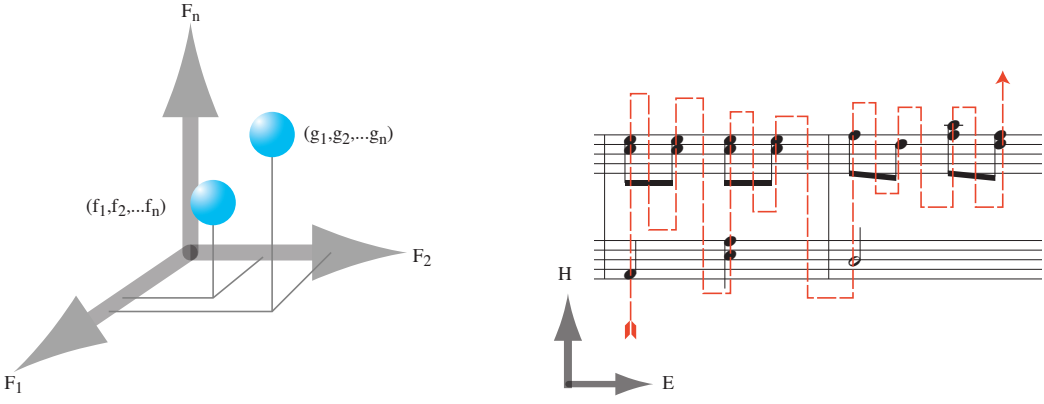


Figure 6.6: Linear ordering on product type. To the right, we have an example of piano notes in their linear ordering indicated by the dotted arrow.

$$Type(F_1, ...F_n) < Type(G_1, ...G_m) \quad \text{iff} \quad F_1, ...F_n < G_1, ...G_m \tag{6.13}$$

for the lexicographic ordering on the words on the alphabet of forms. This settles the order relation between forms, and we are left with the order relation $Form(Coordinates1) <^* Form(Coordinates2)$. The simple forms are already settled by the introductory remark on the standard orderings on $BIT$, **CHR**, $\mathbb{Z}$, and $\mathbb{R}$.

We now turn to compound denotators and make the general recursion hypothesis: On coordinates, a linear ordering is defined. For **Syn**, we just inherit the ordering on the coordinator. For $\prod$, we take the lexicographic ordering: $\prod(f_1, \dots f_n) < \prod(g_1, \dots g_n)$ iff we have $f_i < g_i$ for the first index $i$ where the coordinates differ, see Figure 6.6. For the coproduct $\coprod$, we set $\coprod(f_i) < \coprod(g_j)$ iff either $i < j$ or $i = j$, and then $f_i < g_i$. Intuitively, this is the situation we know from a library, see Figure 6.7.

[17] "iff" is mathematical shorthand for "if and only if".

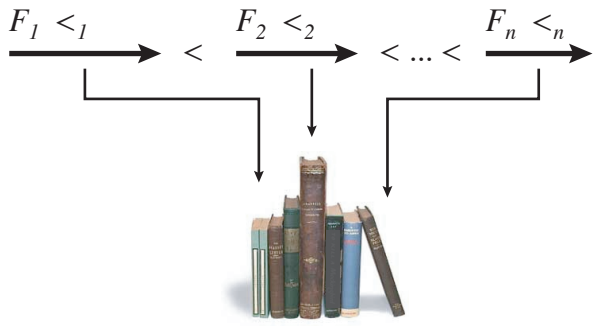


Figure 6.7: Coproduct ordering is similar to the ordering within a library. The books $F_i$ are numbered from 1 to $n$ and arranged in this order on the shelf. Within each book, we have a given linear ordering of entities.

Finally, the powerset ordering[18] between different sets

$$S = \{\}(f_1, \ldots f_n) \text{ and } T = \{\}(g_1, \ldots g_m)$$

of given form $F$ is defined as follows (see Figure 6.8):

$$S < T$$

iff either

$$S \text{ is a proper subset of } T$$

or both difference sets

$$S - T \text{ and } T - S$$

are non-empty, and

$$Max(S - T) < Max(T - S),$$

the maxima being taken on the linear ordering of the form $F$.

Intuitively, we may interpret the linear ordering among denotators as being a tool for conceptual leafing, the usual running through key-words being extended to typically leafing through geometrically shaped coordinator spaces with many coordinate axes (like vector spaces). Further, the user can "dive" into deeper coordinator layers, though never getting lost within hierarchical ramifications since linear ordering is a much stronger ordering principle than hierarchies.

[18] See appendix C, proposition 65, for a proof that the poweset ordering is linear.

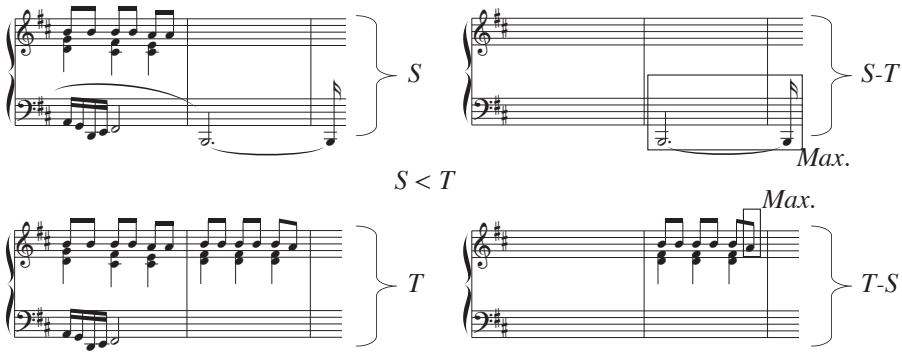


Figure 6.8: The linear order relation between two sets S and T of notes; we have S < T.

## 6.2 Forms

**Summary.** Forms are introduced as the universal format of spaces. They are defined in a recursive way, carry a *name*, a recursive ramification *type*, a diagram of recursive references, the *coordinator*, and the so-called *space functor identifier* of the form. Simple types define the beginning of recursive ramification. The ramification type is either simple, synonym, limit, colimit, or powerset. The coordinator diagram is defined and discussed in detail.

– Σ –

The rest of this chapter is devoted to the generic and formal setup for forms and denotators. We shall, however, not change the surface of our terminology. Denotators and forms are the naive concepts for those readers who have no necessity to view the generic setup. The latter is intimately related to the so-called functorial point of view (see appendix G.1.2). We shall nonetheless adopt the standard attitude of modern algebraic geometry and talk about "points" even though the space where the points are positioned is not a fixed one but may vary as a function of an entire parameter system of 'addresses'. For mathematical reasons which will become clear soon, the naive perspective will be called the *zero (address) perspective* if a clear-cut distinction from the functorial perspective is necessary.

### 6.2.1 Variable Addresses

**Summary.** The functorial approach is introduced as a variable address question.

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Let us start with an elementary example. In section 6.1.1, we have introduced the form

$$\text{Pitch} \to \text{INTEGER}$$

of simple type INTEGER = INTEGER($\mathbb{Z}$) and corresponding denotators which look like

$$c\text{-Pitch} \rightsquigarrow \text{Pitch}(c), c \in \mathbb{Z}.$$

The reason, why we say that such a denotator has "zero address" is that the set $0_{\mathbb{Z}}@\mathbb{Z}$ of affine homomorphisms from the zero module $0_{\mathbb{Z}}$ to the integers $\mathbb{Z}$ (appendix E) reduces to the translations in $\mathbb{Z}$ and therefore is in bijection[19] with $\mathbb{Z}$, each $z \in \mathbb{Z}$ giving rise to the translation $e^z : 0 \mapsto z$. Whence the expression "zero address" to specify the set of denotators $c$-pitch with coordinates in $c \in \mathbb{Z}$. What is the interest in generalizing this zero address which presently looks like an algebraic overhead to catch the elements in modules?

First, there are deep reasons relying on universal constructions of new denotators from given ones: Without the functorial perspective some of these constructions would not be possible; we shall discuss this issue in section 8.3.3. Second, there are important approaches in mathematical music theory which already make use of variable addresses, for example in harmony, see chapters 24 through 26. But let us give a simple example as an ad hoc justification.

The above Pitch form describes integer-valued pitch denotators sitting on the zero address. This would be sufficient to grasp the chromatic pitches if we took the integers for labels of semitone steps, on a well-tempered[20] keyboard, say. But this is a somewhat reduced view of pitch in music. In score notation, we not only have the effective key number but additional information about alterations stemming from tonal frames. For example, if the denotator 66-pitch were attached to the 6th semitone above middle C, this could be denoted by F♯ or G♭, depending upon the tonal context. In order to express this refinement, an adequate non-zero address provides us the necessary data. In fact, if we take $\mathbb{Z}$ instead of $0_{\mathbb{Z}}$, the affine homomorphisms $f = e^a \cdot b : \mathbb{Z} \to \mathbb{Z}$ in $\mathbb{Z}@\mathbb{Z}$ are in one-to-one correspondence with couples $(a, b) \in \mathbb{Z}^2$ where $a = f(0), b = f(1) - f(0)$. In other words, a $\mathbb{Z}$-addressed object $f \in \mathbb{Z}@\mathbb{Z}$ gives rise to two data: the "base pitch" $a = f(0)$ and the "shifted pitch" $f(1) = b + f(0)$ staying b semitones above the base pitch, see Figure 6.9. Intuitively speaking, the $\mathbb{Z}$-addressed objects

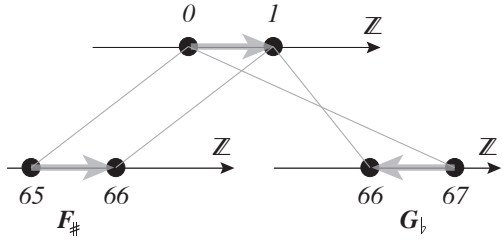


Figure 6.9: Two different $\mathbb{Z}$-addressed points define the musical notations F♯ and G♭, respectively, as "arrow" objects in $\mathbb{Z}$.

are the affine images of the unit arrow $0 \Rightarrow 1$ in $\mathbb{Z}$. And these denote precisely what we were looking for: A base pitch (= arrow tail) and a shifted pitch (= arrow head). In musical terms, our example 66-pitch could be refined to the pitch alteration F♯ by the arrow object $f = e^{65} \cdot 1$

[19] This is true in complete generality: For any module $M$, we have a canonical bijection $0_{\mathbb{Z}}@M \xrightarrow{\sim} M$ (appendix E.3).

[20] Recall that all musicological terms will be introduced with care later. Here we only use examples for motivating the denotator concept.

or to the pitch alteration G♭ with arrow object $f = e^{67} \cdot -1$. This means that we have refined the zero-address object

$$0 \qquad \mapsto \qquad 66$$

to the $\mathbb{Z}$-addressed "arrow" object

$$0 \Rightarrow 1 \qquad \mapsto \qquad 65 \Rightarrow 66$$

for F♯ or

$$0 \Rightarrow 1 \qquad \mapsto \qquad 67 \Rightarrow 66$$

for G♭, respectively[21]. Moreover, zero-addressed and $\mathbb{Z}$-addressed points are canonically related: On one hand, the unique affine homomorphism $zero : \mathbb{Z} \to 0_{\mathbb{Z}}$ transforms every zero-addressed point $z : 0_{\mathbb{Z}} \to \mathbb{Z}$ into the composed $\mathbb{Z}$-addressed point $z \circ zero : \mathbb{Z} \to \mathbb{Z}$. If z stands for the denotator $c$-pitch, $z \circ zero$ is the arrow object which sends the arrow $0 \Rightarrow 1$ to the zero-length arrow $c \Rightarrow c$. On the other hand, if we look at the zero-addressed points $tail := e^0 \cdot 0$ and $head := e^1 \cdot 0$, a $\mathbb{Z}$-addressed point $f = e^a \cdot b$ yields $f$'s tail point $f \circ tail$ and $f$'s head point $f \circ head$, respectively.

Summarizing, adding variable addresses do enrich the expressivity of denotators, the relations between different addresses add canonical connections to the objects which are introduced on different address levels.

### 6.2.2 Formal Definition

**Summary.** The definition of a form is given, including name, type, coordinator, and the space functor identifier.

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Recall (appendix G) that $\mathbf{Mod}^{@}$ denotes the category of contravariant functors $Fu : \mathbf{Mod} \to \mathbf{Sets}$ from the category **Mod** of modules to the category **Sets** of sets. Recall also that for functors $Fu$ in $\mathbf{Mod}^{@}$, a module $M$ which is an argument of $Fu$ is called an *address of* $Fu$, and a morphism $f : M \to N$ of modules is called an *address change*. Recall the notation $M@Fu$ for the value of $Fu$ at address $M$. Recall finally the subobject classifier $\Omega$ in the topos $\mathbf{Mod}^{@}$[107].

**Definition 3** *A form*[22] $F$ *is a quatruple* $F = (NF, TF, CF, IF)$ *where*

(i) $NF$ *is a string of ASCII characters; it is called the* name *of* $F$ *and denoted by* $N(F)$.

[21] Again, for any module $M$, the $\mathbb{Z}$-addressed points of $M$, i.e. the elements of $\mathbb{Z}@M$, are in bijection with the "arrows" $m \Rightarrow n$ in $M$, see appendix E.

[22] A completely general definition of a form which implies names which are also denotators, is given in appendix G.5.3. This generalization is necessary for more flexible "global" name spaces. Here, we stick to a naming convention which restricts to $ASCII$ names.

(ii) *TF is one of the symbols*[23]

1. **Simple**,
2. **Syn**,[24]
3. **Power**,
4. **Limit**,
5. **Colimit**;

*it is called the* type *of F and denoted by* $T(F)$.

(iii) *CF is one of the following objects according to the previous symbols:*

A. *For* **Simple**, $CF$ *is a module* $M$,

B. *for* **Syn**, *and* **Power**, $CF$ *is a form,*

C. *for* **Limit** *and* **Colimit**, $CF$ *is a diagram* $\mathbf{D}$ *of forms;*

*it is called the* coordinator *of F and denoted by* $C(F)$. *The diagram* $\mathbf{D}$ *is a diagram of functors* $Fun(F_i)$, *as defined in (iv), for a family* $(F_i)_i$ *of forms.*

(iv) *IF is a monomorphism of functors* $IF : Fu \rightarrowtail X$ *in* $\mathbf{Mod}^{@}$, *with this data:*

1. *For* **Simple**, $X = @M$,
2. *for* **Syn**, $X = Fun(CF)$,
3. *for* **Power**, $X = \Omega^{Fun(CF)}$,
4. *for* **Limit**, $X = lim(\mathbf{D})$,
5. *for* **Colimit**, $X = colim(\mathbf{D})$;

*it is called the* identifier *of F and denoted by* $I(F)$, *whereas its domain Fu is called the* space (functor) *of F and denoted by* $Fun(F)$. *The codomain of the identifier is called the* frame space *of the form.*

To denote a form $F$, we inherit the notation of the naive setup and add the identifier below the arrow:

$$Name \underset{Identifier}{\longrightarrow} Type(Coordinator). \tag{6.14}$$

**Naming.** Evidently, the naming formalism is identical with the naive one. We shall also adopt the policy that different forms should bear different names.

[23] They turn out to symbolize five types of operators, but in the definition, we just need the symbols, i.e., the character strings.

[24] In the generic definition of a form, the synonym form type is superfluous. We however maintain this type for semiotic reasons: synonymy is a proper type of reference which is meant to be different from any other reference mode.

**Typology.** The typology is somewhat more complicated: For the naive approach, simple types were restrained to a selection of four sample coordinators (**CHR**, $BIT, \mathbb{Z}, and\ \mathbb{R}$) whereas here, simple type means selecting any coordinator module $M$. We shall make this more explicit in section 6.4. Philosophically, this means that the basic substance spaces are now taken from the category of modules and not merely from four types STRING, BOOLE, INTEGER, and FLOAT. It is plausible that the vast domain of modules[25] is large enough to cover once for all the needs of musicology, mathematics, and computer science for grounding space substance. As to the other types, synonymy looks quite the same as with the naive approach: We just have to go back one step in recursion. The real typological difference resides in the last three types, **Power**, **Limit**, and **Colimit** which generalize the naive types powerset, product, and coproduct. We are going to describe all the types and their characteristics in section 6.2.3.

**Coordinator.** The coordinator looks quite familiar for simple type, instead of one of the four simple coordinators in the naive setting, we are given a module. For synonymy and power types, we have the same data as in the naive setting. For limits and colimits, we are given a diagram of forms. This generalizes the naive setup where we just had a finite sequence of forms in the coordinator, whereas a diagram would contain a number of "arrows" between members of such a sequence. See appendix G for the concept of a diagram in a category. In our context, a form diagram has its vertexes in the set of form names and evaluates to the corresponding form functors, but see appendix G.5.3 for a rigorous formalism.

**Identifier.** There is, however, a remarkable difference in that we add the new instance of an "identifier" to the naive setup. Why? In the naive setup, we had a unique choice of simple spaces: the four coordinators (**CHR**, $BIT, \mathbb{Z}, \mathbb{R}$). In general, this cannot be guaranteed, many equivalent modules or higher constructs (functors) may intervene, so we have to integrate them without losing their structural identification. The latter is taken care of by the identifier monomorphism whereas the functor $Fun(F)$ of the form, i.e. the identifier's domain designates the concrete structure. For example, we may encounter several one-dimensional real vector spaces V, but essentially, they are all isomorphic to $\mathbb{R}$. Such an isomorphism $I(F) : V \xrightarrow{\sim} \mathbb{R}$ is denoted by the identifier[26].

A basic comment on the structure of definition 3 is necessary. The concept of a "form" refers to four ingredients: Name, Type, Coordinator, and Identifier. Whereas the first two specifications are inoffensive, the last two share *circularity*: They refer to the form concept to be defined, at least in the cases (iii), B. and C. as well as (iv), 2. to 5. In mathematical definitions, circularity

[25] Observe that modules can have any commutative or non-commutative coefficient rings!

[26] Historically, the identifier problem arose when transcribing the software-oriented PrediBase data base management system [589] to an abstract formalism. In fact, object-oriented programming languages ask for default values of instance variables in order to initialize objects. This initialization process means that you have to write down a first concrete value to be instantiated without further activities from the user's part. If you cannot stick to a unique domain of values, for decimal numbers (floats), say, the default value will tell what kind of representative of the domain you have selected. If you write down 0.0 as a default value, you make precise which kind of zero representation you have taken. What you are really doing is pointing at a particular representative of "the" $0_{\mathbb{R}} \in \mathbb{R}$, i.e. you sort of point at the particular 0.0: $0_{\mathbb{R}} \mapsto 0.0$, and this is what our monomorphism takes care of: It identifies a representative with "the abstract object". Indeed, quite often the identifier is not only mono but even iso. However, isomorphisms would create serious existence problems for circular forms, see appendix G.2.2.1 for this subject.

is admitted only if it is embedded in a recursive definition structure, i.e. if one is given an ordinal number associated with every instance of the concept in question that appears within the definition. For example, the product of natural numbers $a * b$ is defined according to the size of $b$: For $b = 0$, we set $a * 0 = 0$, and $a * (b+1)$ is defined to be $(a * b) + a$. But in set theory, a set $Z$ defined by $Z := \{Z\}$ is a problematic object, since no ordinal "size" number is given. In "normal" set theory, such non-regular, circular sets are excluded by the axiom of regularity [281]. We come back to this issue in section 6.5. For a formal definition of logically consistent domains of forms, see appendix G.5.3.

### 6.2.3 Discussion of the Form Typology

**Summary.** We discuss the characteristics of the form types and compare them to the naive variants.

– Σ –

**Simple.** A form $F$ of simple type consists of a name $N(F)$, a coordinator module $M$, a functor $Fun(F)$ in $\mathbf{Mod}^{@}$, and an identifier monomorphism[27] $I(F) : Fun(F) \rightarrowtail @M$. Since in the naive setup, we did not have to change addresses, the module $M$ (one of four sample modules) was all we needed to describe simple type forms.

**Syn.** This form type adds a new name $N(F)$ to an already known coordinator form $C(F)$, and a functor monomorphism $Fun(F) \rightarrowtail Fun(C(F))$.

**Power.** To relate the naive powerset type to **Power**, let $Fu$ be a functor of $\mathbf{Mod}^{@}$ and consider the functor $2^{Fu}$ of $\mathbf{Mod}^{@}$, defined by $M@2^{Fu} := 2^{M@Fu}$, the powerset of the $Fu$-value at address $M$. For an address change $f : N \to M$, we use the set map $f@Fu : M@Fu \to N@Fu$ to define $f@2^{Fu} : M@2^{Fu} \to N@2^{Fu} : S \mapsto f@Fu(S) =: S \cdot f$.

Recall (appendix G.3, examples 99 and 97) that $\Omega^{Fu}$ evaluates to $M@\Omega^{Fu} \xrightarrow{\sim} Sub(@M \times Fu)$ at address $M$. To connect the naive powerset type and the general one, we define a natural transformation

$$\hat{?} : 2^{Fu} \to \Omega^{Fu} \tag{6.15}$$

by the following subfunctor $\hat{S} \subset @M \times Fu$ for $S \in M@2^{Fu}$. For a given address $N$, the subfunctor $\hat{S}$ is defined by

$$N@\hat{S} := \{(u \in N@M, v \in N@Fu) | v \in S \cdot u\}. \tag{6.16}$$

It is easily seen that the transformation (6.15) is also natural in $Fu$, i.e. if $h : Fu \to Gu$ is a natural transformation, the following canonical diagram is commutative:

$$\begin{array}{ccc} 2^{Fu} & \xrightarrow{\hat{?}} & \Omega^{Fu} \\ {\scriptstyle 2^h}\downarrow & & \downarrow{\scriptstyle \Omega^h} \\ 2^{Gu} & \xrightarrow{\hat{?}} & \Omega^{Gu} \end{array} \tag{6.17}$$

[27] If the identifier is iso, this means that $Fu$ is representable, and that it is represented by $I(F)$. In general, $Fu$ is isomorphic to a sieve (appendix G.3.1) in $M$.

We shall see in section 6.3.1, item "Coordinates", that this transformation connects naive powerset denotators to those of **Power** form.

**Limit.** A "diagram of forms" is just a diagram in the category $\mathbf{Mod}^{@}$, via the functor $Fun(F)$ of a form $F$ (see appendix G.1.2, definition 151 for diagrams in categories). We can also speak of the *category of forms*, the morphisms being the morphisms of the forms' functors, so the terminology "diagram of forms" is also formally correct. The relation to the naive setting is that a product $\prod F_i$ of forms is a limit for a discrete diagram (no arrows) whose vertexes are the indices $i$ of the product's family $(F_i)_{i=1,...n}$.

**Colimit.** Equally, a coproduct $\coprod F_i$ of forms is a colimit for a discrete diagram whose vertexes are the indices $i$ of the product's family $(F_i)_{i=1,...n}$.

We would like to stress that all these constructions presuppose that the reference to the coordinator's forms and their functors is possible, i.e. that these objects all do exist[28]. We come back to this problem in section 6.5.

## 6.3 Denotators

**Summary.** The formal concept of a denotator as being a form plus a substance point is introduced. Implicitly, the point concept is also recursively defined by its reference to the recursive structure of the underlying form space that was introduced in the preceding section.

– Σ –

### 6.3.1 Formal Definition of a Denotator

**Summary.** Based on the definition of a form (subsection 6.2.2), we define denotators.

– Σ –

**Definition 4** *Let $M$ be an address. A* $M$-addressed denotator *is a triple $D = (ND, FD, CD)$ where*

(i) *$ND$ is a string of ASCII characters; it is called the* name of $D$ *and denoted by $N(D)$.*

(ii) *$FD$ is a form; it is called the* form of $D$ *and denoted by $F(D)$.*

(iii) *$CD$ is an element of $M@Fun(F(D))$; it is called the* coordinates[29] of $D$ *and denoted by $CT(D)$.*

According to the naive setup, a denotator is denoted by

$$Name : Address \rightsquigarrow Form(Coordinates) \tag{6.18}$$

[28] Following Paul Finsler [153], an object's existence is equivalent to the possibility to think the objects without provoking any contradiction to classical principles of logic: identity, contradiction, and excluded third.

[29] This is a plural in singular mode.

So, in the full notation, a denotator $D$ with form $F$ is symbolized as follows:

$$N(D) : Address \rightsquigarrow N(F) \underset{Identifier}{\longrightarrow} Type(Coordinator)(Coordinates), \tag{6.19}$$

but this clumsy writing is only used if absolutely unavoidable. Again, as already described for the naive setup, if the denotator name is empty (which happens often), we shall write

$$myAddress \rightsquigarrow AForm(myCoordinates) \tag{6.20}$$

for a denotator of form $AForm$ and coordinates $myCoordinates$, or even

$$AForm(myCoordinates) \tag{6.21}$$

if $myAddress$ is also clear (in fact, the zero address in the naive context).

**Naming.** The naming policy is identical with the naive situation. Recall that we can have the empty denotator name ″.

**Address.** The address module $M$ is an important generalization, however, we shall not overstress it since in the everyday language we speak of denotators independently of their address, as if there were just one ambient space instead of an entire space functor.

**Form.** The form of a denotator englobes the whole recursion information as well as the form's functor. It is the latter which contains the coordinates.

**Coordinates.** The coordinates are one "point"[30] or element of the form's functor *at* the given address $M$. Let us look at the shape of coordinates as a function of the particular form:

1. For **Simple**, the coordinator is a module $N$, and the coordinates $CT(D)$ are identified (via the identifier!) with an element of $M@N$, i.e. a morphism $CT(D) : M \to N$ of modules. In the case of the zero address $M = 0_{\mathbb{Z}}$, such a morphism clearly identifies with a 'real' element of $N$ (appendix E.3).
2. For **Syn**, the coordinates $CT(D)$ identify with an element of $M@Fun(C(F(D)))$. So the coordinates are described by recursion to the coordinator $C(F(D))$.
3. For **Power**, the coordinates $CT(D)$ identify with an element of $M@\Omega^{Fun(C(F))} \xrightarrow{\sim} Sub(@M \times Fun(C(F)))$, i.e. a subfunctor of $@M \times Fun(C(F))$. But we know from the discussion of form typology in section 6.2.3 that each set $S \in M@2^{Fun(C(F))}$ of $M$-addressed points in $Fun(C(F))$ gives rise to a subfunctor $\hat{S}$ of $@M \times Fun(C(F))$. So the zero-addressed coordinates in the naive sense give rise to coordinates in the general setup. We also see that an $M$-addressed naive coordinate set $S \subset M@Fun(C(F))$ is just a set of "points" at a general but *fixed* address $M$. Passing to the functorial setup means switching to an entire system of sets $?@\hat{S}$, where $\hat{S}$ is a 'functor of subsets' in $M@\Omega^{Fun(C(F))}$.

[30] In algebraic geometry, an element of $M@F$ for a functor $F \in \mathbf{Mod}^{@}$ is called an "$M$-valued point of F".

4. For **Limit**, suppose that the diagram **D** of forms has vertex forms $F_i$ and vertex functors $Fu_i := Fun(F_i)$ at vertexes $i$, as well as natural transformations $m(f) : Fu_{tail(f)} \to Fu_{head(f)}$ for arrows $f$ in **D**. Recall (appendix G.2.1, theorem 60) that at a given address $M$, we have a natural isomorphism

$$M@lim(\mathbf{D}) \xrightarrow{\sim}$$
$$\{x \in \prod_i M@Fu_i \mid m(f)(x_{tail(f)}) = x_{head(f)}, \text{ all arrows } f \text{of } \mathbf{D}\}$$

so that the coordinates of **Limit** denotators are special tuples in the product of all $M@Fu_i$. In other words, the limit denotators are canonically related to the product denotators from the naive setup. Therefore the general setup does resemble the naive one as being focused on a general, but fixed address.

5. For **Colimit**, let us inherit the notation of the preceding situation. Recall (appendix G.2.1, theorem 60) that for a given address $M$, we have the equivalence relation $\sim$ on the coproduct $\coprod_i M@Fu_i$, generated by the relation $x \sim y$, $x \in M@Fu_i$ and $y \in M@Fu_j$, iff there is an arrow $f$ in **D** with $i = tail(f)$, $j = head(f)$, and $m(f)(x) = y$. We then have a natural isomorphism

$$M@colim(\mathbf{D}) \xrightarrow{\sim} \coprod_i M@Fu_i/\sim$$

and recognize that the naive coproduct is just the basic space of the **Colimit** denotator space before dividing through the equivalence relation $\sim$, and again, selecting a general, but fixed address $M$. So—up to general addressing—the coordinates of a **Colimit** denotator are those of the coproduct *modulo* an equivalence relation defined by the diagram's arrows.

If $f : F \to G$ is a morphism of forms, and if $D : X \rightsquigarrow F(C)$ is a denotator, we canonically have the *f-image of D* $f(D) : X \rightsquigarrow G(X@f(C))$. Superficially the denotator definition is not recursive, but we recognize from the above discussion of the coordinates that a denotator's coordinates are just as recursive as the involved forms. In the naive setup this was less hidden because of the missing identifier (the identifier was 'set to identity'). Again, a denotator bears a fundamental pointer character: It can only be understood by pointing down to the coordinates which are distributed among the various recursive forms.

## 6.4 Anchoring Forms in Modules

**Summary.** This section deals with recursive foundation of forms in simple spaces. It presents a justification for choosing modules and affine transformations as basic space types. First examples are presented and discussed in the light of musical and musicological requirements.

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Since modules are anchor structures for denotators, we should comment on this decision: Why is the module structure a good choice? There are mathematical and musicological, practical

and theoretical reasons. We want to expose them generically and by use of common examples. We have transferred the purely mathematical description of a module to appendix E, we have also compiled a number of common parameter spaces for musical objects in appendix A. Here, we refer to these chapters and concentrate on the justification discourse.

To begin with, let us shortly digress on the relation between mathematical structure and musical meaning. One could require that a determinate structural instance (such as additive closure of modules) has to play a musical or musicological role in all situations where the structure is present. However, such a requirement would be too restrictive since mathematics should make available to music what may possibly happen in music, and not what happens in any case. In other words, mathematics must furnish a minimal conceptual and theoretical region where the possible applications can be located.

We shall encounter cases where mathematical structure overhead seems to appear. We then obviously have to reflect on possible musical or musicological interpretations of this overhead. This is a delicate question since it may lead to contrary answers depending on its topographic context. In particular with regard to poietics, more concretely: in the process of generating a determinate composition, a mathematical structure may serve as a vehicle of the sound material's organization that otherwise has never been used, e.g. aleatorics with John Cage [277], Fibonacci numbers[31] with Karlheinz Stockhausen [218], or fractals with Mesias Maiguashca [220]. It cannot be decided a priori which mathematical structures carry musical meaning. In this sense, a module structure is but an adequate formal frame for an empirical material which *presently* guarantees a broad field of applications—in the future it may even be enlarged. It is however not true that reflecting musical and musicological thinking via mathematical structuring may lead to "foreign infiltration in music" as it was argued in [394]. Rather does it catalyze the fundamental dialectic process of how music sees itself.

Before looking at concrete examples, let us view the general point of taking modules. In order to handle denotators in music, we need not only consider "points in spaces" but also algebraic operations which relate points to each other and allow creation of new points out of given ones. In music, sound objects are not understood as being isolated points in amorphous spaces but parts of an entire relational system, and this insight is the reason for introducing modules. They are the adequate structure to grasp this formal mechanism of building new from old points. We should also keep in mind in the following that it is not necessary (and sometimes not possible) to look at all addresses for simple denotators, but we will pick up some of them to illustrate their role.

### 6.4.1 First Examples and Comments on Modules in Music

**Summary.** Musicological aspects of the algebraic structure of a module and of affine transformations among modules are discussed. We presuppose the knowledge of module structure and of the examples from the appendices A and E. We also illustrate the use of modules for more general denotators which may intervene for not strictly musical purposes, such as text elements, for example.

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[31] Named after Italian mathematician Leonardo Fibonacci (1180-1250 (?)), the Fibonacci sequence $(x_i)_{i=1,2,3,\ldots}$ is defined by $x_1 = x_2 = 1$ and recursively by $x_{i+2} = x_i + x_{i+1}$. The limit number $\lim_{i\to\infty} x_i/x_{i+1} \approx 0.618036$ is the famous *golden section*.

1. $BIT$. This set is also known as $BIT = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, the integers modulo 2, see appendix D, example 75. It has an additive structure which turns the elements into logical units used in music instrument editing. In fact, look at the simple 'self-addressed' denotators $D(a,b) : BIT \rightsquigarrow Switch(e^a \cdot b)$ of form $Switch \xrightarrow[Id]{} \mathbf{Simple}(BIT)$, having coordinates $e^a \cdot b \in BIT@BIT$. Such a denotator $D(a,1)$ is the $a$-translation on $BIT$ and represents its additive structure. If we view the elements s of $BIT$ as being the states of an ON-OFF switch on a musical instrument, the denotators $D(a,b)$ can be viewed as operators on the states of the switch: We have[32]

$$D(a,1)(s) = \begin{cases} s & \text{if } a = 0, \\ 1 + s = \neg s & \text{if } a = 1. \end{cases} \tag{6.22}$$

So $a = 0$ leaves the state $s$ of the button whereas $a = 1$ changes it. Observe that we also have

$$D(a,0)(s) = a, \tag{6.23}$$

and this means that the operator $D(a,0)$ is a *reset* to the state $a$ of the switch. More generally, the theory of switching circuits and automata [236] can be executed on denotators which are built upon $BIT$ as a simple 'module basis'.

2. **CHR**. In the naive setup, the set of character strings $'x_1x_2\ldots x_n'$ from a character alphabet (the set $ASCII$ of ASCII characters in our case) is not a module, but it can be embedded in a module according to the very comfortable and common mathematical structure of a monoid algebra (see appendix D.1.1). Its construction presupposes an alphabet $A$ (generalizing $A = ASCII$) and a commutative ring $R$ of 'coefficients'. We first build the free monoid $\langle A\rangle$ of words $'x_1x_2\ldots x_n'$ over the alphabet $A$. The monoid algebra $R\langle A\rangle$ consists of all formal sums $\sum_{w\in\langle A\rangle} r_w w$ with only finitely many coefficients $r_w \neq 0$. In particular, the words $w = 1w$ are elements of the monoid algebra, and we have $\mathbf{CHR} \subset \mathbb{Z}\langle ASCII\rangle$.

   The structure of $\mathbb{Z}\langle ASCII\rangle$ as a $\mathbb{Z}$-module is more than the simple names of denotators and forms—for example—require. But the use of formal sums is straightforward: We can take the names of all forms which appear in a determined description of the denotators (notes, bar-lines, pauses, time and key signature, composer's name, etc.) of a score $S$ and attribute to these names $N$ the multiplicity $a_N \in \mathbb{Z}$ of their appearance in denotators of $S$. This defines a multiplicity-sensitive *list of forms* in $S$, i.e.

$$FormList(S) := \sum_{N=\text{form name in S}} a_N N. \tag{6.24}$$

   We obtain the effective calculation of the score's form list by successive addition: If $S$ splits into two disjoint parts $S_1$ and $S_2$, we have $FormList(S) = FormList(S_1) + FormList(S_2)$. Further, if $FormList(S)$ and $FormList(T)$ are two such lists, the difference $FormList(S) - FormList(T)$ in general produces negative coefficients which measure the relative amount of form names in $S$, when compared to $T$. From this we learn that account of words from a determined alphabet can make use of the corresponding monoid algebra over some appropriate ring to perform its calculations.

[32] The symbol $\neg s$ means the negation of state s, see also appendix G.5.

3. The real numbers $\mathbb{R}$. Recall (appendix E.2.1, example 77) that $\mathbb{R}$ is a module[33] over the subfield of rational numbers $\mathbb{Q}$ which we denote by $\mathbb{R}_{[\mathbb{Q}]}$. We want to view this module as being the coordinator of a simple pitch form

$$MathPitch \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}_{[\mathbb{Q}]}), \tag{6.25}$$

as discussed in appendix A.2.3. In accordance with that discussion, we interpret denotators $MathPitch(h)$ as having the mathematical pitch coordinate

$$h = h(f) = u \cdot \ln(f) + v$$

associated with frequency $f$ and a couple of normalization coefficients $u, v$. In this view, the ratio

$$h(f) - h(g) = u \cdot \ln(f/g)$$

of two frequencies $f$ and $g$ becomes a difference of MathPitch coordinates. This transformation reflects the auditory experience that the frequency ratio $f/g$ is perceived as a "pitch distance":

**Fact 1** *The additive structure of $\mathbb{R}_{[\mathbb{Q}]}$ is the basis of thinking in pitch distances.*

Further, scalar multiplication with a rational number $a/b$ splits into a product of three special cases: multiplication by

- $-1$,
- a natural number $p \geq 0$,
- $1/q$, $q$ a positive natural number.

What is the musical meaning of these three operations?

- Multiplication of a pitch difference $h(f) - h(g)$ by $-1$ yields $-1 \cdot (h(f) - h(g)) = h(g) - h(f)$, we look from the second pitch to the first instead of looking from the first to the second, i.e. the roles of the two pitches are *exchanged*.
- For $p \geq 0$, we have $p \cdot (h(f) - h(g)) = p$-fold *juxtaposition* of $h(f) - h(g)$; the special case $p = 0$ means that we switch to the zero distance.
- $1/q \cdot (h(f) - h(g))$ means that we seek a distance which—after a $q$-fold juxtaposition—yields the original distance $h(f) - h(g)$.

**Fact 2** *Scalar multiplication in the pitch module splits into role exchange, p-fold juxtaposition, and q-fold division of pitch distances.*

**Remark 1** The signification of a mathematical operation often becomes evident after a 'split' into special cases from which the general case results via appropriate conjunction. This is based on the principle that musical meaning is conserved after such 'recombination'. This is genuine musical thinking! The example of the contrapuntal operation of

[33] In fact an infinite-dimensional vector space.

"retrograde inversion" rightly demonstrates that the operation is only 'understood' via its decomposition into "retrograde" and "inversion", and *not* as such. We come back to this very important subject in section 8.3.

4. The Euler module $\mathbb{Q}^3$ over the field $\mathbb{Q}$ of rationals. This module is introduced in appendix A.2.3 and represents one of the most classical mathematical structures in mathematical music theory. We also view this module as being the coordinator of the simple form

$$EulerModule \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Q}^3). \tag{6.26}$$

Recall that we have a $\mathbb{Q}$-linear embedding

$$\langle pv| : \mathbb{Q}^3 \to \mathbb{R}_{[\mathbb{Q}]} : x \mapsto \langle pv|x\rangle \tag{6.27}$$

of the Euler module in the pitch module, where $pv = (\ln_b(2), \ln_b(3), \ln_b(5))$ is the "prime vector" relative to a logarithm base $b$, and where the scalar product $\langle pv|x\rangle$ is the above pitch $h(f)$ associated with frequency $f$. Therefore we have a corresponding morphism of forms

$$E2M : EulerModule \to MathPitch \tag{6.28}$$

induced by the above linear embedding and its canonical extension to the forms' functors

$$@\langle pv| : @\mathbb{Q}^3 \to @\mathbb{R}_{[\mathbb{Q}]}. \tag{6.29}$$

As in the preceding example, the additive structure becomes musically significant via differences $x - y$ of Euler points $x$ and $y$ (Euler points are just elements of the Euler module). In fact, we have

$$\langle pv|x\rangle - \langle pv|y\rangle = \langle pv|x - y\rangle, \tag{6.30}$$

and the difference of pitch corresponds to the difference of Euler points. With the same transformation, we can carry over the musical signification of scalar multiplication from pitch to Euler points.

But the Euler module has more in its structure, and this stems from music history. If $o = (1, 0, 0)$, $q = (0, 1, 0)$, and $t = (0, 0, 1)$ are the canonical base vectors, every Euler point $d = (r, s, u)$ can be written uniquely as a linear combination

$$d = r \cdot o + s \cdot q + u \cdot t. \tag{6.31}$$

On the other hand, the fifth and major third (appendix A.2.3) are associated with the Euler points $q - o$ and $t - 2 \cdot o$. Hence, every difference $x - y$ can be written as

$$d = (r + s + 2u) \cdot o + s \cdot (q - o) + u \cdot (t - 2 \cdot o), \tag{6.32}$$

and hence be generated from the octave, fifth, and major third by role exchange, juxtaposition and division, according to fact 2. Here, exchange of roles means that distances are juxtaposed 'downward' instead of 'upward'. Division $1/12 \cdot o$ defines the distance for

semitones in the 12-tempered system, whereas $1/2 \cdot (t - 2 \cdot o)$ means whole tone distance between tonic and third in mediante tuning (see appendix A.2.3). If we work within just tuning, any distance $d = x - y$ can be produced by successive juxtaposition of octave, fifth, and major third, upward or downward.

In the tradition of just tuning, the following Euler point differences have special names [547]:

$$\begin{aligned} &\text{fifth comma } (= \text{Pythagorean comma}) \\ &\qquad Kq = 12 \cdot (q - o) - 7 \cdot o \\ &\qquad\quad = 12 \cdot q - 19 \cdot o \qquad (6.33) \\ &\text{third comma } (= \text{syntonic comma}) \\ &\qquad Kt = (t - 2 \cdot o) + 2 \cdot o - 4 \cdot (q - o) \\ &\qquad\quad = t - 4 \cdot q + 4 \cdot o \qquad (6.34) \end{aligned}$$

Historically, these commata arose from the 12-division of the octave in semitone steps. In this context, an octave equals 12 semitone steps, a fifth equals 7 semitone steps, and a major third equals 4 semitone steps. Therefore there was interest in the difference between least common multiple of the numbers of octave steps and fifth steps (= fifth comma) as well as of third and fifth steps module octave (=third comma). With these identifications we have:

$$12 \text{ fifths} = 7 \text{ octaves (semitone steps)} \qquad (6.35)$$

$$4 \text{ fifths} = 2 \text{ octaves } + 1 \text{ third (semitone steps)} \qquad (6.36)$$

One finds the following approximate values:

$$\text{fifth comma } \approx 23.46 \text{ Cent} \qquad (6.37)$$

$$\text{third comma } \approx -21.51 \text{ Cent} \qquad (6.38)$$

Mathematically, these commata figures are not interesting since we know from appendix A.2.3 that arbitrary small pitch differences can be produced by integer linear combinations of the three vectors $o$, $q$, and $t$. Commata will only become interesting in harmony of just tuning (see section 24.1).

5. Direct sums. Often, new forms are built from simple forms by products, i.e. limits without arrows. These are no longer simple forms, but they are very similar to simple forms. Let us make this precise for the prototypical case of the product of two forms $F \underset{Id}{\longrightarrow} \mathbf{Simple}(M)$ and $G \underset{Id}{\longrightarrow} \mathbf{Simple}(N)$ with coordinator $R$-module $M$ and $S$-module $N$. We can then build two forms

$$F \times G \underset{@(M \oplus N) \xrightarrow{\sim} @M \times @N}{\longrightarrow} \mathbf{Limit}(M, N) \qquad (6.39)$$

$$F \oplus G \underset{Id_{@(M \oplus N)}}{\longrightarrow} \mathbf{Simple}(M \oplus N) \qquad (6.40)$$

where the first identifier is the canonical isomorphism related to the universal property of direct sums of modules. Clearly, these forms are isomorphic under the identity of $@(M \oplus N)$. Under this canonical isomorphism, we may identify these forms and also identify[34] $F_1 \times \dots F_n$ with $F_1 \oplus \dots F_n$ if no confusion is likely.

With this in mind, we may reinterpret the additive structure induced by direct sums of modules: If a product $F \times G$ of simple forms $F$ and $G$ is built as in (6.39), the algebraic a priori isolation of its factors is overridden by a common module where the factors can be compared on an algebraic level. For example, if $n,\ m$ are two positive natural numbers, we can take

$$OnMod_m \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}_m) \tag{6.41}$$

$$PiMod_n \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}_n) \tag{6.42}$$

with $\mathbb{Z}$-modules $\mathbb{Z}_n$ and $\mathbb{Z}_m$ as coordinators. Consider the simplified form

$$OnPiMod_{m,n} = OnMod_m \times PiMod_n. \tag{6.43}$$

Recall from appendix A.2.3 that $PiMod_n$ can be viewed as the form whose coordinator $\mathbb{Z}_n$ denotes the pitch classes in $n$-tempered tuning, whereas $OnMod_m$ can be viewed as the form whose coordinator $\mathbb{Z}_m$ denotes the $m$-cyclic metrical onsets. Putting these two a priori autonomous aspects of a musical note together, we obtain a common representation of onset and pitch within the module $\mathbb{Z}_m \oplus \mathbb{Z}_n$. The coordinator of the form $OnPiMod_{m,n}$ is positioned on a torus and denotes a type of motivic points, i.e. its denotators $OnPiMod_{m,n}(x, y)$ are points of what musicology considers in motif analysis (see section 7.2.3). Now, the algebraic connection of onset and pitch data in this symbolization becomes evident from the self-addressed denotators

$$S(f) : \mathbb{Z}_m \oplus \mathbb{Z}_n \rightsquigarrow OnPiMod_{m,n}(f), \tag{6.44}$$

which evaluate to affine endomorphisms $f$ of $\mathbb{Z}_m \oplus \mathbb{Z}_n$. These denotators $S(f)$ put into affine relation both, onset and pitch data, and thus initiate a *comparative theory of motivic points and consequently of motives* as we shall see in chapter 11, section 11.3.8, and more in detail in section 22.4. The paradigm of comparison will be furnished by the denotators $S(f)$.

Concluding this section, we should review the role of mathematics and semantics in the form concept. The mathematical anchoring of forms in modules is a clear creed to mathematics, but it is also a relativization of mathematical structures in that they need to be enriched: Mathematics is only the 'significate' of the form sign, and the type, identifier and name are equally important to build the entire sign. The conceptual genealogy can be anchored in mathematics, but it is substantially more than that. For instance, the naming of forms is completely different from naming mathematical modules. If we have two module namings $M := \mathbb{Z}^4$, and $N := \mathbb{Z}^4$, then it can be stated that "$M$ and $N$ are identical modules". But for two forms $F \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}^4)$

[34] We adopt the saying that "a compound form $F$ *simplifies* to a simple form $G$" each time we build a simple form $G$ which is isomorphic to a compound form $F$.

and $G \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}^4)$, we cannot say that "$F$ and $G$ are identical forms". However, we could say that, with the above notations, $F \underset{Id}{\longrightarrow} \mathbf{Simple}(M)$ and $F \underset{Id}{\longrightarrow} \mathbf{Simple}(N)$ are identical forms: The naming difference on the mathematical level is irrelevant.

The names are essential, and this stems from semiotics, not from mathematics. We shall see in section 6.5 that there are fundamental differences between forms and straight mathematics. They transcend simple naming conventions in that form names are the only access to forms in circular definitions. Their very existence can be verified only via their name's function as a sign's significant.

## 6.5 Regular and Circular Forms

**Summary.** We discuss the existence problem of a form if in its definition recursion is non-circular, i.e. terminates on the level of simple forms, or if such a recursion is not possible.

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As already announced, the existence question of a form is not trivial. We have to distinguish two cases: Regular and circular forms. Here is the definition of a regular form:

**Definition 5** *Let $\omega$ be an ordinal number*[35]*.*

(i) *A form is* regular of level $\omega = 0$ *iff it is simple.*

(ii) *Suppose that regular forms of all levels $\mu < \omega$ have been defined. Then a form is* regular of level $\omega$ *iff all its coordinator forms are regular of levels $\mu < \omega$, and if $\omega$ is the successor of all the coordinators' levels.*

(iii) *A form is* regular *iff there is an $\omega$ such that the form is regular of level $\omega$.*

(iv) *A form is called* circular *iff it is not regular.*

So the recursive definition of a regular form is built upon its level, and we may define regular forms without any further complication. This can however not work for circular forms. Let us look at some catastrophes which may or may not occur in these cases:

**Circular Synonymy.** The simplest example of circular synonymy is a form of this shape:

$$'SynCirc' \underset{f:Fu \rightarrowtail Fu}{\longrightarrow} \mathbf{Syn}(SynCirc), \tag{6.45}$$

where a form $SynCirc$ is synonymous to itself and therefore the identifier is a monomorphism $f$ of the form's functor $Fu$. Such a form is identified by its name and identifier, i.e. it reduces to a triple $('SynCirc', Fu, f)$ of a character string, an object in $\mathbf{Mod}^{@}$, and a mono endomorphism of this object.

[35] See [281], but in most cases, we may just think of $\omega$ being a natural number.

**Circular Limits.** The prototype of a circular form of limit type is this:

$$'LimCirc' \underset{f:Fu\rightarrowtail Fu\times Fun(G)}{\longrightarrow} \mathbf{Limit}(LimCirc, G), \tag{6.46}$$

where the circular form $LimCirc$ appears in a product with another (supposedly inoffensive) form $G$. If we define the form's identifier by

$$Fu := Fun(G)^{\mathbb{N}}, \text{ and} \tag{6.47}$$

$$f : Fu \xrightarrow{\sim} Fu \times Fun(G), \tag{6.48}$$

where, for an address $M$,

$$\begin{aligned} M@f :& M@Fu \rightarrow M@Fu \times M@Fun(G) \\ & (g_i)_{0\leq i} \mapsto \qquad ((g_{i+1})_{0\leq i}, g_0). \end{aligned} \tag{6.49}$$

From this example we learn that there are infinitely many[36] realizations—on the level of $f$ or of $Fu$—of the above circular limit form.

**Circular Colimits.** In complete analogy to the preceding limit case, we may look at this form[37]:

$$'ColimCirc' \underset{f:Fu\xrightarrow{\sim}Fu\sqcup Fun(G)}{\longrightarrow} \mathbf{Colimit}(ColimCirc, G), \tag{6.50}$$

with the disjoint union $\sqcup$ of functors in its identifier codomain. This form can be realized by

$$Fu := \coprod_{\mathbb{N}} Fun(G), \text{ and} \tag{6.51}$$

$$f : Fu \xrightarrow{\sim} Fu \sqcup Fun(G), \tag{6.52}$$

which is induced by the cofactor isomorphisms

$$f|_i = \begin{cases} Id : Fun(G)_i \xrightarrow{\sim} Fun(G)_{i-1} & \text{for } i > 0, \\ Id : Fun(G)_i \xrightarrow{\sim} Fun(G) & \text{for } i = 0, \end{cases} \tag{6.53}$$

$Fun(G)_i$ being the cofactor of index $i$ in the coproduct. Again, clearly, there are infinitely many realizations of the above circular colimit form.

**Circular Power.** This one can be inspected on a prototype of shape

$$'PowerCirc' \underset{f:Fu\rightarrowtail \Omega^{Fu}}{\longrightarrow} \mathbf{Power}(PowerCirc). \tag{6.54}$$

But we have the canonical monomorphism

$$f = sg = \hat{?} \cdot \{\} : Fu \rightarrowtail 2^{Fu} \rightarrowtail \Omega^{Fu} \tag{6.55}$$

[36] There are even infinitely many mutually non-isomorphic realizations.

[37] If a diagram has no arrows and is finite, we may denote the factors as for products in the naive approach.

which is composed from the canonical singleton morphism $\{\}$ (appendix G.2.2.1, lemma 91) and $\hat{?}$ (see equation (6.15)) and serves as identifier for any functor $Fu$. But observe that this situation forbids that the identifier becomes an isomorphism. In fact, such an isomorphisms $f$ cannot exist since the calculus of cardinalities [281] categorically excludes set injections of type $2^A \rightarrowtail A$ for any set $A$.

Let us look at the example 6.8 from the naive setup. To restate that example in the formal context, we consider

$$'book' \underset{f}{\longrightarrow} \mathbf{Limit}(No., title, text, chapters) \tag{6.56}$$

with identifier

$$f : Fu \rightarrowtail Fun(No.) \times Fun(title) \times Fun(text) \times Fun(chapters) \tag{6.57}$$

and the factor forms

$$'No.' \underset{Fun(No.)\rightarrowtail @\mathbb{Z}}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}), \tag{6.58}$$

$$'title' \underset{Fun(title)\rightarrowtail @\mathbb{Z}\langle ASCII\rangle}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}\langle ASCII\rangle), \tag{6.59}$$

$$'text' \underset{Fun(text)\rightarrowtail @\mathbb{Z}\langle ASCII\rangle}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}\langle ASCII\rangle), \tag{6.60}$$

$$'chapters' \underset{Fin(Fu)\rightarrowtail \Omega^{Fu}}{\longrightarrow} \mathbf{Power}(book), \tag{6.61}$$

where $Fin(Fu)$ is the subfunctor of $2^{Fu}$ of all finite subsets of $N@Fu$ at address $N$, and which canonically injects to $\Omega^{Fu}$. Setting

$$H := Fun(No.) \times Fun(title) \times Fun(text), \tag{6.62}$$

we want to define a functor $Fu$ such that $f$ is an isomorphism

$$f : Fu \xrightarrow{\sim} H \times Fin(Fu), \tag{6.63}$$

this would guarantee that any text data (codified in $H$) and any finite set of chapters (codified in $Fin(Fu)$) can be reached by a book-form denotator. The point is that with the naive setup, there was just the zero address. In our situation, we have to find a solution which is functorial, and this is guaranteed by the second isomorphism in proposition 103 in appendix G. But notice that we do not have uniqueness of such a solution!

However, the above limitation technique to prevent the power sets to appear in full is not always sufficient to avoid non-standard situations with circular forms. Consider the example defined by these data:

$$F \underset{f}{\longrightarrow} \mathbf{Limit}(\mathbf{D}), \tag{6.64}$$

$$\mathbf{D} = \bullet \underset{g}{\rightarrow} P, \text{ where} \tag{6.65}$$

$$P \underset{Id}{\longrightarrow} \mathbf{Power}(F) \text{ and} \tag{6.66}$$

$$N@g : \bullet \mapsto N@\widehat{Fun}(F) \in Sub(@N \times Fun(F)) \tag{6.67}$$

for the final functor $\bullet$ and an address $N$. We deduce the functorial set equation

$$N@f : N@Fun(F) \xrightarrow{\sim} \{N@Fun(F)\} \tag{6.68}$$

which is fulfilled by any circular set $C = \{C\}$ and the constant functor $N@Fun(F) := C$.

## 6.6 Regular Denotators

**Summary.** This section is devoted to a first example set of denotators, including most of the common object types encountered in music scores and in music-theoretical contexts. They are structurally characterized by non-circular recursivity: These denotators do not make recurrent reuse of reference denotators—forms or coordinates—within their coordinators.

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**Definition 6** *A* regular *denotator is a denotator of a regular form.*

It cannot be the scope of this chapter to present an exhaustive list of musicologically relevant regular denotators. This would even not be the philosophy of the denotator formalism. We have to insist that denotators can and should be introduced again and again according to the specific need. There is no reason whatsoever to believe that a terminology should be limited in its concrete exemplifications: Music is a dynamic field of human culture and cannot be understood upon any historically fixed conceptual flashes. After all, denotators have been introduced to cope with all the options of dynamic conceptualization. Nonetheless, we should present a fairly representative list of denotators for a reasonable arsenal of musicological concepts. As theory evolves, we shall add more and more examples to meet the specific problematic. The lesson from this chapter is above all to learn the technique of building concepts via denotators.

Throughout the following explications the reader should bear in mind the prerequisites about common spaces in musicology as discussed in appendix A.

RegDen-1. To begin with, we need a common numeric representation of elementary *mental* tone parameters as they appear in common European music *score* notation. These are three *basis* parameters onset, pitch, and loudness and three corresponding *pianola* parameters duration, glissando, and crescendo. A common space functor for each of them is the real numbers $\mathbb{R}$ (viewed as a real vector space). We then have this arsenal of simple forms:

$$Onset \xrightarrow[Id]{} \mathbf{Simple}(\mathbb{R}), \tag{6.69}$$

$$Pitch \xrightarrow[Id]{} \mathbf{Simple}(\mathbb{R}), \tag{6.70}$$

$$Loudness \xrightarrow[Id]{} \mathbf{Simple}(\mathbb{R}), \tag{6.71}$$

$$Duration \xrightarrow[Id]{} \mathbf{Simple}(\mathbb{R}), \tag{6.72}$$

$$Glissando \xrightarrow[Id]{} \mathbf{Simple}(\mathbb{R}), \tag{6.73}$$

$$Crescendo \xrightarrow[Id]{} \mathbf{Simple}(\mathbb{R}). \tag{6.74}$$

We inherit the notation from naive denotators and write $Onset(ON)$, etc. for an onset with coordinate $ON$ if the address is clear. Without further comment, the address is always zero. Also we shall often use the names *E,H,L,D,G,C* for denotators of forms *Onset, Pitch, Loudness, Duration, Glissando, Crescendo* and also write $E = 3.57$ or even $Onset = 3.57$ for an *Onset* denotator of coordinate $3.57 \in 0_{\mathbb{Z}}@\mathbb{R}$, for example. But pay attention in distinguishing this sloppy notation from an equation between forms!

If we want to restrict the space functors of such simple forms, we specify the subspace[38] $S$ of $@\mathbb{R}$ and then write

$$Onset|_S \underset{S \rightarrowtail @\mathbb{R}}{\longrightarrow} \mathbf{Syn}(Onset), \text{ etc.} \tag{6.75}$$

For example, if we restrict pitch to integer values, we simply write $Pitch|_{\mathbb{Z}} \longrightarrow \mathbf{Syn}(Pitch)$ since the other details are clear. In this notation, the middle C pitch of the MIDI code is $Pitch|_{\mathbb{Z}} = 60$ or $H = 60$ if the form $Pitch|_{\mathbb{Z}}$ is clear. Observe that an inclusion $T \rightarrowtail S$ of subspaces of $@\mathbb{R}$ implies inclusions of respective forms

$$Onset|_T \rightarrowtail Onset|_S \rightarrowtail Onset, \text{ etc.} \tag{6.76}$$

From section 6.3.1 we know that a form morphism also transplants the respective denotators. In our situation, if we are given a denotator $Onset|_{\mathbb{Z}} = 123$, for example, then we may view it as a rational denotator and simply write $Onset|_{\mathbb{Q}} = 123/1 = 123$ since no confusion is likely.

In particular, the canonical (di)morphism $\mathbb{R}_{[\mathbb{Q}]} \rightarrowtail \mathbb{R}$ defines an isomorphism

$$MathPitch \xrightarrow{\sim} Pitch|_{@\mathbb{R}_{[\mathbb{Q}]}} \tag{6.77}$$

with the mathematical pitch space defined in (6.25).

With the identifications discussed in section 6.4.1, item 5 in mind, we shall simplify any direct product of the above forms to the associated simple form, e.g.

$$\begin{aligned} &Onset \times Pitch \times Loudness \times Duration \xrightarrow{\sim} \\ &Onset \oplus Pitch \oplus Loudness \oplus Duration, \end{aligned} \tag{6.78}$$

the latter being a form with coordinator $\mathbb{R}^4$.

RegDen-2. *Symbolic pitch* is often used, e.g. in the notation "$A_\sharp$". A quantitative pitch may be associated as a function of specific tuning paradigms. Thus we need a "neutral" symbolic notation which we realize by the obvious form

$$PitchSymb \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}\langle ASCII \rangle). \tag{6.79}$$

[38] We often omit the "functor" specification and more intuitively speak about "spaces" when we mean "space functors".

RegDen-3. For performance purposes, musical *tone parameters in the physical domain* which correspond to the above mental parameters are required. We denote them by the following self-explanatory forms:

$$PhysOnset \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}), \tag{6.80}$$

$$PhysPitch \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}), \tag{6.81}$$

$$PhysLoudness \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}), \tag{6.82}$$

$$PhysDuration \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}), \tag{6.83}$$

$$PhysGlissando \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}), \tag{6.84}$$

$$PhysCrescendo \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}). \tag{6.85}$$

Corresponding restrictions as shown in the mental situation above are obvious, though not so commonly used.

RegDen-4. For *symbolic absolute* or *relative dynamics* such as "*mf*" or "*crescendo*", we have to take a dynamic symbol (word), onset, and duration parameters. First, define the form for dynamic symbols:

$$DynSymb \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}\langle ASCII\rangle) \tag{6.86}$$

This gives these product forms:

$$AbsDyn \longrightarrow \mathbf{Limit}(DynSymb, Onset) \tag{6.87}$$

$$RelDyn \longrightarrow \mathbf{Limit}(DynSymb, Onset, Duration) \tag{6.88}$$

for absolute and relative dynamics. We shall see in section 18.4 that the quantitative meaning of absolute and relative symbolic dynamics is quite non-trivial.

RegDen-5. *Bar-lines* are onsets with no other properties, the only specificity is the name. So we have

$$BarLine \underset{Id}{\longrightarrow} \mathbf{Syn}(Onset), \tag{6.89}$$

and the reader will recognize that we could have set

$$BarLine^* \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}), \tag{6.90}$$

and get an isomorphic form. However, synonymy guarantees the common ground of onset which yields the bar-line concept. This type of sophistication is specific to the formalism of forms and allows musicologically adequate constructs; we therefore reject the unspecific form BarLine*.

RegDen-6. A *pause* is determined by its onset and duration. Thus, we may set

$$Pause \underset{Id}{\longrightarrow} \mathbf{Syn}(Onset \oplus Duration). \tag{6.91}$$

RegDen-7. The *time signature* $x/y$ is defined by numerator $x$ and denominator $y$ and has a determined onset origin. It is, however, not a mathematical fraction, since the denominator relates to metrical meaning. We thus may set the product form

$$TimeSig \underset{Id}{\longrightarrow} \mathbf{Limit}(Duration|_{\mathbb{Z}}, Duration|_{\mathbb{Z}}, Onset), \tag{6.92}$$

where a denotator $TimeSig(x, y, e)$ means that the time signature $x/y$ has been positioned at origin $e$, usually identified with the first bar-line onset.

RegDen-8. To denote *instrument names*, we may construct a generic naming form

$$InstruName \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}\langle ASCII\rangle), \tag{6.93}$$

and we are in state of defining an orchestral instrumentation

$$myOrchestra : 0_{\mathbb{Z}} \rightsquigarrow InstruName(\sum_{i}^{n} a_i Instr_i) \tag{6.94}$$

or shorter $myOrchestra = \sum_i^n a_i Instr_i$, with $a_i$ copies of instrument named $Instr_i$. For example, a string quartet has the orchestral instrumentation denotator

$$myStringQuartet = 2'Violin' + 1'Viola' + 1'Violoncello'. \tag{6.95}$$

If we do not want to specify one orchestral configuration, but leave it to the user to be more specific, a generic form of the following type

$$OrchSet \underset{Fin(\mathbb{Z}\langle ASCII\rangle)\rightarrowtail\Omega^{@\mathbb{Z}\langle ASCII\rangle}}{\longrightarrow} \mathbf{Power}(InstruName) \tag{6.96}$$

is adequate. A zero-addressed denotator then is a finite set

$$myOrchSet = \{O_1, O_2, \ldots O_k\} \tag{6.97}$$

consisting of $k$ orchestra configurations.

RegDen-9. Let us give two "small" examples for *non-zero-addressed denotators*. Later (e.g. in section 8.3.4), we shall devote in depth discussions to this subject. The first small example relates to pitch modulo octave, i.e. to the form $PiMod_{12}$ introduced in (6.42). Whereas zero-addressed denotators

$$PC_{12}(r) : 0_{\mathbb{Z}} \rightsquigarrow PiMod_{12}(r) \tag{6.98}$$

are evidently viewed as 12-pitch classes, more generally addressed denotators deserve special interpretation. They were extensively studied by Thomas Noll [400] in the frame of generalized function theory, see chapter 25 for a detailed discussion of this theory. This study concentrated on $\mathbb{Z}_{12}$- resp. "self"-addressed denotators

$$[a, b] : \mathbb{Z}_{12} \rightsquigarrow PiMod_{12}(e^a \cdot b) \tag{6.99}$$

for $a, b \in \mathbb{Z}_{12}$, i.e. $e^a \cdot b \in \mathbb{Z}_{12}@\mathbb{Z}_{12}$. Self-addressed pitch classes generalize the 12-pitch classes by the unique projection $! : \mathbb{Z}_{12} \to 0_{\mathbb{Z}}$ of $\mathbb{Z}$-modules. We have an injection

$$!@\mathbb{Z}_{12} : 0_{\mathbb{Z}}@\mathbb{Z}_{12} \rightarrowtail \mathbb{Z}_{12}@\mathbb{Z}_{12} \text{ which induces} \tag{6.100}$$

$$PC_{12}(r) \mapsto [r, 0]. \tag{6.101}$$

In other words, pitch classes are interpreted as constant self-addressed pitch classes. This change-of-address technique is the germ of a far-reaching generalization of harmony, see chapter 24. We shall see there that self-addressed pitch denotators appear also in the context of Euler modules (6.26), i.e. as denotators of shape

$$[a, b] : \mathbb{Q}^3 \rightsquigarrow EulerModule(e^a \cdot b) \tag{6.102}$$

with $e^a \cdot b \in \mathbb{Q}^3@\mathbb{Q}^3$.

The second small example relates to the onset space *Onset* and its rational restriction $Onset|_{\mathbb{Q}}$ as considered in the theories of David Lewin [300] and Dan Tudor Vuza [552]. In [300], Lewin considers *time spans*. These are pairs $(a, x) \in \mathbb{R} \times \mathbb{R}_+$ in the sense of time objects. The first coordinate is a zero-addressed onset in *Onset*, the second is a kind of onset difference, but viewed as a multiplicative quantity of relative increase. The structural meaning of this object is best understood via the transformation rules set up by Lewin. Given two time spans $(a, x), (b, y)$, they are compared by the "interval" which connects them, i.e. the function

$$int((a, x), (b, y)) = ((b - a)/x, y/x).$$

In other words, the first time span $(a, x)$ is transformed into the second one $(b, y)$ by an interval $int = (i, p) = ((b - a)/x, y/x)$, and the law is

$$(b, y) = (a, x)(i, p) = (a + xi, xp).$$

This is the known law of composition of affine transformations, i.e. we should rewrite it under the form

$$e^b \cdot y = e^a \cdot x \cdot e^i \cdot p = e^{a+xi} \cdot xp.$$

In other words, Lewin's time spans $t = (a, x)$ are just self-addressed onset denotators $t : \mathbb{R} \rightsquigarrow Onset(t)$. Later, when we know morphisms between denotators (8.2), the above interval action will be recognized as an address change, whereas Lewin's remark that the interval $int((a, x), (b, y))$ is invariant under time dilatation is a special case of the fact that for any affine automorphism $f = e^u \cdot v : \mathbb{R} \to \mathbb{R}$ of the ambient space $\mathbb{R}$ of the *Onset* form, the interval does not change if applied to the transformed objects, more precisely:

$$int((a, x), (b, y)) = int((u, v)(a, x), (u, v)(b, y)).$$

But this is clear since

$$e^b \cdot y = e^a \cdot x \cdot e^i \cdot p$$

implies

$$f \cdot e^b \cdot y = f \cdot e^a \cdot x \cdot e^i \cdot p,$$

the required equation with transformed objects and the same interval $int = e^i \cdot p$. Anticipating the notation of morphisms between denotators in (8.2), the above equation rewrites as $1/int(fe^b \cdot y) = fe^a \cdot x$, for all automorphisms $f$.

RegDen-10. To describe a *Fourier* decomposition of a periodic sound of frequency $f$, we may consider the simple form

$$Fourier \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R} \oplus \mathbb{R}^{2\times\mathbb{N}}). \tag{6.103}$$

We then have zero-addressed denotators

$$myFourier = (f, (A_i, Ph_i)_{0\leq i}) \tag{6.104}$$

which represent the periodic sound function[39]

$$sd(myFourier)(t) = \sum_{0\leq i}^{\infty} A_i \sin(2\pi i f t + Ph_i) \tag{6.105}$$

of physical time $t$. Notice that denotators do not represent the physical meaning of an information unit. For example, the instrument names are only symbols, essentially ASCII words, and that the physical nature of the denoted instruments are not included. The same holds for the *Fourier* form. The transformation from "symbolic" to physical or technological meaning relates to the paratextual signification processes described in section 18.4.

RegDen-11. To denote an *envelope function* on the unit interval, we may use the real vector space $C^0[0,1]$ of continuous functions on the real unit interval $[0,1]$ and set

$$Envelope \underset{Id}{\longrightarrow} \mathbf{Simple}(C^0[0,1]). \tag{6.106}$$

For $Envelope = V$, $PhysOnset = e$, and non-vanishing duration $PhysDuration = d$ we obtain the deformed envelope

$$V_{e,d}(t) = \begin{cases} V((t-e)/d) & \text{for } t \in [e, e+d], \\ 0 & \text{else} \end{cases} \tag{6.107}$$

which vanishes outside $[e, e+d]$ and has its support between $e$ and $e+d$ if $V$ has its support between 0 and 1, see figure 6.10.

RegDen-12. *Combining elementary data* such as onset, duration, Fourier synthesis, and envelope, we can introduce the corresponding physical sound form by

$$\begin{aligned} &FourierSound \underset{Id}{\longrightarrow} \mathbf{Limit}(\mathbf{D}), \\ &\mathbf{D} = (PhysOnset, PhysDuration, Fourier, Envelope). \end{aligned} \tag{6.108}$$

[39] With the necessary convergence conditions [127]. Often, one also normalizes a periodic sound function to a maximum value $Max(sd(myFourier)) = 1$.

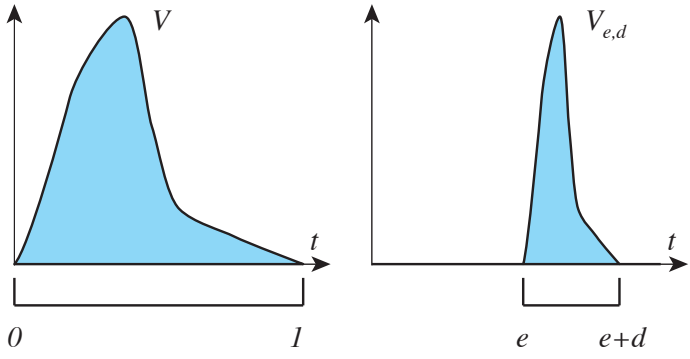


Figure 6.10: Affine deformation of an envelope denotator $V$ on the unit interval. The deformed envelope $V_{e,d}$ "starts" at onset $e$ and "ends" at onset $e + d$.

A denotator

$$myFourierSound = (e, d, myFourier, V) \tag{6.109}$$

is associated with the physical sound function

$$sound(myFourierSound) = V_{e,d} \cdot sd(myFourier). \tag{6.110}$$

## 6.7 Circular Denotators

**Summary.** This second set of denotators is devoted to important constructs of musical objects which are the foundation of powerful musicological concepts. They make use of circular definitions which do reuse forms or denotators on different recurrent coordinator levels.

– Σ –

Already for the example of Fourier denotators, it was felt that there was a sort of superfluous conceptualization. The inductive repetition of partials was not quite the most economic way to think of forms. In this section we shall take care of a more efficient way of conceptualization: circular forms and denotators. More precisely we have

**Definition 7** *A* circular *denotator is a denotator of a circular form.*

CircDen-1. Recall the *book* form discussed in (6.56). An $M$-addressed denotator *myBook* of

this form looks as follows:

$$
\begin{aligned}
&myBook : M \rightsquigarrow book(myNumber, myTitle, myText, myChapters),\\
&myNumber : M \rightsquigarrow No.(N : M \to \mathbb{Z}),\\
&myTitle : M \rightsquigarrow title(Tt : M \to \mathbb{Z}\langle ASCII\rangle),\\
&myText : M \rightsquigarrow text(Tx : M \to \mathbb{Z}\langle ASCII\rangle),\\
&myChapters : M \rightsquigarrow chapters(chapter_1, \ldots chapter_k),\\
&chapter_1 : M \rightsquigarrow book(\ldots),\\
&\ldots\\
&chapter_k : M \rightsquigarrow book(\ldots).
\end{aligned}
$$

This means that we have to specify number, title, and text explicitly, and that for each collection of chapters, we have to reenter in the book form until we reach the situation where the chapters build the *empty set*. If we know this recurrent information until its empty roots, the book is completely determined. This means that it is not necessary to refer explicitly to the space functor which guarantees the form. We just have to know that it exists. And this is what we know from category theory. Thus, the naive setup is justified by the categorical background for determined classes of forms which are covered by the categorical existence theorems.

CircDen-2. Let us have a second look at the *Fourier* form. The infinite partials can also be defined by the following product construction:

$$
\begin{aligned}
&CcFourier \underset{f:Fu \xrightarrow{\sim} P \times Fu}{\longrightarrow} \mathbf{Limit}(Partial, CcFourier), \text{ where}\\
&Fu = Fun(CcFourier), P = Fun(Partial), \text{ and}\\
&Partial \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}^3).
\end{aligned}
\tag{6.111}
$$

We know from 6.47 that such functors exist. Denotators $myCcFourier$ of form $CcFourier$ are identified by sequences

$$
myCcFourier = ((Partial(A_i, f_i, Ph_i))_{0 \le i}, \tag{6.112}
$$

and this can be interpreted by the Fourier expansion

$$
ccsd(myCcFourier) = \sum_{0 \le i}^{\infty} A_i \sin(2\pi f_i + Ph_i). \tag{6.113}
$$

For the special values $f_i = if$ we get back the classical case. Of course, the numerical information is the same as for the non-circular concept of the Fourier decomposition, but conceptually, the circular form is much simpler—once we have settled the functor existence question.

CircDen-3. We want to look at the *FM synthesis* (see appendix A.1.2.2 for historical and technical discussion) which looks much more complicated than the Fourier situation if

defined without circular forms. In fact, it essentially consists of a sinoidal "carrier" function which in its argument adds other sinoidal "modulator" functions which in turn refer to other modulator functions, etc. The point is that the modulator typology is not fixed and may vary from stage to stage. We may, however, introduce directed graphs (so-called "algorithms" (A.1.2.2)) but the spirit is more obscured than evidenced by such a construction.

Let us first define the FM form and then make a picture of the form. The form is called *FM-Object* and looks as follows:

$$\begin{aligned} &FM\text{-}Object \underset{f:F\xrightarrow{\sim}Fin(FK)\rightarrowtail\Omega^{FK}}{\longrightarrow} \mathbf{Power}(Knot) \text{ with} \\ &F = Fun(FM\text{-}Object), FK = Fun(Knot) \text{ and the product form} \\ &Knot \underset{Id}{\longrightarrow} \mathbf{Limit}(Partial, FM\text{-}Object). \end{aligned} \tag{6.114}$$

We need to solve the functor equation $F \xrightarrow{\sim} Fin(P \times F)$, but according to proposition 103 in appendix G, a solution $F$ always exists.

For each *FM-Object* denotator *myFMObject*, its *graph* $\Gamma(myFMObject)$ is defined as follows: Draw a vertex for each knot, and then an arrow from each knot vertex to all the knot vertexes of its modulator, etc. by induction (including circularity!).

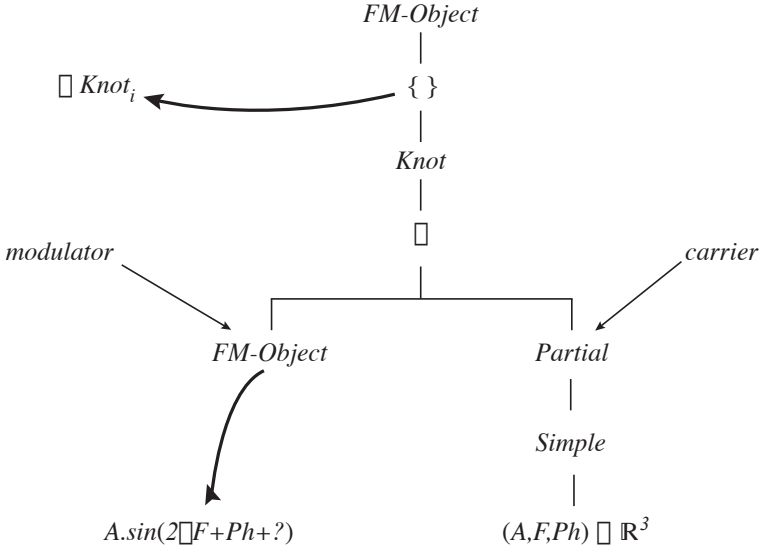


Figure 6.11: The graphical representation of an *FM-Object* denotator.

What does this mean for the sound synthesis function? Intuitively (see figure 6.11), an *FM-Object* denotator *myFMObject* is a finite set of knots $Knot_i$ which form the basic carrier of the FM sound. Each knot has it partial denotator which defines the sinoidal function of the knot. In addition, each knot has its modulator which is another FM sound obtained like the basic carrier, etc. Starting from $FMsound(\emptyset) = 0$, the sound function is recursively defined as follows:

$$FMsound(myFMObject)(t) = \sum_{i}^{n} A_i \sin(2\pi F_i t + Ph_i + FMsound(myModulator_i)(t)) \tag{6.115}$$

where $myModulator_i$ is the *FM-Object* factor of $Knot_i$.

Recursion here means that after a finite number of downward steps from *FM-Object* factor to *FM-Object* factor, we end up with the empty factor, and the sound function is well-defined. However, under appropriate convergence conditions, infinite descent may also yield well-defined sound functions.

CircDen-4. In music, it is common to consider sound events which share a specific grouping behavior, for example when dealing with arpeggios or trills. We want to deal with this phenomenon in defining *Makro* forms. These forms follow the same schema as discussed for *FM-Object* forms. We only have to replace the partial by an event of a particular type. Put generically, let *Basic* be a form which describes a sound event type, for example the piano specific event type $Basic = Onset \oplus Pitch \oplus Loudness \oplus Duration$ introduced in (6.78). We then set

$$Makro_{Basic} \underset{f:F\tilde{\to}Fin(FK)\to\Omega^{FK}}{\longrightarrow} \mathbf{Power}(Knot_{Basic})$$

with $F = Fun(Makro_{Basic}), FK = Fun(Knot_{Basic})$

and the product form

$$Knot_{Basic} \underset{Id}{\longrightarrow} \mathbf{Limit}(Makro_{Basic}, Basic). \tag{6.116}$$

How must we interpret this construction to get 'real' events back from this grouping structure? Suppose that we have a module structure on the space of the *Basic* form (e.g. a vector space $\mathbb{R}^4$ of the piano events). To begin with, let a denotator $Knot_{Basic} = D$ of form $Knot_{Basic}$ have coordinates $(myEvent, \emptyset)$ with the denotator *myEvent* in the event space, e.g. in $\mathbb{R}^4$ in the above piano form. We introduce a *flattening* operation and define the $Makro_{Basic}$-formed denotator $Flatten(D) = \{D\}$. For a denotator $Knot_{Basic} = D = (myEvent, myMakro)$ with non-empty Makro coordinate $myMakro = \{Knot_1, \ldots Knot_r\}$ and knot events resp. Makro coordinates $myEvent_i$ resp. $myMakro_i$ for $Knot_i$, we set

$$Flatten(D) = \{(myEvent + myEvent_i, myMakro_i), i = 1, \ldots r\}. \tag{6.117}$$

The general definition of the flattening operation for a denotator $Makro_{Basic} = myMakro = \{D_1, \ldots D_k\}$ is

$$Flatten(myMakro) = \bigcup_{i=1,\ldots k} Flatten(D_i). \tag{6.118}$$

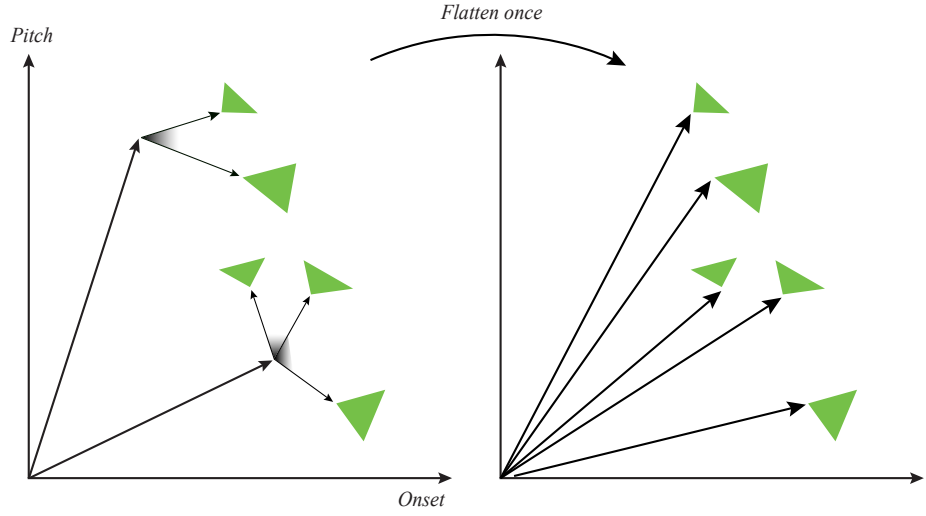


Figure 6.12: The flattening of a macro-event in the mental onset-pitch domain.

If a Makro event *myMakro* ends up with empty Makro coordinates after a finite descent, the flattening operation also ends up, i.e. becomes stationary after an $n$-fold concatenation:

$$Flatten^n(myMakro) =$$
$$Flatten^{n+1}(myMakro) = \ldots$$
$$Flatten^{\infty}(myMakro).$$

Since each element of $Flatten^{\infty}(myMakro)$ has empty Makro coordinate, we may view this as a finite set of *Basic*-formed events, i.e., a *local composition* in the terminology to be introduced later in chapter 7. We shall come back to this construction in section 21.3 and section 39.2.5; refer to figure 6.12 for a first intuition about Makro events.

## 6.8 Ordering on Forms and Denotators

**Summary.** In a number of important foundation modules, (linear and partial) orderings on systems of denotators can be defined. These orderings induce canonical (linear and partial) orderings on denotators, founded on these types of modules. The option is fundamental in defining receptive navigation tools. These tools are applied to representation and retrieval problems of general musical objects.

– Σ –

In this section, we want to look for generalizations of the naive order relation among forms and denotators. As in the "naive discussion" in section 6.1.3, we consider exclusively regular forms here. This means that we are allowed to define order relation recursively on the level[40] of forms.

Recall from section 6.1.3 that naive forms were ordered lexicographically as three-letter words, i.e., if two forms $F_1 = Nm_1 \longrightarrow Tp_1(Co_1)$ and $F_2 = Nm_2 \longrightarrow Tp_2(Co_2)$ were given, we had $F_1 < F_2$ iff for the first position in the "words" $Tp_1Co_1Nm_1$ and $Tp_2Co_2Nm_2$ where the letters differ, the letter of the first form is smaller than the corresponding letter of the second. We understood that the coordinator was itself presented as a word $C_1C_2 \ldots C_m$ of forms in the compound case and the respective module name in the simple case. Functorial forms

$$F = Nm \underset{Id}{\longrightarrow} Tp(Co)$$

are a bit more complicated. We want to define orderings on forms relating to their names in a generative way. Recall that we require that form names be a key to forms, i.e., different forms must bear different names. Since names are strings of characters, they are linearly ordered, and we may order forms according to their names. But we do not want to do this in a completely arbitrary way. The form names should reflect the recursively given orderings of the form components: type, identifier, and coordinator. This is the general policy behind the following procedure. *So whenever we are given already ordered forms, the ordering must be read from the forms' names.*

In this spirit, we would like to interpret them as four-letter words $TpCoIdNm$ with the lexicographic ordering[41] as above. The naming policy then means that if ever the three first letters define an ordering among two forms, then the form names should also follow that ordering. However, if the first three letters are fixed, synymous naming is of course still allowed. Let us inspect the different orderings on the four positions of $TpCoIdNm$. The name $Nm$ poses no problem, we have the lexicographic ordering on usual $ASCII$ strings. The types are ordered by

$$\textbf{Simple} < \textbf{Syn} < \textbf{Limit} < \textbf{Colimit} < \textbf{Power}, \tag{6.119}$$

similar to the ordering defined in (6.12). The difference is that for **Simple** type, we have all the modules as coordinators, whereas in the naive case, we only had four modules **CHR**, $BIT, \mathbb{Z}$, and $\mathbb{R}$, giving rise to the simple types STRING, BOOLE, INTEGER, and FLOAT. Since we now have all the modules of **Mod** as coordinators of simple forms, we need an order relation $<_{\textbf{Mod}}$ on these modules, more precisely: on all morphisms of this category, thereby englobing the modules as identities. This is supposed to extend the "naive" ordering, i.e.

$$\mathbb{Z}_2 <_{\textbf{Mod}} \mathbb{Z} <_{\textbf{Mod}} \mathbb{Z}\langle ASCII \rangle <_{\textbf{Mod}} \mathbb{R}. \tag{6.120}$$

In other words: *The simple forms are ordered according to their coordinator modules.* We suppose to be given such an ordering $<_{\textbf{Mod}}$ which extends the naive ordering. This is part of the basic, purely module-theoretic "level of reality"; see section 6.8.1 for such an ordering.

Next, we want to look at the identifier, supposing by construction that we have fixed a form type $Tp$ and a coordinator $Co$. This means that the form's identifier is a monomorphism

[40] See definition 5 in section 6.5.

[41] Concerning lexicographic orderings, see appendix C, example 64.

$f : Fu \rightarrowtail Uf$, where $Fu = Fun(F)$ and the frame space $Uf = Uni(Tp, Co)$ is the universal functor construction associated with $Tp$ and $Co$. If we have two forms of fixed form type $Tp$ and coordinator $Co$, we have two monomorphisms $f_1 : Fu_1 \rightarrowtail Uf$ and $f_2 : Fu_2 \rightarrowtail Uf$ and these are canonically compared by the subobject relation[42], i.e. we set $Id_1 < Id_2$ iff the subobjects verify $f_1 < f_2$, i.e. iff there is a (necessarily mono) morphism $g : Fu_1 \rightarrowtail Fu_2$ such that $f_2 \cdot g = f_1$. Observe, however, that the latter relation is neither antisymmetric nor total in general, it is only a partial ordering[43]. The naive case appears here as related to the identity identifier. When generating new form names, one should however respect the identifier orderings. This is always possible since the form genealogy is a question of finitely many forms, and therefore, we may suppose that identifier ordering is carried over to name ordering.

We are left with the ordering of the coordinator in case it is not a module (compound type). If the fixed type is either **Syn** or **Power**, the coordinator is a form of smaller level, and we are done by recursion. We are thus left with a diagram coordinator $\mathbf{D}$. The naive setup was concerned with discrete diagrams where the vertex family $V(\mathbf{D}) = (C_1, \dots C_m)$ was finite and linearly ordered by its indices. Comparing diagrams in the naive case was reduced to the lexicographic comparison of words $C_1 \dots C_m$ of forms, where the ordering among the forms $C_i$ was supposed to be given by recursion. To generalize that situation, we suppose that

- by recursion, the vertex forms are already linearly ordered by their names—let us index them for notational comfort by $C_i$.

We are therefore able to compare two such diagrams $\mathbf{D}_1$ and $\mathbf{D}_2$ with respect to their vertex forms $C_{1,i}$ and $C_{2,j}$: We build the words $C_{1,0}C_{1,1}\dots$ and $C_{2,0}C_{2,1}\dots$ and compare them lexicographically. Upon this ordering, we further differentiate according to the involved diagram arrows. We may thus suppose that $V(\mathbf{D}_1) = V(\mathbf{D}_2) = (C_i)_{0 \leq i}$.

We further suppose that the set of morphisms $f : C_i \rightarrow C_j$ is linearly ordered for all vertex indices $i$ and $j$. This suffices to terminate the construction. Fix a pair $i, j$ of vertex indexes. We are going to build a word with the morphisms $f : C_i \rightarrow C_j$ of our diagrams. To begin with, sort these words by the number of their arrows, and therefore suppose that the arrow number $s(i, j)$ is the same for both diagrams. Let $mor^d_{i,j} = f^d_{i,j,1} f^d_{i,j,2} \dots$ be the word of diagram morphisms $f^d_{i,j,s} : C_i \rightarrow C_j, s = 1, \dots s(i, j)$ in the given linear morphism ordering of diagram $\mathbf{D}_d, d = 1, 2$. We may now arrange the words $mor^1_{i,j}, mor^1_{i,j}$ in their lexicographic order. Since the couples $i, j$ of vertex indices are ordered lexicographically, we can build a word for each diagram $\mathbf{D}_d, d = 1, 2$ whose letters are the $mor^d_{i,j}$, in the given double index ordering: $mor^d = mor^d_{0,0} mor^d_{0,1} \dots$. It is now clear how to compare the words $mor^1$ and $mor^2$ lexicographically, and we are done with the ordering of forms.

Let us now look at possible orderings among denotators. In view of the components $N(D) : A(D) \rightsquigarrow F(D)(CT(D))$ of a denotator $D$, we have a similar situation as with forms. We again order denotators lexicographically according to the four-letter wording $F(D)A(D)CT(D)N(D)$. Since by the preceding, forms, addresses i.e. modules, and names are already ordered, we are left with the question of ordering coordinates.

According to the lexicographic ordering, we may thus look at two denotators $D_1$ and $D_2$ with identical form $F$ and address $M$ and with coordinates $Co_1$ and $Co_2$, both elements of

[42] See appendix G.

[43] It is reflexive and transitive, see appendix C.2

the functor value set $M@Fun(F)$. Since the latter functor is identified to a subfunctor of the universal construction $U = U(Type, Coordinator)$ of the form, we can restrict our question to orderings on value sets of such universal construction functors. We have to investigate the situations $M@N$, $M@Fun(\mathbf{Syn}(C))$, $M@Lim(\mathbf{D})$, $M@Colim(\mathbf{D})$, and $M@\Omega^{Fun(C)}$ for a module $N$, a diagram $\mathbf{D}$, or a coordinator form $C$.

In case of simple type, $M@N$ bears an ordering by hypothesis on the category **Mod**. For **Syn** type, we have $M@Fun(\mathbf{Syn}(C)) = M@Fun(C)$, and we are done by recursion. For the power type $\Omega^{Fun(C)}$, we have

$$M@\Omega^{Fun(C)} = Sub(@M \times Fun(C))$$

which is naturally provided with the ordering among subobjects as already discussed above. However, this ordering is not related to the recursively given ordering on (the value sets of) $Fun(C)$. For the frequently used subfunctor $Fin(Fun(C)) \rightarrowtail \Omega^{Fun(C)}$, the subobject ordering can be refined to an ordering which is also linear in case the ordering on $Fun(C)$ is. In fact, by proposition 65 in appendix C.2, every set $A@Fin(Fun(C))$ is ordered by a partial/linear ordering $\preceq$ induced from the partial/linear ordering on $A@Fun(C)$. The ordering $\preceq$ refines the inclusion ordering among finite subsets, in other words, the natural inclusion $Fin(Fun(C)) \rightarrowtail \Omega^{Fun(C)}$ with the subobject orderings admits a refinement on $Fin(Fun(C))$ which is linear if the ordering on $Fun(C)$ is. Whether this ordering on $Fin(Fun(C))$ can be extended to a partial ordering on all of $\Omega^{Fun(C)}$ that includes the subobject ordering, is an open question. If the contrary is not stressed we shall always use the $\preceq$ ordering on $Fin(H)$ for a functor $H \in \mathbf{Mod}^{@}$ whose value sets are ordered.

For **Limit** type, we know from appendix G.2.1 that

$$M@Lim(\mathbf{D}) \subset \prod_i M@Fun(C_i)$$

with the above notation of the diagram's vertexes $C_i$. The functor values $M@Fun(C_i)$ are ordered by recursion, and we may consider the lexicographic ordering on the product, restricted to the limit subset, under the hypothesis that the index set is also linearly ordered, see example 64 in appendix C.

We are left with the **Colimit** type which is less comfortable to handle than the other types. In fact, we have a surjection

$$\coprod_i M@Fun(C_i) \twoheadrightarrow M@Colim(\mathbf{D})$$

defined by the equivalence relation that is generated by the diagram's arrows. The left set carries a canonical ordering as described in the naive discussion in section 6.1.3. We define it by use of the linear ordering among the coordinator forms of $\mathbf{D}$, and by the recursively given orderings on the functor sets $M@Fun(C_i)$. If we are to carry over this data to the colimit, we have to turn the above projection into an order-preserving map, i.e. we define the ordering on the colimit set as the partial ordering generated by the image relation of the coproduct's ordering. But observe that this will—in general—be something much larger than the image relation since the latter is not transitive, in general! This terminates the construction of orderings among denotators as far as this is possible.

We should make a final comment on the sense of ordering forms and denotators. This is not so much a purely mathematical exercise. After all, mathematicians know that any set can be well-ordered, at least in acceptance of certain common axioms (see appendix C.1). The point is much more a constructive approach which lends itself to data base management systems and the related successive growth of denotator arsenals—as a function of a dynamically expanding knowledge base. This perspective stresses the "natural" ordering and not an artificial one which may be built exclusively upon the axiom of choice. The only presence of choice in our context is the historical appearance of concepts, i.e. of forms and denotators, in the development of theory and empirical material, but see chapter 18 for an extensive study of "what is the case" and what is not. This is taken into account on the level of the ordering within the fundamental category **Mod** of modules and affine morphisms. In this sense we should also understand the following set of applications.

### 6.8.1 Concretizations and Applications

**Summary.** We give concrete order relations and applications to musical situations.

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**Modules and Morphisms.** We start this section with a discussion of possible order relations on the the category **Mod** of modules and their affine morphisms. Modules in **Mod** appear in two functions: as coordinators of simple forms and as addresses of denotators. When building compound forms and denotators on these forms, we may apply existing orderings on the already used modules for the compound structures. Only if a new module intervenes in the knowledge extension process is it necessary to insert it relatively to the already ordered module set. In other words, the recursive construction of partial orderings is also dynamic. Before we really need a module, it is not necessary to care about its order position. The only essential point here is that one should not reorder modules after dynamical extension: Once an order relation $M <_{\mathbf{Mod}} N$ is defined, it should persist for ever. Otherwise, reusable retrieval on database management systems break down.

For the purpose of ordering modules, we want to observe the following principle:

**Principle 1** *Order* first *modules from a skeleton*[44] *of* **Mod**, *and* then *the objects within each isomorphism class. Then order modules lexicographically according to the skeleton and the member of the isomorphism class, i.e. as an ordered coproduct of ordered isomorphism classes.*

For example, we start the skeleton by the few examples of hitherto basic modules and

[44] See appendix G, example 87.

order them as follows[45]:

$$\begin{aligned}\emptyset <_{\mathbf{Mod}} 0_{\mathbb{Z}}(=\mathbb{Z}_1) <_{\mathbf{Mod}} \mathbb{Z}_2 \ldots <_{\mathbf{Mod}} \mathbb{Z}_n <_{\mathbf{Mod}} \mathbb{Z}_{n+1} \ldots \\ <_{\mathbf{Mod}} \mathbb{Z} <_{\mathbf{Mod}} \mathbb{Z}\langle ASCII\rangle \\ <_{\mathbf{Mod}} \mathbb{Q} <_{\mathbf{Mod}} \mathbb{Q}^2 \ldots <_{\mathbf{Mod}} \mathbb{Q}^n <_{\mathbf{Mod}} \mathbb{Q}^{n+1} \ldots \\ <_{\mathbf{Mod}} \mathbb{R} <_{\mathbf{Mod}} \mathbb{R}^2 \ldots <_{\mathbf{Mod}} \mathbb{R}^n <_{\mathbf{Mod}} \mathbb{R}^{n+1} \ldots \end{aligned} \tag{6.121}$$

To give an example of the ordering within an isomorphism class, we look at a fixed positive power $\mathbb{R}^n$. As usual in linear algebra, the vectors of this module are viewed as columns, i.e. $n \times 1$ matrices. We thus write $n = n \times 1$ and identify $\mathbb{R}^n$ with $\mathbb{R}^{n\times 1}$. In this spirit, we arrange the factorizations $n = r \times s$ of $n$ by increasing second factors $1 < s_1 < s_2 < \ldots n$ and obtain an ordering among the following matrix $\mathbb{R}$-modules isomorphic to $\mathbb{R}^n$, inserted between (classes of) modules $\mathbb{R}^{n-1}$ and $\mathbb{R}^{n+1}$:

$$\mathbb{R}^{1\times(n-1)} <_{\mathbf{Mod}} \mathbb{R}^{n\times 1} <_{\mathbf{Mod}} \mathbb{R}^{r_1\times s_1} <_{\mathbf{Mod}} \ldots \mathbb{R}^{1\times n} <_{\mathbf{Mod}} \mathbb{R}^{(n+1)\times 1}. \tag{6.122}$$

Before further discussing orderings on forms we have to look at orderings on individual modules since orderings on diagrams depend on orderings on sets of morphisms between functors, and this includes ordering sets of affine homomorphisms between modules which are also, under certain conditions, modules[46].

Ordering on individual modules is quite problematic since nothing functorial can be expected. We have to select specific modules and give them *individual* orderings (without regard to compatibility with affine transformations such as automorphisms). However, a minimum of universality should be observed since one should not produce more incompatibilities than necessary. For example, the identification of forms related to direct sums of modules in point 5 of section 6.4.1 should be compatible with order relations[47]. Suppose therefore that we have a product form $F \times G$ of two simple forms $F \underset{Id}{\longrightarrow} \mathbf{Simple}(M)$ and $G \underset{Id}{\longrightarrow} \mathbf{Simple}(N)$ over modules $M, N$ with respective orderings[48] $A@ <_M, A@ <_N$ at an address $A$. Then, according to the preceding discussion, $A$-addressed denotators $D : A{\rightsquigarrow}F \times G(Co)$ of $F \times G$ with $Co \in A@M \times A@N$ are ordered lexicographically following $A@ <_M$ and then $A@ <_N$. Hence, if these denotators are identified with $A$-addressed denotators in $F \oplus G$, we should order the direct sum $A@(M \oplus N)$ by the lexicographic ordering of its summands under the canonical bijection $A@M \oplus N \xrightarrow{\sim} A@M \times A@N$. We therefore have to take the lexicographic ordering from the summands $A@M, A@N$ whenever we construct an ordering on a direct sum $A@(M \oplus N)$. The common case is zero address, and then, $0@M$ identifies to $M$. So we have to take care of orderings on modules $M$ and their direct sums first. To begin with, we have to order coefficient rings[49].

[45] In this sequence, the modules $\mathbb{Z}_n$ are viewed as $\mathbb{Z}$-modules, i.e. we should write more correctly $(\mathbb{Z}_n)_{[\mathbb{Z}]}$, but we avoid this clumsy notation whenever possible.

[46] see appendix E for more details on this somewhat delicate subject!

[47] Observe, however, that switching from **Limit** to **Simple** type changes the order position of the underlying form.

[48] The following symbolism $A@ <_F$ for an ordering on a value set $A@F$ of a functor $F$ in $\mathbf{Mod}^{@}$ does not mean that the ordering is functorial, i.e. the functor does not necessarily evaluate on the *category* of ordered sets.

[49] Mathematicians should pay attention not to confuse the orderings in this section with the ordering concept for ordered rings.

We have already listed some important rings in (6.121) and will now define orderings on these prototypes:

- $\mathbb{Z}_0$ and $\mathbb{Z}_{p^k} = \{\overline{0}, \dots \overline{p^k - 1}\}$, $p =$ prime and $k = 1, 2, 3, \dots : \overline{i} < \overline{j}$ iff $i < j$.
- For general $n$, let $n = p_1^{k_1} \cdots p_r^{k_r}$ be the (unique) prime decomposition with increasing prime factors and non-zero exponents (see appendix E.2). Then we have the Sylow decomposition (appendix C.3.4) and a canonical isomorphism $\mathbb{Z}_n \xrightarrow{\sim} \mathbb{Z}_{p_1^{k_1}} \oplus \dots \mathbb{Z}_{p_r^{k_r}}$. On this direct sum, we take the lexicographic ordering based on the previous definition of the orderings on the direct factors.
- $\mathbb{Z}$: usual ordering.
- $\mathbb{Z}\langle ASCII\rangle$: Take the lexicographic ordering on the basis set $W$ of $ASCII$-words, then the lexicographic ordering as defined in appendix C, example 64, via the identification[50] of $\mathbb{Z}$-modules $\mathbb{Z}\langle ASCII\rangle = \mathbb{Z}^{(W)}$.
- $\mathbb{Q}$: usual ordering.
- $\mathbb{R}$: usual ordering.

**Direct Sums.** Generalizing the above case with $\mathbb{Z}\langle ASCII\rangle$, we suppose given a family of indices $I$, well-ordered by $<$, as well as for each index $i \in I$ a module $M_i$ with an ordering $<_i$. Then the direct sum $\bigoplus_I M_i$ is ordered lexicographically according to the index ordering. This includes the finite direct sum $R^n$ of n copies of an ordered ring, $R = \mathbb{Z}_n, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ for example. Again we should pay attention to the fact that isomorphic modules may not have isomorphic orderings. The latter are strictly related to the identity and construction of specific modules.

**Matrix Modules.** In (6.122), we have ordered isomorphic copies of $\mathbb{R}^n$ for different factorizations $n = r \times s$ as matrix modules $\mathbb{R}^{r\times s}$. Here, we have to order these copies themselves, and this will be done in the canonical way: Identify $\mathbb{R}^{r\times s}$ canonically with $(\mathbb{R}^r)^s$, order the column summands $\mathbb{R}^r$ and then the total sum as defined above for direct sums. The same ordering procedure works for any matrix module over a ring $R$ which is provided with an ordering, instead of $\mathbb{R}$.

**Ordering on Morphisms.** If we are given two forms $F$ and $G$ with functors which are represented by modules, $Fu(F) = @M$ and $Fu(G) = @N$, then the morphisms $F \to G$ correspond canonically[51] to affine morphisms $e^n \cdot f : M \to N$. In particular, if $M$ and $N$ are the vector spaces $M = \mathbb{R}^r$ and $N = \mathbb{R}^s$, we canonically identify $M@N = e^N \cdot Lin(M, N)$ with $\mathbb{R}^{r\times s} \oplus \mathbb{R}^s$, ordered as a direct sum of the ordered module $\mathbb{R}^{r\times s}$ with the ordered matrix space $\mathbb{R}^s$. The same construction is valid for any free modules $M = R^r$ and $N = R^s$ over a ring $R$ which is provided with an ordering, instead of $\mathbb{R}$.

With the previous constructions we are in a state of ordering diagrams of forms having functors which are represented in free modules over rings which already carry an ordering. And the last construction yields orderings on functor values $A@M$ at address $A$ if both, $A$ and $M$, are free over the coefficient ring which is provided with an ordering.

[50] See appendix E.
[51] By Yoneda's lemma, see appendix G.2.

This being true, it is a good exercise to apply the mathematical constructions to concrete musical examples and to give them interpretations in terms of musical meaning.

**Example 3** Ordering $\mathbb{Z}_n$.
Consider the form $PiMod_n$ of n-pitch classes as defined in (6.41). Its $A$-addressed denotators live in the set $A@\mathbb{Z}_n$ and we have the ordering of $\mathbb{Z}_n$ on the zero-addressed denotators. For $n = 12$, we have the prime decomposition $12 = 2^2 3$ and therefore inherit the lexicographic ordering of $\mathbb{Z}_4 \oplus \mathbb{Z}_3$ via the canonical bijection

$$\mathbb{Z}_{12} \xrightarrow{\sim} \mathbb{Z}_4 \oplus \mathbb{Z}_3 : x \mod 12 \mapsto (x \mod 4, x \mod 3).$$

This gives the following ordering on zero-addressed denotators

$$\begin{array}{rl} \mathbb{Z}_4 \oplus \mathbb{Z}_3\text{:} & (0{,}0)\ (0{,}1)\ (0{,}2)\ (1{,}0)\ (1{,}1)\ (1{,}2)\ (2{,}0)\ (2{,}1)\ (2{,}2)\ (3{,}0)\ (3{,}1)\ (3{,}2) \\ \mathbb{Z}_{12}\text{:} & 0 <\ 4 <\ 8 <\ 9 <\ 1 <\ 5 <\ 6 <\ 10 <\ 2 <\ 3 <\ 7 <\ 11 \end{array}$$

which starts from 0 with two major third steps, then jumps by one semitone and repeats the major third steps from there etc. If played in notes from middle C on, this look like four arpeggiated augmented triads[52], as shown in figure 6.13.
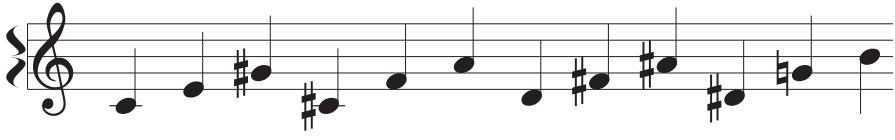
Figure 6.13: The ordering on 12-pitch classes according to the standard procedure described in this section.

If instead we look at self-addressed denotators of $PiMod_n$, we have to consider the elements $(b, a)$ of the ordered module $\mathbb{Z}_n \oplus \mathbb{Z}_n$ which parametrize the affine morphisms $e^a \cdot b$. In case of $n = 12$, we have the self-addressed tones studied in [400]. The ordering thus starts with the so-called *constant* self-addressed tones $e^0 \cdot 0 < e^4 \cdot 0 < \ldots e^{11} \cdot 0$ and ends with the *inversion* tones $e^0 \cdot 11 < e^4 \cdot 11 < \ldots e^{11} \cdot 11$, the last being the inversion between $B$ and $C$, see section 7.2.1.

**Example 4** Ordering Note Groups from a "Chord Perspective".
Intuitively speaking, we may look at (finite) sets of notes on a score and try to order them for lexical retrieval or/and for listing them in a data base management system. But we may also look at these objects from a harmonic or metrical point of view. This means that we do not only consider them as sets of tone events, but with respect to their harmonic or metric content. Let us develop on this scope by use of forms, orderings on forms and the corresponding denotator systems. For the sake of transparency we only look at two parameters, onset and pitch. We

[52] See section 11.3.5 for chord classification.

need the three corresponding forms introduced in section 6.6:

$$Onset \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}),$$
$$Pitch \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}),$$
$$Onset \oplus Pitch \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}^2).$$

Corresponding to first and second projection $p_1 : \mathbb{R}^2 \to \mathbb{R}$ and $p_2 : \mathbb{R}^2 \to \mathbb{R}$, we have two morphisms of forms

$$p_1 : Onset \oplus Pitch \to Onset \text{ and} \tag{6.123}$$
$$p_2 : Onset \oplus Pitch \to Pitch. \tag{6.124}$$

Recall that $Fun(Onset) = Fun(Pitch) = @\mathbb{R}$ and $Fun(Onset \oplus Pitch) = @\mathbb{R}^2$ and consider the forms of finite sets in the three preceding forms:

$$Fin(Onset) \underset{Fin(@\mathbb{R}) \rightarrowtail \Omega^{@\mathbb{R}}}{\longrightarrow} \mathbf{Power}(Onset), \tag{6.125}$$
$$Fin(Pitch) \underset{Fin(@\mathbb{R}) \rightarrowtail \Omega^{@\mathbb{R}}}{\longrightarrow} \mathbf{Power}(Pitch), \tag{6.126}$$
$$Fin(Onset \oplus Pitch) \underset{Fin(@\mathbb{R}^2) \rightarrowtail \Omega^{@\mathbb{R}^2}}{\longrightarrow} \mathbf{Power}(Onset \oplus Pitch). \tag{6.127}$$

Corresponding to the above projections and by lemma 90 in appendix G.2.1, we have two projections

$$Fin(p_1) : Fin(Onset \oplus Pitch) \to Fin(Onset), \tag{6.128}$$
$$Fin(p_1) : Fin(Onset \oplus Pitch) \to Fin(Pitch), \tag{6.129}$$

and corresponding diagrams $\mathbf{D}_{p_1}$ and $\mathbf{D}_{p_2}$. We then have limit forms defined by these diagrams:

$$L(\mathbf{D}_{p_1}) \underset{Id}{\longrightarrow} \mathbf{Limit}(\mathbf{D}_{p_1}), \tag{6.130}$$
$$L(\mathbf{D}_{p_2}) \underset{Id}{\longrightarrow} \mathbf{Limit}(\mathbf{D}_{p_2}). \tag{6.131}$$

Let us consider order relations of denotators in this context. According to the general theory, we first need an ordering on the diagram's vertexes and arrows. Our diagram scheme is just two points and one connecting arrow. We order the points as follows: $\underset{0}{\bullet} \leftarrow \underset{1}{\bullet}$, i.e. we have $C_0^1 = Fin(Onset), C_1^1 = Fin(Onset \oplus Pitch)$ in $\mathbf{D}_{p_1}$ and $C_0^2 = Fin(Pitch), C_1^2 = Fin(Onset \oplus Pitch)$ in $\mathbf{D}_{p_2}$. Therefore, the vertex words satisfy $C_0^1 C_1^1 < C_0^2 C_1^2$ since forms verify $Onset < Pitch$; in fact, the latter differ only by their names, and then it is clear.

This implies that $L(\mathbf{D}_{p_1}) < L(\mathbf{D}_{p_2})$, and therefore denotators on $L(\mathbf{D}_{p_1})$ precede denotators on $L(\mathbf{D}_{p_2})$. For the first form $L(\mathbf{D}_{p_1})$, an $A$-addressed denotator $D \in A@L(\mathbf{D}_{p_1})$ looks like this: We have $D = (D_0, D_1)$ with these specifications:

$$D_0 \in A@Fin(@\mathbb{R}) = Fin(A@\mathbb{R}), \tag{6.132}$$
$$D_1 \in A@Fin(@\mathbb{R}^2) = Fin(A@\mathbb{R}^2), \tag{6.133}$$
$$A@Fin(p_1)(D_1) = D_0. \tag{6.134}$$

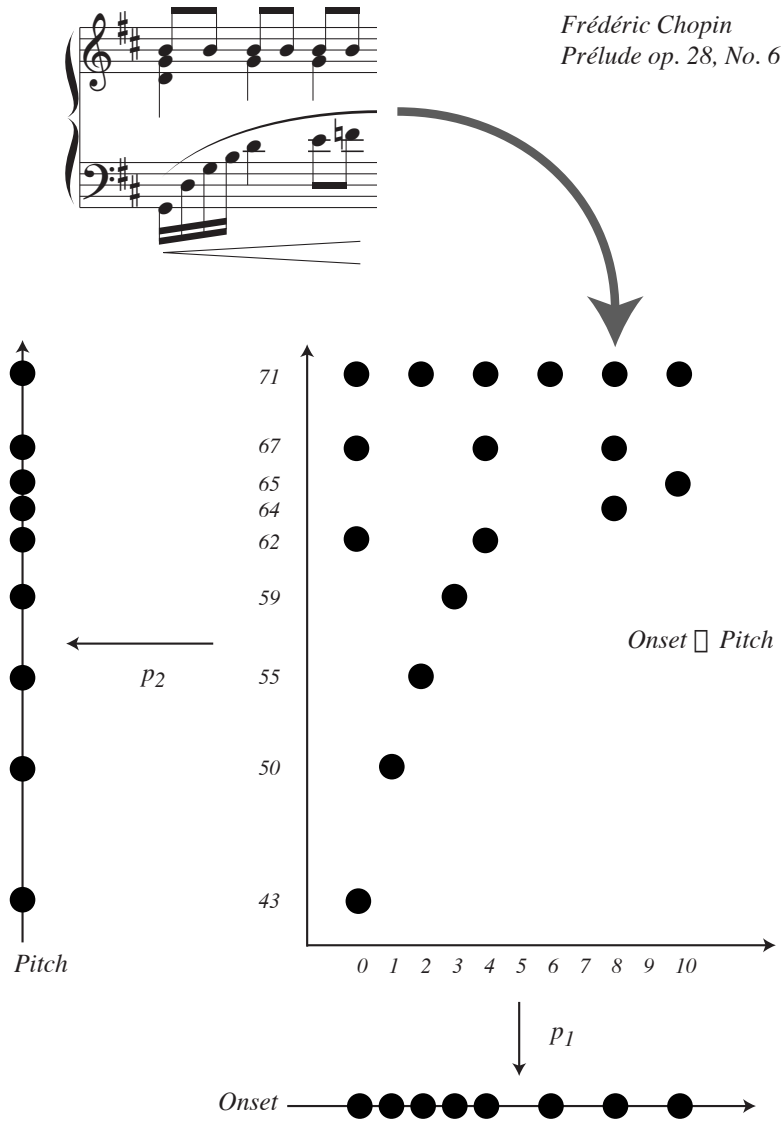


Figure 6.14: A bar of Frédéric Chopin's Prélude op. 28, No. 6 is projected to an $Onset \oplus Pitch$-formed denotator in the plane. Its two one-dimensional projections onto the *Onset* and *Pitch* forms reveal metrical/rhythmical and harmonic aspects of the plane denotator, respectively.

Correspondingly, in the second form $L(\mathbf{D}_{p_2})$, a denotator $D = (D_0, D_1)$ stays in the same mathematical space but verifies the relation

$$A@Fin(p_2)(D_1) = D_0. \tag{6.135}$$

To understand the meaning of such denotators, let us—as usual—start with zero-addressed denotators and see figure 6.14 for an illustration. In this case, the denotator $D \in 0_{\mathbb{Z}}@L(\mathbf{D}_{p_1})$ consists of two finite sets $D_0 \subset \mathbb{R}$, $D_1 \subset \mathbb{R}^2$ such that $D_1$ projects onto $D_0$ when we forget about the *Pitch* coordinates. Correspondingly, a denotator $D \in 0_{\mathbb{Z}}@L(\mathbf{D}_{p_2})$ consists of two finite sets $D_0 \subset \mathbb{R}$, $D_1 \subset \mathbb{R}^2$ such that $D_1$ projects onto $D_0$ when we forget about the *Onset* coordinates.

We have therefore two denotators from one subset $D_1$ in the onset-pitch plane. The first gives information about the relation between the onset configuration of $D_1$ via first projection $p_1(D_1)$ to the onset axis. The second tells about the pitch configuration of $D_1$ via second projection $p_2(D_1)$ to the pitch axis. Roughly speaking, the first deals with metrical or rhythmical aspects of $D_1$, whereas the second deals with its harmonic aspects. Our order principles imply that metrical or rhythmical aspects precede harmonic aspects if ever we are going to allocate such denotators in a library or in a data base system. So we are left with comparison of denotators within one fixed form.

Consider now two $A$-addressed denotators $D^i = (D^i_0, D^i_1)$, $i = 1, 2$ of form $L(\mathbf{D}_{p_1})$ (for the second projection form $L(\mathbf{D}_{p_2})$, we have analogous explications). According to the general discussion of **Limit** and **Power** orderings, we have $D^1 < D^2$ iff either $D^1_1 \prec D^2_1$ or $D^1_1 = D^2_1$ and $D^1_2 \prec D^2_2$. Each of these orderings $\prec$ happens to be the type defined according to proposition 65 in appendix C.2. This means that we first look at the ordering among metrical/rhythmic aspects of denotators, and after they have been identified, we turn to the tones behind these aspects. Recall that $D^1_1 \prec D^2_1$ means that on the onset axis either $D^1_1 \subset D^2_1$ or $max(D^1_1 - D^2_1) < max(D^2_1 - D^1_1)$.

There may be many different configurations with the same metrical/rhythmic aspect. What does it mean that $D^1 < D^2$ when their metrical/rhythmic aspects coincide? We then have $D^1_1 = D^2_1$ and $D^1_2 \prec D^2_2$, where the latter is induced by the lexicographic ordering on $A@\mathbb{R}^2$. This means that for any $x = (x_1, x_2) \in (A@\mathbb{R})^2$ which stays in $D^1_1 - D^2_1$, there is $y = (y_1, y_2)$ in $D^2_1 - D^1_1$ such that either its onsets coincide and pitches verify $x_2 < y_2$, or onsets verify $x_1 < y_1$.

## 6.9 Concept Surgery and Denotator Semantics

**Summary.** Denotators can be used as dynamic tools for concept production. This shows that denotator semantics must be understood as an open-ended specification.

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Denotators are very useful for formally repertorizing ethnomusicological data and facts. The possibility to apply character strings as coordinates opens the way to incomplete semantics. For example, it is not necessary to misrepresent or over-interpret the pitch data of a note in an ethnological context in favor of the European lexemata of pitch when denotating an Indian melody. However, when building a pitch model for that context, it is still possible to map the character strings to numbers, such as European pitch classes, MIDI numbers or whatever may

be relevant to the given scientific project[53]. Whereas a European score symbol for dynamics is recorded as a character string coordinates by *'mf'*, *'pp'*, the user may interpret these coordinates by numerical evaluation for performance purposes. But until that realization, the denotators keep their full potential of "European ethnological context of notation". This is no luxury for ethnomusicological research, quite the contrary: Without such an open denotator system, the retrieval of ethnomusicological data would fail from the beginning since the original information would be distorted in an irreversible way if the record was meant to be a reliable first order source.

This procedure may be generalized to the end of substituting character strings by entire denotators in order to 'fill up symbolic strings' with more explicit denotators. This substitution process is understood as yielding a 'decoding' method for 'deepening meaning' of signs. Let us explicate the technical aspects. To begin with, one is looking for forms that express the above 'symbolic' representation of semantically open data. Suppose that the symbols we are interested in stem from the set **CHR** of $ASCII$-character strings. We want to consider European dynamic signs with a minimum of superfluous information, more precisely, with an intrinsic character of a non-resolved semantics. This is given by circular forms! Let [**CHR**] be the constant functor[54] of **CHR** and $\circledcirc\mathbb{R}$ the Yoneda functor of the *ring* $\mathbb{R}$, applied to the coefficient ring of a given address $A_R$, i.e. $A_R@\circledcirc\mathbb{R} = R\circledcirc\mathbb{R}$. We consider the form

$$'EuroDyn' \underset{Id_{\circledcirc\mathbb{R}\times[\mathbf{CHR}]}}{\longrightarrow} Syn(EuroDyn). \tag{6.136}$$

Next, let us consider the technical aspect of modeling this symbolic status on more concrete parameters. We still stick to our *EuroDyn* example to make the general idea clear. A more concrete parametrization of loudness would be the simple form $Loudness \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R})$ (6.69). Giving the symbolic denotators of form *EuroDyn* a more concrete meaning in terms of form *Loudness* means defining a "meaning" or "modeling" morphism of forms $m : EuroDyn \to Loudness$. This means defining a map $A@m : R\circledcirc\mathbb{R}\times\mathbf{CHR} \to A@\mathbb{R}$ which is functorial in address $A$ over the ring $R$, i.e., for any address change $c : B \to A$ over the scalar restriction $\gamma : S \to R$, we must have a commutative diagram

$$\begin{array}{ccc} R\circledcirc\mathbb{R}\times\mathbf{CHR} & \xrightarrow{A@m} & A@\mathbb{R} \\ {\scriptstyle \gamma\circledcirc\mathbb{R}\times Id}\downarrow & & \downarrow{\scriptstyle c@\mathbb{R}} \\ S\circledcirc\mathbb{R}\times\mathbf{CHR} & \xrightarrow{B@m} & B@\mathbb{R} \end{array} \tag{6.137}$$

of set maps. For example, such a morphism can be defined if we know just what happens on the zero address. Since we have a unique address change $! : B \to 0_S$ of $S$-modules, we get the map $B@m$ by commutativity of the diagram

$$\begin{array}{ccc} S\circledcirc\mathbb{R}\times\mathbf{CHR} & \xrightarrow{0_S@m} & 0_S@\mathbb{R} \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle !@\mathbb{R}} \\ S\circledcirc\mathbb{R}\times\mathbf{CHR} & \xrightarrow{B@m} & B@\mathbb{R} \end{array} \tag{6.138}$$

[53]This is precisely what the RUBATO analysis and performance platform makes possible in the case of European score denotators such as dynamics *mf*, *pp* etc., see section 41.5 for details.

[54]See example 93 in section G.1.3.

and therefore, any modeling transformation is determined by its data on the zero addresses. But we have an isomorphism of functors $\circledcirc\mathbb{R} \times [\mathbb{R}] \xrightarrow{\sim} 0_?@\mathbb{R}$ on the category **Rings**, the latter evaluating to $0_S@\mathbb{R}$ at ring $S$. But then, the morphism $m$, restricted to the zero addresses defines a morphism between set-valued functors on **Rings**. Since the identity is the only endomorphism of $\mathbb{R}$, we easily see that $m$ is defined by the product $Id_{\circledcirc\mathbb{R}} \times m_0$ for a set map $m_0 : \mathbf{CHR} \to \mathbb{R}$. In other words, we have this result:

**Proposition 1** *The modeling morphisms $m : EuroDyn \to Loudness$ are in one-to-one correspondence with the set maps $m_0 : \mathbf{CHR} \to \mathbb{R}$, defined by commutative diagram* (6.138).

The set maps in proposition 1 are precisely what one would define in an ethnomusicological modeling of dynamics. The functorial extension is for free in this case. Conceptually, what happens is that we have mapped a circular denotator into a regular one and thereby deepened the semantic short circuit of circularity into regular meaning in terms of real numbers.

This example shows that though it is necessary to work with incomplete concepts such as *EuroDyn*, we should also be capable of making their signification more precise *on demand* in the language of denotators. We want to complete this technique in order to have the generic procedure at hand. What follows will be called *concept surgery* since it deals with replacement of given forms by "better" ones. We want to start with the general situation of a given corpus of forms. All these forms are constructed by use of more or less large coordinators which are diagrams of forms if they are compound, i.e. built upon other forms. Suppose now that a particular form *ArchaicForm* is connected to a "better" form *ModelForm* by means of a "modeling" morphism, i.e. we are given a morphism $model : ArchaicForm \to ModelForm$. Denote by $\mathbf{D}_{model}$ the diagram defined by this morphism. Define a new form.

$$BetterForm \xrightarrow[Id]{} \mathbf{Limit}(\mathbf{D}_{model}) \tag{6.139}$$

which displays two projections

$$\begin{aligned} &p_{archaic} : BetterForm \to ArchaicForm, \\ &p_{model} : BetterForm \to ModelForm, \end{aligned} \tag{6.140}$$

related by $model \cdot p_{archaic} = p_{model}$. We then replace every morphism of forms $g : G \to ArchaicForm$ by the universal morphism $Better^g : G \to BetterForm$ defined by the pair $(g, model \cdot g)$ of morphisms into $\mathbf{D}_{model}$:

$$\begin{array}{ccc} G & \xrightarrow{g} & ArchaicForm \\ {\scriptstyle Better^g}\downarrow & & \downarrow{\scriptstyle model} \\ BetterForm & \xrightarrow{p_{model}} & ModelForm \end{array} \tag{6.141}$$

Similarly, we replace every morphism $g : ArchaicForm \to G$ of forms by its composition $Better_g = g \cdot p_{archaic}$, and we obtain a commutative (!) diagram

$$\begin{array}{ccc} H & \xrightarrow{h} & ArchaicForm \\ {\scriptstyle Better^h}\downarrow & & \downarrow{\scriptstyle g} \\ BetterForm & \xrightarrow{Better_g} & G \end{array} \tag{6.142}$$

of "bad and better" morphisms of forms for any morphisms $g : ArchaicForm \to G$ and $h : H \to ArchaicForm$, i.e. $g \cdot h = Better_g \cdot Better^h$ which means that the "deviation" of morphisms to $BetterForm$ does not disturb any of the given factorizations through the old $ArchaicForm$. In this surgical intervention, we have excised the $ArchaicForm$ from all its appearances in other forms, and replaced its role by that of a new $BetterForm$. So this surgery makes it possible to improve the whole concept body of forms and denotators once one form has found an improved version. Set more formally, we may now replace every form $OldForm$ where $ArchaicForm$ is involved somewhere in its recursive tree by a new form

$$model \Cap OldForm, \tag{6.143}$$

defined by the surgically inserted $BetterForm$ from the $model$ morphism.

*Despite this semantic surgery, the denotator system has its very strict limits and requires a supersystem of connotation to grasp deeper layers of meaning.* Already with the above example of ethnomusicological signification it became evident that the denotator system is only a signifier surface pointing at interpretative, performative, and emotional, resp. social, meaning.

Moreover, denotators are not sensible to what really exists in music, as a member of the repertoire of a given culture, for example. In fact, this is an essential fact about music: In the real world, denotators are far from realized in their complete potentiality. Beethoven's Fifth Symphony is not just one of a large repertoire of given denotators! Its existence has far-reaching consequences on the theory and practice of music and its meaning. This is the reason why denotators must be deepened to yield this existential aspect underlying music. We shall develop this subject in full detail in chapter 18.

# Part III

# Local Theory

# Chapter 7

# Local Compositions

*Ich kann ein lokales Objekt, eine lokale Struktur nicht* abstrakt *schaffen;*
*so elementar ich sie auch immer denke: ich bin bereits gezwungen,*
*sie zu kodieren, um sie niederschreiben zu können*
*– im Grunde fällt der Lokalstruktur auch*
*eine alphabetische Rolle zu (. . . )*
Pierre Boulez [60, II, p.66]

**Summary.** Local compositions are introduced as elementary objects of music. They derive from powerset denotators. It is shown that all denotators may be transformed into local compositions. A special type of local compositions can be defined from a fixed set of denotators of a given "ambient space". With these so-called *objective* local compositions the problem of universal construction of new concepts from given ones cannot be solved. This imposes a deeper theory of non-objective *functorial* local compositions. The basic vocabulary as well as an introductory list of common local compositions—such as scales, ordinary and fractal chords, meters, rhythms, and motives—are presented. The chapter concludes with a discussion of tangent objects in local theory, a concept framework which leads to alterations and related results by Mason and Mazzola.

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The previous chapter was devoted to the universal concept and navigation formalism of forms, denotators and orders among these objects. The attentive reader could observe that the category of forms has been defined, but no categories of denotators were considered. More down-to-earth speaking, our attention to *relations among denotators* has not been set up, denotators are—until now—entities which are only related by orders and not by more in-depth relations which take care of the precise positions of denotators within their space functors.

In this chapter, we want to set up the *objects* of the category *Loc* of local compositions, i.e. of the type of denotators which are the core of musical conceptualization. The next chapter will be devoted to a thorough discussion of the morphisms among local compositions. It is however not straightforward in which generality such objects should be established. There are two directions of generalization which must be observed. The first one relates to the dichotomy

of *fixed versus variable addresses* and has been introduced previously. The second relates to the dichotomy of *local versus global structure*. This chapter will only deal with the local aspect and introduce and justify the global one in chapter 13.

## 7.1 The Objects of Local Theory

**Summary.** After a heuristic introduction, objective and functorial local compositions are introduced.

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According to Arnold Schönberg [478], tones are "the material of music". In this spirit, modules and the form space functors deduced by universal constructions are the mathematical spaces where this material is surveyed; we did already introduce the basic role of modules insection 6.4.1. But it should be recognized that single tones do not constitute what one would recognize as a musical unit. Musical material needs an elementary structuring in order to build elements of music. What Pierre Boulez refers to in this chapter's header is an alphabet of music, built from local, encoded entities. We agree with Boulez' encoding requirement, and we contend that positioning musical material in the shape of a denotator-point within its form space functor at a determined address does meet encoding. But it is not true that a singular tone is the only elementary unit which one will encounter. Besides singular 'points', triadic chords, intervals, motives or rhythms can also be 'elementary' *as* musical units. We cannot prescribe that elementary objects, i.e. letters of the musical alphabet be reduced to single events. For example, it is known from harmony of the $18^{th}$ century that single notes were only harmonically significant as material parts of triads.

On the other hand, we will certainly require from an elementary system of sounds that it is viewed in a common context which enables us to determine the relations of the constitutive sounds. In the language of gestalt psychology [136] one could restate our project in that "musical gestalts" should be defined which are elementary in the sense that no relations among their material points remain indeterminate. The gestalt's parts, the single sounds, have to be linked in a 'rigid' way.

We illustrate this rigidity requirement by an example. Look at the 12-tempered tuning in the Euler module pitch space, i.e. at zero-addressed denotators $p : 0 \rightsquigarrow EulerModule(p/12 \cdot o)$, $p \in \mathbb{Z}$. It represents pitch $c$ by $p = 0$, $c\sharp$ by $p = 1$, etc. In this context, a triad is determined by three pitch numbers $u, v, w$. We then consider not explicitly selected triples $(u, v, w)$ but only the following relations among $u$, $v$, and $w$:

$$|u - v| = 4, |v - w| = 3, \text{ and } u = 0. \tag{7.1}$$

Then there are four solutions

$$\begin{aligned} (u, v, w) = &(0, 4, 1) && \text{corresponding to} && (c, e, c\sharp) \\ &(0, 4, 7) && \text{corresponding to} && (c, e, g) \\ &(0, -4, -1) && \text{corresponding to} && (c, A\flat, B) \\ &(0, -4, -7) && \text{corresponding to} && (c, A\flat, F) \end{aligned} \tag{7.2}$$

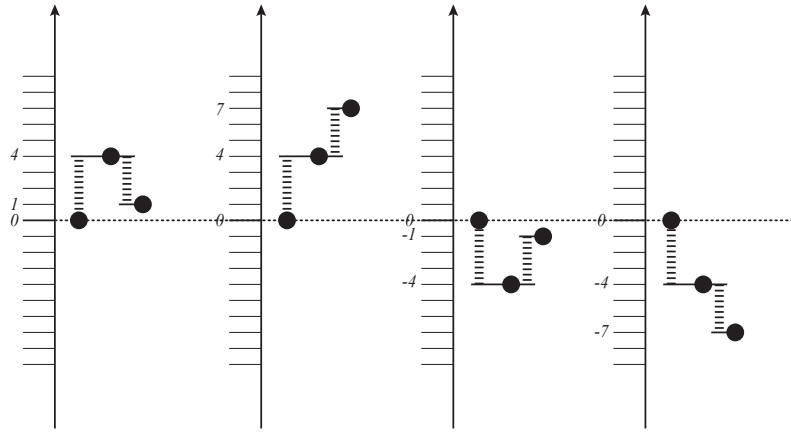


Figure 7.1: Example of a non-elementary pitch configuration in 12-tempered pitch (parametrized by integers) with non-fixed pitches. One only asks that the first pitch be zero whereas the first stays four semitones from the first, and the third stays three semitones from the second. We then obtain four solutions as shown in the quasi-mechanical 'hinge' system.

with triads which verify our conditions (7.1), see also figure 7.1. By these relations, the triad is not uniquely determined. Intuitively speaking, it looks like a hinged mechanical system which consists of two rigid rods $(u, v)$ and $(v, w)$ whose ends are tied by hinge $v$ whereas hinge $u$ is fixed at 0. Such a configuration would *not* be termed "elementary". It is composed of different loosely related parts.

Evidently, it would be difficult if not impossible to relate the concept of an elementary musical gestalt to relations as above in the general case. Therefore, we shall restrict the concept of such a structure to the really 'indecomposable' configuration of a *local composition* and introduce to non-elementary structures later.

**Definition 8** *A* local composition *is a denotator* $D : A \rightsquigarrow F(x)$ *whose form* $F$ *is of* **Power** *type. The coordinator* $S$ *of form* $F$ *is called the* ambient space *of* $D$*; the coordinates*[1] $x$ *is called its* support*. The local composition* $D$ *is called* objective *iff* $x \in 2^{Fun(S)}$ *i.e. iff there is* $X \subset A@Fun(S)$*, with* $x = \hat{X}$*, in this case* $X$ *is also called a support of* $D$*. The* cardinality $card(D)$ *of an objective denotator* $D$ *is the cardinality*[2] *of its coordinates set* $X$*. In particular, an objective denotator* $D$ *is called* empty, finite, infinite*, iff its cardinality is zero, finite, or infinite. Non-objective local compositions are also called* functorial*.*

Up to names, a local composition is a subfunctor $x \hookrightarrow @A \times Fun(S)$; and an objective local composition is a subset $X \hookrightarrow A@Fun(S)$. A bit more sloppily, we often write $D : x \hookrightarrow @A \times S$ or $D : X \hookrightarrow A@S$ resp. $D = (x, @A \times S)$ and $D = (X, A@S)$.

[1] Recall that "coordinates" is a plural in singular mode and as a such covers the muliplicity of coordinate values within such an object. However, sometimes we also use the common language if the strict terminology sounds too far-out.

[2] See appendix C.2 for the concept of cardinality.

If we are given a form $F$, any denotator $D$ of this form can be wrapped as a singleton $\{D\}$ of the form named $Loc^F$ and defined by

$$Loc^F \underset{Id_{\Omega Fun(F)}}{\longrightarrow} \mathbf{Power}(F)$$

We shall always apply this wrapping formalism if talking of a general denotator $D$ of form $F$ as being "wrapped as local composition" or simply "wrapped"; we then consider the objective local composition $\{D\}\hat{}$ in $Loc^F$. Evidently, the construction of $Loc^F$ is a kind of "generic" since any local composition of ambient space $F$ can be derived from $Loc^F$ by synonymy and subfunctors, but we shall not systematically exploit this rigid approach.

Before entering into the very generality of functorial local compositions, we should like to get acquainted with the objective local compositions and try to understand the yoga of this concept. At first sight the semantic potential of a simple structure such as $D = (X, A@S)$ seems to be quite poor. This impression is faulty and stems from the fact that we have packed a lot of information in the preparation of this concept. Recall that this includes:

- selection of the ambient space and address
- the recursively hidden module structures to the ambient space
- the relations among the elements $\xi$ of the support $X$.

Further, the classical Ehrenfels criterion of "invariance under transpositions" [136] seems to be lacking. We shall introduce this requirement *in extenso*, but not in the much too narrow frame of simple pitch transposition. In fact, the latter is a very elementary special case of general symmetries which are embedded in the concept of morphisms of denotators, to be introduced in chapter 8. Under this general perspective, Ehrenfels' criterion is but one of a large number of possibilities to define symmetry invariance for the gestalt concept. Not least is it historical development of music which does force an opening of the symmetry invariance. Fixing a symmetry (such as transposition) would cast the concept of gestalt in a too rigid framework, but see sections 11.7.2 and 18.1.1 for more in-depth discussions of the historical dimension of music, its science, and time-invariant laws of music.

## 7.2 First Local Music Objects

**Summary.** We introduce and discuss concepts of scales, chords, rhythms, motives in the language of objective local compositions.

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Musical resources are indispensable for the musicological discourse, they provide us with the basic vocabulary of music where concepts such as scale, chord etc. should be declared. Whoever tries to establish a standard of basic musicological concepts is unavoidably confronted with the ambiguity of the existing usage and with the problem of arbitrarily normative and materially overrestrictive or even dogmatic "standard language".

But ambiguity of musicological concepts is not so much a defect but rather expression of a complex situation. Should we—for example—include in the concept of "scale" any temporal

order of its tones? Is it then c, d, e, f, g, a, b for the "ascending" C-major scale and b, a, g, f, e, d, c for the "descending" variant? And which "c" do we intend? Any "c" or 'the' "c"? How long does such a tone last? What is its loudness, tuning, etc. And if you play in C-major, does this mean that all tones are contained in the C-major scale? And do they even have to be played in the order indicated by the ascending or descending scale?—We immediately recognize that it is easier to tell what a scale is not than what it is.

We do take into account this ambiguity by usage of a variety of forms and denotators and therefore local compositions (later also global ones). And this is done in full consciousness of the variability of topographic perspectives (discussed in chapter 2). This provides us with the possibility to build successively operative concept fields from precise aspects and to realize inner coherence by precise relations. For the backing principle of "Yoneda philosophy", roughly stating that knowledge comes through the sum of aspects or perspectives, we refer to chapter 9. The price for this gain in conceptual profoundness and universality is a certain slowness of ascent from elementary to more complex concept of practical usage. This procedure has, however, the methodological advantage to exhibit a general scheme of musicological conceptualization by which the reader can easily build new concepts according to individual needs. This scheme will be fully developed in the frame of the global theory (chapter 13). It should be stressed that this scheme is in perfect harmony with the knowledge and data base requirements alluded to in section 2.5 and chapter 5 and realized more down-to-earth in chapter 40.

Let us now introduce some basic concepts which are elementary in the sense that they pertain to the language level of objective local compositions. Momently, these concepts are presented in an unsystematic way; the concept of paradigmatic classification (chapter 10) will lead to a more systematic treatment in chapter 11.

### 7.2.1 Chords and Scales

**Summary.** Thissection defines pitch-related local compositions, such as chords and scales as local compositions and gives examples from music and music theory.

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**Example 5 Chords**
Although there are many ways to view a configuration of tones as being a chord, the essential is that intuitively it is a finite collection of pitches. Duration, onset, sound color etc. play a secondary role. We therefore make this definition:

**Definition 9** *A* chord *is a finite local composition with EulerModule as ambient space, more precisely*[3]*, if we abbreviate F= Fun(EulerModule), it is a denotator of form*

$$EulerChord \underset{Fin(F)\rightarrowtail\Omega^F}{\longrightarrow} \mathbf{Power}(EulerModule) \tag{7.3}$$

*If* $Cr : A \rightsquigarrow EulerChord(Cr)$ *is a chord at address A, we say that it is an* $n$-chord *if its cardinality* $card(Cr) = n$.

[3]Precision in fact regards the naming, nothing more.

Observe that—as always with denotators—the address is relevant, and therefore we have a 0-chord for each address! To distinguish chords from their instantiation in time, one adds onset to pitch and sets

**Definition 10** *A* chord event *is a denotator of form*

$$EulerChordEvent \underset{Id}{\longrightarrow} \mathbf{Limit}(Onset, EulerChord) \tag{7.4}$$

*where limit means product of the two factor coordinators.*

Thus, a chord event is not a local composition, but:

**Exercise 1** Redefine chord events as being special local compositions with ambient space $Onset \oplus EulerModule$, see (6.69) for the $Onset$ form.

Often, pitches of chords are considered modulo a pitch period $p$. Usually, this period is the octave $o : 0_{\mathbb{Z}} \rightsquigarrow EulerModule(o)$. More generally, in order to include Wyschnegradsky's non-octave-periodic constructions [131], Bernhard Stopper's tuning [512] or contemporary jazz composition principles [359], period $p$ may be any non-zero denotator $p : 0_{\mathbb{Z}} \rightsquigarrow EulerModule(p)$. One then works in the quotient Euler module[4] $\mathbb{Q}^3/\mathbb{Z} \cdot p$ and then defines as follows:

**Definition 11** *Let $p$ be a period in the Euler module. The* $p$-class chord *is a finite local composition with ambient space*

$$p\text{-}EulerClass \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Q}^3/\mathbb{Z} \cdot p), \tag{7.5}$$

*more precisely, setting $F = Fun(p\text{-}EulerClass)$, it is a denotator of form*

$$p\text{-}ClassChord \underset{Fin(F) \rightarrowtail \Omega^F}{\longrightarrow} \mathbf{Power}(p\text{-}EulerClass). \tag{7.6}$$

The octave period $o$ is particularly common, and we have this representation of a direct sum of two modules:

$$\mathbb{Q}^3/\mathbb{Z} \cdot o \xrightarrow{\sim} \mathbb{Q} \cdot o/\mathbb{Z} \cdot o \oplus (\mathbb{Q} \cdot q \oplus \mathbb{Q} \cdot t) \tag{7.7}$$

where the first is the circle of rational cosets modulo $o$, and the second is the Euler plane. Hence the points $x : A \rightsquigarrow p\text{-}EulerClass(x)$ of a $o$-class chord consist of two components $x = (x_{temp}, x_{just})$ which live in the corresponding space forms:

$$\begin{aligned} &x_{temp} : A \rightsquigarrow p\text{-}TempClass(x_{temp}), \text{ with form} \\ &o\text{-}TempClass \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Q} \cdot o/\mathbb{Z} \cdot o), \end{aligned} \tag{7.8}$$

$$\begin{aligned} &x_{just} : A \rightsquigarrow EulerPlane(x_{just}), \text{ with form} \\ &EulerPlane \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Q} \cdot q \oplus \mathbb{Q} \cdot t). \end{aligned} \tag{7.9}$$

[4] To be precise, the underlying module is a $\mathbb{Z}$-module, i.e. we are taking $\mathbb{Q}^3_{[\mathbb{Z}]}/\mathbb{Z} \cdot p$.

Accordingly, if a $o$-class chord has all its points in the 'tempered' ambient space $o\text{-}TempClass$ (resp. in the 'just' ambient space $EulerPlane$) we say that it is tempered (resp. just). If there is no danger of confusion we shall omit the octave period in this context and simply speak of *pitch classes* or *class chords* when we deal with points or chords in ambient space $o\text{-}EulerClass$.

On the (octave-)tempered ambient space $TempClass$, one often considers pitch classes in a regular subdivision of these pitch classes, defined by a positive natural number $w$. We then look at the tempered ambient subspace

$$w\text{-}PitchClass \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z} \cdot \frac{1}{w} \cdot o/\mathbb{Z} \cdot o) \tag{7.10}$$

of $TempClass$. When representing its circle module coordinator as a circle, the submodule $\mathbb{Z} \cdot \frac{1}{w} \cdot o/\mathbb{Z} \cdot o \overset{\sim}{\to} \mathbb{Z}_w$ appears as a regular $w$-sided polygon, and a tempered class chord is called $w\text{-}tempered$ if its points are contained in the ambient space $w\text{-}PitchClass$. Of course, the ambient space $w\text{-}PitchClass$ is isomorphic to the form $PiMod_w$ introduced in (6.41). The difference here is that we have embedded the pitch classes in the general Euler module context. We shall often identify the underlying modules and the associated forms.

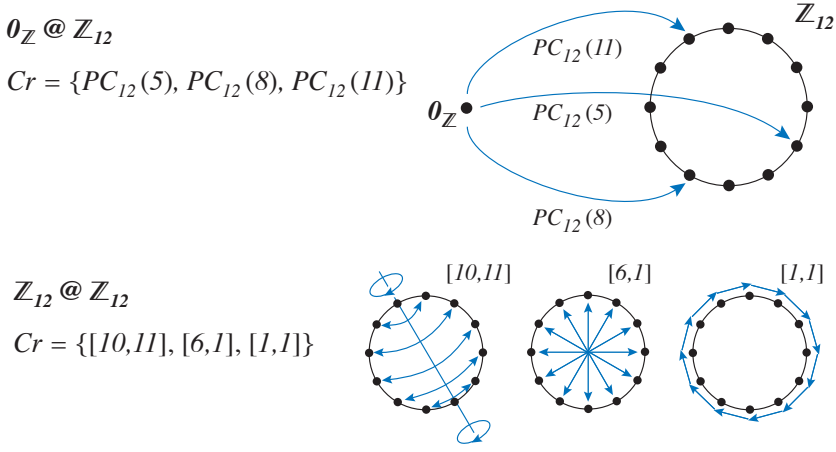


Figure 7.2: Above, a zero-addressed 12-tempered class 3-chord, below a self-addressed 12-tempered class 3-chord.

Whereas there are infinitely many (coordinate sets for) just class chords, there are only $2^w$ (resp. $\binom{w}{n}$) different (coordinate sets for) $w$-tempered class chords (resp. $n$-element class chords). For every couple $Cr_1, Cr_2$ of $A$-addressed chords or class chords, we can build their Boolean combinations: *union* $Cr_1 \cup Cr_2$, *intersection* $Cr_1 \cap Cr_2$, and *difference* $Cr_1 - Cr_2$. For a $w$-tempered class chord $Cr$, one may also build its *complementary chord* $Cr\hat{} = \chi_w - Cr$, i.e. the difference from the $w\text{-}chromatic$ class chord $\chi_w$ of support $\mathbb{Z}_w$.

We conclude with a remark on different addresses for chords. With the identification from (6.41) and notation from chapter 6, RegDen-9, a 0-addressed $w$-tempered class $n$-chord can be

written as a set

$$Cr = \{PC_w(r_1), \dots PC_w(r_n)\},$$

and a self-addressed $w$-tempered class $n$-chord can be written as set

$$Cr = \{[a_1, b_1], \dots [a_n, b_n])\}$$

with $r_n, a_n, b_n \in \mathbb{Z}_w$. Figure 7.2 shows two class chords with different addresses.

**Example 6 Scales** This concept regards special configurations in pitch spaces. We shall work in the $EulerModule$ form as an ambient space. To define a scale, one has to fix a periodicity in order to express the repetition of scale tones under a particular period. For the same reasons as explained for chords in example 5, we select a non-zero period denotator $p : 0 \rightsquigarrow EulerModule(p)$. We have a morphism

$$mod_p : EulerModule \twoheadrightarrow p\text{-}EulerClass \tag{7.11}$$

which is induced by the canonical projection $\pi : \mathbb{Q}^3 \twoheadrightarrow \mathbb{Q}^3/\mathbb{Z} \cdot p$. This associates with every objective local composition $L : A \hookrightarrow A@EulerModule$ its pointwise defined projection $mod_p(L) : A \hookrightarrow A@p\text{-}EulerClass$. We then define

**Definition 12** *Given a period p, a* $p$-scale *is a non-empty local composition $S$ of ambient space $EulerModule$, and such that*

(i) *its projection $mod_p(S)$ is finite, i.e. a p-class chord,*

(ii) *$S = e^p S$, i.e. $S$ is p-periodic.*

*The associated p-class chord is called* the scale's chord.

Conversely, the scale $S$ may be recovered from its $p$-class chord by inverse image:

$$S = mod_p^{-1}(mod_p(S)),$$

see also figure 7.3. But it is not true in general that each non-empty $p$-class chord $Cr$ defines a scale. In fact, for a critical address such as $A = \mathbb{Z}_{12}$, there is no point $x : \mathbb{Z}_{12} \rightsquigarrow EulerModule(x)$ which projects to the canonical injection $\xi : \mathbb{Z}_{12} \rightsquigarrow o\text{-}EulerClass(\xi)$ in the tempered part. One may rightly define a form which comprises exactly the scales as its denotators. We have

**Lemma 1** *Let $F = Fun(EulerModule)$. Then, for addresses $A$, the map*

$$A \mapsto Sc(A) = \{S \in A@2^F \mid S \text{ is a } p\text{-scale}\} \tag{7.12}$$

*defines a subfunctor of $2^F$.*

**Exercise 2** Give a proof of lemma 1.

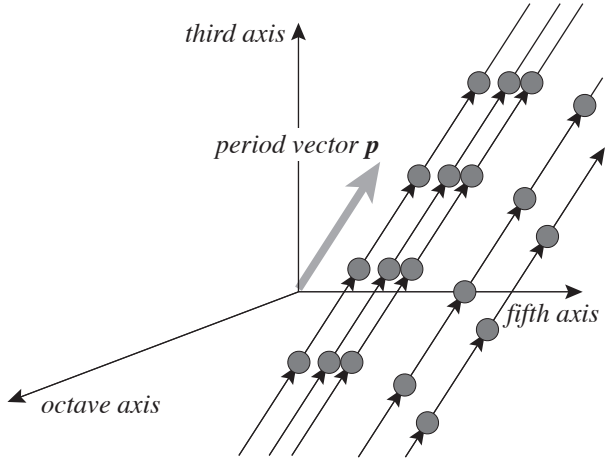


Figure 7.3: A scale with period $p$ in Euler space.

With lemma 1 we can introduce the scale form

$$p\text{-}Scale \underset{Sc \rightarrowtail \Omega^F}{\longrightarrow} \mathbf{Power}(EulerModule) \tag{7.13}$$

and we have a morphism $scalemod_p : p\text{-}Scale \rightarrow p\text{-}ClassChord$ of forms. According to the above, this morphism allows us to recover all $p$-scales by inverse images of non-empty $p$-class chords. We therefore stick to the latter in the following discussion of common scales. Again, these scales will refer to the octave period. In practice, we encounter two main types of $o$-scales, $w$-tempered and just scales.

#### 7.2.1.1 $w$-Tempered Scales

Let $w$ be a positive natural number. A scale, represented by its class chord $S$, is said to be *$w$-tempered* iff the scale's class chord $S$ is $w$-tempered. This also means that the scale is contained in the *$w$-tempered scale space*

$$w\text{-}TemperedScale \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z} \cdot \frac{1}{w} \cdot o).$$

In this case, a zero-addressed scale can be represented as a subsets of this polygon, see figure 7.4 for common scales in 12-tempered tuning.

#### 7.2.1.2 Just Scales

A scale is said to be *just* if its class chord is just. We may then represent the scale by a class chord living in the *EulerPlane* ambient space. When representing zero-addressed pitches on the *EulerPlane*, the horizontal axis is the fifth axis, and the vertical one is the third axis. We show three common scales via their class chords in figure 7.5.

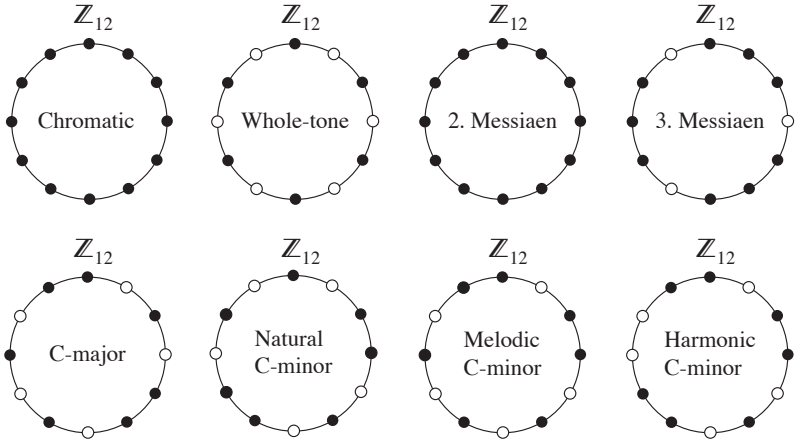


Figure 7.4: Some of the most common 12-tempered scales, related to pitch $c = \overline{0}$ which is shown as the highest point of the polygon; the other numbers $\overline{1}, \overline{2}, \ldots \overline{11}$ follow in clock-wise orientation.

This gives a first impression of how different tempered and just tuning look from the geometric point of view. This insight becomes even more dramatic if we visualize the 12-tempered and just chromatic scale pitches within one octave, see figure 7.6. Whereas the 12-tempered configuration looks like a pearl chain, the just configuration seems to be completely random. We shall learn in section 8.2 that this is a wrong impression. It can be shown that both, the 12-tempered and the just octave configurations, admit a distinguished inner symmetry each, and that this symmetry is intimately related with the theory of consonances and dissonances (see part VII).

### 7.2.2 Local Meters and Local Rhythms

**Summary.** Thissection deals with onset time specific local compositions, such as local meters and local rhythms.

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Before discussing the precise concepts we should recall that meters and rhythms are among the most fluffy concepts in musicology. This is partly due to the extremely complex phenomenon of time-based regularities, but it is also due to a remarkable (and not forgivable) lack of corresponding theory in European musicology. This lack is not only due to the traditional preponderance of harmonic considerations since the Pythagorean school was founded, it is also an effect of the incapability of musicological theorists to construe complex concepts from simple ones in order to grasp complex phenomena. There is a strange belief that either a concept encompasses the intended phenomenon—or it cannot and *should not* be defined[5]! We come back to this special issue in section 10.4.

[5]See the problem of defining a motif in [444, p.11/12, footnote], for example.

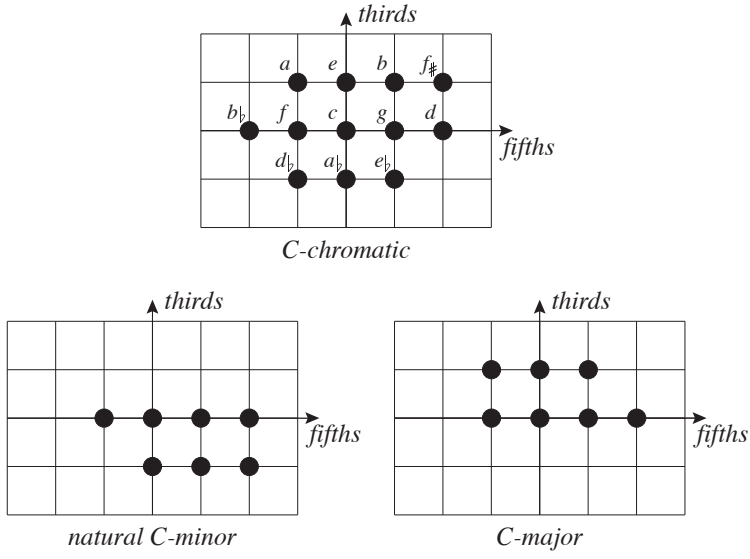


Figure 7.5: The just C-major, natural C-minor, and C-chromatic scale, the latter according to Martin Vogel [547], as represented by their class chords in the *EulerPlane*. Observe that natural C-minor contains another pitch class for $b_\flat$ than C-chromatic. The difference is the class of the syntonic comma, see (6.34).

We start this concept framework with the elementary concept of a local meter, see also figure 7.7 for an illustration.

**Definition 13** *Given an address $A$, let $p : 0_{\mathbb{Z}} \rightsquigarrow Duration(p)$ and $o : A \rightsquigarrow Onset(o)$ be a positive duration period and an onset origin, respectively. Let $-\infty \leq a \leq b \leq \infty$ be two extended integers. Then a (A-addressed)* local meter *is an objective local composition $L \hookrightarrow A@Onset$ of ambient space $Onset$ defined as the $p$-periodic interval*

$$L = \{e^{tp} \cdot o |\ t \in [a, b]\} \tag{7.14}$$

*of $A$-addressed onsets. The difference $b - a$ is called the* length *of $L$; in case of positive length, the uniquely determined duration $p$ is called the* period *of $L$.*

In particular, if we are given a time signature $TimeSig(x, y, e)$ (6.92), two canonical local meters are associated:

- The *barline meter* is defined by *barline* period $Duration(x/y)$, origin $Onset(e)$, and interval numbers $[a, b] = [-\infty, \infty]$.
- The *beat meter* is defined by *beat* period $Duration(1/y)$, origin $Onset(e)$, and interval numbers $[a, b] = [-\infty, \infty]$.

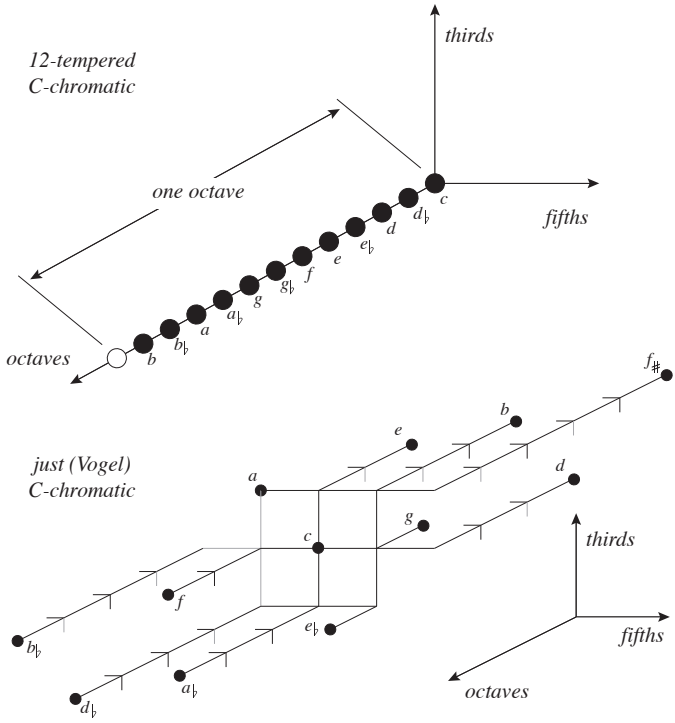


Figure 7.6: One octave of the 12-tempered chromatic C-scale (above) and the just (Vogel) chromatic scale (below) The images seem to be very different but share an important symmetry each.

Observe that the barline meter is independent of the time signature representation of the fraction $x/y$ whereas the beat meter isn't. Later, we shall introduce finite length meters for bars and beats to take care of the limits of concrete scores.

From the music(ologic)al point of view, a local meter is the *minimum of time regularity* that can be observed: a simple, periodic repetition of onsets. In the understanding of Hugo Riemann [453], metrical structures deal with onset weights. We may thus view the concept of a local meter as a basic account of the weight of selected onsets, the selection being driven by a period of regular distribution over time. This approach has a psychological interpretation insofar as it suggests that elementary perception and/or expression of temporal stress or "weighting" relies on a repeated appearance of the same object in regular distances.

There are several generalizations and complexifications of this conceptual germ which build a viable tool to understand metrical and rhythmical phenomena. The *second* one deals with "globalization" issues, which means that local meters are only the elements of a compound time structure. We shall deal with global meters insection 13.4.3, more in depth in chapter 21, and more concretely in section 41.1 which deals with software for metrical analysis.

The *first* generalization concerns local rhythms and will be dealt with now. A local meter is

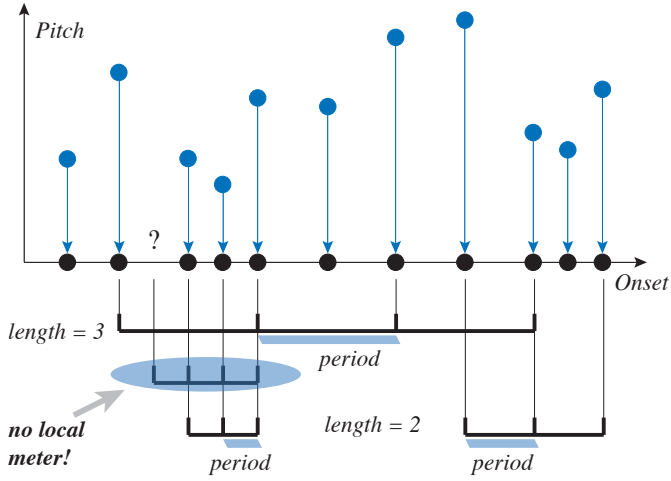


Figure 7.7: Three local meters as appearing in a local composition in pitch-onset space. Observe that we will not consider local meters within the given local composition if their onsets are not realized as projections of events to the onset axis.

only concerned with periodicity on the onset axis. But normal sounds do have lots of additional parameters to make them "audible". And these parameters are responsible for producing rich shapes which are beyond pure onset structures. In [453], Riemann describes rhythm as being related to a periodicity of higher level which means that a melodic movement may involve short onset differences and durations but can nonetheless produce a "rhythmical" period of larger extent. We want to give this idea a first conceptualization, a more general and powerful one will be added when morphisms between local compositions are available. To do so, observe that for a denotator $D : A \rightsquigarrow (Onset \times F)(D)$ in a product space of $Onset$ with a form $F$ and a period $p : 0_{\mathbb{Z}} \rightsquigarrow Duration(p)$, we have a shift $e^p \cdot D$ of $D$ by $e^p$ as above. In fact, since $A@(Onset \times F) \xrightarrow{\sim} A@Onset \times A@F$, we may let $e^p$ act on the onset component of $D$, leaving the $F$-component invariant, and this is it. Carrying over this action point by point to objective local compositions with ambient space $Onset \times F$, we obtain a shift operation on such local compositions.

**Definition 14** *Given an address $A$, let $p : 0_{\mathbb{Z}} \rightsquigarrow Duration(p)$ be a positive duration period and $-\infty \leq a \leq b \leq \infty$ two extended integers. Given a local composition germ $\hookrightarrow A@(Onset \times Para)$ in a product ambient space of onset and some additional "parameter" space $Para$. Then a (A-addressed)* local rhythm *is a local composition $R$ of ambient space $Onset \times Para$ of shape*

$$R = \bigcup_{t=a}^{b} e^{tp} \cdot germ. \tag{7.15}$$

Observe that now, period and length are no longer uniquely determined from the datum of $R$ alone. Intuitively speaking, a local rhythm is a local meter whose recurrent unit has been blown up to a whole set of points with additional parameters, see also figure 7.8.

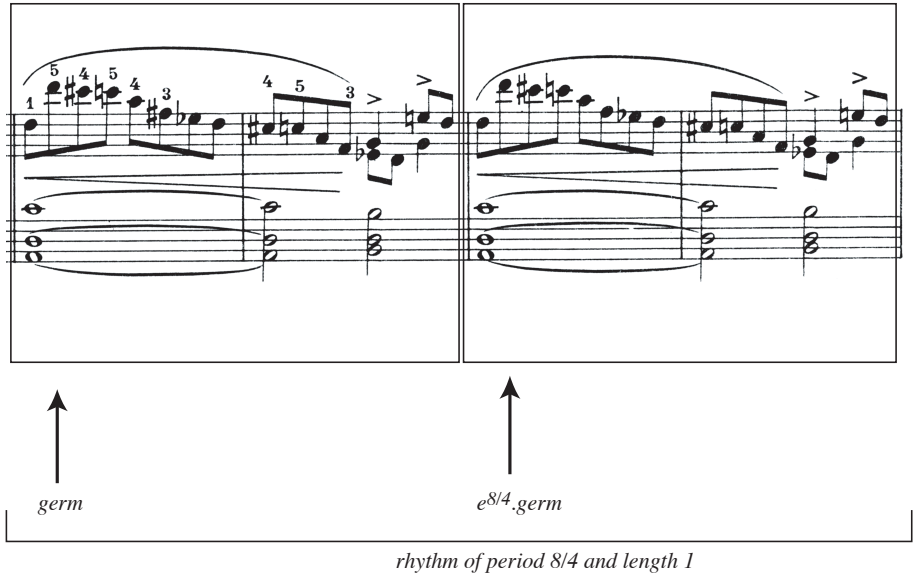


Figure 7.8: A local rhythm Beethoven's op.106, Allegro, bars Nr.75 to Nr.78. The parameters which are added to onset are numerous, including duration, loudness, legati, crescendi, and accents. The germ consists of the entire bars Nr. 75-76, period is $2 = 8/4$, and length is 1.

**Exercise 3** Give an exact description of the space *Para* in figure 7.8.

**Exercise 4** Define a metronome as a local rhythm with space $Para = Loudness$.

Evidently, the purely local approach to meter and rhythm is unsatisfactory since it is felt that real phenomena of "time grouping" are much more involved. Whereas we shall extensively develop the global perspectives and prove that this satisfies many of the present urges, it is already now possible to construct local "polyrhythms" from given collections of local rhythms, see section 7.4.

### 7.2.3 Motives

**Summary.** Motives are introduced as a type of local compositions which share (among others) pitch and onset time in their coordinators.

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In a rather generic setting we could define a motif as a local composition whose ambient space projects to a product of onset and pitch. Let us, however, be more precise in order to make the ideas transparent:

**Definition 15** *If $A$ is an address, a* (A-addressed) motif $M$ *is an objective local composition $M \hookrightarrow A@F$ whose ambient space $F = Onset \oplus Pitch \times Para$, for any form $Para$ of additional "parameters", and such that for any pair $m, n$ of different elements in $M$, projections $m_{Onset} = p_{Onset}(m)$, $n_{Onset}(n) = p_{Onset}$ to the onset space are different; $p_{Onset} : F \to Onset$ is the canonical projection. If the cardinality of a motif is $c$ we say that it is a $c$-motif.*

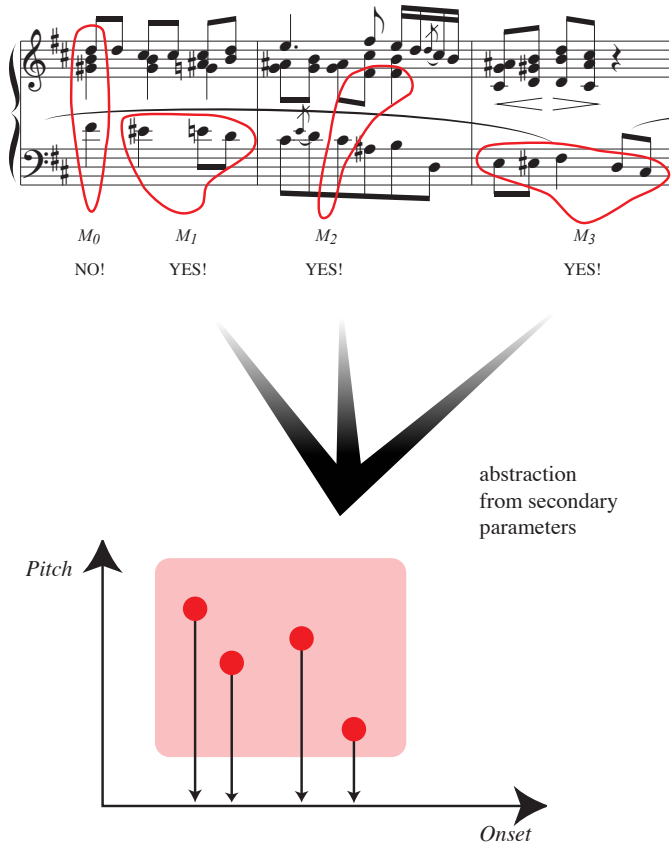


Figure 7.9: Three examples of motives: $M_1, M_2, M_3$, and one local composition, $M_0$, which is not a motif. Though these local compositions live in spaces with possibly many parameters, their relevant 'shape' for motivic analysis is centered around onset and pitch.

To see examples of motives, refer to figure 7.9. Notice that this concept is very restrictive in that it does not allow for motives which contain chords or several contrapuntal voices, say. Notice also that in contrast to local meters and rhythms, motives are not attributed by "local". The reason is that naming tradition views meters and rhythms as being something complex whereas motives are conceived as being "cells" of melodic structures. Whenever we want to make the

situation unambiguous, the attribute "local" is inserted.

Let us shortly comment on the address question here. Naively, we always think of zero address. So it is useful to study a less elementary address situation. If we take $A = \mathbb{Z}$, a motif at this address means a set of denotators $D : \mathbb{Z} \rightsquigarrow (D_{Onset}, D_{Pitch}, D_{Para})$ from which the onset coordinate is an affine homomorphism $D_{Onset} : \mathbb{Z} \to \mathbb{R}$. As insection 6.2.1, this denotator $D_{Onset}$ can be symbolized by an arrow $ON_D \Rightarrow OFF_D$ in *Onset* space, see also figure 7.10. The motif condition of separation of motif elements by their onset projection then means that for any two denotators $D$, $E$ of the motif, the arrows $ON_D \Rightarrow OFF_D$ and $ON_E \Rightarrow OFF_E$ are different, i.e. either their *'ON'* values or their *'OFF'* values are different.

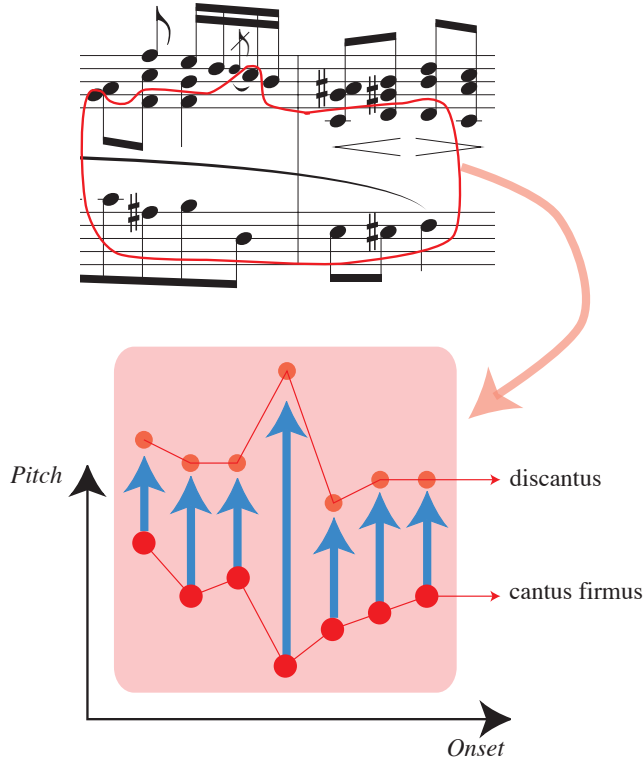


Figure 7.10: A motif at address $\mathbb{Z}$ can grasp contrapuntal structure of 'arrows' targetting from cantus firmus notes to discantus notes.

Musically speaking such a motif consists of 'intervals' of zero-addressed denotators such that no two such intervals intervene twice with same *'ON'* and *'OFF'* values. A prototypical case is a two-voice counterpoint with cantus firmus and discantus, the latter being added simultaneously to each note of the cantus firmus voice. In this case, a $\mathbb{Z}$-addressed denotator $D$ consists of the $ON$-denotator of the cantus firmus and the $OFF$-denotator of the discantus voice. They

differ in pitch but not in onset coordinates. Hence $ON_D \Rightarrow OFF_D$ is the 'zero'-length arrow $ON_D \Rightarrow ON_D$, and a motif just distinguishes its points by the $ON_D$-value.

## 7.3 Functorial Local Compositions

**Summary.** The generalized conceptual framework of non-objective, proper functorial local compositions is presented. It settles problems of universal constructions, such as fiber products of local compositions, which do not always exist in the frame of objective local compositions.

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Recall fromsection 6.3.1 that an objective denotator $D \hookrightarrow A@F$ at address $A$ in ambient space $F$ is identified with its functorial 'version' $\hat{D} \hookrightarrow @A \times F$. Conversely, we may associate to each functorial local composition $S \hookrightarrow @A \times F$ at address $A$ and in ambient space $F$ its *objective trace* $\check{S}$.

**Definition 16** *Let $S \hookrightarrow @A \times F$ be a local composition at address $A$ and in ambient space $F$. Its* objective trace *is defined as the following objective local composition:*

$$\check{S} = \{s \in A@F |\ (Id_A, s) \in A@S\} \tag{7.16}$$

*at address $A$ and in ambient space $F$. The* associated objectivized composition qua functor *will be defined and denoted by*

$$S^\diamond = \widehat{(\check{S})} \tag{7.17}$$

The following is evident:

**Sorite 1** *Let $S \hookrightarrow @A \times F$ be a local composition at address $A$ and in ambient space $F$. Then $S^\diamond \subset S$, and $S = S^\diamond$ iff $S$ is objective.*

We now want to make more explicit pictures of functorial local compositions in order to understand the specific difference to objective local compositions. To this end, let $f : B \to A$ be an address change and let $S \hookrightarrow @A \times F$ be a local composition at address $A$ and in ambient space $F$. Then we set

$$f@S = \{s \in B@S |\ s = (f, t)\}, \tag{7.18}$$

and call this set the *$f$-slice of $S$* whence

$$B@S = \coprod_{f:B\to A} f@S, \tag{7.19}$$

see figure 7.11 for the visual representation which we shall adopt henceforth. In particular, the notation of section 6.2.3 give

$$f@\hat{T} = \{f\} \times T \cdot f \tag{7.20}$$

for an objective local composition $T$ at address $A$ and address change $f : B \to A$. It is now evident that in general, local compositions are not objective.

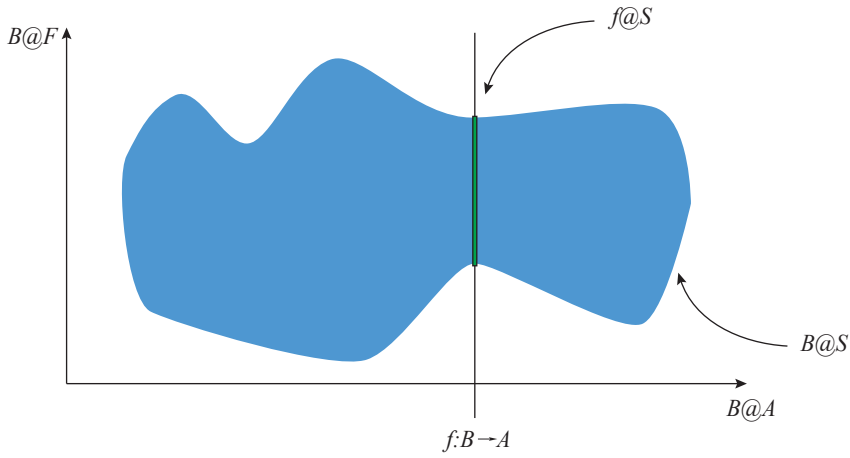


Figure 7.11: Graphical representation of a functorial local composition $S$ at address $A$, as evaluated at address $B$ with its 'slice' $f@S$ at address change $f : B \to A$.

For example, the full local composition $S = @A \times F$ gives the cartesian product $A@S = A@A \times A@F$ at address $A$, i.e. $f@S = \{f\} \times A@F$ for all $f : A \to A$ whereas $f@S^\diamond = \{f\} \times A@F \cdot f$. Take a simple form of functor $@M$ and the zero morphism $z : A \to A$. Then we have $z@S^\diamond = \{z\} \times A@M \cdot z = \{z\} \times e^M \cdot z$, i.e. the constant morphisms in $A@M$ which is much less then the full set $A@M$ of affine morphisms. Since full local compositions play an essential role in the theory of coordinate functions for musical parameters of global compositions—the deep question of realization of abstract compositional ideas, see chapters 15 and 19—we cannot refrain from non-objective constructs.

Other examples of limitations for objective local compositions will appear in questions of universal constructions such as fiber products, seesection 8.3.2. Since fiber products are fundamental tools of global theory and also in all questions of Grothendieck topologies, this is one further motivation to transgress the frame of objective local compositions which a decade ago were the core of mathematical music theory [340].

## 7.4 First Elements of Local Theory

**Summary.** The elementary concepts in local theory are discussed: cardinality, associated module of a local composition, dimension, general position of points, sub-compositions, Boolean constructs, products, coproducts, projections and injections.

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In the following discourse we deal with many different local compositions at a time and with their recombinations under specific operations. From the perspective of rigorous denotator

theory, one should always specify the names of forms where these denotators belong, and not only the names of the ambient spaces of these denotators. However, for the description of our recombination tools, this would be an 'overdressed' formalism. Rather would this hide what we are doing than clarify anything. Hence we shall not specify these names and just name denotators and ambient spaces. However, for a thoroughly unambiguous discourse, e.g. within an implementation of the theory in data base management systems (viz. the implementation for the RUBATO software described in chapter 40), the strict formalism must be followed. Also it is necessary in a conceptually full-fledged musicological context where names are essential.

**Heyting Constructions.** We fix an address $A$ and an ambient space $F$. It makes sense to invoke the Heyting algebra $Sub(@A \times F)$ of subfunctors[6] of $@A \times F$. On the other hand, we have the Boolean algebra $2^{A@F}$ of subsets of $A@F$. Here are some facts which relate the two algebras:

**Lemma 2** *With the preceding notation, for two objective local compositions $U, V \in 2^{A@F}$ and two functorial local compositions $D, E \in Sub(@A \times F)$ we have:*

(i) *If $U \subset V$ then $\hat{U} \subset \hat{V}$.*

(ii) *If $D \subset E$ then $\check{D} \subset \check{E}$.*

(iii) *If $D \subset E$ then $D^\diamond \subset E^\diamond$.*

(iv) *Union preserves objectivity, i.e. $\widehat{U \cup V} = \hat{U} \vee \hat{V}$.*

(v) *Union preserves associated objective local compositions, i.e. $(D \vee E)^\diamond = D^\diamond \vee E^\diamond$.*

(vi) *We have $\widehat{U \cap V} \subset \hat{U} \wedge \hat{V}$.*

(vii) *We have $\neg\hat{U} \subset \widehat{-U}$.*

(viii) *With respect to the Boolean and Heyting implications, respectively, we have*

$$\hat{U} \Rightarrow \hat{V} \subseteq \widehat{U \Rightarrow V}.$$

The proof is left as an exercise. Observe that negation, intersection, and implication behave in a rather complicated way. For example, $\widehat{(-\emptyset)}$ is objective while $\neg\hat{\emptyset} = @A \times F$ is not. Also, if $\hat{U} \wedge (A@F - U)\hat{}$ were objective, equal to $\hat{R}$, $R \subset A@F$, then we would have

$$Id_A@\hat{R} = Id_A@(\hat{U} \wedge (A@F - U)\hat{} = Id_A@\hat{U} \cap Id_A@(A@F - U)\hat{},$$

in other words

$$R = R \cdot Id_A = U \cdot Id_A \cap (A@F - U) \cdot Id_A = U \cap (A@F - U) = \emptyset.$$

But $\hat{U} \wedge (A@F - U)\hat{}$ is clearly not empty in general. This also shows that the inclusion in point (vi) above is strict in general. This is a very special case of a fiber product which is not objective, as announced above.

[6] See appendix G.5.1, proposition 113.

**Products, Coproducts, Projections and Restrictions.** If we are given two local compositions $S \hookrightarrow @A \times F$ and $T \hookrightarrow @B \times G$, the product functor[7] $S \times T$ can be identified with a local composition in the following way. Since limits and colimits of set-valued functors in $\mathbf{Mod}^{@}$ are calculated as corresponding constructs in **Sets**, for each address $A$, $S \times T$ is a subfunctor of $(@A \times F) \times (@B \times G)$. But we have a canonical isomorphism[8] of functors $@A \times @B \xrightarrow{\sim} @(A \times B)$. Therefore, $S \times T$ is isomorphic to a local composition in $@(A \times B) \times F \times G$. Observe that we take the new ambient space with the name $'F \times G'$, called the *product ambient space*, but observe that the new address is also the product $A \times B$ of the old ones. We shall henceforth take this local composition when speaking of the *product* $S \times T$ of local compositions. More concretely, if $f : C \to A \times B$ is an address change, the $f$-slice of $S \times T$ is calculated as follows. Let $f_A : C \to A$ and $f_B : C \to B$ be the two components of $f$. Then we have an isomorphism of sets

$$f@(S \times T) \xrightarrow{\sim} f_A@S \times f_B@T \tag{7.21}$$

which is functorial in the variable address $C$. This formula also defines two projections $p_S : S \times T \to S$ and $p_T : S \times T \to T$ induced by the first projections $p_A : A \times B \to A$, $p_F : F \times G \to F$ and the second projections $p_B : A \times B \to B$, $p_G : F \times G \to G$. In other words, we have two commutative diagrams of functors

$$\begin{array}{ccc} S \times T & \xrightarrow{\text{subfunctor}} & @(A \times B) \times F \times G \\ {\scriptstyle p_S}\downarrow & & \downarrow{\scriptstyle @p_A \times p_F} \\ S & \xrightarrow{\text{subfunctor}} & @A \times F \end{array} \tag{7.22}$$

$$\begin{array}{ccc} S \times T & \xrightarrow{\text{subfunctor}} & @(A \times B) \times F \times G \\ {\scriptstyle p_T}\downarrow & & \downarrow{\scriptstyle @p_B \times p_G} \\ T & \xrightarrow{\text{subfunctor}} & @B \times G \end{array} \tag{7.23}$$

which yield a model for general morphisms between local composition to be introduced in chapter 8.

Coproducts of local compositions $S$ and $T$ can be built in case they pertain to the same address $A = B$, i.e. $S \hookrightarrow @A \times F$ and $T \hookrightarrow @A \times G$. The coproduct functor[9] $S \amalg T$ is a subfunctor of $@A \times F \amalg @B \times G$ which in turn is canonically isomorphic to $@A \times (F \amalg G)$; as with the product we identify $S \amalg T$ with the corresponding local composition of $@A \times (F \amalg G)$, including the name $'F \amalg G'$ of the *coproduct ambient space*. We also have $f@(S \amalg T) \xrightarrow{\sim} f@S \amalg f@T$, an equation which is functorial in the domain $C$ of address change $f : C \to A$ and induced by the canonical injections $i_F : F \to F \amalg G$,

[7] See appendix G.2.1.
[8] See section E.3.8: take $M = 0$ and $C = 0$ in theorem 54.
[9] See appendix G.2.1.

$i_G : G \to F \amalg G$. We again deduce two commutative diagrams of functors:

$$\begin{array}{ccc} S & \xrightarrow{\text{subfunctor}} & @A \times F \\ {\scriptstyle i_S}\downarrow & & \downarrow{\scriptstyle Id \times i_F} \\ S \amalg T & \xrightarrow{\text{subfunctor}} & @A \times (F \amalg G) \end{array} \tag{7.24}$$

$$\begin{array}{ccc} T & \xrightarrow{\text{subfunctor}} & @A \times G \\ {\scriptstyle i_T}\downarrow & & \downarrow{\scriptstyle Id \times i_G} \\ S \amalg T & \xrightarrow{\text{subfunctor}} & @A \times (F \amalg G) \end{array} \tag{7.25}$$

which also anticipate the concept of morphisms between local compositions.

**Cardinality.** The cardinality of a local composition $S$ at address $A$ is defined by $card(Id_A @ S) = card(\check{S})$, and $S$ is called finite, infinite, etc. iff its cardinality is so.

**Module of a commutative local composition.** In general, the space functors of local compositions do not carry algebraic structures. But in the special and important case of simple ambient space $F$ with $Fun(F) \xrightarrow{\sim} @M$ for a module $M$ over a commutative ring $R$, we can say much more about objective local compositions $S$ which are addressed at $R$-module $A$ and their objective trace $\check{S}$ is contained in the subset $A@_R M$ of $R$-affine[10] morphisms $f : A \to M$. Such objective local compositions are called *commutative* (*over* $R$ if this specification is required).

Let us fix a commutative local composition $S$ over ring $R$, address $A$, and ambient space $@M$. Identify $S$ with its objective trace $\check{S}$. We then have a subset $S \subset A@_R M$ of the $R$-module[11] $A@_R M$.

**Lemma 3** *Let $(S, A@_R M)$ be a non-empty commutative local composition over ring $R$ and let $x, y \in S$ be two elements of $S$ (more precisely: of $\check{S}$). Then the sub-$R$-modules $\langle d_x \rangle$ and $\langle d_y \rangle$ of $A@_R M$ generated by the sets $d_x = \{s - x |\ s \in S\}$ and $d_y = \{s - y |\ s \in S\}$ of differences to $x$ and to $y$, respectively, are equal. Further, the non-zero elements of $d_x$ form a basis of $\langle d_x \rangle$ iff the non-zero elements of $d_y$ do so.*

Proof. We have $s - y = (s - x) - (y - x)$, so $\langle d_y \rangle \subset \langle d_x \rangle$, and a symmetric argument shows the other inclusion. Suppose that $\{s - x |\ s \in S, s \neq x\}$ is a set of linearly independent vectors, and let $\sum_{s \in S - \{y\}} \lambda_s (s - y) = 0$ be a vanishing linear combination. We then have

$$\begin{aligned} 0 &= \sum_{s \in S - \{y\}} \lambda_s (s - y) = \sum_{s \in S - \{y\}} \lambda_s ((s - x) - (y - x)) \\ &= -\lambda_x (y - x) + \sum_{s \in S - \{x,y\}} \lambda_s (s - x) - \Big( \sum_{s \in S - \{x,y\}} \lambda_s \Big) (y - x). \end{aligned}$$

[10] See section E.3 for notation.
[11] See sorite 13 in appendix E.3.

By linear independence of the elements in $\{s - x |\ s \in S, s \neq x\}$, we have $\lambda_s = 0$ for $s \in S - \{x, y\}$, and therefore also $\lambda_x = 0$. The symmetric argument yields the other implication. QED.

**Definition 17** *For a commutative local composition $(S, A@_R M)$ over ring $R$, an $R$-module $R.S$ is defined by*

$$R.S = \langle d_x \rangle \text{ with } d_x = \{s - x |\ s \in S\} \tag{7.26}$$

*for any element $x$ of $S$ if $S$ is non-empty, otherwise we set $R.\emptyset = \emptyset$. By lemma 3, this module*[12] *is well-defined. It is called* the module of $S$. *We say that $S$ is* generating *iff $R.S = A@_R M$, and that $S$ is* embedded *iff $S \subset R.S$.*

*If for a non-empty $S$, $d_x - \{0\}$ forms a basis of $R.S$ we say that the points of $S$ are* in general position[13]. *Again, by lemma 3, this statement is independent of the chosen element $x$.*

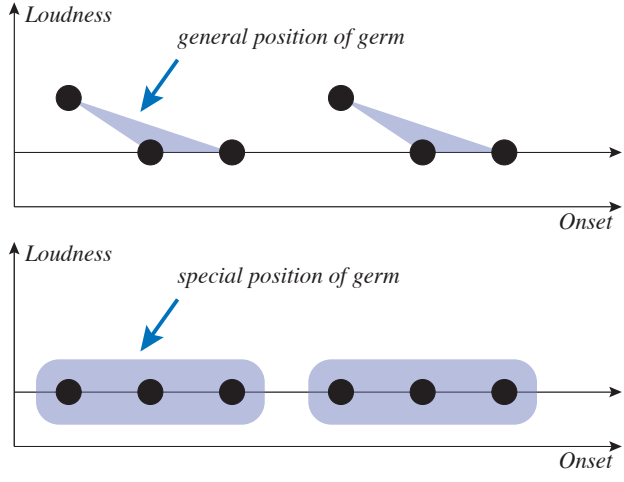


Figure 7.12: Below, we have a germ of a local rhythm in *Onset* $\oplus$ *Loudness* space. Since the germ's elements are in special position, in fact on a horizontal line, the germ is no more retractable from the local rhythm. Above, we see the new germ with first event in higher loudness, and thus giving general position and evidence of the germ, together with the rhythm's period.

**Sorite 2** *If $S$ is a finite commutative local composition, its points are in general position iff its module is free of rank $card(S) - 1$.*

[12] Recall that in the category **Mod** the empty module is admitted, see appendix E.3.
[13] Including the zero-dimensional case $d_x = \{0\}$ and $R.S = 0$.

Proof. The condition is necessary by definition. Conversely, the characterization of projective finitely-generated modules in appendix F.2, theorem 58, guarantees that the points are in general position if the module $R.S$ is free of rank $card(S) - 1$. QED.

Local commutative compositions with points in general position play a fundamental role in classification theory and for understanding performance strategies. We should therefore give one preliminary illustration to make evident the musical meaning of general position. Refer to figure 7.12 for the following comments. Let us consider the zero-addressed local three-element composition $S = \{(0, 10), (1, 10), (2, 10)\}$ in ambient space $Onset \oplus Loudness$. Evidently, $\mathbb{R}.S = \mathbb{R}.(1, 0)$, and the points of $S$ are not in general position but aligned on a horizontal straight line of height 10. Musically, we have to play a regularly spaced sequence of three notes of equal loudness. If we use this local composition as a germ for a local rhythm with period 3, this yields a uniform sequence of events. However, if we want to stress the period and play the first event of this germ louder than the others, at value 18 say, we obtain a germ $S^* = \{(0, 18), (1, 10), (2, 10)\}$. Here, we have $\mathbb{R}.S = \mathbb{R}.(1, 0) \oplus \mathbb{R}.(1, -8)$, and this is a general position situation. Musically, putting the germ events into general position makes a hidden (purely mental) idea—in this case: the periodicity and the germ—evident, i.e., audible.

## 7.5 Alterations Are Tangents

**Summary.** By use of dual numbers, alterations of tones in pitch or any other coordinate are interpreted as 'tangent' objects. Associated transformations between tangents and base-points are introduced.

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The technique of alterated notes is more than a notational formalism. For example, take a piano note denoted by "e-sharp". On a well-tempered piano, the key it denotes is just f, and no deeper reason exists to write "e-sharp" instead of "f" to define a particular key and pitch. But for variable temperament such as is present on string instruments, this cannot be done. Whatever the concrete meaning of such alterations, they do point to a refined situation: We have to keep in mind two notes: the origin and the amount of shift from the origin, to speak geometrically.

From appendix E.3, (E.5), we know that any restriction $r : R \to S$ of commutative scalars gives rise to a functor $S \otimes_R ? : \mathbf{Mod}_R \to \mathbf{Mod}_S : N \mapsto S \otimes_R N$ which carries an $R$-affine transformation $f = e^k \cdot f_0 : N \to K$ to the $S$-affine $S \otimes_R f = e^{1 \otimes k} \cdot S \otimes_R f_0$. To describe alterations we consider the dual number restrictions

$$i : R \to R[\varepsilon] : r \mapsto r + 0\epsilon$$
$$p : R[\varepsilon] \to R : a + b\varepsilon \mapsto a$$

and set $M[\varepsilon] = R[\varepsilon] \otimes_R M$ and $f[\varepsilon] = R[\varepsilon] \otimes_R f$ for corresponding functor values on objects and morphisms in $\mathbf{Mod}_R$. Recall from appendix E.2.2 that over $R$, $M[\varepsilon] = M + \varepsilon M \xrightarrow{\sim} M^2$ such that scalar multiplication on dual numbers is given by $\varepsilon(a, b) = (0, a)$.

In this section we concentrate on (not necessarily objective) $A$-addressed local compositions in simple ambient spaces $@M$, $M$ modules over commutative rings. Let $T \hookrightarrow @A \times @M$

be such an object for the ring $R$. Call $T_\varepsilon$ the image of $T$ under the embedding $Id \times @i : @A \times @M \rightarrowtail @A \times @M[\varepsilon]$ such that we obtain a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\text{subfunctor}} & @A \times @M \\ {\scriptstyle i_T}\downarrow & & \downarrow{\scriptstyle Id\times @i} \\ T_\varepsilon & \xrightarrow{\text{subfunctor}} & @A \times @M[\varepsilon] \end{array} \tag{7.27}$$

of functors. Conversely, let $T \hookrightarrow @A \times @M[\varepsilon]$, and call $T_{red}$ the image of $T$ under the projection $Id \times @p : @A \times @M[\varepsilon] \twoheadrightarrow @A \times @M$. Then we have a corresponding commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\text{subfunctor}} & @A \times @M[\varepsilon] \\ {\scriptstyle p_T}\downarrow & & \downarrow{\scriptstyle Id\times @p} \\ T_{red} & \xrightarrow{\text{subfunctor}} & @A \times @M \end{array} \tag{7.28}$$

of functors; both diagrams remind us of the previous diagrams which we have encountered in (7.22) and (7.23). Evidently, there is need for more systematic treatment of such relations among local compositions, this will be done in chapter 8.

If we suppose that the elements of a local composition with ambient space $@M$ are musically reasonable entities, this is no longer the case with *dual ambient space* $@M[\varepsilon]$. We view this artifact as follows. If $x : A \to M[\varepsilon]$ is a denotator in this ambient space, it can be written as $x = x_{red} + \varepsilon x_{alt}$ with two components $x_{red}, x_{alt} : A \to M$. Musically speaking, the *reduced* component $x_{red}$ is the meaningful musical event whereas the *alteration* component $x_{alt}$ symbolizes a shift from the reduced value. This suggests that we view alterations as tangents[14]. If, for example, $M = Pitch$, we may view $x$ as being a 'real' pitch $x_{red}$ plus an 'pitch alteration quantity' $x_{alt}$. For instance, if pitch is codified such that a semitone corresponds to real number 1, and 60 to middle $C$, then the zero-addressed $x = 60 - \varepsilon 2$ codifies $C_{\flat\flat}$.

To recover 'real music events' from this formalism, consider two $R$-linear projections $\alpha_+, \alpha_- : M[\varepsilon] \to M$, defined by $\alpha_+(x) = x_{red} + x_{alt}$ and $\alpha_-(x) = x_{red} - x_{alt}$, respectively. Clearly, in the above example, $\alpha_+$ gives the altered pitch number $58 = \alpha_+(60 - \varepsilon 2)$ from the dual representation, whereas $\alpha_-$ gives 62.

**Definition 18** *Transformation $\alpha_+$ and the corresponding natural transformations $@\alpha_+$ is called* sweeping orientation*; the transformation $\alpha_-$ and the corresponding natural transformation $@\alpha_-$ is called* hanging orientation.

These concepts will become central in the counterpoint theory of part VII. Again, we have corresponding commutative diagrams on $A$-addressed local compositions $T$ in dual ambient space $M[\varepsilon]$:

$$\begin{array}{ccc} T & \xrightarrow{\text{subfunctor}} & @A \times @M[\varepsilon] \\ \downarrow & & \downarrow{\scriptstyle Id\times @\alpha_\pm} \\ \alpha_\pm T & \xrightarrow{\text{subfunctor}} & @A \times @M \end{array} \tag{7.29}$$

[14] In algebraic geometry, points in dual modules $M[\varepsilon]$ are effectively identified with Zariski tangents, see appendix F.4.1 and [386, p.25].

with $\alpha_{\pm}T$ being the image of $T$ under the 'ambient morphism' $Id \times @\alpha_{\pm}$. The consideration of both, positive and negative alterations is a well known—though not very frequent—tool to produce two alterated pitches from one 'abstract' datum, i.e. viewing simultaneously pitch 65 as $\alpha_{-}(66 + \varepsilon 1)$, and pitch 67 as $\alpha_{+}(66 + \varepsilon 1)$, a so-called "di-alteration".

### 7.5.1 The Theorem of Mason–Mazzola

**Summary.** The use of alterated pitch is a standard argument in music analysis to associate neighboring tones. The theorem of Mason deals with the richness of possible alterations within 12-chromatic pitch classes between determined scales; Mason's theorem is generalized to n-chromatic classes. Musicologically, the theorem gives information about the significance or insignificance of alteration arguments in traditional analysis.

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In classical music theory, alterations are very frequent. Most chords are loaded with alteration signs, thereby reinterpreting given pitch as an alteration of a different 'pitch origin'. This is often inevitable since in real scores, very rarely do we find the pitch arsenal within a determined diatonic scale. But it is nevertheless "clear" that one essentially stays in a given tonality, as represented by the basic diatonic scale. The problem of such an approach is that reinterpreting any given set of pitch values as being the result of a specific alteration process provokes a huge amount of polysemy. In the worst case, one could argue that this technique may be abused to 'demonstrate' anything by construction of appropriate alterations. The following results deal with this problem and give a general answer to the question of how broad an alteration process can become in terms of ambiguities. Our results generalize the theorem of Robert M. Mason [321] concerning alteration with 12-tempered pitch classes.

In the following discourse we shall only deal with commutative local compositions in ambient spaces $@M$ of modules $M$ over a ring $R$ and the associated dual version. *Commutative local compositions of ambient space $M[\varepsilon]$ are also called* alterations.

**Definition 19** *Let $T$ be an $A$-addressed alteration of ambient space $M[\varepsilon]$, and let $0 \neq m \in A@_R M$. Then $T$ is called $m$-elementary iff it contains one element $k = k_{red} + \varepsilon m$ such that $T - \{k\} = (T - \{k\})_{red}$ and $T - \{k\} \cap \{\alpha_{+}(k), p(k)\} = \emptyset$.*

This means that there is a bijection $T_{red} \xrightarrow{\sim} \alpha_{+}T$ which is induced by two bijective projections $T \xrightarrow{\sim} T_{red}$, and $T \xrightarrow{\sim} \alpha_{+}T$, see also figure 7.13. This bijection consists in an $m$-shift of one element $p(k)$ to $\alpha_{+}(k) = m + p(k)$, whereas the other elements remain fixed. In particular, if $M = R$ and $m = e^{\pm 1}.0$ (constant shift), we say that $T$ is *elementary.*

**Definition 20** *If $E$ and $F$ are two local compositions such that there is an elementary alteration $T$ over ring $R$ with $E = T_{red}$ and $F = \alpha_{+}T$, we write $E \rightsquigarrow F$ and say that $F$* is an elementary shift of $E$. *If there is a sequence $E_0 = E \rightsquigarrow E_1, E_1 \rightsquigarrow E_2, \ldots E_{i-1} \rightsquigarrow E_i = F$ of elementary shifts, we again write $E \rightsquigarrow F$ and say that $F$* is a shift of $E$.

If we have an elementary shift $E \rightsquigarrow F$, we also have an *associated bijection* $E \xrightarrow{\sim} F$. The following is then immediate[15]:

[15] Here, the summand $e^{\mathbb{Z}1_R}.0$ relates to the constant shift morphisms $e^{t.1_R}.0 : A \to R, t \in \mathbb{Z}$.

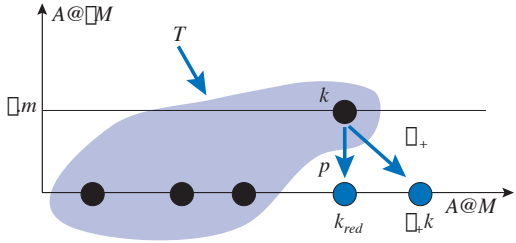


Figure 7.13: An $m$-elementary alteration $T$ controls a shift between its reduced and its alterated images.

**Sorite 3** *If there is a shift $E \rightsquigarrow F$, there is a unique alteration $T$ in $A@R + \varepsilon e^{\mathbb{Z}1_R}.0$ such that the bijection $E \xrightarrow{\sim} F$ associated with the shift is defined by two bijections $T \xrightarrow{\sim} T_{red} = E$ and $T \xrightarrow{\sim} \alpha_+ T = F$.*

The following results which were solicited by Mason's investigation [321] show that the existence of shifts is a strong condition on couples of *equipollent*[16] local compositions.

**Theorem 1** *Let $n$ be a positive natural number, and let $E, F$ be two equipollent zero-addressed local compositions in $\mathbb{Z}_n$ with cardinality $m$. Then there are two non-negative integers $k, l$ with $k + l = n - m$ and a shift $E \rightsquigarrow F$ which is induced by an alteration $K \subset \mathbb{Z}_n + \varepsilon[-l, k]$, where $[-l, k] = \{-l, -l + 1, \ldots k - 1, k\}$. We can even choose $K$ as being defined by a sequence of* elementary *shifts $E_i \rightsquigarrow E_{i+1}$ which all satisfy the condition imposed on $K$.*

See figure 7.14 for an illustration. From the proof of theorem 1 we will also obtain Mason's result:

**Proposition 2** (Mason, [321]) *For $n = 12$ and $7 \leq m$, we can choose $|k - l| \leq 1$ in the above theorem.*

For classical European music, the special case $m = 7$ yields $k = 3, l = 2$ or $k = 2, l = 3$. This means that any scale with seven-element class chord $F$ can be shifted to any other equipollent scale with class chord $E$ by an alteration $K$ whose elements have at most three sharps and two flats or at most two sharps and three flats. In particular, any class chord $Ch$ with $card(Ch) \leq 7$ is the shift of another equipollent class chord in any fixed diatonic scale. This fact is very important to judge musicological argumentations built upon alterations. It could happen that reinterpreting a chord as a shift of another, distinguished, chord (such as one sitting in a particular scale as above) is always possible and therefore does not mean anything to the actual statement!

For the proof of theorem 1 we need some notation. Let $k, l$ be as in the theorem, and let $x \in \mathbb{Z}_n$. Write $[l|x|k] = e^x \cdot [-l, k]$ for the $x$-shift of the interval $[-l, k]$ as defined in the theorem.

[16] Equipollence means that they have equal cardinality.

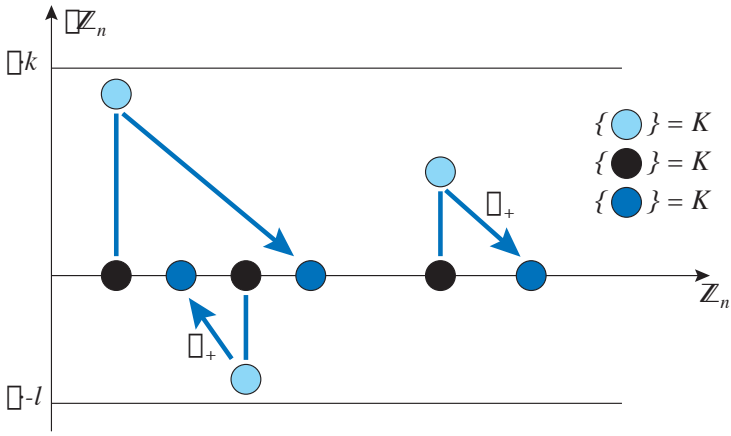


Figure 7.14: The condition imposed on on alteration $K$ in theorem 1.

If $k, l$ are clear we also write $x_+ = x + k$ and $x_- = x - l$, whence $[l|x|k] = [x_-, x_+]$. Since $k + l = n - m < n$, $[l|x|k]$ has exactly $n - m + 1$ elements, and it makes sense to say that an element $y \in [l|x|k]$ is to the right or to the left of $x$ according to whether $y$ appears before of after $x$ in the sequence

$$x_-, x_- + 1, \dots x, \dots x_+ - 1, x_+.$$

Consider now the 'second' $\sigma = \{y, y + 1\}$. If $y$ stays to the left of $x$, we write $x_-|\sigma|x$, if $y + 1$ stays to the right of $x$ we write $x|\sigma|x_+$.

**Lemma 4** *Let $E \rightsquigarrow F$ be a shift whose alteration $K$ verifies the theorem's conditions with $k, l$ and let $s \in E$.*

(i) *If $\sigma$ is a 'second' such that $\alpha_+ p^{-1} s = s_-$ and $s_-|\sigma|s$, then there is no $t \in E$ such that $\alpha_+ p^{-1} t = t_+$ and $t|\sigma|t_+$.*

(ii) *If $\sigma$ is a 'second' such that $\alpha_+ p^{-1} s = s_+$ and $s|\sigma|s_+$, then there is no $t \in E$ such that $\alpha_+ p^{-1} t = t_-$ and $t_-|\sigma|t$.*

Proof of lemma. The proof scheme is the same for both cases, and we may restrict to case (i). Look at the subset $[t_-, t_+] \times [s_-, s_+]$ of $\mathbb{Z}_n \times \mathbb{Z}_n$ and refer to figure 7.15. The diagonal of $\mathbb{Z}_n \times \mathbb{Z}_n$ is denoted by $\Delta$. If we had a $t$, which is forbidden by statement (i) of the lemma, then evidently $(t, s) \notin \Delta$. Under elementary shifts the couple $(t, s)$ is shifted one unit in horizontal or vertical direction within $[t_-, t_+] \times [s_-, s_+]$, but it cannot hit $\Delta$. Since according to our premises, the couple $(t, s)$ stays to the left of $\Delta$, it cannot reach the bottom right edge $(t_+, s_-)$. QED.

Proof of theorem. We make an induction on $n - m$ and observe that the cases $n - m = 0, 1, 2, 3, 4$ are easily verified under the stronger hypothesis $|k - l| \leq 1$. So suppose $5 \leq n - m$. Suppose first that $E \cup F \subsetneq \mathbb{Z}_n$, and then WLOG[17] we can suppose that $z = \overline{n-1} \notin E \cup F$.

[17] Mathematicians' short form of "without loss of generality'.

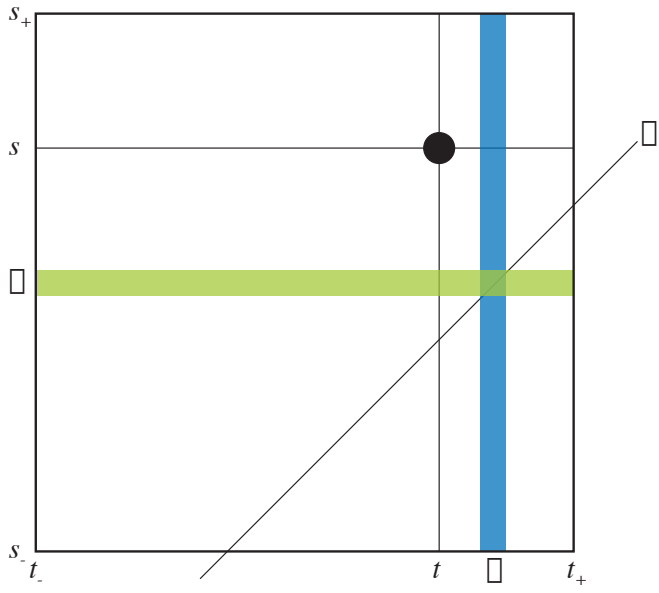


Figure 7.15: The product set $[t_-, t_+] \times [s_-, s_+]$ and the relative position of $(t, s)$ with respect to the diagonal $\Delta$ and the 'second' diagonal $(\sigma, \sigma)$.

We may therefore suppose that $E, F \subset \mathbb{Z}_{n-1}$ by contracting the 'second' $\{\overline{n-1}, \overline{0}\}$ to point $\overline{0} \in \mathbb{Z}$ and creating the 'second' $\sigma = \{\overline{n-2}, \overline{0}\}$ in $\mathbb{Z}_{n-1}$. By induction hypothesis, there are non-negative integers $k', l'$ with $k' + l' = n - m - 1$ and a sequence $E_0 = E \rightsquigarrow E'_1, E'_1 \rightsquigarrow E'_2, \dots E'_{i-1} \rightsquigarrow E'_i = F$ of elementary shifts in $\mathbb{Z}_{n-1}$ which shift every element $s \in E$ within $[s_-, s_+]$. According to lemma 4, we cannot have simultaneously a shift of $s \in E$ to $s_-$ and a shift of $t \in E$ to $t_+$ with $s_-|\sigma|s$ and $t|\sigma|t_+$. At worst, either some $s$ are shifted to $s_-$ with $s_-|\sigma|s$ or some $t$ are shifted to $t_+$ with $t|\sigma|t_+$. After reinsertion of $z$ we have a required shift with $k = k', l = l' + 1$ in the first case; in the second case we have a shift with $k = k' + 1, l = l'$, and we are done. The case $E \cup F = \mathbb{Z}_{n-1}$ is a bit more involved. Take a point $z \in \mathbb{Z}_n$ where $z - 1 \in F$ and $z \in E - F$. Suppose that here, we have $z, z + 1, \dots z' \in E$ whereas $z' + 1 \notin E$.

To begin with, we shift $z, z + 1, \dots z'$ by one unit to the right and obtain $\tilde{E}$. Let $\sigma = \{z - 1, z + 1\}$ be the 'second' in $\mathbb{Z}_{n-1}$ as above from eliminating $z$. By induction hypothesis, there are $0 \leq k', l'$ with $k' + l' = n - m - 1$ and a sequence of elementary shifts starting from $\tilde{E}$ and ending in $F$, which shift the elements $s \in \tilde{E}$ within $[l'|s|k']$. The point here is to understand what happens to the shift quantities if we reinsert $z$ 'in the middle of' $\sigma$. If $s \in \tilde{E}$, we denote the points $s_-, s_+$ with respect to $k', l'$. There are six cases:

1. Point $z$ is unshifted and neither $s_-|\sigma|s$ nor $s|\sigma|s_+$.

2. Point $z$ is shifted and neither $s_-|\sigma|s$ nor $s|\sigma|s_+$.

3. Point $z$ is unshifted and $s_-|\sigma|s$.

4. Point $z$ is unshifted and $s|\sigma|s_+$.
5. Point $z$ is shifted and $s_-|\sigma|s$.
6. Point $z$ is shifted and $s|\sigma|s_+$.

The last possibility cannot happen since $s-(z+1)+k' \leq m-1+n-m-1 = n-2$, and therefore, $\sigma$ cannot be to the right of $s$. We shall do the following: Whenever an element $s$ is shifted to jump from $z-1$ to $z+1$ in $\sigma$ or conversely within one of the existing elementary shifts, we shall replace this elementary shift by the succession $z-1 \rightsquigarrow z \rightsquigarrow z+1$ or $z+1 \rightsquigarrow z \rightsquigarrow z-1$ of two elementary shifts. So we are left with the question of what happens to the possible total shift intervals. *After reinsertion* of $z$ there is no change in the shift intervals in cases 1. and 2. since $z$ does not intervene in those intervals. Set $\tau = \{z, z+1\}$. In case 3. we get $(s-(l'+1))|\tau|s$, in case 4. we have $s|\tau|(s+(k'+1))$, and in case 5. we have $(s-(l'+1))|\tau|s$. If we go back to the *original* positions, we have these total shift intervals

1. $[l'|s|k']$
2. $[l'-1|s|k'+1]$
3. $[l'+1|s|k']$
4. $[l'|s|k'+1]$
5. $[l'|s|k']$

for (unshifted) elements $s \in E$ in the five possible cases. It remains to be shown that case 3. versus cases 2. and 4. are mutually exclusive. But this follows rightly from lemma 4. QED.

**Remark 2** It is not known whether the choice $|k-l| \leq 1$ is always possible.

# Chapter 8

# Symmetries and Morphisms

*Hoffentlich habe ich diejenigen beschwichtigt,*
*die ein gewisses Unbehagen vor dem Worte Symmetrie,*
*das zu phantasievollen Spekulationen Anlaß gegeben hat,*
*nicht verbergen können, die darin Formalismus, Schematisums sehen und es*
*am liebsten ganz aus der 'göttlichen' Vernunft verbannen möchten.*
Wolfgang Graeser [194]

**Summary.** After having introduced the objects of local theory, their relations are discussed and formalized. Thereby, symmetry is a key concept—however in its modern version which by far exceeds traditional axial or rotational symmetries. The core process of symmetry leads to structure-conserving transformations which we rebuild in the context of local compositions and the underlying forms.

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The basically 'rigid' structure of a local composition as it was discussed in section 7.1 requires an additional ingredient of conceptualization to be able to compare such 'rigid' objects. Such a concept of comparison must then conserve the interior correspondences among parts of the local compositions, it must be a concept of "morphological" comparison which is able to grasp these correspondences as expressions of underlying gestalts.

Such a type of comparative paradigm leads to the concept of *symmetry* as it is used in its modern form [574] and may be rephrased[1] more generically as follows:

**Definition 21** *Symmetry is correspondence of parts as an expression of a whole.*

On one hand, this definition by far exceeds simple axial symmetry, on the other it recognizes the semantical function of symmetry and is no longer limited to formal aspects. The modern concept of symmetry is in fact a semiotical one: Symmetry bears the character of a sign. The *form* (the significant) of a symmetry is a mathematical transformation which specifies

[1] Rudolf Wille's definition in [574, p.444] is this: "Symmetrie ist Gleichheit von Teilen als Ausdruck eines Ganzen."

the correspondence, whereas the function (signification) of a symmetry consists of a production mode of meaning (the significate) from the relation between meanings (significates) of the form's parts (expressions) which are transformed into each other under the given transformation. For example, the dynamical symmetry of a ropedancer produces an equilibrium of his body which prevents him from crashing down and risking his life. The physical symmetry of forces means life to the ropedancer.

We insist on replacing the word "Gleichheit" (equality) in Wille's definition by "correspondence" in the above definition since there is no a priori replication under correspondence, and this is essential to avoid the widespread belief that symmetry is an 'equalizer' concept. The concept of symmetry does in fact *not* imply equality of symmetrically related parts, only correspondence! In a symmetry of moral qualities, for example, good and evil are not equalized, but they do correspond.

We also feel obliged to contradict the stubborn simplistic argument against symmetries in fine arts, literature, and music which debates that symmetries would cast the artistic freedom of shaping highly sophisticated works of human expression into a trivial or even fascistoid Procrustean bed and thereby fail to explain the very essence of artistic expression. The point is that

- First, *symmetries are not the only instance* of artistic expression, local-global strategies are at least as important (see also chapter 13.1 for this subject).

- Second, symmetries often appear as broken or *hidden symmetries* which means that they are not immediately visible, and only appear on the phenomenological surface when investigated with powerful tools. In elementary particle physics, this fact is well known and—hélas—responsible for exorbitant costs of particle accelerators: Fundamental laws of physics are virtually always driven by hidden symmetries which are only visible under extremely high energies.

- Third, the stratification of symmetries into form and content initiates *complex relations of form and content*. For example, Kepler's second law states that a planet sweeps out area at a constant rate[2]. Hence it is not the distance between a planet and the sun which remains constant but the area which is swept out by the straight line connecting planet and sun. The visible symmetry of a "perfect" circle motion is replaced by a more abstract symmetry where "distance" is replaced by "area per time unit". The meaning of planetary circle motion is lifted to a more involved physical concept, area per time unit relates to angular momentum[3].

As a final argument for using non-trivial symmetries in music, we should admit that deep musical facts cannot be understood by use of trivial tools, this would be as absurd as to develop physical theories without infinitesimal calculus. If the great composers and musicians participate in a sphere of high or even divine sophistication it is only logical to approach their works by use of adequate, equally skilled tools of human insight. We therefore hope that the reader will not reject conceptually involved investigations because of a conflicting belief that there should be easy ways to understand the genius in music.

[2] See [307, p.524].
[3] It is the law $\mathbf{x} \wedge \mathbf{p} = const.$ of conservation of the angular momentum but see [307, p.523]

## 8.1 Symmetries in Music

**Summary.** This introductory section reviews the multiple and sometimes controversial presence of symmetry in music. We give a short historical account of the quasi-simultaneous appearance of explicit symmetry paradigms in music and mathematics in the works of Schönberg, Graeser, and Noether around 1928. The discussion is complemented by reference to Jakobson's poetic function which reveals a dominant role of symmetries in every poetic effort.

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A classical "counterexample" against symmetries in music is the retrograde of a melody. In music psychology it is traditionally argued that the retrograde (see example 9 below) is not recognized (heard) as being essentially the "same" underlying melody. Such a claim fails to grasp the very concept of symmetry which in this case only refers to a *privileged relation* between the original and the retrograde version. The question would rather be whether the listener feels a privileged relation between melody and its retrograde, as compared to any random melody.

This example reveals the essential role of the *topographic* approach to symmetry in music in the sense of section 2.4. Questioning the role of symmetries in music must always begin with the topographic initial "Where?", otherwise the effort is futile. Whatever its vagueness, the problem whether symmetries can be "heard" is above all an esthesic one and has—to our knowledge—not been investigated systematically in music psychology. On the other hand, from the neutral and poietic point of view, symmetries play a prominent role. We come back to this aspect in chapter 11. Let us just remark here that first, symmetries have been applied (among many others) from Johann Sebastian Bach in fugal composition to Pierre Boulez in serial techniques. Following music theorist Wolfgang Graeser we may further ask whether the incessant structural fascination of Bach's music cannot be explained from a network of locally present symmetries. Second, neutral signification of symmetries is explicitly given in Arthur von Oettingen's [406] and in Sigfrid Karg-Elert's [259] modulation theories. Nowadays, symmetries are widely used in musical composition, for instance in the various transformation tools of sequencers, and explicitly in the author's *presto*®(see chapter 49) and in Opcode's MAX or in music software with fractal[4] composition tools as realized by Mesias Maiguashca [220] or by Peter Stone on his ingenious "Symbolic Composer" software, see figure 8.1.

From the historical point of view there is an interesting coincidence of musicological and mathematical development regarding symmetries. Around 1924–1929, the paradigm of symmetries grew into its first order prominence in music as well as in mathematics. In 1928, Arnold Schönberg's (figure 8.3) "Orchestervariationen" op. 31 had its premiere, realizing the thoroughly symmetric dodecaphonic composition principle (see example 12 below). It is very likely the first time in the history of western music that an entire mathematical group orbit (the orbit of a given series under the full contrapuntal affine group generated by inversion, retrograde, and transposition (see subsection 8.1.1, example 12, for precise definitions) of a kernel motif (the composition's fundamental series) was *consciously* designed to define a motivic paradigm. In music analysis, it was the aforementioned young Wolfgang Graeser (figure 8.2) who in 1924 applied symmetries with great success to Johann Sebastian Bach's "Kunst der Fuge" [194]. Here is his summary:

[4]Observe that fractals are generated by symmetries in the sense of iterated application of one fixed transformation.

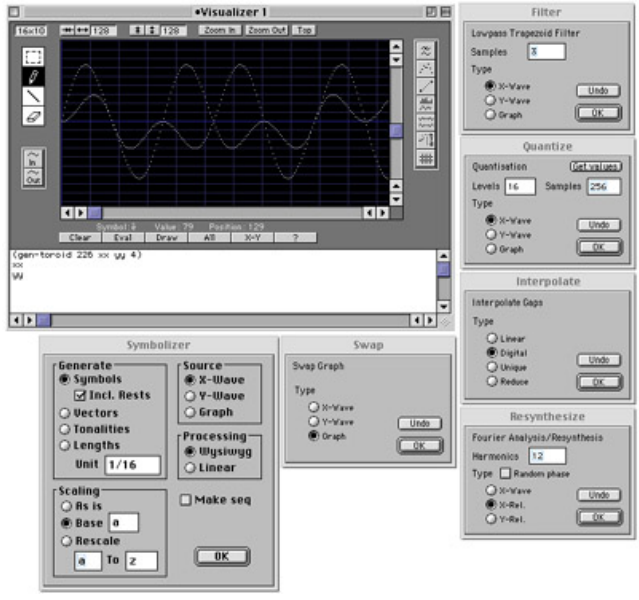


Figure 8.1: Peter Stone's "Symbolic Composer" [511] software includes a large number of composition operators, in particular for fractal techniques.

> *Die Eigenschaft der Symmetrie spielt in der Musik eine so ungeheure Rolle, daß sie verdient, an erster Stelle betrachtet zu werden. Wir werden in der "Kunst der Fuge" ihre fast uneingeschränkte Herrschaft besonders deutlich erkennen.*

In modern abstract algebra, it was Emmy Noether who accomplished a persuasive theory of general symmetries, i.e., the theory of group representations in vector spaces, in the paper "Hyperkomplexe Größen und Darstellungstheorie", written in 1929 [398]. Needless to say that symmetries play a crucial role in contemporary mathematics and therefore in mathematical physics.

Let us shortly digress on poesy whose structural substrate has a strong musical character. Here, symmetries play a prominent role, and this is remarkable since conversely, the poetical moment in music—as it is mediated by meter and rhythm—historically grew from the linguistic poetical form, and since by these facts the apparently unpoetical nature of symmetries is defeated. In his famous statement, Roman Jakobson has defined poeticity [245]:

> *The poetical function projects the principle of equivalence from the axis of selection to the axis of combination.*

This means that associations between concepts in poesy are controlled by strict correspondence within a rhythmically segmented thread of language. According to Jakobson, the symmetries which appear in these configurations of correspondences are a dominant means of poetical structuration [248]. In its study of classics such as Friedrich Hölderlin [247], Dante Alighieri [207] or François Villon [312], modern poetology has indeed discovered an astonishing density and

Figure 8.2: Wolfgang Graeser (1906–1928) was the first music theorist to apply systematically symmetry groups to music analysis.

Figure 8.3: Arnold Schönberg (1874–1951) was the first composer who consciously applied full orbits of symmetry groups to musical composition.

even dominance of symmetrical constructions; we come back to Jakobson's poetical function in subsection 11.6.1.—Also in traditional European musicological analysis, symmetries have been recognized with growing success [395], see section 11.5.2 for an extensive discussion in symmetry-oriented musicological analysis as cultivated in the American tradition of "musical set theory".

### 8.1.1 Elementary Examples

**Summary.** Before starting the general theory of morphisms between local compositions, which encompasses a large portion of what can be thought of being a symmetry phenomenon in music, we should include a number of elementary examples of symmetries in music.

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A word of caution: In the course of these examples we use words such as symmetry, transformation etc. in an informal way. Only in section 8.2 will we introduce the corresponding technical terms.

**Example 7** *Transposition.*
The symmetry of transposition is fundamental for thinking in terms of pitch classes. More precisely, we are given a "translation vector" $t \in \mathbb{Q}^3$ which defines a translation transformation $e^t : \mathbb{Q}^3 \to \mathbb{Q}^3$ and therefore a transformation of functors $@e^t : @\mathbb{Q}^3 \to @\mathbb{Q}^3$ which transforms any EulerModule-denotator $x : A \rightsquigarrow EulerModule(x)$ into its $t$-translate $e^t(x) : A \rightsquigarrow EulerModule(e^t \cdot x)$. If $x$ is even a zero-addressed point in the *EulerModule* space, it may be identified with an element $x \in \mathbb{Q}^3$, and we have $e^t \cdot x = t + x$. Applying all translates $e^{no}, n \in \mathbb{Z}$ of integer multiples of the octave $o$ to $x$ yields

$$mod_o(x) : 0 \rightsquigarrow o\text{-}EulerClass(e^{\mathbb{Z}o} \cdot x),$$

the *octave class* of $x$. For such an octave class $mod_o(x)$ and an octave class $k$-chord

$$Ch : 0 \rightsquigarrow o\text{-}ClassChord(c_1, c_2, \ldots c_k),$$

we can consider the translate $e^{mod_o(x)} \cdot Ch = \{e^{mod_o(x)} \cdot c_1, e^{mod_o(x)} \cdot c_2, \dots e^{mod_o(x)} \cdot c_k\}$. If for example $Ch$ is the 12-tempered C-major scale $C_{maj} = \{\overline{0}, \overline{2}, \overline{4}, \overline{5}, \overline{7}, \overline{9}, \overline{11}\}$, the translate $e^{mod_o(11)} \cdot C_{maj}$ is equal to the 12-tempered B-major scale $B_{maj} = \{\overline{11}, \overline{1}, \overline{3}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\}$, whereas in just temperament, the just B-major scale (chord) is obtained from the just C-major scale (chord) via a translation $e^{q+t} = e^q \cdot e^t$ of a fifth (class[5] ) $q$ plus a major third (class) $t$.

**Example 8** *Repetition, da capo, canon, rhythm.*
Translation $e^t$ on the simple space $Onset$ of onsets means something very different from the same mathematical transformation on pitch domain. We have four different common meanings in this case:

- If we view the onset axis as being a time line, translation of a local composition $G$ in an ambient space $S = Onset \oplus Para$ by $e^t$, i.e., the translation $e^t \cdot G$ in the sense of rhythm theory as introduced in section 7.2.2, means playing $G$ a second time, repeating it later in the course of world time. This seemingly trivial fact is the case if we *replay a composition* on a media such as tape, LP, CD, etc. In the technically ideal case, every replay is a translation on the physical time axis. We have interiorized this symmetry to such a degree that we do not even realize it as being essentially different from identity; it is often and erroneously felt that any such reproduction is the piece itself; this is a point where Sergiu Celibidache [476] was right.

- We can also repeat parts of a composition *within* the composition in the sense of a *da capo*.

- A special case of a repetition within one and the same composition is the duplication of a melody in the *canon* construction.

- Finally, we may view the looping, i.e., periodic repetition of a given "percussive pattern" as a generation principle for *rhythmical* structures, this was already introduced in section 7.2.2.

**Example 9** *Inversion, retrograde, retrograde inversion.*
These are classical symmetries in composition and theory. Let us have a closer look at the highly instructive analysis of these concepts and their multiple meaning and corresponding mathematical interpretations.

Intuitively, *inversion*[6] is a reflection of the pitch axis at a fixed pitch. However, its visual shape depends on the selected pitch space. Everyone of the hitherto discussed pitch spaces $Ptch$

[5] Since the fifth and third coordinates are not touched by factorization through the octave in just classes of the Euler plane (see equation (7.7)) we can refrain from adding the class attribute here.

[6] Pay attention to distinguish the symmetry of inversion from chord inversion which just means moving some of their notes one octave up or down.

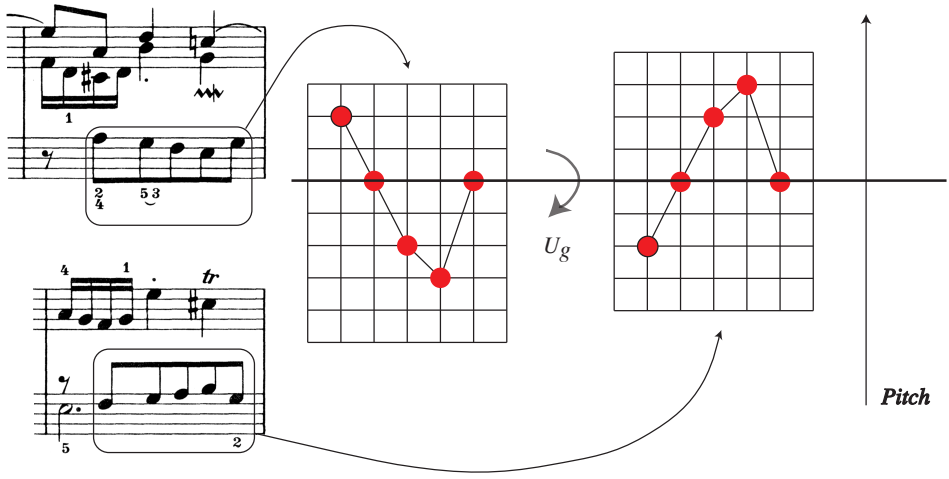



Figure 8.4: This example from Fuga 6, D minor, in Johann Sebastian Bach's "Wohltemperiertes Klavier" I, BWV 851, shows an inversion at $g$ in 12-$TemperedScale$. The symbol $U_g$ is written in the traditional notation, i.e., the fixpoint is indicated. For fixpointless inversions between pitch $g$ and $g\sharp$, for example, that notation would yield. The grid shows semitone steps and quarter notes onset units. The motivic note groups shown here (above: bar 29, below: bar 33) represent the characteristic motif of this fugue.

were of simple form:

$Pitch$ has the $\mathbb{R}$-vector space coordinator $\mathbb{R}$,
$MathPitch$ has the $\mathbb{Q}$-vector space coordinator $\mathbb{R}_{[\mathbb{Q}]}$,
$EulerModule$ has the $\mathbb{Q}$-vector space coordinator $\mathbb{Q} \cdot o \oplus \mathbb{Q} \cdot q \oplus \mathbb{Q} \cdot t = \mathbb{Q}^3$,
$EulerPlane$ has the $\mathbb{Q}$-vector space coordinator $\mathbb{Q} \cdot q \oplus \mathbb{Q} \cdot t = \mathbb{Q}^2$,
$w$-$TemperedScale$, the $w$-tempered scale from origin 0,

has the $\mathbb{Z}$-module coordinator $\mathbb{Z} \cdot \frac{1}{w} \cdot o$, and

$w$-$PitchClass$ has the $\mathbb{Z}$-module coordinator $\mathbb{Z} \cdot \frac{1}{w} \cdot o / \mathbb{Z} \cdot o \xrightarrow{\sim} \mathbb{Z}_w$.

Then inversion is an affine symmetry of shape $U_s = e^s \cdot -1$ which sends a denotator $D : A \rightsquigarrow Ptch(x)$ to $U_s D : A \rightsquigarrow Ptch(U_s(x) = s - x)$. This is inversion at the origin of the coordinator module, followed by a shift $e^s$. The inversion has a fixpoint $x = U_s(x)$ iff with the unique decomposition $x = e^t \cdot x_0$ into translation and linear factor, we have $2t = s$ and $2x_0 = 0$. For real or rational coefficients in the first four forms this yields a unique solution $t = \frac{1}{2}s$ and $x_0 = 0$, whereas in case of $w$-$TemperedScale$, we have at most one solution, and there is one iff $s$ is an even multiple of $\frac{1}{w}$; otherwise, the inversion is a reflection in the

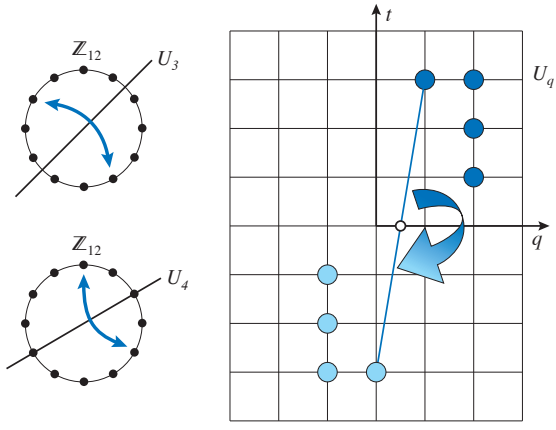



Figure 8.5: To the left, we show two inversions $U_3$ and $U_4$ on zero-addressed denotators in 12-*PitchClass*. The first has no fixpoint and is also denotated by $U_{1/2}$ or $U_{c\sharp/d}$ in traditional notation since its virtual fixpoints are between $c\sharp$ and $d$ as well as between $g$ and $g\sharp$. The second inversion is $U_4$ has fixpoint $2 = \frac{4}{2}$ and would traditionally be denoted by $U_d = U_{g\sharp}$. To the right we have inversion $U_q$ on the *EulerPlane* space, showing zero-addressed denotators again for simplicity. It leaves invariant the middle $q/2$ between origin and $q$. This symmetry can be viewed as a 180°-rotation around the fixpoint. If the inversion's shift vector $p$ has integer coordinates with respect to the grid's basis $q, t$, the fixpoint will stay either on one vertex of the integer grid, or in the middle of one mesh side (as in the case of this figure), or in the middle of one mesh.

middle between two adjacent points $\eta\frac{1}{w}o, (\eta+1)\frac{1}{w}o$. For form $w$-*PitchClass*, we are dealing with affine di-homomorphisms in $x \in A@\mathbb{Z}_w$, and we are looking for numbers $s \in 2\mathbb{Z}_w$ and di-linear homomorphisms $x_0 : A \to \mathbb{Z}_w$ with $2im(x_0) = 0$. If $w$ is odd, 2 has an inverse in $\mathbb{Z}_w$, and fixpoints are given by $t = \frac{1}{2}s$ and the zero-di-homomorphism $x_0 = 0$. If $w = 2q$ is even, the condition $s \in 2\mathbb{Z}_w$ is not automatic, whereas the di-linear factor must map into the subgroup $q\mathbb{Z}_w \xrightarrow{\sim} \mathbb{Z}_2$ and is parametrized by $Lin(A, \mathbb{Z}_2)$. Figures 8.4 and 8.5 show the picture of an inversion in the pitch space 12-*TemperedScale* and in pitch class spaces for 12-tempered and just tuning.

**Exercise 5** Calculate all fixpoints of an inversion $U_s$ for self-addressed denotators

$$D : \mathbb{Z}_{12} \rightsquigarrow 12\text{-}PitchClass(x).$$

Intuitively, the *retrograde* symmetry is a reflection of time's run down. Since several attributes are related to time, this idea is not quite clear. Time intervenes in onset, duration, sound color, and envelope of a sound. If we try to invert time naively as it is done by inversion of a tape, then obviously, something different from musicological "retrograde" happens. So let us have a more detailed look at the retrograde phenomenon.

To begin with, onset is concerned—independently of whether we consider mental or physical time. Let us select the mental onset space *Onset* with simple coordinator $\mathbb{R}$. On

this space, retrograde symmetry means the complete analogy to pitch inversion and can be written[7] as $k_s = e^s \cdot -1$ for a shift number $s$. Whereas $U_s$ has the same formula and acts on pitch, $k_s$ acts on onset. However, this one-dimensional symmetry is somewhat strange if we look at its effect on two events at times *onset* and *offset* = *onset* + *onsetdistance* which are *onsetdistance* apart. If we apply $k_s$ to such a couple $\{onset, offset\}$, we obtain $k_s\{onset, offset\} = \{k_s(onset), k_s(offset)\} = \{s - offset, s - onset\}$, and the "retrograde" of the second event will start *onsetdistance* before the "retrograde" of the first event, see bottom part of figure 8.6. Such a symmetry may be interesting if we disregard durations, such as

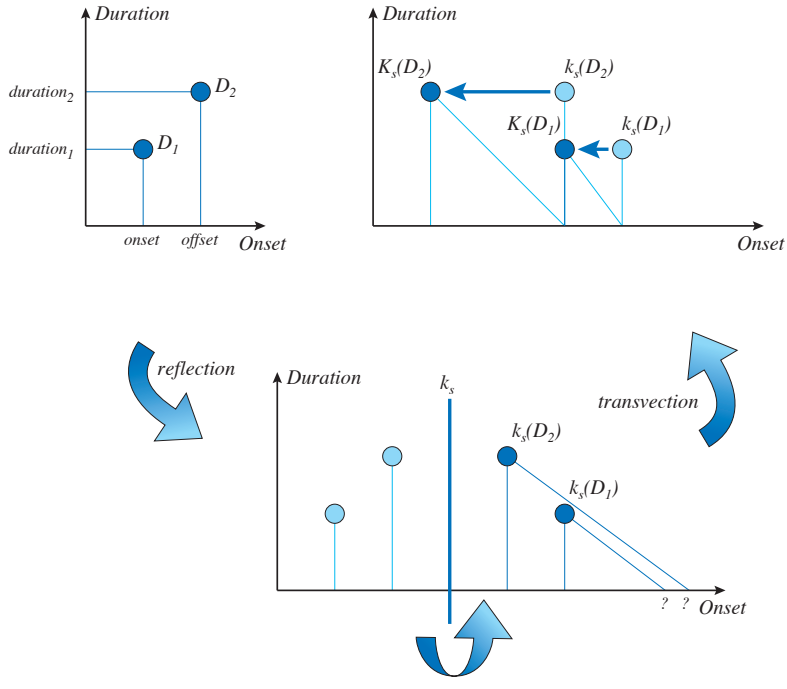



Figure 8.6: In contrast to pitch inversion, usual retrograde concerns onset as well as duration. It is composed from a horizontal reflection (resulting in the lower configuration) and a horizontal transvection (shown in the right upper part).

in drum patterns. However, if the duration of the second event at time *offset* is longer than *onsetdistance*, the transformed couple will have a disturbing effect because $k_s(offset)$ will not end when $k_s(offset)$ starts. In fact, this transformation is not what usually happens when a score is played from the right to the left in reversed order.

In order to include the role of duration in defining retrograde symmetry, we have to work in the two-dimensional space $OD = Onset \oplus Duration$ with coordinator $\mathbb{R}^2$ as defined in formulas (6.39) ff. and section 6.6. So we have to take two events $OD(onset, duration_1)$

[7] The letter "k" stands for German "Krebs".

and $OD(\mathit{offset}, \mathit{duration}_2)$, see left upper part in figure 8.6. We first perform $k_s$ on their onset parameters and obtain an intermediate transformation $OD(k_s(\mathit{onset}), \mathit{duration}_1)$ and $OD(k_s(\mathit{offset}), \mathit{duration}_2)$ as shown in the lower part of figure 8.6. To avoid overlaps due to durations, we go one step further and shift the intermediate onset by the event's duration to the left and get the events $OD(k_s(\mathit{onset}) - \mathit{duration}_1, \mathit{duration}_1)$ and $OD(k_s(\mathit{offset}) - \mathit{duration}_2, \mathit{duration}_2)$ as shown in the upper right part of figure 8.6. Ths latter is a horizontal transvection or 'shearing' operation[8]. Putting all together we have defined a duration-sensitive retrograde which is given by the affine map

$$K_s : \mathbb{Q}^2 \to \mathbb{Q}^2 : (o, d) \mapsto (s - o - d, d)$$

which means

$$K_s = e^{(s,0)} \cdot \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \tag{8.1}$$

and extends functorially to any $A$-addressed denotator $D : A \rightsquigarrow OD(D : A \to \mathbb{Q}^2)$ if we set $K_s(D) : A \rightsquigarrow OD(K_s \cdot D)$. Examples of this duration-sensitive symmetry are abundant in classical European literature, see figure 8.7. Observe that sound colors are not altered by this symmetry. We do obtain what is required in a fugue for *piano*, for example!

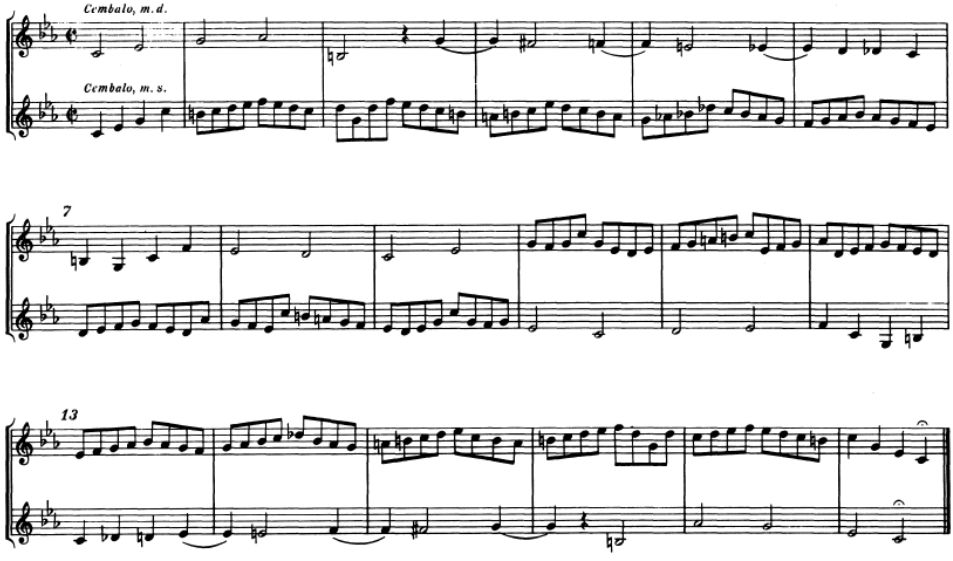



Figure 8.7: J.S. Bach: retrograde canon from "Musikalisches Opfer", BWV 1079 (with kind permission of Bärenreiter Publishers).

Finally, *retrograde inversion* $KU_{s,t}$ is defined as composition of a retrograde symmetry $K_s$ and a pitch inversion $U_t$. It involves three parameters: pitch, onset, and duration, and we

[8]See appendix E.3.4

may consider the typical three-dimensional space[9] $OPD = Onset \oplus Pitch \oplus Duration$ with coordinator $\mathbb{Q}^3$. We then obtain the formula

$$KU_{s,t} = e^{(s,t,0)} \cdot \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8.2}$$

which means $K_s \cdot U_t = U_t \cdot K_s$ if these symmetries are canonically lifted to the three-dimensional space $OPD$. More practically interpreted, $KU_{s,t}$ is a rotation of 180° on the score plane. It is reported that Beethoven, in one of those then famous competitions among pianists, forced his contrahent nicknamed "Tremolo Steibelt" to leave the salon steamed up. Beethoven had just turned around 180° Steibelt's string quartet score and played that symmetric transformation of the composition—evidently turning it into garbage (in those days it very probably *was* garbage).

**Example 10** *Sound transformations.*
Applying symmetries to abstract sound descriptions as they occur in score notation, and which we have dealt with above, is completely different from applying symmetries directly to sound events on the physical level or reality. Let us consider this latter situation to review some details of what happens when a tape is reversed, to name a typical operation. This type of transformation on the sound level is a standard processing tool for analog and digital audio sampling technology.

We stick to the sound description in Fourier synthesis space

$$FourierSound \underset{Id}{\longrightarrow} Limit(PhysOnset, PhysDuration, Fourier, Envelope)$$

introduced in (6.108). The space contains denotators $myFourierSound = (e, d, myFourier, V)$ which give rise to physical sound function

$$p = sound(myFourierSound).$$

For all triples $\beta, \sigma \neq 0, \mu$ of real numbers, we obtain a new time function $p_{\beta\sigma\mu}$ which at physical time $t$ evaluates to

$$p_{\beta\sigma\mu}(t) = \beta p(\sigma t + \mu). \tag{8.3}$$

For $\sigma = -1$, we obtain a reversed run though time and therefore the well-known tape reversion. Let us look at the effect of this symmetry type on the underlying Fourier denotators. Let the Fourier denotator be given by $myFourier = (f, (A_i, Ph_i)_{0 \leq i})$. Then the transformed sound function evaluates to

$$\begin{aligned} p_{\beta\sigma\mu}(t) =& \beta V(((\sigma t + \mu) - e)/d) \cdot \sum_{0 \leq i}^{\infty} A_i \sin(2\pi i f(\sigma t + \mu) + Ph_i) \\ =& V((t + \frac{\mu - e}{\sigma})/\frac{d}{\sigma}) \cdot \sum_{0 \leq i}^{\infty} (\beta A_i) \sin(2\pi i (f\sigma)t + (2\pi i f\mu + Ph_i)), \end{aligned}$$

[9] We choose this sequence of parameters instead of $POD$, for example, since one normally groups the basis parameters and then adds the pianola parameters.

and we have the affine transformation

$$Q_{\beta\sigma\mu} : (e, d, (f, (A_i, Ph_i)_{0\leq i}, V) \mapsto (\frac{\mu - e}{\sigma}, \frac{d}{\sigma}, (f\sigma, (\beta A_i, 2\pi i f\mu + Ph_i)_{0\leq i}, V) \tag{8.4}$$

which relates the symmetry on the sound level to the underlying parametrization. Let us make this a bit more transparent for later constructions. If we interpret the sound functions $p, p_{\beta\sigma\mu}$ as being zero-addressed denotators of form

$$Sound \underset{Id}{\longrightarrow} \mathbf{Simple}(C^0_{cp})$$

where the coordinator $C^0_{cp}$ is the real vector space of functions $f : \mathbb{R} \to \mathbb{R}$ which are continuous and have compact support[10]. With this, the map

$$sound : myFourierSound \mapsto sound(myFourierSound)$$

commutes with the above symmetries, i.e., we have

$$sound(Q_{\beta\sigma\mu}(myFourierSound)) = sound(myFourierSound)_{\beta\sigma\mu} \tag{8.5}$$

which means that *sound* is an equivariant map[11] for the two symmetry actions $Q_{\beta\sigma\mu}$ and $?_{\beta\sigma\mu}$. We also see that the envelope is left invariant by $Q_{\beta\sigma\mu}$ which means that it can even be taken as an envelope normed to maximal value 1, say, which is the situation in various sound synthesizer drivers. The loudness change is solely attributed to the Fourier sum expression. The prize for this invariance is in particular that duration may become negative for negative $\sigma$. This is the case where the sample will run in reversed direction, see also figure 8.8.

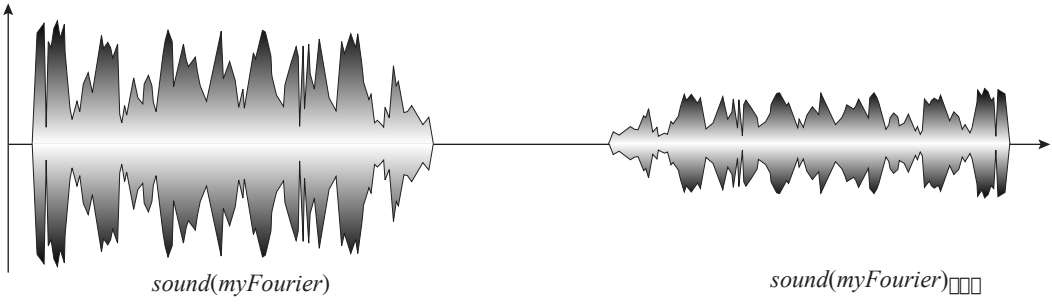



Figure 8.8: A sound sample and its transform by a symmetry $Q_{\beta\sigma\mu}$, showing time inversion.

**Example 11** *The major-minor problem.*
The major-minor problem is one of the classical schisms in occidental music theory. The problem is to understand the relation between the "major" and "minor" paradigms. We are not going to shed philosophical light on this conceptual field but will concentrate on the concrete and most

[10] The support $supp(f)$ is the topological closure of the set $\{x \in \mathbb{R} |\ f(x) \neq 0\}$.
[11] See appendix C.3.1, example 70 for this concept.

common context of just temperament where this discussion is incessantly virulent. Historically and theoretically, it was Gioseffo Zarlino [591] who initiated the subject by the observation that there is a *dualism* between major and minor. In fact, he realized that in just tuning, the major chord $EulerChord(0, t-2\cdot o, q-o)$ becomes a minor chord $EulerChord(0, -t+o, q-o)$ after a symmetry of $EulerModule$.

**Exercise 6** Describe all affine endomorphisms $f$ of $EulerModule$ which transform the major chord into the minor chord, i.e.,

$$f(EulerChord(0, t-2\cdot o, q-o)) = EulerChord(0, -t+o, q-o).$$

To discuss this subject in simpler terms, let us look at the situation in the $EulerPlane$ space. Here, we have the inversion symmetry $U_q$ which converts the major class chord $\{c, e, g\}$ into the minor class chord $\{c, e_\flat, g\}$, more precisely, $U_q(c) = g, U_q(e) = e_\flat$,and $U_q(g) = c$. This symmetry extends to a transformation of the C-major scale into the C-minor scale:

$$U_q(C\text{-}major) = C\text{-}minor,$$

but see figure 7.5 to evidence this fact. This fact was used by Arthur von Oettingen [406] and by Hugo Riemann [450] to deduce the minor cadence from dualism. Under the "dualism" $U_q$ the major cadence[12] sequence $I = \{c, e, g\}, IV = \{f, a, c\}, V = \{g, b, d\}, I = \{c, e, g\}$ of major triadic degrees is transformed into the sequence $I = \{c, e_\flat, g\}, V = \{g, b_\flat, d\}, IV = \{f, a_\flat, c\}, I = \{c, e_\flat, g\}$ of minor triadic degrees. This contradicts the function of dominant and subdominant in Riemann's harmony in that the fourth degree in minor should have a subdominant function whereas the fifth degree in minor should have dominant character. Carl Dahlhaus [100] has interpreted this fact as contributing to the erosion of the concept of a harmonic function[13].

The theory of dualism is confronted with the theory of *turbidity*[14] which was prominently forwarded by Paul Hindemith [224]. It states that the minor triad is derived from the major triad by a "turbidating" lowering shift of the major third ($c-e$ in $\{c, e, g\}$, for example) to the minor third ($c-e_\flat$ in $\{c, e_\flat, g\}$). This view is charged by the emotional connotation of gloom as being conveyed by the minor triad.

Already Johann Wolfgang von Goethe did not accept this psychologically charged theory which had its roots long before Hindemith. We should however ask whether turbidity has nothing to do with symmetry and must therefore be structurally inferior to dualism as being founded on a purely emotional level. We want to show that this is not true: The theory of turbidity is based on symmetry just as dualism is. The key to this unexpected symmetry lies in the inner symmetry of the major scale itself. Let us consider the C-major scale (scale chord, to be precise) to fix the ideas, see figure 8.9.

The inner symmetry of C-major runs as follows: It is a reflection of the $EulerPlane$ space at the vertical line through $q/2$. Then we add a shearing by 45° in direction $-q$. We thus obtain the symmetry

$$A = e^q \cdot \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \tag{8.6}$$

[12] We shall discuss the concept of cadence in more detail in chapter 26.
[13] We come back to Riemann's function theory in section 25.3.
[14] German: "Trübung".

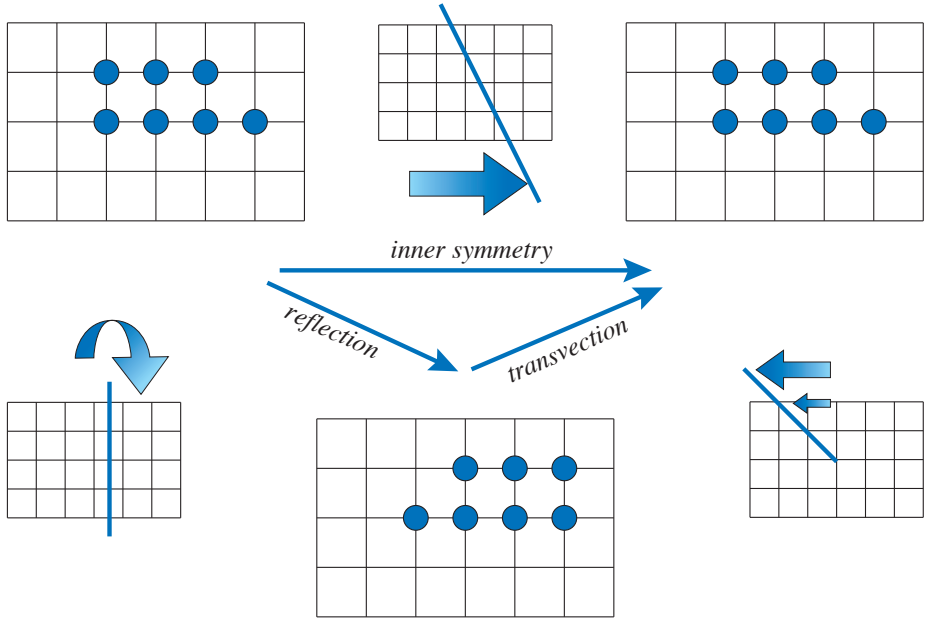



Figure 8.9: The inner symmetry of a major scale in just tuning. This horizontal reflection at a skew axis is composed of a horizontal reflection at a vertical axis through $q/2$, followed by a horizontal transvection of 45°, and is the "tonal inversion" at the scale's third from the tonic on the keyboard (e in C-major).

with $A(C\text{-}major) = C\text{-}major$ where $e$ remains fixed whereas we have exchanges $c \Leftrightarrow g$, $f \Leftrightarrow d$, and $a \Leftrightarrow b$. If we execute this symmetry on a keyboard, it turns out that we obtain a reflection of white keys (C-major) at key e. The black keys are so to speak eliminated. In music theory such a reflection is called *tonal inversion* since the transformation is performed within the given "tonality" if we think of the white keys as being equidistant (which they are not). To distinguish the terms, the above inversion $U$ is termed *real inversion*. We shall see in connection to the formalism of alterations (section 49.3, example 60) how real and tonal inversion relate to each other.

Besides identity, there is only the above $A$ as an "inner" symmetry[15] of C-major. The interesting fact about this non-trivial symmetry is that this one mediates between the theories of dualism and turbidity! In fact, if $U_q$ defines the dualism, then $A \cdot U_q$ evidently defines another symmetry from C-major to C-minor. And this does the following:

$$\begin{array}{llll} U_q \cdot A(c) = c & U_q \cdot A(g) = g & U_q \cdot A(f) = f & U_q \cdot A(d) = d \\ U_q \cdot A(e) = e_\flat & U_q \cdot A(a) = a_\flat & U_q \cdot A(b) = b_\flat. & \end{array} \tag{8.7}$$

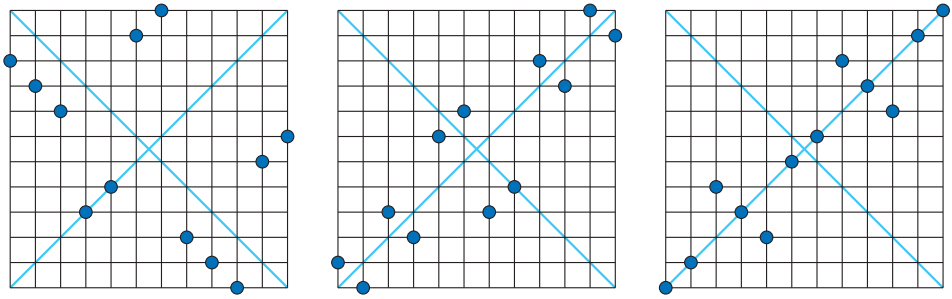
[15] We shall make this statement precise in section 8.2.

We recognize that $A \cdot U_q$ lowers exactly the three tones $e, a, b$ by a semitone, and *this is the process which turbidity theory describes.* This makes evident that turbidity is just the other symmetry $A \cdot U_q$ from major to minor, and we have learned that the unique inner symmetry $A$ of major mediates between dualism and turbidity. Summarizing:

**Scholion 1** *The schism between dualism and turbidity is only apparent. It reduces to understanding the inner symmetry $A$ of the major scale. This one mediates between dualism $U_q$ and turbidity $A \cdot U_q$ so that we are allowed to call both approaches equivalent from the symmetry point of view.*

We shall also encounter the fundamental inner symmetry $A$ when developing modulation theory in chapter 27.

**Example 12** *Schönberg, Berg, and Webern.*



*op. 17.2* *op. 28, string quartet* *op. 30, var.*

Figure 8.10: Three dodecaphonic series by Webern, each showing diagonal or/and codiagonal symmetry.

In the framework of dodecaphonic composition, symmetries play a fundamental role. They define the paradigm associated with a *dodecaphonic series*[16]. This means that they generate a field of twelve-note series which are deduced from a basic form of the series. To make this compositional framework more precise, we invoke the ambient space $PiMod_{12}$ from (6.43) and look for local compositions which are motives in this ambient space:

**Definition 22** *Let $k, n$ be positive naturals. A* $(k, n)$-serial motif *is a $\mathbb{Z}^{k-1}$-addressed denotator*

$$Ser : \mathbb{Z}^{k-1} \rightsquigarrow PiMod_n(Ser_0, Ser_1, \dots Ser_{k-1})$$

*with coordinates*[17] *$Ser_i = Ser(e_i)$. If these coordinates are pairwise distinct, the serial motif is called a* $(k, n)$-series. *An* $(n, n)$-series *is called* $n$-phonic series, *in particular a* dodecaphonic *series for $n = 12$.*

[16] from German: "Zwölftonreihe".

[17] This abuse of language with respect to denotator formalism is fairly acceptable.

*The set of $(k,n)$-serial motifs is denoted by $SERM_{k,n}$, and we abbreviate $SERM_{n,n} = SERM_n$. The subset of $(k,n)$-series is denoted by $SER_{k,n}$, and we abbreviate $SER_{n,n} = SER_n$.*

See figure 8.10 for historic examples. The coordinate index $i$ is viewed as being an 'abstract' onset time line which controls the compositional process. In general, this axis will not be present in a straightforward form within a concrete score. A general rule in twelve-note composition is that within a local realization of the basic series in the particular score, a note with larger onset index should not appear earlier than a note with smaller index, it may however occur simultaneously in a chord. We discuss this issue in more detail in section 11.5.1.

The genealogical principle in twelve-note music starts with a particular dodecaphonic series $Ser$—called basic series[18]—and then creates 48 transformed versions according to a complete group of symmetries. In our generic setup, this group is the direct product group $D_{k,n} = \langle rev_k \rangle \times TI_n$ where $TI_n = e^{\mathbb{Z}_n} \rtimes \langle -1 \rangle$, and where $rev_k$ is the 'retrograde' address involution defined by $rev_k(e_i) = e_{k-1-i}$.

This group consists of transformations known from classical counterpoint: transpositions $e^s$, pitch inversions $U_s$, onset retrograde $rev_k$, and their combinations[19]. A typical element $g = (h, f)$ of the group acts on a serial motif $Ser$ by $(g.Ser)_i = f(Ser_{h(i)})$. The 48 versions are defined by the set

$$D_{12,12} \cdot Ser = \{c.Ser |\ c \in D_{12,12}\} \tag{8.8}$$

where, however, several elements may coincide. According to the general theory of group actions (see appendix C.3.1, example 70), the set $SER_{12}$ of all twelve-note series splits into a disjoint union of twelve-note orbits, also called "paradigms"

$$SER_{12} = \coprod_r D_{12,12} \cdot Ser^r \tag{8.9}$$

running over a set $\{Ser^r\}$ of representative series.

The musical meaning of symmetries in Schönberg's, Webern's and Berg's twelve-note composition is a poietical and mental principle, it gives you a procedure to fabricate a composition. Joseph Rufer has exposed this approach in detail in [465]; the dominant role of symmetries is made explicit. Semiotically, the symmetries from $D_{12,12}$ have the task to build a paradigmatic field around the basic series. There is an unsolved problem in this approach: It is not clear how far the poietical approach in Schönberg's method may be transferred and evidenced on the neutral and/or esthesic levels. As a matter of fact, the historically problematic and poor reception of dodecaphonism shows that a poietical paradigm cannot only be practised, its protagonists should also take care of mediating it for the sake of public's recognition and acceptance on the esthesic level.

**Example 13** *Messiaen modi and non-invertible rhythms.*
Essentially, Olivier Messiaen's composition technique [370] intends to treat all parameters of time and pitch equally. We shall come back to the systematic aspects of this method in section 10.2. Here we merely make evident the structural context and the role of symmetries.

[18] German: "Grundgestalt".
[19] To simplify notation, we denote $U_s$ and $k_{11}$ without adding the identity components in the respective other coordinate.

In the dimension of pitch, Messiaen introduced *modes with limited transpositions*. These are special scales in $\mathbb{Z}_{12}$, i.e., in 12-tempered tuning. Messiaen's list includes seven modes which are however not selected according to any evident logic. We come back to this selection problem when dealing with local classification in section 11.3.5. The explicit selection criterium which is recommended and felt by Messiaen is limited transposition. This means that there is a non-zero pitch class $t \in \mathbb{Z}_{12}$ such that a Messiaen mode $M \hookrightarrow 0@PiMod(12)$ remains invariant under the transposition by $t$: $e^t \cdot M = M$. For a given mode $M$, the smallest candidate for such an invariant transposition is $t = 1, 2, 3, 4, 6$. Further, if $M$ has this limited transposition, its complement $M\hat{}$shares this property, and we may concentrate on $M$ having $6 \leq card(M)$. finally, the full chromatic "mother" scale $M = \mathbb{Z}_{12}$ can be omitted as a trivial case. From the classification theory in section 11.3.5 we learn that—up to transposition—there are ten such scales, among which Messiaen for obscure reasons omits $M_9 = \{c, c_\sharp, d_\sharp, g, a\}$ as well as its inversion $M_{10}$, and $M_8 = \{c, c_\sharp, e, f, g_\sharp, a\}$. Here are the remaining seven scales $M_i, i = 1, \ldots 7$ as listed by Messiaen:

1. $M_1$ = whole-tone scale = $\{c, d, e, f_\sharp, g_\sharp, b_\flat\}$
2. $M_2$ = (diminished seventh chord)$\hat{}$= $\{c, c_\sharp, d_\sharp, e, f_\sharp, g, a, b_\flat\}$
3. $M_3$ = (augmented triad)$\hat{}$= $\{c, c_\sharp, d, e, f, f_\sharp, g_\sharp, a, b_\flat\}$
4. $M_4 = \{e, f, b_\flat, b\}\hat{}$= $\{c, c_\sharp, d, d_\sharp, f_\sharp, g, g_\sharp, a\}$
5. $M_5 = \{c, c_\sharp, d, f_\sharp, g, g_\sharp\}$
6. $M_6 = \{d_\sharp, f, a, b\}\hat{}$= $\{c, c_\sharp, d, e, f_\sharp, g, g_\sharp, b_\flat\}$
7. $M_7$ = (tritone)$\hat{}$= $\{c, c_\sharp, d, d_\sharp, e, f_\sharp, g, g_\sharp, a, b_\flat\}$

Whereas Messiaen's modes are not problematic to define, the way Messiaen uses them to introduce non-invertible rhythms is quite disquieting. Let us anticipate the discussion by the original text [370]:

> *Diese Modi realisieren im Vertikalen (Transposition), was die nicht-umkehrbaren Rhythmen im Horizontalen (Umkehrung) realisieren. Tatsächlich können diese Modi nicht über eine gewisse Anzahl von Transpositionen hinaus transponiert werden, ohne daß man wieder in dieselben Töne hineingerät (enharmonisch gesprochen); ebenso können diese Rhythmen nicht rückläufig gelesen werden, ohne daß man genau dieselbe Anordnung der Werte wiederfindet wie in der Grundform. Diese Modi können nicht transponiert werden, weil sie sich — ohne Polytonalität — in der modalen Atmosphäre mehrerer Tonarten zugleich bewegen und in sich keine Transpositionen enthalten; diese Rhythmen können nicht umgekehrt werden, weil sie in sich selbst keine Umkehrungen enthalten. Diese Modi lassen sich in symmetrische Gruppen teilen; diese Rhythmen auch, mit dem Unterschied, dass die Symmetrie der rhythmischen Gruppen eine rückläufige ist. Endlich hat jede Gruppe dieser Modi jeweils ihren letzten Ton gemeinsam mit dem ersten Ton der folgenden Gruppe; und die Gruppen dieses Rhythmus umrahmen einen für beide gemeinsamen Zentralwert. Die Analogie ist also vollkommen.*

The last sentence is as typical for mathematically oriented phrasings by musicians as it is erroneous. The analogy is anything but perfect. Let us start with the underlying modules for modes and rhythms, respectively. For modes, we clearly have pitch classes referring to coordinator $\mathbb{Z}_{12}$. But to deal with rhythms, we cannot stick to local meters $L \hookrightarrow 0@Onset$ of period $p$ since durations play an important role with Messiaen. Further, it is not sufficient to consider exclusively durations since we have to include pauses in Messiaen's rhythms, and this means that we deal with rhythms in the technical sense of definition 14 in subsection 7.2.2 where the parameter space $Para$ includes duration as well as possibly some additional characterization to differentiate pauses from non-pauses. The least we can say is that we have $Para = Duration \oplus Para_0$, and therefore, the coordinator where Messiaen's rhythms are defined is $Onset \oplus Duration \oplus Para_0$. Now, not only the rhythm concept but also the concept of symmetry includes onset and duration. In fact, non-invertibility is defined in terms of onset *and* duration. There is not the least reason to invoke "perfect analogy" here. The retrograde which Messiaen refers to in [370] is precisely the one which we have denoted by $K_s$ in formula (8.1): It is the symmetry of "reading from right to left what is normally read from left to right". And this symmetry is very different from the simple transposition as we know from the preceding discussion. Under these conditions, non-invertibility means this: Messiaen limits himself to indicate just one *germ* of a rhythm, and to ask from *germ* that it be symmetric under a retrograde symmetry, i.e., $germ = K_s \cdot germ$.

Apart from the fact that like transposition, retrograde transformation is also a symmetry in the mathematical sense (to be discussed in detail in section 8.2), no such analogy as suggested by Messiaen can be discovered. Albeit such an analogy would be feasible, Messiaen seems satisfied with the fact that there is symmetry at all, a fact which is responsible for establishing criteria of compositional esthetics. This does nothing else than reconfirm the dominant role of symmetries in Messiaen's approach.

**Example 14** *Serial techniques: Eimert, Stockhausen, Kagel, and Boulez.* As instances of the compositional process, symmetries have been considered by exponents of the serial school. Serialism is derived from the idea that Schönberg's series should be extended to all sound parameters. As a point of departure for this approach we have to consider Messiaen's construction of non-invertible rhythms as a codex for durations in analogy to his scales (see Example 13). However, serialists start from treating all parameters equally, in other words: to elaborate Messiaen's analogy into a symmetry of the parameters' roles. Another example is Webern's symmetrization of the roles of onset and pitch in his twelve-note series. Figure 8.10 shows three such series; each provides *exchange of pitch and onset as an inner symmetry.*

The idea of a parameter exchange was adopted by Herbert Eimert, Maurizio Kagel, Karlheinz Stockhausen, Pierre Boulez, and other composers. It has been extended to the degree of admitting more general geometric transformations on parameter spaces in order to generate new sound configurations from given ones. For instance [138], such serial transformations include rotation by 90° or rotation by 45°, followed by dilatation by $\sqrt{2}$, see figure 8.11. This more general type of symmetries leads to the general concept of morphisms of denotators to be introduced in section 8.2. Examples of compositions using such methods include Kagel's "Transición II", Boulez' "Structures pour piano", or Stockhausen's "Kontra-Punkte". The musical meaning of symmetries is again of poietical and mental topography and serves—like dodecaphonism—the generation of a structurally specified paradigmatism of musical gestalts wherein hitherto

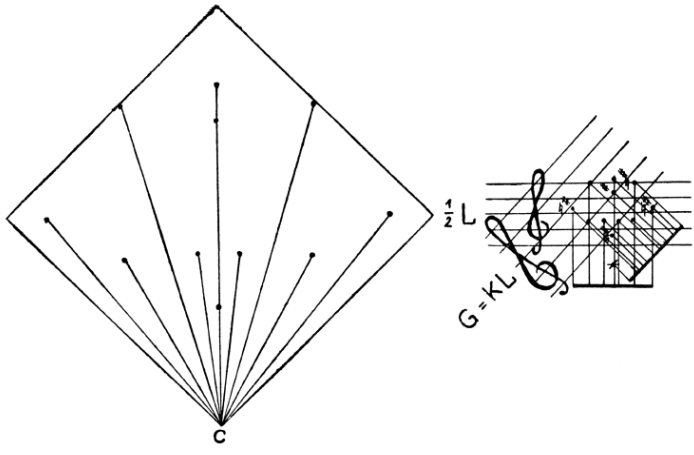



Figure 8.11: (From [138], with kind permission of Universal Edition, Vienna, ©1964.) Example of a serial transformation described by Eimert in [138]. It results from a counter-clockwise rotation by 45°, followed by dilatation by $\sqrt{2}$.

separate musical parameters become freely interchangeable.

**Example 15** *Non-invertible symmetries.*
There is one point which we must make clear before leaving the level of examples. In definition 21 of symmetry as given at the top of this chapter, correspondence of parts is mentioned. But it is not specified what should be the nature of this correspondence. In particular, it is not clear whether such a correspondence should be one-to-one. Could it be that in special situations several parts correspond to one single part? From traditional mathematical terminology, symmetry transformations are understood as being one-to-one. But this is not mandatory.

Let us in fact give two easy examples of symmetries which do not involve one-to-one correspondences. The first one is projection. In twelve-note composition, it is allowed to play a selected part $P$ of the fundamental series $Ser$ or of its transforms $c(Ser)$ as a chord, i.e., simultaneously. Mathematically, we have a subset $P \subset c(Ser)$, and we consider its projection $\pi_2(P) \subset Pitch$ which is a chord. Clearly, no two elements of $P$ are identified under $\pi_2$, but we can no longer tell the temporal order of the original elements. The correspondence $p \mapsto \pi_2(p)$ for $p \in P$ does not conserve temporal order. But the dodecaphonic paradigm precisely means expressing the compositional parts (such as chords) as instances of a whole, and this is rightly done by projection in this case: the projection $\pi_2(P)$ is in correspondence with $P$ (and indirectly with $Ser$) as an expression of a unifying principle.

The second example is the nearly invisible symmetry of projection $mod_o$ from the pitch space $EulerModule$ onto the classical octave pitch class space $o{-}EulerClass$ (7.11). In this context, a chord $Cr : A \rightsquigarrow EulerChord(Cr)$ projects onto a class chord $mod_o(Cr)$ in $o{-}classChord$, but elements of the original chord which are multiples of the octave apart are identified under $mod_o$. Here, symmetry acts as an identifier and is far from one-to-one. Nonetheless, the correspondence under $mod_o$ creates a semantic added value since on the side of $o - classChord$,

harmony furnishes structural insight which is not visible on the original chord. In fact, harmonic progressions only become visible under such a correspondence.

This short discussion yields another problem which underlies symmetry theory: the codification of a symmetry transformation within a determined correspondence. It is not the same to write down an explicit correspondence among elements of two local compositions and to write down a mathematical transformation rule which is responsible for that correspondence. This is a delicate subject since it makes visible interface problems between what is a constructive background and what is on the surface of a given composition. We already saw in the example of dodecaphonic composition that the relation between compositional parts on the score and poietic genealogy (preliminary and subsidiary constructions by use of symmetries) is of fundamental importance to the communication of the composers message. We come back to this issue in chapter 47.

## 8.2 Morphisms of Local Compositions

**Summary.** This section introduces the formal definition of morphisms between local compositions. The definition is motivated by representative examples from music as discussed in subsection 8.1.1.

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From the representative examples in subsection 8.1.1 we have to draw three essential construction basics for general morphisms between local compositions (and whatever will be derived from this nucleus):

- As a realization of the general idea of symmetry, a morphism has to deal with a correspondence between the denotator 'points' which are involved in local compositions.
- The shape of this correspondence must rely on a generic type of structural transformation which operates on the 'background' of the local composition's ambient space.
- The background transformation is only subsidiary and should not be confused with foreground correspondence which really identifies the morphism. There may exist many different background transformations which yield the same correspondence, the essential is only that we find at least one of them which induces the visible correspondence.

We shall now give the formal definition of a morphism between local compositions on two levels of abstraction: first for objective local compositions, second on proper functorial local compositions. We shall then establish the connection between the two levels and show that everything fits well. In a last step, we shall show how to integrate morphisms among completely general denotators in this concept framework.

**Definition 23** *(Morphisms for objective local compositions) Let* $K \hookrightarrow A@F$ *and* $L \hookrightarrow B@G$ *be two objective local compositions*[20] *at addresses* $A$ *and* $B$ *and ambient spaces*[21] $F$ *and* $G$*,*

[20] It is understood that the notation $K$ stands for $K \hookrightarrow A@F$ and has to be explicated if ever necessary.

[21] With their functors being extracted for the functorial calculations, i.e., we use the usual shorthand $A@F$ to denote $A@Fun(F)$.

*respectively. A* morphism *from $K$ to $L$ is a quadruple $(K, L, f, \alpha)$ composed of $K, L$, an address change $\alpha : A \to B$ and a set map $f : K \to L$, such that there exists a morphism of forms $h : F \to G$ with $\alpha.f = A@h|_K$, i.e., the diagram*

$$\begin{array}{ccc} K & \longrightarrow & A@F \\ {\scriptstyle \alpha.f}\downarrow & & \downarrow{\scriptstyle A@h} \\ L\alpha & \longrightarrow & A@G \end{array} \tag{8.10}$$

*commutes ($L\alpha$ being the image of $L$ under the functorial map $B@G \to A@G$, also denoted by $\alpha$ if no confusion is possible). This morphism is denoted by a fraction*

$$f/\alpha : K \to L, \tag{8.11}$$

*if the underlying ambient spaces are clear. Any morphism $h$ of forms which gives rise to $f/\alpha$ as above is called an* underlying symmetry *of this morphism.*

If the morphism's address change denominator $\alpha$ is the identity $1 = Id_A$, we also write $f$ instead of $f/1$.

**Exercise 7** Review the examples of symmetries in subsection 8.1.1 and give interpretations in terms of the above definition. Explicate all possible underlying symmetries.

**Lemma 5** *If $f/\alpha : K \to L$ and $g/\beta : L \to M$ are two morphisms of objective local compositions, then so is $g/\beta \cdot f/\alpha = gf/\beta\alpha : K \to M$. This morphism is called the* composition *of $f/\alpha$ and $g/\beta$.*

Proof. Let $K \hookrightarrow A@F$, $L \hookrightarrow B@G$, and $M \hookrightarrow C@H$ be the complete data, and suppose that $\alpha.f$ is induced by $h : F \to G$, whereas $\beta.g$ is induced by $k : G \to H$. To begin with, naturality of $k$ with respect to $\alpha$ defines a set map $\alpha(\beta.g)$ which is uniquely determined by $\beta.g$ and by $\alpha$ and makes the diagram

$$\begin{array}{ccc} L & \longrightarrow & L\alpha \\ {\scriptstyle \beta.g}\downarrow & & \downarrow{\scriptstyle \alpha(\beta.g)} \\ M\beta & \longrightarrow & M\beta\alpha \end{array} \tag{8.12}$$

commute. We also have these commutative diagrams

$$\begin{array}{ccc} K & \longrightarrow & A@F \\ {\scriptstyle \alpha.f}\downarrow & & \downarrow{\scriptstyle A@h} \\ L\alpha & \longrightarrow & A@G \end{array} \qquad \begin{array}{ccc} L\alpha & \longrightarrow & A@G \\ {\scriptstyle \alpha(\beta.g)}\downarrow & & \downarrow{\scriptstyle A@k} \\ M\beta\alpha & \longrightarrow & A@H \end{array}$$

and hence the composed set map $\beta\alpha.gf = \alpha(\beta.g)\alpha.f : K \to L\alpha \to M\beta\alpha$ which is induced by $A@h \cdot A@k = A@h \cdot k$, QED.

The following is straightforward from the above definitions:

**Sorite 4** *(Existence of identity, associativity)*

(i) *For an $A$-addressed objective local composition $K$, the identity $1_K = 1/1 = Id_K/Id_A : K \to K$ is a morphism.*

(ii) *If $u/\alpha : K \to L$, we have $1_L \cdot u/\alpha = u/\alpha = u/\alpha \cdot 1_K$.*

(iii) *If $u/\alpha : K \to L$, $v/\beta : L \to M$, and $w/\gamma : M \to N$ are three morphisms, then we have associativity of composition: $(w/\gamma \cdot v/\beta) \cdot u/\alpha = w/\gamma \cdot (v/\beta \cdot u/\alpha) = w/\gamma \cdot v/\beta \cdot u/\alpha$.*

For simple ambient spaces and zero-addressed local compositions, the definition of a morphism between objective local compositions boils down to the classical definition of morphisms of local compositions in [340]. In fact, these local compositions can be viewed as subsets of their coordinator modules, and the underlying morphisms by the Yoneda lemma (appendix G.2) are precisely the affine homomorphisms between these coordinator modules inducing set maps in the classical definition.

Let us next look at morphisms in the functorial setup.

**Definition 24** *(Morphisms for functorial local compositions) Let $A, B$ be two addresses, and let $K \hookrightarrow @A \times F$ and $L \hookrightarrow @B \times G$ be two (functorial) local compositions. A morphism from $K$ to $L$ is a couple $(f, \alpha)$ where $\alpha : A \to B$ is an address change, and where $f : K \to L$ is a natural transformation such that there exists a morphism of forms $h : F \to G$ such that the diagram*

$$\begin{array}{ccc} K & \longrightarrow & @A \times F \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle @\alpha \times h} \\ L & \longrightarrow & @B \times G \end{array} \tag{8.13}$$

commutes. As with objective local compositions such a morphism is denoted by $f/\alpha : K \to L$.

**Exercise 8** Equivalently, we could ask for a natural transformation $f_\alpha : K \to L\alpha$ where $L\alpha$ is the fiber product[22] subfunctor of $@A \times G$ defined by the pullback diagram

$$\begin{array}{ccc} L\alpha & \longrightarrow & @A \times G \\ \downarrow & & \downarrow{\scriptstyle @\alpha \times Id_G} \\ L & \longrightarrow & @B \times G \end{array}$$

and where we ask for an underlying $h : F \to G$ such that

$$\begin{array}{ccc} K & \longrightarrow & @A \times H \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle @1_A \times h} \\ L\alpha & \longrightarrow & @A \times G \end{array}$$

commutes.

[22] See appendix G.2.1 for fiber products or pullbacks.

This latter definition reminds us more of the previous definition for objective local compositions. However, it is less elegant than the one we gave in definition 24; this is already evident from the definition of composition of morphisms:

**Definition 25** *If $f/\alpha : K \to L$ and $g/\beta : L \to M$ are two morphisms of functorial local compositions, then their composition $g/\beta \cdot f/\alpha : K \to M$ is defined by $g/\beta \cdot f/\alpha = gf/\beta\alpha$.*

This time, the fact that the composition is also a morphism is immediate. We have a similar sorite to the above one for objective local compositions:

**Sorite 5** *(Existence of identity, associativity, and canonical factorization for functorial local compositions) The statements of sorite 4 are literally true if we replace local compositions and morphisms by their functorial homonyms. Moreover, we have a factorization $f/\alpha = 1/\alpha \cdot f_\alpha/1$ via the fiber product composition $L\alpha$ and the morphism $f_\alpha$ defined in the previous exercise.*

Of course this terminological duplication is not accidental and we can in fact establish a complete embedding of the objective setup in the functorial one as follows. We already know from section 7.3 that every objective local composition $K \hookrightarrow A@F$ has its functorial counterpart $\hat{K}$ with slice $f@\hat{K} = \{f\} \times K \cdot f$ for any address change $f : X \to A$. Suppose that we are given a morphism $f/\alpha : K \to L$ of objective local compositions with $L \hookrightarrow B@G$ and an underlying $h : F \to G$. The obvious claim is that this same $h$ and $\alpha$ induce a commutative diagram

$$\begin{array}{ccc} \hat{K} & \longrightarrow & @A \times F \\ \downarrow & & \downarrow{\scriptstyle @\alpha \times h} \\ \hat{L} & \longrightarrow & @B \times G \end{array}$$

of functors, i.e., that the embedding of $\hat{K}$, followed by the product $@\alpha \times h$, factorizes (necessarily uniquely) through $\hat{L}$.

So let us calculate the image of an element $(g, k \cdot g) \in g@\hat{K}$ for $g : X \to A$. We have $X@\alpha \times X@h(g, k \cdot g) = (\alpha g, X@h(k \cdot g)) = (\alpha g, (A@h(k)) \cdot g)$. But we know that $A@h(k) \in L\alpha$, i.e., $A@h(k) = l \cdot \alpha$. Therefore $(A@h(k)) \cdot g = l \cdot \alpha g$ and therefore $X@\alpha \times X@h(g, k \cdot g) = (\alpha g, l \cdot \alpha g)$, i.e., this is an element of $\alpha g@\hat{L}$, a subset of $X@\hat{L}$. This functorial morphism $\hat{K} \to \hat{L}$ is denoted by $\hat{f}$. It remains to be shown that $\hat{f}$ is well-defined, i.e., depends only on $f$ and not on $h$. In fact, the evaluation of $\hat{f}$ at $Id_A$ sends $Id_A@\hat{K}$ to $\alpha@\hat{L}$ and yields $\alpha.f$ on the second coordinates of these sets. Further, for any address change $x : X \to A$, we have a commutative diagram

$$\begin{array}{ccc} Id_A@\hat{K} & \xrightarrow{\text{surjection}} & x@\hat{K} \\ {\scriptstyle \alpha.f}\downarrow & & \downarrow{\scriptstyle \hat{f}} \\ \alpha@\hat{L} & \xrightarrow{\text{surjection}} & \alpha x@\hat{L} \end{array} \tag{8.14}$$

with surjective horizontal arrows. Therefore the right vertical arrow is uniquely determined by $f$, and we have defined a map

$$\hat{?} : f/\alpha \mapsto \hat{f}/\alpha \tag{8.15}$$

which transforms an objective morphism into a functorial one.

**Exercise 9** For objective local compositions, there is no factorization $f/\alpha = 1/\alpha.f_\alpha/1$ into a pure address change morphism and a morphism with fixed address. We shall see later (proposition 3 in subsection 8.3.2), that the map $\hat{?}$ defines a full functor from objective morphisms to functorial ones, but it is not faithful, although it reflects isomorphisms. Therefore, the factorization of the image $\hat{f}/\alpha$ does not, in general, imply a corresponding factorization on the objective side. However, if $\alpha$ is an isomorphism, one has the factorization $f/\alpha = \alpha^{-1}/\alpha.f_\alpha/1$, where $f_\alpha/1 : K \to L.\alpha$ and $\alpha^{-1}/\alpha : L.\alpha \to L$. Prove this statement.

Before turning to the overall discussion of categories of local compositions we should show how to integrate more morphisms for general denotators. As already introduced in special situations (see also example 4 in section 6.8) a denotator $D : Address \rightsquigarrow Form(Coordinates)$ could be wrapped in the following local composition: We first set up a new **Power**-typed form

$$Form^{Form} \underset{Id}{\longrightarrow} \mathbf{Power}(Form) \tag{8.16}$$

and then define the singleton objective local composition

$$\{D\} : Address \rightsquigarrow Form^{Form}(D). \tag{8.17}$$

What does it mean to have a morphism of two such singleton local compositions? We are just given two elements $D_1 \in A@Form_1$ and $D_2 \in B@Form_2$ and a form morphism $h : Form_1 \to Form_2$ as well as an address change $\alpha : A \to B$ such that we have the equation[23] $hD_1 = D_2\alpha$. We now recognize that this is a standard concept in category theory: the category of elements[24] for set-valued functors. In that context, we have $h = Id_F$ for the underlying functor $F = Fun(Form) : \mathbf{Mod}^{opp} \to \mathbf{Sets}$, and the category of elements is $\int_{\mathbf{Mod}} F$. In other words, we have considered denotators as "elements" and morphisms between such "elements" $D_1, D_2$ as couples of a unique set map of singletons $! : \{D_1\} \to \{D_2\}$ which is induced by a functorial morphism of underlying spaces, and an address change $\alpha$. We shall denote this type of morphisms of singleton local compositions by $!/\alpha$ and also write $!/\alpha : D_1 \to D_2$ to ease notation. This means that the category of local compositions generalizes the category of elements of each of its forms.

## 8.3 Categories of Local Compositions

**Summary.** The formal definition of the category *ObLoc* of *objective* local compositions and morphisms is given and discussed with respect to its musicological relevance. A semantic model of general symmetries over the n-dimensional integer module $\mathbb{Z}^n$ is discussed.

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After having prepared all necessary prerequisites, we are ready to define the category of objective local compositions:

[23] We are a bit sloppy in identifying the denotator names with their coordinates and also omitting the address in the natural transformation $h$, but this makes notation less clumsy.

[24] See appendix G.2.1.

**Definition 26** *The category ObLoc of objective local compositions has as its object collection* ${}_0ObLoc$ *the set of all objective local compositions as declared in definition 8. They are usually denoted by their names (often capital letters* $L, K, M_1$*, etc.) if no ambiguities are likely. The set* ${}_1ObLoc$ *of morphisms is the disjoint union of the sets* $ObLoc(L_1, L_2)$ *of morphisms* $f/\alpha : L_1 \to L_2$ *as declared in definition 23. The composition of morphisms is that declared in lemma 5. The identity morphism of an objective local composition* $L$ *is the morphism* $1_L$ *as declared in point* (i) *of sorite 4.*

Mathematically, there is no further point in this definition, once its ingredients have been introduced. However, there is one delicate remark to be made as to the musical signification of the entire construct. The remark is that the categorical framework not only suggests general ambient spaces, viz. modules for **Simple** Type, but also that any 'wild' morphisms may intervene, and that any concatenation, i.e., composition[25] of morphisms is allowed—in fact a critical musical and musicological point.

Let us therefore digress on a semiotic paradigm for general symmetries, i.e., affine transformations, on 'prototypical' modules $\mathbb{Z}^n$ which underly morphisms of objective local compositions in such ambient spaces. Of course we know that underlying symmetries are only the background transformations for visible correspondences on the supports of local compositions. But we have to ask whether symmetries—as a structural potential for *possible* morphisms—could be musically significant—independently of the possible points which could be moved around between such spaces. This question becomes of primordial importance not later than when local compositions are transformed by automatic processes as they are implemented in music software, such as in *presto*®, see chapter 49. For this case we should have a certain theoretical guarantee for the musical meaning of what is being done.

Therefore we are going to discuss a semantical paradigm for symmetries between modules which play an important role in practice. Very often practice deals with gauged and then digitized parameters such that we may suppose that they are integers. We may therefore work on $\mathbb{Z}^n$ and consider symmetries $f : \mathbb{Z}^n \to \mathbb{Z}^m$. Further any such symmetry can be interpreted as an endomorphism of a large power $\mathbb{Z}^N$, for example by viewing both spaces as direct summands of a sufficiently large superspace, i.e., we may suppose $n = m$ without loss of generality. In this case, a symmetry $e^t \cdot P : \mathbb{Z}^n \to \mathbb{Z}^n$ is described by $n + n^2$ integers via vector $t = (t_1, \dots t_n)$ and matrix $P = (P_{ij})_{1 \leq i,j \leq n}$. This manifold of symmetries can be described in a very reduced language by the following theorem which is proven in appendix E.3.5 and E.3.6, and which we restate in terms that fit with the musical context:

**Theorem 2** *Consider the set*

$$\mathfrak{MusGen} = \{T, D_m(m \in \mathbb{N}), K, S, P_s(s = 2, \dots n)\}$$

*of affine endomorphisms on* $\mathbb{Z}^n$ *which consists of the following elements:*

1. $T = e^t, t = (0, 1, 0, \dots 0)$*, the translation in the direction of the second coordinate axis.*

[25] We should apologize for maintaining the mathematical terminology of "composition" of morphisms; we believe however that no confusion will arise thereby. In delicate contexts we switch to the synonym "concatenation" to make things more distinct.

2. *If* $m \in \mathbb{N}$, *let* $D_m = (d_{ij})$ *be the* $m$*-fold dilatation in the direction of the first coordinate axis. This means* $d_{11} = m, d_{22} = \ldots d_{nn} = 1$, *and* $d_{ij} = 0$ *else.*

3. $K = D_{-1}$, *dilatation by* $-1$ *or reflection in the first coordinate.*

4. $S = (d_{ij})$ *the transvection or "shearing" of the second coordinate in the direction of the first coordinate axis, i.e.,* $d_{11} = d_{12} = d_{22} = \ldots d_{nn} = 1, d_{ij} = 0$ *else.*

5. $P_s = (d_{ij})$, $s = 2, \ldots n$, *parameter exchange of first and* $s$*th coordinates, i.e.,* $d_{ij} = 0$ *for* $ij = 11, ss$, *and for* $i \neq j$ *and* $ij \neq 1s, s1$, *whereas* $d_{1s} = d_{s1} = d_{33} = \ldots d_{nn}$.

*Then every affine endomorphism on* $\mathbb{Z}^n$ *can be written as a concatenation (composition) of some of the elements of* $\mathfrak{MusGen}$ *(including repetitions of the same element).*

*Affine automorphisms (i.e. the invertible symmetries) on* $\mathbb{Z}^n$ *can be written as a concatenation of elements of* $\mathfrak{MusGen} - \{D_m, 0 \leq m\}$, *i.e., no* $m$*-fold dilatations for* $0 \leq m$.

In particular, each of the generators from $\mathfrak{MusGen}$ is concerned with affecting at most two coordinates. Except generators of type parameter exchange $P_s, 3 \leq s$, they affect only the first two coordinates!

The following *concatenation principle* yields a semantical interpretation of the preceding result. This principle can obviously not be demonstrated in full generality (this would be a test of an infinite number of musical situations—God beware!), but it states, in its very generality, a cornerstone of generic musical semantics:

**Principle 2** *Concatenation of two musically meaningful symmetries or morphisms yields a musically meaningful result.*

Before discussing this principle in subsection 8.3.1, let us apply it to theorem 2. Accordingly, every symmetry $f : \mathbb{Z}^n \to \mathbb{Z}^n$ can be written as a product

$$f = e_1 \cdot e_2 \cdot \ldots e_s$$

all factors being taken from $\mathfrak{MusGen}$. In order to turn this purely mathematical fact into a musicological one, we have to view the modules $\mathbb{Z}_n$ as a coordinator for a reasonably chosen form space. Let us take a direct sum space of shape $Onset|_{\mathbb{Z}} \oplus w\text{-}Pitch \oplus Para$ where $w\text{-}Pitch$ is the $w$-tempered scale space $w\text{-}TemperedScale$ (see section 7.2.1.1) with canonical identifier $@\mathbb{Z} \xrightarrow{\sim} @\mathbb{Z} \cdot \frac{1}{w} \cdot o$, and the space $Para$ has coordinator $\mathbb{Z}^{n-2}$ for any 'accessory' parameters such as loudness, duration, whatever. But it should be stressed that this selection is a function of the selection of the special generator set $\mathfrak{MusGen}$, and that there are many other semantical interpretations of symmetries. The subject here is not coverage of all possible meanings but evidence of the very existence of musical meaning behind such a priori constructs from mathematics. On this space, the generating symmetries from $\mathfrak{MusGen}$ have the following musical meaning:

1. Translation $T$ means **transposition** (example 7 in subsection 8.1.1) by one pitch unit upwards in $w$-tempered pitch space, a classical musical process.

2. Reflection $D_{-1}$ means **retrograde** of onsets (example 9 in subsection 8.1.1), a classical technique in counterpoint.

3. Shearing $S$ in onset direction means **arpeggio** since the notes of a chord instance at a determined onset are played one after another from low to high pitch.

4. Dilatations $D_m$ are classical **augmentations** in onset directions: distances of onsets are augmented by factor $m$.

5. Operations $P_s$ of **parameter exchange** are related to classical compositional techniques or established musical thinking. concerning exchange of onset and pitch we had already seen Messiaen's—unfortunately not very consistent—exchange principle (example 13 in subsection 8.1.1). Already in the middle ages (to which Messiaen refers [156]) we can observe a symmetry and exchangeability of treatment of modes in time and pitch. Here, $P_2$ is not applied on a given compositional material but as structural 'isomorphism of meaning', in fact a semiotical symmetry. Further we should recall the $P_2$-symmetry in twelve-note series from example 12 in subsection 8.1.1, as well as rotation by 90° which can be viewed as concatenation of retrograde and parameter exchange.

   Parameter exchange of onset with other parameters in $P_3$, $P_4$ etc. is less investigated by composers and theorists. Serial technique makes use of such symmetries in some cases [550]; a systematic application can however only be traced in nowadays computer-assisted compositional environments, see chapter 49.

Let us conclude by a remark on the group $\overrightarrow{GL}(\mathbb{Z}^n)$ of invertible symmetries on $\mathbb{Z}^n$ which is generated by transposition $T$, retrograde $K$, arpeggio $S$, and parameter exchange $P_s, s = 2, 3, \ldots n$. By such symmetries, local compositions are always transformed in a 'reversible' way. All relations among their tones are preserved without loss. In other words, every local $A$-addressed composition $K$ in ambient space $Onset|_{\mathbb{Z}} \oplus w\text{-}Pitch \oplus Para$ is embedded in its *orbit* $\overrightarrow{GL}(\mathbb{Z}^n) \cdot K = \{F(K) |\ F \in \overrightarrow{GL}(\mathbb{Z}^n)\}$ which we call $K$'s *general affine paradigm*. This concept, together with specific selections of (parametrized families of) subgroups of the full symmetry group $\overrightarrow{GL}(\mathbb{Z}^n)$ will lead to a far-reaching conceptualization of paradigmatic phenomena in music and music history, see section 10.2.

### 8.3.1 Commenting the Concatenation Principle

**Summary.** We give some musicological comments on the concatenation principle by means of two examples.

– Σ –

A. *Counterpoint.* In classical counterpoint, one considers translations, retrogrades, inversions, and retrograde inversions. This collection meets the requirement that concatenation (composition) of two of its members is musically meaningful. But there is more: Musically, the retrograde inversion (see example 9) KU isn't understood as a 180°-rotation, this does not make any sense in classical terms of European music theory. If Beethoven rightly operated on Steibelt's score by such a rotation, this was nothing more than destruction of its very intention, rotation was—in those days—simply ridiculous. The only explanatory access to this symmetry is its decomposition *qua* concatenation of retrograde $K$ and inversion $U$! In this interpretation, retrograde inversion is composed by a reasonable action in time and one in pitch, there is no genuine

theory and semantics in two-dimensional space *Onset* $\oplus$ *Pitch*. The concatenation principle precisely generalizes this classical mechanism of musical semantics.

B. *Dodecaphonism* (see also [465], [550]). Concerning semantics of symmetries on $\mathbb{Z}^2$, twelve-note composition offers an interesting example. Schönberg's exegete Joseph Rufer comments on the esthetic principles of dodecaphonism [465]:

> *Die beiden an der Entstehung einer musikalischen Form vornehmlich beteiligten Gestaltungsprinzipien sind: die* Wiederholung *und die* Variation. *Wiederholung ist das Anfangsstadium, Variation und Entwicklung die hohere Entwicklungsstufe musikalischer Formtechnik.*

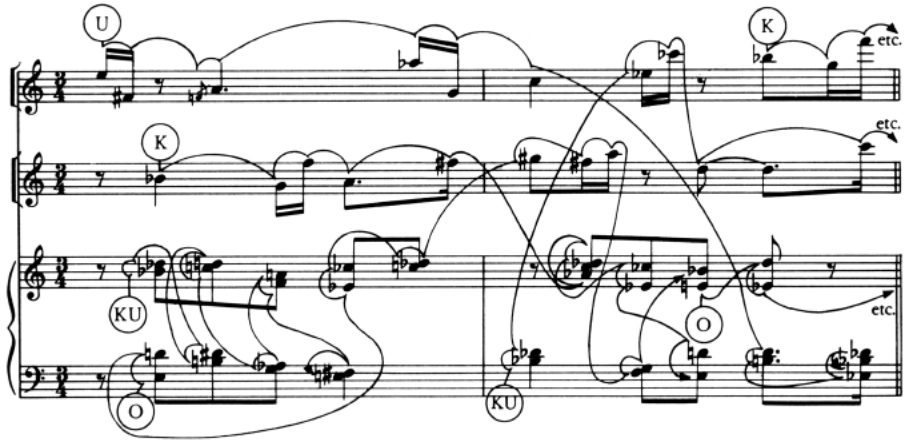


Figure 8.12: (With kind permission of Reclam Publishers.) This graphics shows complex combinations of the fundamental series and its variations in twelve-note technique.

The musical space where these transformations take place is described as follows by Schönberg [479], see also figure 8.12:

> *Der zwei- oder mehrdimensionale Raum, in dem musikalische Gedanken dargestellt werden, ist eine Einheit. Obwohl die Elemente dieser Gedanken dem Auge und dem Ohr getrennt und unabhängig voneinander erscheinen, enthüllen sie ihre wahre Bedeutung nur durch ihr Zusammenwirken, so wie kein Wort allein, ohne Beziehung zu anderen Worten, einen Gedanken ausdrücken kann. Alles was an irgend einer Stelle dieses musikalischen Raumes geschieht, hat mehr als nur lokale Wirkung. Es wirkt nicht nur in seiner eigenen Ebene, sondern auch in allen Richtungen und Flächen und zeigt seinen Einfluß selbst an entfernten Stellen.*

We pick up from this very modern text that Schönberg thinks at once in two dimensions and not twice in one dimension. This is quite a progress with respect to classical counterpoint. With this example not only semantic relevance of symmetry groups is exemplified, one also learns to understand the communicative problem implied by dodecaphonism. It seems that

dodecaphonists had realized that symmetry groups could unfold their presence to the listener only 'subcutaneusly'. Neither the "ideal" listener nor diligent analysis are in a state to decode the massive ambiguities and intertwinements in twelve-note composition. This compositional process is strictly poietic. From this we have to abandon hope to transmit the message more than in a diffuse emotional sphere. The fact that the hoped-for emotional echo has mainly failed does not prove conservative auditory habits (though this is true to a certain degree) but rather a deficiency of compositional strategy.

Let us make this more explicit. The contrapuntal symmetries which de facto embed the basic series in its paradigm is instantiated by symmetries which as musical thoughts exist on the poietical and—at best—neutral level (the latter via analytical activities). However, the temporal course of a composition does not allow the listener to reflect symmetry relations. Here the composer has to bring symmetry relations into audible shape, for instance by means of compositional parts which make these symmetries evident, as inner symmetries, say. The sonata principle has been so successful since its basic components—exposition, development, and recapitulation—build a thoroughly communicative concept. Explicating a musical thought within a musical composition, i.e., on the neutral level, is no compromise to the listener but a genuine ingredient of whatever art claims to communicate.

The concatenation principle attributes musical meaning to a concatenation of symmetries or morphisms if the factors share such a meaning. This does not provide us with a justification automatism for twelve-note composition. In fact, if the factors $e_1, \dots e_m$ are immediately evident to the listener, their product $e = e_1, \dots e_m$ does not automatically bear musical evidence, it could as well be hidden to the listener. The turning point from construction to audible evidence is the evidence of concatenation, and, even more prominently, evidence of the underlying factors since they are not at all unique. It is as if you were listening to a spoken sentence in a foreign language but you do not know its word grouping. Somebody has to make the grouping evident by means of prosodic tools like stress or pauses. A step-by-step construction of a local composition $L$ from $K$ via

$$K \mapsto e_1(K) \mapsto e_2(e_1(K)), \dots e_{n-1}(\dots e_1(K)\dots) \mapsto e(K) = L$$

must be evidenced. The entire chain of intermediate compositions must be integrated into the composition to become part of communication. We conclude that the concatenation principle is a metamusical thought. The correspondent semantic paradigmatics, especially in twelve-note and serial composition, must be turned into neutral musical 'material' in order to become musical reality. This insight is intimately related to Einstein's principle that physical information—for instance on a reference frame's time—can only be transmitted by physical support, such as light, and therefore at most at light's speed. *Each media has to unfold (communication) in its proper terms.*

### 8.3.2 Embedding and Addressed Adjointness

**Summary.** The category *ObLoc* of objective local compositions is embedded in a category *Loc* of functorial local compositions which admits sufficiently general universal constructions. It is shown that the embedding of *ObLoc* in *Loc* is full and reflects isomorphisms.

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After discussion of some topics pertaining to objective local compositions, we are ready to define the category of local compositions:

**Definition 27** *The category Loc of local compositions has as its object the collection* ${}_0Loc$ *the set of all local compositions as declared in definition 8 of chapter 7. They are usually denoted by their names (often capital letters* $L, K, M_1$*, etc.) if no ambiguities are likely. The set* ${}_1Loc$ *of morphisms is the disjoint union of the sets* $Loc(L_1, L_2)$ *of morphisms* $f/\alpha : L_1 \to L_2$ *as declared in definition 24 of this chapter. The composition of morphisms was declared in definition 25 of this chapter. The identity morphism of a local composition* $L$ *is the morphism* $1_L$ *as declared in point* (i) *of sorite 5.*

From definition 8 and formula (8.15) we know that objects and morphisms of *ObLoc* are mapped into objects and morphisms of *Loc* under a map $\hat{?}$. Recall from section 7.3 and definition 16 that for a functorial local composition $K$ at address $A$, the objective trace $\check{K}$ is given by the set $Id_A@K$. We now have this embedding proposition:

**Proposition 3** *The map*

$$\hat{?} : ObLoc \to Loc \tag{8.18}$$

*is a full functor which reflects isomorphisms.*

Proof. Functoriality of $\hat{?}$: Let $(K, A@F), (L, B@G), (M, C@H)$ be three objective local compositions, $\alpha : A \to B, \beta : B \to C$ two address changes, and $f/\alpha : K \to L, g/\beta : L \to M$ two morphisms of objective local compositions. We have to show that $\hat{g}/\beta \cdot \hat{f}/\alpha = \widehat{gf}/\beta\alpha$, i.e., $\hat{g} \cdot \hat{f} = \widehat{gf}$. Let $q : X \to A$ be a point at address $A$, and look at the composed map on the $q$-slice $q@\hat{K} = \{q\} \times K \cdot q$. Take $(q, k \cdot q) \in q@\hat{K}$, then

$$\begin{aligned}
\hat{g}(\hat{f}(q, k \cdot q)) =& \hat{g}((\alpha q, f(k) \cdot \alpha q)) \\
=& \hat{g}((\alpha q, l \cdot \alpha q)) \text{ for an } l \in L, \\
=& (\beta\alpha q, g(l) \cdot \beta\alpha q) \\
=& (\beta\alpha q, g(f(k)) \cdot \beta\alpha q) \\
=& \widehat{gf}(q, k \cdot q),
\end{aligned}$$

and functoriality is established. To see that the functor reflects isomorphisms, look at the evaluation of $\hat{f}/\alpha$ at $Id_A@\hat{K}$: We have $\hat{f}(Id_A, k) = (\alpha, f(k).\alpha)$ which uniquely determines $\alpha.f$ and $\alpha$. But if $\alpha$ is an isomorphism, so $f$ is also determined. So we are done if we can show that the functor is full. Take any morphism $f/\alpha : \hat{K} \to \hat{L}$. Its evaluation $f/\alpha : Id_A@\hat{K} \to \alpha@\hat{L}$ defines a commutative diagram

$$\begin{array}{ccc}
K & \longrightarrow & A@F \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle A@h} \\
L.\alpha & \longrightarrow & A@G
\end{array} \tag{8.19}$$

of set maps where the right vertical arrow is the natural transformation $h$ underlying $\alpha.f$. Take any $r : K \to L$ such that $s = \alpha.r$. We contend that $f/\alpha = \hat{r}/\alpha$. Let $(q, k \cdot q) \in q@\hat{K}$. Then

$$\begin{aligned} f/\alpha(q, k \cdot q) =& (\alpha q, h(k \cdot q)) \\ =& (\alpha q, h(k) \cdot q), \text{ by functoriality of } h, \\ =& (\alpha q, s(k) \cdot q) \\ =& (\alpha q, r(k).\alpha q) \\ =& \hat{r}/\alpha(q, k \cdot q), \end{aligned}$$

and we are done with the claims. QED.

In other words, the objective local compositions can be classified up to isomorphism in their proper category *ObLoc* or in *Loc*, it does not matter. Further, every local composition at address $A$ can be 'frozen' to the 'identity point' $Id_A$ and thereby reduced to its objective trace at $A$.

So the extension to functorial local compositions is seamless and can be 'traced back' to the objective framework. The deeper reason why we have built the functorial point of view upon a universe of objective local compositions is a double one. First, from the geometric and logical point of view, one needs to compare local compositions by use of universal constructions such as limits, or colimits. These not only have generic interest from category theory but are essential to model logical derivates of given ensembles of sounds and other musical objects. We shall see in chapter 18 that predicate logic on the topos theoretic level needs substantially universal constructs. But there is also a more musicological reason for our extension policy: Musical objects are not only local in their spatial nature, we shall see in the global concept frame work of part IV that gluing local objects to global patchworks is genuine musicology. However, the mathematical description of global objects has to make use of fiber products of local compositions, and it is precisely this type of universal construction which fails within the purely objective setup.

Although we can build the objective trace $\check{K}$ for any object in *Loc*, this construction does not carry over to morphisms. To achieve such a goal, we have to restrict to a fixed address. Let us therefore define addressed comma categories as follows. Fix an address $A$. When talking of $@A$ as of a local composition we identify it with the functor—in fact the objective local composition—$@A \times @0_0$, where $0_0$ is the zero module over the zero ring, i.e., the terminal[26] object of **Mod**. We have $X@(A \times @0_0) = X@A \times \{0\} \xrightarrow{\sim} X@A$. With this convention, we denote by $ObLoc_{@A}$ and $Loc_{@A}$ the comma categories over the local composition $@A$ where the structural address change is the identity on the address, i.e., the unique morphism $!/Id_A = !/1 : K \to @A$. On $Loc_{@A}$, we have an objective trace functor

$$\check{?}_{@A} : Loc_{@A} \to ObLoc_{@A}$$

which carries $K$ to $\check{K}$ and $f/Id_A : K \to L$ to $\check{f}/Id_A : \check{K} \to \check{L}$ via $Id_A@f : Id_A@K \to Id_A@L$. This gives us an evident restriction of the embedding functor and the addressed objective trace functor

$$ObLoc_{@A} \underset{\check{?}_{@A}}{\overset{\hat{?}_{@A}}{\rightleftarrows}} Loc_{@A} \tag{8.20}$$

[26] See appendix G.2.1.

since the basis object $@A$ and the structural morphisms $!/Id_A$ remain fixed under both functors.

**Proposition 4** *The addressed restriction functor $\check{?}_{@A}$ is a left inverse to the addressed embedding $\hat{?}_{@A}$*

**Exercise 10** Give a proof of the preceding proposition.

We terminate this subsection with a short discussion of adjointness [27] of the embedding functor $\hat{?}_{@A}$ and the objective trace functor $\check{?}_{@A}$, a subject which is central for the topos-logical approach, see for example [314]. We have adjointness for fixed address, i.e., 'addressed' adjointness:

**Proposition 5** *The morphisms $\hat{?}_{@A}$ and $\check{?}_{@A}$ build an adjoint pair $\hat{?}_{@A} \dashv \check{?}_{@A}$.*

Proof. By definition 16 in section 7.3 and proposition 3, we have an isomorphism

$$Hom_{ObLoc_{@A}}(K, \check{L}) \xrightarrow{\sim} Hom_{Loc_{@A}}(\hat{K}, L^\diamond)$$

and by the injection $L^\diamond \rightarrowtail L$ from sorite 1, we have an injection

$$Hom_{Loc_{@A}}(\hat{K}, L^\diamond) \rightarrowtail Hom_{Loc_{@A}}(\hat{K}, L).$$

So all we have to show is that every morphism $f/1 : \hat{K} \to L$ over $A$ factorizes through $L^\diamond$. Consider the effect of a morphism $f/1 : \hat{K} \to L$ at a point $x : X \to A$, i.e., the application $x@f : x@\hat{K} \to x@L$. By definition, $Id_A@L = Id_A@L^\diamond$. So we have the commutative diagram

$$\begin{array}{ccc} Id_A@\hat{K} & \xrightarrow{Id_A@f} & Id_A@L^\diamond \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x} \\ x@\hat{K} & \xrightarrow{x@f} & x@L \end{array} \tag{8.21}$$

But the very definition of the embedding $\hat{?}$ (see (6.16)) implies that the image of $x : Id_A@L^\diamond \to x@L$ stays in $x@L^\diamond$. By the surjectivity of the left vertical $x$-arrow, the image of $x@f$ also lives in $x@L^\diamond$, QED.

### 8.3.3 Universal Constructions on Local Compositions

**Summary.** We show the finite completeness of *Loc* and its interpretation in musical terms.

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It is well-known [314] that $\mathbf{Mod}^@$ is a topos, and in particular admits universal constructions such as finite limits, colimits and a power object. Since the denotator space functors are in $\mathbf{Mod}^@$, there is a good chance that local compositions too admit such universal constructions. We want to investigate the specific construction in this subsection. Recall from appendix G.2.1 that in $\mathbf{Mod}^@$, limits and colimits are "calculated pointwise".

[27] See appendix G.2.1 for the concept of adjointness.

**Theorem 3** *The category Loc has fiber products.*

Proof. Given three local compositions $K \hookrightarrow @A \times F, L \hookrightarrow @B \times G, M \hookrightarrow @C \times H$ and two morphisms $f/\alpha : K \to M,\ g/\beta : L \to M$, we are looking for a local composition $P \hookrightarrow @D \times R$ and two morphisms $q/\kappa : P \to K,\ p/\lambda : P \to L$ such that we have a pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{p/\lambda} & L \\ {\scriptstyle q/\kappa}\downarrow & & \downarrow{\scriptstyle g/\beta} \\ K & \xrightarrow{f/\alpha} & M \end{array} \tag{8.22}$$

of local compositions. From appendix E.3.8 we know that the underlying address changes have a fiber product

$$\begin{array}{ccc} D & \xrightarrow{\lambda} & B \\ {\scriptstyle \kappa}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha} & C \end{array} \tag{8.23}$$

of modules and di-affine morphisms in **Mod**, i.e., $D = A \times_C B$. Further, suppose that $f, g$ are induced by the functor morphisms $h : F \to H, k : G \to H$. Then we have two cartesian diagrams in $\mathbf{Mod}^{@}$:

$$\begin{array}{ccc} R & \xrightarrow{s} & G \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle k} \\ F & \xrightarrow{h} & H \end{array} \tag{8.24}$$

with $R = F \times_H G$ and

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\tilde{p}} & L \\ {\scriptstyle \tilde{q}}\downarrow & & \downarrow{\scriptstyle g} \\ K & \xrightarrow{f} & M \end{array} \tag{8.25}$$

with $\tilde{P} = K \times_M L$ *as set-valued functors*. But then, the product diagram

$$\begin{array}{ccc} @(A \times_C B) \times F \times_H G & \xrightarrow{@\lambda \times s} & @B \times G \\ {\scriptstyle @\kappa \times r}\downarrow & & \downarrow{\scriptstyle @\beta \times k} \\ @A \times F & \xrightarrow{@\alpha \times h} & @C \times H \end{array} \tag{8.26}$$

is also cartesian. Since the cartesian diagram (8.25) has its functors $K, L, M$ and their morphisms $f, g$ as subfunctors of the corresponding ambient space functors in diagram (8.26), the fiber product $\tilde{P}$ clearly injects into $@(A \times_C B) \times F \times_H G$, and we can replace $\tilde{P}$ by its image $P = im(\tilde{P})$, and the morphisms $\tilde{q}, \tilde{p}$ by the corresponding morphisms $q, p$ from $P$, without losing the fiber product properties of the functor $\tilde{P}$. In other words, we have a commutative

diagram of local compositions with ambient spaces from diagram (8.26). We now have to show that this diagram is cartesian. To this end, give any commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{v/\nu} & L \\ {\scriptstyle u/\tau}\downarrow & & \downarrow{\scriptstyle g/\beta} \\ K & \xrightarrow{f/\alpha} & M \end{array} \tag{8.27}$$

of local compositions with $T \hookrightarrow @X \times W$, where $u/\tau$ is induced by $m : W \to F$, and where $v/\nu$ by $n : W \to G$. Then we have a uniquely determined functor morphism $t : T \to P$ and a uniquely determined di-affine morphism $\rho : X \to D$ such that $q \cdot t = u, p \cdot t = v$ and $\kappa \cdot \rho = \tau, \lambda \cdot \rho = \nu$. So if $t/\rho$ is a morphism of local compositions, it is uniquely determined and we are done.

If $z : W \to R$ is the universal arrow induced by $r, s$, we have a space functor morphism $@\rho \times z : @X \times W \to @D \times R$, and this one evidently induces $t$, QED.

**Corollary 1** *The category Loc is finitely complete, i.e., has finite limits.*

Proof. We use the fact (see appendix G.2.2) that a category has finite limits iff it has fiber products and a terminal object. Let $0_0$ be the terminal object (see appendix G.2.1) in **Mod**. The local composition $1_{Loc} = @0_0 \times @0_0$ then is terminal since any $A$-addressed local composition $K$ has a unique morphism $!/! : K \to 1_{Loc}$. Together with theorem 3 we can apply the criterion for finite limits, QED.

**Corollary 2** *For a morphism $f/\alpha : K \to L$ in Loc the following statements are equivalent:*

(i) *$f/\alpha$ is mono.*

(ii) *$f$ and $\alpha$ are mono.*

(iii) *$X@f$ is injective for every address $X$ and $\alpha$ is diinjective.*

Proof. It is well-known (see appendix G.2.2) that $f/\alpha$ is mono iff in the cartesian product

$$\begin{array}{ccc} \Delta & \xrightarrow{f_2/\alpha_2} & K \\ {\scriptstyle f_1/\alpha_1}\downarrow & & \downarrow{\scriptstyle f/\alpha} \\ K & \xrightarrow{f/\alpha} & L \end{array} \tag{8.28}$$

both projections $f_1/\alpha_1, f_2/\alpha_2$ coincide and are isomorphisms. But clearly, this is equivalent to $f_1 = f_2$ being an isomorphism, and $\alpha_1 = \alpha_2$ being an isomorphism, and this in turn means that $f$ and $\alpha$ are mono. Therefore (i) and (ii) are equivalent. Clearly, (iii) implies (ii). Conversely, we know from appendix G.2.2 that a natural transformation $f$ in $\mathbf{Mod}^{@}$ is mono iff its evaluation at every address $X$ is injective. By proposition 86 in appendix E, the required characterization of mono diaffine morphisms is given, QED.

Remark. The dual situation is less simple, in fact, general fiber sums do not exist in *Loc*. This implies that we cannot glue together arbitrary local compositions to a new local composition. In other words, 'global compositions' as a result of gluing together local compositions present a proper extension of the local concept framework. This is the *mathematical* reason for the entire global theory to be developed later in part IV.

Let us now evaluate the finite completeness of *Loc* in more concrete terms and with regard to the subcategory of objective local compositions. To begin with, we should discuss an example where the fiber product of objective local compositions is a proper functorial local composition.

To understand this phenomenon we want to construct a fiber product which does not fulfill the following evident requirement for objective local compositions: For any objective local compositions $K$ at address $A$, and for any $X$-valued point $x : X \to A$, the map $x : Id_A@K \to x@K$ is surjective. We shall in fact exhibit a pair $f/\alpha : K \to M,\ g/\beta : L \to M$ of objective local compositions for address changes $\alpha : A \to C,\ \beta : B \to C$ such that the fiber product map $u : Id_{A\times_C B}@(K \times_M L) \to u@(K \times_M L)$ is not surjective for a selected address change $u : W \to A \times_C B$.

Take $A = B = C = \mathbb{Z}_{\mathbb{Z}}$ and take the zero point $x = 0 : 0_{\mathbb{Z}} \to A$. Consider two non-empty objective local compositions $K, L \subset A@Pitch$ in the pitch form with ambient space $F = @\mathbb{R}$. This means that we take sets of intervals $i = i_0 \Rightarrow i_1$, corresponding to morphism $e^{i_0} \cdot (i_1 - i_0)$ as in subsection 6.2.1. Suppose that $K \cap L = \emptyset$, more precisely that $Kx = Lx$, i.e., that the base points of the intervals in $K, L$ are the same, but the interval arrow heads $i_1$ never coincide. Take $M = A@F$, and look at the natural inclusions $K \hookrightarrow M,\ L \hookrightarrow M$. Then the evaluation of the fiber product $T = \hat{K} \times_{\hat{M}} \hat{L}$ gives $Id_A@T = \emptyset$, and $x@T \cong K$ so that the morphism $x : Id_A@T \to x@T$ cannot be surjective and the fiber product $T$ not objective. In other words, the objective perspective at address $A$ yields no common points whereas the zero-addressed functorial perspective at $x$ shows common base points, see also figure 8.13.

### 8.3.4 The Address Question

**Summary.** We make the address concept explicit: Variation of addresses in music and musicology. We approach categories as point spaces which are sorted and structured by addresses.

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We should now review the entire universe of denotators, objective and functorial local compositions and the related address navigation in order to understand the overall 'geography' of addresses.

From an overall point of view, we have built a comprehensive category *Loc* of local compositions. It extends the full subcategory *ObLoc* of objective local composition which are essentially sets of denotators of address-fixed "ambient space". The latter include the *singletons* which are the very general denotators and take care of the classical construction of "categories of elements". Let us denote this latter category by *SinLoc*. So we have the chain

$$SinLoc \hookrightarrow ObLoc \hookrightarrow Loc \tag{8.29}$$

of full subcategories.

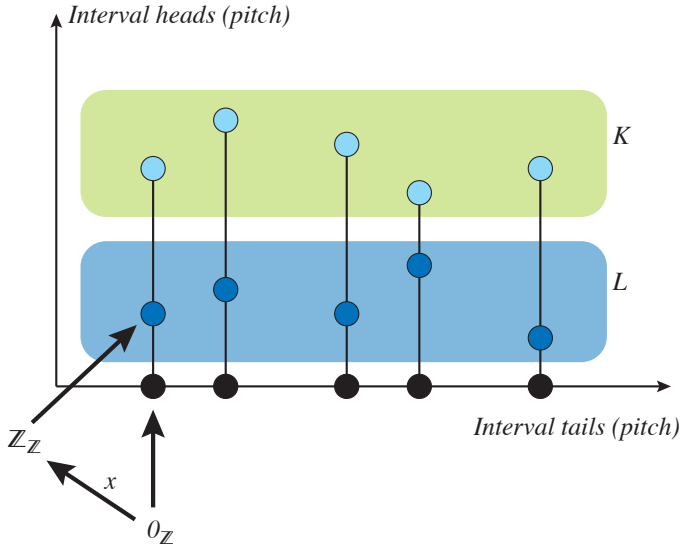


Figure 8.13: Two objective local compositions $K, L$ in pitch space $M$ at address $\mathbb{Z}$ with identical 'perspective' at the zero-addressed point $x = 0 : 0_{\mathbb{Z}} \to \mathbb{Z}$. The functorial fiber product $\hat{K} \times_{\hat{M}} \hat{L}$ identifies these common 'base points' whereas the objective fiber product (in fact the intersection) is empty.

On the other hand, every denotator $D$ has an address $Ad(D)$, i.e., a module object in **Mod**. This defines a map

$$Ad : Loc \to \mathbf{Mod} : D \mapsto Ad(D) \tag{8.30}$$

which is indeed a functor, the *address functor*, since morphisms $f/\alpha$ of local compositions include address change $\alpha$. Within this framework, we have different fibers which are of interest in determined contexts. For example, if we consider the full subcategory $End(A) \hookrightarrow \mathbf{Mod}$ whose only object is $A$, and its 'identity' subcategory $Id_A$ with the identity of $A$ as its only arrow, then we have two corresponding fiber categories

$$
\begin{array}{ccccc}
SinLoc_{@A} & \longrightarrow & SinLoc_{End(A)} & \longrightarrow & SinLoc \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \text{subcategory}} \\
ObLoc_{@A} & \longrightarrow & ObLoc_{End(A)} & \longrightarrow & ObLoc \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \text{full and reflexive}} \\
Loc_{@A} & \longrightarrow & Loc_{End(A)} & \longrightarrow & Loc \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle Ad} \\
Id_A & \xrightarrow[\text{subcategory}]{} & End(A) & \xrightarrow[\text{subcategory}]{} & \mathbf{Mod}
\end{array}
\tag{8.31}
$$

defined by this evident system of cartesian squares. In principle, every subcategory of **Mod** defines an associated theory of musical objects on its *Ad*-fiber. The indicated fixations of $A$ are only the most obvious constructs. But there is more: If we are given an objective local composition $K \subset A@F$ in the $A$-fiber $ObLoc_{@A}$, the passage to its associated functorial 'copy' $\hat{K} \hookrightarrow @A \times F$ is indeed an extension of the fixed address to variable address perspectives. The evaluation of $\hat{K}$ at a morphism $x : X \to A$ is a kind of '$K$'s view as seen from address $X$ under the perspective $x$'. What is seen is the 'slice' $x@\hat{K}$, the fiber of $X@\hat{K}$ at point $x$ (see also figure 7.11 for this interpretation). We shall see in chapter 9 that the Yoneda lemma essentially instantiates this 'philosophy of perspectives': We look at an address $A$ from all other addresses and try to understand what's happing at $A$ by collecting all these relative views.

But the address question is not only a matter of relativization of perspectives, it also deals with ontology. Let us ask for the existentiality of a local composition. Viewed as a functor it is not one naive set but an entire connected collection of sets, viz all its evaluations at addresses $X$. Even an objective local composition is solicited by an underlying space functor whose evaluation at the composition's address produces the foreground set structure. The evaluation at a determined address generates a set which is specifically related to this address. The functor can only be 'seen' under such an evaluation.

But what part of a functor can be seen at a determined address is not only mathematically different, it is also ontologically different. For instance, in our example from figure 8.13, we have $\mathbb{Z}$-valued points which reflect concepts around intervals in pitch space. Musicologically this perspective traces an ontology of different flavor compared to the poor reduction to base points under the zero address. Intervals are not just an ordered couple of zero-valued points! For instance, they have a radically different perceptual quality in music psychology, esthetics and in compositional contexts. In this spirit we can make the point as follows:

**Principle 3** *The fibration of Loc and its subcategories via the address functor constitutes a 'bundle of ontologies', one for each address—or even for each subcategory of* **Mod**. *Address change therefore is an expression of an ontological shift and thereby constitutes a fundamental reconstruction tool of musical reality from a bundle of addressed ontologies.*

### 8.3.5 Categories of Commutative Local Compositions

**Summary.** This subsection deals with the structure of the practically very important category

of commutative (objective) local compositions.

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**Definition 28** *Let $A$ be an address which is an $R$-module for a commutative ring $R$. Then we denote by $ComLoc_A$ the subcategory of $ObLoc_{@A}$ whose objects are commutative, $A$-addressed local compositions*[28] *as defined in section 7.4. Its morphisms are all morphisms $f = f/1$ of $ObLoc_{@A}$, which means that no address changes are admitted. Hence a morphism $f : K \to L$ is a set map $f$ such that there exists an $R$-affine morphism $h : M \to N$ of the respective coordinator modules such that the diagram*

$$\begin{array}{ccc} K & \longrightarrow & A@_R M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle A@_R h} \\ L & \longrightarrow & A@_R N \end{array} \tag{8.32}$$

*commutes*[29]*.*

**Lemma 6** *With the preceding notations, if a morphism $f : K \to L$ in $ComLoc_A$ has underlying morphism $h : M \to N$ of the supporting coordinator modules, and if $h_0$ is the $R$-linear part of $h$, then the $R$-linear application*

$$R.f : R.K \to R.L : x \mapsto h_0.x \tag{8.33}$$

*is well-defined, it only depends on $f$. The application $R.f$ is called the* linear map associated with *$f$. The association $f \mapsto R.f$ is functorial, i.e., $R.1_K = 1_{R.K}$ and $R.(g \cdot f) = R.g \cdot R.f$. We therefore have a covariant functor*

$$R : ComLoc_A \to \mathbf{Mod}_R \tag{8.34}$$

*called the* module functor*.*

Proof. Once the first assertion is proven, it is clear that $f \mapsto R.f$ is functorial. If $K$ is empty, the first assertion is clear since, by definition, $R.\emptyset = \emptyset$. But if $k_0 \in K$, we have $x = k - k_0$, and $h_0.x = h_0.k - h_0.k_0 = h.k - h.k_0 = f(k) - f(k_0)$, so $h$ does not really matter, and the result lives in $R.L$. QED.

Consider the following subcategories:

- The full subcategory $ComLoc_A^{emb}$ of $ComLoc_A$ whose objects are the embedded commutative local compositions.
- The full subcategory $ComLoc_A^{gen}$ of $ComLoc_A^{emb}$ whose objects are the generating commutative local compositions.

[28] Recall that we often suppress form names and simply write $(L, A@M)$ or $(L, A@_R F)$ for a simple ambient space $F$ with Functor $Fun(F) = @_R M$ for such an object since the names do not intervene for classification (= determination of isomorphism classes in a category). They however play a role for identification, but this is a straightforward aspect which may be suppressed for the majority of mathematically oriented considerations.

[29] The notation $A@_R h$ with index $R$ means that di-morphisms are restricted to the identity of the coefficient ring $R$.

- The full subcategory $ComLoc_A^{in}$ of $ComLoc_A^{emb}$ whose objects have injective $R$-modules as coordinators.

**Fact 3** *For an address $A$, there is a faithful functor*

$$gen : ComLoc_A^{emb} \to ComLoc_0^{gen}$$

*defined by $gen(K, A@_R M) = (K, 0_R@R.K)$ and $gen(f : K \to L) = R@f$ according to appendix E, lemma 82.*

**Sorite 6** *For zero address $A = 0_R$, a local composition $(K, 0_R@M)$ may be identified with $(K, M)$. With this notation, we have the following results:*

(i) *The embedding $ComLoc_0^{emb} \rightarrowtail ComLoc_0$ is an equivalence of categories.*

(ii) *Let $ingen : ComLoc_0^{in} \to ComLoc_0^{gen}$ be the functor which maps $(K, M)$ to $(K, R.K)$ and a morphism $f : (K, M) \to (L, N)$ to $R@f$ as defined in the above fact 3 for embedded local commutative compositions. Then ingen is an equivalence of categories.*

(iii) *Let $(K, M)$ and $(L, M)$ be two objects in $ComLoc_0$ such that their coordinator has finite length[30] $l(M)$. Then each isomorphism $f : K \xrightarrow{\sim} L$ is induced by an invertible symmetry $F \in \overrightarrow{GL}(M)$, i.e., the orbit $\overrightarrow{GL}(M) \cdot K$ describes all local compositions with coordinator $M$ which are isomorphic to $K$.*

Proof. (i) If $K$ is an object of $ComLoc_0$, and if $k_0 \in K$, we have an isomorphism $e^{-k_0} : K \xrightarrow{\sim} e^{-k_0}(K) = L$, and $L$ is evidently embedded. Since the embedding of embedded objects is fully-faithful, we are done. (ii) Since every generating $(K, M)$ is the image of $(K, I(M))$ for the injective envelope[31] $I(M)$ of $M$, *ingen* is surjective. Since $R@f$ is an extension of $f$ to the compositions's modules, *ingen* is faithful. Let $f : K \to L$ be a morphism in $ComLoc_0^{gen}$. Then any underlying symmetry $F : R.K \to R.L$ extends to a symmetry $H : I(R.K) \to I(R.L)$, i.e., the diagram of modules and $R$-affine morphisms

$$\begin{array}{ccc} R.K & \xrightarrow{i_K} & I(R.K) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle H} \\ R.K & \xrightarrow{i_L} & I(R.L) \end{array} \tag{8.35}$$

commutes. In fact, we may suppose without loss of generality that $F$ is linear, and then, by injectivity of $I(R.L)$, the map $i_L \cdot F$ factorizes through injection $i_K$, and *ingen* is full. Therefore *ingen* is fully faithful and surjective, i.e., an equivalence.

(iii) Clearly, we may suppose that $K$ and $L$ are embedded and that the isomorphism $f : K \xrightarrow{\sim} L$ and its inverse $f^{-1} : L \xrightarrow{\sim} K$ are induced by linear symmetries $G$ and $H$, in particular $f = R.f|_K$. According to Fitting's lemma (appendix E.2.4, lemma 77), there are a natural number $n$ and two direct decompositions $M = S_1 \oplus T_1 = S_2 \oplus T_2$ such that:

[30] See appendix E.2.4
[31] See appendix E.4.3.

1. $RK \subset S_1, R.L \subset S_2$,
2. $(HG)^n|_{S_1} \in GL(S_1), (GH)^n|_{S_2} \in GL(S_2)$,
3. $T_1 = Ker(HG)^n, T_2 = Ker(GH)^n$.

Since $(GH)^n$ kills $T_2$, $G(HG)^n|_{S_1}$ maps $S_1$ into $S_2$, and by a symmetric argument, $H(GH)^n|_{S_2}$ maps $S_2$ into $S_1$. But the composition $H(GH)^n|_{S_2} \cdot G(HG)^n|_{S_1} = (HG)^{2n+1}|_{S_1}$ has its $n$th power in $GL(S_1)$, so it is also in $GL(S_1)$. By a symmetric argument on $S_2$ we conclude that $G(HG)^n|_{S_1} : S_1 \xrightarrow{\sim} S_2$; also $G(HG)^n|_K = f$. Since $l(M) < \infty$, the theorem of Krull–Remak–Schmidt (appendix E.2.5) implies that $T_1 \xrightarrow{\sim} T_2$, and we obtain an invertible extension of $G(HG)^n|_{S_1}$ to all of $M$, QED.

**Corollary 3** *If $R$ is a semi-simple commutative ring, i.e., a finite direct product of commutative fields (see appendix E.2), then the categories $ComLoc_0$ and $ComLoc_0^{gen}$ are equivalent.*

Proof. Since in this case, every module is injective, we have $ComLoc_0^{in} = ComLoc_0^{emb}$, and the corollary follows from sorite 6, (i) and (ii).

Statement (iii) of sorite 6 can be restated in the following way. For an $A$-addressed commutative local composition $(K, A@_R M)$ in $ComLoc_A$, we denote by $Sym(K)$ the *symmetry group of* $K$ of all invertible symmetries $h \in \overrightarrow{GL}(M)$ which induce an automorphism of $K$. By definition, we have a group homomorphism

$$r : Sym(K) \to Aut(K) : h \mapsto h|K \tag{8.36}$$

into the automorphism group of $K$. This is neither surjective nor injective in general, but *for a ambient module $M$ of finite length, the homomorphism is surjective*, by sorite 6, (iii), and in this case, if we denote $Triv(K) = Ker(r)$ the group of those symmetries of $K$ which act trivially (as identity) on $K$, we have

$$Sym(K)/Triv(K) \xrightarrow{\sim} Aut(K) \tag{8.37}$$

if $K$'s ambient module $M$ is of finite length. But be aware that the isomorphism class of $Sym(K)$ is *not* an invariant of the class of $K$ while the automorphism group $Aut(K)$ clearly is.

# Chapter 9

# Yoneda Perspectives

*Kant m'apprit*
*qu'il n'y a point de nombres,*
*et qu'il faut faire les nombres*
*chaque fois qu'il faut les penser.*
Alain [13]

**Summary.** This chapter reviews and completes the inherent paradigm change from "object-based" to functorial mathematics which was initiated by category theory, and completed in the celebrated Yoneda lemma. Beyond an apparant technical innocence, this lemma introduced a revolution in understanding structures of general types, and transcending pure mathematics. Its implications touch general hermeneutics as well as esthetics of art, and—quite paradoxically!—principles of "object-oriented" programming.

– Σ –

At first sight Nobuo Yoneda's lemma[1] is a technical tool of category theory. In algebraic geometry as it was developed by Alexander Grothendieck in his Paris School, the lemma serves as a background to rebuild the very concept of a point as an elementary structure of generic geometry. But the deeper impact of this lemma is a radical review of the nature of human conceptualization. This is what we should keep in mind when tracing the role of what is called the "Yoneda philosophy" since its introduction in mathematical music theory [332], [335].

The drama behind the Yoneda revolution in mathematical sciences is that it proposes a thorough geometrization of human concepts—without limitations. And it does this in an approach of infinite recursive descent: a point is a point is a point... there is no lowest level, and this means that there is no basic level of concepts; thinking in this style is bottomless and creates existence from the very fact of performing logically consistent mental processes of conceptualization. This is a modern form of Paul Finsler's principle [152]:

[1] The contribution of Yoneda to this lemma is not completely clear, but the name has been commonly accepted.

**Principle 4** *Mathematical existence of an object means that it can be thought of in a logically consistent*[2] *way.*

It resembles Murray Gell-Mann's "Eleventh Commandment" which states that whatever is not forbidden is mandatory, it has to happen somewhere and sometimes. To Finsler (see figure 9.1), mathematics was not the usual formal game with empty symbols[3], to him, mathematical objects were entities that existed like physical objects, though in a more "platonic" mental reality. Existence of such "logical" objects meant consistence with absolute logic, and this was not only the usual concept framework: Finsler was extremely concerned with circular definitions as an important device for creating basic objects of mathematics. To him, the definition of a circular set $M = \{M\}$ is acceptable as long as it does not create logical inconsistencies.

Usually, we are not aware that *conceptual circularity is unavoidable*. For instance, the set concept is circular: Set theory is defined by a fundamental concept "set" and a binary element relation "$\in$" among instances of the set concept. However, to bring set theory to existence, we need examples of sets which are generated by special definitions or definition schemes. To do so, we have the well-known identification of a set $S$: It is a reference to the elements of $S$, i.e. those sets $X$ such that $X \in S$. This is a circular concept: We ask "What is $S$?" And you answer: "It is the collection of those *sets* $X$ which are elements of $S$." Your set $S$ is only identified via a pointer to a well-defined collection of instances of the *same* concept. The only set which (in the usual setting, see C.1) exists from scratch is the empty set $\emptyset$: It is defined by the statement that "for every set $X$, $X \notin \emptyset$", or shorter: "$S$ refers to no element". The empty set is the pure reduction to the concept. It is the reference to all instances of the concept, together with the overall negation of the element relation.

To make this evident, we could as well restart set theory with a new name for the basic concept: Call the new objects "newset", and define a "newset" theory with everything except the name unaltered. We could then start with the "newemptyset" $\emptyset_{new}$ defined by "for every newset $X$, $X \notin \emptyset_{new}$". From the perspective of mathematical model theory, we would say that the newset theory is just another model for set theory. But this is not the point since we are not dealing with formal languages which are mapped to some models in a given reality: The question is about the very concept building: What do we think when we think of mental objects named "sets"? The point then is that we think of self-referential, circularly defined objects and that, while doing so, we have to investigate a universe of such objects which can be built without logical inconsistency. The set $\emptyset$ is such an object for the concept type "set". And $\emptyset_{new}$ is such an object for the concept type "newset".

This is exactly the situation of denotators and their forms: Up to form names, the denotators are the same—one species lives in the form named "set", the other in the form named "newset". These spaces are isomorphic, the difference is reduced to their names. The first form is $set \underset{Id}{\longrightarrow} \mathbf{Power}(set)$, the second is $newset \underset{Id}{\longrightarrow} \mathbf{Power}(newset)$, and the **Power** type is nothing but the element relation in its category-theoretic transfiguration. These reflections make clear

[2]To Finsler logic was not a formal game but "absolute logic" which reaches as far as the three fundamental laws: A is identical to A, A and Non-A are mutually exclusive and a third possibility is excluded.

[3]He used to call the formal games "paper science", and we adhere to this verdict. There is no substitute for really performing thoughts; no arsenal of strings, no memory storage device can replace mental activity since the latter is semantic. It is not important which organism does the job—a computer could as well—but it is not legitimate to "delegate thoughts to philosophers", which mathematicians often prefer for shear comfort of passing to their beloved game of empty forms, see [281], for example.

Figure 9.1: The mathematician Paul Finsler (1894–1970) was a pioneer of circular and non-formal concept design.

that we understand concepts as being pointers, i.e. structured recursive references with possibly circular recurrence. Quite radically, there is nothing behind concepts except the paths and ramification modes leading down to their referenced 'coordinate concepts'. This is a thorough geometrization of the concept structure: Concepts are points that point to other points...

Now this is very much the same idea tah you learn from Yoneda's lemma: It tells mathematicians that all they can understand about mathematical objects is from the way they point at other objects, from these to still others, and so forth. Replacing a "real" object $X$ by its functor $@X$ means forgetting about the "point as a such" and being exclusively concerned with its reference to other points. Of course, it is not obvious that this methodology gives you back all you need to know about an object, viz its isomorphism class. But this is Yoneda's kick: It is indeed sufficient to know about the pointer chains. And this in turn means:

**Thesis 1** *Concepts are nothing more than this recurrent pointing. Their semantics is nothing more than this thorough instantiation of the coordinates of your concepts.*

We should however not enter a more detailed discussion after this bunch of provocations of mathematical catechism[4] without stressing that Yoneda's basic epistemological insight is fundamentally the same as it is recognized in the theory of fine arts and of musical performance. We shall discuss this topic in section 9.4 below and hope that the reader may now feel and then understand that the previous remarks about concept architectures are only apparently limited to mathematics.

[4]We should remark that Finsler's point of view was passionately attacked by the establishment of "paper math" logicians [153]. I have participated in the last Finsler seminar before his death, and I admit that his anti-formalist point of view never failed, he had a unique instinct in unveiling hidden formalia.

## 9.1 Morphisms Are Points

**Summary.** Reviewing Euclid: points, addresses, pointers, and arrows. Why morphisms are elementary.

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In the first book of his elements [215], Euclid defines point, line, and surface. Definition one reads: "A point is that which has no part." Definition three: "A line is breadthless length." So a line has a part: length, but no part called "breadth". And definition five reads: "A surface is that which has length and breadth only." In contrast to a point, a line has an attribute and a surface has two of them. This does not mean that a point is nothing, and that two lines are equal if their lengths are. Points, lines, and surfaces have an identity, but points have no further specification of their identity whereas lines and surfaces have[5].

So we are confronted with the identification of points. What can be the identification criterion if the intended point object has no part or attribute which can be used to make its identity more explicit? In analytical geometry, a point would have its coordinates as parts, and we were done, but here, the concept is irreducible. Why is it imaginable, and at least intuitively acceptable, that a point object could have an identity and nothing else? Evidently, the identification act seems a *secure* process which is not further problematic. In the etymology of the word "point", we indeed find a strong argument for the acceptance of the said identification mechanism: "Point" stems from latin "punctum", the past participle of "pungere", to prick. The latter is defined by the circular image "to pierce slightly with a sharp point"[6]. The important fact is that a point is a result of a movement with an arrow-like, acute (sharpened) instrument, its identification is established by a pointing action—the English expression "to point at" is essential: a point is what we point at. It is the result of a pointing gesture which in German reads "zeigen", to show.

**Thesis 2** *Identification of what has no parts is that pointing gesture.*

In Euclid's identificatory approach, two particularities were unspecified: The pointing subject and the variety of identification types. First, the pointing subject was uniquely there to 'throw the identification arrows', it was hidden as a subsidiary instance behind the targeted points, had to hold a quiver filled with identification arrows, nothing else. The second lacuna is the variety of arrows that is offered in Euclid's geometry. There is only the 'point' type of arrows. No other irreducible targets are considered.

For a long time, the problem of identification arrows was not recognized. From the naive Platonic point of view, mathematical objects were cast somewhere in a Platonic topos and one had to recognize them, to describe their eternal properties. The turning point came through a complex, but fascinating, review of the theory of polynomial equations which eventually ended in modern algebraic geometry. In the vein of analytical geometry as initiated by Descartes, a point $p$ is a compound concept, it is identified via its numerical coordinates: $p = (x, y, z)$.

[5] We are not discussing synthetic geometry here where points as well as lines are formal objects of an axiomatic system, but Euclid's attempt to introduce meaningful objects via attributes. The fact that Euclid did virtually not use his fundamental definitions in his axiomatic system of geometry is not our argument.

[6] This is Webster's explanation. The explanation of the German Brockhaus is not less circular: "Punkt" refers to "Geometrie"/"Raum", and there, "Raum" is explained as "Menge von Dingen, die *Punkte* genannt werden".

In the classical setting, the coordinates are 'given' numbers, there is no question of inventing whatsoever; either we have the coordinates or we don't. But the algebraic approach suggested a radical review of this view. The point $p$ can be seen as a solution of three polynomial equations $X-x=0, Y-y=0, Z-z=0$. The three linear polynomials $X-x, Y-y, Z-z$ intervene in the following restatement: We look at the ring homomorphism[7]

$$ev_p : \mathbb{R}[X,Y,Z] \to \mathbb{R} : f(X,Y,Z) \mapsto f(x,y,z) \tag{9.1}$$

on the ring of polynomials $f(X,Y,Z)$ with real coefficients in the variables $X,Y,Z$. The point $p$ is then associated with the kernel of $ev_p$ which is the (maximal) ideal[8] $(X-x, Y-y, Z-z)$ generated by the three polynomials $X-x, Y-y, Z-z$. So the coordinates of $p$ are recovered by generators of $Ker(ev_p)$.

This means that we can restate a point $p$ by the evaluation homomorphism $ev_p$. This seemingly complicated restatement has a far-reaching advantage. We may now look at homomorphisms $e : \mathbb{R}[X,Y,Z] \to B$ with codomain *any* commutative ring $B$ and identify it with a kind of 'generalized point'. This point has coordinates $x = e(X), y = e(Y), z = e(Z)$, but these values are no longer visible in the naive context of real coordinates. Let us see why this is completely natural! Take the canonical factorization homomorphisms

$$e : \mathbb{R}[X,Y,Z] \to \mathbb{R}[X,Y,Z]/(X,Y,Z^2+1)$$

with $x = X \pmod{X,Y,Z^2+1}, y = Y \pmod{X,Y,Z^2+1}, z = Z \pmod{X,Y,Z^2+1}$. Thus we have $x = y = 0$ and $z^2 = -1$. Clearly the third equation cannot be fulfilled on $\mathbb{R}$. In other words, this 'generalized point' $e$ is something very exotic! We have not found a classical coordinate triple, but created a new one *uniquely through the homomorphism* $e$. What is the key argument here? *The introduction of 'points' via homomorphisms allows solution of equations which were not solvable in the previous real-valued coordinate context.* So the creation of such generalized points identifies with the construction of solutions of algebraic equations.

In this setup, finally, the Euclidean pointing subject and the variety of arrows are instantiated. In algebraic geometry, a homomorphism $d : A \to B$ of rings is read as an arrow with reversed direction $Spec(d) : Spec(B) \to Spec(A)$ of associated geometric objects[9], the spectra $Spec(R)$ of rings $R$. To us the only point is that algebraic geometry views our homomorphism $e$ as a morphism $Spec(e) : Spec(B) \to Spec(\mathbb{R}[X,Y,Z])$ which in the spirit of our general denotator terminology starts at 'address' $B$ instead of the usual address $\mathbb{R}$ of real coordinates. So the Euclidean subject is instantiated by this address selection whereas the Euclidean 'arrow variety' is precisely the set of arrows from $Spec(B)$ to $Spec(\mathbb{R}[X,Y,Z])$. In algebraic geometry, these arrows are called "$B$-valued points of $\mathbb{R}[X,Y,Z]$", see figure 9.2.

In our above case, the quotient ring $B = \mathbb{R}[X,Y,Z]/(X,Y,Z^2+1)$ is isomorphic to the domain $\mathbb{C}$ of complex numbers via the identity on the reals and $z \mapsto i = \sqrt{-1}$. And this is a crucial situation in the development of modern mathematics: Construction of solution spaces for algebraic equations! The complex numbers are the solution space whose points are built around the 'square root of $-1$ point'. In other words, the relativization of the point's subject domain

[7] See appendix D.1.
[8] See appendix D.1.
[9] See appendix F.2; the essence is that such an object is really controlled by its functor **Rings** $\to$ **Sets** : $B \mapsto$ **Rings**$(R,B)$ on the 'addresses' $B$.

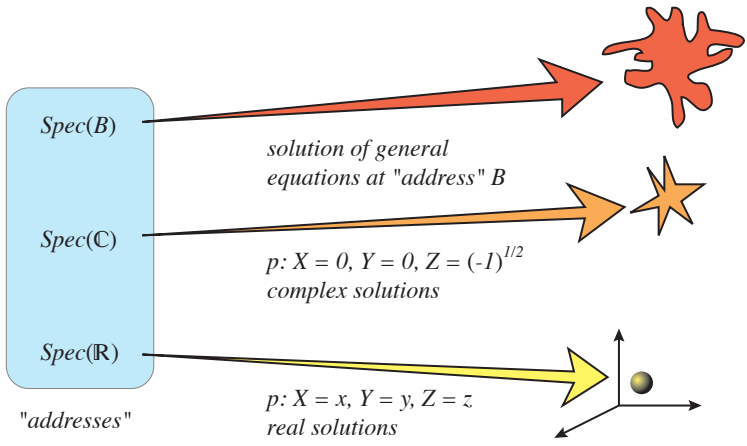


Figure 9.2: In modern algebraic geometry, points are arrows of spectra, corresponding to ring homomorphisms in reversed direction. Real-valued points can be drawn in usual three-space by three coordinates $x, y, z$. Complex points define a different space which is no more visible on the naive level and is built upon the solution of equation $Z^2 + 1 = 0$. General rings $B$ in the role of 'point addresses' yield solutions for general algebraic equations.

adds new points to the usual ones from naive real geometry. At this point it also becomes evident that the view of addresses as ontological parameters such as was described in principle 3 is completely canonical for mathematical ontology: New numbers provide new ontologies of mathematical objects. *The reality of real numbers is only one of a huge universe of ontological addresses.* In electromagnetic theory and engineering, this is standard: things such as imaginary currents are common practice.

It is not the place here to pursue in depth the deep consequences of this change of paradigm in mathematics which Alexander Grothendieck introduced around 1963, we should however notice that the very concept of a space was relativized according to the address variable. In fact, a Grothendieck topology (see appendix G.4 and our more specific coverage of the subject in chapter 19) on a category $C$ is a collection of point sets where points are *arrows* on all possible addresses in $C$ with special properties. Unlike classical topology, the Grothendieck concept works on the point concept which we have introduced above: Points are arrows in categories.

This is completed in the 1963 revolutionary approach to mathematics of William Lawvere in joint work with Myles Tierny [290]: He started his successful foundation program to lay the basis of mathematics not on set theory but on category theory. From now on, the elementary objects are no longer sets as given by their element points, but arrows which replace the old points and are now parametrized by their 'domain addresses'. What then is left in this new paradigm from the old pointing gesture? Exactly what was the essence in concept construction: You have a domain address for your arrows and for any arrow $f : X \to Y$, if its codomain $Y$ is the domain of another arrow $g : Y \to Z$, you can concatenate the arrows to $X \to Y \to Z$ and end up with a composed pointing gesture $g \cdot f : X \to Z$. This axiomatics of category theory reduces to the very nature of 'pointing at': the only thing which can be done in the universe of

arrows is to put them together if ever possible.

**Thesis 3** *This is semantics, reduced to its very essence. There is no more than pointing and pursuing the arrows' path as long as there is a way out...*[10]

## 9.2 Yoneda's Fundamental Lemma

**Summary.** The formal statement, a comprehensive interpretation, and corollaries.

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See also appendix G.2 for the formal statement. Here, we try to understand its contents. This is the statement:

**Lemma 7** *Let $C$ be a category, $X \in {}_0C$ an object, and $F \in {}_0C^{@}$ a contravariant set-valued functor on $C$. Then the applications*

$$\iota : Hom(@X, F) \to X@F : \quad g \mapsto g(X)(Id_X), \tag{9.2}$$

$$\eta : X@F \to Hom(@X, F) : \quad \gamma \mapsto (f : Y \to X \mapsto f@F(\gamma)) \tag{9.3}$$

*are inverse to each other. In particular, if $Y \in {}_0C$ is another object in $C$, we have a bijection*

$$\eta : X@Y \to Hom(@X, @Y) : f \mapsto @f, \tag{9.4}$$

*i.e. the Yoneda functor $@ : C \to C^{@} : X \mapsto @X$ is fully faithful.*

We want to give a comprehensible interpretation of the last statement concerning the Yoneda functor. To begin with, what does it mean to replace an object $X$ by its functor $@X$? We are indeed creating an overall perspective to $X$ since $@X$ means selecting all possible 'addresses' $Y$ in $C$ and then pointing at[11] $X$ via all possible $Y$-addressed arrows $f : Y \to X$. We also call such arrows $f$ 'perspectives of $X$ viewed from $Y$'. Let us make an example to learn what such perspectives can teach us about $X$, and why this terminology is adequate, see also figures 9.3 and 9.4. To this end, take the category $ComLoc_{0_{\mathbb{R}}}$ of commutative zero-addressed local compositions over the reals. With the notation of section 8.3.5, sorite 6, take two three-element motives $(M_1, Onset \oplus Pitch), (M_2, Onset \oplus Pitch)$ with each three elements: $M_1 = \{(0, 40), (1, 39), (2, 43)\}, M_2 = \{(-1, 41), (0, 40), (1, 39)\}$. Let us look at different 'addresses' from which we look at our motives. The first address is the singleton motif $S$. The arrows $c : S \to M_i$ correspond to the elements in motives $M_i$. Since we have three of them in each target motif, the arrows are in one-to-one correspondence. So, intuitively, $M_1$ and $M_2$ look the same when viewed from $S$.

Observe that if we were dealing with set theory, the singleton perspective would already do: The perspectives from the singleton just define the cardinality of a set, and this means fixing

[10] The reduction of semantics to 'pure arrows' in cognitive science is approached in a remarkable work by Daniel Dennett [124].

[11] This is the reason for selecting @ as a symbol for the arrows $f : Y \to X$ in $Y@X$

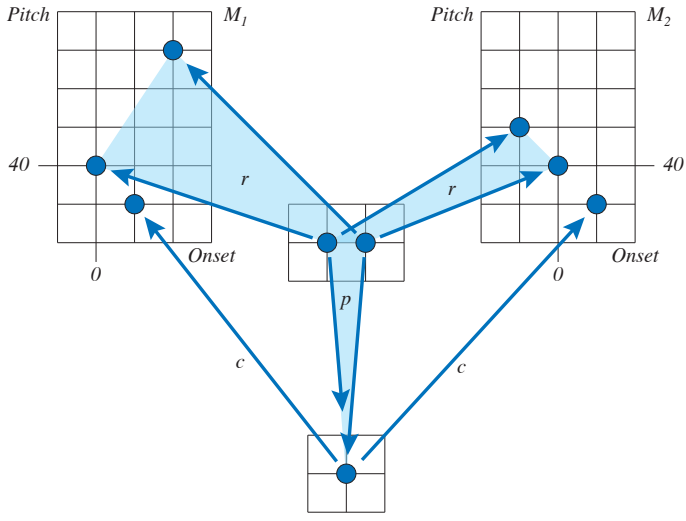


Figure 9.3: Two three-element motives $M_1, M_2$ and points addressed in the singleton and in a two-element motif. On this level, $M_1$ and $M_2$ look the same: For both, $M_1$ and $M_2$, we have the three arrows $c$ from the singleton and three arrows factorizing through the singleton from the two-element motif, as well as six injective arrows $r$ from the two-element motif.

the isomorphism class of the set! But we are not in set theory and want to vary perspectives. The next standpoint of perspective is a typical two-element motif as shown in figure 9.3. We have two types of arrows $r$ here: First those which factorize through the singleton: $r = c \cdot p$. Then we have injective arrows hitting two different points in the target motives. These can be any set-theoretically defined maps, and we get six arrows which again, unfortunately, do not yield any visible difference between the motives.

So let us try a reciprocal confrontation of these motives, see figure 9.3! We have a bijective morphism $p \cdot s : M_1 \to M_2$ through the projection $p$ onto the first coordinate and the shearing $s$ in pitch direction (transvection). But in the converse direction, every morphism must leave the collinear elements of $M_2$ collinear. But the points in $M_1$ are not collinear and we have no bijection. In other words, you can never see all of $M_1$ from $M_2$. This breaks the symmetry of roles between the candidates.

**Exercise 11** Another argument would have been to inspect the module functor introduced in section 8.3.5, definition 28. Use the fact that a bijection in this situation must be an isomorphism and the fact that a functor maps isomorphisms to isomorphisms.

From this easy example from motif theory we learn that replacing an object $X$ (a motif in our case) by its functor $@X$ does not yield a full picture of the object at any one of its arrow perspectives. If we restrict to the singleton perspective, we only see the set-theoretic aspect, but not the full affine geometry of the category of local compositions. However, a sufficiently general variation of the perspective can reveal subtler structures of arrow point sets.

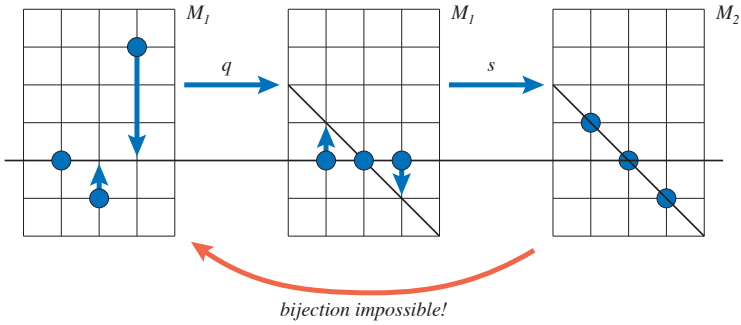


Figure 9.4: Whereas $M_2$ can be pointed-at from $M_2$, there is no *bijective* $M_2$-valued point at $M_1$: In its totality, $M_1$ is literally 'invisible' to $M_2$.

In a more intuitive setup[12], we may look at 2D and 3D objects and perspective maps from 3D objects $X$ to 2D objects $Y$ which occur in painting and photography. If we view such maps as arrows (in an appropriate category) $p : Y \to X$, we may ask about the information on the 3D object $X$ which is obtained through its photographs $Y$ under the viewpoint $p$. For instance: Is it possible to reconstruct $X$ from a bunch of photographs $Y_1, \dots Y_n$ without knowledge of the perspectives $p_i : Y_i \to X$? A historically famous example for this situation is the discovery by Edward Whymper of an easy route to the Matterhorn's top. The common view from Zermatt showed no evident path, every possibility seemed too steep to be attacked by mountain climbers of those early days. However, a change of the 'dogmatic' perspective unveiled the solution to Whymper: Viewed from the eastern perspective of the Theodul glacier, the northern crest appeared as much less steep than they believed from the common 'knowledge' down in Zermatt. The moral of this lesson is that, whereas a single 'address' may blur a lot of a particular object, variation of address and perspective may afford a more complete view of our object.

With this in mind, we now have the crucial statement of Yoneda's lemma: Passing from the objects to their functors yields not only a rich system of perspectives but the entire information we need to tell whether objects are isomorphic or not. In fact, by bijection (9.4), $\xi : @X \xrightarrow{\sim} @Y$ is an isomorphism of functors iff $\xi$ stems from an isomorphism $x : X \xrightarrow{\sim} Y$, $\xi = \eta(x)$. Mathematically, the isomorphism $\xi$ means that we are given a commutative diagram

$$\begin{array}{ccc} B@X & \xrightarrow{B@\xi} & B@Y \\ {\scriptstyle g@X}\downarrow & & \downarrow{\scriptstyle g@Y} \\ A@X & \xrightarrow{A@\xi} & A@Y \end{array} \tag{9.5}$$

with horizontal bijections, for every 'address change' $g : A \to B$. The horizontal bijections are correspondences among point sets for any addresses $A, B$, and the vertical arrows connect these

[12]It can be made rigorous by use of projective geometry, but this is beyond our concern.

bijections of variable address points. This intuitively means that if we give a system of bijections between points or 'perspectives' at objects $X, Y$, which copes with diagrams of type (9.5), we can be sure that they indeed generate a 'real' isomorphism between $X$ and $Y$.

As a corollary of Yoneda's lemma we now understand why we always called "points in a space functor" the denotators. Indeed, the denotator's $D$ coordinates $CT(D)$ is an element of the denotator's functor $F(D) \in {}_0\mathbf{Mod}^{@}$. By Yoneda's statement 9.4, if $M$ is the address of $D$, we may identify $CT(D)$ with the arrow $\eta(CT(D)) : @M \to F(D)$. And therefore, we have this statement:

**Corollary 4** *The coordinates of denotator $D$ is a point of the denotator's space $F(D)$ with values in the denotator's address $@M$.*

Notice, however, that we are not given all possible 'addresses' to be able to calculate $F(D)$ from its denotators since our addresses $@M$ are only the *representable* functors, see appendix G.2.

## 9.3 The Yoneda Philosophy

**Summary.** The Yoneda revolution in mathematics: functors instead of objects. "The functorial point of view is the geometric one". Perspective and truth, objectivity and dialog.

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As this is not a book on mathematics, we cannot explain all the proper mathematical consequences of the Yoneda lemma. But it is essential to understand its impact on musicology as it is exposed in the present viewpoint based upon functors and points in the form of spaces and denotators.

We shall call the following body of insights the "Yoneda philosophy" because it is more than a technical approach, it is a paradigm of far-reaching power which reshapes not only mathematics but also concept architectures and in particular software engineering as well as methodologies in fine and other arts.

This philosophy is built around the restatement of the elementary character of Euclid's points in terms of arrows: points are addressed perspectives, and on each address, we have an ontology in the sense that the point space of this address is a specific layer of spatiality.

We further know from Yoneda's isomorphism (9.4) that classification of objects is equivalent to classification of their functors, i.e. we can discover everything about an object via its point spaces on variable addresses: The object is recognized through variation of perspectives. Knowledge is the result of a network of (ontological) perspectives.

Therefore, all we know from an object is determined via its behavior (its functorial point system). Whatever the 'real ontology' of an object, it is completely manageable on the behavioral level.

In particular, an object may also be defined by means of its behavior! In mathematical terms, an object can be first defined by its functorial properties, and then, if required, retrieved among 'real' objects. But even if a functorial definition fails corresponding to a real object, i.e. if the functor $F \in C^{@}$ is not representable (see appendix G.2), it can be shown (see appendix G.2.1, proposition 97) that it is the colimit of a diagram of representable functors $F_\iota = @X_\iota$. In

other words: In the colimit, all behavioral definitions are reachable via 'objective' definitions, i.e. representable functors[13].

Points are not only mathematical details, they are also technical realizations of concept constructs, this is the philosophy of denotators. Since denotators are basically defined by recursion to ramification instances (via universal constructs such as limits etc.), building concept points is also a concatenation of points qua arrows.

In particular, circular constructs in concept architectures are not only possible, they are even necessary on a basic building level since rooted axiomatics can not yield all the conceptual foundations which are needed to think the concepts—instead of just playing around without knowing what instance is really doing the hardware of thinking.

The Yoneda embedding as functorial review of objects yields a bundle of addressed ontologies which are by no means abstract: "The functorial point of view *is* the geometric one!"[14] This profiled statement by Alexander Grothendieck's pupil Peter Gabriel claims that geometry cannot terminate on the limits of everyday's poor intuition and experience. Of course, such a statement is more general if we are not only looking at instances of geometric provenance. It essentially states that the behavioral reality *is* the essential one! In music this has another interpretation which we shall review in section 9.4.

In order to understand an object via its functor, there is a basic research agenda to pursue:

- Exhibit a minimal selection of 'ontological addresses and perspectives' which are sufficient to classify an object (in which case we say that they are "classifying").
- Understand what are "equations", and what their "solution" means on a particular address.

In traditional architecture it is known that three mutually orthogonal projections are classifying. We have seen that in the theory of local compositions, singleton perspectives are not classifying. We shall see later, in the global theory, to what degree local compositions are classifying.

Understanding "equations" and "solutions" really is a deep problem: What does a particular address unveil about the object in question? What problem does a particular 'geometric' space perspective address? This agenda item drives us towards the approach of *viewing the Yoneda philosophy as a fundamental concern of understanding things in spaces which are created to make problems evident.*

## 9.4 Understanding Fine and Other Arts

### 9.4.1 Painting and Music

**Summary.** Understanding painting and music is a synthesis of perspective variations. The year 1954. Comparison of Adorno's, Bätschmann's, and Valéry's esthetic principles in view of the Yoneda philosophy. Interpretation as an identification process.

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[13] In mathematics it has become common to use this technique to introduce objects since often, the very object is less important than its behavior!

[14] "Der funktorielle Gesichtspunkt *ist* der geometrische!" [175]

In the fine arts, above all in painting, the consciousness of a particular view has always been part of the art because of its technical requirement to map a higher reality onto a canvas or another plane surface. With the development of the central perspective in the early renaissance, the explicit point of view became even more prominent in that the divine position "sub specie aeternitatis" was replaced by an explicit human place. It is about the same time when in music too, the passage from the representation of metaphysical harmonies as conveyed from Pythagorean tradition to the individual, human composition, the musical work as we understand it in modern times, took place. *The musical work is the discovery of the central perspective in music.*

It is a commonplace to conceive the artist as the first observer or listener. The work is never neutral and independent of its perception, interpretation and understanding. It is virtually never created without its message to an esthetic instance. And it is always clear that the multiplicity of views and interpretations is either desired or at least accepted by the artist.

But it was much less evident and acceptable to the artists and even less to their religious and political patrons that a work of fine art cannot be understood except through a complex and long-term process of variation of the perspective. Perspective variation was—on the contrary—a well-protected tabou of presentation. A perspectivic view was accepted, but not as a possibility, rather as a mandatory presentation of dignity, symmetry and devotion. The selection of a determined perspective, be it in painting or sacral or political ceremonies, was an expression of a weltanschauung or dogma for the humble addressee[15].

On one hand, the prescription of a fixed authoritative perspective was explication of a rigid ontology: "This is the world, this is, how we all should see and accept it!" On the other, this would obscure the relative position of the work of art in a variety of possible alternatives. You cannot really understand if you cannot see the special choice among a variety of—possibly worse or questionable—alternatives. We refer to our computer-aided graphical analysis of Raffael's "School of Athens" which is documented in [329] and which was highly controversial because the computer simulation admitted free perspective variation and thereby destroyed the hitherto dogmatic frontal view of the monumental fresco in the Vatican's sacred rooms, see also figures 9.5 and 9.6.

Whereas in sculpture, walking around the work which is positioned in free space is mandatory to understand it[16], perspective variation is practically forbidden in musical compositions of classical European tradition. To the contrary: Following the composer's indications has for a long time been a must and was even a severe criterion of success in the European tradition of music critique. We come back to this subject in chapter 45. But there is one important point which broke this music tradition, initiated a new approach to understanding musical composition, and which must be seen in the context of Yoneda's lemma: It was Theodor Wiesengrund Adorno who recapitulated the question of identity of a musical work with respect to performance. According to Adorno [6] we have to recognize this fact: "Die Idee der Interpretation gehört zur Musik selber und ist ihr nicht adkzidenziell." Performance is a substantial part of a

[15] I remember that a computer-aided graphical analysis of Raffael's *School of Athens* during the Darmstadt symmetry exposition [329] was vehemently attacked by conservative historians because the computer simulation included arbitrary variation of the original frontal central perspective. This freedom of manipulation was felt as a serious disrespect against Rome's Sancta Ecclesia.

[16] Here, Yoneda's statement that the object is understood via the totality of its perspectivic views—its functor—is completely obvious.

Figure 9.5: Original frontal perspective of Raffael's "School of Athens" in a computer-aided simulation. The simulation includes 58 human figures and architectural essentials.

Figure 9.6: Perspective from the right side of Raffael's "School of Athens" in a computer-aided simulation. It reveals hidden symmetries of the work of art, see [329] for details.

composition's identity. This means that the work of art is not fully existential before or without performance. Of course, this addendum is a critical one since *now, performance with all its variations would no longer provide us with a unique perspective but with an a priori infinite bunch of alternatives which share the composition's essence.* We could still argue that the best one is the only admitted one, but this is no longer the unique one, we must inspect a variety before selecting the optimal alternative.

In poetry, Adorno's insight was anticipated by Paul Valéry's reflections on literature which culminated in the famous dictum: "C'est l'exécution du poeme qui est le poème." According to Hans Robert Jauss [250], Valéry's approach was a rupture with the classical tradition that viewed a work of literature as an autonomous unit, "ens causa sui" which appeared as one and the same object at any time and to everybody. Jauss concludes that a work of art only unveils its substance in the course of its historical life. This is quite the same as to say—with Adorno—that the literary work realizes its identity in the light of a time chain of performance perspectives.

It seems that Adorno and Valéry have uniquely stressed the exterior performance activity in music and poetry, but their concern is much deeper. We should keep in mind that performance is strongly tied to its rhetoric function as a means to express understanding, and in this respect, performance is not only a perspective of action but instantiation of understanding, of interpretation of given structures. In other words, the Adorno and Valéry approach is classificatory on the level of functorial representation of works of art.

In the hermeneutics of fine arts, Oskar Bätschmann has explicitly thematized the multiplicity of interpretative esthesic perspectives in his systematic treatise [43]. Bätschmann not only presents a picture of multiple interpretations where the artist's is only the very first, he also argues for an *interesthetic dialog* in a community of argumentation[17]. This is nothing less than the requirement of a coherent network of single interpretations, in fact the functorial interconnectivity of the collection of address ontologies, to use our terminology from Yoneda philosophy as developed in section 9.3. There is, however, a critical point in Bätschmann's system. He does accept mutual contradiction within a set of interpretations which—each in itself—are consistent

[17]"Argumentationsgemeinschaft", [43, p.160].

in argumentation and method. It is not necessary to reject the total set in this case, nor is it of any particular quality if an interpretation is materially complete and consistent. We cannot accept such a "tolerance" which in fact denies any overall system of comprehension. Rather should the fact of contradiction within a system of interpretations (and not within the proper art work where contradictions are common) be a source of refined research.

It is a remarkable fact that Adorno's approach was virtually simultaneous with Yoneda's, namely around 1954–1956. This is exactly a hundred years after the equally simultaneous emergence of the local/global paradigm as proposed by Hanslick and Riemann. Let us summarize that interpretation in the sense that functorial point perspectives distributed among a variety of ontologies is not a subsidiary task of understanding objects of the fine arts or of mathematics or even conceptual constructs, it is the very identification process which is solicited in so doing. Conversely, it should not be argued that we already know everything about classical works of art, be this Bach's, Scriabin's, Bartok's compositions, Raphael's, van Gogh's, Picasso's paintings, or Villon's, Poe's, Benn's poetry. The complete identification is not yet, and will for a long time not be, settled as long as the full functorial poly-ontology of point arrows is not developed to maturity.

### 9.4.2 The Art of Object-Oriented Programming

**Summary.** In contrast to its name, object-oriented programming is essentially a categorical access to programming. Characteristics, such as encapsulation, inheritance, methods, and class or instance variables do realize what the Yoneda lemma suggests: to replace program entities by the behavior they can show under determined conditions, i.e., identification by behavior.

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Concluding this chapter, we feel obliged to complete the circle from concept frameworks through mathematics and philosophy of fine arts to the core of our knowledge society and its cultural impact: computer science. We would in particular like to discuss the most advanced constructive paradigm of theoretical informatics: object-oriented programming.

Without explaining technicalities which are of no interest in this chapter[18] we can characterize object-oriented programming languages as being built on programming units called objects. These have an identity (in fact a pointer to their data structures) and are accessible through so-called methods which can be performed in a messaging action. All you know about objects is through their response upon messaging to their methods. For instance, if an object describes a point on the graphical interface, the point's coordinates are not automatically accessible. A priori, they pertain to the object's privacy and must be accessed via a special method. This is the encapsulation principle. So an object has an interior structure which is hidden to the program's context, you can only understand the object by its response to method messaging. Methods can be formalized in the language of categories [358]. There, roughly speaking, a method is an arrow from the given object's space functor (technically speaking: its class) to a second object's space functor which is responsible for the method's output[19].

[18] But they will intervene in the chapters concerning design and implementation of software, see parts X and XIII

[19] The dot-notation for method actions on objects in the Java language is quite near to this category-theoretic formalization.

Object-oriented programming is also an application of the point arrow approach to semantics which we discussed in the context of concept architectures: As such the identity of an object is fairly insignificant, it is just an arrow to an address where you can access further data. In terms of denotator theory, the identity is the denotator's arrow $@A \to Fun(Form)$, it will only give you further information upon request of coordinate arrows which—in this programming paradigm—correspond to the object's methods. It is remarkable that the implementation of the denotator theory in the Java-based RUBATO®software environment (see chapter 40) has realized the full functorial point of view exposed in this chapter.

# Chapter 10

# Paradigmatic Classification

*The poetic function projects*
*the principle of equivalence*
*from the axis of selection*
*to the axis of combination.*
Roman Jakobson [245]

**Summary.** Paradigmatic classification deals with formation of classes of objects which belong to specific paradigms. Here, paradigms are fields of equivalence or association. In a more general setting than in mathematics, equivalence is *not necessarily* understood as being a transitive relation. We give motivation of the paradigm concept from musicology, semiotics and poetology, and mathematics. Our taxonomy yields two types of paradigms: by transformations and by similarity—however, in practice, they often appear in mixed form. In a mathematical perspective, the first type is covered by group theory, the second by topology. This means that fuzzy concepts in the humanities are not a priori useless, they can be incorporated into exact reasoning by means of a refined paradigmatic reconstruction.

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This chapter is a preliminary discourse on local classification theory. Since chapters 11 and 12 will have more technical flavor, we want to give an overview of the principles of local classification and the corresponding references to sciences which are sensitive to this type of classification. We do not deal with lexical classification here; this was already discussed in section 6.8 dealing with orders on denotators. Concept taxonomy following classical Porphyrean trees and similar constructs are completely absorbed in that discussion. Rather are we concerned with paradigmata, i.e. "fields of equivalence", whatever this may signify—we have to review the differentiations of this subject. Its essence is very clear: In most human activities, from concept building and abstraction to concrete gestural actions, identity is not easy to get under control. Rather is this onto-logical cornerstone a necessary evil which in virtually all cases is overridden by grouping of large sets of objects whose identity is much less relevant than some common features, be they precise or only fuzzy. We are going to discuss the actual types of fuzziness and their genealogy in musicology, linguistics, and mathematics.

We should however stress that classification is highly controversial in musicology for three reasons:

- It is commonly believed that classifying musical objects on whatever level is contrary to the individual expressivity of compositions as they have been cultivated since the renaissance, but see our discussion in section 9.4.
- Classification is misunderstood as a purely bureaucratic activity of list compilation.
- Due to a catastrophical lack of technical tools, traditional musicology has only rarely been able to control the variety of their objects. A disdain of detailed technical work which is psychologically comprehensible must scientifically be blamed for a major scientific retardation even with respect to other humanities such as linguistics.

We shall vaporize these mystifications in the following chapters, the second yet seems already partly settled from our previous discussion on the Yoneda philosophy: classification in the sense of a determination of isomorphism classes is nothing else than the overall task of understanding an object as we have identified it in the functorial approach in the spirit of Adorno, Valéry, and Bätschmann in chapter 9.

## 10.1 Paradigmata in Musicology, Linguistics, and Mathematics

**Summary.** Transformation and similarity in music: Neumes, melodic contours, variations, and counterpoint. De Saussure's paradigmatic and syntactic axes. Characteristic combination of transformation and similarity in poetics. Group theory and topology: the poetics of mathematics.

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The etymology of paradigm is Greek $\pi\alpha\rho\grave{\alpha}\delta\varepsilon\iota\gamma\mu\alpha$: example, model, that thing side by side which you show or point at. Already this Greek root has a double meaning for the relation of that thing: On one hand it is "side by side", on the other, you "point at" it. The first suggests a space where neigborhood makes sense, the second is just a connecting action from me to that thing. It is however open whether the object is in my neighborhood because I am pointing at it or vice versa: whether I am pointing at something because it is in my neighborhood anyway. This ambiguity is after all not a defect, it shows a multiple specification which is essential to the word's explanatory power.

The difference between "side by side" and "point at" as explanatory roots is that in the first case, the model object is situated in a spatial neigborhood without my intervention, all I do is to pick it from a multitude of available objects in that neighborhood. We could also say that the model object and its neighboring specimens are taken into account because they are similar to the point where I am standing, so that we can better understand our point when recognizing similar points. We call this meaning the *topological* one.

In the second case, the model object is "side by side" because I have shown it in a pointing gesture. Its neighborhood to my own position is not given from an a priori ambient space but by

a construction of this arrow whose head is my model object. The explanatory power here results from a transformation of my position to the target position, i.e. the model is a transformed object issued from my operation upon the given proper position. In this case, the resulting model object is not random as in the topological case, it is a well-defined functional value. The examples can help understanding of our point because they are related under a specific operation. We call this meaning the *transformational* one.

**Fact 4** *Both topological and transformational paradigms help understanding of the proper position by the formation of classes of equivalent objects: the "example set" of what is to be explained.*

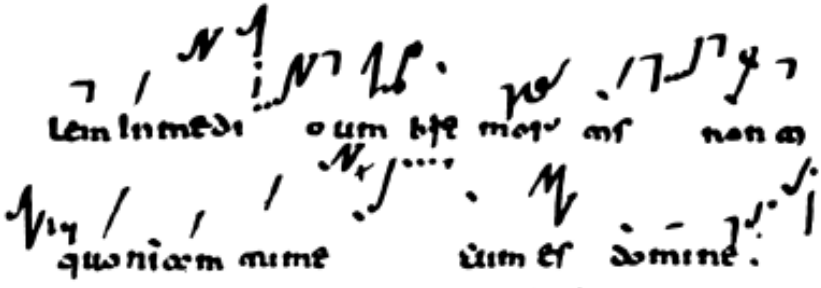

Figure 10.1: Neumes from the Beneventan writing (Rome, Bibl. Vat. Ms. lat. 10673; 11th century).

Let us first see how both these paradigmatic explications arise in **music**. The topological type means that we are given a musical object which is similar to the model object. This similarity approach in comparison of musical objects is widespread, we shall deal with it in several contexts and can restrict here to a characteristic summary. To begin with, before written notation of European music was developed, the cheironomic practice of hand gestures served as a paradigmatic sign system to indicate melodic movements in choral singing. This gestural system was then abstracted in the neume system with its different writing styles and developmental completion, tracing articulation, accentuation and diastematic movement in Gregorian choral, see figure 10.1. The word neume is rooted in Greek $\nu\varepsilon\tilde{\upsilon}\mu\alpha$, hint, and by this very etymology alludes to a movement of the voice that is only defined in the vague similarity paradigm. This rooting of melodic conceptualization is not only historic, it is—strangely enough—still a fundamental attitude towards melody: Music theory has not yet been able to really define a musical motif or melody apart from a priori vague concept fields. It is understood that the precise identity of a melody is less important than its—however ill-defined—similarity class which may be termed "contour" to indicate some type of topological paradigmatics[1].

After a long period of hidden or even patent refusal to work out explicit definitions of terms such as (melodic) "motif" [2], the vagueness of the contour concept has been attacked

[1] Contour theory is an interesting object for the problematic building of valuable paradigmatic concepts in musicology.

[2] See our discussion of Rudolph Reti's theory of motives in chapter 22.8.

by what lends itself to the imprecise: fuzzy theory [428]. We shall come back to this approach but should remark here that fuzzy relations are a poor imitation of what general topology, and even theory of pseudometric spaces, could grasp much better. The interesting point is that the fashionable expression "fuzzy" was necessary to introduce instances of topological reasoning to musicology, the imprecise as a scientific category was the real thing which musicologists seemed to wait for.

The consciousness of similarity is also present in variational methods. Variation does not mean to apply a determined transformation but to move around in a given topological neigborhood. In this sense a tonal variant of a melody or harmony, from major to minor tonality, say, is just a restatement of a structure with neighboring values; the theory of turbidity (see our discussion in example 11 of section 8.1.1) is indeed thought in this topological spirit.

A huge field of extremely topological paradigmatics opens in the subject of sound colors. However, the decision of whether sound $S_1$ is more similar to a reference sound than sound $S_2$ is very difficult and in fact points at one of the most complex problems in sound synthesis and analysis as well as associated topologies. We shall discuss it more carefully in section 12.3.2.

In contrast, the contrapuntal technique of fugue and canon constructions typically recurs to the second meaning of transformational paradigmatics: The construction of a *comes* answer from the *dux'* thematic germ makes use (among others) of contrapuntal symmetries, such as inversion, retrograde, retrograde inversion, or augmentation. These secondary instantiations of the original object are transformations and not similar objects. On the contrary: It is not true that these "models" are topologically similar to the original. A retrograde theme may look very *dissimilar* from the original; we have already explained this fact in the course of the general symmetry discussion in section 8.1. This is also completely logical from the construction guideline of a fugue or canon: Answering to the dux melody is a "mental reflection", a correspondence that cannot be similar at random. Note that, in general, an answer need not be a faithful image of the original, it can as well be a poor shadow. But it has to be a functional consequence and not just something vaguely deformed.

Also in the vein of the transformational paradigm is the classification of chords according to transposition and/or inversion: A chord $B$ is a model of a given chord $A$ if it is deduced from $A$ by a transposition and/or pitch inversion. On a more basic level, the formation of pitch classes also creates a pitch paradigm by transpositions of multiples of an octave. Clearly, pitch classes or chord classes are far from similar in the topological sense, but they are essential in the understanding of chord or pitch concepts and structures. Classification is often based upon transformational paradigms. For example, harmony is based upon these classifications in the pitch domain; without comparison of "equivalent" chords, harmony would be an impossible task. So much the more is it remarkable that the entire Riemann program of functional harmony [100] which was designed to include the entire zoo of chord classes did not incite musicologists to write down complete lists of chord classes once for all.

In structuralist **linguistics**, paradigmatic relations intervene in the basic dichotomic system, as sketched by Ferdinand de Saussure in his famous "Cours de Linguistique" [471]. To Saussure, language is spanned within a twofold dimensionality: the syntagmatic and the paradigmatic axes, see the introductory section 2.3.4. The paradigmatic axis covers relations between language units, but these relations are not present in the textual syntagm, they occur on a more abstract level "in absentia", see figure 10.2.

What Saussure had called an associative field has been reviewed in different theories by

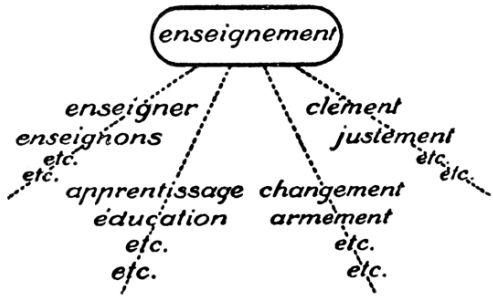


Figure 10.2: A visualization of the paradigm of the word "enseignement" in Saussure's "Cours de Linguistique". Observe that the paradigmatic relation is defined on different levels: phonological, grammatical, semantical, etc.

subsequent research. The differentiation among these theories also reflects the double meaning of "paradigm". In a more transformational spirit[3], relations among instances of one paradigm are termed "in opposition" and *suggest* a binary operationality which of course does not cover the general phenomenology, as Roland Barthes has rightly criticized in [41]. Both situations do occur: Grammatical paradigmatics is transformational in the sense that the inflection of a word, for example, obeys the transformational paradigm whereas phonological vicinity in "minimal pairs" [12] such as "vine"/"fine", for example, is topological.

The classificatory function of paradigmatic activity is perhaps best understood in the linguistic context. Understanding a word as a member of a language is virtually never a question of abstract and isolated identification. Understanding means identifying not the individuum but its role, its position in the overall language environment. This position can be realized in two ways: topological deformation and transformation. In each, you learn rules of the language game which determine what you can do to an instance of the game without altering its essence.

In **mathematics**, paradigmatic processes and methods are well known but not in the semiotic interpretation, rather it is a basic mathematical activity of classification to boil down the overwhelming multitude of individual mathematical objects and structures. Topology, originally termed "analysis situs", deals with invariants of geometric objects under shape deformation: What is preserved if we stretch a rubber band? What is the common property of all kinds of holes? Transformations are commonly dealt with in group theory: We want to describe the nature and essence of systematically performing transformations upon a set of objects: Is it the concrete mechanism or are we given a general context which is present each time we proceed to a transformational activity? Is it the case that there are no plane ornaments with fivefold rotational symmetry? Why are there exactly five platonic, i.e. perfectly regular bodies?

Before the advent of these paradigmatic theories mathematics was a thoroughly sober, anti-poetical science. It was about geometric, numerical and functional quantification. There are still uninformed persons—often in the humanities—who believe that mathematics is this type of mechanical quantification business. But it should be recognized that paradigmatic theories have not only introduced a strong qualitative profile of mathematical concerns, it is also true

[3]Observe that we in no sense allude to Noam Chomsky's transformational theory here.

that linguistic poetology has strongly profited from the group-theoretic approach, for example in Jakobson's identification of the poeticity as a projection from the paradigmatic axis to the syntagmatic axis [245]. In fact, this projection introduces group theory in poetology since the functional connection of paradigmatics to syntagmatics uses syntactical symmetries, typically realized in rhyme constructs, to accumulate paradigmatic connectivity. Rhyme periodicity, the most common syntactical symmetry transformation, acts on the association of rhyming units, be it on the phonological or on the semantic level. Such an entanglement of mathematical and poetical roots induces a reversal of perspective: introduction of poetical aspects in mathematics. Even among conservative mathematicians it is commonly felt that works of art such as Cornelis Maurits Escher's or Johann Sebastian Bach's or Dante Alighieri's approaches are a kind of incarnation of mathematics in the fine arts, music, and literature, and *that this is a substantial correlation*[4]. The recent impact of fractal theory and quasi-crystal theory on mathematical rationales of esthetics has confirmed Hermann Weyl's symmetry-oriented approach to the arts [564] as well as René Thom's topological catastrophe theory of cognitive processes [526].

**Fact 5** *The poetical character of qualitative theories in mathematics is a restitution of a common root of artistic and scientific thinking in the tradition of renaissance universalism as issued from medieval constructivism*[5] *which goes back to the Pythagorean tradition of operationalized thinking in music and mathematics.*

As may be expected, the double meaning of the word paradigm is often realized in combined form. The model object is then either a topological deformation of a transformed specimen or vice versa: a deformed version of a transformed specimen. For example, a contrapuntal answer may be the inversion of the leading theme followed by a tonal alteration to a new scale. We shall deal with this combined paradigmatics in chapter 22 on motif gestalts. A very intuitive application of mixed paradigmatics is realized in Escher's metamorphosis works, such as shown in figure 10.3

## 10.2 Transformation

**Summary.** Transformation as a transfer of association to generative rules. Regular and degenerate rules: symmetries, morphisms, specializations, alterations, and fractals.

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There is a fundamental and far-reaching difference between topological and transformational paradigmatics. The former not only refers to an ambient space topology but also remains much more passive than the latter. You simply have to recognize the model object as compared to the original object, and the rest is a plain verification of similarity. Everything is already given by inherent structures, the only 'activity' which is left to the observer is a cognitive act. Of course this one need not be completely evident, the psychology of topological pattern recognition is not trivial. But once a general acquaintance with the underlying ambient space topology is present, nothing has to be added beyond perception of what is there.

[4] This has been popularized in [229].
[5] See [477] and [363] for this background.

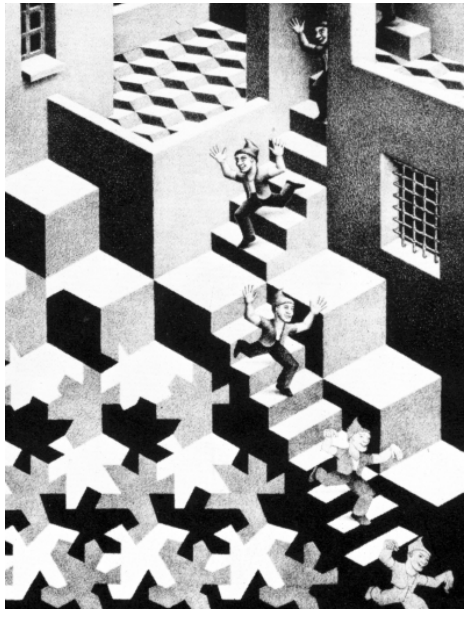

Figure 10.3: Detail from Maurits Cornelis Escher's "Cycle" (©2002 Cordon Art B.V. – Baarn – Holland. All rights reserved). The generating cube is positioned to the left middle, then iterated according to a plane ornament symmetry, the ornament is then deformed to organic shapes, ending up with the garden gnome figuration.

Transformational paradigmatics is much more complex. Not only are we asked to retrieve the actual arsenal of admitted transformations and therefore recall a non-automatic competence, the level of understanding is also quite different from the topological one. In fact, a transformation is a process which unfolds on a metalevel, the objects which are acted upon will only be results of this activity, the activity as such pertains to a hidden operating system. So, whenever transformational paradigmatic association is required, not only Saussure's "absence" is felt (in contrast to the syntagmatic presence of manifest contiguity), but also the highly demanding activation of our transformational competence. In other words:

**Fact 6** *On the level of transformational paradigmatics, the simple association of manifest objects is transmuted to a complex construction activity of the subject who is concerned with the paradigm, and therefore the phenomenon is degraded to a secondary level which can only be understood by a competence on the generative rules rather then their effects.*

There are several degrees of complexity of generative rules as related to their effect upon objects. In the simplest case of an axial symmetry, the relation is evident and the operation as such is not difficult to retrace. Also the reflected object is a faithful image, double reflection restores the starting point. As a transformation becomes irreversible it is important to know the underlying transformation, its 'objective trace' no longer enables the transformation's reconstruction. For example, the projection of a part of a dodecaphonic series onto a chord is

a "degenerate map" and destroys the temporal order of the series: the chord is a very special 'shadow' of the original series. This is why the morphisms have become the core of structure theories, such as category theory and its crystallizations in the theory of local composition as presented and commented on in chapters 8 and 9. In the theory of alterations which we introduced in section 7.5, transformations which are associated with alterations virtually disappear in the relative local compositions, altered local compositions are nearly topological deformations. The extremal situation of degeneracy is fractal theory. Here, the naive visibility of symmetry transformations has completely disappeared or at least been disguised in a fascinating esthetics. A transformation of type $f_c : \mathbb{C} \to \mathbb{C} : z \mapsto z^2 + c$ is used to distinguish those complex numbers $x$ which go to infinity with successive iteration $x, f_c(x), f_c^2(x), \ldots f_c^n(x)$ for $n \to \infty$ from those $x$ which have all their iterations $f_c^n(x)$ in a finite region of the complex numbers. So we are not looking at a repeated shift of a point, i.e. $x, a + x, a + a + x, \ldots a + \ldots a + x, \ldots$ but of an operation: $Id, f = f \cdot Id, f^2 = f \cdot f, f^3 = f \cdot f^2, \ldots, f^{n+1} = f \cdot f^n, \ldots$. The set of points which approach infinity is separated from the rest by a boundary known as the Julia set[6], and which shows a breathtaking beauty, see figure 10.4. This trace of the hidden transformation—an iterated application of a fixed function—shows no simple evidence of the transformation; on the contrary, *without its generating transformation rule* the knowledge of the Julia set per se is completely mysterious.

It may seem that the entire discussion of transformational paradigms is mainly an analytical affair, but it is equally concerned with composition and poiesis, as the attentive reader will have noticed. Paradigmatic methods in counterpoint and serial composition are transformational techniques which *create* ways of understanding the musical ideas as paradigmatic germs of an organic universe of sounds; Anton Webern's compositional techniques following Goethe's organic principles realize these techniques, see section 17.2 for the corresponding discussion. In view of this, a composer has to pay especial attention to the mentioned vanishing degree of transformations because the communicative evidence of a composition depends strongly on the emergence of the operational metalevel *within* the unfolding composition material, see section 47.1 for this subject.

## 10.3 Similarity

**Summary.** Similarity as a quantification of comparison. Metrical and topological similarity. Quantities and qualities: What is a parametrization?

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The scientific goal of comparison is a difficult when the specific difference of the comparanda has to be objectivized. Whereas exhibiting logical *differentiae specificae* is an easy task, comparison of objects which pertain to floating paradigms is much more delicate. In psychometrics this problem is classically tackled by numerical quantification of similarity, for example by Charles Osgood's *semantic differentials* [409]. The same spirit controls the theory of fuzzy sets [37], or fuzzy relations in musical contour theory [428], where the classical logical differentiae specificae are refined by a distance measure of fuzzy validation.

[6] The set is named after French mathematician Gaston Julia [254].

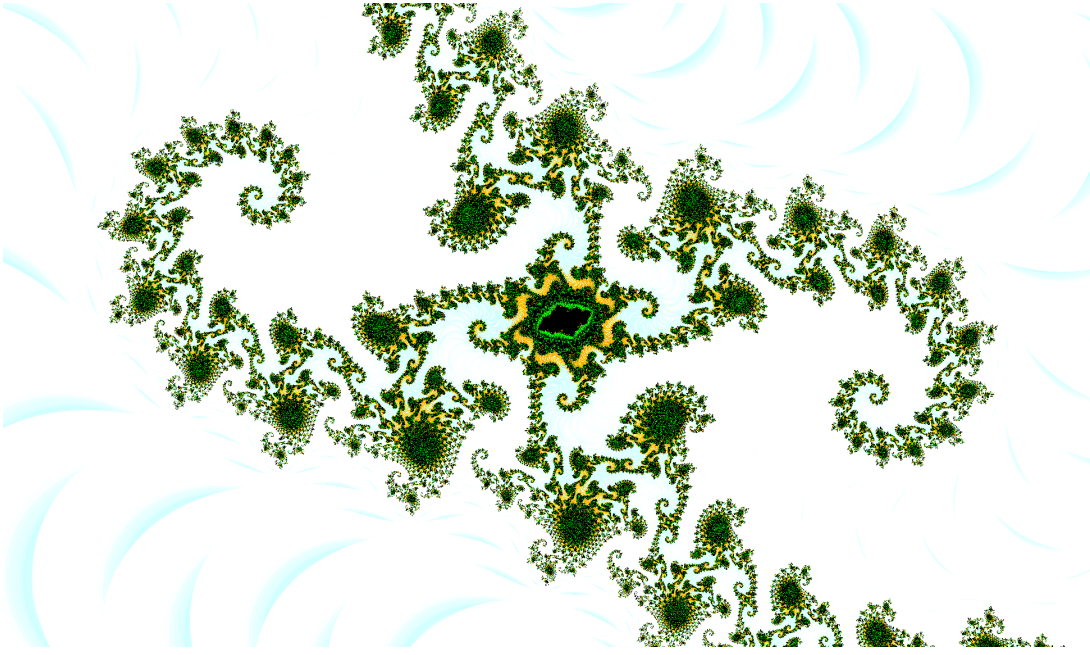

Figure 10.4: The Julia set, of which a detail is shown here, is a trace of a transformation $f_c = z^2 + c$ on the complex plane which completely disappears in favor of the visible result. Powers of $f_c$ produce chains of associated points which tend to infinity or stay in a finite region, a paradigm of highly complex overall appearance.

This approach is completely natural since experimental verification is primarily concerned with numerical measurement, and metrical distance functions are the right thing to invoke when such quantification is required. However, music is not a branch of psychometrics, metrical similarity is a very special realization of the topological paradigm. In fact, the original concern of paradigmatic neigborhood was to understand what it means to stay in the vicinity of some given point, and *no allusion to metrical measurement* of vicinity was implicit. In mathematical topology the generic concept of "vicinity" or "neigborhood" has been analyzed, and this "analysis situs" has revealed an axiomatic construction where no reference to metrical data is left, see appendix H.1. The topological neigborhood concept has shifted to a pure *quality* where the naive "epsilon neigborhood of a point $x$" is replaced by an abstract open set containing $x$. In this context, there is a theoretical branch which deals with metrization theorems, i.e. with topological spaces which can be deduced from the metrical neighborhood paradigm.

But there are important classes of topological spaces which are far from metrizable, for example Zariski topologies in algebraic geometry. We shall make substantial use of this type of non-metrizable spaces in the theory of musical motives, see chapter 22. These "exotic" topologies share the remarkable property that there are points of different "size", "thick" and "thin" points of any extension. Whereas a thick point may have many thin points in its closure, these thin points do not have the thick point in their closures. This is an important asymmetry which

would not happen in metrical contexts: distances are symmetric quantities, and whenever we have two different points in a metric space, we can mutually separate them by sufficiently small disjoint neighborhoods. This is an important advantage of non-metrical spaces which can be used in, and is a central feature of, topological motif theory. On the basis of metrical spaces one could never arrive at non-symmetrical neighborhoods and therefore never grasp the phenomenon of special and generic examples of a paradigm! For example, if we have to give examples of the *triangle* concept, the equilateral triangle is a very special example which we should never draw in classroom since equilaterality is not the typical property of a triangle; giving examples of equilateral triangles would never produce other special types of triangles. To understand this difference, or the difference between generic and special motives in a musical composition, topologies of the "exotic" type are necessary.

So topological paradigmatics is a huge extension of traditional metrical similarity. It nevertheless remains within the field of objectivized description of paradigmatic vicinity, it is a scientific refinement of the common language concept of comparison. The characteristic difference lies in the renouncement from plain quantification. Topology is the qualitative theory of neighborhoods, this is a piece of mathematics without a priori quantification which helps building powerful concepts beyond measurement. We should however notice that the absence of immanent quantification has its prize: Classification of topological spaces is a hard program which is described in the so-called *algebraic topology*, see appendix H for more details.

The common denominator of topological paradigmatics is to distribute comparable objects in a space which allows the objects

- attributes of spatial location which share the property of
- being objective, explicit, adequate, and highly differentiated.

So the idea is to create spatial parameters for paradigmatic fields, parameters which include but go far beyond numerical quantizations. Parametrization is an auxiliary measurement system but not necessarily by numbers, rather by precise concepts of spatial, i.e. topological character. This is the decisive difference between old-style and new-style precision.

## 10.4 Fuzzy Concepts in the Humanities

**Summary.** In the humanities, fuzzy concepts are a frequent phenomenon. Paradigmatic reconstruction of such concepts by use of precise similarity and/or transformation concepts can help in validating amphibological discourses instead of rejecting them.

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There are two objections to precise terminology in the humanities:

- historically and culturally generated polysemy,
- uncontrolled paradigmatics.

**Polysemy** has been a major objection to scientific discourse beyond what Hermann Hesse called "feuilleton science" [221]. It is contended that polysemy is in contradiction to precision.

But there is no logical argument against a precise description of polysemy. A well-known example of an exact theory of polysemy is the mathematical theory of algebraic equations. In general, a solution $X$ of an equation $f(x) = 0$ is a sign having multiple significates. The description of the variety of solutions, i.e. the collection of all valid semioses of the signifier "solution $X$ of $f(x) = 0$", is a successful branch of mathematics, viz algebraic geometry. The reason for this success is that the signification mechanism specified in the equation $f(x) = 0$ is explicit and precise. *Hence it is not the problem to eliminate polysemy, but to analyze it in a powerful language.* The theory of global compositions developed in part IV is an example of adequate scientific formalization of polysemy for the analysis of musical texts. Quite radically, the neutral identification of a musical work (see section 2.2.2) is precisely the organized ensemble of polysemies on the analytical level.

The more serious obstruction to scientific precision in the humanities is a fuzzy concept in a non-technical sense, i.e. **uncontrolled paradigmatics**. These are typically produced as follows: Initially, a specific research yields a field of conceptual variants which seems to point to a coherent phenomenon. But any attempt to build a well-defined paradigma from the vague concept field fails. Usually this happens for one or more of three reasons:

- blurred consciousness of the layer of reality or—more generally—the topographic location of the bunch of phenomena, i.e. a smeared topographical analysis,
- taking blurred reflections for blurred facts;
- defectuous concept tools for building well-defined paradigmata.

Whereas the first two reasons are simple defects in analytical mastery, detection of the third is more delicate. In the argumentation the defectuous concept tools are not recognized. Rather does the discourse either seek a deep mystery in the phenomenon itself, or the addressee is persuaded that an uncontrollable infinity of cases and variants simply rules out any conceptual effort. In short: the complexity of the phenomenon transcends human intellectual power.

We shall discuss this latter attitude at length in section 22.8 on Rudolph Reti's concept sketches dealing with musical motives, themes and melodies. Here, we want to give a short illustration of how failure of paradigmatic tools can create defectous results in the humanities: In the course of an exhibition project on the Darmstadt Mathildenhöhe [329] and with respect to the "School of Athens", Raphael's famous fresco in the Vatican, the question concerning the significance of the star being constructed by the Bramante figure in the fresco's lower right was raised. To date it had been assumed that the star represented a star of David [149], [539]. However, this semantic was never examined with respect to the precise star shape. A star of David must comprise two regular triangles. But the central perspective in which this masterpiece was demonstrably painted permits more precise conclusions on the form of the geometric configurations in the painting.

In collaboration with the Institute for Architecture of the Technische Hochschule Darmstadt and the Fraunhofer-Gesellschaft computer graphics working group, a computer graphics reconstruction was made, including the 20 most important among the total of 58 human figures, as well as important geometric elements (steps, blocks, spheres, Bramante star etc.). Visualization of the object configurations from all perspectives with the aid of computer graphics software permitted new insights due to visualization of previously concealed features. It came to light that the star does not comprise equilateral but right-angled triangles which configure

in relation to the lateral faces of Platonic bodies and thus also create a link to Plato, a central figure of the composition. At the same time it turned out that the points of these triangles are set with amazing precision in relation to the base-points of the principal human figures in the work [328]. Thus a new immanent semantic has become apparent in the fresco by means of precise geometric analysis, and in sharp contradiction to previous interpretations.

**Result 1** *The moral of this anecdote is that the paradigmatic tools for analyzing the deformed version of the star in the spatial paradigm of central perspective were not available in the traditional interpretations, and that only rigorous elaboration of these tools by use of adequate computer software and geometric background theory led to a reliable understanding of the star paradigm in the given context.*

# Chapter 11

# Orbits

*2 230 741 522 540 743 033 415 296 821 609 381 912*
*The number of isomorphism classes (orbits) of*
*72-element motives in* $\mathbb{Z}_{12}^2$.
Harald Fripertinger [169]

*100 000 000 000*
*The average number of stars in a galaxis.*
Hubert Reeves [436]

**Summary.** This chapter deals with groups of symmetries, their action and orbits as musicological and mathematical concepts. Elementary local compositions—chords, self-addressed chords, and motives are classified under group actions. Enumeration theory of orbits of local compositions in finite $\mathbb{Z}$-modules—including traditional pitch class sets and motives—is presented and discussed for its implications towards a "Big Science" in music. Follows a discussion of group-theoretical methods in composition and theory, including a review of the American tradition and recent developments.

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## 11.1 Gestalt and Symmetry Groups

**Summary.** Group orbits and the gestalt concept. Orbits are tools of conceptual abstraction.

– Σ –

Transformational paradigmatics is addressed in Christian von Ehrenfels' concept of a *gestalt* as proposed in [136]. The concept is characterized by *transposability* and *super-summativity*. Ehrenfels's famous example of a melodic gestalt explains these attributes: If a melody is transposed in pitch, we still recognize the melody, and—according to Ehrenfels—this also demonstrates that the melody "is more than the sum of its parts", the single notes. We come back to this second point in chapter 12. Here we concentrate on the first point. The statement is

that although the melody, as an individual local composition, say, changes after transposition, something remains invariant, and this is an attribute of the melody's gestalt. In Ehrenfels' example, this invariant is a transformational paradigm under translation symmetries in the pitch domain (see example 7 in section 8.1.1) and could as well be extended to temporal translation (see example 8 in section 8.1.1). The very concept of gestalt refers to the entire set of transpositions of a given melody object. The melody's gestalt is something which is identically the same in all the melody's transpositions. But then, it is obvious to identify the set of all transpositions $e^t(M)$ of the melody $M$ with an attribute of this gestalt of $M$. This means that the orbit set $\{e^t(M)|\ t \in \text{ambient space of } M\}$ is an attribute of the gestalt of $M$.

This motivates the conclusion that building classes of local compositions under actions of determined groups of symmetries is a conceptual abstraction tool for constructing the gestalt concept. Again, as discussed in chapter 10, we recognize paradigmatic classification as a classical subject of the humanities, here: cognitive science in music and music psychology. The concrete selection of the acting symmetry group is not the concern of our calculations, these must be justified by other criteria. But the a priori calculations of orbits is a technical prerequisite without which no effective cognitive science can emerge. Ehrenfels' approach can only be a prototypical one, since the question of which "transpositions", i.e., symmetries, acting on a melody are really responsible for the cognitive construction of a gestalt cannot be answered ante rem, before cognitive experiments are performed to answer empirically to the empirical question in cognitive science.

**Principle 5** *So mathematical music theory has to furnish the orbits for all possible—or at least all reasonable—applications, be it for cognitive science, for compositional purposes, or for musicological analysis.*

## 11.2 The Framework for Local Classification

**Summary.** We delimit the categories of local compositions where one presently has significant classification results.

– Σ –

In this chapter we make the general hypothesis that all coefficient rings $R$ of modules are *commutative*. We also restrict to commutative local compositions, i.e., those having $R$-modules as simple spaces and identify them whenever reasonable with their objective trace (see also section 7.4 and section 8.3.5 for these local compositions and their categories). So our commutative local compositions are addressed at an $R$-module; call the category of these commutative local compositions, together with $R$-affine module morphisms $ComLoc_R$. In terms of the general denotator theory of form semiotics exposed in G.5.3, we are working over the topos $\mathbf{Mod}_R^{@}$ instead of the larger topos $\mathbf{Mod}^{@}$. However, address changes including different commutative rings may occur. But such exceptional situations will be expressely mentioned.

We should also notice a canonical technique of 'address killing' which occurs in the following situation: We are given a functor $F \in {}_0\mathbf{Mod}_R^{@}$ and an address $B \in {}_0\mathbf{Mod}_R$. Using the affine tensor product $\boxtimes$ as defined in appendix E, we define a new contravariant, set-valued functor

on $R$-modules $B\tilde{@}F \in {}_0\mathbf{Mod}_R^{@}$ as follows. Let $X, Y \in {}_0\mathbf{Mod}_R$ and $f : X \to Y \in {}_1\mathbf{Mod}_R$. Then

$$X@B\tilde{@}F = X \boxtimes B@F, \tag{11.1}$$

$$f@B\tilde{@}F = f \boxtimes Id_B@F. \tag{11.2}$$

By the equation $0_R \boxtimes B \xrightarrow{\sim} B$ (appendix E, sorite 14) this functor has the property that $0_R@B\tilde{@}F \xrightarrow{\sim} B@F$, i.e., it gives us back the $B$-addressed points of $F$ as zero-addressed points. We therefore call it the *$B$-address killer of $F$*. For $F = @_R M$, $M \in {}_0\mathbf{Mod}_R$, we get the well-known identification $X@B\tilde{@}M \xrightarrow{\sim} (X \boxtimes B)@_R M \xrightarrow{\sim} X@_R(B@_R M)$, see appendix E, lemma 79 and proposition 83.

Finally, we shall exclusively deal with finite, non-empty commutative local compositions here.

## 11.3 Orbits of Elementary Structures

**Summary.** This section deals with different common types of local compositions and their classifications.

– Σ –

In complex categories such as *Loc*, *ComLoc*, and other categories as displayed in diagram 8.31, classification is a multi-threaded task. At present we are far from a complete classification for three reasons: First, classification is far from settled in the module categories which underlie local compositions. Second, as we shall see, even if module category had solved the classification problem the algebro-geometric classification problem would be open, and the latter is essential for local compositions. Third, even if we had a theoretical solution of a classification problem, it would be an unsolved problem to find algorithmic solutions for concrete cases. Already the strict enumeration of isomorphism classes is, as we shall see in section 11.4, an interesting problem of computer algebra.

So the following sections are only the state of the art and not complete classification. However, there is a considerable number of cases where complete classification is feasible, even on the algorithmic level. This means that musicology disposes of important complete lists of representatives of isomorphism classes, in particular for zero-addressed and self-addressed chords and rhythms, and of small cardinality motives. We have added such lists in the appendices and the working musicologist may consult such lists if she or he does not want to plug in these technicalities. However, we repeat it, classification is much more than the naked list of representatives. If ever any random local composition is given, it has to be attributed to one of the list's representatives, and this requires understanding the candidate in depth! So what we suggest is at least a passive lecture on the following classification discourse in order to trace the conceptual path and to memorize it in case deeper investigations should occur.

### 11.3.1 Classification Techniques

The first step in any classification program is to turn the given objects into points of a space where isomorphism classes are controllable by canonical parameters. To begin with, let us

recapitulate the isomorphisms we will deal with. We fix an address $B \in {}_0Mod_R$ and an ambient module $M \in {}_0Mod_R$; we look at local compositions $K \subset B@_RM$ of positive cardinality $card(K) = n+1$. We further concentrate on generating local compositions, however admitting all endomorphisms of $B$ as address changes, i.e., we extend from the category $ComLoc_B^{gen}$ as defined in section 8.3.5 to $ComLoc_{End(B)}^{gen}$ as discussed in diagram 8.31, but with fixed coefficient ring $R$. Let $K \subset B@_RM, L \subset B@_RN$ be two objects of $ComLoc_{End(B)}^{gen}$. An isomorphism $f/\alpha : K \xrightarrow{\sim} L$ in this category is given by a diagram

$$\begin{array}{ccc} K & \longrightarrow & B@_RM \\ {\scriptstyle \alpha.f}\downarrow & & \downarrow{\scriptstyle B@_Rh} \\ L.\alpha & \longrightarrow & B@_RN \end{array} \tag{11.3}$$

with $\alpha$ an automorphism of $B$ and an underlying morphism $h \in M@_RN$ of ambient spaces[1]. This means that we have an associated linear isomorphism $R.(\alpha.f) : R.K \xrightarrow{\sim} R.L.\alpha$. But since both $K, L$ are generating, we are given an automorphism $B@_Rh$ which sends the constant elements $0_R@_RM = M$ bijectively onto the constant elements $0_R@_RN = N$, and this is $h$, i.e., $h$ is an affine isomorphism. This implies that we may select one representative of every (linear) isomorphism class of $\mathbf{Mod}_R$ and then concentrate on the local compositions within one fixed such module $M$ as ambient space. So we are dealing with the simultaneous action of the groups $\overrightarrow{GL}(B)$ from the right and $\overrightarrow{GL}(M)$ from the left on the set $ComLoc_{End(B)}^{gen,M}$ of objects of $ComLoc_{End(B)}^{gen}$ with ambient space $M$. If $h \in \overrightarrow{GL}(M)$, $\alpha \in \overrightarrow{GL}(B)$, and if $K$ is such an object, the operation is

$$K \mapsto h.K.\alpha. \tag{11.4}$$

We next transform these local compositions into singleton denotators, i.e., points in ambient spaces, as follows. We view $K$ as a zero-addressed local composition in ambient space $B\tilde{@}M$. But then, this is also defined by a singleton (denotator) $K : R^n \rightsquigarrow B\tilde{@}M(k)$ in the following way: Write the elements of $K$ as a sequence $k = (k_0, \dots k_n)$ and observe that $R^n@B\tilde{@}M = R^n \boxtimes B@_RM \xrightarrow{\sim} R^n@_R(B@_RM) \xrightarrow{\sim} (B@_RM)^{n+1} = (0_R@B\tilde{@}M)^{n+1}$. So $k$ is a zero-addressed point of $(B\tilde{@}M)^{n+1}$ which is identified with an injective $R^n$-addressed denotator, i.e., an $R^n$-addressed generating[2] point in $B\tilde{@}M$ with pairwise different coordinates; we denote this point set by $R^n\dot{@}B\tilde{@}M$. Identify the symmetric group $\mathfrak{S}_{n+1}$ with the group of affine automorphisms of $R^n$ which permute the canonical affine basis $(e_0 = 0, e_1 = (1, 0, \dots 0), \dots e_n = (0, \dots 0, 1))$ of $R^n$. Then the address-change orbit $k.\mathfrak{S}_{n+1}$ of the denotator $k$ corresponds 1-1 to $K$ since it abolishes the arbitrary choice of indices of $K$-elements. So the local $(n+1)$-element compositions $K$ correspond to the elements of the orbit space $R^n\dot{@}B\tilde{@}M/\mathfrak{S}_{n+1}$.

Putting these actions together, we have a left action of $\overrightarrow{GL}(M)$, and two commuting right actions, one by $\mathfrak{S}_{n+1}$ and one by $\overrightarrow{GL}(B)$. This gives the following theorem:

[1] And we have a canonical linear isomorphism $R.L \xrightarrow{\sim} R.(L.\alpha)$ since the action $x \mapsto x.\alpha$ is linear and $\alpha$ is auto.

[2] i.e., k defines a generating local composition.

**Theorem 4** *With the above notation, let* $ComLoClass^{gen,M}_{n+1,End(B)}$ *be the set of isomorphism classes of* $(n+1)$*-element generating commutative local compositions in* $ComLoc^{gen,M}_{End(B)}$*, and consider the set*

$$DenOrb(R,n,B,M) = \overrightarrow{GL}(M) \setminus R^n \dot{@} B \tilde{@} M / \overrightarrow{GL}(B) \times \mathfrak{S}_{n+1}$$

*of denotator orbits. Then we have a canonical bijection*

$$ComLoClass^{gen,M}_{n+1,End(B)} \xrightarrow{\sim} DenOrb(R,n,B,M). \tag{11.5}$$

### 11.3.2 The Local Classification Theorem

**Summary.** Classifying orbits of group actions on such standard morphisms. The geometric framework.

– Σ –

The next step is concerned with geometric parameters for the denotator orbit space $DenOrb(R,n,B,M)$.

**Definition 29** *With the preceding notation, let*

$$\begin{aligned} X(R,n,B,M) = \{(V,W) |\ & V \subset W \text{ submodules of } R^n \text{ with } e_i - e_j \notin V, \\ & \text{all } 0 \le i < j \le n, \text{ and there are } R\text{-linear isomorphisms} \\ & \text{such that diagram 11.6 commutes.}\} \end{aligned}$$

$$\begin{array}{ccc} W/V & \xrightarrow{iso.} & 0@_R M \\ {\scriptstyle can.}\downarrow & & \downarrow{\scriptstyle can.} \\ R^n/V & \xrightarrow{iso.} & B@_R M \end{array} \tag{11.6}$$

*and define a right action of the symmetric group* $\mathfrak{S}_{n+1}$*,*

$$X(R,n,B,M) \times \mathfrak{S}_{n+1} \to X(R,n,B,M) : ((V,W),\sigma) \mapsto (\sigma_0^{-1}V, \sigma_0^{-1}W) \tag{11.7}$$

*where* $\sigma_0$ *is the linear part of* $\sigma$*.*

We have a projection of the denotator orbit space into the orbit space $X(R,n,B,M)/\mathfrak{S}_{n+1}$ in the following sense. For $k \in R^n \dot{@} B \tilde{@} M$, we have an element

$$drap(k) = (Ker(k_0), k_0^{-1}(0_R@_R M)) \tag{11.8}$$

of $X(R,n,B,M)$.

Before proving the next proposition we need a technical lemma,

**Lemma 8** *With the above notation, the linear part* $(\alpha@g)_0$ *of the affine map* $(\alpha@g) : B@_R M \to B@_R M$ *is* $\alpha@g_0$*.*

Proof. If $e^\gamma \cdot g_0$ is the decomposition of $g$, and if $f \in B@_R M$, then $\alpha@g(f) = e^\gamma \cdot g_0 \cdot f \cdot \alpha$ whereas $\alpha@g_0(f) = g_0 \cdot f \cdot \alpha$ is linear in $f$ and $\alpha@g(0) = e^\gamma$. In other words, $\alpha@g(f) = (e^{e^\gamma} \cdot (\alpha@g_0))(f)$, whence the claim.

**Proposition 6** *With the above notation, there is a (necessarily unique) surjective map dr such that the diagram*

$$\begin{array}{ccc} R^n \dot{@} B \tilde{@} M & \xrightarrow{p} & DenOrb(R,n,B,M) \\ {\scriptstyle drap}\downarrow & & \downarrow{\scriptstyle dr} \\ X(R,n,B,M) & \xrightarrow{q} & X(R,n,B,M)/\mathfrak{S}_{n+1} \end{array} \tag{11.9}$$

*with canonical horizontal surjections commutes. The map dr is bijective if B is the zero-address.*

Proof. Suppose we are given two denotators $k, l \in R^n \dot{@} B \tilde{@} M$. Let us first see that $dr$ is well-defined, i.e., that if they have same orbit in $DenOrb(R,n,B,M)$, then their images $drap(k) \cdot \mathfrak{S}_{n+1}$ and $drap(l) \cdot \mathfrak{S}_{n+1}$ coincide. If $p(k) = p(l)$, there is a triple $\sigma \in \mathfrak{S}_{n+1}, \alpha \in \overrightarrow{GL}(B), g \in \overrightarrow{GL}(M)$ such that the diagram

$$\begin{array}{ccc} R^n & \xrightarrow{k} & B@_R M \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \alpha@g} \\ R^n & \xrightarrow{l} & B@_R M \end{array} \tag{11.10}$$

commutes. Therefore, the corresponding diagram

$$\begin{array}{ccc} R^n & \xrightarrow{k_0} & B@_R M \\ {\scriptstyle \sigma_0}\downarrow & & \downarrow{\scriptstyle \alpha@g_0} \\ R^n & \xrightarrow{l_0} & B@_R M \end{array} \tag{11.11}$$

of linear parts according to lemma 8 also commutes. Since the right arrow of the latter diagram is iso, the kernels of $k_0, l_0$ verify $\sigma_0(Ker(k_0)) = Ker(l_0)$. Further, the subspace $0_R@_R M$ of constant maps in $B@_R M$ is invariant under $\alpha@g_0$, and therefore, $\sigma_0(k_0^{-1}(0_R@_R M) = l_0^{-1}(0_R@_R M)$. This means that $q(drap(k)) = q(drap(l))$. Surjectivity of $dr$ follows since $drap$ and $q$ are evidently surjective.

Last, we want to see that $dr$ is injective for $B = 0$. In this case, $B@_R M \xrightarrow{\sim} M$, $drap(k)$ specializes to $(Ker(k_0), R^n)$, and the action of $\overrightarrow{GL}(B) = 1$ becomes trivial. Therefore, if for $\sigma \in \mathfrak{S}_{n+1}$, we have $\sigma_0^{-1} \cdot drap(k) = drap(l)$, $\sigma_0$ induces a linear automorphism of $M$ by the quotient automorphism from $\sigma_0$. So $k$ and $l$ have the same orbit in $DenOrb(R,n,B,M)$. QED.

For the rest of this section, we suppose that $B$ is the zero-address[3], and we want to present algebro-geometric classification results in this case.

For the zero-address we have several simplifications of the above formalism. To begin with, consider the short exact sequence

$$0 \longrightarrow M \xrightarrow{\Delta} M^{n+1} \xrightarrow{d} M^n \longrightarrow 0 \tag{11.12}$$

[3]It is an open problem to find a canonical generalization of the *drap* map for general address $B$ such that $dr$ becomes a bijection.

of $R$-modules and linear maps with the diagonal embedding $\Delta(m) = (m, m, \dots m)$ and the difference formula $d(m_0, m_1, \dots m_n) = (m_1 - m_0, \dots m_n - m_0)$. We have a right linear action of $\mathfrak{S}_{n+1}$ on $M^{n+1}$ defined by $(m_0, m_1, \dots m_n) \cdot \sigma = (m_{\sigma 0}, m_{\sigma 1}, \dots m_{\sigma n})$ which leaves $\Delta(M)$ invariant and therefore induces a linear action on the quotient $M^n$. We also have the left diagonal actions of $\overrightarrow{GL}(M)$ on $M^{n+1}$ and of $GL(M)$ on $M^n$. Further:

**Lemma 9** *The projection d is equivariant with respect to the above actions of the symmetric group $\mathfrak{S}_{n+1}$ and the canonical group homomorphism $\overrightarrow{GL}(M) \to GL(M)$.*

Proof. The first claim is true by construction. For the second, let $m_\cdot \in M^{n+1}$ and $e^m \cdot g \in \overrightarrow{GL}(M)$. Then $d(e^u \cdot g(m_\cdot)) = g(d(m_\cdot))$ since u cancels out by the difference homomorphism $d$.

Let $M^{\overline{n}}$ be the subset of $M^n$ consisting of the $n$-tuples with pairwise different *and non-vanishing* coordinates and such that these coordinates generate the module $M$. Set $M^{\underline{n+1}} = d^{-1}M^{\overline{n}}$. Clearly, both sets are invariant under the above actions, and d projects $M^{\underline{n+1}}$ onto $M^{\overline{n}}$. Moreover:

**Lemma 10** *The canonical map of orbits*

$$\overrightarrow{GL}(M)\backslash M^{\underline{n+1}}/\mathfrak{S}_{n+1} \to GL(M)\backslash M^{\overline{n}}/\mathfrak{S}_{n+1} \tag{11.13}$$

*is a bijection. The left orbit set (and therefore also the right one) identifies to the set of isomorphism classes of $(n+1)$-element generating local compositions in $M$.*

Proof. The last claim is clear by construction. The map of orbits is also surjective since $d : M^{\underline{n+1}} \to M^{\overline{n}}$ is surjective. Let $m_\cdot, n_\cdot \in M^{\underline{n+1}}$ such that there are $g \in GL(M), \sigma \in \mathfrak{S}_{n+1}$ with $g \cdot d(m_\cdot) \cdot \sigma = d(n_\cdot)$. Then for all indices $i = 1, \dots n$, we have $n_i - n_0 = g(m_{\sigma i} - m_{\sigma 0}) = g(m_{\sigma i}) - g(m_{\sigma 0})$ and hence the affine equation

$$n_i = (e^{n_0 - g(m_{\sigma 0})} \cdot g)(m_{\sigma i}),$$

and this shows that the orbits of $m_\cdot$ and $n_\cdot$ in the left orbit set coincide. QED.

Therefore we may observe the linear action of $GL(M)$ and of $\mathfrak{S}_{n+1}$ on $M^{\overline{n}}$. Clearly, by the canonical isomorphism $M^n \xrightarrow{\sim} Lin_R(R^n, M)$, this action identifies to the left-right actions of these groups on $Lin_R(R^n, M)$, i.e., the canonical left action of $GL(M)$ and the right action of $\mathfrak{S}_{n+1}$ on $R^n$ via a group homomorphism $D : \mathfrak{S}_{n+1} \to GL_n(R)$ which on transpositions $(1, i)$ is given by the matrix

$$D(1,i) = \begin{pmatrix} 1 & & & & & 0 & \\ 0 & 1 & & & & & \\ \dots & \dots & 1 & & & & \\ -1 & -1 & \dots & -1 & \dots & -1 & \\ & & & & & & \\ & & 0 & & & 1 & \end{pmatrix} \tag{11.14}$$

with $-1$ entries overriding the $n \times n$ identity matrix on row $i - 1$.

Through this identification $M^{\overline{n}}$ identifies to a stable subset $\overline{Lin}_R(R^n, M)$ of $Lin_R(R^n, M)$, and this yields the following zero-address version of theorem 4. This reduction from affine to linear structures is only possible for the zero-address.

**Theorem 5** *There is a canonical bijection*

$$ComLoClass^{gen,M}_{n+1,O_R} \xrightarrow{\sim} GL(M)\backslash\overline{Lin}_R(R^n,M)/\mathfrak{S}_{n+1}.$$

We now proceed to the linear analogy of the flag space $X(R,n,B,M)$ for zero-address. We set

$$\begin{aligned} X_n(R) &= \{V|\ V \text{ a submodule of } R^n \text{ with } e_i, e_i - e_j \notin V, \\ &\qquad \text{all } 1 \leq i < j \leq n\}, \qquad (11.15) \\ X_n^M(R) &= \{V|\ V \in X_n(R) \text{ and there is an } R\text{-linear} \\ &\qquad \text{isomorphism } R^n/V \xrightarrow{\sim} M\}. \qquad (11.16) \end{aligned}$$

Clearly, the symmetric group $\mathfrak{S}_{n+1}$ acts from the right on both, $X_n(R)$ and $X_n^M(R)$, via $V\cdot\sigma = D(\sigma^{-1})\cdot(V)$, an action which we recognize as being the linear analog to the above action on $X(R,n,B,M)$. Let $\mathcal{M}$ be a system of representatives of isomorphism classes in $\mathbf{Mod}_R$. Then we have a disjoint union

$$X_n(R) = \coprod_{M\in\mathcal{M}} X_n^M(R) \qquad (11.17)$$

(most of these co-factors are empty, of course) and its corresponding orbit space union

$$X_n(R)/\mathfrak{S}_{n+1} = \coprod_{M\in\mathcal{M}} X_n^M(R)/\mathfrak{S}_{n+1}, \qquad (11.18)$$

**Lemma 11** *The orbit set $X_n^M(R)/\mathfrak{S}_{n+1}$ is in canonical bijection with the class set*

$$ComLoClass^{gen,M}_{n+1,O_R}.$$

Proof. We know from theorem 5 that this class set is in bijection with the orbit set

$$GL(M)\backslash\overline{Lin}_R(R^n,M)/\mathfrak{S}_{n+1}.$$

In the diagram

$$\begin{array}{ccc} \overline{Lin}_R(R^n,M) & \xrightarrow{Ker} & X_n^M(R) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ GL(M)\backslash\overline{Lin}_R(R^n,M)/\mathfrak{S}_{n+1} & \xrightarrow{f} & X_n^M(R)/\mathfrak{S}_{n+1} \end{array} \qquad (11.19)$$

with canonical vertical arrows and the upper horizontal arrow $R^n \xrightarrow{g} M \mapsto Ker(g)$, there is a unique bijection $f$ which makes the diagram commute. In fact, the kernel of $g$ does not change under an automorphism of $M$, and we may factorize through the $GL(M)$-orbits. Since the horizontal map $Ker$ is $\mathfrak{S}_{n+1}$-equivariant, the factorization through $f$ is defined, and $f$ is surjective since both, $Ker$ and $p$, are. Finally, if two $R^n$-addressed denotators $k,l : R^n \to M$ in $\overline{Lin}_R(R^n,M)$ have kernels which differ by a permutation, then their quotients which identify to $M$ are related by an automorphism of $M$ and therefore, $q(k) = q(l)$, so $f$ is mono. QED.

Let $ComLoClass^{gen}_{n+1,O_R}$ be the set of all isomorphism classes (any ambient $R$-module $M$ allowed). Then the above lemma gives

**Theorem 6** (Classification of n+1-element, generating, zero-addressed local compositions over ring $R$) *There is a canonical bijection*

$$X_n(R)/\mathfrak{S}_{n+1} \xrightarrow{\sim} ComLoClass^{gen}_{n+1,O_R}.$$

We therefore may concentrate on the left space and look to situations which may lead to geometric classification spaces, i.e., where the classes appear as points on appropriate geometric spaces.

Before delving into techniques of algebraic geometry we should try to understand the geometric aspect of theorem 6 and its musical interpretation. Let us concentrate on the intuitive situation of local compositions over the rationals. Observe that for $\mathbb{Q}$, every such composition is isomorphic to a generating one[4] and we may forget about this specification and even concentrate on ambient spaces of coordinator $\mathbb{Q}^r$.

To begin with, replacing a generating local $(n+1)$-element composition $(K, \mathbb{Q}^r)$ by a $\mathbb{Q}^n$-addressed denotator $k : \mathbb{Q}^n \to \mathbb{Q}^r$ means that we really have an instance of a more general approach which is based on standard local compositions (see figure 11.1):

**Definition 30** *Given the ring $R$ and a natural number $n$, we denote by $\Delta_n$ the zero-addressed commutative local composition of cardinality $n+1$ in ambient space $R^{\oplus n}$ consisting of the canonical affine base, i.e.,*

$$\Delta_n = \{e_i |\ 0 \leq i \leq N\} \tag{11.20}$$

*with the standard notation $e_0 = 0$ and $e_i = (\delta_{ij})_{1 \leq j \leq N}$ for $i > 0$ and the identification of $R^{\oplus n}$ with $0_R @ R^{\oplus n}$. The local composition $\Delta_n$ is called* standard simplex composition of dimension *$n$.*

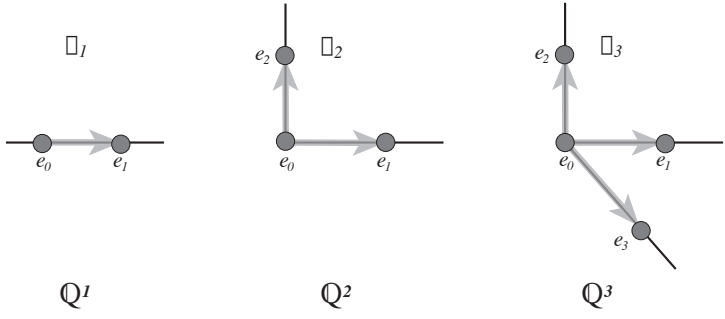


Figure 11.1: Local standard simplex compositions over $\mathbb{Q}$ of dimension 1, 2, and 3.

With this in mind, the denotator $k : \mathbb{Q}^n \to \mathbb{Q}^r$ corresponds 1-1 to the morphism $k_\Delta : \Delta_n \to \mathbb{Q}^r$ with generating image, and $K$ identifies to the orbit $k_\Delta \cdot \mathfrak{S}_{n+1}$. So we have abstracted from $K$ and concentrated on *one distinguished local composition*: the standard simplex composition $\Delta_n$. The variability of $K$ has been transferred to that of $k : \Delta_n \to \mathbb{Q}^r$, and finally, by the *drap*

[4]Simply because every local composition $(K, M)$ over a field $F$ is isomorphic to $(K, F.K)$.

function, that of $k$ has been reduced to the kernel $V = Ker(k_0)$. But what does this restatement mean? We have a finite orbit $V \cdot \mathfrak{S}_{n+1}$ and therefore[5], there is a *single* submodule $W \subset \mathbb{Q}^r$ which is a linear complement for all elements $V \cdot \sigma$ of this orbit: $\mathbb{Q}^n = V \cdot \sigma \oplus W$, *all* $\sigma \in \mathfrak{S}_{n+1}$. By construction, projection $p_{V,\sigma} : V \cdot \sigma \oplus W \to W$ gives us $K$ as the bijective image of the standard composition, i.e., $K = p_{V,\sigma}(\Delta_n)$, see figure 11.2.

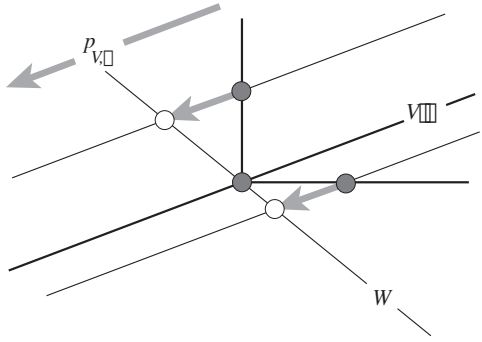


Figure 11.2: Each kernel space $p_{V,\sigma}$ that is complementary to a given subspace $W$ in $\mathbb{Q}^n$ gives rise to a specific projection of the standard composition $\Delta_n$ whose image defines a special instance of our ensemble of local compositions. This may be viewed as a variable coordinate system and leads to musical interpretation of the classification technique.

In other words: the ensemble of local $n+1$-element local compositions $K$ with $dim(K) = r$ is transformed into *one single* "standard representative" $\Delta_n$, together with an ensemble of projections $p_{V,\sigma}$ indexed by $V$ and the symmetric group, and defined by the variable "coordinate system" $\mathbb{Q}^n = V \cdot \sigma \oplus W$ which works for all orbit members. This means that we deal with one "generic" local composition whose points are *in general position*[6] and then project this generic specimen onto specialized images defined by *selection of special coordinate functions*. We shall come back to this classification approach which also solves the problem for global classification in section 15.1.1. Musically speaking, this process means to give an "abstract, generic composition" $\Delta_n$ and to find a set of coordinates to realize a special variant of this generic data. Ontologically, this passage can be viewed as a realization of a symbolic composition in special, musically meaningful coordinates. We shall work out this now somewhat artificially blown up interpretation in 15.1.1, but want to mention it already now since this sheds light on the entire classification business.

For the time being, the more important point is that the passage to kernel spaces via *drap* turns the seemingly dispersed appearance of local compositions in their ensemble into a geometric situation because subspaces can easily be compared with each other by continuous movement. This means that we may now look for class-invariant geometric relations among

[5]See appendix F.5. Observe also that the orbit $V \cdot \mathfrak{S}_{n+1}$ is contained in the $\mathfrak{S}_{n+1}$-invariant set of the $n-r$-dimensional subspaces which are complementary to all $W \cdot \sigma$. This is a stable open set in the corresponding Grassmannian variety (appendix F.5).

[6]See appendix H.1.1.

local compositions, for example linear incidence among their points. In other words, the present efforts aim at *a geometrically sophisticated understanding of local compositions*, a scope which allows a very refined comparison of musical structures—such as motives or chords—in concrete compositions; applications of this view are given in section 11.6.2.

To construct a geometric parametrization of classes, we restrict to local compositions $(K, M)$ which are *locally free of rank* $r$, i.e., their module $R.K$ is locally free of rank $r$ (see appendix F.2, theorem 144). Denote by $\mathcal{M}_r$ the representative subsystem of modules in $\mathcal{M}$ which are locally free of rank $r \in \mathbb{N}$ and set $X_n^r(R) = \coprod_{M\in\mathcal{M}_r} X_n^M(R)$. In other words,

$$X_n^r(R) \subset \{V \subset R^n |\ R^n/V = \text{ locally free of rank } r\},$$

but the latter is by definition[7] the evaluation $Grass_{n,r}(R)$ of the Grassmann functor

$$Grass_{n,r} : \mathbf{ComRings} \to \mathbf{Sets} \tag{11.21}$$

at ring $R$. More precisely, if $x \in R^n$, let$V_{n,x}^r(R) = \{V |\ V \in Grass_{n,r}(R), x \in V\}$. Further, if $f : R \to S$ is a ring homomorphism, the functorial map

$$Grass_{n,r}(f) : Grass_{n,r}(R) \to Grass_{n,r}(S) : V \mapsto S \otimes V \tag{11.22}$$

restricts to

$$V_{n,x}^r(f) : V_{n,x}^r(R) \to V_{n,1\otimes x}^r(S).$$

In fact, we know from appendix E.4, lemma 83 that $V \in Grass_{n,r}(R)$ is in $V_{n,x}^r(R)$ iff $\wedge^r x = 0$ for the direct summand $V \subset R^n$. But scalar extension commutes with formation of exterior powers ([63, ch.III, §7, no.5, Prop. 8]) and so $\wedge^r 1 \otimes x = 0$ with respect to the direct summand $S \otimes V$ of $S^n$. In particular, setting $e_{ij} = e_i - e_j$ for $i \neq j$ and $e_{ii} = e_i$ for the canonical basis vectors $e_i$ in $\mathbb{Z}^n$, we have subfunctors $V_{n,e_{ij}}^r$ of $Grass_{n,r}$, defined by $V_{n,e_{ij}}^r(R) = V_{n,1\otimes e_{ij}}^r(R)$. The above action of the symmetric group $\mathfrak{S}_{n+1}$ permutes the subfunctors $V_{n,e_{ij}}^r$, and the complement of their union

$$V_n^r(R) = \bigcup_{1\leq i\leq j\leq n} V_{n,e_{ij}}^r(R)$$

gives us the $\mathfrak{S}_{n+1}$-invariant set

$$X_n^r(R) = Grass_{n,r}(R) - V_n^r(R).$$

In contrast to the $V_{n,e_{ij}}^r(R)$, to $V_n^r(R)$, and to $Grass_{n,r}(R)$, $X_n^r(R)$ is not functorial in $R$.

**Exercise 12** Prove this last claim with $R = \mathbb{Z}$.

We need to define a subfunctor $\tilde{X}_n^r(R) \subset Grass_{n,r}(R)$ which is disjoint from $V_n^r(R)$ and "essentially" gives us $X_n^r(R)$. The idea is that $V_n^r(R)$ is representable by a closed subscheme of the Grassmannian, and that we want to find a functorial expression $\check{X}_n^r(R) \subset X_n^r(R)$ of its open complement. We can show the following:

[7] See section F.5 for the theory of Grassmann schemes.

**Lemma 12** *The subfunctors $V^r_{n,e_{ij}}(R) \to Grass_{n,r}(R)$ are representable by closed subschemes $V_{n,e_{ij}} \to Grass_{n,r}$. Each of the open complement schemes $Grass_{n,r} - V^r_{n,e_{ij}}$ is covered by the open subfunctor $\tilde{X}^r_{n,e_{ij}}(R) \subset Grass_{n,r}(R)$ consisting of the direct factors $V \subset R^n$ such that $\wedge^r e_{ij} : \bigwedge^r V \to \bigwedge^{r+1} R^n$ splits (has a right inverse).*

Proof. Let $i_\cdot = i_1, i_2, \dots i_r$ be an increasing subsequence $1 \leq i_1 < i_2 < \dots i_r \leq n$. We know from appendix F.5 that the functor $Grass_{n,r}(R)$ is covered by the affine open subfunctors $Grass_{n,r,i_\cdot}(R)$ of those direct factors $V \subset R^n$ such that $R^{i_\cdot}$ projects isomorphically onto $R^n/V$. If $i'_\cdot$ denotes the complementary index sequence of $i_\cdot$, then $Grass_{n,r,i_\cdot}(R)$ identifies to the set of graphs $\Gamma f$ of linear maps $f \in Lin_R(R^{i'_\cdot}, R^{i_\cdot})$, i.e., to the $n \times (n-r)$ matrices with columns $(e_i, f(e_i))^t$, $i$ in $i'_\cdot$. Restricted to such an open affine subfunctor, $V^r_{n,e_{kl}}(R)$ identifies to the set of $n \times (n-r+1)$-matrices $M_{f,kl}$ with $n-r$ columns $(e_i, f(e_i))^t$ for $j = 1, \dots n-r$ and an $(n-r+1)^{th}$ column $e_{kl}$ such that all $(r+1) \times (n-r+1)$-determinants of $M_{f,kl}$ vanish. This is a closed condition defined by the ideal which is generated by these determinants. We then look for the intersection of $Grass_{n,r,i_\cdot}(R)$ with the subfunctor $\tilde{X}^r_{n,e_{ij}}(R)$. To see that this is an open subfunctor, observe that an $R$-valued point

$$M_{f,kl} \in Grass_{n,r,i_\cdot}(R) \cap \tilde{X}^r_{n,e_{ij}}(R) \tag{11.23}$$

is characterized by the property that the $(n-r+1)\times(n-r+1)$-determinants $d_b, b = 1, \dots \binom{n}{n-r+1}$ of $M_{f,kl}$ define the first column of an invertible $\binom{n}{n-r+1} \times \binom{n}{n-r+1}$-matrix over $R$. This set in affine $\binom{n}{n-r+1}$-space is open by proposition 96 in appendix F.5, and the determinants are polynomial functions of the coordinates of $\Gamma f$, so the intersection $Grass_{n,r,i_\cdot} \cap \tilde{X}^r_{n,e_{ij}}$ is an open subscheme of $Grass_{n,r,i_\cdot}$. To see that the open complements $Grass_{n,r} - V^r_{n,e_{ij}}$ are covered by $\tilde{X}^r_{n,e_{ij}}$, observe that if $R$ is a field, the intersections (11.23) coincide with the complements of the sets $V^r_{n,e_{kl}}(R)$ since non-vanishing of the determinants creates a basis vector by the exchange theorem for vector spaces. QED.

**Lemma 13** *The open subscheme*

$$\tilde{X}^r_n = \bigcap_{1 \leq i \leq j \leq n} \tilde{X}^r_{n,e_{ij}} \tag{11.24}$$

*is invariant under the given action of $\mathfrak{S}_{n+1}$, and every orbit in $\tilde{X}^r_n$ is contained in an affine neighborhood. We have $\tilde{X}^r_n(R) \subset X^r_n(R)$, and if $R$ is semi-simple we have $\tilde{X}^r_n(R) = X^r_n(R)$.*

Proof. Since the action of the symmetric group permutes the elements $e_{ij}$, it permutes the members of the intersection (11.24). Further, since the Grassmannian is projective, every finite orbit in $\tilde{X}^r_n$ is contained in an affine neighborhood by [199, ch.II, 4.5.4]. The inclusion $\tilde{X}^r_n(R) \subset X^r_n(R)$ follows from construction. The last statement is clear for a field, and hence for a semi-simple ring which, by Wedderburn's theorem is a finite direct product of fields (see appendix E.2.3, theorem 48). QED.

**Theorem 7** *With the above notation, there is a quotient scheme*[8] $\tilde{X}_n^r/\mathfrak{S}_{n+1}$, *i.e., there is an exact sequence of schemes*

$$\tilde{X}_n^r \times \mathfrak{S}_{n+1} \underset{\mu}{\overset{pr_1}{\rightrightarrows}} \tilde{X}_n^r \to \tilde{X}_n^r/\mathfrak{S}_{n+1}$$

*with group action* $\mu$ *and first projection* $pr_1$. *Its* $R$*-valued points are the orbits of* $\tilde{X}_n^r(R)$ *and of* $X_n^r(R)$ *in case* $R$ *is a semi-simple ring.*

Proof. The exactness property follows from lemma 13 and [123, ch.III, §2, no.6.1]. QED.

The translation of the defining property for the scheme $\tilde{X}_n^r$ is that for such a generating local composition $(K, M)$,

$$\left\{\begin{array}{l} \text{all module homomorphisms} \\ .(k-l) : R \to M : r \mapsto r(k-l) \\ \text{for any two different elements } k, l \in K \\ \text{are split injections.} \end{array}\right. \tag{11.25}$$

**Definition 31** *If a zero-addressed commutative local composition has the splitting property* (11.25) *we say that it is split.*

**Exercise 13** Verify that the splitting property (11.25) is characteristic for membership in $\tilde{X}_n^r$.

**Corollary 5** *The set* $ComLoClass_{n+1,0_R}^{gen,lf,sp}$ *of isomorphism classes of zero-addressed, local, split, locally free, and generating* $(n+1)$*-element compositions with any* $R$*-module as ambient space is in canonical bijection to the set of* $R$*-valued points of the scheme*

$$Cl_n = \coprod_{1 \leq r \leq n} \tilde{X}_n^r/\mathfrak{S}_{n+1}.$$

*If* $R$ *is a semi-simple ring, the set* $ComLoClass_{n+1,0_R}$ *of isomorphism classes of zero-addressed, local (not necessarily generating, locally free, or split)* $(n+1)$*-element compositions with any* $R$*-module as ambient space is in canonical bijection to the set of* $R$*-valued points of the scheme* $Cl_n$.

Let us have a closer look at the set $ComLoClass_{n+1,0_R}^{gen,lf,sp}$ in a special case which is useful in practice. If the coefficient ring $R$ is self-injective[9], the condition that the linear homomorphism $.(k-l) : R \to M : r \mapsto r(k-l)$ is split boils down to the vanishing of the annulator of $k-l$: $Ann(k-l) = 0$. In fact, we have $Ker(.(k-l)) = Ann(k-l)$ and in that case, $R(k-l) \xrightarrow{\sim} R$ means by appendix E.4.3, proposition 88, that $R(k-l)$ is a direct factor of any module which contains it.

[8] In the sense of [123, ch.III, §2, no.1.3], see theorem 59 in appendix F.6.

[9] See appendix E.4.3.

### 11.3.3 The Finite Case

**Summary.** Recursive classification on finite modules via socles and geometric classification.

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Although a general geometric classification of local compositions in the above sense is not accomplished, we may use geometric classification to settle local compositions in finite ambient spaces, a situation which is particularly important in music theory. We shall make this more explicit in sections 11.3.5, 11.3.7, and 11.3.8.

More precisely, we will consider zero-addressed local compositions $(K, \mathbb{Z}_N^n)$ for $1 < N, 0 < n$. Since $\mathbb{Z}_N$ is self-injective by appendix E, proposition 78, we may apply sorite 6 in section 8.3.5 to classify such local compositions: Since the categories $ComLoc_{0_R}^{in}$ and $ComLoc_{0_R}^{gen}$ are equivalent, we may concentrate on the latter.

If $N = p_1^{n_1} \cdot \ldots p_r^{n_r}$ is the prime factorization (see appendix D.2), we have $\mathbb{Z}_N \xrightarrow{\sim} \mathbb{Z}_{p_1^{n_1}} \times \ldots \mathbb{Z}_{p_r^{n_r}}$. So we first have to deal with direct products of rings and corresponding categories of local compositions. Recall the following from appendix D.1.1, definition 129. Let $R = S \times T$ be a direct product ring, $M$ an $R$-module. Denote by $x = x_S + x_T$ the decomposition of element $x \in M$ into its $S$-component $x_S = 1_S x$ and $T$-component $x_T = 1_T x$, and by $M_S = 1_S M,\ M_T = 1_T M$ the corresponding decomposition of $M$ such that $M = M_S \oplus M_T$ is a direct sum of an $S$- and of a $T$-module. For an address $A$ in $\mathbf{Mod}_R$, we also have a corresponding canonical decomposition $A@_R M \xrightarrow{\sim} A_S@_S M_S \oplus A_T@_T M_T$ which carries over to local compositions in these spaces. With any object $(K, A@_R M) \in ComLoc_A$, we then associate the pair

$$\times(K, A@_R M) = ((K_S, A_S@_S M_S), (K_T, A_T@_T M_T)) \in ComLoc_{A_S} \times ComLoc_{A_T}.$$

For a morphism $f : (K, A@_R M) \to (L, A@_R N)$ in $ComLoc_A$ we have an obvious decomposition into two morphisms $f_S : (K_S, A_S@_S M_S) \to (L_S, A_S@_S N_S)$ and $f_T : (K_T, A_T@_T M_T) \to (L_T, A_T@_T N_T)$ and therefore a map $f \mapsto \times f = (f_S, f_T)$.

**Lemma 14** *The above map*

$$\times : ComLoc_A \to ComLoc_{A_S} \times ComLoc_{A_T}$$

*is a functor, has well-defined restrictions*

$$\begin{aligned}
\times &: ComLoc_A^{emb} \to ComLoc_{A_S}^{emb} \times ComLoc_{A_T}^{emb}, \\
\times &: ComLoc_A^{gen} \to ComLoc_{A_S}^{gen} \times ComLoc_{A_T}^{gen}, \\
\times &: ComLoc_A^{in} \to ComLoc_{A_S}^{in} \times ComLoc_{A_T}^{in},
\end{aligned}$$

*and all these functors are surjective on the objects.*

We leave the proof as an exercise. Observe that for a pair

$$((K_1, A_S@_S M_S), (K_2, A_T@_T M_T)) \in {}_0ComLoc_{A_S} \times {}_0ComLoc_{A_T},$$

there is a unique maximal element $((K_1 \times K_2, A@_R M)) \in \times^{-1}(K_1, K_2)$. So the fiber consists of all local subcompositions $U \subset K_1 \times K_2$ with $U_S = K_1, U_T = K_2$. We call them $K_1$-$K_2$-*projecting*.

Suppose that classification is settled for the factors and that we are given a pair $K_1, K_2$ of representatives of each class. Then classification of the set $K_1 \curlywedge K_2$ of $K_1$-$K_2$-projecting local compositions amounts to the calculation of the orbit set

$$Aut(K_1) \times Aut(K_2) \backslash K_1 \curlywedge K_2$$

under the obvious left action of $Aut(K_1) \times Aut(K_2)$. Observe that the latter groups are subsets of the symmetric groups of the underlying sets $K_1$ and $K_2$. If these are finite, the problem boils down to finite product group actions on finite set products, a subject of pure combinatorics, see section 11.4.1 on this subject.

Let us now return to the more substantial problem of classifying the factor objects, i.e., objects in the category $ComLoc^{gen}_{0_{\mathbb{Z}_{p^n}}}$. We want to settle this problem by means of a recursive procedure that runs on the set of orbits of the set

$$U_r(R) = \{V |\ V \text{ a submodule of } R^r\} \tag{11.26}$$

under the action of a subgroup $G \subset \mathfrak{S}_{r+1}$ which extends the known linear action on $X_r(R)$ defined in 11.17. Recursion runs on the cardinality $card(\mathbb{Z}^r_{p^n})$ of our free rank $r$ module over the ground ring $R = \mathbb{Z}_{p^n}$. Let $V \in U_r(R)$.

1. Let $soc(R^r) \subset V$ (see appendix E.2.4 for the socle concept). Since the socle is invariant under any linear action, we may as well calculate the orbit of $V/soc(R^r)$ in $U_r(R/soc(R))$ which is recursively earlier.

2. Let $soc(V) \subsetneq soc(R^r)$. If $soc(V) = V$, then $V \in U_r(R/Rad(R))$, and we are in the case of the characteristic $p$ prime field $R/Rad(R) = \mathbb{Z}_p$ (see appendix E.2.4 for the radical concept). This is the geometric situation which is settled in theorem 7.

3. Let $soc(V) \subsetneq soc(R^r)$ and $soc(V) \subsetneq V$. Since $R^r$ is injective, there is an injective envelope $I \subset R^r$ of $V$. By appendix E.4.3, proposition 88, it is a direct factor of $R^r$, and since $R$ is local, it is free, i.e., $I \xrightarrow{\sim} R^s$, $s < r$. This direct factor of $R^r$ is an element of the Grassmannian $Grass_{r,r-s}(R)$ and can be classified by the geometric method which is standard for Grassmannians. Let $G = (\mathfrak{S}_{r+1})_I \subset \mathfrak{S}_{r+1}$ be the isotropy group (see appendix C.3.1) of $I$. Then we have $V \in U_s(R)$, and we have to calculate its $G$-orbit for smaller cardinality. This completes the recursion process.

This process settles many of the practically important cases as we shall see in a moment. But we have to stress that we do not control the general geometric classification with address changes—a fortiori we do not control the deeper musicological consequences of this general geometric setup. We shall however give several partial answers in the following sections, and in section 11.6, the latter concerning the esthetic aspect of classification.

### 11.3.4 Dimension

**Summary.** We discuss the problematic concept of dimension of local compositions.

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We already learned in section 6.4.1, formula (6.26), that the pitch space is not the seemingly one-dimensional real space but the subtler construction of Euler module which in the common setting (octave, fifth, third) has three dimensions over the rationals although it injects into the real line. Music theory thinks of octave, fifth, and third as being three independent musical directions, and restriction to rational scalars does the job in the mathematical sense.

This is even more delicate for progressive abstraction as it happens for pitch and onset classes modulo some positive periods. And this has a dramatic impact on the musical consequences of classification. To fix the ideas, look at a zero-addressed motif (7.2.3) $(M, Onset|\mathbb{Z} \oplus Pitch|\mathbb{Z})$ with integer coordinates. In this case, one agrees that the dimension of the underlying space is the rank 2 of a space isomorphic to $\mathbb{Z}^2$. But if we proceed to an identification of integer onsets modulo $m$ and integer pitch modulo $n$—a completely common practice, we recall it—, then the residual motif $\overline{M} \subset OnPiMod_{m,n}$ (defined in (6.44)) shares a more complex dimensionality. To understand what happens, we compare the residual motives for $m, n = 12, 12$ (typical to dodecaphonism) with those for $m, n = 5, 12$. So in both cases, pitch is taken modulo octave, but in the first case, onset is taken modulo 12 and therefore, rhythmic phenomena are understood to share period 12 whereas in the second case, the period has to be 5.

There are $\binom{144}{3} = 487'344$ zero-addressed three-element motive residues in $OnPiMod_{12,12}$ whereas there are only $\binom{60}{3} = 34'220$ three-element motive residues in $OnPiMod_{5,12}$. However, anticipating classification of motive residues in these ambient spaces (see section 11.3.8, and appendix M), we have 26 isomorphism classes in case $m, n = 12, 12$ against 45 classes for $m, n = 5, 12$ which at first glance is very surprising.

The surprise vanishes if we look at the Sylow groups of these modules. We have $\mathbb{Z}_{12}^2 \xrightarrow{\sim} \mathbb{Z}_4^2 \times \mathbb{Z}_3^2$ whereas $\mathbb{Z}_5 \times \mathbb{Z}_{12} \xrightarrow{\sim} \mathbb{Z}_5 \times \mathbb{Z}_4 \times \mathbb{Z}_3$. The first space has a 2- and a 3-Sylow group, the second has a 2-, a 3-, and a 5-Sylow group. This corresponds to the general affine groups

$$\begin{aligned} \overrightarrow{GL}(\mathbb{Z}_{12}^2) &\xrightarrow{\sim} \overrightarrow{GL}(\mathbb{Z}_4^2) \times \overrightarrow{GL}(\mathbb{Z}_3^2), \\ \overrightarrow{GL}(\mathbb{Z}_5 \times \mathbb{Z}_{12}) &\xrightarrow{\sim} \overrightarrow{GL}(\mathbb{Z}_4) \times \overrightarrow{GL}(\mathbb{Z}_3) \times \overrightarrow{GL}(\mathbb{Z}_5). \end{aligned}$$

The orders are $card(\overrightarrow{GL}(\mathbb{Z}_{12}^2)) = 663'552$, and $card(\overrightarrow{GL}(\mathbb{Z}_5 \times \mathbb{Z}_{12})) = 960$ (see appendix C.3.5 for the calculation). We have $\approx 14.2$ more motives in $OnPiMod_{12,12}$ than in $OnPiMod_{5,12}$, but the corresponding operating group $\overrightarrow{GL}(\mathbb{Z}_{12}^2)$ is $\approx 691.2$ times larger than $\overrightarrow{GL}(\mathbb{Z}_5 \times \mathbb{Z}_{12})$. This is due to the larger number of Sylow groups in the second module and therefore produces $\approx 1.7$ times more isomorphism classes for the 5-periodic motive residues.

This difference has a remarkable interpretation in terms of music theory and even music practice. If we want to classify motives modulo octave and in onset periods dividing 12, e.eg in 2/4, 3/4, 4/4, 6/8 or 12/8 time signatures, this is $\approx 1.7$ times easier than to do so modulo octave and in onset period 5, i.e., in 5/8 time signature. This means that our cognitive effort to identify motive classes is nearly double for 5/8 time signature compared to the other situations.

It is of course not known—but to find out would be an important experiment in cognitive psychology—whether and to what degree humans do perform classification tasks in the above sense. There is however strong evidence from practice that this could be the case in light of the above results: Among jazz musicians it is well known how difficult it is to improvise in 5/8 time signature compared to 3/4, 4/4, or 6/8. For example, coherent, bar-exceeding improvisation in Desmond's *Take Five* is much more difficult than improvisation in a waltz such as Sherman's

*Chim Chim Cheree*. So the increase in difficulty is not due to odd against even time signature, but the fact that 5 and 12 are relatively prime, whereas 3 and 12 are not.

These observations suggest that cognitive dimension in music is rather related to Sylow groups than to the more obvious length of a finite ambient space (which would be 6 vs. 3 in 12-periodic vs. 5-periodic rhythms). The systematic problem behind these observations is to investigate a general understanding of what is structural "independence" among musical attributes and how this one could be related to cognitive "independence". Structural/cognitive dimension would be a measure for the amount of structural/cognitive independence. *The systematic investigation of dimension as a degree of freedom in a musical system is undoubtedly a central issue for the description of global strategies in musical composition and cognition.*

### 11.3.5 Chords

**Summary.** Classification of zero-addressed chords in $PiMod_{12}$, their symmetry groups, automorphism groups, and conjugacy classes of endomorphisms in 12-chromatic pitch class space $\mathbb{Z}_{12}$.

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We shall not discuss historic questions of this basic classification issue here, but refer to section 11.5.2 on the American musical set theory tradition for more detailed remarks. But we should stress right here that, besides this American tradition, no systematic research on classification of chords has ever been undertaken. European classical musicology and music theory has only dealt with some important chords (such as minor, major, diminished, and augmented triads) and has not dealt with classification in the sense of determination of isomorphism classes under any non-trivial group beyond transposition. Even in recent times [38], so-called chord dictionaries are published which are anything but complete. This does not only testify to a fragmentary practice but much more (as a foundation of practice) a fragmentary *theory*. The complete list of isomorphism classes of zero-addressed chords in $PiMod_{12}$, together with additional determinants and class invariants, is accessible in appendix L. This list was first discussed in a series of University lectures [327] in 1981, and published in [328]. In [400], more information concerning conjugacy classes of endomorphisms has been added.

As the chord class list can be calculated by a computer program, we concentrate on the description of the information which specifies the isomorphism classes. The list has five columns:

- The first column denotes the *class number*. The numbering first reflects the cardinality, starting with the special case of the full, 12-element chord, then the one-element class, then the two-element classes, and so forth until the six-element classes. Classes with more than six elements, seven,..., eleven, and the zero-element, the empty class, are not listed explicitly, since they appear as complements of the explicitly listed classes. There are, however, exceptions to this rule: There are six-element classes which are isomorphic to their complements. These classes have a star exponent with their class number, for example number 82*. So when counting the isomorphism classes, those 18 classes with star count only once, all others count double. This gives us a total of 157 non-empty classes.

The numbering for fixed cardinality is as follows: We order chords according to the power-set order defined in 6.8, but with two (more historically motivated) differences: The order on $\mathbb{Z}_{12}$ is the clock order $\bar{0} < \bar{1} < \bar{2} < \ldots \bar{11}$, and we take the minimal elements $x_{min}, y_{min}$ of the differences $X - Y$ and $Y - X$ instead of the maximal ones to define the ordering. This gives the lexicographic ordering of subsets of $\mathbb{Z}_{12}$ where a bullet is prior to a circle, as drawn in column two of the list. For example, we have

●●○●○○○○○○○○ < ●○●●○○○○○○○○

According to this order, the first representatives of an orbit are taken, and the numbering of the classes for given cardinality is the ordering among the first representatives.

- The second column shows the first representatives as discussed in the previous item, indicated by sets of bullets. The complements are shown in circles. In particular, the classes of the complements (circle representatives) are ordered according to the order of the bullet representatives.

- The third column shows the symmetry group of $Sym(K)$ of the representative $K$, as defined in 8.3.5. Although, in general, this group is not an invariant of the class, if we stick to local compositions in $\mathbb{Z}_{12}$, and if we recall that all isomorphisms of local compositions in this ambient space are induced by affine isomorphisms, the symmetry group is unique up to conjugation.

- The conjugation class of the symmetry group from column three is indicated in the fourth column, symbolized by numbers as explained in the list's preamble.

- The fifth column shows the pair $a|b$ where $a$ is the number of conjugacy classes of endomorphisms modulo automorphisms of the given local composition, and $b$ is the corresponding number for the complement.

Contemplation of this list shows that the following data nearly defines a complete set of invariants:

1. Cardinality
2. Conjugacy class of the symmetry group
3. Pair of numbers $a|b$ of numbers of conjugacy classes of endomorphisms
4. Autocomplementarity, i.e., the star exponent in the class number

The only exceptions are four pairs of classes which are not distinguished:

- 18, 19
- 40, 41
- 64*, 68*
- 75*, 82*

We shall learn in chapter 30 on counterpoint theory that the classes of the last two couples can also be separated by the span and diameter invariants, so essentially, only the first two couples are indistinguishable with the above data.

We cannot discuss the richness of the chord classification here, this will, however be addressed in more specialized and musically motivated chapters. We should instead make a final remark on the musicological range of the above list. Evidently, the pitch parameter was only relevant from the denotator perspective here. Mathematically, we have simultaneously classified all zero-addressed local compositions which implement the same ambient space, for example local compositions of form $OnMod_{12}$ as defined in formula (6.41), i.e., local compositions in 12-cyclic metrical onsets. According to definition 14 in section 7.2.2, *we have classified 12-periodic local rhythms in the (reduced) onset domain.*

### 11.3.6 Empirical Harmonic Vocabularies

**Summary.** The systematic classification is compared to some empirical material.

– Σ –

Unfortunately, to this day no systematic repertorization of chords in classical literature has been realized. We shall see in chapter 41.3 that this project is also due to some conceptual difficulties concerning the very definition of what can be identified as a chord in a given score. It is, however, not a serious problem and this repertorization should absolutely be realized in order to know the empirical distribution of chord types. The following exposition—initiated by Thomas Noll in [402]—is a first sketch of this project, but it already shows some remarkable results.

The material in [402] was taken from Nikolai Garbusow's work [193] on harmony. The difference of Grabusow's chord notation to the original works he refers to is not relevant to our objective since we only look at pitch classes in $PiMod_{12}$, including enharmonic identification. Our examples split into three lists.

The first list (from [193, chapter IV]) refers to 85 examples which boil down to 23 chord classes. Except for class 78 which has a top number of 23 conjugacy classes of endomorphisms, these (22) classes are shown in figure 11.3.

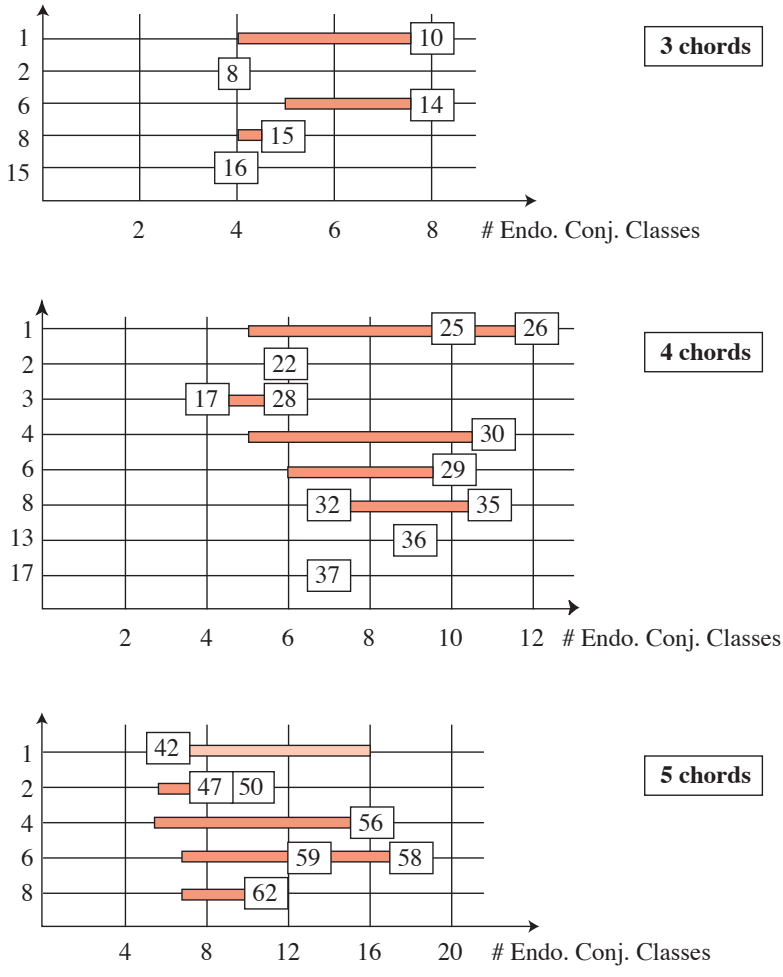


Figure 11.3: The 22 chord classes from Garbusow's list from [193, chapter IV] in their distribution among cardinalities 3,4,5, symmetry group conjugacy classes (ordinate axis), and cardinality of endomorphism conjugacy classes. The total range of the latter cardinalities is shown as a dark bar. We recognize that except for class 42, all classes range among the top numbers, i.e., have large numbers of endomorphisms.

The second list refers to 28 examples from Alexander Scriabin's sonatas 6 to 10. These examples boil down to 15 classes, one of which is a 4-chord of class 29 with maximal cardinality of endomorphism conjugacy classes. The other 14 classes are all 5-, 6-, and 7-chords and are shown in figure 11.4.

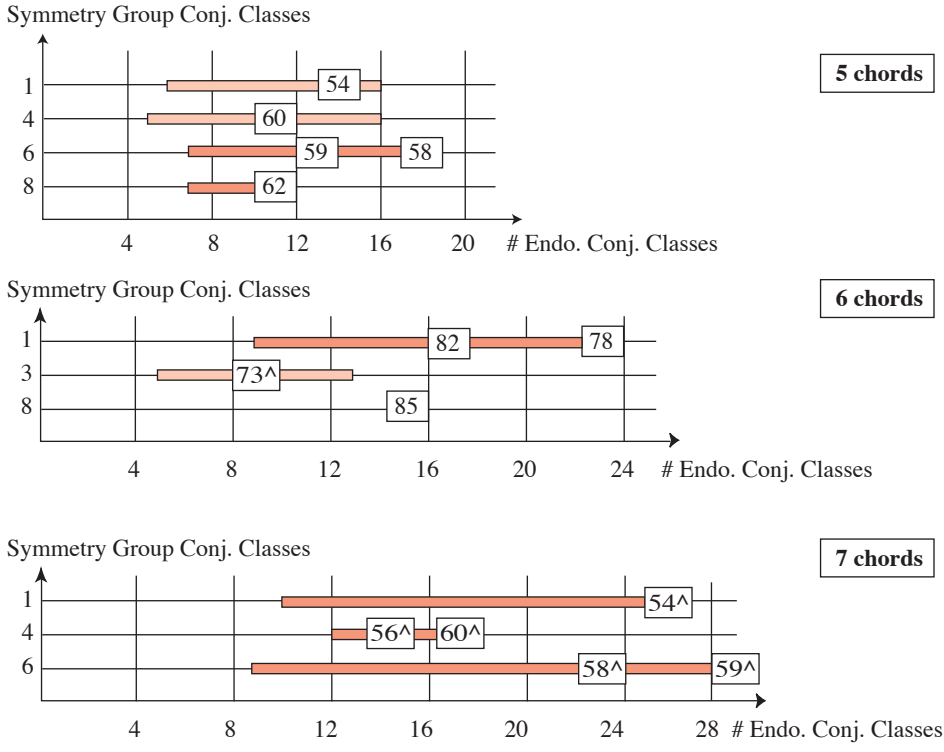


Figure 11.4: The 14 chord classes of Garbusow's list from Scriabin's sonatas 6 to 10. Again, many classes have large numbers of endomorphisms.

Table 11.1: Examples from $20^{th}$ Century Composers

| | |
|---|---|
| Debussy: "Ibérie" | $62, 56, 80\hat{}, 47\hat{}$ |
| Debussy: "L'isle joyeuse" | $42\hat{}, 80\hat{}, 62, 78, 88$ |
| Ravel: "Ma Mère l'Oye" | $40\hat{}, 85, 53\hat{}, 62$ |
| Prokofiev: "Sarcasmes" | $(8), 62, 58, 78, 85\hat{}, 86$ |
| Prokofiev: "Le sacre du printemps" | $54\hat{}, 17\hat{}, 82, 54\hat{}, 70$ |
| Schönberg: "Erwartung" | $20\hat{}, 45\hat{}, 54\hat{}, 33\hat{}, 62\hat{}, 36\hat{}, 65\hat{}, 12\hat{}, 10\hat{}, 26\hat{}, 26\hat{}, (2\hat{})$ |
| Casella: "Sonatine" | $40\hat{}, 54\hat{}, 39\hat{}, (1), 10\hat{}, 4\hat{}$ |

The third list refers to 45 examples from $20^{th}$ century composers, table 11.1 shows the class list, together with the composers. Our third "statistics" of classes is shown in figures 11.5 and 11.6.

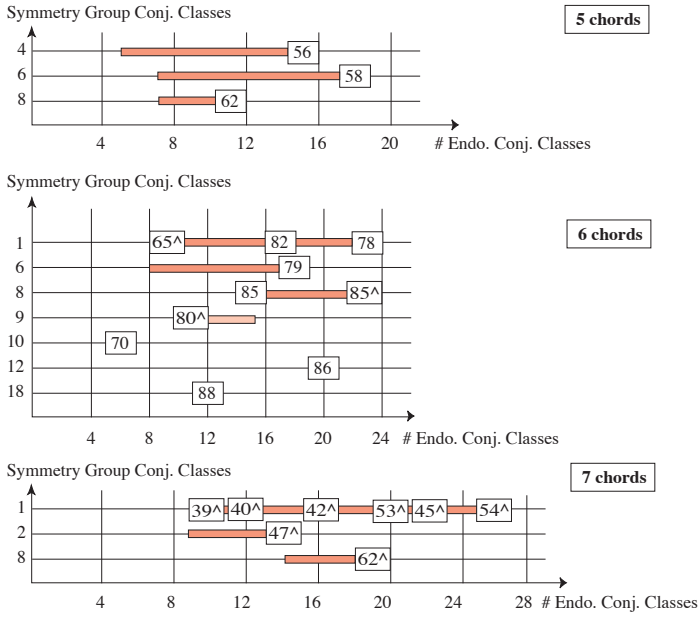


Figure 11.5: The 5-, 6-, 7-chord classes of Garbusow's list from $20^{th}$ century composers listed in table 11.1. We again have large numbers of endomorphisms.

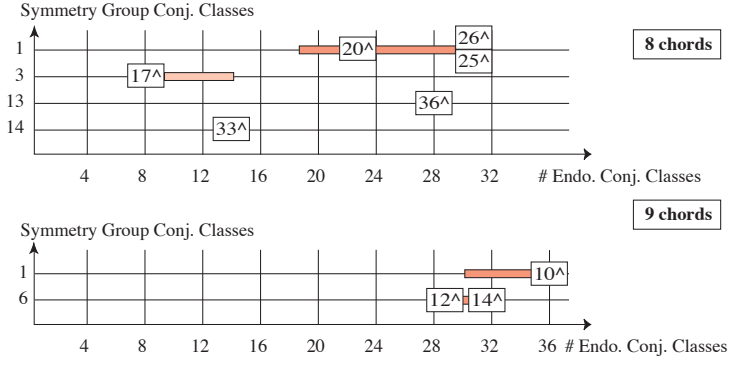


Figure 11.6: The 8, 9 chord classes of Garbusow's list from $20^{th}$ century composers listed in table 11.1. We again have large numbers of endomorphisms.

From the figures 11.3, 11.4, 11.5, and 11.6, we see that *there is a strong tendency in this empirical repertoire to be placed where the numbers of conjugacy classes of endomorphisms for a given symmetry group within a determined cardinality is high.* This observation is an empirical hint to the approach introduced in [402] which interprets endomorphisms as generalized tones.

In our language this is, of course, the situation of $\mathbb{Z}_{12}$-addressed chords in $PiMod_{12}$ which will be treated in section 11.3.7. This approach also suggests that harmony could be based to the differentiated richness of $\mathbb{Z}_{12}$-addressed chords in $PiMod_{12}$ which are canonically associated with zero-addressed chords.

### 11.3.7 Self-addressed Chords

**Summary.** We discuss classification of self-addressed local compositions.

– Σ –

The reflections from the previous section 11.3.6 suggest that endomorphisms of ambient spaces could play a major role in harmony. This idea was pursued in [400] and will be taken up in our part VI on harmony, in particular chapters 24 through 26. In this section we want to discuss classification of these endomorphisms.

More precisely, we have learned that sets of affine endomorphisms $F = e^t \cdot s : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ (modulo certain automorphisms) are relevant to important chords in European tradition. In order to restate this context we introduce a type of denotators in simple forms, the self-addressed denotators[10]:

**Definition 32** *If $F$ is a simple form with coordinator module $M$, a denotator $S : M \rightsquigarrow F(s)$ is called* self-addressed *(at address $M$). An objective local composition is called self-addressed if its elements are. A functorial local composition is self-addressed (at $M$) if it is a subfunctor of the product functor $@M \times F$.*

Observe the conceptual difference between endomorphisms of ambient spaces, local compositions and self-addressed denotators or local compositions. For local compositions, address change makes sense, but for endomorphisms this does not make sense.—So we want to discuss classification of finite local compositions $K \subset M@M$. As before, we look for a classification for self-addressed local compositions at the fixed address $M$. Quite generally, given two $A$-addressed local compositions $K \subset A@M, L \subset A@M$, let

$$f/\alpha : K \xrightarrow{\sim} L$$

be an isomorphism with underlying endomorphism $F = e^m \cdot F_0$ of $M$. If $M$ is indecomposable of finite length, and if $1 < card(K)$, we know from Fitting's lemma 77 in appendix E that $F$ must also be an isomorphism. In fact, looking at the associated linear isomorphism $R.\alpha f : R.K \xrightarrow{\sim} R.(L.\alpha)$, we conclude that $F_0$ is an isomorphism. If $1 = card(K)$, it can only be isomorphic to $L$ under a non-invertible endomorphism if it is a constant, since the linear part of the isomorphism on $M$ must be nilpotent. But then, clearly, it can be shifted to the constant $L$ by a translation isomorphism. Since the denominator $\alpha$ is an isomorphism by definition, we are then left with the calculation of orbits under the left and right action of $\overrightarrow{GL}(M)$ on $M@M$. Now, if we are given a ring $R$ which is of finite length and which is a product of local rings, classification of self-addressed compositions at $R$ also boils down to the above orbit calculation since we are then essentially dealing with each local factor of $R$ which is indecomposable and therefore

[10] In [400], such objects were called "fractal" because of evident self-similarity structure.

endomorphisms are induced by automorphisms. This is what is required when classifying self-addressed local compositions at address $\mathbb{Z}_{12}$, for example, or at address $\mathbb{Z}_{12}[\varepsilon]$ for counterpoint theory.

In the following discussion we first concentrate on local rings $R$ of finite length. We are given local compositions $K \subset R@R$ of finite cardinality $n$. The group action is the following. Let $e^s \cdot t \in R@R$ be a self-addressed point which we identify with the row vector $(s,t) \in R^2$. If $(u,v) \in R \times R^\times$ and $(k,l) \in R \times R^\times$ represent an automorphism $e^u \cdot v$ and a base change $e^k \cdot l$ of $R$, respectively, their action $e^u \cdot v \cdot e^s \cdot t \cdot e^k \cdot l$ is represented by $(u+v.s+v.t.k, v.t.l)$. So, since the elements $u,k$ are arbitrary and $v,l$ are arbitrarily invertible, our action is also described[11] by

$$G^u_{v,k,l} = e^{(u,0)} \cdot G_{v,k,l} : (s,t) \mapsto e^{(u,0)}(G_{v,k,l}(s,t)) = (u + v.s + k.t, l.t)$$

where $e^{(u,0)} \cdot G_{v,k,l}$ is in the affine triangular group $\overrightarrow{T}_2(R)$, i.e., by definition of $\overrightarrow{T}_2(R)$, it has linear part

$$G_{v,k,l} = \begin{pmatrix} v & k \\ 0 & l \end{pmatrix}$$

in the group $T_2(R)$ of upper triangular $2 \times 2$-matrices over $R$ (with invertible diagonal coefficients). Taking up the methods of section 11.3.2, fix a positive natural number $n$ and look at the short exact sequence

$$0 \longrightarrow R \xrightarrow{\Delta_1} R^{2n} \xrightarrow{d_1} R^{2n-1} \longrightarrow 0 \tag{11.27}$$

of $R$-modules defined as follows. We denote the $2n$-tuples in the middle module by

$$(s_1, s_2, \dots s_n; t_1, t_2, \dots t_n)$$

and the $(2n-1)$-tuples in the right module by $(s_2, \dots s_n; t_1, t_2, \dots t_n)$. Then the sequence is defined by

$$\Delta_1(u) = (u, u, \dots u; 0, 0, \dots 0) \text{ and}$$
$$d_1(s_1, s_2, \dots s_n; t_1, t_2, \dots t_n) = (\sigma_2, \dots \sigma_n; \tau_1, \tau_2, \dots \tau_n)$$

where

$$\begin{aligned} \sigma_i &= s_i - s_1, 1 < i \leq n, \\ \tau_1 &= t_1, \\ \tau_i &= t_i - t_1, 1 < i \leq n. \end{aligned}$$

If we let the symmetric group $\mathfrak{S}_n$ act linearly from the right on $R^{2n}$ by

$$(s_1, s_2, \dots s_n; t_1, t_2, \dots t_n) \cdot \pi = (s_{\pi 1}, s_{\pi 2}, \dots s_{\pi n}; t_{\pi 1}, t_{\pi 2}, \dots t_{\pi n})$$

for $\pi \in \mathfrak{S}_n$, we get local compositions from sequences $(s.;t.)$ via orbits under $\mathfrak{S}_n$ (and as before, also restricting to the sequences with $n$ pairwise different couples $(s_i, t_i)$. Again, the diagonal

[11] We make the common abuse of language and write a matrix multiplication $A.(x,y)$, meaning $(A.(x,y)^t)^t$ with transposed rows.

image $Im(\Delta_1) = R.\delta_1$, with $\delta_1 = (1.;0.)$, is invariant under this action and therefore, we have an induced action on the cokernel $R^{2n-1}$. We have the diagonal left action of the affine triangular group $\overrightarrow{T}_2(R)$. Let $G = G^u_{v,k,l}$. Then $G.(s_1, s_2, \dots s_n; t_1, t_2, \dots t_n)$ leaves the diagonal image $R.\delta_1$ invariant and commutes with the action of the symmetric group. Moreover, the translational part of the action of $\overrightarrow{T}_2(R)$ has orbits which include $R.\delta_1$-cosets , i.e., it induces a linear action of the upper triangular group $T_2(R)$ on the cokernel $R^{2n-1}$. If $G_{v,k,l} \in T_2(R)$, we have

$$G_{v,k,l}(\sigma_2, \dots \sigma_n; \tau_1, \tau_2, \dots \tau_n) = (v.\sigma_2 + k.\tau_2, \dots v.\sigma_n + k.\tau_n; l.\tau_1, l.\tau_2, \dots l.\tau_n), \tag{11.28}$$

and this means the following: First, the action of the invertible elements $l \in R^\times$ generates orbits $R^\times.\tau.$ on the second half $R^n$ of $R^{2n-1}$ which, according to appendix F, lemma 87, is equivalent to considering monogenous subspaces $R.\tau.$ of $R^n$. Moreover, the contribution of the $\tau$-component to the first half is only a function of these monogenous spaces. We therefore have a first invariant: the monogenous subspaces $R.\tau. \subset R^n$. When fixing one of these spaces, $V$, say, and denote the projection $pr_2V$ onto the second to last coordinates, we have the induced action of $v \in R^\times$ on the quotient space $R^{n-1}/pr_2V$. This again gives monogenous subspaces as invariants. Of course, only those generators $(\sigma_2, \dots \sigma_n; \tau_1, \tau_2, \dots \tau_n)$ with

$$(\sigma_i, \tau_i) \neq (0,0), 1 < i \text{ and} \tag{11.29}$$

$$(\sigma_i, \tau_i) \neq (\sigma_j, \tau_j), 1 < i < j \tag{11.30}$$

correspond to local compositions. Summarizing, we have these invariants of the left action of the triangular affine group:

$$U^1_n = \{(V, W) |\ V \subset R^n, W \subset R^n/pr_2V \text{ both monogenous}\}. \tag{11.31}$$

On this space, the symmetric group acts from the right and we are in a similar situation as for the finite classification discussed in section 11.3.3. More precisely, we first have to classify the monogenous subspaces of $R^n$ under the action of $\mathfrak{S}_n$. Again, let us suppose that $R$ is self-injective. Then we may proceed by literal copy of the recursive algorithm from 11.3.3. If we fix a representative $V$ from this classification, and if $G_V \subset \mathfrak{S}_n$ is its isotropy group, we are left with the action of this group on monogenous spaces $W \subset R^{n-1}/pr_2V$, and recursion can go on until the algorithm stops. This may be summarized in

**Theorem 8** *If $R$ is a local, self-injective commutative ring of finite length and maximal ideal $m$, then there is an algorithm which classifies self-addressed local compositions of cardinality $n$ in $R$, and the invariants from this algorithm are parametrized by $S$-rational points of quotient schemes from actions of finite subgroups of $\mathfrak{S}_n$ on Grassmann schemes, where the occurring rings $S = R/m^r$ are quotients of the ground ring $R$.*

In particular, if we have a finite ring $\mathbb{Z}_N$, or an infinitesimal extension $R[\varepsilon]$ of a local, self-injective commutative ring $R$ of finite length, this one has the same property, and we therefore have classified the self-addressed local compositions in $\mathbb{Z}_N[\varepsilon]$ which will be used in counterpoint, see chapter 29.

### 11.3.8 Motives

**Summary.** Classification of motives of small cardinality.

– Σ –

The classification techniques which we discussed in the previous sections will now be applied to describe explicitly classification of small motives. The following case of three-element motives is far from general classification but it gives a complete description of the algorithms and, in contrast to the more abstract performance of enumeration theory which is discussed in 11.4, yields explicit lists of representatives of isomorphism classes. We shall also need this kind of list in sections 11.6.2, 11.6.3, so it is useful to pay them a short moment of attention. The list which we shall calculate now is shown in appendix M.3.

We want to classify the three-element commutative, zero-addressed local compositions $K$ in $OnPiMod_{12,12}$, i.e., mathematically speaking, $K \subset \mathbb{Z}_{12}^2$ with $card(K) = 3$. But we know that $\mathbb{Z}_{12}^2$ is injective, and therefore we know from 11.3.3 that we may concentrate on three-element objects in $ComLoc_{3,0_{\mathbb{Z}_{12}}}^{gen}$. Conversely, according to the theory which yields theorem 6, any such object $K$ has a subspace $V \subset \mathbb{Z}_{12}^2$ such that $\mathbb{Z}_{12}^2/V \xrightarrow{\sim} \mathbb{Z}_{12}.K$. But then, the module $\mathbb{Z}_{12}.K$, as a quotient of $\mathbb{Z}_{12}^2$ is also a submodule[12] of $\mathbb{Z}_{12}^2$, and therefore, $K$ is already realized in $\mathbb{Z}_{12}^2$.

So we are left with the the calculation of $X_2(\mathbb{Z}_{12})/\mathfrak{S}_3$, and we may start by the calculation of orbits of the larger space $U_2(\mathbb{Z}_{12})$ under a subgroup of $\mathfrak{S}_3$. Let us first describe the representation

$$D^{opp} : \mathfrak{S}_3 \to GL_2(\mathbb{Z}_{12})^{opp} : \pi \mapsto D^{opp}(\pi) = D(\pi^{-1})$$

of the right group action in terms of $2 \times 2$-matrices over $\mathbb{Z}_{12}$:

$$D^{opp}(Id) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad D^{opp}(Id)(x,y) = (x,y) \tag{11.32}$$

$$D^{opp}(1\ 2) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \qquad D^{opp}(1\ 2)(x,y) = (-x-y,y) \tag{11.33}$$

$$D^{opp}(1\ 3) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \qquad D^{opp}(1\ 3)(x,y) = (x,-x-y) \tag{11.34}$$

$$D^{opp}(2\ 3) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad D^{opp}(2\ 3)(x,y) = (y,x) \tag{11.35}$$

$$D^{opp}(1\ 2\ 3) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \qquad D^{opp}(1\ 2\ 3)(x,y) = (-x-y,x) \tag{11.36}$$

$$D^{opp}(1\ 3\ 2) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \qquad D^{opp}(1\ 3\ 2)(x,y) = (y,-x-y) \tag{11.37}$$

Let us now calculate the $\mathfrak{S}_3$-orbits of subspaces $V$ of $\mathbb{Z}_4^2$.

[12] This follows from the main theorem on finitely generated abelian groups, see section C.3.4.2.

$V$ **is semi-simple.** It therefore is a $\mathbb{Z}_2$-vector space, i.e., $V \subset 2\mathbb{Z}_4^2$. So we have $V = 0$ for $dim(V) = 0$, $V = 2\mathbb{Z}_4^2$ for $dim(V) = 2$, and the $\mathfrak{S}_3$-orbit of the one-dimensional space $V = \mathbb{Z}_4.(2,0)$ is the set of all one-dimensional spaces in $2\mathbb{Z}_4^2$. So we have three orbits of semi-simple submodules.

**Suppose** $2.V \neq 0$**.** Then the injective envelope $W$ of $V$ in $\mathbb{Z}_4^2$ must be either free of rank one or rank two by proposition 90 in appendix E.

In the rank two case, $W = \mathbb{Z}_4^2$, and we claim that in this case, $2\mathbb{Z}_4^2 \subset V$. In fact, otherwise, there is $X \subset W$ with $X \xrightarrow{\sim} \mathbb{Z}_2$ and $X \cap V = 0$ and $W$ cannot be the injective envelope of $V$. But then, we are in the semi-simple situation since the socle is invariant and we have to look for the orbits of subspaces in $\mathbb{Z}_4^2/2\mathbb{Z}_4^2$. The zero space has already been recognized above. Hence we have the two cases $V = \mathbb{Z}_4 \times 2\mathbb{Z}_4$ and $V = \mathbb{Z}_4^2$.

In the rank one case, we have $V = \mathbb{Z}_4.x \xrightarrow{\sim} \mathbb{Z}_4$ since otherwise, $V$ would be semi-simple. According to the group action table above, we have two orbits, one for $x = (1,0)$, one for $x = (1,2)$.

Summarizing, we have these subspace orbit representatives, including their isotropy groups which we need later:

| Representative | Isotropy Group | |
|---|---|---|
| $0$ | $\mathfrak{S}_3$ | semi-simple |
| $2\mathbb{Z}_4^2$ | $\mathfrak{S}_3$ | semi-simple |
| $\mathbb{Z}_4(2,0)$ | $\langle D^{opp}(1,2)\rangle$ | semi-simple |
| $\mathbb{Z}_4(1,0)$ | $\langle D^{opp}(1,2)\rangle$ | injective |
| $\mathbb{Z}_4(1,2)$ | $\langle D^{opp}(1,2)\rangle$ | injective |
| $\mathbb{Z}_4^2$ | $\mathfrak{S}_3$ | injective |
| $\mathbb{Z}_4 \times 2\mathbb{Z}_4$ | $\langle D^{opp}(1,2)\rangle$ | |

We have the following orbits of subspaces of $\mathbb{Z}_3^2$ under the two isotropy groups of the above list:

| Isotropy Group | Representatives |
|---|---|
| $\mathfrak{S}_3$ | $0, \mathbb{Z}_3^2, \mathbb{Z}_3.(1,0), \mathbb{Z}_3.(1,1)$ |
| $\langle D^{opp}(1,2)\rangle$ | $0, \mathbb{Z}_3^2, \mathbb{Z}_3.(1,0), \mathbb{Z}_3.(1,1), \mathbb{Z}_3.(0,1)$ |

From this repertory, we have to select those products $V_4 \times V_3 \subset \mathbb{Z}_4^2 \times \mathbb{Z}_3^2$ which do not contain the vectors $e_1 = ((1,0),(1,0)), e_2 = ((0,1),(0,1))$, and $e_1 - e_2$. This gives us the list of kernel spaces in column three of table M.3, and we are done with the subspaces.

**Exercise 14** Recapitulate the general algorithm described in 11.3.3 along the lines of the previous example.

**Exercise 15** Calculate the representatives of motives in that table from the spaces. Recall that these kernel spaces $Ker(f)$ stem from canonical linear maps $f : \mathbb{Z}_{12}^2 \to \mathbb{Z}_{12}.K$ which is associated with the motif $K$.

Table M.3 also contains two more invariants, the *class weight* in column four, and the *volume* in column five. Whereas the class weight will be defined and discussed later in section 14.5, the volume of a local composition is defined as follows:

**Definition 33** *Let $K \subset R^n$ be a commutative local composition in the free $R$-module of rank $n$. Then the image of the $n$-fold exterior power $\bigwedge^n R.K$ in $\bigwedge^n R^n$ is an ideal $Vol(K)$ of $R$. The ideal $Vol(K)$ is called the* volume *of $K$.*

If the ring $R$ is of finite length, two isomorphic local compositions $K \subset R^n, L \subset R^n$ are related by an invertible symmetry $F$, according to sorite 6 in section 8.3.5. This symmetry's linear part yields an automorphism of the $n^{th}$ exterior power $\bigwedge^n R^n \xrightarrow{\sim}_R R$, and this leaves the ideal invariant. Therefore:

**Proposition 7** *With the above notation, and if $R$ is of finite length, the volume is an invariant among all local compositions in the same ambient space $R^n$.*

In particular, if every ideal in $R$ is principal, and if $R$ is a product of local rings of finite length, the generator of $Vol(K)$ is unique up to invertible factors in $R$. By abuse of language, we also call this generator the volume of $K$. This number is included in the last column of table M.3.

In [513], Hans Straub has classified of four-element motives in $OnPiMod_{12,12}$ by use of a software written in C. This work made use of the methods introduced in the previous sections and confirmed the list of Egmont Köhler [266] which was communicated without any specification of the used methods. It also completed the redundant list in [328] which applied the present methods, but without computer support.

The procedure in [513] first uses classification of two-element motives to build a reasonably small set of motives which includes all isomorphism classes. From this classification, which is listed in table M.1, we can fix two of the four tones to one of the five representatives of two-element motives of our table. The other two tones are free, and we obtain a total list $List_4$ of $5 \cdot 142 \cdot 141/2 = 50055$ motives to be investigated. The elements of this list are ordered by an index: $List_4 = \{K_i |\ i = 1, \ldots 50055\}$. The program picks each of the list's elements in this order, decides whether its class has already been found and then adds it to the the found class representatives or else throws it away and picks the next element.

To decide upon class membership, the program first calculates the volume and class weight for a selected motif $K_i \in List_4$. If this data is not yet present in the list of class representatives, we have found a new representative and proceed to the next element $K_{i+1}$. Else, if volume and class weight is already present, we concentrate on those representatives for the following comparison: For each four-element motif $K_i \in List_4$, the program calculates a set $G(i)$ of generators of the kernel $Ker(f_i)$ of the linear map $f_i : \mathbb{Z}_{12}^2 \to \mathbb{Z}_{12}^2$ which is canonically associated with $K_i$ in our theory, since the elements of $K_i$ are listed in a determined order (ordered representation is automatic in computer programs).

Then the program calculates the orbit of $G(i)$ under the action of the symmetric group $\mathfrak{S}_3$ and checks whether an element of this orbit is included in the kernel $Ker(f_j)$ of a motif $K_j \in List_4$ which has already been selected as a representative and which has the same volume and class weight as $K_i$. If this is the case, the orbit $Ker(f_i) \cdot \mathfrak{S}_3$ contains $Ker(f_j)$ and we can proceed to the next candidate, otherwise, $K_i$ is a new representative. By the preselection

via volume and class weight, the orbit calculations happen less frequently and the program terminates quickly.

This program also settles the question of finding the class of any given motif and therefore can and has been used for experimental purposes. In [513], three- and four-element motif classification was used to compare melodies. The selection includes: Johann Sebastian Bach/Gigue Nr. 32, Johann Sebastian Bach/Sarabande Nr. 52, John Lord[13]/Sarabande, Wolfgang Amadeus Mozart/KV 449/T.1-8. These Melodies were stratified into motives built from three- and four-element groups of successive notes. The classification of these motives shows rather strong commonalities for three-element motif classes:

| Composer/Piece | Number of Occurring Classes | Most Frequent Class (Nr.) |
|---|---|---|
| J.S. Bach/Gigue Nr. 32 | 13 | 10 |
| J.S. Bach/Sarabande Nr. 52 | 12 | 10 |
| J. Lord/Sarabande | 15 | 10 |
| W. A. Mozart/KV 449 | 14 | 10 |

The pieces differ significantly for the refined four-element motif stratification:

| Composer/Piece | Number of Occurring Classes | Most Frequent Class (Nr.) |
|---|---|---|
| J.S. Bach/Gigue Nr. 32 | 52 | 19 |
| J.S. Bach/Sarabande Nr. 52 | 66 | 14 |
| J. Lord/Sarabande | 75 | 15 |
| W. A. Mozart/KV 449 | 65 | 23 |

We see that there are major differences in the latter data. However, as it is pointed out in [513], the second number is not very reliable since the orbit cardinalities should be included in these figures, i.e., frequency should be divided by orbit cardinality. This type of analysis should be investigated systematically and with statistical skill. Without doubt, it yields a refined melody classification method which can be used for stylistic classification. We shall perform a detailed and poetological melodic analysis built on three-element motives in section 11.6.2.

## 11.4 Enumeration Theory

**Summary.** Enumeration theory is a quantitative approach to classification of local compositions via actions of permutation groups. We review Harald Fripertinger's work on this subject.

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[13] John Lord is the organist of the rock group "Deep Purple".

Enumeration theory deals with counting of types of structures, i.e., orbits of finite group actions on finite sets. These sets represent particular species of objects. In musical enumeration theory, these structures are chords, partitions of pitch class sets, motives, twelve-tone rows and the like. The groups consist of selected systems of musically interesting transformations, such as transposition or inversion. Evidently, counting orbits is much less than listing them, or even describing them as rational points of schemes or similar classifying spaces of geometric nature. Why is there any interest to know such numbers—for example the incredible number of

$$2\ 230\ 741\ 522\ 540\ 743\ 033\ 415\ 296\ 821\ 609\ 381\ 912 = 2.23\ldots \cdot 10^{36}$$

isomorphism classes of 72-element motives in $\mathbb{Z}_{12}^2$? We shall deal with the more philosophical aspect and the question concerning big science in musicology in section 11.4.2. Here, we can anticipate that first, a good part of the enumeration work automatically yields or eases explicit lists of representatives, and second, knowing the class numbers is a strong prerequisite to apply probabilistic and statistical reasonment for classification of large music repertories. We gave a small account to this direction in the final words of section 11.3.8.

In musical enumeration theory, scholars often deal with enumeration of local and global compositions without clear conceptual distinction. We only deal with *local enumeration theory* here and postpone global enumeration to section 16.2.

### 11.4.1 Pólya and de Bruijn Theory

**Summary.** Pólya and de Bruijn theory yields a crucial tools for enumeration of classes of local compositions in the case of finite abelian groups as ambient modules.

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This section mainly refers to Harald Fripertinger's work [168, 169, 170, 171]. It rests on combinatorics of finite group actions on finite sets[14]. In order to apply the Pólya theory to the enumeration of isomorphism classes of zero-addressed local compositions $K \subset \mathbb{Z}_n$, the set $K$ is identified with its characteristic function[15] $\chi_K : \mathbb{Z}_n \to 2$ with the identification $2 = \{0, 1\}$. So the set of zero-addressed local compositions $K \subset \mathbb{Z}_n$ is identified with the powerset $2^{\mathbb{Z}_n}$. If a group $G \subset \overrightarrow{GL}(\mathbb{Z}_n)$ of affine automorphisms acts on the set of zero-addressed local compositions $K \subset \mathbb{Z}_n$, this action is reflected on the powerset via the obvious right action $(\chi_K, g) \mapsto \chi_K \cdot g$. To count the $G$-orbits in $2^{\mathbb{Z}_n}$, we introduce weight functions $w : 2 \to R$ with values in a ring $R$ which express special sets of local compositions.

Let us shortly digress on the general Pólya context to understand the ideas. One looks at powersets $F^P$ of finite sets $F, P$ and canonical right group actions of permutation groups $G \subset \mathfrak{S}_P$ on $F^P$ via $(f, g) \mapsto f \cdot g$ as above for $P = \mathbb{Z}_n, F = 2$. A *Pólya weight function* is a function $w : F \to R$ with values in a ring $R$ which contains the field of rational numbers. This induces a *product weight function*

$$\Pi_w : F^P \to R : f \mapsto \prod_{p \in P} w(f(p)). \tag{11.38}$$

[14] For another approach to enumeration theory of local compositions in finite abelian groups, see [204].
[15] See appendix C.1, lemma 63.

Clearly, the product weight is invariant on a $G$-orbit and we may define the product weight on the orbit space via

$$\Pi_w : F^P/G \to R : f \cdot G \mapsto \Pi_w(f). \tag{11.39}$$

For our above example, we can take the ring $R = \mathbb{Q}[x]$ of rational polynomials over the indeterminate $x$ and the weight

$$w(i) = \begin{cases} 1 & \text{if } i = 0, \\ x & \text{if } i = 1. \end{cases} \tag{11.40}$$

Whence the product weight $\Pi_w(\chi_K) = x^{card(K)}$. The *configuration counting series* $C(G, F, P, w)$ of this weight is defined by

$$C(G, F, P, w) = \sum_{\omega \in F^P/G} \Pi_w(\omega) \tag{11.41}$$

and we see that *this polynomial in* $\mathbb{Q}[x]$ *has the number of* $G$*-orbits of local, zero-addressed* $k$*-element compositions* $K \subset \mathbb{Z}_n$ *as its* $x^k$*-coefficient* in the above case $P = \mathbb{Z}_n, F = 2$.

The Pólya theory offers tools to calculate the configuration counting series. The central object of the theory is the *cycle index*, a rational polynomial which depends on the subgroup $G \subset \mathfrak{S}_P$. It gives a formal account of the cycles[16] of the elements of $G$ in $P$. Then:

**Definition 34** *With the above notation, the* cycle index *of the permutation group* $G$ *is the rational polynomial*

$$Z(G) = card(G)^{-1} \sum_{g \in G} X_{\cdot}^{cyc(g)}. \tag{11.42}$$

For several common permutation groups, the cycle index is known, see appendix C.3.6 for the explicit formulas. The relation between the configuration counting series and the cycle index is set up by the main theorems of Pólya's enumeration theory:

**Theorem 9** *With the above notation, we have*

$$C(G, F, P, w) = Z(G)(\sum_{y \in F} w(y), \sum_{y \in F} w(y)^2, \dots \sum_{y \in F} w(y)^d). \tag{11.43}$$

**Corollary 6** *With the above notation, if the weight* $w = 1$ *is the constant weight with value* 1, $C(G, F, P, 1) = card(F^P/G)$ *is the number of orbits and this evaluates to*

$$card(F^P/G) = Z(G)(card(F), card(F), \dots card(F)). \tag{11.44}$$

*In particular, if we have the above action of a group* $G \subset \overrightarrow{GL}(\mathbb{Z}_n)$ *on the set of zero-addressed local compositions* $K \subset \mathbb{Z}_n$*, we have the orbit number*

$$card(2^{\mathbb{Z}_n}/G) = Z(G)(2, 2, \dots 2). \tag{11.45}$$

[16] See appendix C.3.6, definition 127 for the concepts of cycles and the cycle type $cyc(g)$ of a permutation $g$.

These results yield formulas for counting the number of classes of zero-addressed local compositions with respect to isomorphism types which are common in chord classification. This is a straightforward application of cycle index formulas for these groups.

For the translation group $T_n = e^{\mathbb{Z}_n}$, and for the group of translations and inversions $TI_n = e^{\mathbb{Z}_n} \rtimes \langle -1 \rangle$, the number of $G$-orbits of zero-addressed local compositions in $\mathbb{Z}_n$ is calculated from the coefficient of the $k$-th power $x^k$ in theorem 9 and from the formulas for $Z(T_n)$ and for $Z(TI_n)$. We have

**Proposition 8** [170] *The number of $T_n$-orbits of zero-addressed $k$-element local compositions in $\mathbb{Z}_n$ is*

$$\frac{1}{n} \sum_{j|gcd(n,k)} \varphi(j) \binom{n/j}{k/j}, \tag{11.46}$$

*and the number of $TI_n$-orbits of zero-addressed $k$-element local compositions in $\mathbb{Z}_n$ is*

$$\begin{cases} \frac{1}{2n}\left(\sum_{j|gcd(n,k)} \varphi(j) \binom{n/j}{k/j} + n\binom{(n-1)/2}{[k/2]}\right) & n \text{ odd}, \\ \frac{1}{2n}\left(\sum_{j|gcd(n,k)} \varphi(j) \binom{n/j}{k/j} + n\binom{n/2}{k/2}\right) & n, k \text{ even}, \\ \frac{1}{2n}\left(\sum_{j|gcd(n,k)} \varphi(j) \binom{n/j}{k/j} + n\binom{(n-2)/2}{[k/2]}\right) & n \text{ even}, k \text{ odd}, \end{cases} \tag{11.47}$$

*where $\varphi$ is Euler's $\varphi$ function*[17]*, and where $[t]$ denotes $t$'s integer part*[18]*.*

For the special case $n = 12$, the cycle index $Z(\overrightarrow{GL}(\mathbb{Z}_{12}))$ and theorem 9 yield the $\overrightarrow{GL}(\mathbb{Z}_{12})$-orbits [170], i.e., the isomorphism classes, and we obtain this orbit number list:

| $k =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{12}$ | 1 | 1 | 6 | 19 | 43 | 66 | 80 | 66 | 43 | 19 | 6 | 1 | 1 |
| $TI_{12}$ | 1 | 1 | 6 | 12 | 29 | 38 | 50 | 38 | 29 | 12 | 6 | 1 | 1 |
| $\overrightarrow{GL}(\mathbb{Z}_{12})$ | 1 | 1 | 5 | 9 | 21 | 25 | 34 | 25 | 21 | 9 | 5 | 1 | 1 |

#### 11.4.1.1 Enumeration of Interspace Structures

For special groups of affine automorphisms of $\mathbb{Z}_n$, *interspace structures*[19] of zero-addressed local compositions in $\mathbb{Z}_n$ are useful; they realize a special instance associated with our above construction of difference structures, for example in short exact sequence (11.12). For a non-empty local composition $K \subset \mathbb{Z}_n$, write $(K) = (k_1, k_2, \ldots k_t)$ for the ordered sequence (also called the *sequence of* $K$) of the $t = card(K)$ elements of $K$ in the canonical linear order $0 < 1 < 2 < \ldots n-1$ on the canonical representatives of $\mathbb{Z}_n$. The *interspace sequence of* $K$ is defined to be the image $\delta(K)$ of the sequence $(K)$ under the linear endomorphism

$$\delta : \mathbb{Z}_n^t \to \mathbb{Z}_n^t : (x_1, \ldots x_t) \mapsto (x_2 - x_1, \ldots x_t - x_{t-1}, x_1 - x_t) \tag{11.48}$$

with the evident kernel $Ker(\delta) = \Delta\mathbb{Z}_n$.

[17] See appendix, C.3.4.1.
[18] The largest integer $\leq t$.
[19] Terminology of Fripertinger [170].

**Exercise 16** [170] Consider the subset $IS_{t,n} \subset [1, n-1]^t$ of those sequences of positive integer $t$-tuples $(i_1, i_2, \dots i_t)$ such that $i_1 + i_2 + \dots i_t = n$. Then the canonical map $IS_{t,n} \to \mathbb{Z}_n^t$ is a bijection onto the set of *interspace sequences* of $t$-element local compositions in $\mathbb{Z}_n$.

We next consider three linear automorphisms of $\mathbb{Z}_n^t$:

$$\begin{aligned}
\varphi: &\quad \mathbb{Z}_n^t \xrightarrow{\sim} \mathbb{Z}_n^t: \quad (x_1, \dots x_t) \mapsto (x_t, x_1, \dots x_{t-1}) \\
\psi: &\quad \mathbb{Z}_n^t \xrightarrow{\sim} \mathbb{Z}_n^t: \quad (x_1, \dots x_t) \mapsto (x_t, x_{t-1}, \dots x_1) \\
\rho: &\quad \mathbb{Z}_n^t \xrightarrow{\sim} \mathbb{Z}_n^t: \quad (x_1, \dots x_t) \mapsto (-x_t, -x_{t-1}, \dots - x_1)
\end{aligned}$$

and observe that $\delta \cdot \varphi = \varphi \cdot \delta$ and $\varphi \cdot \delta \cdot \rho = \psi \cdot \delta$. The *interspace structure* $[K]$ of $K$ is defined as the orbit of $\delta(K)$ under the dihedral group $\langle \varphi, \psi \rangle \xrightarrow{\sim} TI_t$. We then have

**Lemma 15** [170] *If $K, L \subset \mathbb{Z}_n$ are non-empty local compositions, their orbits under the group $TI_n$ are equal iff $[K] = [L]$.*

Proof. Clearly, the translation $e^1(K)$ and the inversion $-K$ have the same interspace structure as $K$. Conversely, if $K, L$ are such that $\delta(K) = \varphi \cdot \delta(L)$, then $(K) - \varphi(L) \in Ker(\delta) = \Delta\mathbb{Z}_n$, and we conclude that $K$ is a translation of $L$. If $\delta(K) = \psi \cdot \delta(L)$, the above equation gives $\delta(K) = \psi \cdot \delta(L) = \varphi \cdot \delta \cdot \rho(L)$, and by the preceding case, the $TI_n$-orbits of $K$ and $-L$ coincide, so $K$ and $L$ have same $TI_n$-orbit. QED.

Together with exercise 16, lemma 15 yields the following technique for enumerating classes of local compositions which have special interspace configurations. If $x, y_1, y_2, \dots y_n$ are indeterminates, consider the weight function

$$w : [1, n-1] \to \mathbb{Q}[x, y_1, y_2, \dots y_n] : i \mapsto x^i y_i \tag{11.49}$$

which is defined on the interval $F = [1, n-1]$ of positive integers. Therefore, with the formalism introduced in formula 11.38, we have the product weight

$$\Pi_w : F^t \to \mathbb{Q}[x, y_1, y_2, \dots y_n] : g \mapsto x^{\sum g(i)} \prod y_{g(i)}.$$

Since we can identify the set of interspace sequences of $t$-element local compositions in $\mathbb{Z}_n$ with $IS_{t,n}$, the orbits of such local compositions are identified with $TI_n$-orbits of sequences $g$ in $IS_{t,n}$ under the same action. Then the above weight function is invariant under the action of the above dihedral group $\langle \varphi, \psi \rangle \xrightarrow{\sim} TI_t$, and since the set $IS_{t,n}$ is characterized by the sum of coordinates being $n$, we have a weight function

$$W : TI_t \backslash IS_{t,n} \to \mathbb{Q}[x, y_1, y_2, \dots y_n] : g \mapsto x^n \prod y_{g(i)} \tag{11.50}$$

where the indices of the $y$ give information about the intervals between successive elements in these local compositions. We then have

**Theorem 10** *With the above notation, the number of interspace structures of $t$-element local compositions in $\mathbb{Z}_n$ is given via the $x^n$-coefficient of the cycle index evaluation*

$$Z(TI_t)(\sum_{i=1}^{n-1} x^i y_i, \sum_{i=1}^{n-1} x^{2i} y_i^2, \dots \sum_{i=1}^{n-1} x^{ti} y_i^t), \tag{11.51}$$

*i.e., the coefficients of the monomials in the indeterminates $y_i$ of the $x^n$-coefficient are the numbers of interspace structures of given interval distribution.*

**Example 16** For example, if we look for interspace structures of local 3-element compositions in $\mathbb{Z}_{12}$, the coefficient of $x^{12}$ is

$$y_1^2 y_{10} + y_1(y_2 y_9 + y_3 y_8 + y_4 y_7 + y_5 y_6) + y_2^2 y_8 \\ + y_2(y_3 y_7 + y_4 y_6 + y_5^2) + y_3^2 y_6 + y_3 y_4 y_5 + y_4^3$$

and we can calculate, for example, those interspace structures with all intervals being at least $k$ units by setting $y_1 = y_2 = \ldots y_{k-1} = 0$ and $y_i = 1$ else. For $k = 2$ we get seven such structures.

#### 11.4.1.2 Enumeration of Series

Recall from definition 22 in section 8.1.1 that a $n$-phonic series is a denotator

$$Ser : \mathbb{Z}^{n-1} \rightsquigarrow PiMod_n(Ser_0, \ldots Ser_{n-1})$$

with pairwise distinct coordinates. As we know from the discussion of $n$-phonic series following definition 22, the group $D_{n,n}$ acts on the set $SER_n$ of $n$-phonic series. Enumeration theory yields this number of orbits:

**Proposition 9** *For $3 \leq n$, the number of orbits in $D_{n,n} \backslash SER_n$ is as follows. Set*

$$n!! = \begin{cases} n(n-2)(n-4) \cdot \ldots 2 & \text{if } n \equiv 0 \mod 2, \\ n(n-2)(n-4) \cdot \ldots 1 & \text{else.} \end{cases} \tag{11.52}$$

*Then the numbers are*

$$\begin{cases} \frac{1}{4}((n-1)! + (n-2)!!(\frac{n}{2}+1)) & \text{if } n \equiv 0 \mod 2, \\ \frac{1}{4}((n-1)! + (n-1)!!) & \text{else.} \end{cases} \tag{11.53}$$

A proof can be found in [168].

**Example 17** For example, the number of $D_{12,12}$-orbits of a dodecaphonic series is 9 985 920. So there is still some uncovered material for dodecaphonic compositions.

More generally, the Bruijn extension of Pólya's enumeration theory yields formulas for orbits of $(k, n)$-series [170, Satz 2.2.5]. We omit the details here and just reproduce the particularly interesting list of orbits for $n = 12$:

| $k$ | number of orbits of $(k,12)$-series |
|---|---|
| 2 | 6 |
| 3 | 30 |
| 4 | 275 |
| 5 | 2 000 |
| 6 | 14 060 |
| 7 | 83 280 |
| 8 | 416 880 |
| 9 | 1 663 680 |
| 10 | 4 993 440 |
| 11 | 9 980 160 |
| 12 | (dodecaphonic series) 9 985 920 |

To deal with all-interval series, we invoke the context of definition 22 in section 8.1.1. For natural numbers $0 < k \leq n$, we have the set $SERM_{k,n}$ of $(k,n)$-serial motives which is an additive abelian group by pointwise addition, i.e., $(Ser + Ser')_i = Ser_i + Ser'_i$ for $Ser, Ser' \in SERM_{k,n}$. We have a projection homomorphism

$$D : SERM_{k+1,n} \to SERM_{k,n} \tag{11.54}$$

defined by $D(Ser)_i = Ser_{i+1} - Ser_i,\ 0 \leq i < k$, call $D(Ser)$ the *derived serial motif*. This projection has a right inverse injection

$$I : SERM_{k,n} \to SERM_{k+1,n} \tag{11.55}$$

defined by $I(Ser)_0 = 0$ and $I(Ser)_i = Ser_0 + \ldots Ser_{i-1}, 0 < i \leq k$, i.e., $D \cdot I = Id_{SERM_{k,n}}$. Call $I(Ser)$ the *integrated serial motif.*

By definition, the set of *all-interval n-phonic series* is the intersection set $ALLSER_n$ in the cartesian diagram

$$\begin{array}{ccc} ALLSER_n & \longrightarrow & SER_n \\ \downarrow & & \downarrow \\ D^{-1}SER_{(n-1,n)} & \longrightarrow & SERM_n \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D} \\ SER_{(n-1,n)} & \longrightarrow & SERM_{(n-1,n)} \end{array}$$

of sets of serial motives with all horizontal arrows being inclusions. Consider the left action of the group $D_{n,n}$ on $SERM_n$. Then, $SER_n$ is invariant, and for any affine automorphism $g : \mathbb{Z}_n \xrightarrow{\sim} \mathbb{Z}_n$, and the reversion operation $rev$, we have

$$D \cdot g = g_0 \cdot D,\ D \cdot rev_n = -rev_{n-1} \cdot D. \tag{11.56}$$

Therefore $D^{-1}SER_{(n-1,n)}$ is also invariant under $D_{n,n}$ and even under the larger group where we take all affine pitch automorphisms instead of transpositions and inversions only. So the set $ALLSER_n$ of all-interval series is invariant under $D_{n,n}$ and under the larger group $G_n$ generated by the reversion and all affine pitch automorphisms.

In [170], these properties of all-interval series, as well as specific weight function (as considered above) are used to calculate the size of the orbit spaces of all-interval series under the actions of different, musically indicated subgroups $H \subseteq G_n$. In particular, for $n = 12$, we have

**Proposition 10** [170, Satz 2.3.5] *$card(G_{12}\backslash ALLSER_{12}) = 267$.*

#### 11.4.1.3 Enumeration of Motives

We have discussed the classification techniques for zero-addressed local compositions $(K, \mathbb{Z}_N^n)$ in section 11.3.3, and more concretely of local compositions $K$ in $OnPiMod_{12,12}$ in section 11.3.8. Here, we deal with purely enumerative aspects for local compositions $K$ in $OnPiMod_{n,m}$, as discussed in [170]. We are again in the situation exposed in theorem 9, however, this time, $F = 2, G = \mathbb{Z}_n \oplus \mathbb{Z}_m$, and the weight is again given by formula (11.40). Therefore,

**Theorem 11** [170, Satz 2.5.2] *The number of orbits of zero-addressed local compositions $K$ in $OnPiMod_{n,m}$ under a group $H$ of affine automorphisms of $\mathbb{Z}_n \oplus \mathbb{Z}_m$ is the coefficient of $x^k$ in the cycle index polynomial*

$$Z(H)(1 + x, 1 + x^2, \ldots 1 + x^{nm}).$$

For the special case $n = m = 12$ and $H = \overrightarrow{GL}(\mathbb{Z}_{12}^2)$, the evaluation has been calculated by use of a Turbo Pascal program in [170]. We write down the first few monomials of the cycle index polynomial in theorem 11:

$$Z(\overrightarrow{GL}(\mathbb{Z}_{12}^2)(1 + x, 1 + x^2, \ldots 1 + x^{144}) = x^{144} + x^{143} + 5x^{142} + 26x^{141} + \ldots,$$

the entire formula is written down in appendix C.3.6. It shows that, in particular, the number of 72-element classes is

$$2\ 230\ 741\ 522\ 540\ 743\ 033\ 415\ 296\ 821\ 609\ 381\ 912 = 2.23\ldots \cdot 10^{36},$$

so there are still some motives to be introduced into musical composition!—In [170], the enumeration of some other dodecaphonic structures is calculated. The reader is referred to this work for more details.

### 11.4.2 Big Science for Big Numbers

**Summary.** Signification of the exorbitant growth of numbers of classes of local compositions for computational complexity in musical analysis. Complexity and depth in music revisited.

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Big Science is the type of science created by physicists since the Los Alamos nuclear bomb project during world war II. Nowadays, it is best illustrated by the research environment of Europe's CERN[20] near Geneva. With a total of roughly 10 000 employees, among others a circular LEP[21], an electricity consumption equivalent to that of the city of Geneva and a financial volume of about one billion CHF [83] in 1995. Big Science is recognized by the following characteristics:

- Scientific language
- Models and theorems
- Experimental paradigms and operationalization
- Universal collaboration and communication
- Adequate laboratories and machines
- Political acceptance and corresponding resources

Why does the exorbitant number of isomorphism classes of local compositions which we have exposed in the previous sections suggest the research context of Big Science? Let us look at the above characteristics and their realization in centers such as CERN, see figure 11.7.

Figure 11.7: (©cern) The 27 km long LEP ring on the CERN site, viewed from above. The ring is situated 100 m underground.

[20] CERN = Centre Européen pour la Recherche Nucléaire.
[21] LEP = Large Electron Positron Collider.

Evidently, the first two items are within the reach of mathematical music theorists. The third item is also attainable, at least in principle. We are given an enormous repertoire of musical works and know how to do experiments on this material. For example, one could calculate the distribution of isomorphism classes of chords, or motives, of serial motives, of local meters and so on within a determined material selection. We have already given small examples of such investigations in 11.3.6 and 11.3.8. And one could then compare this data to the models of harmony etc. in order to verify/falsify corresponding statements, etc. One also disposes of good operationalization techniques, see the annual ICMC[22] meetings, or also the relevant parts and chapters in this book (e.g., parts X, XIII). What is less evolved is universal collaboration, as well as political acceptance and corresponding resources.

However, the negative extremum is the serious lack of adequate laboratories and machines. What is "adequate" in this case? There are several requirements to be fulfilled. To begin with, any serious processing of relevant musical material, a classical sonata with its note size of order $10^4$, say, must be based on powerful computers, precisely because of class numbers such as the above number $2.23\ldots\cdot 10^{36}$ of 72-element motive classes. Second, harmonic, rhythmic, and motivic analysis of such class data requires enormous combinatoric calculations. For example, as we shall see in section 41.3, the calculation of harmonic paths is proportional to $2^{\text{number of onsets of a composition}}$, and this is practically infinity. Further, navigating on data bases of relevant musical works must be based on powerful visual and auditive representation paradigms, for example 3D-tables, see figure 11.8, or 3D-caves. See also chapter 20 for the visualization of musical data.

Finally, it is absolutely necessary to have auditory representations of whatever music or analysis data exists in order to control the complex material. All this combines to require a huge apparatus for calculation and representation.

Now, can this be compared to the CERN equipment which deals with experimental necessities deduced from high energy conditions required to unveil hidden symmetries of suspected laws of external nature? But we have already discussed this topic in chapter 4: Experiments with inner nature require high spiritual energies in a very rational sense, we cannot understand mental event systems having the quality of high-ranked music without adequate analysis, representation, and navigation tools. And these must be set up in technologically sophisticated and collaborative environments. It is not clear and we would not hope that a musicological CERN would be as expensive and monumental as the physicists' display. We would nonetheless like to stress that music is of comparable significance to humans, and this justifies much of such an effort which is not likely to produce destruction devices such as nuclear bombs.

Whatever the technological apparatus which becomes necessary to control this complex data, it has become evident that depth now is no more a question of empty or censored encapsulation and knowledge hiding. Depth can now completely be recovered in its explicable form of a complex universe which is overwhelming but not disclosed from understanding. The situation is much the same as with the early days of anatomy where opening a human body was prohibited by authorities who feared that detailed and rational access to the subject would annihilate the mumbo-jumbo of Aristotelian and Christian mystification.

[22] ICMC = International Computer Music Conference.

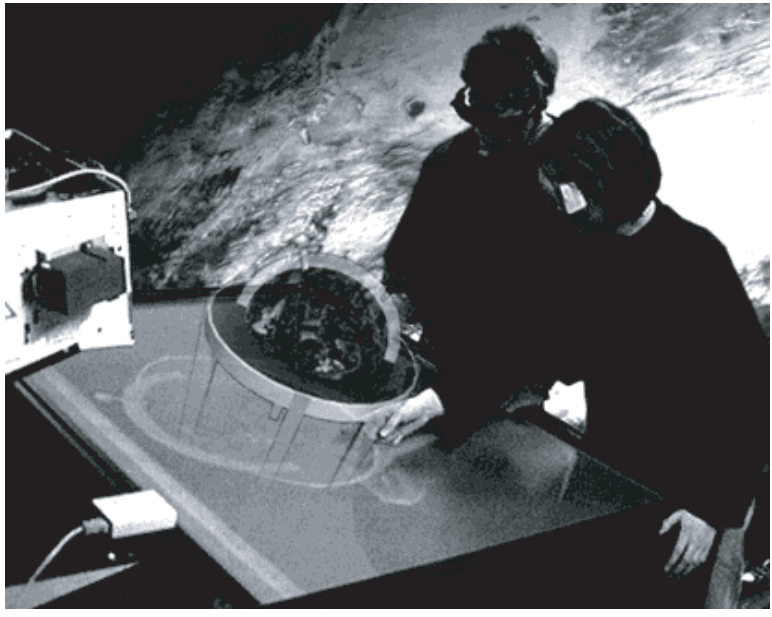

Figure 11.8: The 3D table [282] as a necessary virtual reality tool for visual navigation in music data bases.

## 11.5 Group-theoretical Methods in Composition and Theory

**Summary.** This section reviews the group theoretic methods in composition from Bach to Schönberg, and contemporary serialists. In analysis, approaches from Graeser to Forte and the American set theory school are reviewed.

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The wide-spread presence of groups of symmetries in music has already been discussed in chapters 8 and 10. Here, we want to relate the quantitative and qualitative aspect of orbit spaces to historically traced approaches in composition and analysis. We do, however, refer to further symmetry-related harmonic analysis because this aspect will also be treated in due detail in part VI.

Knowing that paradigmatic classification is a fundamental poetic device that must be taken into account in the analytical perspective, too, what is the specific power of group orbits of musical structures such as motives, chords, rhythms? De facto, once the paradigm and a concrete object have been related either by compositional or analytical efforts, we are thrown into a space of "prototypes" as orbits may be called in a more intuitive language.

So what help is given by moving within a space of prototypes? First of all, this space is a reduction to the paradigmatically essential information, we have abstracted from contingencies which blur the core information. But then, in many cases, as the previous classification discourse

has shown, the resulting list is often finite and—for some classical cases such as chord classes—of manageable size. In the tradition of medieval artistic poetics (such as François Villon's poesy [312]) or in the baroque combinatorial rhetoric (such as Athanasius Kircher's classificatory work [477]) such lists of reasonable size are germs of combinatorial approaches to universal classification or artistic construction. This is one essential aspect: Disposing of an alphabet of structural ingredients reduces much of the work to a task of combination or recombination. In other words, synthetical and analytical creativity are projected onto a field of combinatorial activity.

We should nonetheless recognize that lists which turn out to be infinite or so large that even the best computers cannot search them in reasonable time (e.g., the roughly $10^{36}$ classes of 72-element motives in $\mathbb{Z}_{12}^2$, see section 11.4.1.3) give a slightly different approach to combinatorics: We have to introduce statistical procedures since systematic combination would fail. So here, aleatoric components cannot be avoided, and this is one aspect which indeed is historically prominent as we shall see.

Whatever the size of combinatorial arsenals, the overall situation is *a massive shift of creative quality*. On one hand, combinatorics gives a feeling of divine power: Everything is reachable, the whole universe at your fingertips. You can play God with the molecular combinatorics of chemical substances from a small number of roughly a hundred atom types. On the other hand, this same "total control" not unexpectedly excites one's boredom with a once-for-all limited alphabet of creativity.

Let us first concentrate on the positive side: complete reachability of classes in an orbit space. To begin with, the specific nature of this space lends itself to an enrichment of semantic depth. More precisely, the classes being points in a particular space type—such as a geometric parameter space for the complete set of class invariants—may be related to each other qua points and therefore induce an additional instrument of poetic construction or analytical investigation. For example, the analysis of historical chord repertories such as discussed in section 11.3.7 used the size of automorphism groups of classes as a measure for historical distinction between important and marginal chord classes. In other words, the enrichment of having placed classes in a coherent space, and not only in an insignificant list array, yields a deeper principle of unification of the instance of these classes in a given composition. On the one hand, this is a plain added value, on the other, it may also happen that either in composition or in analysis, the space specifics are not or only in part taken into account. So there is no automatic integration of classifying space attributes, but the potential must be observed.

Second, the negative aspect of a well-defined classificatory arsenal is the limitation of the "game's rules" in a rigid framework. We know from music semiotics [361] that a significant difference between the music sign system and the linguistic system is the weak conventional aspect termed "langue" vs. the spoken "parole" in Saussure's dichotomic setting. Vocabularies tend to englobe small portions of space-time in musical cultures. This fact means that the "game's rules" may de facto vary without creating much ado, at least is this a fact without a priori negative consequences. So we observe that each determined combinatorial setup on the basis of a classification arsenal provokes the antithesis of progressive negation of the classification's basics, i.e., the paradigma as such.

In this dialectic dynamics of combinatorics and its negation, of power and creative expansion, history unfolds and lets us recognize a *coupled* syn- and diachronic development of the musical sign system. We shall come back to this overall phenomenon in section 11.7 after a

closer look at representative examples.

### 11.5.1 Aspects of Serialism

**Summary.** Extending transformation groups as a developmental parameter: Counterpoint, dodecaphonic composition technique, and genuine serialism.

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We have already seen symmetries with Bach, Schönberg and serialists in composition in section 8.1.1. Here, we reflect their roles with regard to paradigmatic classification via group orbits. In the baroque fugue technique as completed by Bach, the rhetoric scheme of a dialog between dux and comes is not specific enough to define a thematic vocabulary, it only gives empty boxes to be filled with musical structure. But the contrapuntal tradition had prepared a generic procedure of relating voices by a "switch" between counterparts, punctus contra punctum meaning that two opposites are confronted. We shall see in chapter 29 that this idea can be made completely precise in modeling Fux' contrapuntal rule system. Moreover, the artistic tradition of medieval poesy was highly sensitive towards construction of explicit correspondences between poetic instances: Recall François Villon's poesy [312] which abounds with retrograde, inversion and their combinations in verse and rhyme construction.

In this spirit of combinatorial artistry, the orbit of themes under inversion, retrograde, transposition and their combinations was a perfect completion of the said empty rhetoric scheme. On the one hand, this was however not the only principle of compositional construction. The harmonic level was to be developed into a dominant syntactical force which evolved until its dispersion at the end of the $19^{th}$ century for reasons we shall discuss in chapter 23. This led to an apparantly secondary role of the contrapuntal dialog of voices during the harmonic homophony. On the other hand, the construction of themes was not completely explicit in the sense that the interrelation between thematic/motivic and harmonic/contrapuntal constraints had not been made evident as a composed phenomenon.

Only after the dispersion of harmony in the first years of the $20^{th}$ century was the counterpoint group rediscoverd by Schönberg and his followers. This rediscovery followed Schönberg's declared program of rebuilding the compositional paradigm ab ovo and without any implicit interrelations to harmonic constraints. So the really new message in Schönberg's attempt was *the exclusive basing of local compositional coherence* upon the orbit of a given dodecaphonic series. In contrast to the baroque usage of such orbits, this time, the orbit *defined* the a priori alphabet of local compositions to be distributed over the syntagm of the composition.

So the progress against fugue tradition and the harmonic vocabulary was to make available a proper alphabet for each composition instead of a fixed alphabet being perpetuated along a large number of compositions. The common denominator of the dodecaphonic vocabularies was exactly this meta-vocabulary, i.e., a procedure to build vocabularies, a new one for each composition. Moreover, the corpus of such compositions was classified by a fundamental invariant: the orbit of the governing dodecaphonic series[23]. So the logic of orbits forced a thorough inves-

[23] Of course are there other characteristics than this of dodecaphonic composition technique. They are situated in the global configuration of the alphabet defined by the orbit $D_{12,12} \cdot Ser$ of series $Ser$ (see section 8.1.1). We shall come back this subject in chapter 13.

tigation of classes of dodecaphonic series, such as all-interval series and other 'special' types. Under such aspects was it no surprise that Schönberg even patented one of his series.

We know that a dodecaphonic series is a denotator

$$Ser : \mathbb{Z}^{11} \rightsquigarrow PiMod_{12}(Ser_0, Ser_1, \ldots Ser_{11})$$

at the address $\mathbb{Z}^{11}$ which essentially defines a sequence or "row" of pitch values. We could also say that a series is a "discrete curve" with values in a determined parameter space (see figure 11.9). Then a dodecaphonic series is by no means a generic concept: There is no deeper reason for disclosing other musical parameters, such as duration, loudness, onset, whatever, from such a concept. In other words, if a 12-series is a curve with values in a pitch domain, i.e., form $PiMod_{12}$, then a generic series would be a $\mathbb{Z}^{11}$-addressed denotator with values in a more general form such as $Duration \oplus Pitch \oplus Loudness \oplus \ldots$ or a variant with (partial) modular division $Duration_n \oplus Pitch_m \oplus Loudness \oplus \ldots$.

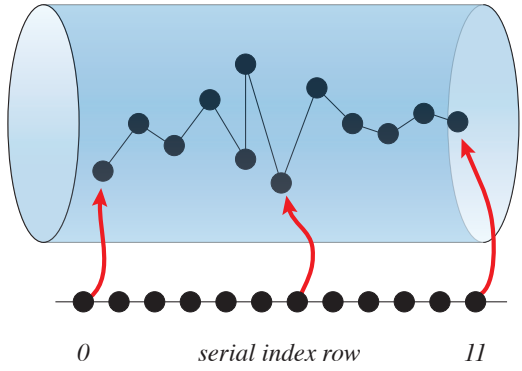


Figure 11.9: A generic series as viewed by serialists can be interpreted as a discrete curve. It takes values in a parameter space form; here, for example, in the tubular form $Duration \oplus Pitch_m$, for each serial index $0, \ldots 11$.

Asking such a series to fulfill the requirement of unique representation of a pitch, for example, amounts to looking at its projection onto the corresponding form and to making the requirement explicit on this projected parameter space.

In view of this canonical generalizatio,n which is immediate from the universal denotator language, the historical realization and handling of serial perspectives has been a poor attempt, and even failure. More precisely, the deficiency has occurred on two levels: Object *description* and control of *transformations* together with the attached orbit spaces.

The first one could be observed when composers were dealing with the question of realizing a given series on the level of a concrete composition. No systematic treatment of admitted series has ever been published. The formal power of composers and musicologists (in serialism often in personal union) was evidently too poor for grasping the complexity of a systematic serialism. So, when reviewing the serial compositions and their interpretations and analyses (see for example [94, 138, 502, 550]), we notice a permanent struggle with the fundamental

discipline of what parameter should be included, ordered, related to others, and then realized in a concrete composition or its analysis, respectively. The impression is that of a chaotic usage of undigested form and concept fragments from combinatorial and group theory. This chaotic approach is further alimented by an old and strong numerological tradition, such as Fibonacci numbers or the mumbo jumbo of magic squares. It seems that we are participating in a painful process of conceptualization, comparable to the birth of modern score notation.

The second deficiency is a consequence of the first one: Lack of adequate conceptualization makes control of transformations virtually impossible. So there is no evidence for mental control of which transformations of series should be allowed or how they should be formalized, not to mention the question of classification via group orbits. So the chaos of concepts intertwines with the chaos of transformations and results in a wild—euphemistically termed "creative"—conglomerate of operations and listings. We come back to this subject in the introduction of global compositions in chapter 13. Intuitively, the situation is that of constructing a new musical instrument and having nobody who is capable of playing it in a controlled way, but instead, treating it according to one's inherited capacities.

It should however be emphasized that there is no moral or artistic ground for refuting such a development. It rather turns out to be a typical effect in the history of music: Already it had happened with classical harmony, in the line of Riemann, say, that a fragmentary theory and blurred practice had led to its abandon in the name of (a true or desperate) creative innovation. Schönberg's dodecaphonism was in a strong sense this kind of reaction to fragmentary and deficient harmony. In this spirit of positive redefinition of a crisis, serialism has even been *defined* as a precursor of post-serialism [502]. History of music is also a history of failures to control results of creative extension.

Though adaptation of classification techniques from mathematics as described above solves some of these deficiencies, the very complexity of explicit control and manipulation of large ensembles of orbits and transformation groups requires more than just conceptual control, it requires auxiliary tools to *operationalize mental handling*. We have already mentioned this perspective in section 11.4.2. The computer is precisely the tool we are asking for. In chapter 49, we shall discuss more extensively the computer program *presto*® developed for this type of paradigmatic composition based on structural units and their innumerable transforms (in fact *presto*® allows more than 126 billion affine transformations in each parameter plane).

A typical application of the computer-controlled paradigmatic composition will be discussed in chapter 50. And we should also mention the impressive Piano concert No.1 by Jan Beran [49] which is the broad instantiation of transformations of a short, but important[24] motif (figure 11.10) from Beethoven's Hammerklavier sonata op 106.

This motif is a local composition of form $OnMod_{71} \oplus PiMod_{71} \oplus LoudMod_{71} \oplus DurMod_{71}$ and is transformed according to the general affine group of $\mathbb{Z}_{71}^4$, a group of cardinality[25]

$$10\ 445\ 260\ 466\ 832\ 483\ 579\ 436\ 191\ 905\ 936\ 640\ 000$$

or $1.0445 \cdot 10^{37}$. This orbit defines the a priori local alphabet of Beran's piano concert. Clearly, no human memory can manage this amount of *a priori* elements in a single orbit.

[24] See the motivic analysis in [328].

[25] See appendix C.3.5 for the cardinality of general affine groups over finite fields.
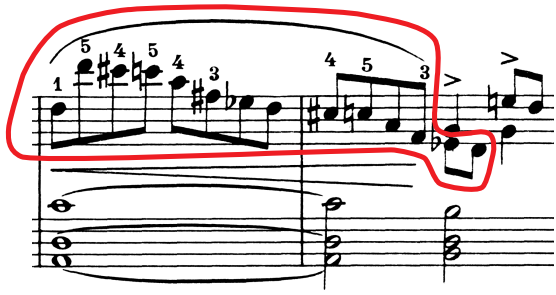
Figure 11.10: The fundamental "series" in Beran's Piano Concert Nr. 1: an important motif from Beethoven's Hammerklavier sonata op. 106.

#### 11.5.1.1 Serial Strategies in Jan Beran's Piano Concerto Nr.1

In the following, the main structural principles used for the piano concert are described briefly. It should be noted that the equally fundamental 'principle' used for the composition was 'musical intuition'. The structural principles were only used as a tool that helped 'translate' intuition into a large coherent composition, and thus create a 'skeleton' that was a starting point for the actual musical composition.

The technical tool for the effective realization of the composition was the composition application *presto*®. See chapter 49 for technical details of the *presto*® functionality and sound representation.

The construction consisted of the following steps that were partially carried out jointly and repeatedly, not necessarily in the sequence given here:

**Step A:** Selection of the basic theme $M$: The basic theme consists of the upper voice in Beethoven's Hammerklavier sonata op. 106, bars 75 and 76, see figure 11.10. The note events were put as zero-addressed denotators in general (geometric) position in the four-dimensional space

$$P = Onset \oplus Pitch_{71} \oplus Loudness \oplus Duration,$$

by assigning different values of Loudness and Duration to different notes. (The embedding of *presto*®'s half-tone pitch modulo 71 into a MIDI pitch space is discussed in chapter 49.)

**Step B:** Basic pattern in $P$: A basic pattern (ornament) was created by copying theme $M$ to a large number of starting points in the two-dimensional projection $P_o = Onset \oplus PitchMod_{71}$ of $P$. The starting points were defined by $(onset, pitch) = ja + lb$ where $a$ and $b$ are two (nonparallel) vectors in $P_o$ and $j, l$ are positive integers in a certain range. (Recall that here all values of pitch are understood modulo 71.) The vectors $a$ and $b$ were chosen such that repetitions of the same pattern occurred only after a very long period.

**Step C:** Most frequent transformations: A set $T$ of basic transformations (matrices) was defined. These consisted mainly of combinations of slight rotation, stretching and shearing.

**Step D:** Application of transformations: The pattern created in Step B was varied by local application of transformations, most of them being from set $T$. The transformations were applied to all two-dimensional projections of $P$. Of particular interest were transformations of projections that excluded either onset time or pitch. Transforming the pattern led to new musical motives that were used subsequently in the composition.

**Step E:** Instrumentation: Instruments were assigned (and created) to polygonal areas of the composition such that musically interesting structures in $P$ are emphasized.

**Step F:** Tempo curve: A detailed tempo curve was defined.

The sequence of the steps was as follows: Steps A through D were done using a piano score only. Steps E and F were done subsequently together with a second application of Step D. Most of the effort was spent with steps D through F, since this is where the actual creative compositional work took place.

## 11.5.2 The American Tradition

**Summary.** The purpose of this section is essentially to account for and discuss some crucial contributions of the American tradition to the emergence and proliferation of what Babbitt termed 'professional music theory' [31]. Our account will be split into a genealogical discussion (11.5.2.1) and a contribution to a vocabulary switch (11.5.2.2) between the American set theory (for short: AST) and our present concept environment.

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### 11.5.2.1 Genealogy

The impossibility of giving even a partial (ordered) description of the topics dealt with in the American music-theoretical literature since the 1950s leads us to look for historically and methodologically pertinent 'segmentations' in the domain of contemporary music theory. Perhaps one of the most fruitful approaches is based on the underlying dichotomy between an apparently more compositional attitude (Milton Babbitt) and a radically analytical perspective (Allen Forte) towards music theory. This distinction appears, more or less explicitly, in many studies on the subject, from the important "Perspectives on Contemporary Music Theory" (1972, [59]) to the recent "Music Theory in Concept and Practice" (1997, [35]). Both of these works are divided in two formal parts, meta-theory and methodology/compositional theory and historical and theoretical essays/analytical studies, respectively. The former work points to "the engagement by composers in fundamental music-theoretical explications" ([59, p.vii]), while the latter suggests that "the very fact that Forte is not himself a composer has changed the field of theory considerably" ([35, p.50]).

We may also suggest here that this distinction is not only relevant for an historical discussion on pitch-class set theory, but that it also helps in understanding how this theory successively enlarged its field of applicability thanks to important works by John Rahn [430], Robert Morris [380], and David Lewin ([300, 301]).

As pointed out by George Perle in his comprehensive study on serial and atonal music, "the most important influence of Schönberg's method is not the 12-note idea in itself, but along with it the individual concept of permutation, inversional symmetry and complementation, invariance under transformation, aggregate construction, closed systems, properties of adjacency as compositional determinants (...)" ([414, p.x]). This 'Babbittian' presentation of twelve-tone problematics constitutes perhaps the most appropriate introduction to the composer and theorist Babbitt. It is widely accepted that he first provided "twelve-tone theory with a consistent technical vocabulary" ([414, p.xiv]) and suggested that the relevance and "the force of any 'musical system' was not as universal constraints for all music but as alternative theoretical constructs, rooted in a commonality of shared empirical principles and assumptions validated by tradition, experience, and experiment" ([59, p.ix]).

Reflecting on the problem of "an adequately reconstructed terminology" ([30, p.10]) is a necessary preliminary step for the description of structural characteristics of the twelve-tone system, as Babbitt's four crucial articles make clear [26, 27, 28, 29]. Detailed discussions of Babbitt's terminology also feature in ([26], repr.) and [31]. One cannot emphasize enough that, for Babbitt, a "set" is an ordered collection of pitch classes and it is used as a perfect synonym for row and series. In contrast, the very predecessor of Allen Forte's "pitch-class set" is Babbitt's "source set", a set "considered only in terms of the content of its hexachords, and whose combinatorial characteristics are independent of the ordering imposed on this content" ([26, p.57n]). A synonym for it is the term "collection", first introduced by Lewin in [295] and widely discussed for its analytical pertinence in [296].

Subsequently, a vast body of American literature was devoted to the study of the specific properties of sets and collections, particularly combinatoriality and partitioning [319, 235, 178, 45, 506, 507]. Partition problems connected with Babbitt's original idea have also largely proved their relevance to mathematics with their natural embedding into the theory of groups [300, 380]. But even the idea of applying the mathematical concept of group for modeling musical systems can be regarded as one of Babbitt's most fruitful intuitions[26], provided that "the rules of formation and transformation of the twelve-tone system are interpretable as defining a group element (a permutation of order of set numbers) and a group operation (composition of permutations)" ([30, p.20]). This equivalence of structures, first introduced in [27], has important compositional consequences which are "directly derivable from the theorems of finite group theory" ([28], p.8). It is perhaps no exaggeration to see the introduction of groups by Graeser and then Babbitt as the 'Copernican Revolution' of modern music theory, especially if one considers the proliferation of group-theoretical methods applied to music, from Lewin's GIS structure to Clough and Agmon's modern theory of diatonicism. Suggestions for further reading in this area may be found in Rahn's review of Lewin's "Generalized Musical Intervals and Transformation" ([431]).

The most important representative of the analytical approach in the American music-theoretical literature is Allen Forte who is the author of a theory of set complexes [157] and of a book primarily devoted to the atonal music of Schönberg, Webern, and Berg in the first 20 years of the $20^{th}$century [159]. Forte's main purpose is to "provide a general theoretical framework, with reference to which the processes underlying atonal music may be systematically described" ([159, p.ix]). Forte's starting-point is "firmly analytical, springing from a truly fervent desire to

[26]However anticipated in 1924 by Wolfgang Graeser in his study on Bach's "Kunst der Fuge" [194].

uncover the secrets of an (...) enigmatic repertoire" ([56, p.50]). In his most recent article, Forte particularly emphasized this aspect, stating that "The structure of atonal music (is) above all the study of a musical repertoire rather than a theoretical presentation" ([160, p.83]). However, one of his most striking merits was the introduction of a "consistent terminology for pitch-class collections based on the mathematical properties of the set" ([56, p.49]) together with the elaboration of the "set complex", a topic[27] recently developed by Robert Morris [382]. But it is probably true that "Allen Forte's real success lies in the developments he inspired: beyond his theorization of atonal music, his work convinced many of the interest of a formal study of chromatic space" ([369, p.90]).

An instance of this is Rahn's pedagogically oriented introduction to some problematics common to the atonal and serial repertoire [430]. The book also prepares "its reader for the professional literature in the field" ([430, p.v]) and gives accurate references for further specialized topics including advanced serialism and combinatoriality. Babbitt's "general formative role", together with Forte's terminological heritage are widely recognized in the US-American sphere and the book has become a standard reference for further discussions in set theory. Robert Morris for example developed some "relations between ordering, aggregate completion, and pitch-structure based on groups of operations acting on pitch-classes" ([380], p.xiii).

The following section is dedicated to a detailed discussion of many of the topics dealt with by Morris by means of the theory of local compositions. Morris' recent formalization of his so-called "compositional spaces" provides a new theoretical tool which compensates for some of the weaknesses of his original compositional model. As defined by Morris, "compositional spaces are out-of-time structures from which the more specific and temporally oriented compositional design can be composed" ([381, p.330]). A crucial point here is its intimate connection with Lewin's transformational approach. Like Babbitt and Forte, Lewin's contribution to the field of music theory is both terminological and methodological. His work could also be regarded as one of the clearest attempts to go beyond the music-theoretical dichotomy of compositional vs. analytical procedures. His very first article concerning generalized approaches in describing intervallic collections of notes [295], which is almost in the spirit of Babbitt's compositional attitude, was soon followed by an analytical application of some formal constructions, like the interval function, to Schönberg's hexachordal pieces [296].

As pointed out by Jonathan W. Bernard, the fact that Lewin's function deals with the interval content of a single collection of notes, makes it "the conceptual predecessor of the interval-class vector (...) in Forte's work" ([56, p.44]). Lewin later studied the specific differences between his own "interval function" and other similar theoretical constructions, like Forte's interval vector or Regener's common-note function [297]. Most of the areas developed by Lewin since the sixties find their natural place in his fundamental treatise [300]. Lewin's most important abstract construction is probably the GIS (Generalized Interval System), a more elaborated version of his previous "Formal Interval System" or FIS [299]. The study of such a structure leads to a natural generalization of Forte's Set Theory, also by means of a transformational model that includes transformational graphs, networks, and isographies. Lewin's GIS model is algebraically equivalent to the structure of principal homogeneous $G$-sets ([553, p.271]). By taking $G$ to be the additive group of a vector space, one is formally lead to the same family of objects (i.e., affine spaces) that are amply discussed in the present book. An example

[27] See also our discussion of global compositions in chapter 13, where the natural generalization of this and other global concepts in AST are treated.

of the way in which Lewin's theoretical constructions could be described in terms of forms and denotators was considered in section 6.6, the second small example of 9.

Transformational networks and "their pertinence for music theory and analysis" are largely discussed in Lewin's second book [301], dedicated to Babbitt and Forte. For further applications to Lewin's transformational approach to the problem of voice leading in atonal music see [515].

We conclude this brief genealogy of the American music-theoretical tradition with the mention of a few noteworthy works on modern diatonic theory. This would be in perfect tune with Babbitt and Lewin's main concepts, for the topic leads to important developments in the field of mathematical music theory. As suggested by Eytan Agmon in his recent comprehensive study in the subject [10], the modern theory of diatonicism originated in some problems first raised by Babbitt [29, 30]. The study of combinatorial properties of generalized well-tempered musical systems [178] leads in a natural way to a formalization of musical structures by means of group-theoretical methods [36] and the elementary theory of numbers [88, 9, 89, 90]. A special issue of JMT[28] confirms the amount of interesting mathematical and musicological problems concerned with diatonic theory and the so-called Neo-Riemannian Theory. Both, by paraphrasing John Rahn, have transformed in the USA the field of music theory into a modern mathematical study.

#### 11.5.2.2 Concepts and Theory—A Vocabulary Switch

This section is devoted to an embedding of the Musical Set Theory constructs of *local* structures into the theory of local compositions with their forms and denotators. The global aspects will be dealt with in section 16.2. We shall comment on the structure and performance of American Music Set Theory (the abbreviation AST also in honor of its characteristic abbreviation overhead) in section 11.5.2.4. To be consistent with conceptual cross relations, we keep track with Robert Morris' lucid book [380]; for other references, see [82, **?**], for example. In fact, Morris' presentation is quite generic, explicit and also recognized among the AST theorists.

The scheme of our proposed embedding is a translation of the AST concepts into forms and denotators, including comments on the semantic background of AST constructs. We do not claim coverage of theorems and definitions, they may be traced back in the AST literature. But a vocabulary switch of the basics is the indispensable minimum for intercultural communication.

A first large class of AST concepts is built around pitch spaces. The basic space form is $n$-modular pitch, i.e., the form

$$PiMod_n = PiMod_n \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}_n)$$

defined in formula (6.42). This explicitly includes the plain integer pitch module for $n = 0$, i.e., a synonymy to $Pitch|_{\mathbb{Z}}$, but we refrain from this identification since we do not want to connotate too much of the context of $Pitch|_{\mathbb{Z}}$. Morris calls the cases $n \neq 0$ the cyclic cases, whereas $n = 0$ is called the linear case.

If form spaces are deduced from these basic spaces, and if they are related by modular identification with $0 < n$, this always works via the common projection $mod_n : \mathbb{Z}_0 = \mathbb{Z} \to \mathbb{Z}_n$ and its associated natural transformation and form morphism

$$mod_n : PiMod_0 \to PiMod_n$$

[28] Spring 1998, Vol. 42/2.

which will be assumed to be clear in the present AST context. Very often, AST deals with $(k, n)$-serial motives or synonymous objects; we shall explicitly include serial motives with integer pitch values, i.e., consider $(k, 0)$-serial motives

$$Ser : \mathbb{Z}^{k-1} \rightsquigarrow PiMod_0(Ser_0, \ldots Ser_{k-1}).$$

**C-Spaces.** For a positive natural number $n$, a contour space or *c-space* $X$ of order $n$ has form

$$c\text{-}space_n(X) \underset{Id}{\longrightarrow} \mathbf{Syn}(PiMod_0), \tag{11.57}$$

so that the form name is characterized by the generic "c-space" head and the specification of order and name $X$. A c-pitch (= cp, plural: cps) of $X$ a zero-addressed denotator $cp : 0 \rightsquigarrow c\text{-}space_n(X)(x)$ with coordinate pitch numbers $0 \leq x < n$. So the order does only affect the possible denotators. The musical background of such a space is encoded in the name "$X$". In particular, the material nature of the cps is hidden in this name. Contour spaces just parametrize pitch in some generic context where precise pitch is not relevant, but only comparative pitch, and this is numbered by integers from 0 to $n-1$.

A zero-addressed local composition in ambient space $c\text{-}space_n(X)$ (whose elements are c-pitches!) is called a *cpset*[29]. A pcset is always of cardinality at most $n$.

A *contour* for $c\text{-}space_n(X)$ is defined as a serial motif which takes values in $c\text{-}space_n(X)$, i.e., a denotator

$$Cont : \mathbb{Z}^{k-1} \rightsquigarrow c\text{-}space_n(X)(Cont_0, \ldots Cont_{k-1})$$

whose coordinates are cps, i.e., evaluate to pitch numbers between 0 and $n-1$. The number $k$ is the *length* $l(Cont)$ of $Cont$. We denote the set of these contours by $CONT_{n,k}(X)$, its cardinality is $n^k$. As with dodecaphonism, the index of a contour is an abstract substitute for onset time. In this sense a contour formalizes an abstract motif structure. We shall deal with the general approach to motif theory in chapter 22, here, we just describe the AST approach. The AST group operations of contours are those in the Kleinian subgroup $K_4$ of $D_{k,0}$ (see the discussion of dodecaphonism following definition 22) generated by the address retrograde $R = rev_k$ and the inversion $I = e^{n-1} \cdot -1$ which flips the interval $[0, n-1]$.

The topological classification paradigm for contours is defined by an "abstract motif" (see chapter 22), the *comparison matrix* $COM(Cont)$ associated with a contour $Cont$ of length $k$. $COM(Cont)$ is the integer $k \times k$-matrix with coefficients $COM(Cont)_{i,j} = sign(Cont_{j-1} - Cont_{i-1})$. Clearly, $COM$ is skew-symmetric, i.e.,

$$COM(Cont)^\tau = -COM(Cont).$$

The effect of the $K_4$-action on $CONT_{n,k}(X)$ upon the comparison matrix is this:

$$COM(I \cdot Cont) = -COM(Cont),$$

$$COM(R \cdot Cont) = COM(Cont)^\rho,$$

[29] In the AST, sets are called unordered sets, and sequences are called ordered sets. We do not inherit this unmathematical terminology.

the 180-degree rotation $\rho$ of the matrix around its center. We have

$$COM(IR \cdot Cont) = COM(R \cdot Cont)^{\tau} = COM(Cont)^{\rho\tau} = COM(Cont)^{\kappa},$$

the co-diagonal transposition $\kappa$. In other words, the $COM$-map is $K_4$-equivariant for the above actions of $K_4$. We have a commutative diagram

$$\begin{array}{ccc} CONT_{n,k}(X) & \xrightarrow{COM} & \mathbb{Z}^{k\times k} \\ {\scriptstyle K_4\backslash}\downarrow & & \downarrow{\scriptstyle K_4\backslash} \\ K_4\backslash CONT_{n,k}(X) & \longrightarrow & K_4\backslash\mathbb{Z}^{k\times k} \end{array} \tag{11.58}$$

and can define the AST *segment classes* or *contour classes* as fibers of the composed map, this is a typical situation in motif theory (see chapter 22).

**P-Spaces.** For a more precise pitch information, AST uses the concept of p-space. For positive natural numbers $n, M$, a pitch space or *p-space* $[M^{1/n}]$ has form

$$[M^{1/n}] \xrightarrow[Id]{} \mathbf{Syn}(PiMod_0), \tag{11.59}$$

where the form name is characterized by the $n$th root of $M$ which *defines a physical connotation of the pitch numbers*. More precisely, a *p-pitch* (or shorter: a *pitch*) of $[M^{1/n}]$ is a zero-addressed denotator $x : 0 \rightsquigarrow [M^{1/n}](x)$ with coordinate pitch numbers $x$ that stands for a physical frequency $F_0 M^{x/n}$, the case $M = 2, n = 12$ being the 12-tempered tuning starting from $F_0$.

The *ordered p-space interval* $ip < a, b >$ is the integer $a - b$; the *unordered p-space interval* is defined by $ip\{a, b\} = |ip < a, b > |$.

A zero-addressed finite local composition in ambient space $[M^{1/n}]$ is called (unordered) pitch set and abbreviated by *pset*, a serial motif

$$Seg : \mathbb{Z}^{k-1} \rightsquigarrow [M^{1/n}](Seg_0, \ldots Seg_{k-1})$$

is called *pitch segment* and abbreviated by *pseg*; we denote the space of length $k$ pitch segments in $[M^{1/n}]$ by $SEG_k[M^{1/n}]$. *Pitch cycles* are the same as pitch segments, the AST only distinguishes them by different admitted group operations, see below.

The *interval content* of a pset $P$ is the function

$$int : P \times P \to \mathbb{Z} : (x, y) \mapsto ip\{x, y\}$$

which is also written as an equivalence class of matrices according to the permutations of the pset elements.

The *interval succession* of a pitch segment $Seg$ is precisely its derivation $INT(Ser) = D(Ser)$ defined in (11.54), the *iterated interval succession* is denoted by$INT_m(Ser) = D^m(Ser)$.

The *cyclic interval succession* $CINT(Ser)$ is $INT(C(Ser))$, the interval succession of the "cyclic extension"[30] $C(Ser) = (Ser_0, \ldots Ser_{k-1}, Ser_0)$ of $Ser$; $CINT_m$ is just the iteration $CINT^m$.

In p-space, AST considers *translation* $T_n = e^n$ and *multiplication* $M_n = (n)$ of pitch, and $I = (-1)$ represents *inversion*, whereas general inversions are of shape $T_nI$. The affine transformations $T_nM_m = e^n \cdot (m)$ on the pitch coordinator $\mathbb{Z}$ define the set of *canonical operators*, i.e., $\mathbb{Z}@\mathbb{Z}$.

*Retrogression* $R$ on pitch segments is defined as with any serial motif. *Rotation* $r_t$ of a pitch segment $Ser$ is the serial motif $r_t(Ser)$ defined by cyclic permutation address change $i \mapsto i + t \mod k$, i.e., $r_t(Ser)_i = Ser_{i+t}$. We know from (11.54) that $INT$ is invariant under translations, commutes with multiplications and has $INT(R(Ser)) = -R(INT(Ser))$.

**Pc-Spaces.** The pitch class spaces in AST are thought of as being a modular derivative from pitch spaces, and the standard interpretation of passing from the p-space $[2^{1/12}]$ to a 12-modular pc space makes it useless to introduce a more refined terminology than just "the" $n$-pc space, i.e.,

$$pc\text{-}space_n \underset{Id}{\longrightarrow} \mathbf{Syn}(PiMod_n)$$

for positive natural number $n$. A pitch class (abbreviated *pc*) is a zero-addressed denotator

$$x : 0 \rightsquigarrow pc\text{-}space_n(x)$$

in $pc\text{-}space_n$. The set of all pc is called the *aggregate* and denoted by $U_n$ or just $U$ for the usual $n = 12$.

An *ordered pc interval* between pc $x$ and pc $y$ is $i < x, y >= y - x$; the *unordered pc interval* $i\{x, y\}$ is the smallest nonnegative representative in $\mathbb{Z}_n$ of the differences $i < x, y >, i < y, x >$. AST writes $ic\{k\}$ resp. $ic < k >$ for the set $\{(x, y) \in U^2 |\ i\{x, y\} = k\}$ resp. the set $\{(x, y) \in U^2 |\ i < x, y >= k\}$ and says "$i\{x, y\}$ is $ic\{k\}$" resp. "$i < x, y >$ is $ic < k >$"[31].

An (unordered) pitch class set *pcset* is a local composition in ambient space $pc\text{-}space_n$. A pitch class segment *pcseg* is a serial motif with coordinates in $pc\text{-}space_n$.

*Operations and TTOs (TTO = Twelve Tone Operators).* Operations on $pc\text{-}space_n$ are, as already with the p-spaces, the affine transformations on the ambient space, i.e., $\mathbb{Z}_n@\mathbb{Z}_n$, mostly used for $n = 12$. Notation is again $T_m$ and $M_m$ for translation and multiplication. The group $\overrightarrow{GL}(\mathbb{Z}_{12}^2)$ is denoted by $TTO$, and its 48 elements are called *TTO operators*, multiplication $M_5$ is notated by $M$. A *set class SC* is an orbit $TTO \cdot A$ of a pcset $A$.

AST fixes $n = 12$. Given two pcsets $A, B$ the *interval class content vector* is the 7-tuple in square brackets $ICV(A, B) = [ICV(A, B)_0, \ldots ICV(A, B)_6]$ with $ICV(A, B)_0 = card(A \cap B)$, $ICV(A, B)_j = \frac{1}{2} card(A \times B \cap ic\{j\})$ for $0 < j < 7$. The *interval vector* is the 12-tuple in square brackets $IV(A, B) = [IV(A, B)_0, \ldots IV(A, B)_{11}]$ with $IV(A, B)_j =$

[30] This is not AST terminology.

[31] The difference between unordered and ordered interval notation is not AST, but useful.

$card(A \times B \cap ic < j >)$, also called the *multiplicity* of position $j$ and denoted by $Mul(A, B, j)$. The "complement theorem" is the formula

$$MUL(A\hat{}, B\hat{}, n) = MUL(A, B) + 12 - (card(A) + card(B))$$

which is a trivial consequence of the facts

$$MUL(X, X, n) = MUL(X, X, -n)$$

and

$$MUL(X, Y, n) + MUL(X\hat{}, Y, n) = card(Y).$$

The special case $ICV(A, A)$ is denoted $ICV(A)$. Clearly, the interval class vector of a pcset $A$ is invariant under inversions and translations. The interval vector $IV(A, B)$ is invariant under translations and exchanges its arguments under Inversion.

A pcset $A$ is called *invariant under a TTO* $K$ if it is a symmetry of this local composition. The order of $Sym(A)$ is called the *degree of symmetry* of $A$. Define[32] $Sym_i(A, B) = \{x = e^t \cdot (i) \in TTO | \ x(A) \subset B\}$ and $sym_i(A, B) = card(Sym_i(A, B))$. Then the *invariance vector* of $A$ is the 8-tuple

$$\begin{aligned} sym(A) = <& sym_1(A, A), sym_{11}(A, A), sym_5(A, A), sym_7(A, A), \\ & sym_1(A, A\hat{}), sym_{11}(A, A\hat{}), sym_5(A, A\hat{}), sym_7(A, A\hat{}) > \end{aligned}$$

which is an invariant of the $TTO$-orbit of a pcset. The $T_n/I$ *set class* of a pcset is its orbit under $TI_{12}$. For a given subgroup $G$ of $TTO$, The *G-prime form* of a pcset is the first representative of its $G$-orbit in the lexicographic order defined in section 11.3.5.

*Abstract inclusion* of pcset $A$ in pcset $B$ means that there is a member in the orbit (SC) $TTO \cdot A$ being included in $B$, i.e., $TTO$ contains a morphism $f : A \to B$ of local compositions. The number of (categorical) subobjects defined by a TTO $f : A \to B$ is the *"embedding number"*[33] $EMB(A, B)$.

**Exercise 17** Prove that the last assertion is equivalent to Morris' definition "$EMB(A, B)$ = number of pcsets in the SC of $A$ which are contained in a pcset member of the SC of $B$".

SC X is the *abstract complement* of SC Y iff they are the orbits of complementary pcsets.

**Pcset Similarity.** This subject will be dealt with in chapter 22.

**Pc Segments.** As with psegs, the $m^{th}$ *interval succession* of a pcseg $S$ is the $m$th derived serial motif $INT_m(S) = D^m(S)$. The *cyclic interval succession* $CINT(S)$ of a pcseg $S$ is the derived serial motif $INT_m(C(S))$ of its cyclic extension as defined above for psegs; as above, $CINT_m(S)$ is the $m$th iteration of $CINT(S)$.

*Retrograde* $R$ and *rotations* and pitch operations from group $TTO$ or the monoid of affine operations are defined mutatis mutandis as above for psegs.

[32] This is not AST terminology.
[33] Terminology not explicit in AST.

Given two pcsegs X, Y of lengths $l = length(X) \leq k = length(Y)$, AST calls $X$ *(literally) included in* $Y$ iff there is an order preserving address change monomorphism

$$\iota : \mathbb{Z}^{l-1} \rightarrowtail \mathbb{Z}^{k-1} : e_s \mapsto e_{\iota(s)}$$

such that $X = Y \cdot \iota$. Abstract inclusion means combining address-change monomorphisms with $TTO$-induced morphisms $f$ on pitch space. So abstract inclusion is a fractional morphism $f/\iota$ of denotators (viewed as singleton local compositions).

A *segment class* $SGC$ of a pcseg $S$ of length $k$ is the orbit $D_{k,12} \cdot S$ (see the discussion of $n$-phonic series following definition 22). Orbits generated by dodecaphonic series are called *row-classes*.

**TTOs.** Chapter four of Morris' book [380] is entirely devoted to the study of the group $TTO = \overrightarrow{GL}(\mathbb{Z}_{12}^2)$. The justification for considering TTOs and not any general permutations is precisely the affine character ([380, p.125]). The $TTO$ description includes elementary facts about finite groups, the formula for the inverse of a TTO, and cycle representations of the $TTO$ as a permutation group on aggregate $U_{12}$. The conjugacy classes of subgroups of $TTO$ are discussed and listed ([380, appendix III]).

Finally, *sets of operations*, i.e., self-addressed local compositions in $pc\text{-}space_{12}$ (subsets of $\mathbb{Z}_{12}@pc\text{-}space_{12}$) consisting of TTOs are considered, together with *strings of operations*, i.e., the serial motives

$$OpStrg : \mathbb{Z}^{k-1} \rightsquigarrow \mathbb{Z}_{12}\tilde{@}pc\text{-}space_{12}(OpStrg_0, \ldots OpStrg_{k-1}),$$

see the technique of address killing defined in section 11.2. These constructs are used of compositional design, i.e., the abstract configurations of pcs, pcsets, pcsegs, and operators to build background structures for musical compositions. See our discussion in section 16.3 for more details on global aspects in AST.

#### 11.5.2.3 Software for Musical Set Theory

The thoroughly combinatorial and algorithmic character of AST has led to several implementations of its objects and operations on the level of computer applications, the first ones being already described by Allen Forte [158] in 1970. More recent applications are, for example, the CMAP package developed by Craig Harris together with Alexander Brinkman [208] (see also [82] for further references) and updated by Peter Castine for Macintosh in C++ [82], including MIDI input. His class hierarchy is shown in figure 11.11 Another implementation in LISP has been presented by John Amuedo [15]. The package includes a computing environment SET-SLAVE for general AST calculations in Common LISP for Macintosh computers.

SET-SLAVE computes several "normal forms" of zero-addressed local compositions $c$ of pitch classes from AST:

- Rahn's normal form $NF$: This is a standard sequential presentation of $c$, using a not very systematic cyclic permutation[34] of the ordered sequence of $c$ in $[0, 11]$. (Taking the lexicographically first sequence would be the better solution.)

[34] Take the largest interval of successive pcs on the circle $\mathbb{Z}_{12}$, start the sequence with the right pc with such a maximal interval, and such that this is the lexicographically first among these maximal interval right-hand pcs. Take the smallest pc with these properties.—What a mess!

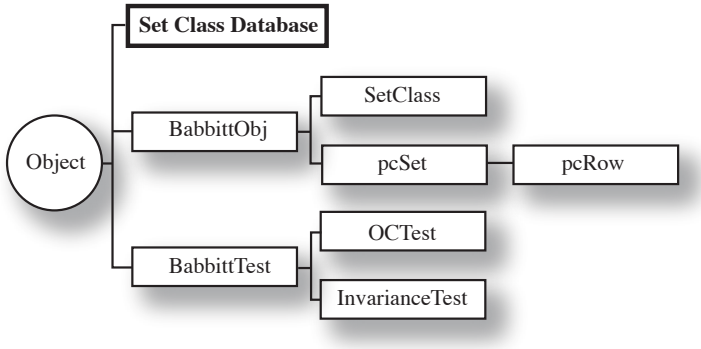


Figure 11.11: Built on the object-oriented Think Class Library, Castine's principal subclasses are (1) a SetClassDatabase class managing database requirements, (2) a BabbitObj class describing the basic objects of AST, and (3) a BabbitTest class which gives a generic framework for operations to be performed on objects.

- Straus' zero normal form $ZNF$. This is just the transposition-orbits, represented by $e^{-t}NF, t =$ first $pc$ of $NF$. Amuedo also includes the evident binary codification ($BNF$) of this normal form via the characteristic function on $\mathbb{Z}_{12}$, and its decimal representation $DNF$.

- Forte's prime form $PF$: This is a standard notation of $TI_{12}$-orbits of zero-addressed local compositions of pitch classes, using the lexicographically first orbit representatives.

- Forte's interval class vector $IVC$, i.e., the sequence of cardinalities

$$\langle i_1, i_2, i_3, i_4, i_5, i_6 \rangle$$

  with $i_j = card(\{$ unordered intervals $(x, y)$ in $c$ with $i\{x, y\} = j\}$

- Amuedo's decimal normal rotation $DNR$. This is the refined $ZNF.x$ from $ZNF$, where the "bass note" is codified by the "rotation index" $x$ for the power of the cyclic permutation ("rotation") $(1, 2, 3, \ldots card(c))$ of the $ZNF$ sequence bringing the "bass note" $pc$ to the first position of the sequence.—This is nothing but the distinction of a particular pitch, stated in terms of inversion of pitches (successive octave increase of a pitch) that yields a new bass note.

Like CMAP, it also computes Forte's "K" and "Kh" inclusion relations (see discussion in section 16.2 for these relations). Amuedo's thesis also includes software for real-time harmonic analysis. The SET-SLAVE is combined with three interactive MAX programs. This package contains a CHORD-CLASSIFIER, a SCALE-FINDER, and a SCALE-MONITOR for analysis of chords, and analysis resp. visual inspection of scales. See chapter 25.2 for more details.

Recently a library specifically devoted to American Set Theory has been integrated in Ircam's visual programming language OpenMusic based on CommonLisp/CLOS (see chapter 51

for a presentation of this computer-aided system for music composition and analysis) by Carlos Agon and Moreno Andreatta. The library, called Dn (from the Dihedral group of order 2n), is a generalization of a previous Lisp library of Janusz Podrazik (http://www.mracpublishing.com) on AST. The originality of the OpenMusic implementation consists in the manipulation of set-theoretical operations in the circular representation, with the possibility of taking into account any division of the octave in a given number $n$ of equal parts. It also allows us to switch from the pitch perspective to the rhythmic content by mapping the circular representation into a 'chord' or into a voice (see section 51.3 for more details). A specific function calculates the numbers of equivalence classes of $k$-chords (up to transposition and/or inversion). Other set-theoretical concepts which have been implemented are the normal order, the prime form, the interval vector, the literal and abstract complement, the Z-relation, combinatoriality and partitioning, the literal and abstract inclusion and the sub-complex Kh. We mention that the Dn library is part of a wider project on implementation of algebraic methods in OpenMusic which originated with the Zn library (devoted to the structure of the cyclic group of order n in the classification of chords and of tiling rhythmic canons; see section 16.2.3 for the presentation of Vuza's model of rhythm) and that will be continued with the study of the affine group and of the symmetric group (in the pitch as well as in the rhythmic domain).

#### 11.5.2.4 Comments

We want to conclude this section with three short comments on the AST, a mathematical, a conceptual, and a model-oriented one.

Mathematically, the AST is a very special achievement. Its concepts are thoroughly out of date from the point of view of $20^{th}$ century mathematical conceptualization. Even the most standard concepts in group theory are ignored, e.g., the index of a subgroup, or the semi-direct product of translations and multiplications which explain a large portion of the $TTO$ by the corresponding short exact sequence

$$1 \to e^{\mathbb{Z}_{12}} \to TTO \to \mathbb{Z}_{12}^{\times} \to 1.$$

Also the concept of a group action and corresponding elementary facts such as orbit cardinalities in relation to isotropy groups does not appear. Although the theory of categories has been around since the early 1940s and is even recognized by computer scientists, no attempt is visible in AST to deal with morphisms between pcsets, for example. We have seen certain germs of this direction in the definition of abstract subsets, but this is not what leads to a powerful theory of relations between local and/or global musical objects. It would be important to apply the above vocabulary switch in order to adopt the findings of AST to workable mathematical formalism such as it has been used by Fripertinger in Pólya and de Bruijn enumeration theory, for example. It would also be necessary to confront the AST approach with the many other parameters which define musical events, such as onset, duration, loudness, glissando, and crescendo, just to name a few important ones. The AST has never dealt with all these parameters in a global conceptual framework. However, the work of Dan Tudor Vuza or Anatol Vieru does heavily recommend such an extension. We shall deal with these directions in chapter 21 on metrical and rhythmical global compositions. Nonetheless, many valid lists of isomorphism classes, such as chord classes under translations, translations and inversions, or the full $TTO$ group as well as conjugacy classes of subgroups of $TTO$ have been established. We also have to recognize

that the operationalization of musicological concepts has been completely realistic insofar as computer programs and algorithms have been provided.

The conceptual comment must take care of a dramatic need for precise musicological concepts as tools for dodecaphonic analysis and of its theoretical extension in atonal theory. European music theory has only very rarely shown up in this domain, perhaps best in the work of Herbert Eimert and Iannis Xenakis. But the mainstream of after world-war II European musicology had turned towards dialectic mumbo-jumbo and far-out aesthetics and transcendental black-box-theories. So Americans had to start from scratch with precise conceptualization of even the simplest concepts such as pcsets, pcsegs and their classes. Whatever the status of a baby theory the AST concept framework might be, it is an indispensable reset of a rotten conceptualization in musicology where even the most elementary things are blurred.

The question of theoretical modeling is a difficult one. It appears that modeling has predominantly been oriented towards and useful for compositional strategies. Morris' composition designs are a real enrichment to grasp the complex construction of precise sound aggregates when starting from pcsets and similar elementary local compositions. Also the analytical use of the AST language is a considerable one. We are happy that finally, it was possible to simply talk in a precise jargon about analytical problems of atonal and also tonal music (e.g., in Amuedo's work). But there remains a big lack of models in the sense that beyond descriptive tasks, the AST language has very seldom led to musicological modeling. Most theorems of AST are of strictly combinatorial nature. We come back to such achievements in chapter 25.3.

So we could summarize the AST achievement as a necessary but far from sufficient attempt to escape decadent and impotent European musicology.

## 11.6 Esthetic Implications of Classification

**Summary.** Group-theoretical classification is a central issue of poetology. It is used as an esthetical tool in analysis and composition.

– Σ –

This subject is hard to understand from scratch. In many talks and discussions, I have experienced the disbelief that something like abstract group theory, together with group operations and orbit set construction, could have the slightest connection with something as intuitive, emotional or fuzzy-like poesy. This section is not written to replace passionate discussions about the legitimation of abstract algebra in poetology. But we can help in channeling them and focusing attention on the critical points.

To begin with, let us localize the problem on musical topography. As groups pertain to mental reality, we do not discuss psychological or physiological aspects of the phenomenon but only mental perspectives. In turn, these latter may or may not relate to actions on other levels of reality. Perhaps the confusion between mental facts and their psychological or physiological correspondences is one of the important sources of irritation. We shall discuss this subject in extenso in especially section 30.2.4.

On the mental level, there are three communicative allocations of poetic instances: poiesis, neutral level, and esthesis. The composer may use certain tools to express esthetic categories,

and the listener or analyzer may apply appropriate tools to perceive esthetical categories. Esthetic properties may also be defined on the neutral level of the proper work, but esthetic categories emerge on the human consciousness and therefore primarily pertain to the poietic or esthesic level to which we stick from now on.

A strong esthetic category is poetic structure or poeticity. In fact, prosaic esthetics could hardly ever be really esthetic in music, even if we deal with so-called musical prose in the sense of Schönberg or Wagner. They characterize it as an antithesis to metrical regularity[35], but metrics is by far not the only instance of poeticity; musical prose can be perfectly poetic in the sense of the word as obtained by Jakobson's revolutionary approach which we introduce below.

The importance of group theory to poeticity is that it makes the instance of paradigmatic transformation equivalence precise, and this is a core characteristic of poeticity of music. In fact, in the linguistic context, paradigmatic transformations are not so rich and complex as with music. Orbits which are generated via the multidimensional forms of local musical compositions (motives, chords etc.) can only be controlled by explicit group theory. This is the crux and blessing of music. Understanding musical poeticity is far more than a business of evident transformations, and this is what perhaps causes another irritation among those who would prefer that the genealogy of poesy be as straightforward as the evidence of its power upon humans is manifest.

After an introduction to Jakobson's poetic function we want to perform a poetological analysis (Schubert/Stolberg) as well as a poetological (re)construction (Mazzola/Baudelaire). We shall stress the group-theoretical aspect and give a critical comment on this procedure.

### 11.6.1 Jakobson's Poetic Function

**Summary.** The turning point of Roman Jakobson's poetic function in poetic analysis. Immanent analysis versus imported knowledge. Construction of poetic structure by use of orbits under symmetry groups.

– Σ –

In his famous paper [245], Roman Jakobson has isolated the poetical function as a central issue of esthetic structure in language. He has also pointed out that the poetic structure is not restricted to poetry, it may be present in mnemotechique or publicity or political rhetorics as well. It is interesting to note on which basic linguistic structures and functions poeticity is located. Jakobson lists six structures: sender, receiver, context, message, contact, and code. He relates these structures to the functions of (in the same order) emotivity, conativity, referentiality, poeticity, phaticity, and metalanguage. So poeticity is the function of language qua message. And poeticity is one important issues of esthetics, together with emotional functionalities, for example.

However, in music, we do not have automatic semantics and therefore, semantical enrichment by techniques of inner-systemic, i.e., autonomous constructions is an essential procedure to elaborate the musical sign system. The message must be strongly constructed in absence of contextual and predefined semantics.

Jakobson defines poeticity—without further empirical deduction—as follows:

[35] "...ihr primäres Kennzeichen ist das Abweichen von Normen klassischer musikalischer Metrik...", [457].

**Definition 35** *The poetical function projects the principle of equivalence from the axis of selection to the axis of combination.*

The terms in this definition have the following meaning. The "axis of selection" is the paradigmatic axis of language, i.e., the relation among its signs by topological or transformational equivalence as discussed in chapter 10. The The "axis of combination" is the relation among signs in their linear ordering of the syntagm. So we are given two spaces (axes): the abstract ensemble of all possible signs and a concrete string of signs which are juxtaposed in a linguistic message. The first shares the associative relation of paradigmatic fields whereas the second shares the adjacency relation of syntagmatic chains. The principle of equivalence on the paradigmatic axis is association whereas this principle on the syntagmatic axis is relative position on the syntagmatic chain. The first one is a given relation, the second one is not. Apart from im-

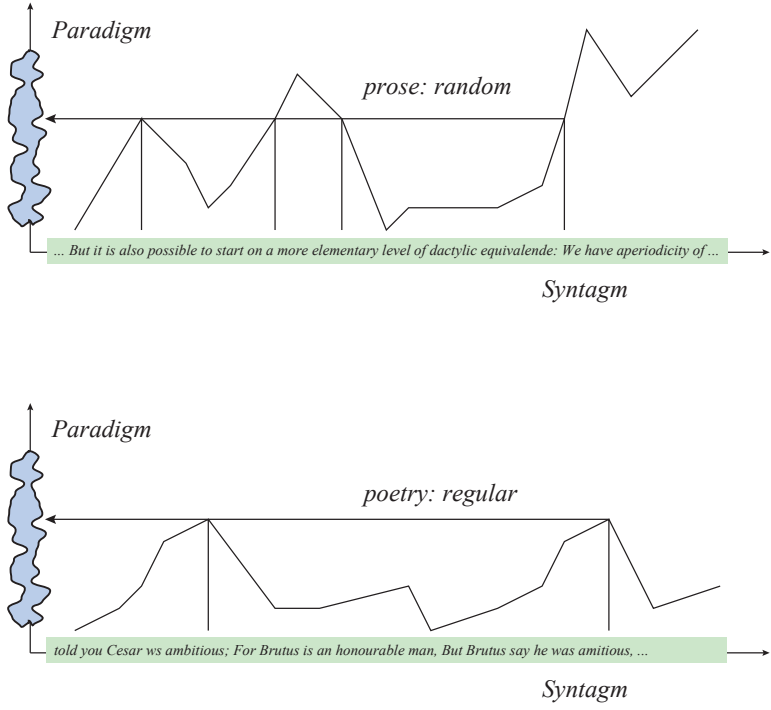


Figure 11.12: The Jakobson function associates the paradigm with each syntagmatic unit. In case of prose, this function is "random" whereas in case of poetry, this function is poetical, i.e., periodic with respect to a selected equivalence between syntagmatic positions, mostly a periodic relation.

mediate adjacency, no a priori relation is given. This enforces the construction of syntagmatic equivalence based on syntagmatic adjacency. The typical example is a periodic distribution of syllables in a verse form, such as a hexameter, the sequence of five dactylic metrical units

(/ ⏑⏑) (stressed/unstressed or long/short) plus the ending trochean unit (/ ⏑):

/ ⏑⏑ / ⏑⏑ / ⏑⏑ / ⏑⏑ / ⏑⏑ / ⏑
/ ⏑⏑ / ⏑⏑ / ⏑⏑ / ⏑⏑ / ⏑⏑ / ⏑
...

In such an arrangement of hexametric rows, equivalent positions in syntagm are the syllables on a determined column of the verse matrix. But it is also possible to start on a more elementary level of dactylic equivalence: We have a periodicity of successive triples of syllables, for each dactylic meter one unit, and then equivalence of metrical position means the first, second or third place within each dactylic meter, as may be shown in a more abstract "metrical versification" scheme:

/ ⏑⏑
/ ⏑⏑
/ ⏑⏑
...

Given such a syntagmatic periodicity (just to fix a concrete syntagmatic equivalence relation), the Jakobson projection is related to a special function $jak$ from the syntagmatic axis to the paradigmatic axis. It associates with each syntagmatic unit $u_s$ its paradigm

$$jak(u_s) = \text{paradigm of } u_s,$$

see also figure 11.12. This Jakobson function $jak$ is always defined, but with prosaic texts, it shows no regularity whereas the function (by definition) realizes a poetical functionality if it is periodic with respect to a syntagmatic (in this case: periodicity) equivalence: Its values are identical for equivalent arguments on the syntagmatic axis.

So the rhyme on the final syllable of corresponding verses is a poetical Jakobson function on the phonological level. If we have the semantic paradigm of "house", including "domicile", "home" etc., we would have two of these words on corresponding arguments. Jakobson wasn't precise either on the level of paradigmatic equivalence. Roland Posner has observed [421] that different levels of poeticity must be distinguished, such that besides the "horizontal" poeticity following Jakobson, a "vertical" poeticity must be considered. This vertical perspective relates to the connections between different semiotic aspects of signs when coupled by poetical functions. See [361] for this topic.

Vertical poeticity is an important tool for semantic enrichment, a fundamental feature of poeticity. It relates to the semantic production mode by symmetries which we have discussed in the beginning of chapter 8. Suppose, for example, that we are given a verse form equivalence on the syntagmatic axis. Suppose that we have alternating verses A,B,A,B,... and that the poetical function is realized on the phonological level by a rhyme at the verse ends, verse A ending with "...God", and the next verse A ending with "mob". This projection of the paradigm of the phonological class /o/ in the final syllable onto the syntagmatically equivalent ends of two verses of type A is a first poetical functionality, but it does more: it correlates the semantic contents of the words "God" and "mob" in a stronger way than in normal prose. We are given a poetic

index to a semantic relation between God and mob. This semantic enrichment is a conditio sine qua non for poetic meaning. It is an added value to the prosaic meaning of syntagmatic units. The following three facts give an upshot of poeticity:

1. Poeticity is based on an "unfolding of correspondences in time" or "a tale of symmetries".
2. By means of poetic coupling, semantics is enriched, not only stressed: We obtain an "added value" of meaning.
3. Classification (building equivalence fields) as a prerequisite to perception of equivalence is a central reference of poeticity.

In the following examples, we shall deal with a concrete and precise realization of the Jakobson principle, and we shall do this for the transformational paradigm, mainly in the sense of group actions.

### 11.6.2 Motivic Analysis: Schubert/Stolberg "Lied auf dem Wasser zu singen..."

**Summary.** This is a prototypical motivic analysis relating motives to dactylic metrics. It reveals a fundamental regularity in the distribution of three-tone motives in Schubert's composition.

– Σ –

We want to analyze Leopold Stolberg's poem *"Lied auf dem Wasser zu singen, für meine Agnes"*, written in 1782 and set to music by Franz Schubert 1823, see figure 11.13 for the vocal score. Science of literature [424, 425] has recognized that in the eighteenth century, no poem realizes symmetry—in the sense of congruence between emotional disposition and esthetic form—so perfectly as these lines. At first sight, the rigorous articulation catches our eye: three strophes of six verses each, ending on identical rhymes, each threefold. Whereas the identical rhyme produces the intended effect of monotony, the dactylic meter achieves another equilibrated movement: the regularity of undulation, again a triple measure.

Whereas in the first strophe, the soul moves rather passively "wie ein Kahn" "auf der Freude sanft schimmernden Wellen", in the second strophe, "atmet die Seele" more actively, namely in a series of four polar word couples: Über/Unter, westlich/östlich, Wipfel/Zweige, Kalmus(säuselt)/Schein (winket). The objects of this world are arranged in a symmetric fashion around something like "meine athmende Seele". The creative power of the poetic Ego incorporates its environment.

What happens to the Ego in the exterior object world, is reflected in the soul of the singer. The way this song accomplishes this identity turns it into a singular artistic achievement [424].

Schubert's setting (op. 72) translates the word's movement into continuous movement of sound. The composer succeeds in establishing a fascinating identity between text and musical transformation, a result which Moritz Bauer [44] has called a masterpiece, one of Schubert's best performances.

After this reference to traditional humanities, we want to start our analysis with the textual poeticity and then try to recognize the corresponding poeticity in music. This is the text:

LIED AUF DEM WASSER ZU SINGEN,
FÜR MEINE AGNES

Mitten im Schimmer der spiegelnden Wellen
Gleitet wie Schwäne der wankende Kahn;
Ach, auf der Freude sanftschimmernden Wellen
Gleitet die Seele dahin wie der Kahn;
Denn von dem Himmel herab auf die Wellen
Tanzet das Abendroth rund um den Kahn.

Über den Wipfeln des westlichen Haines
Winket uns freundlich der röthliche Schein;
Unter den Zweigen des östlichen Haines
Säuselt der Kalmus im röthlichen Schein;
Freude des Himmels und Ruhe des Haines
Athmet die Seel' im erröthenden Schein.

Ach es entschwindet mit thauigem Flügel
Mir auf den wiegenden Wellen die Zeit.
Morgen entschwinde mit schimmerndem Flügel
Wieder wie gestern und heute die Zeit,
Bis ich auf höherem stralendem Flügel
Selber entschwinde der wechselnden Zeit.

We discuss two examples of the Jakobson function. The first relates to the first strophe. We are given the syntagmatic equivalence of feminine and masculine ending syllables of verses 1,3,5, and 2,4,6. On these two classes of positions, we have two phonological paradigms of (1) a trocheic meter / ⏑ with its two identity paradigms /Wel/ and /len/ and (2) the stressed / with identity paradigm /Kahn/. These two equivalences relate the corresponding verses and their corresponding units: The phonological equivalence /Kahn/ points to the semantic unit of the word "Kahn" (boat).

On verse 2, this Kahn's activity of "gleiten" (sliding) is compared to the same activity of "Schwäne" (swans). On verse 4, the same activity (also identified on the first position of the verse) relates "die Seele" to the said "Kahn". Therefore, two different object categories "Seele" and "Schwäne" are loaded with additional semantics: they slide in the same way as the "Kahn" does. So three object categories: a boat, an animal, and a psychic entity are moving in the same way. This metaphoric intensification sets forth a strong semantic enrichment qua effect of this poeticity.

The second example relates to the second strophe. A phonological poeticity is set up which is isomorphic to that described before: equivalence on verses 1,3,5, and 2,4,6. The turning points of "Heins" and "Scheins", respectively, produce symmetry correspondences of the word couples "Über den Wipfeln"/"Unter den Zweigen", "westlichen"/"östlichen", "(Schein) winket"/"(Kalmus) säuselt". This turns out to be more than a syntagmatic coupling of unrelated objects: Each pairing is a paradigmatic symmetry on the semantic level!

The first is a spatial localization pairing of vertical antipodes relating to the extremal positions on a tree, the second is a pairing of geographic antipodes, the third is a pairing of distinct sensorial modalities: visual-showing against auditive-whispering, the latter pairing

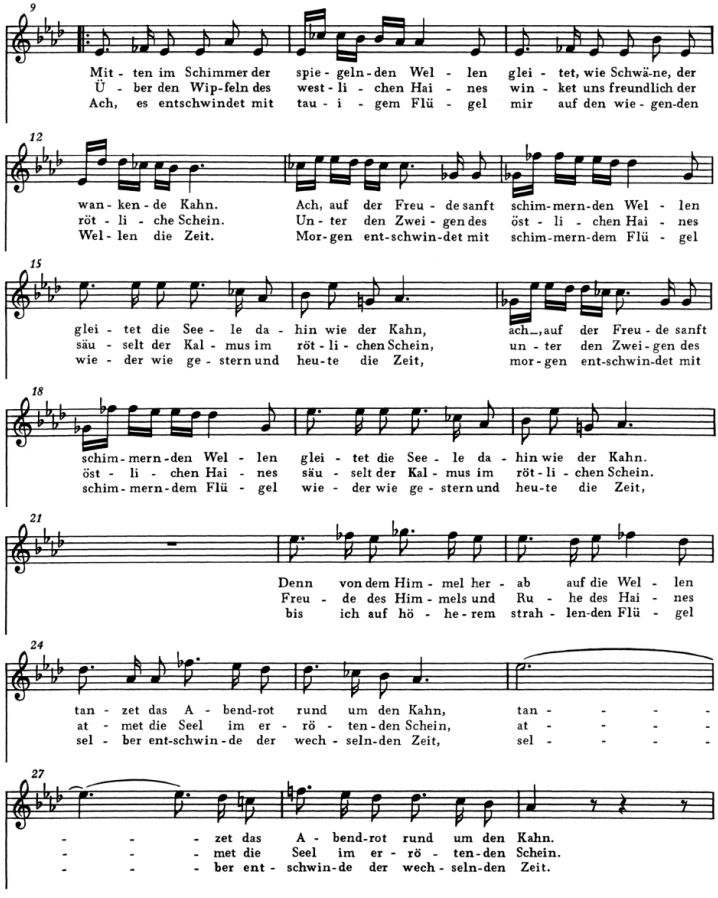


Figure 11.13: Voice part from Franz Schubert: *Lied zu singen auf dem Wasser, für meine Agnes*, op. 72 [480].

again coupled with the other vertical pairing: visual activity is from above ("Über...Winket... der Schein"), auditive activity is from below ("Unter...Säuselt der Kalmus"). This ensemble of polar symmetries presents semantic equivalences. Formally, the pairings are two-element orbits of positions which are arranged by the basic poeticity on the phonological level.

The ensemble's added semantical value is this: We recognize a group of instances being arranged around a central point of rotation which switches the polar positions into one another: west into east and vice versa, above into below, and vice versa, etc. The poetical function sets up a center of rotation, a force which turns things around, but a force which remains unnamed. However, in the last couple of verses, we are finally informed of the moving force: It is "die Seele" which "athmet" "Freude des Himmels" (from above!) and "Ruhe des Heines" (from below); in other words, the center of rotation is the breathing soul, metaphor for the poetic Ego. This one will become evident in the third strophe. Only there—at the end of a successive and dramatic

unvealing process—are we provided with the basic, nominative form of the pronoun "Ich".

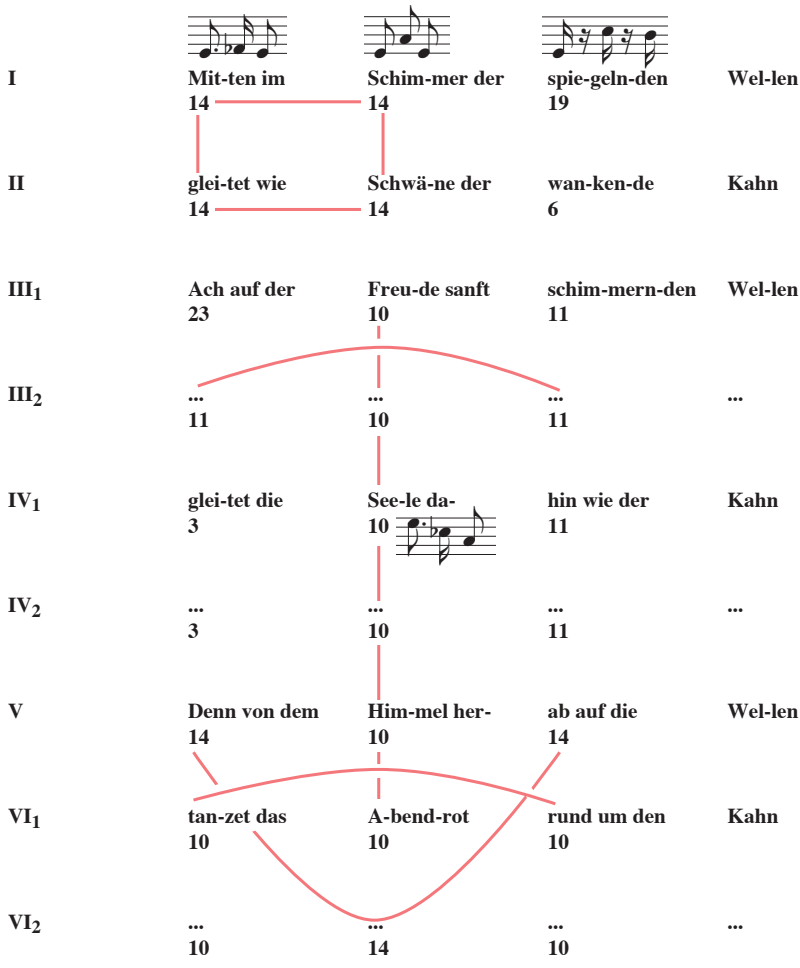


Figure 11.14: The highly symmetric arrangement of motive classes in the dactylus grid of the voice score in Schubert's setting of Stolberg's poem "Lied auf dem Wasser zu singen, für meine Agnes".

Let us now look for *poetical functionality on the musical level*. The Schubert composition is written in 3/4 time, and we may capture all durations in integer multiples of 1/16 notes. So 12/16 time suggests that we introduce onset of form $OnMod_{12}$. Together with octave periodicity, we are led to work in form $OnMod_{12} \oplus PiMod_{12}$. Relating to the dactylic text as discussed above, we want to group melodic parts according to the three dactylus groups in each verse. This gives a grid of 6 times 3 motivic units, corresponding to six verses per strophe and three dactylus meters per verse (the endings are not considered). Each motivic unit will be given

its natural structure of a three-element motif $M$ in $OnMod_{12} \oplus PiMod_{12}$. Where we have the repetitive sixteenth note motives (bars 2, 5, for example), we take the three-element motives with eighth note onset distance, conformal with the accentuation. Figure 11.14 shows the grid as well as the isomorphism class number of the three-element motives as listed in the class list of appendix M.

We take the full affine group $\overrightarrow{GL}(\mathbb{Z}_{12}^2)$ as equivalence principle. This is by no means obvious or mandatory, so we should discuss the choice now. The Jakobson function has to be evaluated along the syntagmatic dactylus grid. The values have to be taken in a paradigmatic space which is, contrary to the linguistic situation, not common or standardized. In music, the idiolect of an individual composer can vary considerably, in fact, it is a highly characteristic feature of a good composer to realize an individual brand of paradigmatic equivalences. The problem is that the analytical work cannot, in this case, rely on known paradigms, and it isn't necessary to copy the poietic standpoint to the esthesic perspective. More precisely, we are not going to judge upon what would be the best analytical choice of paradigmatic equivalence in this piece of music. We just start an experiment (of the mind) and set up a first approach to equivalence.

This forwards an important methodological question: Where should one start on the paradigmatic analysis if no pre-selection is available? Is there the "right" selection against "wrong" alternatives? Globally seen, we have an entire bunch of paradigmata to be applied. However, on the level of group actions there is a hierarchy of orbit size or group size, respectively. In order to decide which group is the best one (if it is any unique one!), we should follow the line of this size hierarchy. One one end, the identity group will very improbably instantiate a poetical functionality (though we have exactly this situation on the phonological level described above). On the other, complete identification (by the full permutation group on $\mathbb{Z}_{12}^2$) of all function values will clearly produce poeticity, but this is a trivial case which is definitely unspecific. So the idea is to start at this end where poeticity is most probable, and then to refine the group choice until non-automatic poeticity emerges.

More systematically one may call poeticity the entire spectrum of Jakobson's poetical functions

$$jak_\pi : Syntagm \to Paradigm_\pi$$

for all paradigms of a certain parametrized arsenal of a priori candidates. This "spectral analysis" would yield a poetic support of the parameter space in question (in fact not one analysis, but a whole variety of analyses), consisting of all parameters where poeticity can be observed.

In this vein, we approach the easiest non-trivial case: the full affine group $\overrightarrow{GL}(\mathbb{Z}_{12}^2)$. Its orbits are 26 in number and we may ask whether this equivalence relation yields structures beyond random functionality on the dactylic grid. Figure 11.14 shows the result. The Jakobson function of this paradigm has highly symmetrical (regular) properties. Above all, the orbit Nr. 10 appears on 10 of the 27 positions, this is 37%, against a probability of 22.7% to hit a representative of this orbit[36]. And it shows a perfect axial symmetry in the dactylic grid. Other symmetric configurations are realized for the Nr. 14 and Nr. 11 orbits, though not globally.

The dominant Nr. 10 "spine" also shares a more refined paradigmatic role in the following sense which will be made more precise in chapter 12.2.2. The paradigm is extend to the

[36]The total of $\binom{144}{3} = 487\ 344$ motives in $\mathbb{Z}_{12}^2$ contains 110 592 motives of the orbit of Nr. 10, this is $card(\overrightarrow{GL}(\mathbb{Z}_{12}^2))/card(Aut(Nr.10)) = 663\ 552/6 = 110\ 592$.

dominance relation between equipollent motives: $X \succ Y$ ($X$ *dominates* $Y$ or $Y$ is a *specialization* of $X$) iff there is a surjective morphism $X \to Y$ of local compositions. This relation is invariant under $\overrightarrow{GL}(\mathbb{Z}_{12}^2)$. The Hasse diagram[37] of this relation on the isomorphism classes of three-element motives is shown in figure M.2 of appendix M. Under the specialization paradigm, Nr. 10 is the generic class, it dominates every other class. Therefore, the spinal class is also the generic specialization paradigm, and it appears with highly significant overhead above a priori probability.

The importance of the spinal motif Nr. 10 is also evidenced by use of motivic weights, numeric functions which measure the presence of motif classes in the total motivic space of this music piece. Details can be found in chapter 22 and section 41.2. Figure 11.15 shows the weight profile of the motives over the dactylic grid, in particular the dominant Nr. 10 towards the end of the poem.

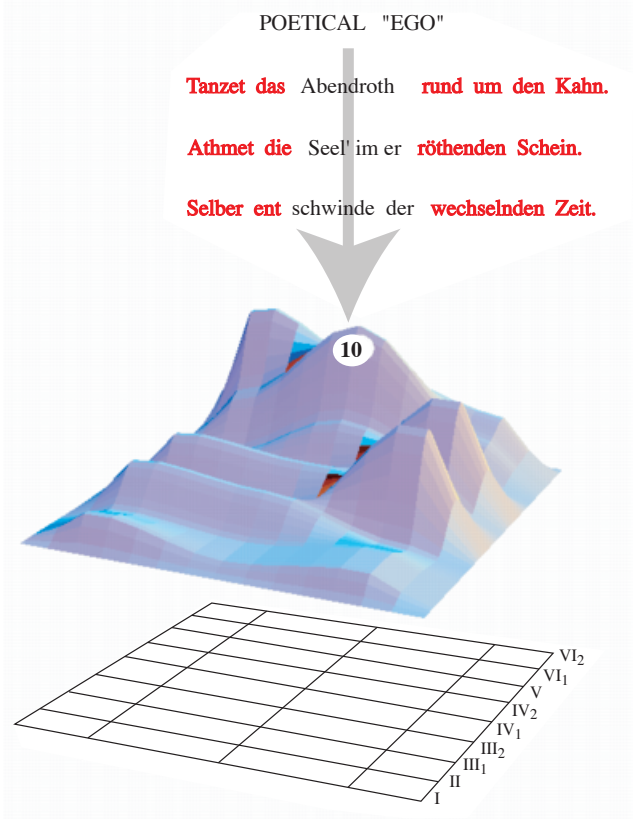


Figure 11.15: The weight profile of the three-element motive classes as a quantification of the musical paradigm of motives, and related to the corresponding textual instances of the poetic Ego.

[37] The minimal diagram whose transitive closure is the given relation, see appendix C.2, definition 121.

So these facts are a concentrated presentation of a highly poetical functionality together with a motif-weight dominance of the spinal class Nr. 10. The words being associated with the spinal dactylus are nouns of the poetic subject: “Freude”, “Seele”, “Himmel”, “Abendroth”, “Kalmus”, “Zweigen”. The climax of the motivic weight is taken on the middle dactylus of the $VI_1$ verse setting, and this is correlated with the middle of the three last verses of strophes 1,2,and 3. Figure 11.15 shows these words: “Abendroth”, “Seele”, and “(Ich) entschwinde”, a locus where also the symmetry of Nr. 10 and Nr. 14 becomes perfect. Within this perfection also appears the subject in its explicit form in verse $V$ of the third strophe:

**Bis** ich auf **hö**- he-rem **stra**-len-dem **Flü**-gel
**Sel**-ber ent-**schwin**-de der **wech**-seln-den **Zeit**.

This analysis is fairly neutral: It is scarcely esthesic and limits itself to a motive structure which is established by means of a classification list and its extension to the specialization Hasse diagram. In particular, no intention from the side of Schubert has been subtended. Rather is the question emerging from this analysis whether Schubert’s motif configuration is an unconscious instantiation of a highly poetic expressivity.

The compactness of this analysis should however not entail that the propedeutical work of classification and specialization data be underestimated or neglected. To understand this analysis, a preliminary understanding of classification is indispensable. Classification in fact has a deep cognitive impact which we do not yet understand: In which way can poetic feeling be generated through group actions and the associated orbits of equivalence? A first answer to this question will be given in chapter 30.

### 11.6.3 Composition: Mazzola/Baudelaire “La mort des artistes”

**Summary.** This section describes the use of the 26 isomorphism classes of three-element motives in the construction of a composition on the basis of Baudelaire’s poem.

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This section deals with paradigmatic composition techniques applied in scherzo movement three “Poem of Wind” of the 45 minute concert for piano, percussion and bass “Synthesis”, composed and performed on CD [339] in 1990 by the author. The composition was realized on the composition software *presto*® which will be dealt with in chapter 49.

“Poem of Wind” is a musical transformation of the first and second strophes of Charles Baudelaire’s poem “La mort des artistes” from his abysmal “Les fleurs du mal” (in the final version of 1861). These strophes are centered around the poetic Ego whereas the third and fourth strophes are written in the third person. They are four verses each, and follow an embraced A,B,B,A rhyme:

LA MORT DES ARTISTES

Combien faut-il de fois secouer mes grelots
Et baiser ton front bas, morne caricature?
Pour piquer dans le but, de mystique nature,
Combien, ô mon carquois, perdre de javelots?

Nous userons notre âme en de subtils complots,
Et nous démolirons mainte lourde armature,
Avant de contempler la grande Créature
Dont l'infernal désir nous remplit de sanglots!

.....

Compared to the Schubert piece, the paradigmatic realization of the 26 motif classes was much denser here. We started from the phonological paradigm realized on the level of letters and spaces. Each letter and inter-word space within a verse were given an onset with onset difference of 12 integer units, relating to $Onset|\mathbb{Z}$. Each inter-verse space was defined by an onset difference of 48 units, and the inter-strophe onset difference was set to 96. So we have an sequence of letter-onset couples, starting like this: (/C/,0), (/o/,12), (/m/,24), (/b/,36), (/i/,48), (/e/,60), (/n/,72), (/␣/,84),(/f/,96),...

On each such onset, a motive class will be allocated following a bijection between the 26 classes and the 26 letters which is motivated[38] by the frequency of letters in the two strophes, and by the hierarchy of motive classes in the specialization Hasse diagram (see appendix M, figure M.2). The most frequent letter is (as expected) /e/, and this is associated with the generic motif Nr. 10. Then, we have letter /n/, corresponding to motif class Nr. 11, etc. For the effectively used 21 letters, the bijection is this: (/e/,10), (/n/,11), (/o/,13), (/r/,12), (/t/,18), (/a/,19), (/s/,17), (/u/,15), (/l/,14), (/i/,16), (/m/,20), (/d/,25), (/c/,2), (/p/,1), (/f/,4), (/q/,3), (/b/,5), (/v/,22), (/g/,23), (/j/,26), (/y/,6).

In the next step, we take care of the selection of a representative for each class. To this end, we make use of a germinal melody which is responsible for the entire piano concert. It was constructed such that it can be covered by representatives of these classes, each once. The melody is a local composition $MelodyGerm$ in $Onset|\mathbb{Z} \oplus Pitch|\mathbb{Z}$ and the covering is a sequence of 26 subcompositions $Motif_t \subset MelodyGerm$ such that each $Motif_t$ reduces to a representative of class Nr. $t$ modulo 12 in onset and pitch .

Figure 11.16 shows the covering and the germinal melody, as a local composition and in conventional notation. This yields the candidates for representing all classes. This covering of a local composition by a system of small sub-compositions will turn out to be a crucial procedure for global structures in music, see the global theory, in particular the theory of "interpretations" discussed in chapter 16. In our context it is only an auxiliary tool to construct a kind of synopsis of all classes as being "charts" of a global connection.

The next step is the concrete realization of these representatives on the level of instrumental voices. In this piano concert, the piano is accompanied by 122 different percussion instruments[39]. In this setup, the pitches are assigned to different percussion instruments, and not to usual pitch. These instruments are grouped into toms, bells, etc., each group being distributed along an interval of pitch values. For example, we have snares for pitch values 8 to 17. Each motif is given a specific pitch range, i.e., a particular percussion sound so that we may distinguish the motives from their percussion sound range.

Figure 11.17 shows the *presto*® score as elaborated for the original piano score. Vertical axis is *presto*® onset (1-71), horizontal axis is integer onset time. The small quadratic icons

[38] The bijection is not unique, we just choose a reasonable candidate.

[39] The instruments are taken from Yamaha RX5 and TX802, and Roland R-8M synthesizers. The latter is charged with the sound chips "Jazz", "Contemporary Percussion", and "Mallets".

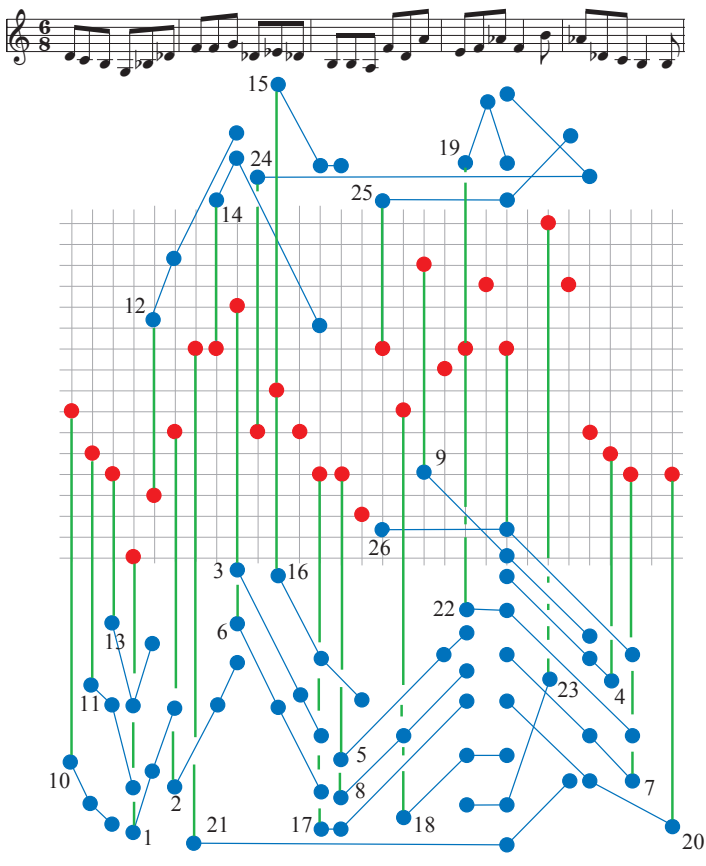


Figure 11.16: The germinal melody of the *Synthesis* concert for piano, percussion, and bass is a patchwork of 26 three-element motives, each representing one motif class in $OnPiMod_{12,12}$. The melody is shown as a score and as local composition (magenta-colored points).

denote elements of the local composition. We recognize not only the letter-related onsets of the different motives, but also some ornamentation which was mainly added for echo-type percussion effects. Below the upper rectangular score, we see the tempo curve which determines the micro timing of the piece. This subject will be treated in detail in section 33.1. Here, we can just remark that the function of tempo—and this is considerable, as one can recognize from the extreme variation of the *presto*® tempo curve—is that of a musical prosody which is parallel to the linguistic one related to the poem's words. So the overall philosophy of this musical construction is a one-to-one transformation of the linguistic text into a musical one, including the prosodic specification by tempo which must be defined here since the *presto*® software is required to furnish a real performance of percussive parts such that the piano can interact as if the synthesizer output would be a human one.

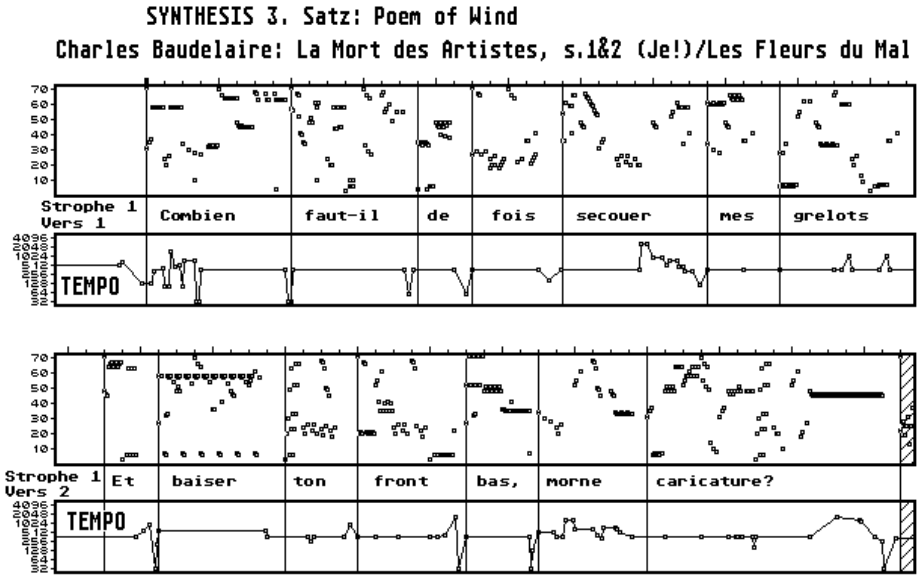


Figure 11.17: The original *Synthesis* score from the *presto*® graphical interface and piano performance prescriptions, the tempo curve below the score shows an extreme variability from 32 onsets/min. to 4096 onsets/min.

## 11.7 Mathematical Reflections on Historicity in Music

**Summary.** Whereas historicity seems abolished from the group-theoretical point of view, it is evidenced that the diachronic line of compositional tools parallels—among others—the size of involved transformation groups. This ideas originate from Jean-Jacques Nattiez' paradigmatic theme.

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The group-theoretical classification theory which we have dealt with in the preceding development is based upon categories of local compositions which are denotator constructs of seemingly exclusive mathematical nature, in other words: Analytical work seems to be reduced to a mathematically oriented systemic process, and as such, apparently contradicts historical perspectives—which are essential to a musical dynamics that parallels the historical dynamics of our physical universe.

This problem arises when we question the program of neutral analysis. Günther Mayer has pointed out this problem in [104]. In fact neutral analysis does not offer tools for historical localization of works. Mathematical structure and diachronic position are independent. In this sense, classification appears to be an ahistorical approach.

We want to discuss this problem in the light of structuralist theory of musical analysis as it has been developed by Ruwet and Nattiez.

### 11.7.1 Jean-Jacques Nattiez' Paradigmatic Theme

**Summary.** This section discusses the structure of Nattiez' paradigmatic theme in describing associations in the frame of Jakobson's function.

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The above singular "mathematical structure" is misleading. Mathematics offers a wide range of structures, and the possibility to distribute this spectrum on the diachronic axis may initiate a reconciliation between structure and history. The germ of such a reconciliation was laid down in the investigations of the Paris school of structuralist linguistics, mainly in the works of Nicolas Ruwet [466] and Jean-Jacques Nattiez [393] regarding applications of methods and insights of Saussure and Jakobson (see our previous discussion 11.6.1 of Jakobson's poetical function) to musical semiology and analysis. Ruwet's concept of *neutral analysis* means partitioning the work into a hierarchical system of units which have to be classified up to equivalence among each other and compared to units within other works, in the words of Ruwet [466, p.134]:

> Les diveres unités ont entre elles des rapports d'équivalence de toutes sortes, rapports qui peuvent unir, par exemple, des segments de longeur inégale — tel segment apparaîtra comme une extension, ou comme une contraction, de tel autre — et aussi des segments empiétinant les uns sur les autres.

The totality of classification criteria of a given analysis have been termed *"paradigmatic theme"* by Nattiez. This theme is realized by a concept of equivalence among units, defined by a *determined species* of geometric transformations [393, p.265]:

> Les unités paradigmatiquement associées sont équivalentes d'un point de vue donné (le thème paradigmatique), rarement identiques, et reliées entre elles par des transformations qui décrivent les variants par rapport à des invariants.

The hierarchical system of units is in fact a special case of what we shall call interpretation of a local composition in the framework of global theory, see 13.4. However, in Ruwet's and Nattiez' approach no systematic account is paid to the structure of this system, in particular no classification tools are discussed on this level: the paradigmatic theme is only a local affair.

On this level, equivalence means selecting a special subcategory of local compositions, more specifically, a special subgroup of automorphisms $P_M \subset \overrightarrow{GL}(M)$ for each ambient module of musical parameters. When comparing two hierarchical systems of units under a paradigmatic theme means admitting only the isomorphisms which stem from the system $P = (P_M)_M$ of automorphism groups. For instance, one could ask for a restriction to translations, i.e., $P_M = e^M$. So we could redefine the paradigmatic theme by the above system $P$, possibly extending the latter to a subcategory of **Mod** which has $P$ as the system of automorphisms.

### 11.7.2 Groups as a Parameter of Historicity

**Summary.** This section terminates the chapter with an abstract of the determinants of size in historical growth of paradigmatic transformation groups.

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By use of the preceding mathematical rephrasing of a paradigmatic theme $P$, we may attach $P$ as a diachronic index to a determined analysis. This means that the a priori variability of admitted isomorphisms—which by default is purely mathematical—could be used as a valuation criterion in view of historical localization; the paradigmatic theme becomes a determinant which traces historical dynamics. And this is by no means restricted to esthesic positions, it can be applied symmetrically on the poietical side of a work, a perspective which is equally affected by historical dynamics.

In fact, the presence of a paradigmatic theme in a defined historical moment—be it in esthesic valuation or in poietic construction—is by no means an invariant of the diachronic axis. On the contrary:

**Principle 6** *The appearance of certain symmetries as transformations in analytical technique or compositional tools is a secure index of a new musical epoch.*

We could even hypothesize that the paradigmatic theme has been monotonically growing since the last five hundred years of European music history.

# Chapter 12

# Topological Specialization

*Musical phenomena come to existence*
*in the constant fluency and motion*
*of compositional creation.*
*Therefore any description of them*
*must finally prove but approximations.*
Rudolph Reti [444, p.12]

**Summary.** In general, transformations will conserve interior relations of objects, but *not* their 'absolute' position, or site, in the ambient space. This aspect is covered by the topological perspective. In its infancy, mathematical topology was in fact called "analysis situs". It deals with the general question of what it means to be in the vicinity of an object. In music, topological considerations are of central importance since slight deformations of objects to neighboring objects are standard identification concepts—though never handled with the necessary care. The point in making these structures precise lies in the sharpening of a fundamental descriptive tool, and in the semantic potential which topology induces. We make explicit the latter topic in a discussion of the problem of topological classification of sounds.

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Although in mathematics, general topology is a basic discipline, musicology and also humanities have not yet understood the deep impact of genuinely topological reasoning. Topology has only penetrated the humanities on the metonymous level of metrical reasoning where topology is not at its best.

Genuine topology is a radical antagonist to transformation or metrics. We want to make this plausible in this preliminary chapter, more details will appear in other chapters, for instance in motive theory (chapter 22) or in inverse performance theory (chapters 45 and 46). The relatively abstract character of topologies may be a reason for their sparse usage in the humanities, but the abstract character is precisely the power of this approach: It is very helpful in creating concepts which are akin to usual fuzzy situations in the humanities.

The following discussion will in particular evidence the advantage of topologies for the delicate and still unsettled problem of sound classification, and we begin with a classic of shape conceptualization: Ehrenfels' gestalt theory.

## 12.1 What Ehrenfels Neglected

**Summary.** Ehrenfels' concept of *gestalt* is characterized by *invariance* under transformations and *super-summativity*. Apart from a marginal remark he did not, however, take into account the question of gestalt stability, i.e., the conservation of the gestalt under more or less small deformations. Without this stability aspect there is no workable concept of gestalt. We discuss the reasons.

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In Christian von Ehrenfels' seminal work on gestalt [136] he has characterized gestalt by two attributes: invariance and super-summativity. The former means that the gestalt of an object does not depend upon the identity of the object. We may replace this object by another specimen which is related to the original object by a determined transformation. In Ehrenfels' example of a musical melody, the transposition of the melody by some semitones does affect its identity but not its gestalt. The second attribute, super-summativity is akin to the old Aristotelian principle that the whole is more than its parts. In the musical example this means that the melodic gestalt is more than the enumeration of its tones, in other words: it is related to the relations among its parts rather than to their isolated presence. We do not discuss this characterization here, more will be said in the global theory and in particular in the topological theory of motives, see chapter 22.

Our present concern is rather that Ehenfels did not give a workable definition of gestalt since he did not specify the problem of gestalt stability. To be clear: we contend that the two well-known Ehrenfels criteria are necessary, but they are not sufficient. Let us discuss this topic on Ehrenfels' example of a musical melody. Suppose that we are given two melodies of equal gestalt, the second being produced from the first by a pitch transposition. Suppose that we have to play these melodies on an ordinary piano. Then, clearly, we will not refuse the gestalt equality of these melodies even if the piano is not tuned in perfect equal temperament. Small deviations of the pitch relations and also small deviations in the temporal reproduction of the second melody against the first one will not bother our gestalt identification.

Now, this invariance against small deformations is neither super-summativity nor transformational invariance. In fact, no difference in the totality of relations is affected. And deformation is not transformation since we do not refer to any transformation rule whatsoever to say that the second melody is 'similar' to the first one. We rather need a concept of distance or neighborhood such that the second melody stays at this small distance or in the neighborhood of the 'prototype'. On the contrary, even the most innocent transformation, such as a transposition, does strongly alter the identity of the melodies constituents: pitch will change dramatically if we transpose by two octaves, for example. Small deformations really have to take care of the original position and small alterations of its coordinates. But the precise quantity of alteration is not relevant, no specification is required—except that it be 'small'. The same observation holds with super-summativity: If gestalt refers to relations among its parts, it is equally true that it is stable under small variations of these relations.

So we may summarize that *gestalt must be preserved under small deformations*. This is not only an argument from cognitive psychology, it is also an argument towards classification in the sense of stable concepts in parameter spaces underlying denotator forms. The point is that not only perception and sensorial processing are subjected to stability requirements.

Abstract conceptual processes are also built upon stability phenomena, and this is one of the valid arguments of René Thom's catastrophe theory, see [526].

## 12.2 Topology

**Summary.** This section introduces the topological argument and its formalism. Basically, music involves two radically different topological processes: comparison of *given* neighboring objects, and "degenerative" specialization of an object into a derived object. Both types of topologies are equally important in music and musicology.

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We refer to appendix H.1 for the mathematical definition of a topological space. Here we suppose that the reader has this prerequisite in his/her mind or that he/she is willing to believe that the following discourse is provided with a rigorous mathematical theory. We concentrate on the characteristics of topological argumentation.

Topology deals with the "logic of toposes"[1], it is a conceptual framework for the "analysis situs", the old title for topology in mathematics. The cornerstone of topology is the concept of a neighborhood. We are given a set $Top$ of 'points' of whatever nature and want to give an axiomatic account of what it means that we stay in a neighborhood of a selected point $x$. All we need is a minimum of properties of neighborhoods. Here are our requirements:

**Axiom 1** *Every point $x \in Top$ has a non-empty system $N_x$ of subsets $U \subset Top$, called* neighborhoods *of $x$, such that:*

1. *For every such neighborhood $U$, we ask that $x \in U$.*
2. *If $U$ is a neighborhood of $x$, then any larger set $U \subset V$ is also a neighborhood of $x$.*
3. *If $U$ is a neighborhood of $x$, there is a special neighborhood $V$ of $x$ such that it also is a neighborhood of every one of its elements $y \in V$, i.e., $V \in N_y$.*
4. *If $U, V \in N_x$, then also $U \cap V$.*

The first requirement is completely natural from common language. The second requirement is not so. It means that if a neighborhood contains the 'very near' points to $x$, then adding other points to these does not alter this specification; the near ones are still there. In other words, if we say that a gestalt of an object x is conserved under small deformations, and if we agree that a neighborhood of $x$ should *by definition* (!) contain those small deformations, then a larger set a fortiori contains those small deformations and therefore still is a neighborhood. So a neighborhood of $x$ is a set of points 'surrounding' $x$ and containing 'small deformation' points around $x$. The third axiom is a 'firewall' against too general a situation as suggested in axiom 2: It says that a neighborhood contains a 'core' neighborhood around that point in the sense that this core neighborhood is also a neighborhood of all its elements. This need not be the case for an arbitrary neighborhood: it is not true that any member $y$ of a

[1] Not toposes in the modern technical sense but in the old, set-theoretic understanding. However, the modern topos theory is nothing else but the functorial globalization of this old approach.

neighborhood of $x$ has this neighborhood as its *own* neighborhood. But the core neighborhood is stable: everyone of its members contains small deformations in this neighborhood. The fourth requirement is evident from common language.

This suggests that we should consider globally 'stable' sets of points in $Top$: By definition, these are the sets $O \subset Top$ which are neighborhoods of all their elements, the so-called *open sets* of the topological space $Top$ defined by the above neighborhood collection. In this language[2], the core neighborhood of $x$ in axiom 3 is an open set (and neighborhood) built around $x$.

The point of this construction is that it is by no means symmetrical with respect to the points of $Top$. More precisely, if we consider any points $x$ in $Top$, the closure $\{x\}^-$ of $x$ is defined as the set of those points $z$ such that $x$ is a member of all their neighborhoods. Intuitively speaking, $x$ is an arbitrarily small deformation such as $z$. We also say that $x$ *dominates* or *specializes to* $z$ iff $z \in \{x\}^-$, in symbols: $x \succ z$. In this setup, it is not true *in general* that specialization is a symmetric relation. This means that there may be points $x$ which are arbitrary small deformations of points $z$, but there are small neighborhoods of $x$ such that $z$ will not appear as a small deformation of $x$ in such neighborhoods. So $x$ is arbitrarily 'near' to $z$, but not vice versa, a situation which is completely pathological if one thinks of naive neighborhoods in normal life (say a person sitting near you in a crowded subway will have the same feeling that you equally sit near this person).

We insist on this abstract point of view since it shows that by no means can we associate topology with transformational paradigms, it is a paradigmatic framework sui generis! And we shall make full use of it. Let us give one concrete and classical example of such a strange

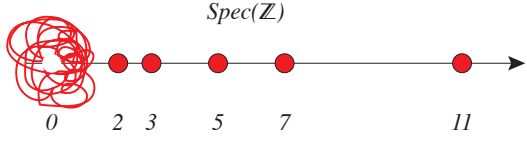


Figure 12.1: The topology of the prime spectrum of the integers has a unique "generic" point, namely the zero, whereas all other primes are "closed" points.

topological space to illustrate the concepts, see figure 12.1. Other examples will be given in the following sections. The example space is the set $Top = Spec(\mathbb{Z}) = \{0, 2, 3, 5, 7, 11, 13, 17, \ldots\}$ consisting of the integer zero and all positive prime numbers, the prime spectrum[3] of $\mathbb{Z}$. We set

$$N_x = \begin{cases} x = 0 & \text{all } 0 \in U \subset Spec(\mathbb{Z}) \text{ with } Spec(\mathbb{Z}) - U = \text{ finite,} \\ x \neq 0 & \text{all } U \in N_0 \text{ with } x \in U. \end{cases} \tag{12.1}$$

So we have a slight asymmetry: All primes are dominated by 0, but 0 is dominated by no prime.

[2] Open sets are a common alternative to neighborhoods to write down the axioms of a topology, see appendix H.1.

[3] See appendix F.2 for this concept from algebraic geometry.

Further, every prime doesn't dominate any other point: We have

$$\{x\}^- = \begin{cases} Spec(\mathbb{Z}) & \text{if } x = 0, \\ \{x\} & \text{else.} \end{cases} \tag{12.2}$$

One says that a point $x$ in a topological space is *closed* iff $\{x\}^- = \{x\}$. So all primes in our space are closed whereas 0 is not, it dominates all other points and is therefore called *generic*.

**Exercise 18** Prove all these claims concerning the preceding example $Spec(\mathbb{Z})$.

This example shows that it is very easy, and mathematically standard, to introduce topologies which are far from the common intuition. The zero is a very 'thick' point compared to the prime numbers which are all topologically closed. Of course it is not a question of introducing artificial topologies, and this one is completely natural as a mirror of well-known facts from number theory. We shall introduce such natural topologies on spaces of motives, for example. But here, we should try to make the a priori point: topology is a very powerful tool for creating similarity paradigms without any allusion to naive distance concepts which are well known to psychologists in the wide-spread polarity profiles, for example.

### 12.2.1 Metrical Comparison

**Summary.** Metrical comparison is based on geometric distances between objects within their ambient spaces. (Not to be confused with temporal metrics of music!) Metrics are quantifications of similarity and generate a very special—thoroughly intuitive—type of topologies. Essentially, this aspect is a neutral comparison of given objects and does not ask for generative procedures to derive objects from each other. The idea of variation is problematic in this topological paradigm.

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In the humanities, above all in psychometrics, metrical comparison arises if a set $S$ of objects is loaded with a non-negative, real-valued distance $d(x, y)$ for every couple of elements $x, y$ in $S$, measuring something like similarity between the couple's objects. In general, such a quantification of similarity can be arbitrarily wild. For instance, if we measure the distance between two pitches in auditory recognition, this can be a dramatic function; for example we may have $d(x, x) > 0$ or $d(x, y) \neq d(y, x)$ if the pitches x and y are presented enough time apart, and if the presentation time is *not* a parameter of the events. See appendix B.2 for more details.

Avoiding such pathological—but realistic—phenomena in psychometrics, let us assume that we have a distance function on a set $S$ of denotators such that it defines a metric [4] on $S$. For example, if $S$ is a local composition in an ambient space which is isomorphic to an $n$-dimensional real vector space $V$, the usual Euclidean metric on $V$ defines[5] a metric on $S$. Such a metric canonically defines a topology[6] on $S$ by the neighborhood sets

$$N_x = \{U | \text{ there exists a positive } \epsilon \text{ such that } B_\epsilon x \subset U\} \tag{12.3}$$

[4] See appendix H.1.

[5] It is however a non-trivial problem to recalculate a possible ambient space for embedding an arbitrary set $S$ which is provided with a metric *on* $S$.

[6] See appendix H.1.

where $B_\epsilon x = \{z|\ d(x,z) < \epsilon\}$ is the $\epsilon$-ball (in $S$) around $x$.

**Example 18** Take the set $S = SERM_{k,0}$ of serial motives of length $k$ with integer values. For such a serial motif $Ser$ we set $\|Ser\|_1 = \frac{1}{k}\sum |Ser_i|$ and then $d(Ser_1, Ser_2) = \|Ser_1 - Ser_2\|_1$. Division by $k$ guarantees that if the series get longer and have only few differences, they will be more similar.

In this setup, if $x$ is in the $\epsilon$-ball neighborhood of $y$, then the same is true $i^{th}$ exchanged roles, the relation is symmetric. However, the relation is not transitive: In general, $y \in B_\epsilon x$ and $z \in B_\epsilon y$ does not imply $z \in B_\epsilon x$.

**Exercise 19** Check this with the above example.

But this means that the "paradigmatic $\epsilon$-equivalence" around $x$ defined by being in $B_\epsilon x$ is not a mathematical equivalence relation. It may happen that an object $z$ is in two different $\epsilon$-paradigms, one of $x_1$, one of a second $x_2$. This is a trivial mathematical fact, but for musicological conceptualization, it is of dramatic impact: If one tries to define pitch by *perceptual* equivalence via differences $\epsilon$ which are just below the famous "smallest just noticeable difference" (jnd), then no pitch concept will emerge since equality will not be an equivalence relation, and this contradicts an elementary requirement of conceptual identification. In other words, there is no such thing as a pitch which can be defined via psychological perception.

So why stick to topologies if one is given metrical functions? What is the added value? The added value is that in general, metrics have an artificial flavor. For example, in the above example, we could have taken another distance function, starting from the Euclidean norm $\|Ser\|_2 = (\sum |Ser_i|^2)^{\frac{1}{2}}$ instead of $\|Ser\|_1$. But the associated topology wouldn't have changed[7]. There are important properties which are intrinsically topological, i.e., they are independent of any chosen metric. For example, the pitches stemming from $(2,3,5)$-just tuning[8] are a dense subset of all pitches: there is no $\epsilon$-paradigm of any pitch without just tuning pitches; this is an important fact for pitch paradigmatics.

Summarizing, the topological paradigms derived from metrics have the symmetry of $\epsilon$-paradigms in common with the transformational, group-theoretical paradigms, but they differ in that transitivity is broken, and therefore, paradigms are not disjoint partitions of the given space. As a rule of thumb, one should observe *that metrics and associated topologies are not good candidates for paradigmatic concept constructions.* However, these topologies are the antagonist of functional relations in paradigms. We have no control of how relata in a given paradigm could have been generated from each other, they are independent, equivalent entities. To put it more topologically: In a topology which is derived from a metric, every point is closed, *dominance is absent.*

Therefore, talking about prototypical objects and variants thereof in this context is a delicate point. Distance does not generate prototypes, nothing is prototypical. The other way round, this says that we lack basic techniques for generating hierarchies among the given objects. A motivic analysis or a sound classification which is built on metrical topologies is completely flat. In particular, memorizing a corpus of objects in this flat world is utterly unattractive, and

[7] see appendix I.1.2, theorem 75 for background.
[8] See appendix A.2.3.

composition (poietical work) from this perspective will never yield more than an associative chaos. If we are to produce basic directives for compositional strategies, metrical topologies cannot be the motor of quality. We come back to this basic fact in the following discourse on sound classification, section 12.3.

### 12.2.2 Specialization Morphisms of Local Compositions

**Summary.** Specialization is related to generative relations between "dominant" and "dominated" objects. The topologies which are instantiated in this context are radically different from the above topologies, associated with metrical structures. The idea of motivic hierarchies is related to this type of topologies. We discuss the specialization graph of three-element motives in $\mathbb{Z}_{12}^2$, and its signification for the Schubert/Stolberg composition exposed in section 11.6.2.

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To make the ideas concrete and intuitive, we start working in the very small category of zero-addressed finite local compositions $K$ in ambient vector space $M = \mathbb{R}^3$. While deforming such a local composition in the sense of metrical neighborhoods discussed in the previous section, say, by moving each point a bit away from its given position, something may happen which is invisible from the metrical point of view, see figure 12.2, namely the relative position of points. This is a problem which does not arise with affine transformation groups since any set of points on a line will also be colinear after the transformation. This is evidently violated by metrical deformation. Our present concern is something between transformation groups

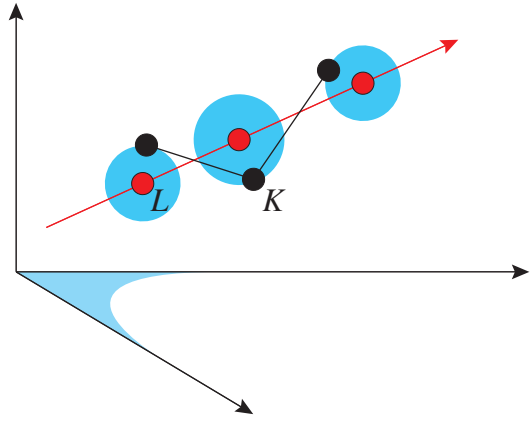


Figure 12.2: Three (light) points in colinear position may lose this property while performing a very small displacement.

and metrical deformations: Transformations from transformation groups are invertible, and therefore sets of colinear points in local compositions are invariants of the group orbits. Look at the topological situation within the space of affine transformations $e^t \cdot m$ which is isomorphic to $\mathbb{R}^3 \oplus \mathbb{R}^{3\times 3}$, the topology being the ordinary metrical topology. Then the set $\overrightarrow{GL}(\mathbb{R}^3)$ of invertible

transformations is open since it identifies to those points $e^t \cdot m$ with nonvanishing determinant, $det(m) \neq 0$. This is a condition which remains true if we change the matrix coefficients of $m$ by a sufficiently small quantity[9]. However, if we approach the boundary of $\overrightarrow{GL}(\mathbb{R}^3)$, we encounter an interesting phenomenon. More concretely, take a curve $m(\lambda)$ of linear transformations

$$m(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

as a function of the real parameter $\lambda$. If $\lambda \neq 0$, $m(\lambda) \in \overrightarrow{GL}(\mathbb{R}^3)$. But for vanishing $\lambda$, we obtain the projection $m(0) = p_{1,2}$, see figure 12.3. The figure shows that if we apply a transformation

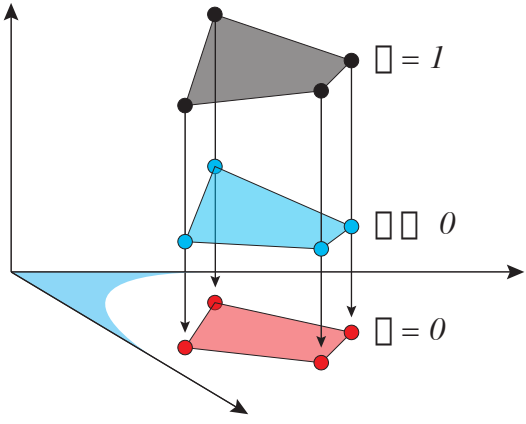


Figure 12.3: A one-parameter family of local compositions, specializing to the plane projection for value $\gamma = 0$.

$m(\lambda)$ of this curve to a given local composition $K$, we obtain a one-parameter family or *curve* of local compositions $K(\lambda) = m(\lambda)(K)$ which approximates the two-dimensional local composition $K(0)$ for $\lambda \to 0$. Whereas all $K(\lambda)$ give us $K$ back via $K = m(1/\lambda)(K(\lambda))$, we cannot retrace $K$ from $K(0)$. The latter is a proper *specialization* of the general member of the curve $(K(\lambda))_\lambda$ of local compositions.

The point is that we have a curve of morphisms $m(\lambda) : K \to K(\lambda)$ which are isomorphisms, except for the special value $\lambda = 0$, where we have a surjective morphism which is not iso. All colinearity relation of $K$ are preserved since we are dealing with affine transformations, but the limit composition $K(0)$ has strictly[10] more colinearity relations than the other $K(\lambda)$, in fact $dim(\mathbb{R}.K) = 2$, in other words, straight lines connecting pairs of points in $K$ which do not intersect and aren't parallel, now, on $K(0)$, do intersect.

[9] The determinant is a polynomial of these coefficients, and this is a continuous function.
[10] At least for our $K$ having its points in general position as in the figure, i.e., $\mathbb{R}.K = \mathbb{R}^3$

The intuition of a 'dominance' of $K$ over the special position projection $K(0)$ can be turned into a topological dominance relation by the following construction: Let us fix a cardinality $n$ of our local compositions $(K, \mathbb{R}^3)$. Then the set $ObLoc^{\mathbb{R}^3}_{n,0_{\mathbb{R}}}$ of all these local compositions bears a reflexive partial order relation $K \succ L$ iff there is a surjective (and therefore bijective) morphism $K \twoheadrightarrow L$, and $K \succ L, L \succ K$ is equivalent to $K, L$ being isomorphic. If $[K) = \{L|\ L \succ K\}$ is the closed halfspace above $K$, we define a neighborhood system on $ObLoc^{\mathbb{R}^3}_{n,0_{\mathbb{R}}}$ by

$$N_K = \{U|\ [K) \subset U\}, \tag{12.4}$$

and thereby obtain the *dominance topology* on $ObLoc^{\mathbb{R}^3}_{n,0_{\mathbb{R}}}$. This definition evidently works on much more general spaces, but the idea is best seen on this concrete example.

**Exercise 20** The core neighborhood of $K$ is uniquely determined, namely $[K)$.

Clearly, the partial order relation $\succ$ is invariant under $\overrightarrow{GL}(\mathbb{R}^3)$ and we may pass to the *quotient dominance topology*[11] on $ObLoClass^{\mathbb{R}^3}_{n,0_{\mathbb{R}}}$. On both these dominance topologies, the *topological* dominance relation $\succ$ introduced in 12.2 turns out to be identical with the synonymous relation introduced in this section.

The dominance topology is an antagonist to the metrical topologies in that it is highly unsymmetrical: the core neighborhoods of two local compositions are equal iff the local compositions are isomorphic. The point of these topologies is that dominance means existence of degenerate transformations specializing the dominant local composition into the dominated one. Musically, this means that we are given a hierarchy of objects which is defined by 'projection' of generally positioned points onto points which can be thought as living in lower dimensional ambient spaces.

A typical example is the harmonic projection of a motif. If we of think our example space $\mathbb{R}^3$ as parametrizing onset, pitch, and duration, in this order, then projecting into the second axis by

$$p = e^{(o,0,d)} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with fixed onset and duration values $o, d$ produces a chord out of a motif. This derivation process is also a standard rule in dodecaphonism when the generic series is instantiated in a score.

We have already alluded to this type of dominance topology in our example 11.6.2 where we worked in the module $\mathbb{Z}_{12,12}$ instead of $\mathbb{R}^3$. There, the dominance was meant as hierarchy among different instances of three-element motives in Schubert's composition. The dominance graph (Hasse diagram shown in figure M.2 of appendix M.3) visualizes all core neighborhoods: for each class of number $x$, the set of classes above $x$ are equal to $[x)$. In general, it is not true that motive classes in $\mathbb{Z}_{12,12}$ have a unique dominant top element. Here, the class number 10 stands for the generic class, and it is also the dominant element in the dactylic grid as shown in 11.6.2. We already saw in 11.6.2 that the presence of the generic class is significantly above probability. So the hierarchy of motive classes is given a remarkable presence in Schubert's setting of Stolberg's poem.

[11] See appendix H.1.

We shall develop the dominance topology of motives in chapter 22. In that discussion, a combination of metrical similarity and dominance topology will be introduced, thereby generating a synthesis of the antagonist views set up in this chapter. So the antagonists are only preliminary stages of a universal topological similarity paradigm.

## 12.3 The Problem of Sound Classification

**Summary.** Sound classification is one of the most complex unsolved problems in musicology. The reasons are triple: First, the communicative determinants of sounds are not clear. Second, sound varieties are—independently of selected sound parametrization/representation—completely unclassified objects. Third, the semantic charge of sounds is a substantial constraint for classification; there is no good classification without semantic constraints.

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We have positioned this subject within the topological paradigmatics because it is above all a topological problem—besides the other aspects, of communication and semiotics, which are also important but cannot be solved without a thorough reflection on types and tools from paradigmatics.

We start our discussion with an account on the topographic determinants of sound description. This is a delicate discussion since its chronically underpinned relevance has led to serious errors in music psychology and psychophysics, such as the catastrophe of the "Fourier paradigm".

We then turn to the problem of describing varieties of sounds, as a function of the chosen sound representation and parametrization. As may be expected, the shape of such varieties changes dramatically depending on these prerequisites. There is no such thing as a universal approach.

The most irritating problem is the topological classification of sounds with respect to its semiotic potential. In fact, good mathematical candidates for metrical or dominance topologies must be valid with respect to meaning of sounds, meaning on poietical and esthesic, on physiological and psychological levels. This coupling of semiotic constraints in sound classification should be the central concern of any classificatory theory. Purely mathematical games are arbitrary and infinite in number.

### 12.3.1 Topographic Determinants of Sound Descriptions

**Summary.** Sound descriptions are not neutral, they depend on the topographic perspective. We discuss the two relevant dimensions: communication and reality.

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In the following sections, we give the descriptive level necessary to deal with the semiotic problem which is the third topographic dimension to be treated subsequently in section 12.3.3.

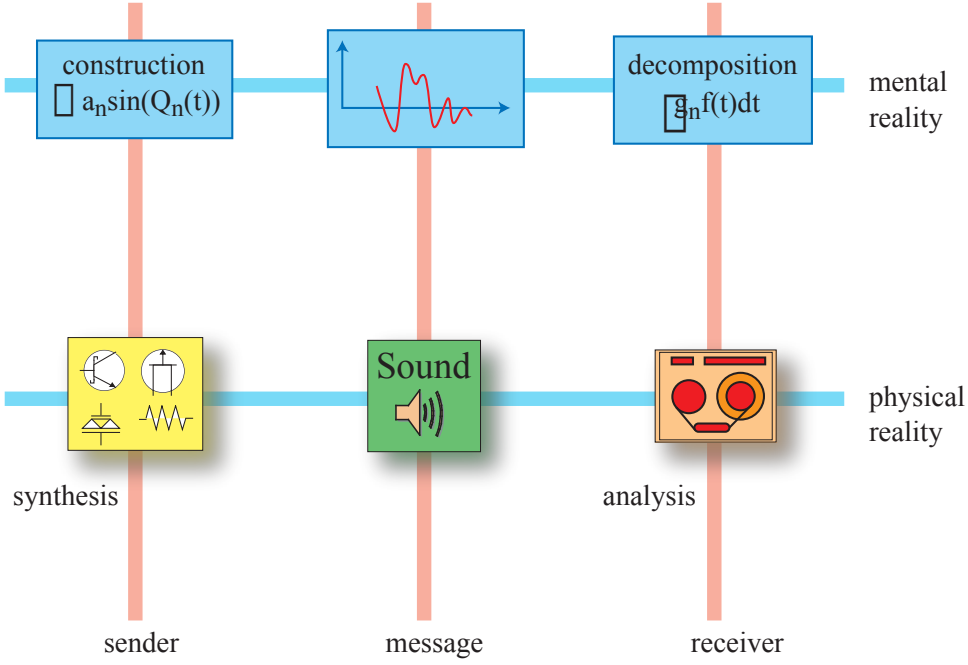


Figure 12.4: The axes of communication and physical/mental realities of sound production (psychological reality is omitted here). The transition from mental to physical reality is more complex than shown here. The technological codification of the purely mental constructs of mathematics are an intermediate layer. Semiotically speaking, this phenomenon corresponds to a connotation/metalanguage structure.

#### 12.3.1.1 Communication

**Summary.** Poietic and esthesic descriptions are independent of each other and may intervene for completely different reasons. The necessity of such a distinction is classically misunderstood in the ubiquitous "Fourier paradigm".

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As a physical object, sound is produced by a specific synthesis machine and received by another analysis machine which is in a completely independent device environment, see figure 12.4. More concretely, the synthesis machine may be a classical instrument, like a violin, or human voice, or an electronic synthesizer built on analog and/or digital principles. The analysis machine may be a human ear, a microphone and a digital or analog analyzer device built on a specific dynamical system or a digital sample recorder. This diversity on the physical level may be understood in its mirror structures on the mental level.

Suppose we are given a process of Fourier synthesis, i.e., the summation of a series of sinoidal curves as described by a circular Fourier denotator as described in section 6.7, example

2. The resulting curve is a time function which can be decomposed by a series of functions which are derived from the sinoidal functions via any isometric linear transformation, for example a rotation in a plane spanned by any two of the sinoidal base curves. The resulting parameter representation of the curve message will then have different coordinates which relate to the original ones by an arbitrary isometry, a rotation, say, in our example.

In order to identify these two aspects, one would have to look at the orbits of Fourier coordinates under the group of isometries in our function space. But this is not the common approach. Fourier ideology (above all on the metaphysical level of so-called "pure" sinoidal sounds, maintained by Stockhausen, for example) does not take orbits but the identical Fourier coordinates. Topologically, this point of view is also unstable: Any tiny rotation of the coordinate system will turn pure waves into impure coordinate sets, or conversely: in every neigborhood of a sinoidal function there are an infinite number of impure functions.

This has dramatic consequences for physical and/or physiological mirrors of this mathematical representation. The synthesis as well as the analysis machine have to be robust under small deformations. For example, if the auditory system really works by Fourier analysis to build its cognitive performance (and not only on the cochlear level, see appendix B for the detailed mechanisms!), then it has to map small deformations of sinoidal waves to sinoidal templates, otherwise, perception of 'pure waves' is illusory.

So communication of sound poiesis to sound esthesis is a topological problem: You have to take care of topologically stable attributes of sound poiesis in order to guarantee its communicability.

A good example of a stable codification of sound phenomena is the phonological system as proposed by Roman Jakobson and Morris Halle in [244] and elaborated to the *Sound Pattern of English* (SPE) system by Halle and Noam Chomsky in [85]. The task of this system is to present the English consonants (e.g., /b/, /m/) and vowels (e.g., /i/, /$\varepsilon$/) as points in a stable coordinate system. The stability has to be a mathematical one and one that can be mapped into acoustic reality.

The $SPE$ is built on a multidimensional digital space. More precisely[12], we set up a form

$$SPE \underset{Id}{\longrightarrow} \mathbf{Colimit}(Consonant, Vovel)$$

with a first cofactor

$$Consonant \underset{Id}{\longrightarrow} \mathbf{Limit}(Syllabic, Consonantal, Sonorant, \ldots Low)$$

consisting of 17 factors of essentially identical form:

$$Syllabic \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}_2), Consonant \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}_2), \text{ etc.}$$

All have the same Bit-type Coordinator $\mathbb{Z}_2$ and differ by name. The values $1, 0$ are traditionally codified by $+, -$. The second cofactor is a six-dimensional product

$$Vowel \underset{Id}{\longrightarrow} \mathbf{Limit}(Syllabic, High, Back, Low, Round, Tense)$$

[12] See [12, p.103ff] for details.

which has the same type

$$Syllabic \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}_2), \text{ etc.},$$

of Bit-typed factors. A *phoneme* is a denotator of form $SPE$, its name is the phoneme's usual name, e.g., the consonant

$$\theta : 0 \rightsquigarrow SPE(-,+,-,-,+,-,-,-,-,-,-,-,+,+,-,-,-)$$

with its coordinates[13] in the cofactor space $Consonant$. This representation is mathematically discrete and the acoustical correspondence to the abstract digital symbols is meant to be produced by a distinctive position/movement of the vocal tract. This implies that any such position is *isolated* with respect to the others in an adequate representation of the vocal tract dynamics. So small displacements do not change the type of physical dynamics in the vocal tract, and there should be no smooth transition between any two of these dimensions. This justifies the digital representation of acoustical dynamics. And it is meant that by identical vocal tract dispositions among humans (at least English speaking individuals) the esthesic interpretation of the symbolism should be unambiguous, i.e., the distinctive nature can be stably replicated on the basis of the receiver's knowledge on personal vocal tract topology.

This latter approach is also interesting because it represents something like a prototype of psychologically oriented topological classification: it is hoped that the continuous space of physical parameters may be split into disjoint regions where discrete values can be attached in a stable way. After all, linguistics has been successful on this path. But music is not linguistics. The variety of musical sounds is far from frozen in a low-dimensional space above $\mathbb{Z}_2$. So communication will have to face dramatic changes on both sides of the message object of neutral sound.

#### 12.3.1.2 Reality

**Summary.** The levels of reality define a strong parameter for sound representations. Psychological, physical, and symbolic descriptions may vary substantially. We discuss the representations inscripted by chronospectra, frequency modulation, wavelet methods, and physical modeling.

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Sound parameters can be symbolized on the string level of pure names. This is the case for the MIDI[14] code, for example. Here, sound types are accessed via program change numbers and these are just numbers for predefined sounds in synthesizers which understand the MIDI code. There have been attempts to give these numbers a certain standard meaning (General Midi, see [399, p.105], for example), but this has nothing to do with the contents of sounds. However, the attempt is oriented towards grouping 'similar' instrumental sounds (such as piano-like sounds in number 0 to 7) on neighboring places on the axis of natural numbers.

To move deeper into the sound classification problem, one has to open a poietic (construction) or esthesic (decomposition) method beyond pure convention. From classical Fourier analysis on real instrumental sounds, it is standard to model the temporal unfolding of a sound

[13] Of course, a grouping of some of these coordinator spaces would have allowed for fewer dimensions and more values ($2^k$ per dimension), but the information is unaltered.

[14] MIDI = Musical Instrument digital Interface, see [379, 399]

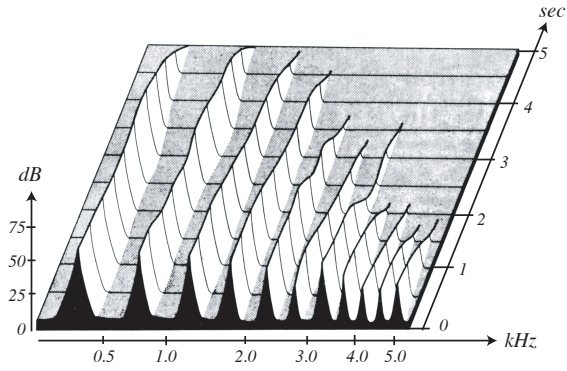


Figure 12.5: The chronospectrum of a trumpet sound, played decrescendo; the dark sections represents the Fourier spectrum at a given time of the sound's execution.

as a function where Fourier components are not constant. Moreover, the amplitude is defined by an envelope curve. We have described a corresponding form *FourierSound* in section 6.6, example 12. In order to generalize that form to variable Fourier spectra, we may replace the factor form *Fourier* by a one-parameter family of such forms $Chrono-Fourier$, parametrized by a variable $\lambda \in [0,1]$ of the unit interval, exactly like the envelope form *Envelope* described in section 6.6, example 11. This defines the form *ChronoFourierSound* as a generalization of *FourierSound*, see figure 12.5 for an illustration.

It is erroneous to think that this is the end of the classical sound description. In terms of instrumental construction, this is only a basis of general instrumental characteristics. Since we shall deal with this subject in chapter 53 on parameters for string quartet, we can stick to a short sketch here. For instruments of the violin family, the form *ChronoFourierSound* is enriched by two classes of parameters: bow application and vibrato.

Bow parameters are split into four groups:

- Bow pressure
- Bow velocity
- Contact point bow to string
- Bow angle.

The vibrato effect is given by four numeric parameters:

- Relative delay time from onset
- Modulation frequency (frequency of finger displacement)
- Pitch modulation (extent of finger displacement on string)
- Amplitude modulation (contact point of finger-tip).

We call these eight parameters the *technical parameters* of a violin sound. They are an additional variety of sound construction superimposed on a given chronospectrum.

The complexity of this poietical perspective of physical sound production is in sharp contrast to what can be analyzed on the esthesic side. However, it shows that the topology of such poietic objects can become quite complex, see next section 12.3.2 for this. The physical modeling technique is related to this situation, it is a strictly poietic construction methodology of instrumental sounds via software modeling of the complete dynamics of instrumental execution. It however goes one step further in that the Fourier chronospectrum is replaced by the simulation of the physical device, classically called a musical instrument. So it models the making of a sound, not the abstraction via mathematical curve synthesis. The high fidelity to natural sounds which physical modeling makes possible has the drawback that the parameter space for sound description becomes increasingly complex. And that a corresponding analysis, i.e., a reconstruction of the defining parameters from the message signal is a very complex mathematical task.

If the Fourier construction is rather easy to decode since the Fourier coefficients can be calculated by classical methods (see appendix A.1.2), frequency modulation (FM) is rather hard, but less hard than physical modeling. FM sound construction is associated with a form *FM-Object* introduced in section 6.7, example 3, see appendix A.1.2.2 for technicalities. It generalizes the classical Fourier construction in that it introduces a hierarchy of sinoidal functions which act as relative modulators in a functional concatenation which can be displayed as a directed graph[15], possibly with cycles. Relating to Fourier construction, there are two important differences in the zoo of these FM-sounds: First, the FM-graph represents a new type of combinatorial variable (with Fourier it was only a discrete set of points). Second, the FM analysis is far from settled: Even if we know about the underlying FM-graph, the FM-coefficients cannot be retraced in the general case, at present, only special graphs show solutions, see appendix A.1.2.2 for this problem. So FM-synthesis is strongly poietic, though less complex than physical modeling.

Wavelets are a last example in our overview. See appendix A.1.2.3 for technical details. In principle, the wavelet method is comparable to the Fourier method in that it parametrizes a given time signal function $f(t)$ by a distinguished system of 'coordinate functions'. We have to fix a generic wavelet $\psi$, for example the Mexican hat function

$$\psi(t) = (1 - t^2)e^{-t^2/2}. \tag{12.5}$$

This furnishes the "wavelet" paradigm prototype of the method. On this function, the affine group $\overrightarrow{GL}(\mathbb{R})$ acts from the left by $\psi_{a,b} = e^b \cdot a \bullet \psi = |a|^{-1/2}\psi \circ (e^b \cdot a)^{-1}$. We then take the orbit $\overrightarrow{GL}(\mathbb{R}) \bullet \psi$ and apply each of the orbit's members as affine deformations of the prototype, thus yielding a system of 'coordinate' wavelets. The coordinate of the test function $f(t)$ with respect to a coordinate wavelet $\psi_{a,b}$ is the scalar product

$$L_\psi f(a, b) = c_\psi \langle f, \psi_{a,b} \rangle_{L^2}$$

(see A.1.2.3). The inversion formula for wavelets says that this representation has an inverse and that it is an isometry, in other words, the wavelet representation of $f(t)$ is faithful and

[15] In YAMAHA's TX802 synthesizer technology, this graphical structure was called the "algorithm" of the FM synthesis. TX802 offered 32 such algorithms.

preserves metrical properties among functions. In contrast to the Fourier representation, no fundamental frequency is needed here, but we have to pay the price of two coordinate variables.

However, the wavelet method also must be concerned with questions of topological stability, in particular the reduction to discrete "frames" which represent the continuous variety of parameters. In fact, the theory of *wavelet frames* (see [308, p.79ff]) deals exactly with this problem. It is, however not evident that the 'cochlear wavelet' yields a faithful wavelet representation of sound (relating to adequate wavelet frames). And it is absolutely unknown whether on the neocortical level, wavelet analysis is performed.

Let us now turn to more psychological sound representations. In the attempt to transpose the linguistic SPE model to music, several authors have described spaces of sound colors. Among others Wayne Slawson [489] and Fred Lerdahl [293]. The common characteristic of these attempts is a shift from the physiologically founded coordinates of the SPE system to frankly psychological perception coordinates such as "acuteness", "openness", "smallness", "vibrato", "inharmonicity", "brilliance", etc.

All these attributes being on the esthesic side, they can hardly transgress the communicative barrier to poiesis. And they are also very fuzzy as such. For example, Lerdahl's "vibrato" is completely wrong as a one-dimensional attribute, we have seen above that four dimensions are needed for a common description. The universe of sounds (let us talk about the poietic construction universe discussed above) is much too broad to be captured by manifestly reductionist words such as "brilliance". The rationale of Lerdahl's construction (see also figure 12.6)

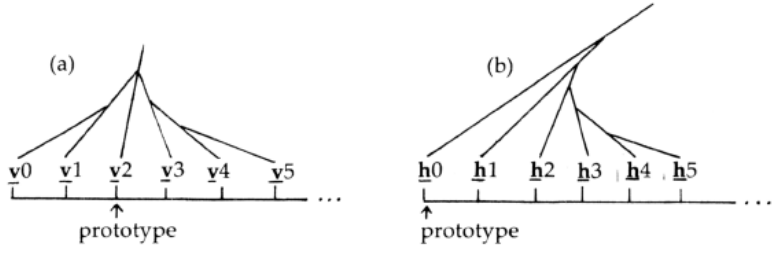


Figure 12.6: Two hierarchical trees from Lerdahl's attempt to impose a linear order and hierarchy on vibrato and inharmonicity.

is a very limited system of ramification trees from that theory which enforces absurd hierarchies without necessity. An example is the similarity between different variants of a vowel described by prolongational ramifications where no hierarchy is given. The generative theory of tonal music is mathematically primitive since it lacks most elementary tools of topology. The vibrato space is at least four-dimensional, and the vibrato types are much richer than an ordered set which is dominated by a "prototype" (left part of figure 12.6). Even if we accept templates of vibrato, these would be trivially conventional for technical reasons of software preferences and could not claim prototypical roles in musical performance.

Even worse is the linearization of "inharmonicity" (right part of figure 12.6) since there is not the least indication of a one-dimensionality in non-harmonic Fourier components (what would be the theorem to which this fuzzy construct refers?). This perceptual model, which is built on a wrong analysis model, is wrong on every single step, and it is induced by systemic

constraints which are completely artificial in this subject. The topological richness would never enforce unique maximal elements since it does not enforce global hierarchies. Topological dominance can have as many maximal elements as necessary. The reduction to hierarchical trees simply destroys facts in favor of graph theory for beginners. Such terrible simplifications from psychology are in dramatic contrast to the complexity of sound description for realistic synthesis and even analysis on the physical (or technological) and physiological level.

We can summarize that the present sound color spaces from psychology are reductionist in their construction, have nothing to do with sound synthesis and analysis whatsoever, have a ridiculously low number (two, three, at most...) of dimensions which are even exceeded by the linguistic system (17 dimensions for consonants in the SPE model), and do not cover any of the fundamental problems in sound description, classification, similarity, dominance, and sound semantics; see also Reinhard Kopiez' review of this situation in music psychology [274].

### 12.3.2 Varieties of Sounds

**Summary.** As a function of the chosen representation, the universe of sounds constitutes a more or less difficult variety. We look at possible shapes of such varieties and their meaning for the classification problem.

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Already for classical Fourier construction, the infinite-dimensional coefficient space of the amplitude and phase spectrum $(A_i, Ph_i)_{0\leq i}$ is difficult to turn into a reasonable manifold. In fact, the naive Euclidean distance does not express more than local similarity, whereas one is more interested in topological description of 'dominant regions', i.e., locally maximal coefficients in the sense of formants which are also responsible for vowel recognition[16]. This means that one is looking for indices $b$ such that the amplitudes have local peaks at index $b$, i.e., $A_i < A_j$ for couples of indices $i, j$ with $a \leq i < j \leq b$ or $b \leq j < i \leq c$. Such a formant condition defines an open formant set $U_{a,b,c}$, and each[17] formant set is given by the intersection

$$U_{a_1,b_1,c_1} \cap U_{a_2,b_2,c_2} \cap \ldots U_{a_k,b_k,c_k}$$

of all local peak neighborhoods of an increasing sequence of index intervals $a_1 < b_1 < c_1 \leq a_2 < b_2 < c_2 \leq \ldots a_k < b_k < c_k$. If we agree that such neighborhoods are regions of reasonable sounds, the relation between different such regions is not clear: Possibly, there are no reasonable sounds relating these regions, possibly are there many different paths from one such region to another. So we are left with complex problems regarding subsets of reasonable candidates and the manifolds which are defined by these subsets, see also figure 12.7. It is not even clear whether these manifolds are of a determined dimension since boundary conditions could define closed submanifolds, e.g. by inverse images of functional equations $F(A.) = 0$ or energy constraints (e.g., upper limits $\sum A_i^2 \leq const.$). A serious look at these problems is exactly what Lerdahl missed in his work [293] at IRCAM where he applied the CHANT program to deduce his broken sound color space. In fact, CHANT is exactly what would be necessary to make experiments which are based on serious theories of formant manifolds.

[16] We stick to discrete Fourier spectra for simplicity

[17] Supposing for simplicity and experience that no two successive amplitudes are equal.

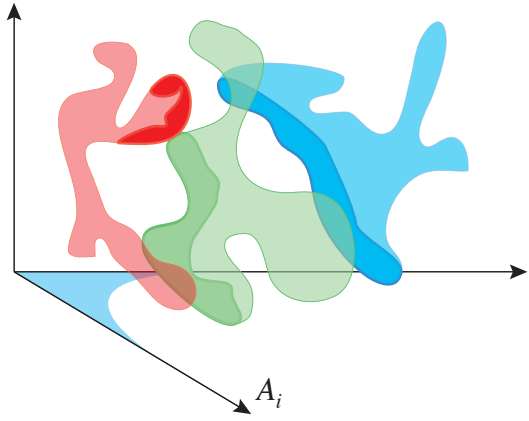


Figure 12.7: In the infinite-dimensional space of Fourier coefficients, the relevant sounds can define a subset which looks like an inhomogeneous variety, a patchwork of open manifolds and closed connections, defined by algebraic or analytical constraints on energy, spectral formant distribution etc.

What was only defined by the selection of relevant candidates is intrinsically built in for FM objects. We have a double description level of these objects. An *FM-Object*-formed denotator $myFM$ has its FM-graph $\Gamma(myFM)$[18] and the partial coefficients $(A_k, F_k, Ph_k) \in \mathbb{R}^3$ for each vertex $k$ of $\Gamma(myFM)$. So the variety of FM-sounds is parametrized by the directed graphs, and is an open set in $\mathbb{R}^{3v_\Gamma}$ for the vertex cardinality $v_\Gamma$ of the parametrizing graph $\Gamma$. This splits the FM-sounds into disjoint sets and is not a workable environment for the totality of FM-sounds since we are not yet able to compare sounds of different graphs.

The point is that graphs are discrete objects which we should embed in a topological space in order to manage deformation of FM-objects. In fact, a graph arrow $a : v_i \to v_j$ in $\Gamma(myFM)$ symbolizes that the partial at $v_i$ acts as one of the modulators on the partial at $v_j$. The strength of this action is not variable, if it does not act, we simply omit it. So this is a digital reduction which generalizes in a canonical way. Instead of sticking to graphs $\Gamma$, we weight their arrows by non-negative real numbers, writing $w(a)$ for the weight of arrow $a$ We therefore consider weighted graphs $\Gamma^w$. If we denote 1 the weight having values 1 for all arrows, then the unweighted situation for FM-sound $myFM$ was equivalent to the situation with $\Gamma(myFM)^1$. For a general weight $w$ on $\Gamma(myFM)$, we get the generalized FM-object $myFM^w$ which refers to $\Gamma(myFM)^w$. We then have an obvious projection

$$p : myFM^w \mapsto myFM_w \tag{12.6}$$

which by definition replaces each partial $(A_v, F_v, Ph_v)$ at the tail $v$ of arrow $a$ by the partial $(w(a)A_v, F_v, Ph_v)$ and generates a normal (weight 1) FM-sound from a weighted object. In other words, we have a $p$-section $\sigma_1$ of the set $\Gamma^1$ of 1-weighted FM-objects of given graph $\Gamma$ in

[18] See section 6.7, example CircDen-3, for the definition of the FM-graph $\Gamma(myFM)$.

the space $\Gamma^-$ of weighted FM-objects of given graph $\Gamma$. The composed map

$$\Gamma^1 \xrightarrow{\sigma_1} \Gamma^- \xrightarrow{p} \Gamma^1$$

is the identity, and the subspace $\Gamma^1$ is defined by the closed condition $w = 1$. But on $\Gamma^-$, we have the action of the multiplicative group $\mu^{a_\Gamma}$, the $a_\Gamma$-th power of the multiplicative group of the base field $\mathbb{R}$, where $a_\Gamma$ is the arrow cardinality of $\Gamma$: An element $(t_a)_{a=\text{arrow}}$ acts factorwise on the weight $w$. If $myFM$ is an object in $\Gamma^1$, its orbit $\mu^{a_\Gamma} \cdot myFM$ is irreducible in the Zariski topology[19]. In this setup, if we are given an object $myFM$, we may ask for objects

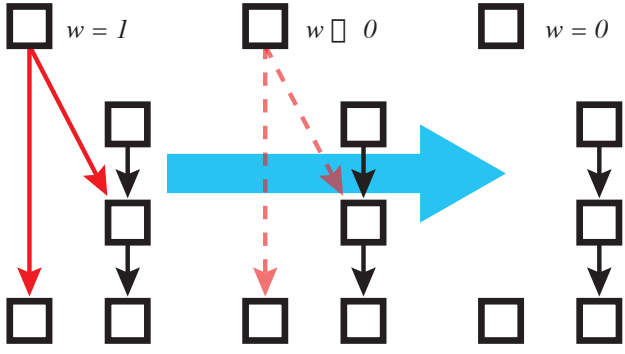


Figure 12.8: Cutting off arrows means specializing weighted graphs, here for two arrows starting at the same vertex.

$myFM^*$ which are dominated by the orbit of $myFM$. This means that the generic point of $\mu^{a_\Gamma} \cdot myFM$ dominates the generic point of $\mu^{a_\Gamma} \cdot myFM^*$. Such a specialization is obtained, for example, if we let the weight of arrow $a$ of $\Gamma(myFM)$ vanish. Then, the specialized FM-object has no contribution from this arrow, see figure 12.8. So we can integrate different graphs into a generic graph in such a way that they are defined by cutting adequate subgraphs. This idea was already introduced in the classification of TX802 sounds in the first software prototype MDZ71 of *presto*®, see [340, chapter 10]. That approach divides the 32 algorithms from TX802 into six *families*, according the number 1-6 of root partials, and then, within each family, a specific graph defines a *species*. Within one species, two graphs $\Gamma_1, \Gamma_2$ are called *kindred* if they are derived from each other by moving one arrow. This means that they are both specializations of a weighted object $\Gamma^w_{1,2}$ in such a way that there are two arrows $a_1, a_2$ in the latter which yield $\Gamma_1$ or $\Gamma_2$ if $w(a_1) \to 0$ or $w(a_2) \to 0$, respectively, see figure 12.9.

Though these methods yield valid and precise topological classification of FM-sounds, the perspective is strictly poietic and mental/technological. It is known that the FM construction is far from evident on the psychological level: Controlling the FM parameters when building such a sound is very difficult if one aims at constructing a sound which one "has in mind". This means that a vocabulary switch between the preceding topology and the psychological intuition on instrumental colors is needed.

[19]This classical algebro-geometric topology is also given by the prime spectrum of a ring, see appendix F.2.

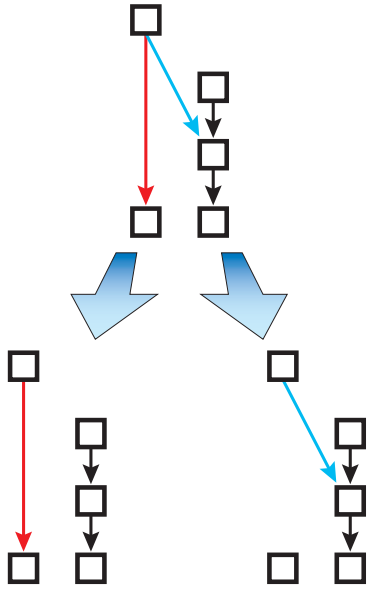

Figure 12.9: Two graphs (from the TX802 algorithms) are in kindred relation if they are specializations of a dominant third graph and are mutually related by displacement of a particular arrow.

To complete this short remark on sound varieties, let us mention an important technique which will be applied in the discourse on string quartet theory 53. Although sound varieties are of infinite dimension, it is often reasonable to select subvarieties of relatively small dimension. In these subvarieties, classification may become easier and more akin to psychological or semantic requirements. It should however not be a selection which is dictated by technical incompetence—such as the broken two-dimensional space constructed by Lerdahl in [293]—it should be defined by rational criteria and as a such a variable selection, depending on the context. For example, for questions concerning a fixed orchestration (a string quartet, a vocal ensemble, etc.), selecting a sound subvariety which contains the corresponding instrumental colors will do, but see [293] for details.

### 12.3.3 Semiotics of Sound Classification

**Summary.** Sound classification has to cope with constraints of meaning of sound. This concerns above all the topological properties of sound varieties. They are a function of the type of musical compositions and their instrumentations. They also depend on the semantic charge which the composer imposes on his/her message.

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Topological classification of sounds is not only a matter of mathematical description of sound varieties on specific levels of reality. The point is that from all possible topologies, metrics,

etc., we are interested in those which are able to carry a semantic function. They are good if they reflect meaning of sounds. Meaning on any level: auditory physiology, perceptual category, emotional quality, or compositional/analytical function.

So we are confronted with a complex problem: To map topological sound varieties (varieties of timbral colors, the name is irrelevant) to semantic manifolds—such as emotional landscapes—with their topologies, in such a way that the mapping becomes continuous or even an isomorphism of topological spaces. Or to define metrics on these manifolds such that geodesics correspond to semantically shortest paths. In the example [293] of the string quartet, it is shown that general position of timbral points in the violin family can help to represent contrapuntal and harmonic structures of classical compositions by Joseph Haydn or Luigi Boccherini.

Unfortunately, no significant research has been done to investigate possible maps and associated timbral topologies. The question is highly non-trivial since it is not only a problem of testing topologies, but of defining them in order to settle semantical constraints. Regarding Jakobson's poetical function, the problem is related to the definition of topological paradigms of sounds which turn emotional syntax of musical compositions into poetical syntax. We know the emotional syntax (not so sure, though...), we know the timbral spaces, we know that the emotional syntax is poetically instantiated on the sound colors of musical compositions. But we do not know which timbral topologies and/or metrics turn music into timbral poetics.

## 12.4 Making the Vague Precise

**Summary.** Summarizing, topological methods of musical paradigmatics are powerful tools for grasping blurred and ambiguous concepts in musicology. Transcending the evident structural task by far, topological considerations unveil deep problems of communication and semantics of music.

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The foregoing reflections are far from systematic. We are at the very beginning of an exact theory of topological classification of musical objects. But we have a good technical basis, and we shall see that in the theory of motives, global metrics, and in the string quartet theory, these techniques can be successfully applied to construct basic concepts, such as "motivic gestalt", "metrical rhythm", and to give good and precise answers to the intriguing question why the string quartet came up to primordial prominence so suddenly in the middle of the eighteenth century.

Topological considerations are not only good technical classification tools, they are also the key to a deeper understanding of many problems of musical communication and—above all—semantics on all levels of reality: mental (composition), physical (auditory perception), and psychological (emotion). It is also clear that the present state of our theory, namely the strictly local point of view, is not sufficient to cope with more refined topological situations. We encountered many situations where the word "manifold" or "variety" was the good pointer, however, this was a more mathematical flavor, and we still lack an intrinsically music-related approach to global views. So we are well prepared to envisage this new challenge: *What are musical manifolds?*

# Part IV

# Global Theory

# Chapter 13

# Global Compositions

*. . . bei der Bestimmung der Großform wirkt*
*über den ganzen Eingliederungsprozeßhinweg*
*die gleiche* Denkweise*, und zwar*
*von der morphologischen Mikrostruktur*
*bis zur rhetorischen Makrostruktur.*
Pierre Boulez [60, II, p.60]

**Summary.** The categories *ObLoc* and *Loc* are representations of local objects, i.e., of objects that are not composed of proper "parts". However, music mainly deals with compound objects. The adequate concept of a global composition is defined. The corresponding vocabulary of elementary global music objects is described, including ecclesiastical modes, tridadic degrees, meters, rhythms, motives and themes.

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This chapter represents a major segmentation point of this book since it introduces the paradigm of global objects into the topos of music. This is a mandatory step as the nature of music is an organically compound one from all topographic points of view. Composition, analysis, interpretation, cognition, understanding, and performance, all of them are immersed in an omnipresent dialectic of analysis into local parts and synthesis of a global whole. However, compared to the investigation of exterior nature, it is very difficult to separate analytical or synthetical activity from the object of investigation. The uncertainty relation in physics, stating that every experimental interaction with nature is subjected to a complementarity between hidden and revealed information, this relation becomes a dominant and even characteristic attribute of music as an expression of human nature.

The investigation of the musical object tends to be an integral part of the object. The composition's germs, the parts of an analysis, the shaping elements of a performance, and the units of cognitive processes, all of them define works and musical systems as multiply varied perspectives of a whole. We know from our previous discussion of Yoneda perspectives in section 9.4 that the identification of a musical work involves an integration of multiple perspectives. This integration is in particular one of local points of view, a "patchwork" of multiply zoomed

micro-views. But the huge variability of such views is not a drawback or defect of music and its theory, on the contrary, is substantial:

**Fact 7** *Music is not the definitely ambiguous, it is the definiteness of ambiguity.*

On the level of music performance, this general insight was pronounced by Theodor Wiesengrund Adorno [6]: "Musik interpretieren: Musik machen." Music is only instantiated *via* its interpretation or performance.

A theory which crashes on the ambiguity of its object in misunderstanding it as an indeterminacy cannot, contrary to Carl Dahlhaus' judgment [100], be an adequate one. Rather the problem is to set up an exact theory of ambiguity. This is the methodological background of the theory of global compositions. There is no way out of making the multiplicity of compositional and interpretative activities a part of the theory's concerns.

The objects of the global theory are mainly global compositions. This concept generalizes that of local compositions in the sense that it takes care of and formalizes ambiguity. In an important special case—where we talk about interpretations—such global objects precisely represent the result of interpretative activity. The possibility of crystallizing selective actions in mathematical objects encompasses their structural comparability, a possibility which will be dealt with in chapter 14 about morphisms between global objects. This achievement is interesting for musicians and musicologists since a theory of ambiguity also favors mediation between various possible explications of a given work or analysis. Moreover, the precise setup of morphisms qua objects of comparative studies implies an objective (meta)discourse upon comparative studies.

The fact that we were only able to define very elementary objects in the previous theory reflects the musicological fact that ambiguity enters very early in music and its science. Most of the relevant concepts, such as tonality, degree, tonal function, etc., refer to objects which are composed of local ingredients. The following global theory will compensate for this defect. The power of the global approach is, however, proportional to the power of the local components: If we had not prepared a universally valid local concept framework, the global theory would also fail. Recall that we had pushed the local theory to the extreme of its potential and then recognized (regarding the non-existence of general colimits, see 8.3.3) that there is a limit of the local approach which must give rise to a global context.

We should stress that this global approach is not intended as a normative standardization of musicological language, but as a precise methodology of language formation. We also believe that musicology cannot survive in the present status quo where essential competences and problems regarding the intertwining of object and subject are not lifted to a scientific level but still reside in the lowlands of feuilletonistic entertainment. The widespread belief that ambiguity and polysemy in music is a firewall against precise science unmasks its representatives: *There is no ambiguous theory, there are only ambiguous theorists.*

## 13.1 The Local-Global Dichotomy in Music

**Summary.** This section motivates the local/global dichotomy in music. Composition, score representation, analysis, and performance do use this dichotomy—virtually without clear-cut

conceptualization or even consciousness. Built on the Aristotelian tradition, Ehrenfels' criterion of super-summativity introduces the dichotomy into psychology. The question of a precise description of the difference between the "sum of the parts" and the "whole" in Ehrenfels' criterion is addressed. More generally, it is evidenced that specification of parts in music objects is not accidental for understanding the very nature of music. We discuss the radical difference between the local/global dichotomies in music and mathematics. A short historical account on the simultaneous appearance (1854) of these approaches in musicology and mathematics with the work of Eduard Hanslick and Bernhard Riemann is given.

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Let us introduce the subject by the summary of a (non-exhaustive) series of approaches to global structures in music(ology), including Boulez/Webern, Ude/Wieland and Marek, Hofmann and Kaiser, Graeser and Ruwet/Nattiez, Jackendoff/Lerdahl, Xenakis and other music program designers.

**Boulez/Webern.** This approach deals with the composer's perspective. Evidently, a musical composition is never created as a local composition. In classical European literature, e.g., for sonatas, string quartets, or symphonies, which consist of $10^4$ to $10^5$ tone events, it would be absurd to start from such immense local compositions. Rather does the composer start from 'small' local compositions, such as motives, themes, chords, rhythms and similar elements as a basis of the 'creative combinatorics' and then merges these parts or recombinations thereof by use of various transformations and deformations in order to build a compound whole.

A good example is Boulez' example of a dodecaphonic series in [60, I]: He describes a series together with its internal structure[1], i.e., its composition from partial series and their transformations, see figure 13.1. Boulez' reflections refer to the local composition *Boul* in ambient space $OnPiMod_{0,12}$ of integer onset and common pitch classes. We see local subcompositions $A, A_1, B, G, H$ of *Boul* and some transformations, drawn as subsets of points which are surrounded by rectangles, defining a total of 15 partial series. These subcompositions are not disjoint in general, they define a rather complex overlapping covering of the given series. We come back to a more systematic and formal discussion of the covering configuration of this example in section 14.4.

In his discussion of compositional principles with Schönberg, Berg, and Webern, the idea of hierarchies of local compositions is made explicit by Boulez [60, I, p. 86]:

> *Bei Webern findet sich der Keim einer äußerst fruchtbaren Idee, die die Reihe als Einigungsfaktor von* Untergruppen *und* Obergruppen *betrachtet. Im Effekt sind alle isomorphen Figuren einer Grundstruktur von der Tatsache abhängig, daß sie sich immer innerhalb der gleichen Ordnung abwickeln, entsprechend den gegebenen Transpositionen und Umkehrungen; sie integrieren sich in die* Conditio sine qua non *der Zwölftonreihe: die chromatische Totalität. Diese isomorphen Figuren bilden die Basis von privilegierten Mengen, die ihrerseits auf einer höheren Ebene wiederum das vorstellen, was die isomorphen Figuren selbst innerhalb der Reihe bedeuten. Durch Verkettung fügen sich*

[1] German: "Binnenstruktur".

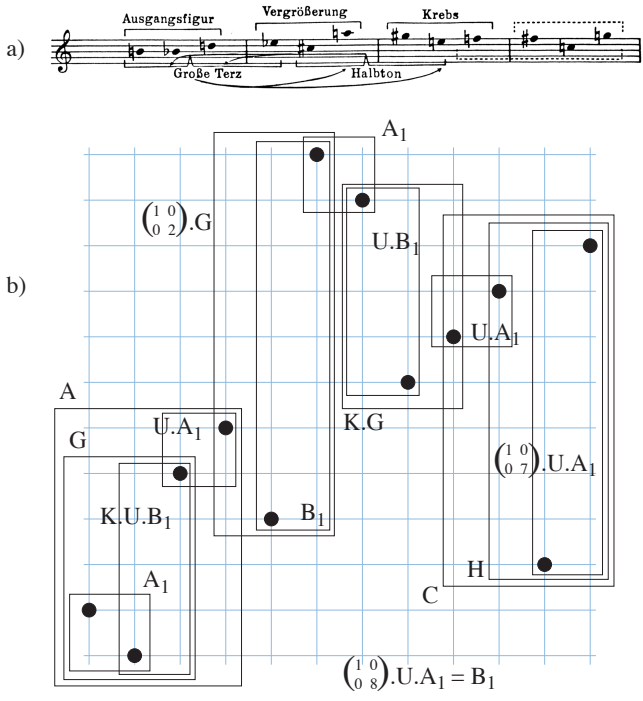


Figure 13.1: a) Internal structure of a dodecaphonic series following Boulez [60, I], with kind permission of Schott-Verlag; b) representation of the series as a local composition in ambient space $OnPiMod_{0,12}$. Frames are drawn around partial series as suggested by Boulez (part a) of this figure), which are related to each other by symmetry transformations (U = inversion, K = retrograde).

> *Reihenformen, welche festgelegte Privilegien besitzen, zu einer Ganzheit, einer in gewisser Weise "höheren Reihe" zusammen. Die Grundreihe kann dann als strukturelle Kraft der* Vermittlung *zwischen Unter- und Obergruppen betrachtet werden.*

So it is evident that Boulez had learned from the second Viennese school that there is a strong local-global principle in serial composition.

**Ude/Wieland and Marek.** The art of performance is not accessory, but essential in the constitution of a musical work; this was already pointed out by Adorno [6]:

> *Musikalische Interpretation ist der Vollzug, der als Synthesis die Sprachähnlichkeit festhält und zugleich alles einzelne Sprach-ähnliche tilgt. Darum gehört die Idee der Interpretation zur Musik selber und ist ihr nicht akzidentiell.*

According to Uhde's and Wieland's comment in [535], the meaning of "alles einzelne Sprach-ähnliche" is a semantic moment[2], the significate of the involved signs. Thus, music need not be decoded, but requires "imitation" of itself [3].

Semiotically speaking, Uhde seems to refer to Jakobson's poetical function (see section 11.6.1) as a projection of the paradigmatic axis to the syntagmatic axis of the sign system. A semantic which is independent of the poetical function is excluded here. This view of musical performance as a poetical oriented activity is essentially *syntactical articulation and paradigmatic intertwining*. It establishes an articulated whole, built from local, elementary parts of the composition.

In practical performance theory, e.g., in Ceslav Marek's standard work "Lehre des Klavierspiels" [317], the basic insights of Adorno and Unde/Wieland are realized in the artisanal details. We choose the example of dynamics and phrasing which is built upon criteria of verse poetics (see figure 13.2). Here, the metrical grouping of the melody creates

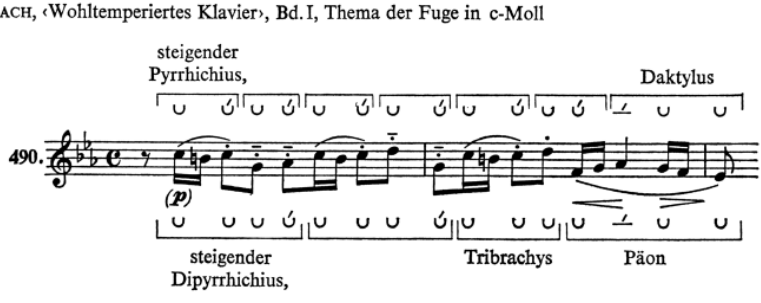


Figure 13.2: Two grouping proposals by Marek for dynamics and phrasing at the beginning of the C-minor fugue of Bach's *Well-Tempered Piano I*, based upon criteria of verse poetics (from [317], with kind permission of Atlantis-Verlag).

successive parts which induce the shaping in performance by dynamical and phrasing (legato/staccato) prescriptions. This level of grouping into local units is also essential for mnemotechnical purposes and in order to give the fingering strategy a support. In fact, it would be bad piano playing to phrase against the fingering strategy.

**Hofmann and Kaiser.** After the foundation of music criticism by Mattheson, Rousseau, or Avison, to name some of the important contributors, it was the merit of Ernst Theodor Amadeus Hoffmann—above all in his famous review of Beethoven's *Fifth symphony* in the Leipzig-based "Allgemeine Musikalische Zeitung" in 1810—of having given music criticism importance as a contribution to the esthesic identification of the work. Since this achievement, music criticism has been cultivated by profiled critics, such as Eduard Hanslick or Joachim Kaiser, and has contributed to the self-estimation of music and to the history of its reception.

[2] Uhde: "meinende Intention".
[3] Uhde: "erheischt Nachahmung ihrer selbst".

It happens only as nonprofessional side-effect that this criticism is eager to celebrate the "uniquely valid" interpretation and performance, or else to lament its vanishing. Although Kaiser may be right in his Beethoven book [257] that whereas it is a rare event in our days when a fundamentally new reading of the 32 Beethoven sonatas is presented, it is a never-ending objective in this field of classical literature to set forth new approaches to the spiritual torso of these sonatas, approaches which add to the given ones new perspectives, variations of explanatory power. In the media and concert business, the music critics should do exactly the job of commenting on this process and of comparing, questioning and evaluating the singular approaches in the spirit of a work in progress towards understanding an infinite evolution.

Now, if the poietic work of the artist is expressed by articulation and correlation of the composition's local parts, then this a fortiori is the work of the critic—only on the level of esthesis. The overall impression of a performance integrates knowledge, prejudice, and personal disposition, which may result in tiny local effects on agogics, dynamics, articulation, and tuning. This is a central effort to open the access of a broader public to the present work.

With the technology of saving works on LP, CD, MC and other media, the articulated listening to music which was founded by music criticism has been enriched by a new aspect: multiply repeated perception. Listening repeatedly to a work on CD changes one's articulation and grouping activity. Every new listening changes or questions the relevant local compositions and their mutual relations, some are eliminated, others are added. Successively, the listener accumulates a *patchwork of elements of comprehension.*

**Graeser and Ruwet/Nattiez.** On the opposite side of the composers stand the musicologists whose efforts for an adequate analysis are characterized by the need to retrace the composer's thoughts. In this respect it is not astonishing that in the analysis of musical works, organically composed hierarchies are common structures. They start from small local elements, such as chords, degrees, tonalities, tonal functions, voice leading, contrapuntal and harmonic progressions, and end up with large local compositions, such as exposition, development, recapitulation, and coda in the sonata form. This principle of hierarchical organisms was explicitly put into evidence in 1924 by Wolfgang Graeser in his analysis of Bach's *Kunst der Fuge* [194]. He describes a contrapuntal form as follows ([194, p.17]):

> *Bezeichnen wir die Zusammenfassung irgendwelcher Dinge zu einem Ganzen als eine Menge dieser Dinge und die Dinge selber als Elemente der Menge, so bekommen wir etwa das folgende Bild einer kontrapunktischen Form: eine kontrapunktische Form ist eine Menge von Mengen von Mengen.*
>
> *Das klingt etwas abstrus, wir wollen aber gleich sehen, was wir uns darunter vorzustellen haben. Bauen wir einmal ein kontrapunktisches Werk auf. Da haben wir zunächst ein Thema. Dies ist eine Zusammenfassung gewisser Töne, also eine Menge, deren Elemente Töne sind. Aus diesem Thema bilden wir eine Durchführung in irgendeiner Form. Immer wird dies Durchführung die Zusammenfassung gewisser Themaeinsätze zu einem Ganzen sein, also eine Menge, deren Elemente Themen sind. Da die Themen selber Mengen von*

> *Tönen sind, so ist die Durchführung eine Menge von Mengen. Und eine kontrapunktische Form, ein kontrapunktisches Musikstück ist die Zusammenfassung gewisser Durchführungen zu einem Ganzen, also ein Menge, deren Elemente Mengen von Mengen sind, wir können also sagen: eine Menge von Mengen von Mengen.*

The explicit reference to set theory is historically interesting since set theory was a relatively new language in mathematics in 1924. The text is somewhat misleading since it suggests that tones are abstract objects. This is however not the case: Graeser views tones as points in a geometric space, and he also recognizes the role of symmetry transformations, like rotations and reflections, in such a space, transformations which may be applied to alter sets of tones or to compare different tone-sets ([194, p.13]):

> *Gegenstand der Untersuchungen sind aber nicht die Töne selbst, denn deren spezielle Beschaffenheit spielt gar keine Rolle, sondern die Verknüpfungen und Verbindungen der Töne untereinander.*
>
> *Es wird uns interessieren, ob wir gewisse Analogien zwischen den Gebilden, die man in der Geometrie aus Punkten aufbaut, und unseren aus Tönen hergestellten erkennen können.*
>
> *Das wichtigste Grundprinzip der festen Körper, und die Geometrie ist nichts anderes als das Studium der festen Körper, ist die Eigenschaft der Symmetrie.*

The Paris school of structuralist linguistics has applied results of semiology after Saussure and Jakobson to musical analysis, above all in the investigations of Nicolas Ruwet [466] and Jean-Jacques Nattiez [393]. The method of *neutral analysis* which was developed by these authors starts from a hierarchical ordering into units and subunits which are a function of the given work. These units are associated with each other by certain equivalence relations:

> *Les divers unités ont entre elles des rapports d'équivalence de toutes sortes, rapports qui peuvent unir, par exemple, des segments de longueur inégale — tel segment apparaîtra comme une expansion, ou comme une contraction, de tel autre — et aussi des segments empiétinant les uns sur les autres.*

The equivalence relations are not arbitrary but realized by specific transformations:

> *Les unites paradigmatiquement associées sont équivalentes d'un point de vue donné (le thème paradigmatique), rarement identiques, et reliées entre elles par des transformations qui décrivent les variants par rapport a des invariants.*

This language resembles the one which Graeser seems to aim at. However the geometric aspect—embedding tones in spaces which admit symmetry transformations—is more radical with Graeser, though less flexible regarding the transformations which may be applied (Graeser limits his approach to transformations of the "rigid" geometry, i.e., isometries).

**Jackendoff/Lerdahl.** Explicit grouping concepts are described by Ray Jackendoff and Fred Lerdahl in [243]. Grouping is described from an esthesic point of view of music psychology. It deals with portions of notes which are heard as building a unit of hearing. In this

approach, a group is always defined by a time interval, i.e., a group consists of strictly time-adjacent notes and cannot be restricted to proper subsets within time-slices which would be defined by voice splitting or parametric splitting of pitch or loudness, for example. Also this grouping is a nearly perfect hierarchy under inclusion: Overlapping neighboring groups may only contain one common onset and are treated as very special situations in this theory. A more general "web of motivic associations" would be beyond the theory *because* it is not hierarchical [243, p.17].

This approach resembles Graeser's concept of a contrapuntal structure which is also a hierarchical grouping of parts, but Graeser's idea was more general insofar as it did not strictly ask for time slices. It is also less paradigmatic than Graeser's, Ruwet's and Nattiez' because no significant statements are made concerning the association of groups under symmetry transformations. From the remarks in [243, p.286] it follows that these authors have no concept of the transformation groups which may be adapted to specific contexts, as it is explicitly proposed by Ruwet and Nattiez under the flag of "paradigmatic theme" (see 11.7.1 for this concept). But it is precisely the psychological claim of grouping—even in its strictly hierarchical appearance of the Jackendoff-Lerdahl theory—as a cognitive basic which undermines the fact of global structures in music.

**Schaeffer and Cage.** In the last years of the 1940s of the $20^{th}$ century, the new technology of tape music became a paradigm for local-global constructs. The American music for tape movement was initiated by Otto Luening and Vladimir Ussachevsky (figure 13.3) and applied by John Cage in the early fifties[4] since his first music for tape *Imaginary landscape*. At the same time, in Paris, Pierre Schaeffer and Pierre Henry initiated the "musique concrète" movement (figure 13.4), also based on tape as a flexible medium of syntactical combination and recombination.

Figure 13.3: The founders of American music for tape movements, Otto Luening and Vladimir Ussachevsky. Their achievement was applied by John Cage and others in the *Project of Music for Magnetic Tape*.

Figure 13.4: Pierre Henry produced his first work in musique concrète, *concerto des ambiguités* in 1950. The photograph shows him while realizing this music to the ballet *Voyage au coeur d'un enfant* by Maurice Béjart

[4] In the *Project of Music for Magnetic Tape*, together with Morton Feldman, David Tudor, and Christian Wolff.

Tape as a new medium had become a tool for concretely cutting and merging time-slices of music. This was a starting point for an entire group of technological realizations of local-global patchworks. At present, it is largely extended and refined in various software for musical composition, notation, and postproduction as compared to hard disk recording. The local parts are termed tracks, parts, global and local scores, etc. We shall review one of these approaches in chapter 49 when discussing the composition software *presto*®.

But the tape music movement had not only consequences for music software, it also changed the concept of a score. In particular, Cage realized compositions, such as his *concert for piano and orchestra* (1957), where the modular structure of the composition, showing a number of relatively autonomous local parts, and an intended ambiguity of identifying such parts, became an explicit and primordial feature of poiesis.

In the light of these rich traces of a local-global paradigm in music, Ehrenfels' approach to gestalt which stresses super-summativity (in fact a warmed-up version of the Aristotelian principle that "the whole is more than the sum of its parts") does not look very original. But it does provoke the question how much more the added value exceeds the sum of the parts. The above examples show that this may be a very complex question, in fact the only interesting point of super-summativity. How is the whole constructed from the simple collection of its parts? When are two wholes different, though having identical parts? How can we compare wholes if we suppose that their parts are comparable?

**Fact 8** *As long as no precise structure theory of wholes qua constructs from local parts is available, no real understanding of music is possible. By the arsenal of the preceding examples, this is a problem which touches all levels of the communicative axis. And it is a problem which involves interpretative activity and its innate ambiguities.*

### 13.1.1 Musical and Mathematical Manifolds

**Summary.** We shortly discuss the question of how music and mathematics differ in their understanding of local and global structures.

– Σ –

The historical point of creating and pronouncing mathematical and musicological concepts of global gestalts is situated around 1854 when Eduard Hanslick (figure 13.5) published his famous treatise "Vom Musikalisch Schönen" [206]. Musical content was recognized as being "tönend bewegte Formen". Hanslick added that these forms are by no means elementary but composed in an artistical way, and building a unity within the manifold, as restated by music theorist Hugo Riemann.

In the same year, the mathematician Bernhard Riemann (figure 13.6) conceived a far reaching generalization of the mathematical concept of space to so-called manifolds [448]. These are understood as being patchworks of locally cartesian charts, similar to geographic atlases. The common ground of both approaches is that locally trivial structures can add up to esthetically valid configurations if glued together in a non-trivial way. A simple and well-known example of such a global shape is the Möbius strip. Its fascination stems from gluing together the ends of an ordinary 'belt'-shaped strip after a rotation of half the full circle of one end. For example,

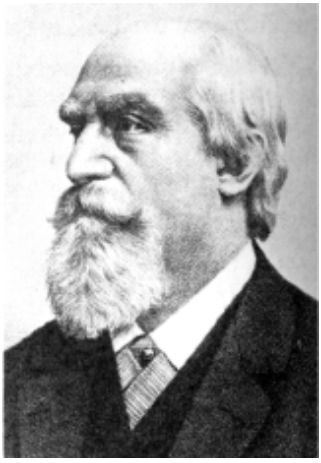

Figure 13.5: Eduard Hanslick described music as a compound structure, built artistically from parts.

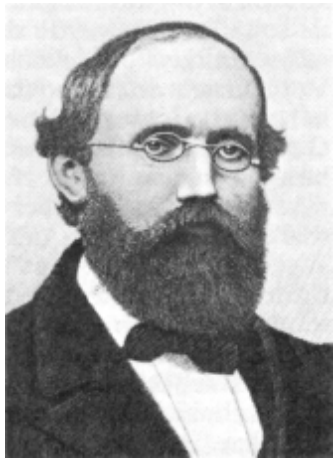

Figure 13.6: Bernhard Riemann introduced global structures (manifolds) as compound mathematical spaces.

in musicology, the Möbius strip is realized by the harmonic strip of triadic degrees within a diatonic scale, see section 13.2.

The characteristic difference between musical and mathematical manifolds is that musical manifolds are defined with a fixed atlas whereas mathematical manifolds are not tied to fixed atlases. Intuitively speaking, the sphere of our globe, viewed as a mathematical manifold, is the same if we add new small or large charts as long as they are compatible with the given ones. If I add a city map, nobody will complain that the geographic identity of the globe has changed. Mathematically speaking, we may go to the colimit of all atlases. In music, this is completely different: A given covering of a composition by a determined set of charts, such as chords, motives, or periods, is essential to the identification of the composition, two different coverings change the composition qua global structure or gestalt, to use Ehrenfels' concept. The colimit is not allowed, the individual interpretational activity is an integral part of the object's identity.

## 13.2 What Are Global Compositions?

**Summary.** This more technical section gives the precise definition of an *objective* global composition. This is a "patchwork" of (objective) local compositions, formally captured by the concept pairing of *charts* and *atlases*, a standard structure in differential and algebraic geometry.

– Σ –

After the preceding propaedeutic reflections, we want to give a precise technical definition of a global composition built upon the objective local compositions as local charts. After these technicalities, we give comments on more intuitive aspects.

**Definition 36** *A (global) objective composition is defined by the following data:*

(i) *A set $G$ and a finite, non-empty covering $I$ of $G$,*

(ii) *an address $A$,*

(iii) *a family $(K_t, A@F_t)_{t\in T}$ of $A$-addressed local objective compositions,*

(iv) *a surjection $I_? : T \to I : t \mapsto I_t$,*

(v) *a bijection $\phi_t : K_t \xrightarrow{\sim} I_t$ for each $t \in T$,*

(vi) *for each couple $s, t \in T$ such that $I_s \cap I_t \neq \emptyset$, the induced bijection*

$$\phi_{s,t} := \phi_t^{-1} \circ \phi_s : \phi_s^{-1}(I_s \cap I_t) \to \phi_t^{-1}(I_s \cap I_t)$$

*(restricted to the respective domains and codomains) defines an isomorphism*

$$\phi_{s,t}/1 : (\phi_s^{-1}(I_s \cap I_t), A@F_s) \xrightarrow{\sim} (\phi_t^{-1}(I_s \cap I_t), A@F_t)$$

*of local compositions.*

*The data (iii) to (v) are called an* $A$-addressed atlas $\Phi$ for the covering $I$ *of $G$. The bijections $\phi_t$ (or—by abuse of language—the local compositions $K_t$) are called the* charts of the atlas $\Phi$.

*Two $A$-addressed atlases $\Phi, \Psi$ for the covering $I$ of $G$ are called* equivalent *iff their disjoint sum*[5] $\Phi \amalg \Psi$ *is an $A$-addressed atlas for the covering $I$ of $G$. An* $A$-addressed objective global composition *is a covering $I$ of $G$ (often abbreviated by $G^I$) together with an equivalence class of $A$-addressed atlases for $G^I$. If no confusion is likely, we may abbreviate the entire data by saying that we are given an objective global composition $G$, i.e., by just naming the* support set *$G$ of the objective global composition.*

**Example 19** Let us start with an illustration of this definition by an example which we have already alluded to: the internal structure of Boulez' dodecaphonic series (see the Boulez/Webern approach in section 13.1 and figure 13.1).

We select as the supporting set the support *Boul* of the zero-addressed local composition $(Boul, 0@OnPiMod_{0,12})$ in that example and abbreviate $OP := OnPiMod_{0,12}$. We consider the covering $I = \{A, B, G, \dots H\}$ by the 15 subsets described in figure 13.1, and which express the internal structure of the series. These subsets of *Boul* define corresponding zero-addressed local compositions

$$K_1 = (A, 0@OP), K_2 = (B, 0@OP), K_3 = (G, 0@OP), \dots K_{15} = (H, 0@OP)$$

and the evident atlas $\Phi$ via the identities $\phi_1 : K_1 \xrightarrow{\sim} A, \dots \phi_{15} : K_{15} \xrightarrow{\sim} H$ on the chart supports. Clearly, the gluing condition (vi) is verified.

Evidently, this is not the most economic atlas since many of the local compositions are isomorphic. For instance, we could replace the identity chart $Id : K.G \xrightarrow{\sim} K.G$ by the retrograde

[5] $\Phi \amalg \Psi$ has index set $T_\Phi \amalg T_\Psi$, and the atlas corresponding to the coproduct surjection $T_\Phi \amalg T_\Psi \to I$.

isomorphism $K : G \xrightarrow{\sim} K.G$, and therefore introduce a new chart domain $(G, 0@OP)$ of this isomorphism. We can proceed in this way until we obtain a minimal set of eight chart domains

$$(A, 0@OP), (A_1, 0@OP), (B, 0@OP), (B_1, 0@OP)$$
$$(G, 0@OP), (\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right) \cdot G, 0@OP), (C, 0@OP), (H, 0@OP)$$

together with 15 corresponding isomorphisms $\psi_i$, defining a second, equivalent atlas $\Psi$ for $Boul^I$. This second atlas makes evident the typology of the internal structure of $Boul$.

**Exercise 21** Show that the relation of atlas equivalence in definition 36 is in fact an equivalence relation

### 13.2.1 The Nerve of an Objective Global Composition

Since the covering $G^I$ is a fixed data of an objective global composition, we should immediately introduce a visualization tool: the nerve of an objective global composition. In the course of the theory, this construction will be extended, but we should look at its most elementary aspect as soon as possible.

Given a global objective composition $G^I$, the *(abstract) nerve* or simplicial complex $n(G^I)$ of $G^I$, as well as its *geometric realization* $N(G^I)$ are defined[6]. Recall that $N(G^I)$ is a union of affine simplexes $|\sigma|$ associated with the (abstract) simplexes $\sigma \in n(G^I)$ such that for any two different simplexes $|\sigma|, |\tau|$, their interiors[7] are disjoint, i.e., $|\sigma|^o \cap |\tau|^o = \emptyset$.

**Example 20** For our example $Boul^I$, we start with the zero-dimensional skeleton $N_0(Boul^I)$ which is a set of 15 points, the 0-simplexes $|J|$, one for each chart $J \in I$, which we distribute in three-space $\mathbb{R}^3$, see figure 13.7 a) for this procedure. For the moment, it is not important where to place these points; the only condition is that they be distinct if their charts[8] $J \in I$ are. Now, we look at all couples $J, J'$ with non-empty intersection. Their 0-simplexes $|J|, |J'|$ are connected by a straight line, a 1-simplex $|J, J'|$, see 13.7 b). This visualization idea stems from combinatorial topology, however, musicology has been aware of such a construction. In his treatise on harmony [479] Schönberg talks about the harmonic strip[9] between two chords which have one or several tones (pitch classes) in common. If we view chords as being charts in the framework of harmony (see also section 13.4.2 and chapter 27), 1-simplexes are precisely the formalization of Schönberg's harmonic strip.

In combinatorial topology, one goes one step further. There is no deeper reason to stop at 1-simplexes and to proceed with a test for common tones in three or more charts. In our next step, we look at all triples $J, J', J''$ of mutually distinct charts such that they have common elements. We then add 2-simplexes $|J, J', J''|$, i.e., triangular surfaces with the 0-simplexes $|J|, |J'|, |J''|$ as vertices, see 13.7 c). And the last step consists in joining each group of four

[6] See appendix H.2 for these concepts.

[7] The interior of an affine simplex is the simplex minus its faces. For example, the interior of an affine 1-simplex is the straight line minus the two endpoints.

[8] This is a typical abuse of language: identification of the chart with its codomain. In fact, we are only looking at the codomains in the covering $I$.

[9] "Harmonisches Band."

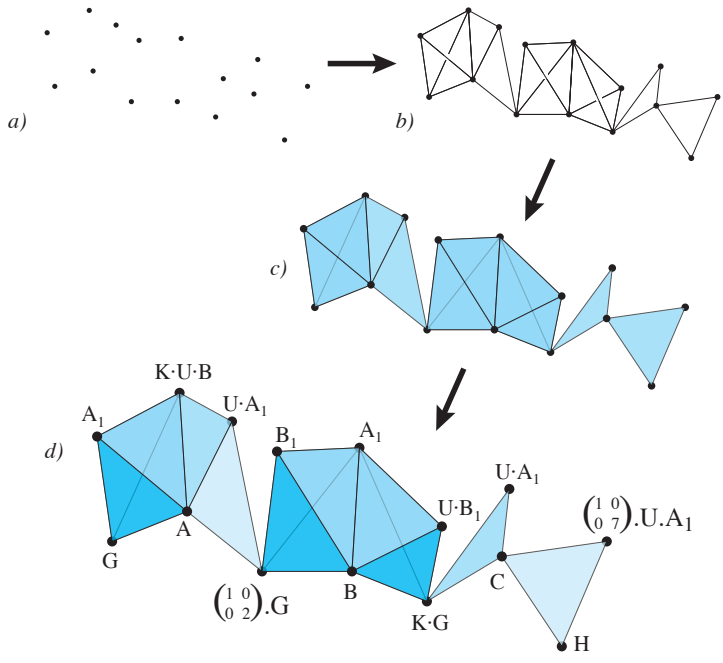


Figure 13.7: Construction of the nerve of the Boulez series *Boul* for its internal structure defined in figure 13.1. a) Every chart corresponds to a point. b) Two points are connected by a straight line iff the corresponding charts have non-empty intersection. c) Three different charts having common points are connected by triangular surfaces. d) four different charts with common tones define a full tetrahedron.

distinct 0-simplexes $|J|, |J'|, |J''|, |J'''|$ by a full tetrahedron if the intersection $J \cap J' \cap J'' \cap J'''$ is non-empty, see 13.7 d).

So the internal structure of Boulez' series appears as a complex intertwining of local charts which overlap as shown by the geometric nerve in 13.7 d). Observe that in general, this procedure will not stop in three-dimensional simplexes. Only here, no five charts have common notes.

The shape of the nerve of an objective global composition is a good measure for the complexity of the global configuration. There are two extremal situations of this perspective: On one hand, a nerve may be discrete, i.e., reduced to the 0-dimensional skeleton. This happens if we just draw disjoint groups of notes on a given composition (as principally suggested by Jackendoff/Lerdahl), which is called a *discrete interpretation*. In this case, there are only isolated local objective compositions around, and the connectivity of the global construction is trivial. The extreme case of such a "strategy" happens if we just draw circles around every tone of a composition; this will completely destroy its gestalt, and we essentially boil down everything to counting notes. This is quite silly (though not superfluous for accounting purposes), and we

shall call such a global composition a *silly interpretation*. On the other hand, we may just take one single local chart to cover the entire composition, and in this case, we have a huge set of notes which does not observe all the more local similarities or symmetries or other relations. This will be a very difficult, cryptic object, also since classification of large local compositions is quite intricate (see our discussion in chapter 11), we call it the *indiscrete interpretation*.

Both extrema are no intelligent solutions for human cognition: Good global structures are somewhere in the middle between intractable monsters and insignificant atomized data. But it would be nonsense to set up artificial limits (like Jackendoff/Lerdahl [243]) to the grouping formalism because it is one of the most important objectives to make all grouping perspectives accessible and formally tractable. The breakdown of the Jackendoff/Lerdahl Generative Theory of Tonal Music (GTTM) happens where their grouping levels have to rely on traditional music theory in order to work. The GTTM is built on a competence which is both, formally and semantically, out of the reach of the GTTM, and therefore boils down to an interface between psychology and music theory which does neither solve the formal inconsistency of music theory nor observe the variability of the psychological grouping: a bad glue of two unresolved components.

Often, an objective global composition is not given in advance, but only results from a compatible gluing of local charts. The next theorem describes a necessary and sufficient condition for a system of local compositions to become an atlas for a global composition.

**Theorem 12** *Let $A$ be an address, $T$ a set of indexes. Suppose we are given a family of $A$-addressed objective local compositions*

$$Rel\Phi = ((K_{i,j}, A@F_j))_{i,j\in T} \tag{13.1}$$

*of forms $F_j$, together with a family*

$$Iso\Phi = (\phi_{i,j}/1 : (K_{i,j}, A@F_j) \tilde{\to} (K_{j,i}, A@F_i))_{i,j\in T} \tag{13.2}$$

*of isomorphisms in ObLoc whenever $K_{i,j}$ is non-empty. And we assume that each $(K_{i,j}, A@F_j)$ is a sub-composition of the diagonal element $(K_{j,j}, A@F_j)$, i.e., we have inclusion morphisms $\rho_{i,j}/1 : K_{i,j} \subset K_{j,j}$, and that for all indexes, $\phi_{i,j} \circ \phi_{j,i} = Id$, and, more specially, $\phi_{i,i} = Id$. Consider the binary relation $\sim$ on the disjoint union of the sets $K_{j,j}$ defined by $x \sim y$ iff $x \in K_{i,j}$ for some index couple $i, j$, and $\phi_{i,j}(x) = y$.*

*Let $U$ be the colimit of the system of set maps $\phi_{i,j}$ and $\rho_{i,j}$. Then the canonical maps $k_j : K_{j,j} \to U$ define an $A$-addressed objective global composition with atlas $((K_{j,j}, A@F_j))_{j\in T}$ and intersections $(K_{i,j}, A@F_j)$ on the charts, iff the relation $\sim$ is an equivalence relation.*

Proof. If the canonical maps define an $A$-addressed objective global composition with atlas $((K_{j,j}, A@F_j))_{j\in T}$, then the relation $\sim$ is the inverse image of the equivalence relation of equality on $U$ for the surjection $\coprod K_{j,j} \to U$, and the claim follows. If, conversely, the relation $\sim$ is an equivalence relation, then the colimit, which is identified[10] to $U = \coprod K_{j,j}/\sim$, defines injective maps (!) $k_j : K_{j,j} \to U$, and the non-empty intersections $k_i(K_{i,i}) \cap k_j(K_{j,j})$ correspond to the local subcompositions $(K_{i,j}, A@F_j), (K_{j,i}, A@F_i)$ which are isomorphic under the morphisms of the system $Iso\Phi$. QED.

[10] See appendix G.2.1.

Observe that objective global compositions need not be derived from given local compositions by coverings as in the above Boulez example. The concept is a proper extension to the local framework, and the general construction method is described by the above theorem. We shall develop necessary and sufficient criteria for covering constructs in section 16.1.

**Exercise 22** Let us define a zero-addressed objective global composition by data corresponding to the above theorem. The index set is $T = \{1, 2, 3\}$. We take a constant space form $F_i = Onset \oplus Pitch$ with functor $@\mathbb{R}^2$. The family $Rel\Phi$ of local compositions is defined as follows: Consider the four (zero-addressed) points $c_1 = (0, 0), c_2 = (0, 1), c_3 = (1, 0), c_4 = (1, 1)$ in $\mathbb{R}^2$ and set

$$
\begin{aligned}
&K_{i,i} = (\{c_1, c_2, c_3, c_4\}, 0@\mathbb{R}^2), i = 1, 2, 3,\\
&K_{1,2} = K_{2,1} = (\{c_3, c_4\}, 0@\mathbb{R}^2),\\
&K_{1,3} = K_{3,1} = (\{c_1, c_2\}, 0@\mathbb{R}^2),\\
&K_{2,3} = (\{c_3, c_4\}, 0@\mathbb{R}^2),\\
&K_{3,2} = (\{c_1, c_2\}, 0@\mathbb{R}^2),\\
&K_{i,j} = \emptyset \text{ else.}
\end{aligned}
$$

The family $Iso\Phi$ is given by

$$
\begin{aligned}
&\phi_{1,2} = Id_{K_{2,1}},\\
&\phi_{1,3} = Id_{K_{3,1}},\\
&\phi_{2,3} = \phi_{3,2} = e^{1,1} \cdot \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right).
\end{aligned}
$$

Show that the conditions of the theorem are satisfied and try to draw a picture of the global composition. We shall see in section 16.1 that this global composition cannot be constructed by a covering of a local composition.

We should observe a hidden subtlety of the global composition context: Global ambient spaces. In fact, every local composition is connected to an ambient space form $F$. But if we are given an atlas of local compositions, the ambient spaces are no longer uniquely determined. In fact, the construction of a global composition makes chart names exchangeable: it is 'homonymous'. More precisely, we have as many names for a given point $x$ of the support as we have chart names on any of the compatible atlases. So the name of the 'ambient space' of x is an entire collection of local chart names. On the level of local supports, univocal naming would not be a good idea since on the overlaps of two different covering sets, what name should one select? Homonymy is unavoidable for global compositions. One should, however, not refrain from naming or sweep names away in favor of anonymy. Global naming is interchangeable, but not irrelevant:

**Definition 37** *The name $Name(x)$ of an element $x \in G$ of an objective global composition $G$ is the set of names of all the local chart forms (in any compatible atlas) which hit this element.*

## 13.3 Functorial Global Compositions

**Summary.** Corresponding to functorial local objects, the (functorial) local compositions in the category *Loc*, *global functorial compositions* are introduced as global objects by an atlas of charts, composed of (functorial) local compositions.

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The definition of functorial global compositions is the transposition of the objective case with everything made functorial:

**Definition 38** *A (global) functorial composition is defined by the following data:*

(i) *A functor* $G \in \mathbf{Mod}^{@}$ *and a finite, non-empty, generating set* $I$ *of subfunctors of* $G$*, i.e.,* $G = \bigcup I$*,*

(ii) *an address* $A$*,*

(iii) *a family* $(K_t, @A \times F_t)_{t\in T}$ *of* $A$*-addressed functorial local compositions,*

(iv) *a surjection* $I_? : T \to I : t \mapsto I_t$*,*

(v) *an isomorphism of functors* $\phi_t : K_t \xrightarrow{\sim} I_t$ *for each* $t \in T$*,*

(vi) *for each couple* $s, t \in T$ *such that* $I_s \cap I_t \neq \emptyset$*, the induced isomorphism*

$$\phi_{s,t} := \phi_t^{-1} \circ \phi_s : \phi_s^{-1}(I_s \cap I_t) \to \phi_t^{-1}(I_s \cap I_t)$$

*(restricted to the respective domains and codomains) defines an isomorphism*

$$\phi_{s,t}/1 : (\phi_s^{-1}(I_s \cap I_t), @A \times F_s) \xrightarrow{\sim} (\phi_t^{-1}(I_s \cap I_t), @A \times F_t)$$

*of functorial local compositions.*

*The data (iii) to (v) are called an* $A$-addressed atlas $\Phi$ for the covering $I$ *of* $G$*. The bijections* $\phi_t$ *(or—by abuse of language—the local compositions* $K_t$*) are called the* charts of the atlas $\Phi$*.*

*Two* $A$*-addressed atlases* $\Phi, \Psi$ *for the covering* $I$ *of* $G$ *are called* equivalent *iff their disjoint sum*[11] $\Phi \amalg \Psi$ *is an* $A$*-addressed atlas for the covering* $I$ *of* $G$*. An* $A$*-addressed functorial global composition is a covering* $I$ *of* $G$ *(often abbreviated by* $G^I$*) together with an equivalence class of* $A$*-addressed atlases for* $G^I$*. If no confusion is likely, we may abbreviate the entire data by saying that we are given a global functorial composition* $G$*, i.e., by just naming the* support $G$ *of the global composition.*

**Example 21** The first and immediate example for this definition is the functorial global composition $\hat{G}$ associated with a given objective global composition $G$, if the latter has some adequate properties. To construct this object, suppose that we are given an $A$-addressed objective global composition $G$ with the notation and data of definition 36, in particular we have an atlas $(K_t, A@F_t)_{t\in T}$ of local compositions.

[11] $\Phi \amalg \Psi$ has index set $T_\Phi \amalg T_\Psi$, and the atlas corresponding to the coproduct surjection $T_\Phi \amalg T_\Psi \to I$.

To begin with, we have a family of functorial local compositions $(\hat{K}_t, @A \times F_t)_{t\in T}$, as well as a system of subcomposition inclusion monomorphisms

$$\hat{\rho}_{s,t} : (\hat{K}_{s,t}, @A \times F_t) \subset (\hat{K}_t, @A \times F_t),$$

defined by the inverse images $K_{s,t} = \phi_t^{-1}(I_s \cap I_t)$ for any two different indexes $s, t$. We also have induced isomorphisms

$$\hat{\phi}_{s,t}/1 : \hat{K}_{t,s} \xrightarrow{\sim} \hat{K}_{s,t}$$

of functorial local compositions. Let $\hat{G}$ be the colimit functor of the system $\hat{\phi}_{s,t}, \hat{\rho}_{s,t}$ of isomorphisms and inclusions. For an address change $x : B \to A$, we have the formula

$$x@\hat{G} \xrightarrow{\sim} \mathrm{colim}(x@\hat{K}_s, x@\hat{K}_{s,t}) = \mathrm{colim}(K_s.x, K_{s,t}.x) \tag{13.3}$$

on the slices of the colimit. Because of the right colimit expression, the slice $x@\hat{G}$ is also denoted by $G.x$.

In general, $\hat{G}$ is not a global functorial composition, in fact, the canonical morphisms $\hat{K}_t \to \hat{G}$ are not even injective. To obtain a global composition, we suppose that for all address changes $f : B \to A$, the maps

$$A@F_s \to B@F_s$$

are injective for all indexes $s$. This is the case, for example, if $A = 0_{\mathbb{Z}}$, or if $F_s$ is constant. Then it follows that the system of isomorphisms and inclusions $\phi_{s,t}, \rho_{s,t}$ satisfies the conditions of theorem 12, and therefore, the canonical morphisms

$$i_t : \hat{K}_t \to \hat{G}$$

are mono, i.e., injective for all addresses. Further, the canonical squares

$$\begin{array}{ccc} \hat{K}_{t,s} & \xrightarrow{\hat{\rho}_{t,s}} & \hat{K}_s \\ {\scriptstyle \hat{\rho}_{s,t}\circ\hat{\phi}_{s,t}}\downarrow & & \downarrow{\scriptstyle i_s} \\ \hat{K}_t & \xrightarrow{i_t} & \hat{G} \end{array} \tag{13.4}$$

are cartesian. Therefore $\hat{G}$ is generated by the isomorphic images $\hat{I}_t$ of $\hat{K}_t$. Furthermore, since the evaluation of the inductive system of functors at the morphism $Id_A$ (the slice at $Id_A$) yields the original objective configuration, the generators $\hat{I}_t$ are different iff the original covering elements $I_t$ are. So we have obtained a finite covering $\hat{I}$ of $\hat{G}$, an atlas $(\hat{K}_t, @A \times F_t)_{t\in T}$ of functorial local compositions, and a surjection $I_? : T \to \hat{I}$; this completes the construction.

**Exercise 23** Explicate this construction in all the details.

**Exercise 24** Establish a theorem for functorial compositions which corresponds to theorem 12.

**Remark 3** Observe that, although the chart morphisms are compatible in the sense of fulfilling the conditions of theorem 12, no compatibility of the underlying functor morphisms on the ambient spaces is required. This means that we will not be able to extend the gluing procedures to the ambient spaces; in other words: global objects are really global, no common ambient space is available in general!

## 13.4 Interpretations and the Vocabulary of Global Concepts

**Summary.** A large set of (objective) global compositions is constructed by "interpretations" of given (objective) local compositions. This construction is a subtlety that tends to escape to common interpretative activities. It is, however, a basic prerequisite for every interpretative activity in music. This is why this type of global construction is called *interpretation*. We give standard examples of such global objects.

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Very often, music analysis and composition explicitly deal with local compositions, but this coarse perspective is not sufficient: implicitly, one specifies local parts of this data and different overlapping relations among such parts. For example, Boulez' series of example 13.1 needs a covering by 15 local parts in order to be fully understood. Many analytical texts (e.g. [243, 393, 466]) also use this kind of local part selection.

As already stressed, this type of chart specification is not temporary, it is a substantial attribute of the analytical work. We shall therefore introduce a special construction method of global compositions which take into account this important technique as an interpretational basic. Here is the general construction method of such global compositions which we therefore call interpretations.

**Definition 39** *Let $(K, A@F)$ be an $A$-addressed objective local composition, and let $I$ be a non-empty covering of the support set $K$. Then the covering $K^I$, together with the obvious atlas $(i, A@F)_{i\in I}$, defines an objective global composition, the* interpretation $K^I$ of $K$ *associated with the covering $I$.*

*Let $(K, @A \times F)$ be an $A$-addressed functorial local composition, and let $I$ be a non-empty covering*[12] *of the support functor $K$. Then the functorial covering $K^I$, together with the obvious atlas $(i, @A \times F)_{i\in I}$, defines a functorial global composition, the* interpretation $K^I$ of $K$ *associated with the functorial covering $I$.*

We shall see in chapter 16 that interpretations define a special case: Global compositions may be far from 'interpretable', i.e., they are not 'isomorphic' to any possible interpretations (we shall make these remarks precise later).

**Example 22** A second example of functorial global compositions is associated with interpretations of objective compositions. Let $(K, A@F)$ be an objective local composition, and take an interpretation $K^I$ by an atlas $I = (K_\iota)_\iota$ of subcompositions. Then we have an interpretation $\hat{I} = (\hat{K_\iota})_\iota$ of the local functorial composition $\hat{K}$ since unions and $\hat{?}$ commute by lemma 2, section 7.4. But observe that the intersection $\hat{K_{\iota_1}} \wedge \hat{K_{\iota_2}}$ is not objective in general. This is however the case for the zero address $A = 0_{\mathbb{Z}}$, and there, the colimit construction and this one coincide.

[12] The functor $K$ is generated by the subfunctors in $I$.

### 13.4.1 Iterated Interpretations

**Summary.** Interpretation is not a one-step process, it may be iterated on an infinity of levels. We present the formal framework and examples.

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If $(K, A@F)$ is an $A$-addressed objective local composition, we can give an interpretation by a covering set $I$ of subcompositions $(K_\iota, A@F)$ of $K$. We then introduce this sequence of forms:

$$G_0 = F,$$
$$G_{i+1} \underset{2^{Fun(G_i)} \rightarrowtail \Omega^{Fun(G_i)}}{\longrightarrow} \mathbf{Power}(G_i), i = 0, \dots.$$

Then, the interpretation $I$ is an $A$-addressed objective local composition in $G_1$ whose union is $K$, $\bigcup I = K$. Here, the union operator is defined functorially as a transformation $\bigcup_i : G_{i+1} \to G_i, i = 1, 2, \dots$, with the initial evaluation $\bigcup_1(I) = K$. So we have an infinite diagram

$$\mathbb{D} = (G_1 \overset{\bigcup_1}{\leftarrow} G_2 \overset{\bigcup_2}{\leftarrow} G_3 \overset{\bigcup_3}{\leftarrow} \dots)$$

and we can define a corresponding limit form

$$Limint(F) \underset{Id}{\longrightarrow} \mathbf{Limit}(\mathbb{D}) \tag{13.5}$$

whose denotators are local compositions $K$ as above, together with coverings $I$ of $K$, and coverings $J$ of $I$ etc. This means that we are given an infinite succession of interpretations and interpretations of interpretations etc., in other words, an infinite sequence of interpretations of constantly increasing level of complexity. Of course there are infinitely many such denotators even if we fix an initial sequence of first $k$ positions.

A very classical 'infinite interpretation' is the one induced by the nerve of an interpretation. In fact, if we start as above, i.e., taking the covering $I$ of $K$, we get an initial sequence $(K, I, \dots)$. The simplicial complex $n(I)$ is an element of $G_3$ and can be iterated ad libitum by $n_{t+1}(I) = n(n_t(I))$. This defines a sequence

$$n_\infty(K^I) = (K, I, n(I), n^2(I), \dots n^t(I), \dots)$$

which indeed is in $Limint(F)$. In particular, if $x \in K$, the star of $x$ $st(x) = \{U \in I \text{ with } x \in U\}$ is an element of $n(I)$, and the star $st^2(x) = st(st(x))$ of $st(x)$ is an element of $n^3(I)$, etc., $st^i(x) \in n^{1+2i}(I)$.

Suppose that each element $x \in K$ is given a weight $w$, for example the constant weight $w(x) = 1$ or some weight stemming from an analysis of $K$ as in motivic or metrical theory. Then each covering set $U \in I$ can be given a numerical weight, for example a power $w(U) = (\sum_{x \in U} w(x) - 1)^p$ as in the metrical theory (see section 21.2) or some other musically motivated evaluation. Then, if $x \in K$, we have a weight $w_1(x) = w(st(x)) = \sum_{U \in st(x)} w(U)$, and then inductively $w_{t+1}(x) = \sum_{u \in st^{t+1}(x)} w(u)$, if we take the above formula $w(u) = (\sum_{x \in u} w(x) - 1)^p$ to calculate recursively the weight of elements of $G_i$. This associates a power series

$$\lambda_w(x) = 1 + \sum_{1 \leq t} \frac{w_t(x)}{t!} T^t.$$

The reader may easily generalize this idea to obtain a numerical evaluation of points in infinite interpretations. This means that infinite interpretations can be boiled down to yield numerical 'coordinates' of points in compositions, and therefore quantitative measures for further processing within performance or analytical contexts.

### 13.4.2 The Pitch Domain: Chains of Thirds, Ecclesiastical Modes, Triadic and Quaternary Degrees

**Summary.** Chains of thirds and modes are a classical domain of interpretation. The construction deals with "refinement" of scales qua local compositions. The refinement specifies determined notes or groups of notes which give the scales an interior profile. More elaborate interpretations of scales concern triadic and quaternary degrees for concerns of harmony.

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Recall from section 7.2.1 that we may identify an octave-periodic scale $S$ by its class chord $S : A \rightsquigarrow \text{o-Scale}(S_1, \dots S_k)$. Many common scales and chords are classified by use of the so-called third chain construction. To understand this construction, consider the two standard cases: 12-tempered and just scales.

For the 12-tempered case, we are effectively working in the ambient pitch class space $PiMod_{12}$ with module functor $@\mathbb{Z}_{12}$. Here, we have the major third $M_{12} = e^4$ and the minor third $m_{12} = e^3$ translation. If the pitch class space is identified with its Sylow decomposition: $@\mathbb{Z}_{12} \xrightarrow{\sim} @(\mathbb{Z}_4 \oplus \mathbb{Z}_3) : x \mapsto (x \mod 4, x \mod 3)$, $M_{12}$ identifies to $M_{12} = e^{(0,1)}$, whereas $m_{12}$ identifies to $e^{(-1,0)}$. See right half of figure 13.8 for the torus representation of $PiMod_{12}$. On the *EulerPlane* ambient space (see example 5 in section 7.2.1), we have the corresponding translations $M_{\text{just}} = e^{(0,1)}$ and $m_{\text{just}} = e^{(1,-1)}$, see left half of figure 13.8.

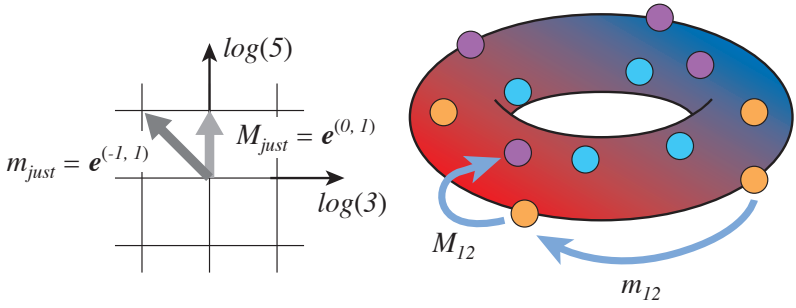


Figure 13.8: Left: Major and minor third transpositions in just tuning pitch classes. Right: The third transpositions on the third torus representing 12-tempered pitch classes.

With these translations in mind, we can define a third chain:

**Definition 40** *A chord $S = \{S_1, \dots S_k\}$ in $PiMod_{12}$ or in EulerPlane, respectively, of cardinality $k$ is called a* third chain *iff the elements of $S$ can be ordered in such a way that*

*$S_{i_{j+1}} = T_j(S_{i_j})$ for all $j = 1, \ldots k-1$ and $T_j$ is either the major third translation $M_{12}$ (resp. $M_{just}$) or the minor major translation $m_{12}$ or $m_{just}$, respectively.*

**Lemma 16** *Every zero-addressed chord in $PiMod_{12}$ is contained in a third chain.*

If we admit positive and negative translation in the minor third direction $m_{\text{just}}$, we obtain the concept of *weak third chains* in just tuning (see also right half of figure 13.8).

**Lemma 17** *Every zero-addressed chord in EulerPlane is contained in a weak third chain, but not necessarily in a third chain.*

**Exercise 25** Give a proof of the preceding lemmata.

The set $3Chains$ of all zero-addressed third chains in $PiMod_{12}$ is known and has been used in computer software *presto*®(see section 25.2.1), as well as in RUBATO®'s HarmoRubette (see chapter 41.3). See appendix L.2 for this list. The (objective) interpretation $0@PiMod_{12}^{3Chains}$ of $0@PiMod_{12}$ by the covering $3Chains$ is called the *third chain interpretation* of the pitch class space.

In the 12-tempered case, the set of third chains containing a given zero-addressed chord $S$ with minimal cardinality are called *the minimal third chains of $S$*, and this set is called the *third chain closure of $S$*, it is denoted by $3Chain(S)$. This set can grow quite dramatically for non-standard chords. For example, the chromatic 3-chord $S = \{0, 1, 2\}$ has 23 minimal (7-element) third chains, including third chains which start with three of the classical triads: major, minor, and augmented.

In order to describe ecclesiastical modes, we make use of the 12-tempered space $PiMod_{12}$, although other pitch class spaces would also do the job. The point here is less the tuning but the construction of an interpretation from a given scale (chord). Our modeling of ecclesiastical modes will follow the (zero-addressed) scale $X = \text{C-major} = \{\overline{0}, \overline{2}, \overline{4}, \overline{5}, \overline{7}, \overline{9}, \overline{11}\}$, the other modes are be derived by evident transposition.

To describe modes, one selects two elements in $X$: the *tenor* or *recitation tone* $t$, as well as the *finalis* or *final* $f$. The first interpretation is the *mode* $X_f$, i.e., the enrichment of the scale by a determined *tonic* $f$. This is an interpretation $X_f = X^{I_f}$ of $X$ by an atlas $I_f$ of two charts: $I = \{X, \{f\}\}$, the scale and the tonic singleton. Observe that this is not a full-fledged concept of tonality. To meet this requirement, a mode may and will be enriched by other structural aspects, such as selected degree chords, etc. According to the position of the tonic, the mode

has the names as listed in table 13.4.2.A.

Table 13.4.2.A

| **Nr.** | **mode and plagal variant** | **finalis $f$** | **tenor $t$** |
|---|---|---|---|
| 1.$a$ | Dorian | $d$ | $a = d + 7$ |
| 1.$p$ | Hypodorian | $d$ | $f = d + 3$ |
| 2.$a$ | Phrygian | $e$ | $c = e + 8$ |
| 2.$p$ | Hypophrygian | $e$ | $a = e + 5$ |
| 3.$a$ | Lydian | $f$ | $c = f + 7$ |
| 3.$p$ | Hypolydian | $f$ | $a = f + 4$ |
| 4.$a$ | Mixolydian | $g$ | $d = g + 7$ |
| 4.$p$ | Hypomixolydian | $g$ | $c = g + 5$ |
| 5.$a$ | Aeolian | $a$ | $e = a + 7$ |
| 5.$p$ | Hypoaeolian | $a$ | $c = a + 3$ |
| 6.$a$ | Locrian | $b$ | $g = b + 8$ |
| 6.$p$ | Hypolorian | $b$ | $d = b + 3$ |
| 7.$a$ | Ionian | $c$ | $g = c + 7$ |
| 7.$p$ | Hypoionian | $c$ | $e = c + 4$ |

The next step introduces an ecclesiastical mode[13] as a refined version of the simple mode interpretation: An *ecclesiastical mode* on scale $X$ is the interpretation $X_{f,t} = X^{I_{f,t}}$ with atlas $I_{f,t} = \{X, \{f\}, \{f, t\}\}$. For every final $f$, there are two modal variants: the *authentic* (.a in table 13.4.2.A) and the *plagal* (.p in table 13.4.2.A), the latter being marked by the prefix "Hypo". For the authentic mode we have either $t = f + 7$ or $t = f + 8$ (sixths), for the plagal one, we have either $t = f + 3$ or $t = f + 4$ or $t = f + 5$ (thirds or the fourth). The modal pair Nr. $6.a, 6.p$ are listed for completeness, however, they are scarcely documented.

The aeolian and ionian modes were only introduced in the 16th century by Glarean, but they are the basis of the modern major-minor system where the tenors have disappeared (and we only consider the modes $X_f$), the final tone being renamed to *tonic*, such that we now have aeolian =cantus mollis ($C_a$ for scale $C$), and ionian = cantus durus ($C_c$ for scale $C$).

In harmony, interpretations are important which associate 3-chords with special harmonic functions. Corresponding interpretations of scales are called triadic interpretations, we shall discuss them in the sequel for the 12-tempered and just tuning. We shall see below that the examples to be discussed here have special automorphism properties. This is why we call them *triadic degree interpretations*.

For 12-tempered pitch classes, we again take the major scales $X$, as well as the melodic and harmonic minor scale $x_m$ and $x_h$ (see figure 7.4). Observe however, that the name of such a scale, $X = C$, for example, does not imply the selection of a tonic. The name is only historically loaded and not in our structural setting. We are going to define the triadic degree interpretations for $X = C$, the others being deduced by transposition.

[13] The *ambitus*, i.e., the octave where the modal melody may move, is omitted here because we work in octave classes.

The triadic degree interpretation $C^{(3)}$ is defined by a seven element atlas

$$(3) = \{I_C, II_C, III_C, IV_C, V_C, VI_C, VII_C\}$$

of three-element charts. The analogous notation works for minor scales, i.e., $c_m^{(3)}$ is defined by seven charts $(3_m) = \{I_{c_m}, II_{c_m}, \dots VII_{c_m}\}$, and $c_h^{(3)}$ by $(3_h) = \{I_{c_h}, II_{c_h}, \dots VII_{c_h}\}$. If ever the scale is clear, we omit it, writing simply $I$ for the first degree, etc. The precise values for all our charts are shown in table 13.4.2.B. The non-trivial automorphisms of the major and melodic minor scales are shown with the notation $U_x$ = inversion (German "Umkehrung") at pitch $x$, and $U_{x/x+1}$ = inversion between neighboring pitches $x, x+1$. The arrows in row two are for the alteration shifts in the melodic and harmonic minor scales against the major scale.

Table 13.4.2.B

| | $C$ | | $c_m$ | | $c_h$ | |
|---|---|---|---|---|---|---|
| | ●○●○●●○●○●○● | | ●○●●←○●○●○● | | ●○●●←○●○●←○● | |
| autom. | \| $= U_d$ | | \| $= U_g$ | | | |
| **deg.** | | **tpe.** | | **tpe.** | | **tpe.** |
| $I$ | ●○○○●○○●○○○○ | maj. | ●○○●○○○●○○○○ | min. | ●○○●○○○●○○○○ | min. |
| $II$ | ○○●○○●○○○●○○ | min. | ○○●○○●○○○●○○ | min. | ○○●○○●○○●○○○ | dim. |
| $III$ | ○○○○●○○●○○○● | min. | ○○○●○○○●○○○● | aug. | ○○○●○○○●○○○● | aug. |
| $IV$ | ●○○○○●○○○●○○ | maj. | ●○○○○●○○○●○○ | maj. | ●○○○○●○○●○○○ | min. |
| $V$ | ○○●○○○○●○○○● | maj. | ○○●○○○○●○○○● | maj. | ○○●○○○○●○○○● | maj. |
| $VI$ | ●○○○●○○○○●○○ | min. | ●○○●○○○○○●○○ | dim. | ●○○●○○○○●○○○ | maj. |
| $VII$ | ○○●○○●○○○○○● | dim. | ○○●○○●○○○○○● | dim. | ○○●○○●○○○○○● | dim. |

Observe that not all major, minor, diminished, or augmented triads in these scales are automatically degrees. For example, the triad $\{a_\flat, b, d\}$ is not a degree in harmonic minor $c_h$.

Referring to the automorphism groups of these scales, as listed in the chord classification table appendix L.1, we observe that the degree atlases $(3), (3_m), (3_h)$ are all invariant under the respective automorphism groups. More precisely, we have these degree orbits in the three scales:

$$X : \{I \leftrightarrow VI\}, \{II \leftrightarrow V\}, \{III \leftrightarrow IV\}, \{VII\}$$
$$x_m : \{I \leftrightarrow V\}, \{II \leftrightarrow IV\}, \{III\}, \{VI \leftrightarrow VII\}$$
$$x_h : \{I\}, \{II\}, \{III\}, \{IV\}, \{V\}, \{VI\}, \{VII\}$$

where the third is so since the automorphism group of harmonic minor is trivial, $x_h$ is "rigid". The geometric nerves $N(X^{(3)}), N(x_m^{(3)}), N(x_h^{(3)})$ of each of these global objective zero-addressed compositions are Möbius strips, as shown in figure 13.9. Following Schönberg's proposal [478], we call it the *harmonic strip*. The harmonic strip has seven simplexes of dimensions zero and two, and fourteen simplexes of dimension one. The 1-skeleton is exactly Schönberg's harmonic strip. The order of the degrees on the strip's boundary is the so-called *fifth sequence*,

$$V \to I \to IV \to VII \to III \to VI \to II \ \ (\to V),$$

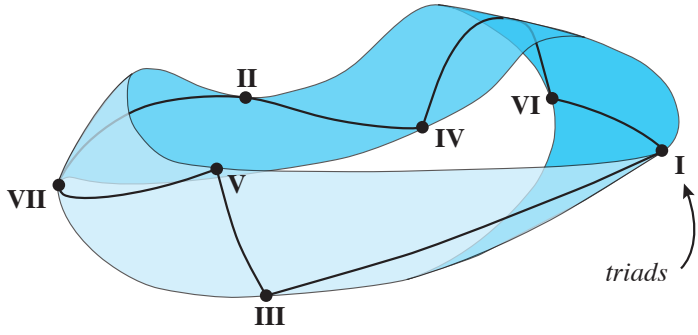


Figure 13.9: The nerve $N(X^{(3)}), N(x_m^{(3)}), N(x_h^{(3)})$ is a Möbius strip. Following Schönberg's proposal, we call it the harmonic strip.

although this is only a diminished fifth (tritone) for the seventh-fourth passage. In other words, the boundary is connected, unlike with a normal strip where we have two connected components. This is due to the lack of orientation on the harmonic strip: One walk around the entire strip changes your upside to downside, see figure 13.10.

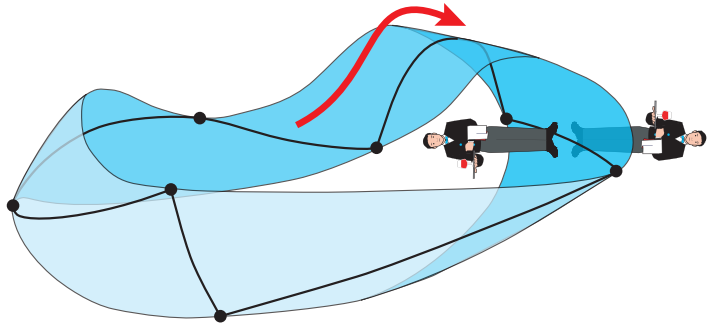

Figure 13.10: The harmonic strip is not orientable. The mutual position of the boundary changes after a round-trip. This property has consequences for Riemann's harmony.

#### 13.4.2.1 Orientation in Riemann Function Theory

We should add a remark on the failure of Riemann's attempt to build a global function theory of harmony, as was discussed by Carl Dahlhaus in [100]. The central concept of this theory is the function which is to be attached to every (!) possible chord, not only to the chords of common usage. Before starting the discussion, we observe that the problem here is not so much one of tunings, but rather one attached to the non-orientability of the triadic interpretation

independently of tuning specifications. Therefore we want to stick to the harmonic strip as discussed above (this nerve will reappear for the triadic interpretation in just tuning which we discuss below).

Riemann's idea was to define "tonality" by use of a function with three possible values:"Tonical" ($T$), "Dominant" ($D$), and "Subdominant" ($S$), which can be attributed to chords. So we have to deal with a function

$$\tau : Ch \to TDS$$

defined on the set $Ch$ of all (zero-addressed objective) chords (in $PiMod_{12}$, to make the ideas precise), with values in the three-element set $TDS = \{T, D, S\}$ of harmonic function values. Moreover, we have to distinguish between two sub-categories of tonal functions: major, and minor, for each pitch class $x$:

$$^{+}\tau_x : Ch \to TDS \ (x\text{-major tonality})$$
$$^{0}\tau_x : Ch \to TDS \ (x\text{-minor tonality})$$

According the the selected tonality, Riemann's program is to attribute a specific value to given chords. For example, one would like to have

$$^{+}\tau_g(\{g, b, d\}) = T$$

which means "$\{g, b, d\}$ is tonical in G-major tonality", or

$$^{0}\tau_c(\{g, b, d\}) = D$$

which means "$\{g, b, d\}$ is dominant in C-minor tonality".

When they define tonality, special chords of the triadic degree interpretation $X^{(3)}$are given special values , i.e.:

$$^{+}\tau_x(I_X) = T, \ ^{+}\tau_x(V_X) = D, \ ^{+}\tau_x(IV_X) = S.$$

However, it is not true that the function concept is not a mathematical one, as Dahlhaus has criticized in [100] relating to an erroneous proposal. Rather is the problem of function theory to extend the values of tonality functions from common first, fifth, and fourth degree triads to any chords in such a way that the harmonic coherence is reflected. This procedure can be termed "musical logic" in the sense of Riemann. This concept is an obscure one, as Dahlhaus has rightly recognized. It is however possible to shed some light on this approach [100, p.96]:"Die Bestimmung der Akkordbedeutungen, der 'harmonischen Logik', ist also mit einer Regel über die Reihenfolge der Stufen verbunden."

According to Dahlhaus, this is the fifth sequence, the fundamental sequence along the harmonic strip's boundary[14]. This sequence lays the basis of the idea of "different" (German:"differente") degrees: the values

$$^{+}\tau(V) = D, \ ^{+}\tau(I) = T, \ ^{+}\tau(IV) = S$$

are different according to the fifth steps $V \to I \to IV$. In order to proceed, the other four triadic degrees must obtain one of the existing values, each.

[14]Dahlhaus starts the sequence in degree $I$, but this is the equivalent to ours since the sequence is cyclic.

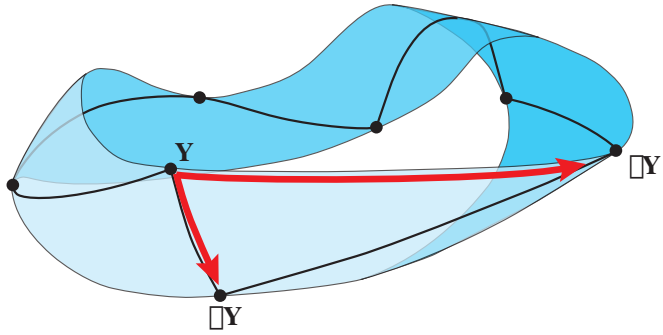


Figure 13.11: The local orientation on the harmonic strip yields the parallel degree as the one staying in front and to the right when moving in fifth sequence direction such that the strip stays to our right.

The second musicological idea relates every degree $Y$ to its "parallel degree" $\pi Y$. On the harmonic strip, the latter is deduced from the fifth sequence in the following way (see figure 13.11): Let $\delta Y$ be the successor of $Y$ in the fifth sequence. Then, $\pi Y$ is the third member of the 2-simplex containing $Y$ and $\delta Y$. If we move along the harmonic strip in such a way that the strip stays to the right when moving from $Y$ to $\delta Y$, then $\pi Y$ is to the right in front of us. For example, if we move from $Y = I$ to $\delta Y = V$, the parallel degree is $\pi Y = III$. We also have $\pi I = VI$, $\pi IV = II$. The wording "parallel" is also geometrically correct since we look for the degree which is parallel to the present position.

The contradiction in function theory comes out from the requirement in function theory that parallel degrees should have equal function values:

$$^{+}\tau(Y) = {}^{+}\tau(\pi Y).$$

In fact, we then must have

$$D = {}^{+}\tau(V) = {}^{+}\tau(\pi V) = {}^{+}\tau(III) = {}^{+}\tau(\pi III) = {}^{+}\tau(I) = T.$$

This contradiction can also be read as follows: When applying the parallel function to every degree on the harmonic strip, the connectedness of the boundardy (=the fifth sequence) leads to the parallelism $\pi\pi Y = \delta Y$ (the "Gegenklang" relation), and therefore the tonal values cannot be different on the vertices of a 2-simplex. The orientation being only a local one, the musicological requirements can only be fulfilled locally and do not glue to yield a global function!

Therefore we cannot follow Dahlhaus [100, p.102] when he says that function theory: "gerade dort versagt, wo auch das Phänomen, das sie erklären soll, ins Vage und Unbestimmte gerät." In reality, the phenomenon is founded in a very precise fact: the non-orientability of the Möbius strip. Any attempt to define a function despite this fact must fail for mathematical reasons.

#### 13.4.2.2 Just Triadic Degree Interpretations

For this situation, recall the just C-major scale in the *EulerPlane* from section 7.2.1.2. The triadic degrees are not completely clear here, see figure 13.12. The problem is that one would like to respect certain pitch relations. Except for degree $II, VII$ we obtain major or minor chords. If we want to have a minor chord for degree $II$ which has two tones in common with degree $IV$, we need to take the tone $d^* = -2q + t$ instead of $d = 2q$, i.e., $d^* - d = Kt$, the third comma—more precisely: its pitch class, see formulas (6.33) for the definitions. In the same sense, the seventh degree must, if we want it to be built from two minor thirds $d - b$ and $f - d$, refer to the second $f$-variant $f^* = f - Kt$. We then have the alternative degrees

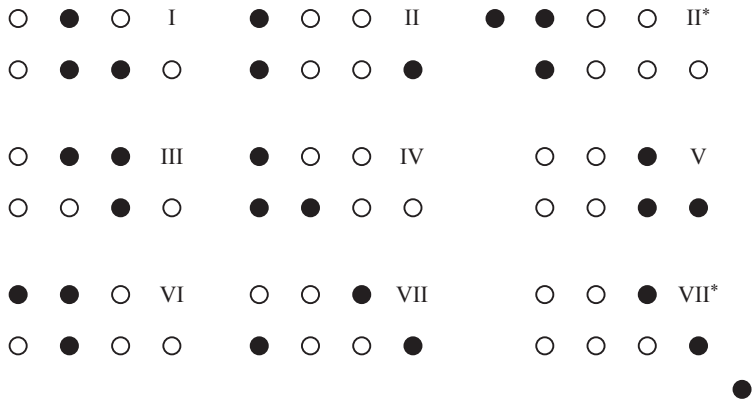


Figure 13.12: The list of just triadic degrees, together with the variants $II^*, VII^*$ which meet standard requirements of interval distances.

$II^* = \{d^*, f, a\}, VII^* = \{b, d, f^*\}$, and these have an empty intersection, in contrast to the canonical degrees $II, VII$.

In order to harmonize this irritating situation—which stems from pitch-distance requirements on degrees—one proceeds as follows: We no longer work with pitch classes $x$ but with *third comma classes*

$$]x[ = x + \mathbb{Z}Kt.$$

The *justest scales* can then be defined as local compositions by the quotient module $J_{Kt} = \mathbb{Z}^2/\mathbb{Z}Kt$, i.e., with ambient space $@J_{Kt}$, a space which can be identified with the Pythagorean tuning subspace $@\mathbb{Z}q$ in the *EulerPlane*. In fact, we have $\mathbb{Z}^2 = \mathbb{Z}q \oplus \mathbb{Z}Kt$, see also figure 13.13. With such a construction, we may transport the triadic degrees of just tuning $C^{(3)}$ to C-major in *justest tuning* , and thereby obtain the interpretation $\mathfrak{C}^{(3)}$ with degrees $\mathfrak{I}, \mathfrak{II}, \mathfrak{III}, \mathfrak{IV}, \mathfrak{V}, \mathfrak{VI}, \mathfrak{VII}$ with $\mathfrak{I} = ]I[ = \{]c[, ]e[, ]g[\}$, etc. In this setup, ambiguities from just tuning disappear, and the nerve $N(\mathfrak{C}^{(3)})$ is a Möbius strip. The second degree now can be realized by selecting one variant of the third comma equivalent just degrees (tone by tone, not as a whole!).

The atlas of degrees of a just triadic degree interpretation also remains invariant under the automorphism group of the just scale. For the *C-major* scale, this group is generated by

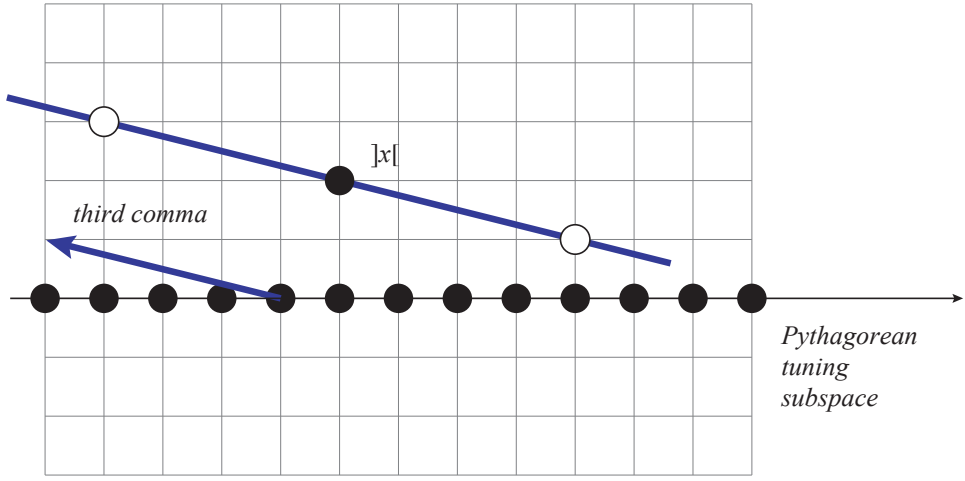


Figure 13.13: The third (or Pythagorean) comma classes cover all just pitch classes when starting from the Pythagorean subspace. Third comma classes solve the problem of the second and seventh degrees, one is allowed to select any one of the possible representatives within one comma class.

the skew reflection

$$A = e^q \cdot \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$

(see also example 11 in section 8.1.1). The orbits of the degrees look as follows:

$$\{I\}, \{II \leftrightarrow VII\}, \{III \leftrightarrow VI\}, \{IV \leftrightarrow V\}.$$

**Exercise 26** Verify the invariance and the orbit structure as described above.

We shall see however in chapter 27.1.6 that modulation in just tuning is also easily modeled without necessarily building justest tuning constructs.

### 13.4.3 Interpreting Time: Global Meters and Rhythms

**Summary.** Global meters and rhythms as interpretation by atlases of local rhythms and meters. We describe macros for rhythmic germs.

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In music, different local metrical structures mostly coexist in the same portion of music, as for example illustrated in figure 13.14. In this section, we want to give an account of global time constructions deduced from this type of phenomena.

Recall from section 7.2.2 that a local rhythm is defined as a union of translates of a local 'germ' composition in an ambient space $Onset \times Para$. We would like to include formally the
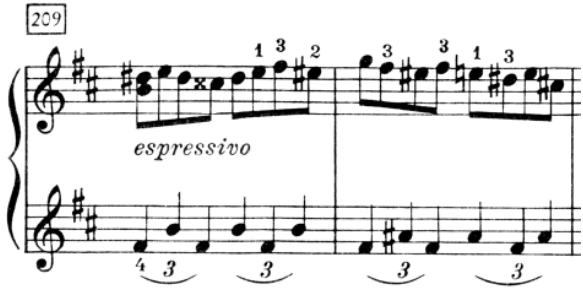
Figure 13.14: Example of a simultaneous presence of local meters in Beethoven's op.106, Allegro, bars 209-210. The right hand plays eight quavers while the left hand plays six triplets per meter.

case where the rhythm is a local meter (empty parameter set for space $Para$) on one hand, and where the parameters may vary, on the other. The first requirement is met by the form

$$Rhythm(Para) \underset{Id}{\longrightarrow} \mathbf{Colimit}(Onset, R(Para))$$

with $R(Para) \underset{Id}{\longrightarrow} \mathbf{Limit}(Onset, Para)$. This means that an objective $A$-addressed local composition $Rh$ in ambient space $Rhythm(Para)$ is a subset of $A@Onset \sqcup A@Onset \times A@Para$ (with corresponding functors of these forms). For any period $p : 0 \rightsquigarrow Duration(p)$, we have the evident translation action $e^p$ on $Rhythm(Para)$, and therefore on the local $Para$-rhythms, in the sense of section 7.2.2.

**Definition 41** *With the above notation, for a rhythm germ*

$$G \subset A@Rhythm(Para)$$

*and an interval* $[a, b]$ *of extended natural numbers,* a local ($A$-addressed, objective) $Para$-rhythm *is an $A$-addressed local composition $R$ defined by*

$$R = \bigcup_{t \in [a,b]} e^{tp} G$$

*and denoted by* $e^{[a,b]p}G$.

We have an obvious projection endomorphism

$$p_{meter} : Rhythm(Para) \to Rhythm(Para)$$

which is the identity on cofactor $Onset$ and projects onto the first factor on $R(Para)$. For a local composition $L$ in $Rhythm(Para)$, the projection $p_{meter}(L)$ is called the *associated metrical rhythm*, and the intersection $L \cap Onset$ is called the *metrical component* of $L$.

A local $Para$-rhythm defined by a singleton germ $G = \{g\}$ is called a *local $Para$-meter.* Conformal with the known terminology, a local $Para$-meter is again called a local meter if its

germ lives in the cofactor *Onset* and is equal to its *metrical component*; it is called local rhythm if its germ lives in the cofactor $R(Para)$.

A finite local $Para$-meter $M$ with germ $\{g\}$ may be written as $M = e^{[0,b]p}\{g\}$, with non-negative $b$; we abbreviate this object by $M = [b, p, g]$. For a local $Para$-meter $M$, $g$ is called the *origin* onset, while $b = l(M)$ its *length*. For length zero, we shall always assume the period to be also zero; the now uniquely defined period of a local $Para$-meter $M$ is denoted by $p(M)$.

The second requirement is met by suitably generalizing the form $Para$: Suppose that we want to take local rhythms from different parameter spaces

$$Para_1, Para_2, \dots Para_n.$$

Then we may collect these options to

$$Para \underset{Id}{\longrightarrow} \mathbf{Colimit}(Para_1, Para_2, \dots Para_n),$$

and the second problem is also solved.

**Lemma 18** *The intersection of any two A-addressed local Para-meters $R, S$ is a A-addressed local Para-meter.*

Proof. Let $R = [b, p, g], S = [c, q, h]$ be a representation by germs, periods, and origins. Suppose that $R \cap S$ is not empty (otherwise, the empty germ will do it), and take $x = e^{sp}g = e^{tq}h$. This implies that either $g, h$ are both onsets, or they have the same $Para$-component. WLOG[15], we may assume the first case, i.e., $g, h \in A@\mathbb{R}$ are $A$-addressed onsets. So $g, h$ have the same linear parts and if their translation parts are $\gamma, \eta \in \mathbb{R}$, respectively, we have the equation $sp+\gamma = tq+\eta$ of real numbers. If there is no other common point, we are done with length zero. Else, if there is another common value $\omega = s^*p + \gamma = t^*q + \eta$, we have $\Delta = (s - s^*)p = (t - t^*)q$. We may suppose that $\Delta$ is the smallest possible positive number of this type. Then we claim that the maximal integer interval $a \leq b$ such that $e^{[a,b]\Delta}\omega \subset R \cap S$ equals $R \cap S$. Clearly, the interval $[0, 0]$ is in $R \cap S$. So, there are such numbers $a, b$. Take a maximal interval. It is evidently unique. Take any element $y = s^{**}p+\gamma = t^{**}q+\eta$ in $R \cap S - e^{[a,b]\Delta}\omega$. It cannot lie between two elements of the local meter $e^{[a,b]\Delta}\omega$ since this would contradict minimality of $\Delta$. So the point lies to the right or left of the set $e^{[a,b]\Delta}\omega$. WLOG, suppose it lies to the right. Then we evidently can increase $b$ until we reach $y$, and this is a contradiction. So the intersection is this local meter. QED.

**Lemma 19** *For two local A-addressed Para-meters $R, S$, if $R \subset S$, then $l(R) \leq l(S)$.*

Proof: Exercise.

With this terminology, the interpretation $G^I$ of an objective, $A$-addressed local composition $G$ in ambient space $Rhythm(Para)$ is said to be *(Para-)rhythmical* iff every chart is a local ($Para$-)rhythm; it is said to be *(Para-)metrical* iff every chart is a local ($Para$-)meter. These definitions englobe Vuza's concepts of rhythms and of canons developed in his works [552, 554, 555, 556, 557].

[15] WLOG = without loss of generality.

In general, it is not easy to decide how to interpret a local composition by (*Para*-)rhythms since the decision between large repetition numbers and large germs has no evident rationale. When taking (*Para*-)meters, the situation is easier. We have several basic examples of (*Para*-)metric interpretations of an $A$-addressed objective local composition $X$ in $Rhythm(Para)$.

1. Consider the set $Max(X)$ of all maximal local $A$-addressed $Para$-meters contained in $X$. Observe that by lemma 19, the set $MaxMet(X)$ of all finite intersections of elements of $Max(X)$ consists of local $A$-addressed $Para$-meters which form a base of the so-called *maximal meter topology* on $X$. We identify it with the corresponding covering and also denote it by the interpretation symbol $X^{MaxMet}$. Its charts are called the *canonical* (local $A$-addressed) $Para$-meters of $X$.

2. Take the atlas of all local $A$-addressed $Para$-meters $U \subset X$ such that their lengths $l(U)$ are at most equal to a limit $L$. This interpretation is denoted by $X^{MetLg[L]}$.

3. Select the set of all local $A$-addressed $Para$-meters $U \subset X$ such that their periods $p(U)$ are at most equal to a limit $P$. This interpretation is denoted by $X^{MetPer[P]}$.

4. For any interpretation $K^I$ of a finite local composition $K$, we may introduce a level function $Lev : I \to \mathbb{N}$ as follows. We define the inverse images $lev_i = lev^{-1}(i)$ by recursion:

   - $lev_o = Max(I)$, the set of (set-theoretically) maximal members of $I$;
   - $lev_{i+1} = lev_0(I - \bigcup_{k \leq i} lev_k)$.

   The charts in $lev_j$ are called *charts of level* $j$, and we set $I|i = \bigcup_{k \leq i} lev_k$. We then evidently have an interpretation $K^{I|i}$, for each $i$.

   The above example of $MaxMet(X)$ has the atlas $Max(X)$ for its interpretation $X^{I|0}$, i.e., $Max(X) = MaxMet(X)|0$.

With the above notation, the *simplicial metrical weight* $sp(x)$ *of a point* $x \in X$ is defined to be the dimension of the simplex $Sp(x)$ of all maximal (=level zero) canonical $Para$-meters in $X^{MaxMet}$ which contain $x$. The *level* $lev(x)$ is defined to be the level of the (uniquely defined) smallest canonical $Para$-meter containing $x$.

**Proposition 11** *With the above notation, we have* $lev(x) \leq sp(x)$.

Proof. In fact, we have a surjective map of partially ordered sets

$$\bigcap : n(X^{MaxMet|0}) \to X^{MaxMet} : \sigma \mapsto \bigcap \sigma, \tag{13.6}$$

i.e., $\sigma \subset \tau$ implies $\bigcap \tau \subset \bigcap \sigma$, whence the claim. QED.

The open set $U(x) = \bigcap Sp(x)$ is the minimal neighborhood of $x$. The dominance relations in this topology read as follows:

**Lemma 20** *With the above notation, if* $x, y$ *are points in the local composition* $X$*, and if* $<$ *is the dominance relation*[16] *on the maximal meter topology, then we have*

$$y < x \text{ iff } Sp(y) \subset Sp(x) \text{ iff } U(x) \subset U(y).$$

[16] See appendix F.2.1.

Proof. We have $y < x$ iff $x \in U(y)$ iff $U(x) \subset U(y)$, whence $Sp(y) \subset Sp(x)$, whence $x \in U(x) \subset U(y)$ and therefore $U(x) \subset U(y)$. QED.

**Proposition 12** *The irreducible closed sets of the maximal meter topology are the closures of their points with maximal simplexes in the nerve.*

Proof. Suppose $F$ is an irreducible closed set and $x$ is a point having maximal simplicial metrical weight among the elements of $F$. If $y$ is any point of $F$, the minimal neighborhoods $U(x), U(y)$ intersect, so take a $z$ in that intersection. This point dominates $x$ and $y$. By maximality of the simplicial metrical weight of $x$, the simplicial metrical weight of $z$ is equal to the weight of $x$, and lemma 20 implies that $Sp(x) = Sp(z)$, so $x$ dominates $y$. The rest is clear. QED.

For example, the closures of points of maximal simplicial weight are irreducible components, but not necessarily vice versa. The irreducible components correspond to the maximal simplexes of the nerve $n(X^{MaxMet})$, and the generic points are the elements of the images of the maximal simplexes under the map $\bigcap$ (13.6). Musically speaking, the generic points are those which participate in maximal collections of local *Para*-meters, and this means that they are present in a maximal collection of regular patterns in onset time. We shall introduce refined metrical weight functions which generalize the above simplicial weight, and take care of the lengths of the local *Para*-meters as well as of the periods and the positions of the points within the maximal local *Para*-meters in chapter 21.

#### 13.4.3.1 Macros for Rhythmic Germs

We have seen that for general rhythmic germs, there is no unique maximality of germs, and that the metrical special case is built upon the singleton germ set. There are however two approaches in order to circumvent such a problem.

The first deals with address change. Suppose we are given a finite germ $G$ which is an $A$-addressed local composition which is completely contained either in the sub-form *Onset* or in the sub-form $R(Para)$ of $Rhythm(Para)$, and where $Para \xrightarrow{\sim} @PARA$ is simple with a module $PARA$. This means that the germ has coordinates in a module. Then the elements of $G$ are in $A@M$, where $M$ is one of the modules $\mathbb{R}, \mathbb{R} \oplus PARA$, and we suppose that the coefficient ring for A is $\mathbb{R}$. In this case, by the construction of fiber sums of modules (appendix E.3.8), we have[17] $(A@M)^n \xrightarrow{\sim} A^{\oplus n} \oplus \mathbb{R}^{n-1}@M$ and we can parametrize an $n$-element germ $G$ by the n-tuple $(G)$ of $A$-addressed points, and then by the corresponding point $\coprod(G)$ with address $\coprod_n A = (A)^n \oplus \mathbb{R}^{n-1}$.

The second solution to ambiguities in germ definitions uses macros and a flattening operation[18], as discussed in section 6.7. We start from the hypothesis of the preceding discussion and therefore may have the germ $G$ living in a simple form index$Makro_{Basic}Basic$ of module $M = \mathbb{R} \oplus PARA$ where—for the sake of mathematical simplicity—the first factor parametrizes the *Onset* coordinates. We need the circular form $Knot_{Basic}$:

[17] The bijection takes the n-tuple $(g_i = e^{t_i} \cdot g_{i,0})_i$ into $e^{t_1} \cdot g$, where $g(a_1, \ldots) = g_1(a_a)$, $g(0, \ldots a_i, \ldots) = g_{i,0}(a_i)$ for $1 < i$, and $g(0, \ldots 1_{n+j}, \ldots) = t_{j+1} - t_1$ for $j = 1, \ldots n-1$. The injections $\iota_i : A \rightarrowtail \coprod_n A$ are these: $\iota_1(a) = (a, 0, \ldots)$, $\iota_i(a) = (0, \ldots a_i = a, 0, \ldots 1_{n+i-1}, \ldots, 0)$ for $1 < i$.

[18] The present flattening operation is just the infinite one from the previous discussion.

$$Knot_{Basic} \underset{Id}{\longrightarrow} \mathbf{Limit}(Basic, Makro_{Basic})$$
$$\text{with } Makro_{Basic} \underset{f:F\tilde{\to}2^{FK}\rightarrowtail\Omega^{FK}}{\longrightarrow} \mathbf{Power}(Knot_{Basic})$$
$$\text{and } F = Fun(Makro_{Basic}), FK = Fun(Knot_{Basic}).$$

We have the flattening operation on $Knot_{Basic}$ Denotators $D = (b, m)$ defined by

1. $Flatten^{\infty}(D) = \{b\}$ if $m = \emptyset$,
2. $Flatten^{\infty}(D) = \bigcup_i b + Flatten^{\infty}(Knot_i)$, if $m = \{Knot_1, \ldots Knot_k\}$.

Then the idea is to define a macro germ $G^*$ such that the given, 'flat' germ $G$ is the flattened version of the former, $G = Flatten^{\infty}(G^*)$. For example, we may take a germ which contains ornaments as $Makro_{basic}$-formed denotators, and/or drum patterns or the like as 'satellites' of the macro. The translation in time is the same as before, only that it operates exclusively on the basic event in the knot macro, i.e., if $e^p$ is a period translation in time, and if the macro germ is $G^* = (b, m)$, we define $e^p(G^*) = (e^p(b), m)$.

An $A$-addressed local $Para$-rhythm can be paraphrased by the local compositions in ambient space $Knot_{Basic}$; they are local compositions of the shape $e^{[a,b]p}G^*$, built upon macros $G^*$. In this language, a global rhythm is an interpretation of a local composition in ambient space $Knot_{Basic}$ by local $Para$-rhythm of macro type. So global macro rhythms can be projected to global rhythms by flattening, and they constitute a hierarchical shaping of the flat rhythmical germs.

We should stress that these reflections are merely 'germs' of a theory of macro-events, and that we are far from having established any systematic treatment of such objects. But there is an evident relation of this approach to the Schenker idea, a perspective which we shall reconsider in chapter 21 when discussing grouping rules in Jackendoff-Lerdahl's GTTM.

### 13.4.4 Motivic Interpretations: Melodies and Themes

**Summary.** Motives are considered as being local ingredients of melodies. We discuss the local/global dichotomy in this subject, and, in particular, the subtle interpretative activity in motivic analysis.

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Concepts such as "motif", "melody", "theme" are among the most fuzzy of musicology. There are three main sources of such fuzziness: On the one hand, the very concepts are not clear, neither in the delimitation against each other, nor in the intrinsic attribution. On the other hand, it is not clear how to compare motives, for example, how to decide when two motives are "similar". Thirdly, the concept of a theme or motive is not only a structural one, but includes a *semiotic* perspective. A theme is a subcomposition of a composition which plays a certain role in the entire construction of the composition qua expression of meaning. We shall deal with this aspect in chapter 22. We should also stress that a theme is not necessarily related to the motif

concept, a theme may as well be a harmonic or rhythmic object. Its characteristic property is the semiotic role, not the structure.

In this section, we shortly focus our attention on the problem of elementary and compound motivic structures. We have already introduced the motif concept in definition 15 (subsection 7.2.3) as an objective local composition with onset and pitch coordinators in its ambient space, and such that the onset projection is bijective.

We should therefore define a "motivic" interpretation as being an interpretation of a local composition in a determined motif ambient space (having onset and pitch factors) such that every chart is a motif in this space. In particular:

**Definition 42** *A* melody *is defined as a motivic interpretation of a motif.*

In fact, a melody is a local composition with successive tones in time. But it is more: it is understood as a compound object with "characteristic" melodic ingredients, the melody's motives. Now, these units are not neutral data, they depend upon the interpretative, analytical, or poietic interaction. In other words: they are a covering of the motivic tone material of the melody by specific selected submotives.

In traditional musicology (see, e.g., the article "Melodie" in the Riemann Musiklexikon [457, p.554]), a melody is also associated with the Ehrenfels gestalt qualities of super-summativity and transformational invariance. The AST has dealt with this aspect in the contour theory. We shall deal with this ramification in chapter 22. Gestalt is a central invariant of melody, but it is not the intrinsic concept, it is a *derived* attribute which comes out from a complex and not uniquely defined abstraction process. Every contour or gestalt or shape concept is derived from the motif structure by elaboration of specific motif aspects, such as interval sequences, onset-pitch-interval angles, length ratios, diastematic ambitus, etc.

Before the gestalt aspect comes in, the motivic interpretation which defines the melody upon a motif has a more elementary—but nonetheless dramatic—function which we discuss now. If we cover a motif by characteristic motifs, as expressed by the melody's nerve, this action reflects a disintegration of the original underlying melody motif into several relatively small parts which may intersect in a rather loose way. For example, the covering may split into several disconnected (disjoint) motives, or motives may have just one tone in common. The crucial question is whether we should keep in mind the original melodic motif (the entire set of tones) or whether the splitting activity is really meant as a decomposition in henceforth in- or inter-dependent parts. So this elementary interpretation activity precedes refined considerations: it deals with the initial statement of which are the parts and which is the whole.

But even if we are given two fixed motivic interpretations, i.e., melodies, it is not clear how one should compare two such objects. Gestalt is a concept of equivalence which should be built upon comparison devices for melodies. The only thing which is clear now is that melodies are not neutral objects, they are results of interpretative interventions. We are urged to set up the basic tools for comparison of global compositions, and this is what we shall do in the next chapter.

# Chapter 14

# Global Perspectives

*The woods are lovely, dark, and deep,*
*But I have promises to keep,*
*And miles to go before I sleep,*
*And miles to go before I sleep.*
Robert Frost (1874–1963)

**Summary.** Global perspectives deal with relations among global music objects. We introduce this subject together with its musical motivation and append the formal definition of morphisms among global compositions. This leads to the categories *ObGlob* of objective and *Glob* of functorial global compositions. We describe the combinatorial aspects of globality and associated functors, as well as corresponding geometric classification tools: nerves and simplicial weights.

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## 14.1 Musical Motivation

**Summary.** Since global compositions are of musical and musicological interest, requiring their comparison is quasi-automatic. However, this is not in the tradition of musical analysis. We discuss this issue with regard to comparative analysis of melodic themes in a composition, and in the more general context of comparative analysis of musical works. It turns out that the latter is virtually non-existent, at least on the level of a reliable scientific language.

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Mathematically, it is straightforward how to extend the categories of local compositions to categories of global compositions. But musicologically, this is not a common situation, on the contrary, The musicological fuzziness of global structures appearing in melodic analysis—which we have made precise via motivic interpretations—makes it impossible to take a step further in the comparative methodology of global structures. In the typical case of a comparison of two melodies, or in the discussion of the motivic-thematic work in, say, Beethoven's compositions, it is intuitively accepted that one deals with smaller and larger units, and that comparison

should take into account comparison of parts, but it is not clear how the relations of parts within a first melody (for example) should be taken over to relations among parts of a second melody. Comparison of the gluing of parts is not conceived. So Ehrenfels' requirement of an "added value" to the "sum of parts" against the "whole" is not understood: How much more do we need to get the whole? How can we compare this specific difference. For example, if we have split a local composition into isolated singleton charts (the 'silly' interpretation), the disintregation is extremal: We essentially have to count the tones, and there is no connection left. So the comparison reduces to counting notes, the cardinality of the original local composition. This is the trivial case where the added value is naught. But in general, for example with the harmonic strip, there is a rich connectivity, in fact the Möbius strip, in the nerve. So the added value is complex: it lies in the single charts (the triads) as well in their intersections (thirds or single notes) and also in the combinatorial configuration. So, how strongly is the harmonic strip related to the original scale? Are there many scales with essentially equal harmonic strips? Same situation for motivic analysis: If we take the main theme of Bach's *Art of Fugue*, is it the 8-tone theme or the 12-tone theme? This still debated question depends on the development of a sophisticated technique of motivic analysis of melodies or themes. And, finally, without answering these questions, a comparison of gestalts is a fortiori impossible.

On the other hand, overall comparative analysis of musical works of course transcends the theory of local or global compositions. In fact, the denotators of musical works, such as European scores, cannot be grasped by compositions (in the technical sense), they deal with more general forms of all possible typologies: limits, colimits, synonymy, and all kinds of simple forms, see the contribution of Mariana Montiel Hernandez [378] concerning a standard form for piano scores. In this chapter we shall introduce only comparative tools (morphisms) for global compositions. The category theory for general denotators is still embryonal.

## 14.2 Global Morphisms

**Summary.** This section introduces the technical definition of morphisms between global compositions. Special aspects, e.g., isomorphisms or composition of morphisms, are discussed. The concept of interpretable global compositions is presented and illustrated.

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The first definition regards objective global compositions:

**Definition 43** *Suppose we are given two objective global compositions $G^I$ at address $A$ and $H^J$ at address $B$. Then a morphism from $G^I$ to $H^J$ is a triple $(f, \iota, \alpha)$ where*

1. *$f : G \to H$ is a set map,*
2. *$\iota : I \to J$ is a set map such that $f(i) \subset \iota(i)$ for all covering sets $i \in I$,*
3. *$\alpha$ is a family $\alpha = (\alpha_i : A \to B)_I$ of address changes, such that*
4. *for any atlases $(K_t, A@E_t)_T$ for $G$ and $(L_s, B@F_s)_S$ for $H$, if we take the chart isomorphisms $\phi_t : K_t \xrightarrow{\sim} I_t$ and $\psi_s : L_s \xrightarrow{\sim} J_s$ for some pair $t, s$ of indexes which correspond under the map $\iota$ (i.e., $\iota(I_t) = J_s$), then the induced maps $f_t : K_t \to L_s$ define morphisms $f_t/\alpha_i : K_t \to L_s$ of objective local compositions.*

*This morphism will be denoted by* $\overset{\iota}{f}/\alpha : G^I \to H^J$ *or, if no misunderstanding is possible, by* $f/\alpha : G \to H$.

Observe that this is the precise generalization of the morphism concept from objective local compositions. For functorial global compositions, the definition of morphisms runs correspondingly:

**Definition 44** *Suppose we are given two functorial global compositions* $G^I$ *at address* $A$ *and* $H^J$ *at address* $B$. *Then a morphism from* $G^I$ *to* $H^J$ *is a triple* $(f, \iota, \alpha)$ *where*

1. $f : G \to H$ *is a natural transformation,*
2. $\iota : I \to J$ *is a set map such that we have subfunctor relations* $im(f|i) \subset \iota(i)$ *for all covering functors* $i \in I$,
3. $\alpha$ *is a family* $\alpha = (\alpha_i : A \to B)_I$ *of address changes, such that*
4. *for any atlases* $(K_t, @A \times E_t)_T$ *for* $G$ *and* $(L_s, @B \times F_s)_S$ *for* $H$, *if we take the chart isomorphisms* $\phi_t : K_t \overset{\sim}{\to} I_t$ *and* $\psi_s : L_s \overset{\sim}{\to} J_s$ *for some pair* $t, s$ *of indexes which correspond under the map* $\iota$ *(i.e.,* $\iota(I_t) = J_s$*), then the induced maps* $f_t : K_t \to L_s$ *define a morphism* $f_t/\alpha_i$ *of functorial local compositions.*

*This morphism will be also be denoted by* $\overset{\iota}{f}/\alpha : G^I \to H^J$ *or, if no misunderstanding is possible, by* $f/\alpha : G \to H$.

If we have $G^I = H^J$, the identity morphism is the triple $(Id_G, Id_I, Id_A)$. This being the case, defining the composition of morphisms of global compositions is straightforward.

**Definition 45** *Given three objective or functorial global compositions* $G^I, H^J, K^L$, *a morphism* $\overset{\iota}{f}/\alpha : G^I \to H^J$, *and a morphism* $\overset{\kappa}{g}/\beta : H^J \to K^L$, *their composition* $\overset{\zeta}{h}/\gamma = \overset{\kappa}{g}/\beta \cdot \overset{\iota}{f}/\alpha : G^I \to K^L$ *is defined by the triple* $(h = g \cdot f, \zeta = \kappa \cdot \iota, \gamma = \beta \cdot \alpha)$, *where* $\beta \cdot \alpha$ *is the family* $(\beta_{\iota(i)} \cdot \alpha_i)_I$.

**Exercise 27** Show that composition of global morphisms is associative whenever corresponding factors are defined. Show that the identity morphisms are in fact left and right identities in the sense of category theory.

**Definition 46** *The category of objective global compositions ObGlob has the objective global compositions as its object set* ${}_0ObGlob$, *as defined in definition 36, whereas the set* ${}_1ObGlob$ *of morphisms is defined by definition 43.*

*The category of functorial global compositions Glob has the functorial global compositions as its object set* ${}_0Glob$, *as defined in definition 38 whereas the set* ${}_1Glob$ *of morphisms is defined by definition 44.*

It is useful to view functorial global compositions as a system of objective global compositions as follows:

**Lemma 21** *Let $G^I$ be a functorial global composition at address $A$. Suppose that $\phi_\kappa : K_\kappa \xrightarrow{\sim} I_\kappa$ and $\phi_\lambda : K_\lambda \xrightarrow{\sim} I_\lambda$ are any two charts for $G^I$. Fix any address change $f : B \to A$, and consider the subsets $f@K_\kappa \subset B@K_\kappa$, $f@K_\lambda \subset B@K_\lambda$. Then the intersection of their images under $B@\phi_\kappa$ and $B@\phi_\lambda$, respectively, with $B@I_\kappa \cap B@I_\lambda$ coincide.*

Proof. In fact, the intersections are the same since the induced isomorphism $\phi_{\kappa,\lambda}/\alpha$ on the inverse images of the intersection have denominator $\alpha = Id_A$, and therefore, the parts related to $f$ correspond.

**Definition 47** *For every address change $f : B \to A$ and a functorial global composition $G^I$ at address $A$, we have a $B$-addressed objective global composition $f@G^I$ which is defined on the subset of $B@G$ which is locally covered by the images $f@I_\kappa$ of the chart subsets $f@K_\kappa$ of any atlas, and which has the atlas defined by the second factors in the charts $f@K_\kappa = \{f\} \times K_{\kappa,f}$, together with the given transition isomorphisms. We denote this covering of $f@G$ by $f@I$, and call the objective global composition the $f$-*slice of $G^I$. *The entire construction is well defined by lemma 21. The identity slice $Id_A@G^I$ is denoted by $(G^I)^\vee$.*

**Example 23** Given two interpretations $G^I, G^J$ of an objective local $A$-addressed composition $G$, and a *refinement map*, i.e., a map $\iota : I \to J$ with $i \subset \iota(i)$, all $i \in I$, then the identity $Id_G$ and $\iota$ induce the *refinement morphism* $Id_A/^\iota Id_A : G^I \to G^J$, also denoted by $\iota$. In particular, any interpretation $G^I$ yields a unique refinement morphism $! : G^I \to G$.

For example, with the notation of 13.4.3, we have a chain of refinements

$$K^{I|0} \to K^{I|1} \to \dots K^{I|n} \to K^{I|n+1} \to K^I \tag{14.1}$$

of an $A$-addressed finite objective local composition $K$.

If the local composition is a major or minor scale scale $X$ as discussed in section 13.4.2, we may consider the different versions of degrees. We have discussed the triadic interpretation $X^{(3)}$, but this is just one possibility. More generally, set $X = \{x_1, x_2, \dots x_7\}$ in the order of increasing position on $\mathbb{Z}_{12}$, i.e., also with indexes mod 7. One starts with the singleton interpretation $X^{(1)}$, where $(1) = \{\{x_i\} |\ i = 1, 2, \dots 7\}$; the third interpretation $X^{(2)}$ is defined by charts of thirds, i.e., $(2) = \{\{x_i, x_{i+2}\} |\ i = 1, 2, \dots 7\}$; the triadic interpretation is defined by the atlas $(3) = \{\{x_i, x_{i+2}, x_{i+4}\} |\ i = 1, 2, \dots 7\}$; and the tetradic interpretation $X^{(4)}$ which is important in jazz harmonics is defined by the atlas $(4) = \{\{x_i, x_{i+2}, x_{i+4}, x_{i+6}\} |\ i = 1, 2, \dots 7\}$.

With this setup, we have $2^7$ refinement maps for each succession of the above atlases. More precisely, if $Z^i$ is the degree $Z$ in $X^{(i)}$, we have two embeddings $Z^i \subset W^{i+1}$ depending on whether we take $Z^i$ as the lower or higher part within $W^{i+1}$. The lower embedding is denoted by 0, the higher one by 1, and the refinement map $\delta = (\delta_1, \delta_2, \dots \delta_7) : (i) \to (i+1)$ denotes the choice of $\delta_i \in BIT = \{0, 1\}$ for the $i$th value. So we have this succession of refinements, ending with the unique morphism to the local scale composition:

$$X^{(1)} \xrightarrow{\delta} X^{(2)} \xrightarrow{\delta} X^{(3)} \xrightarrow{\delta} X^{(4)} \xrightarrow{!} X \tag{14.2}$$

which has, of course an important musicological meaning: If ever we are given a chart of atlas $(i)$, it may be reinterpreted as being one of two charts of the successive refinements, etc., and

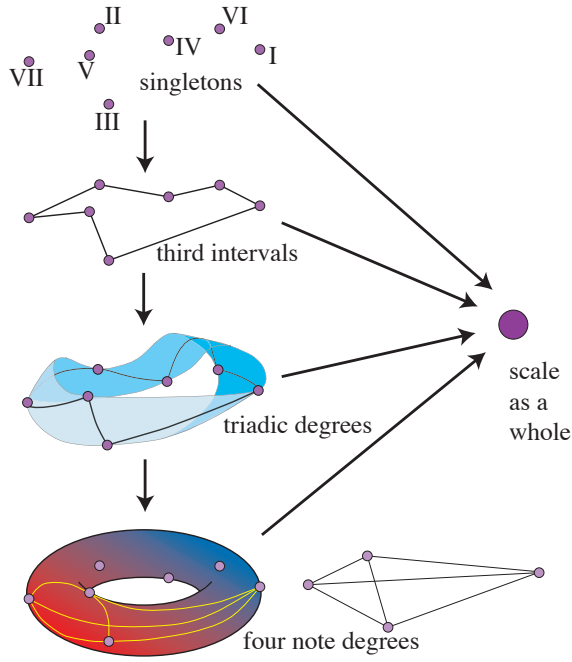


Figure 14.1: The four interpretations of a major or minor scale, starting with the singletons, up to the tetradic interpretation for jazz harmony. We have successive refinement morphisms, and associated maps between nerves (see section 14.4 for this association). Every interpretation uniquely maps to the 'uninterpreted' local scale composition shown to the right. Observe that the tetradic interpretation has a nerve which is a full torus, in fact a union of tetrahedra, the intersection configuration around any fixed tone. The harmonic strip of the triadic interpretation is embedded in the tetradic torus as a strip which is 'entwined' along the torus' interior circle. So the absence of orientation of the harmonic strip vanishes on the tetradic jazz harmony.

this documents the vast ambiguity in elementary harmony of degrees! For the corresponding nerves, see figure 14.1.

This ambiguity is also dramatic with regard to the geometric configuration of the nerve geometry. The absence of orientation on the harmonic strip vanishes on the tetradic interpretation of jazz. This may not only ease jazz harmony, more radically, it throws the context of jazz harmony one dimension higher: Moving in a three-space such as the torus, is more complex, there is more freedom of choice, more improvisational flexibility—at the cost of unambiguous harmonic syntagmatics.

**Example 24** This example refers to the analysis in [328] of the first movement of Beethoven's 'Hammerklavier' sonata Op. 106. The general harmonic principles of this sonata yield a motivic germ $G \subset Onset \times Pitch|_{\mathbb{Z}[1/2]}$ with coefficients in the localization ring $\mathbb{Z}[1/2]$ of fractions with denominators that are powers of 2. This germ is essentially an ascending and then descending

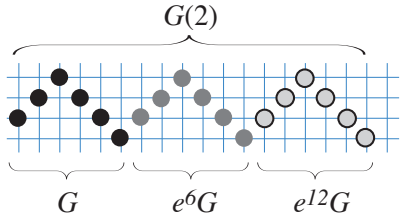


Figure 14.2: The germ $G$ of motivic work in Beethoven's 'Hammerklavier' sonata Op. 106 is essentially an ascending and then descending chromatic scale of three half-tone steps 'octave' period. Here we show the *Pitch*-rhythm with three copies of the germ.

chromatic scale of three half-tone steps 'octave' period. The claim of our analysis is that the motivic work (German: "motivisch-thematische Arbeit") relates to interpretations of the *Pitch*-rhythms of type $G(t) = e^{[0,t]6}G$, see also figure 14.2 for this 'motivic zig-zag'. In [328] it is shown

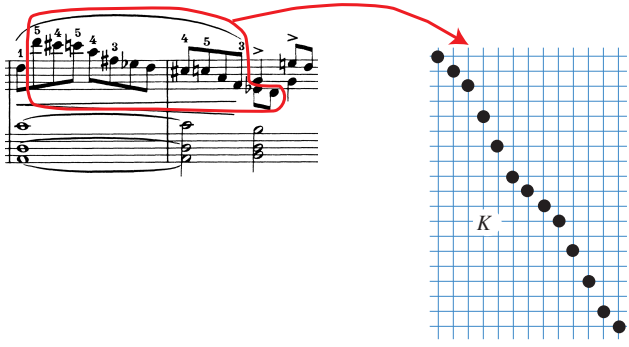


Figure 14.3: The right-hand melody which appears in bars 75-76, and 77-78 of the first movement in op. 106.

that a large number of motivic charts can in fact be interpreted such as to become isomorphic to interpretations of local subcompositions of $G(t)$. Of course, the problem is not the existence of such interpretations since the silly interpretations always fit in such a framwork. We therefore look for good quality interpretations.

**Definition 48** *Let $K^I$ be an interpretation of a local composition $K$. The quality of $K^I$ is the pair $(s, p)$, where $s = card(I)$, and $p$ is the cardinality of connected components of $N(K^I)$. Qualities are compared lexicographically: $(s_1, p_1) < (s_2, p_2)$ iff either $s_1 < s_2$ or $s_1 = s_2$ and $p_1 < p_2$. So the best quality is $(1, 1)$.*

With this in mind, we look for best possible qualities in our interpretations. To begin with, we look at the right-hand melody which appears in bars 75-78 (figure 14.3). The (projected) representation $K$ in *Onset* $\times$ *Pitch* space is shown to the score's right.

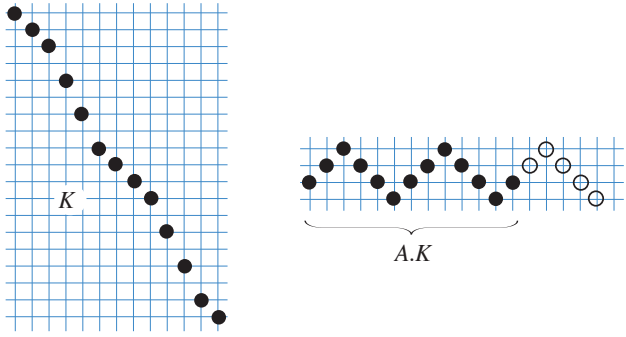


Figure 14.4: The right-hand melody in bars 75-76 and 77-78 is isomorphic to a local subcomposition of the motivic 'zig-zag' under a vertical arpeggio $A$.

In figure 14.4, we see that this melodic part is isomorphic as a local composition (quality $(1,1)$) to a subcomposition of $G(2)$. The isomorphism is given by the matrix

$$A = e^q \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

representing a vertical arpeggio.

The second sample needs a proper interpretation of quality $(3,3)$. The sample, a twelve-element motif, is shown to the left in figure 14.5. To the right we see the (projected) representation and interpretation $K$ of this motif in $Onset \times Pitch$ space.

In figure 14.6 we see the subcomposition $L$ of $G(2)$, as well as its interpretation $L^J$ by three disjoint charts $L_1, L_2, L_3$. This interpretation is isomorphic to the interpretation $K^I$ under the following isomorphisms: $L_2$ and $L_3$ are translations of $K_2$ and $K_3$, respectively. The first chart $L_1$ equals $B \cdot K_1$ with a transformation

$$B = e^q \cdot \begin{pmatrix} 2 & 9 \\ 0 & -1 \end{pmatrix}$$

where the linear part is an onset dilatation by 2, followed by a ninefold horizontal arpeggio, and the pitch inversion, an isomorphism since its determinant $-2$ is invertible in $\mathbb{Z}[1/2]$.

**Example 25** Consider the two following non-isomorphic, zero-addressed three-element motives in $OnPiMod_{12,12}$:

$$K = \{(0,0),(0,1),(1,0)\} \text{ isomorphism class Nr.10,}$$
$$L = \{(0,0),(0,1),(3,11)\} \text{ isomorphism class Nr.13,}$$

and consider the interpretations $K^{(2)}, L^{(2)}$ by the three two-element 'interval' subcompositions. Evidently, every interval is isomorphic to $\{(0,0),(0,1)\}$, and since the intersections are singletons, we have $K^{(2)} \xrightarrow{\sim} L^{(2)}$. So the isomorphism of the interpretations by submaximal charts is no guarantee for the isomorphism of the original local compositions.

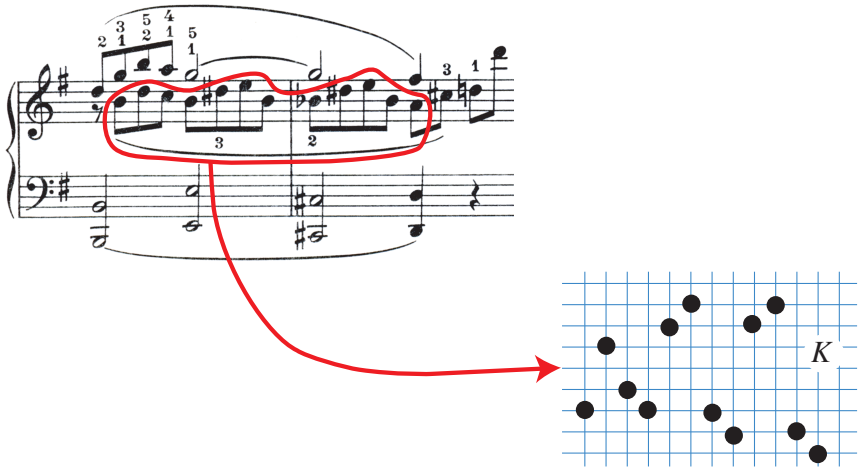


Figure 14.5: A twelve-element alto-voice motif $K$ in bars 79-80 in the first movement of op. 106.

**Exercise 28** Consider the following zero-addressed local compositions in $PiMod_{12}$:

$$C = \{0, 2, 4, 5, 7, 8, 9\} \text{ C-major scale, isomorphism class Nr } 38\widehat{},$$
$$X = \{0, 2, 3, 4, 7, 10, 11\} \text{ isomorphism class Nr } 50\widehat{},$$

which are not isomorphic. Take the triadic interpretation $C^{(3)}$ and the following triadic interpretation of $X$:

$$I_X = \{11, 3, 0\}, II_X = \{7, 10, 2\}, III_X = \{3, 0, 4\},$$
$$IV_X = \{10, 2, 11\}, V_X = \{0, 4, 7\}, VI_X = \{2, 11, 3\},$$
$$VII_X = \{4, 7, 10\}$$

and call the associated interpretation $X^{(3)}$. Show that we have an isomorphism $\overset{\iota}{f}/1 : X^{(3)} \tilde{\to} C^{(3)}$ which is defined by

$$f(0) = 7, f(2) = 9, f(3) = 4, f(4) = 11,$$
$$f(7) = 2, f(10) = 5, f(11) = 0,$$
$$\iota(Y_X) = Y_C, Y = I, \ldots VII.$$

Show that this example fails if we do not allow isomorphisms with fifth or fourth transformations, i.e., if *only* inversions and transpositions are allowed, then we have $C \tilde{\to} X$ as soon as $C^{(3)} \tilde{\to} X^{(3)}$.

**Exercise 29** Show that the ecclesiastic modes (section 13.4.2) are *rigid*, i.e., they have trivial automorphism group: $Aut(X_{f,t}) = 1$; it is even true that any two different ecclesiastic modes

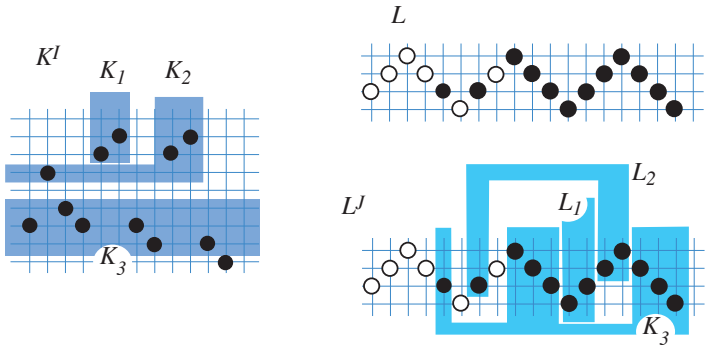


Figure 14.6: The interpretation of the alto voice in bars 79-80 of quality $(3, 3)$.

on the same scale are not isomorphic. This is false for modes $X_f$, in fact, since $Aut(C) = \langle U_d \rangle$, we have a permutation of these modes (we take the authentic mode names):

$$U_d(C_d) = C_d, \text{ dorian} \leftrightarrow \text{dorian}$$
$$U_d(C_e) = C_c, \text{ phrygian} \leftrightarrow \text{ionian}$$
$$U_d(C_f) = C_b, \text{ lydian} \leftrightarrow \text{locrian}$$
$$U_d(C_g) = C_a, \text{ mixolydian} \leftrightarrow \text{aeolian}$$

and this means that modes become rigid when transformed into their ecclesiastic castings, a completely traditional effect...

## 14.3 Local Domains

**Summary.** The morphisms which are addressed in local compositions separate morphisms of global compositions. We discuss the musical consequences.

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The Yoneda lemma (see our discussion in section 9) tells us that we may replace any global composition $X$ in $Glob$ by its functor $@X \in Glob^{@}$ and we shall not lose any information about its isomorphism class, more precisely, for any two global compositions, $X, Y$, we have a canonical bijection $Hom(X, Y) \xrightarrow{\sim} Hom(@X, @Y)$. It is a fundamental question in every category to find out whether some determined subclass of arguments of the functor $@X$ is already rich enough to determine $X$ up to isomorphism. In algebraic geometry, it is well known ([198]) that the restriction $@S|_{Aff}$ of the scheme functor $@S$ of a scheme $S$ to the subcategory $Aff$ of the affine schemes is classifying, i.e.,if $S, T$ are two schemes, we have a bijection $Hom(@S, @T) \xrightarrow{\sim} Hom(@S|_{Aff}, @T|_{Aff})$.

We shall now show that we have a weaker result in the theory of global compositions, i.e., the local arguments are 'separating'.

**Definition 49** *Let $X$ be a global composition in Glob. By $@_{loc}X$ we denote the restriction of the Yoneda functor $@X$ to the subcategory Loc of local compositions. By definition, we have $Y\,@_{loc}X = Y\,@X = Hom(Y, X)$ for any local (functorial) composition $Y$.*

Since a local composition has just one chart, the map on the atlases for a local domain $Y$ reduces to the indication of one chart $I_i$ or chart index $i$ in the target composition $X$. Therefore we write

$$\overset{i}{f/\alpha} : Y \to X$$

to indicate elements of $Y@_{loc}X$.

**Lemma 22** *If $X^I, Y^J$ are local compositions at addresses $A, B$, respectively, the canonical restriction map*

$$loc : Hom(@X^I, @Y^J) \to Hom(@_{loc}X^I, @_{loc}Y^J)$$

*is injective, i.e., local domains are separating. If $X^I = X$ is a local composition, then loc is a bijection.*

Proof. Let $u : @X^I \to @Y^J$ be a natural transformation. We show that $u$ is uniquely determined by its restriction to local arguments. By the Yoneda lemma we know that $u = @\overset{\iota}{f/\alpha}$ for a morphism $\overset{\iota}{f/\alpha} : X \to Y$. So let us show that $\overset{\iota}{f/\alpha}$ is determined by the restriction $loc(u)$ to local arguments. To this end, consider the canonical embeddings $\overset{i}{h_i/Id_A} : X_i \to X$ of the subfunctors $X_i$ of the covering $I$ of $X$. We have $u(\overset{i}{h_i/Id_A}) = \overset{\iota}{f/\alpha} \cdot \overset{i}{h_i/Id_A} = \overset{\iota(i)}{f \cdot h_i \,/\, \alpha_i}$, where $\alpha_i$ is the address change of the $i^{th}$ covering subfunctor. Therefore, the map $\iota : I \to J$ is determined by the local evaluation at $X_i$. Also, the address change $\alpha = (\alpha_i)_i$ is determined, and the restriction $f \cdot h_i$ on the covering $(X_i)$ of $X$ determines $f$. Therefore $\overset{\iota}{f/\alpha}$ and its functorial counterpart $u = @\overset{\iota}{f/\alpha}$ is uniquely determined by the local restrictions.

Let now $X$ be local and let $u : @_{loc}X \to @_{loc}Y^J$ be a natural transformation. We have to extend $u$ to a natural transformation $U : @X \to @Y^J$. Let $W^I$ be a global composition with charts $W_t$. Then the difference kernel $Ker(\prod_t W_t @X \rightrightarrows \prod_{s,t} W_{s,t} @X)$ of the restriction morphisms $W_{s,t} \to W_t, W_{s,t} \to W_s$ identifies[1] to $W^I @X$. By $u$, this kernel is mapped into the kernel $Ker(\prod_t W_t @Y^J \rightrightarrows \prod_{s,t} W_{s,t} @Y^J)$ which is a subset of $W^I @Y^J$, and we are done. QED.

**Lemma 23** *If $X^I, Y^J$ are local compositions at addresses $A, B$, respectively, then there is a bijection*

$$l : Hom(@_{loc}X^I, @_{loc}Y^J) \xrightarrow{\sim} \coprod_{\iota : I \to J} \prod_{i \in I} X_i @Y_{\iota(i)}.$$

Proof. In fact, the contravariant functors $@_{loc}X^I, @_{loc}Y^J : Loc \to \mathbf{Sets}$, respectively, are isomorphic to $\coprod_{i \in I} @_{loc}X_i, \coprod_{j \in J} @_{loc}Y_j$, respectively. Hence,

$$Hom(@_{loc}X^I, @_{loc}Y^J) \xrightarrow{\sim} \prod_{i \in I} Hom(@_{loc}X_i, @_{loc}Y^J)$$

[1] For global codomains, the kernel is strictly smaller than the morphism set because of the index map!

and by lemma 22 and Yoneda's lemma

$$Hom(@_{loc}X_i, @_{loc}Y^J) \xrightarrow{\sim} X_i@_{loc}Y^J \xrightarrow{\sim} \coprod_{j\in J} X_i@Y_j$$

whence our claim follows from the distributivity laws for products and coproducts of set-valued functors. QED.

**Lemma 24** *With the above notation a natural transformation* $u : @_{loc}X^I \to @_{loc}Y^J$ *stems from a morphism* $X^I \to Y^J$ *of global compositions iff its image*

$$l(u) = (f_i/\alpha_i : X_i \to Y_{\iota(i)})_{i\in I}$$

*with index map* $\iota : I \to J$ *has the property that for every couple* $i_1, i_2 \in I$*, the transformations* $f_{i_1}, f_{i_2}$ *coincide on the intersection functor* $X_{i_1,i_2} = X_{i_1} \cap X_{i_2}$*. By* $Hom^{loc}(@_{loc}X^I, @_{loc}Y^J)$*, we denote the set of these natural transformations and call them* localizable*.*

**Exercise 30** Prove these statements.

So we have this proposition:

**Proposition 13** *If* $X^I, Y^J$ *are local compositions at addresses* $A, B$*, respectively, then the canonical restriction map*

$$loc : Hom(X^I, Y^J) \to Hom^{loc}(@_{loc}X^I, @_{loc}Y^J)$$

*is well defined and bijective.*

In other words, we may replace the study of isomorphism classes of global compositions $X^I$ by the study of the isomorphism classes of their local domain functors $@_{loc}X^I$ under localizable natural transformations (which define a non-full subcategory ${}^{loc}Loc^{@}$ on the local domain functors).

The musicological meaning of these results is that the Yoneda philosophy for global musical structures may be restricted to local perspectives, i.e., morphisms on local compositions. With the restriction that the functors of local perspectives only reflect isomorphisms of global compositions if these isomorphisms are localizable.

We are however far from completely understanding the nature of the local domain functors and their category ${}^{loc}Loc^{@}$. But we know that above the local domain argument, the nature of the patchwork of the local aspects is crucial, and this is why we look at the nerve functor now.

## 14.4 Nerves

**Summary.** To each global composition is associated a combinatorial structure, its *nerve*, in fact the simplicial complex of the covering of the composition by its charts. This functorial association retrieves important information on the global object.

– Σ –

Nerves of objective global compositions were introduced in 13.2.1. For functorial global compositions, we have the same procedure. The *abstract nerve* $n(G^I)$ of a global functorial composition is the covering's simplicial complex, i.e., the simplicial complex whose vertexes are the covering subfunctors $G_i$, and whose simplexes are the (finite) vertex sets with non-empty intersection; the *geometric nerve* $N(G^I)$ is defined as the space $|n(G^I)|$ of $n(G^I)$, but see appendix H.2.2, example 103. Since $I$ is finite, the geometric nerve can be realized [2] as a polyhedron in $\mathbb{R}^{2m+1}$ if $dim(n(G^I)) \leq m$. Clearly, every morphism $f\overset{\iota}{/}\alpha : G^I \to H^J$ of objective or functorial global compositions gives rise to a morphism

$$n(f\overset{\iota}{/}\alpha) : n(G^I) \to n(H^J),$$

and this one generates the continuous map

$$N(f\overset{\iota}{/}\alpha) : N(G^I) \to N(H^J),$$

everything in a functorial way, i.e., we have two functors

$$N : Glob \to \mathbf{Top}, N : ObGlob \to \mathbf{Top}$$

into the category **Top** of topological spaces and continuous maps[3]. In particular, the functor $N$ of geometric nerves generates topological invariants for isomorphism classes of global compositions.

**Definition 50** *Two morphisms $f\overset{\iota}{/}\alpha, g\overset{\kappa}{/}\beta : G^I \to H^J$ between global compositions are called* mathematically equivalent *iff they coincide on the functors, i.e., iff $f = g$. An equivalence class of mathematically equivalent morphisms, i.e., a natural transformation $f : G \to H$ which stems from a morphism of global compositions, is called a* mathematical morphism *between global compositions.*

We shall come back to mathematical morphisms in section 19.1.

Evidently, for two mathematically equivalent morphisms $f\overset{\iota}{/}\alpha, g\overset{\kappa}{/}\beta : G^I \to H^J$, the corresponding simplicial maps are contiguous[4]. So, by lemma 96 in appendix H, we have

**Proposition 14** *Any two mathematically equivalent morphisms $f\overset{\iota}{/}\alpha, g\overset{\kappa}{/}\beta : G^I \to H^J$ induce homotopic maps on the corresponding geometric nerves.*

**Exercise 31** Draw a picture of a homotopy between the equivalent morphisms $\delta : X^{(2)} \to X^{(3)}$ from the third interval to the triadic interpretation (yielding the harmonic strip) of a diatonic scale as described in the diagram 14.2.

[2] See appendix H.2.2, theorem 74.
[3] See appendix H.1.
[4] See appendix H.2.3.

Proposition 14 means that the variation of the embedding of a chart—of a refined interpretation, say—in one particular candidate of a larger chart of a coarser interpretation is related to the topological operation of homotopy if we consider the associated geometric nerves. Intuitively, this means that reinterpretation of an embedding is a kind of deformation on the geometric level.

For every address change $f : B \to A$ from the address $A$ of a global functorial composition $G^I$, we have the $f$-slice $f@G^I$, an objective global composition. In general, its nerve has less vertexes and these may collapse. Also some simplexes could vanish. However, for every $k$-simplex $\sigma = \{G_{i_0}, \ldots G_{i_k}\}$ of $n(G^I)$, there is an address change $f : B \to A$ such that $f@ \cap \sigma = \cap_j f@G_{i_j} \neq \emptyset$. But observe that $\emptyset \neq f@ \cap \sigma$ implies $\emptyset \neq f \cdot g@ \cap \sigma$ for any address change $g : C \to B$. So take address changes $f_s : B_s \to A$, one for each maximal simplex $\sigma_s$, such that all intersections $f_s@ \cap \sigma_s$ are non-empty. Take the limit $(g_s : L \to B_s)_s$ of this system of address changes over $A$ (it exists according to appendix E.3.8). Then we have one unique address change $h = f_s \cdot g_s : L \to A$ with nonvanishing intersections in every simplex of the functorial composition. This means:

**Theorem 13** *For every global functorial composition $G^I$, there is an $f$-slice $f@G^I$ such that the canonical simplicial map $n(G^I) \to n(f@G^I)$ is well defined and surjective. On the geometric nerves, it appears as a specialization morphism by collapse of particular vertexes.*

For an interpretation $X^I$ of a local objective composition $X$, we have seen the construction of the global functorial composition $\widehat{X^I}$ associated with the covering of $\hat{X}$ by $\hat{I}$. The nerve $n(\widehat{X^I})$ may be significantly larger than the nerve $n(X^I)$. This can be checked on an adequate slice of $\widehat{X^I}$. We shall discuss such an example relating to the functorial interpretation associated with the harmonic strip in section 25.3.4, example 61.

## 14.5 Simplicial Weights

**Summary.** The structure of a nerve induces weights on the nerve's simplexes with values in isomorphism classes of local compositions. This yields a tool for local and global classification, to be treated in the next chapter. We present the elementary example of simplicial motive weights.

– Σ –

**Lemma 25** *Let $G^I$ be an $A$-addressed objective or functorial global composition, and take a $k$-simplex $\sigma = \{G_{i_0}, \ldots G_{i_k}\}$ of its nerve. Consider the subobject (set or functor) $\cap\sigma \subset G$ and the chart isomorphism $\phi_{i_j} : K_{i_j} \tilde{\to} G_{i_j}$ of a $G$-chart $K_{i_j}$ for the $i_j$th vertex of our simplex. Make $\cap\sigma \subset G$ into a local composition by the induced isomorphism $\phi_{i_j,\sigma} : K_{i_j,\sigma} \tilde{\to} \cap\sigma$ on the inverse image $K_{i_j,\sigma} = \phi_{i_j}^{-1}(\cap\sigma)$. Then all the charts defined by any of the simplex vertexes are compatible, i.e., they form an atlas for the (local) composition imposed on the intersection $\cap\sigma$. This local composition (viewed as a global composition with one chart) is again denoted by $\cap\sigma$.*

Proof. The case of a one-dimensional simplex is just the axiom of gluing charts together by transition isomorphisms of local compositions. Clearly, this transition situation restricts to the intersection of any higher number of covering functors. QED.

If we view the abstract nerve $n(G^I)$ as a category by the inclusion morphisms $i_{\sigma,\tau} : \sigma \hookrightarrow \tau$ among its simplexes, we have a contravariant functor

$$\cap : n(G^I) \to Loc : \sigma \mapsto \cap\sigma \tag{14.3}$$

(with the usual identification of local and one-chart global compositions), the *simplicial weight* of $G^I$.

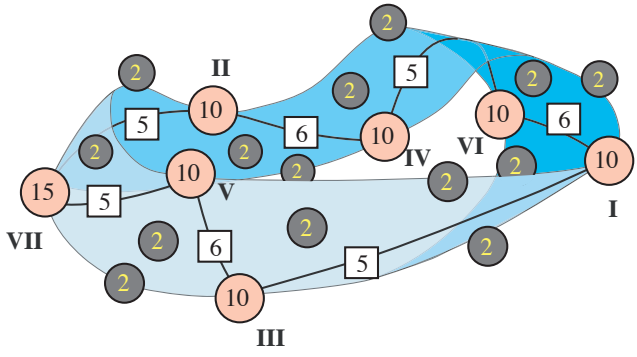


Figure 14.7: The class nerve of the triadic interpretation of a major scale. The numbers are the class numbers from the classification list in appendix L.1.

This functor induces a map

$$iso\cap : n(G^I) \to Loc/\tilde{\to}$$

from the abstract nerve to the set $Loc/\tilde{\to}$ of isomorphism classes of local compositions which we call the *class weight* (function) of the global composition $G^I$.

Summarizing, we have a geometric nerve $N(G^I)$ and a class weight $iso\cap$ for each global composition. If we parametrize the isomorphism classes of local compositions by a determined set of symbols, this means that this information can be visualized by a polyhedron with a class weight attached to each simplex; we call this object the *class nerve* of $G^I$ and denote it by $CN(G^I)$. This is a very intuitive invariant object which we want to illustrate in two examples.

**Example 26** We consider the harmonic strip of $X^{(3)}$ of a major scale $X$. The classes are numbered according to the classification in appendix L.1. For example, this nerve has seven 2-simplexes with weight number 2 for the unique zero-addressed singleton class in $PiMod_{12}$. The class nerve $CN(X^{(3)})$ is shown in figure 14.7. We see that the only automorphism of this interpretation must fix the seventh degree. So we have only the exchange of the boundary around the seventh degree, in fact the induced automorphism from the uninterpreted local scale.

**Example 27** Consider the motif situation, more precisely, the zero-addressed local three-element compositions $M$ in $OnPiMod_{12,12}$. We look at the 'face' interpretation $M^{(2)}$ by maximal subcompositions, i.e., by the three 2-element 'interval' charts in the motif. Clearly, if two

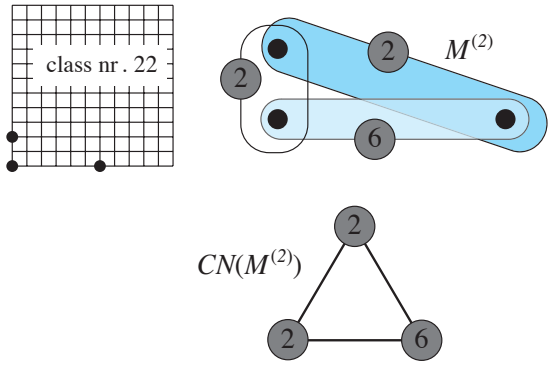


Figure 14.8: A three-element motif of class number 22 and its class nerve.

motives are isomorphic, the class nerves are also. So the class nerve yields invariants of the local classification. Figure 14.8 shows a motif of class number 22 in the classification from appendix M.3, as well as the interpretation, the chart class numbers of the intervals from appendix L.1, and the class weights $(2, 2, 6)$ on the class nerve. From the table in appendix M.3, we see that the class nerve is not classifying, but, together with the motif's volume (see section 11.3.8), the class nerve yields a complete set of class invariants.

For motives with larger cardinality this is, however, no longer true. The four-element motives show counterexamples. The classification list in appendix M.4 which was calculated by Hans Straub [513] shows the class weights on the tetrahedra $CN(M^{(3)})$ of the face interpretation of the four-element motives in $OnPiMod_{12,12}$. The motif classes which are not uniquely determined by volume and class nerve are indicated by a star.

## 14.6 Categories of Commutative Global Compositions

**Summary.** The category $ObLoc_A$ of commutative local compositions introduced in section 8.3.5 has a global counterpart which is discussed here.

– Σ –

We fix an address module $A$ over a commutative ring $R$. Recall from definition 28 in section 8.3.5 that the category of commutative local compositions refers to the fixed address module $A$; the morphisms are of form $f/Id_A$ with $f$ being induced by an $R$-affine morphism of the Coordinator $R$-modules. If we consider global objective $A$-addressed compositions which are glued together by commutative local charts, this means that the transition morphisms are isomorphisms of commutative local compositions. And it means that morphism between commutative global compositions are induced by $R$-affine morphisms on the ambient modules of the local charts. Denote this category of commutative global composition at address $A$ by $ComGlob_A$.

# Chapter 15

# Global Classification

*Wir sehen also, daß in der Musik*
*die Mehrzahl der sogenannten "wissenschaftlichen" Geister*
*fast so naiv ist wie Monsieur Achras – eine Figur von Jarry –,*
*der eine Sammlung von Polyedern anlegte.*
*Nun, es mag dahinstehen, ob in unserem Fall*
*Polyeder wirklich von unerschöpflichem Interesse sind.*
*Mir jedenfalls will die Notwendigkeit solcher*
*pataphysikalischer Spekulationen*
*nur schwerlich einleuchten.*
Pierre Boulez [60, II, p.19]

**Summary.** Global classification relies on two concepts: affine functions and resolutions of global compositions. These constructs are discussed and exemplified. We derive classifying spaces and compare them to the situation in the Dreiding–Dress–Haegi theory of molecules: The latter are deduced from global compositions by additional structures concerning *orientation*, *distances* and *angles* (bilinear and exterior forms). It is therefore possible to view "molecules" as being global compositions with additional constraints; their musical meaning is discussed.

– Σ –

This chapter deals only with objective (more precisely, commutative) global compositions. This is probably not always necessary from the technical point of view, but presently, no more general theory has been elaborated and we prefer dealing with situations where concrete results are available.

From the musicological point of view, this is one of the most difficult chapters since the relation of classification and musicology, in particular esthetics, is quite implicit. But classification is a deep concern since it reveals the a priori extent of a structural framework and therefore its power as an expressive tool of artistic activity, be it in composition, performance, or understanding. So it is a kind of essence of the overall efforts of the art of music.

## 15.1 Module Complexes

**Summary.** Module complexes describe systems of global affine functions in a classical homological language.

– Σ –

For this chapter, we fix a commutative ring $R$. If we do not stress the contrary, we only consider $R$-modules and $R$-affine morphisms between $R$-modules, in particular, addresses $A$ are $R$-modules.

Given a global composition $G^I$, we recall the category $n(G^I)$ of its abstract nerve, see section 14.5.

**Definition 51** *An (R-)module complex over $G^I$ is a covariant functor (a coefficient system, see appendix H.3)*

$$M : n(G^I) \to {}_R\mathbf{Mod} \tag{15.1}$$

*into the category ${}_R\mathbf{Mod}$ of R-modules and R-affine morphisms, where*

$$M_{\sigma,\tau} : M(\sigma) \to M(\tau)$$

*are transition morphisms of modules for the simplex inclusions (morphisms) $\sigma \subset \tau$.*

*As usual in sheaf theory, we put $\Gamma M = \lim_{n(G^I)} M(\sigma)$ and call this the set of* global sections.

**Example 28** Since any morphism of global compositions $\overset{\iota}{f}/\alpha : G^I \to H^J$ yields a natural transformation $n(\overset{\iota}{f}/\alpha) : n(G^I) \to n(H^J)$, every module complex $M$ over $H^J$ induces a module complex on $\overset{\iota}{f}/\alpha \star M$ over $G^I$, with

$$\overset{\iota}{f}/\alpha \star M(\sigma) = M(n(\overset{\iota}{f}/\alpha)(\sigma)).$$

**Example 29** If M is any $R$-module, the *constant module complex* of $M$ is the complex with $M(\sigma) = M$ for all simplexes and identity transition. Observe that its global sections are in bijection with the set $M^c$, if $N(G^I)$ has $c$ connected components.

### 15.1.1 Global Affine Functions

**Summary.** Global affine functions are patchworks of affine functions on charts of atlases. Their role is that of generalized coordinate functions for musical compositions. This approach makes it possible to separate the core process of composing music from its realization in instrumental parameter spaces.

– Σ –

In the sequel of this chapter, we shall work on categories $ComGlob_A$ of global commutative compositions—except when explicitly mentioning the contrary. Suppose that we are given an $A$-addressed commutative global composition $G^I$. Let $A@_R R$ be the $A$-addressed local composition of the full $R$-module $A@_R R$. For a simplex $\sigma$ of $n(G^I)$, a morphism $f/Id_A : \cap\sigma \to A@_R R$ may be identified with the supporting map $f : \cap\sigma \to A@_R R$. We call such a morphism an *$A$-addressed function* on the simplex. The set $n\Gamma(\sigma)$ of these functions is provided with the structure of an $R$-module by the usual addition and scalar multiplication of function values: $(f+g)(x) = f(x) + g(x), (rf)(x) = rf(x), r \in R$.

**Exercise 32** Show that sums and scalar multiples of $A$-addressed functions is again an $A$-addressed function.

Suppose that we have an inclusion of simplexes $\sigma \subset \tau$ of $n(G^I)$. Then the ambient spaces of the charts of these simplexes are the same, i.e., the inclusion of local compositions $\cap\tau \subset \cap\sigma$ are in bijection with an inclusion of local compositions $K_\tau \subset K_\sigma \subset A@_R N$ for a specific module $N$. Since an affine function on $\cap\tau$ is the restriction of an affine morphism $A@h : A@_R N \to A@_R R$, $f$ evidently extends to the restriction $A@h|_{K_\sigma}$, and we have proven that the transition morphisms by restriction of affine functions are surjective. The corresponding complex of *affine functions* is denoted by $n\Gamma(G^I)$. The subcomplex $C = C_{G^I}$ of constant functions is defined by $C(\sigma) = \{f \in n\Gamma(G^I)(\sigma), f = \text{ constant on } \cap\sigma\}$. The set of global functions of the function complex is denoted by $\Gamma(G^I)$.

For the following construction, we select one ambient space $A@_R F_\sigma$ for every simplex $\sigma$ of $n(G^I)$. Take a chart $\cap(\sigma)$ of $\cap\sigma$ in $A@_R F_\sigma$. Then we have the submodule $R.\cap(\sigma)$ of $A@_R F_\sigma$ (see definition 17 in section 7.4). If we have a morphism $\sigma \subset \tau$, we deduce a $R$-linear homomorphism $R.\cap(\tau) \to R.\cap(\sigma)$ which induce these homomorphisms

$$Lin_{\sigma,\tau}(M) : Lin_R(R.\cap(\sigma), M) \to Lin_R(R.\cap(\tau), M)$$

on the homomorphism modules for any given module $M$. This module complex is denoted by $Lin_R(G^I, M)$, although it is only determined up to an isomorphism of module complexes[1] on $G^I$; it evaluates to $Lin_R(G^I, M)(\sigma) = Lin_R(R.\cap(\sigma), M)$. In the particular case $M = A@_R R$, we abbreviate $Lin_R(G^I, A@_R R) = G^{I*}$ and call this *the module complex of $A$-addressed forms on $G^I$*. For an affine function $f : \cap\sigma \to A@_R R$, we have an $R$-linear homomorphism $R.f : R.\cap(\sigma) \to A@_R R$ (see lemma 6 in section 8.3.5), and the map $f \mapsto R.f$ is $R$-linear and functorial in $\sigma$, i.e., we have a morphism of module complexes on $G^I$:

$$R_{G^I} : n\Gamma(G^I) \to G^{I*} \tag{15.2}$$

If we have a subcomplex situation $N \subset M$ on $G^I$, we get the quotient complex $M/N$ with values $M/N(\sigma) = M(\sigma)/N(\sigma)$.

**Lemma 26** *The kernel of $R_{G^I}$ is the module $C_{G^I}$ of constant functions. Therefore, if $N$ is any module complex with $C \subset N \subset n\Gamma(G^I)$, we have an embedding of module complexes*

$$N/C \rightarrowtail G^{I*} \tag{15.3}$$

[1] By definition of module complexes, this is an isomorphism of functors.

Proof. The function $f$ is in the kernel iff its values $R.f(s_i - s_j) = f(s_i) - f(s_j)$ vanish, and this in turn means that $f$ is constant on $\cap\sigma$. QED.

**Definition 52** *Let $P$ be a property of modules. If $M$ is a module complex over $G^I$, it is said to share property $P$ iff all its values $M(\sigma)$ do so. In particular, $M$ is called injective (projective) iff the module $M(\sigma)$ is so for every simplex $\sigma$.*

**Lemma 27** *If $A$ is the zero address $A = 0_R$, and if $n\Gamma$ is injective (in particular, if $R$ is a field), the morphism $R_{G^I}$ is surjective, and therefore, we have an isomorphism $n\Gamma/C \xrightarrow{\sim} G^{I*}$.*

Proof. Since we have the zero address, any $\cap\sigma$ is isomorphic to an embedded local composition in the same ambient module, and any linear map $l : R.\cap\sigma \to R$ extends to a map on the ambient module. QED.

Let $\overset{\iota}{f}/Id_A : G^I \to H^J$ be a morphism of $A$-addressed commutative global compositions. Take a simplex $\sigma$ in $n(G^I)$, and its image $\sigma'$ under the associated simplicial map. Then, each restricted morphism

$$f|_{\cap\sigma}/Id_A : \cap\sigma \to \cap\sigma'$$

gives rise to a map

$$\overset{\iota}{f}/Id_A \star n\Gamma(H^J)(\sigma) = n\Gamma(H^J)(\sigma') \to n\Gamma(G^I)(\sigma)$$

by right composition with this restricted morphism. Moreover, the map is $R$-linear. Therefore, if $M$ is any subcomplex of $n\Gamma(H^J)$, its induced complex $\overset{\iota}{f}/Id_A \star M$ is mapped $R$-linearly onto what is called the *retracted* module complex

$$M|\overset{\iota}{f}/Id_A \subset n\Gamma(G^I). \tag{15.4}$$

In particular, if $M = C_{H^J}$, we have $C_{H^J}|\overset{\iota}{f}/Id_A \subset C_{G^I}$.

If $M$ is any subcomplex of $n\Gamma(H^J)$ which contains the constant complex $C_{H^J}$, we have the short exact sequence

$$0 \to C_{H^J} \to M \to M/C_{H^J} \to 0$$

and therefore the corresponding short exact sequence

$$0 \to \overset{\iota}{f}/Id_A \star C_{H^J} \to \overset{\iota}{f}/Id_A \star M \to \overset{\iota}{f}/Id_A \star M/C_{H^J} \to 0$$

which projects to an exact sequence on $G^I$ as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overset{\iota}{f}/Id_A \star C_{H^J} & \longrightarrow & \overset{\iota}{f}/Id_A \star M & \longrightarrow & \overset{\iota}{f}/Id_A \star M/C_{H^J} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{H^J}|\overset{\iota}{f}/Id_A & \longrightarrow & M|\overset{\iota}{f}/Id_A & \longrightarrow & M/C_{H^J}|\overset{\iota}{f}/Id_A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{G^I} & \longrightarrow & n\Gamma(G^I) & \longrightarrow & n\Gamma(G^I)/C_{G^I} & \longrightarrow & 0
\end{array}$$

including the definition of $M/C_{H^J}|\overset{\iota}{f}/Id_A$. In particular, for $M = n\Gamma(H^J)$, we have this surjective $R$-linear homomorphism

$$\overset{\iota}{f}/Id_A \star n\Gamma(H^J)/C_{H^J} \to n\Gamma(H^J)/C_{H^J}|\overset{\iota}{f}/Id_A. \tag{15.5}$$

Finally, by construction of global sections, there is a canonical projective system of $R$-linear maps

$$\Gamma(H^J) \to \overset{\iota}{f}/Id_A \star n\Gamma(H^J)(\sigma) \to n\Gamma(G^I)(\sigma)$$

and therefore we have a canonical $R$-linear map

$$\Gamma(\overset{\iota}{f}/Id_A) : \Gamma(H^J) \to \Gamma(G^I)$$

which is a contravariant functor

$$\Gamma : ObGlob_A \to {}_R\mathbf{Mod}. \tag{15.6}$$

### 15.1.2 Bilinear and Exterior Forms

**Summary.** Bilinear and exterior forms capture the language of classical geometry of angles, distances, and orientation in Euclidean three-space. This formalism is built on the systems of global functions.

– Σ –

Until now, no considerations regarding angles and similar properties from Euclidean geometry have been made in the theory of local or global compositions. Here is the formal framework for such perspectives.

Suppose that $N$ is a module complex of $A$-addressed affine functions on $G^I$ which contains the constant complex $C = C_{G^I}$. Consider the module complex $B_N$ defined on simplexes $\sigma$ by the expression

$$B_N(\sigma) = Lin_R((N(\sigma)/C)^\star, N(\sigma)/C)) \tag{15.7}$$

with $X^\star$ being the $R$-linear dual of $X$, and with the evident transition homomorphisms.

With the injection 15.3 in mind we have a canonical $R$-linear map

$$Lin_R((N(\sigma)/C)^\star, N(\sigma)/C)) \to Lin_R((G^{I*}(\sigma))^\star, G^{I*}(\sigma))$$

and the right side is

$$Lin_R((Lin_R(R.\cap(\sigma), A@_R R))^\star, Lin_R(R.\cap(\sigma), A@_R R))$$

which, according to appendix E.3.2, lemma 81 identifies to

$$Lin_R((A@_R R.\cap(\sigma)^\star)^\star, A@_R R.\cap(\sigma)^\star).$$

We now assume that $A = R^n$, $0 \leq n$ natural. Then appendix E.3.2, proposition 85 yields linear maps

$$d : A@_R R.\cap(\sigma)^\star \to (A@_R R.\cap(\sigma))^\star,$$
$$u : A@_R R.\cap(\sigma) \to (A@_R R.\cap(\sigma)^\star)^\star,$$

and therefore a map

$$Lin_R((G^{I*}(\sigma))^\star, G^{I*}(\sigma)) \to Lin_R(A@_R R.\cap(\sigma), (A@_R R.\cap(\sigma))^\star)$$

which gives an $R$-linear map

$$\beta_\sigma : B_N(\sigma) \to Lin_R(A@_R R.\cap(\sigma), (A@_R R.\cap(\sigma))^\star). \tag{15.8}$$

The right side describes the bilinear forms on the space of $A$-addressed points in the module $R.\cap(\sigma)$. We therefore associate a bilinear form $\beta(x)$ with each linear map $x : (N(\sigma)/C)^\star \to N(\sigma)/C$, and in this framework, we also speak of the bilinear form $x$, even when we cannot instantiate an associated form $\beta(x)$ (which is the case for general address $A$).

**Definition 53** *With the above notation, the pair $(G^{I*}, \beta)$ with a global section $\beta \in \Gamma B_N$ on $G^{I*}$ is called an $N$-formed* global composition.

Let $\overset{\iota}{f}/Id_A : G^I \to H^J$ be a morphism of $A$-addressed commutative global compositions. Let $M, N$ be submodules of $n\Gamma(G^I), n\Gamma(H^J)$, respectively, both containing the constants. Take global sections $\beta \in \Gamma B_M$, $\gamma \in \Gamma B_N$, respectively. Suppose that we have an inclusion of the restriction of $N$ under $\overset{\iota}{f}/Id_A$ in $M$:

$$N|\overset{\iota}{f}/Id_A \rightarrowtail M \tag{15.9}$$

and therefore an inclusion

$$N/C|\overset{\iota}{f}/Id_A \rightarrowtail M/C$$

of quotients modulo constants. Then we have a canonical map

$$B_f : B_N \to B_M \tag{15.10}$$

over $\overset{\iota}{f}/Id_A$ and therefore one of global sections

$$\Gamma B_f : \Gamma B_N \to \Gamma B_M. \tag{15.11}$$

**Definition 54** *With the preceding notation and hypotheses, we say that the morphism* $\overset{\iota}{f}/Id_A$ *is a* morphism of formed compositions $\overset{\iota}{f}/Id_A : (G^I, \beta) \to (H^J, \gamma)$ *if* $\Gamma B_f(\gamma) = \beta$.

Given a submodule complex $N$ of the complex of affine functions on $G^I$, containing the constants, we have an evident complex $\Lambda^n(N/C)$ of exterior $n$-forms. If $\Omega \in \Gamma\Lambda^n(N/C)$ is a global exterior $n$-form, we call $(G^I, \Omega)$ an oriented global composition. As with bilinear forms, we have the concept of a morphism $\overset{\iota}{f}/Id_A : (G^I, \Omega) \to (H^J, \Delta)$ of oriented global compositions if $\Gamma\Lambda^n_f(\Delta) = \Omega$, with the evident $n$th exterior power morphism whenever we have the above inclusion (15.9).

### 15.1.3 Deviation: Compositions vs. "Molecules"

**Summary.** Molecule structures, as they are considered in the Dreiding–Dress–Haegi theory, are very similar to representable global compositions. They emerge from the latter by adding bilinear and exterior forms. Geometrically this means that compositions may be "deduced" from molecular structures by abstraction from angles, distances, and orientation. We consider the specific difference of such an abstraction in the musical perspective: What does music gain after adding "molecular" information?

– Σ –

For the coefficient field $R = \mathbb{R}$ of real numbers, and for the zero address, special formed and oriented global compositions may represent molecular structures such as have been considered in [129]. We shall see below (16.4) that classification of formed and oriented global compositions, and therefore of molecular structures, is significantly more difficult than classification of global compositions. So this short deviation is not only an application of the general formalism of global compositions, it also shows that enriching the inner structure of a global composition may cause substantial difficulties in the understanding of the corresponding category.

For this situation, the (zero-addressed) points of $G^I$ are viewed as being momentary positions of atoms of a molecule in $\mathbb{R}^3$. The atom species are parametrized by natural numbers, and their distribution on $G^I$ is defined by a marking application $\alpha : G \to \mathbb{N}$. Moreover, we ask that

- $dim(n\Gamma(\sigma)/C) \leq 3$ for all charts $\sigma$ of $G^I$,
- $G^I$ is formed by a symmetric bilinear form $\beta$ which is positive definite on all charts,
- we are given an exterior 3-form $\Omega$ which is $\beta$-normed, i.e., the Gram identity $\Omega^2 = det(\beta)$ is valid[2].

[2]The Gram form verifies the identity $\Omega(x_1, \dots x_n)\Omega(y_1, \dots y_n) = det((\beta(x_i, y_j))_{i,j}$, for all sequences $x_1, \dots x_n, y_1, \dots y_n$ of elements in the chart vector space, see also [196, VII,3].

**Definition 55** *A quadruple $(G^I, \alpha, \beta, \Omega)$ with the above properties, i.e., an $\alpha$-marked, $\beta$-formed, and $\Omega$-oriented (zero-addressed) global composition $G^I$ is called a* global molecule. *A* morphism between global molecules *is a morphism of global compositions which respects the additional structures as discussed above. A global molecule which is isomorphic to an interpretation of a local molecule*[3] *is called interpretable.*

Interpretable molecules are canonically associated with distinguished structures in the theory of Dreiding–Dress–Haegi, the proof may be omitted in this context:

**Theorem 14** *The molecular structures which are associated with interpretable molecules are $(d, \chi)$-defined in the sense of* [129].

## 15.2 The Resolution of a Global Composition

**Summary.** The second device for classification is the global standard composition, canonically associated with the composition's nerve. Special module complexes on such standard objects allow the reconstruction of the original composition.

– Σ –

This section introduces two methods which are used in classification theory: The first is the global standard composition; it is a free object in the category of global compositions and helps standardize the various special objects via corresponding special module complexes of affine functions. The second method is the construction of quotient compositions from given modules of affine functions.

Combining these methods, we shall classify global compositions by the classification of special module complexes of affine functions in free objects, and by a theorem which tells us how to rebuild compositions as quotients from such special modules.

### 15.2.1 Global Standard Compositions

**Summary.** The standard compositions are objects representing compositions with "notes in general position", i.e., their configuration is as 'free' as possible from 'occasional' coincidences. In fact, the standard composition is a geometric realization of the composition's nerve and thus depends only on combinatorial information. There is a natural projection from the standard object onto the generating composition.

– Σ –

We first want to broaden the provisional concept of a standard local composition given in definition 30, section 11.3.2. Given a module $A$ in ${}_R\mathbf{Mod}$ and a natural number $0 \leq n$, we denote by $A^{\sqcup n}$ the $n+1$-fold coproduct $\coprod_{n+1} A$ of $A$. Recall from its construction (appendix E.3.8) that there is an isomorphism $A^{\sqcup n} \xrightarrow{\sim} R^n \oplus A^{n+1}$. We denote the canonical basis of $R^n$ by $(e_1, \dots e_n)$, and for any element $a \in A$ and $0 \leq i \leq n$, we set $a_i = (0, \dots a, \dots 0)$ for the

[3] One with a single chart; on the interpretation's charts, bilinear forms and the orientation are induced from the local data.

$n+1$-tuple in $A^{n+1}$ having $a$ at its $i+1$-th position and zero else; the zero element is denoted by $e_0$. We have the inclusion morphisms

$$\sigma_i : A \to A^{\sqcup n} \tag{15.12}$$

for $0 \leq i \leq n$, with

$$\sigma_i(a) = (e_i, a_i) = \begin{cases} \text{linear if } i = 0, \\ \text{affine if } i > 0. \end{cases} \tag{15.13}$$

This defines a local, $A$-addressed composition ${}_A\Delta_n \subset A@_R A^{\sqcup n}$ which is called the *$A$-addressed local standard composition of dimension $n$*. By construction, it has the following property: If $M$ is any $R$-module, and if $s. = (s_0, \dots s_n)$ is any sequence of $A$-addressed points in $M$, with associated local composition $S = \{s_0, \dots s_n\} \subset A@_R M$, then there is exactly one morphism of local compositions

$$(s.) : {}_A\Delta_n \to S : \sigma_i \mapsto s_i \text{ for } i = 0, \dots n. \tag{15.14}$$

This morphism is in fact defined by the universal property of the coproduct and is mediated by the following affine function $f : A^{\sqcup n} \to M$: Write $s_i = e^{t_i} \cdot s_{i,0}$. Then we have

$$\begin{aligned} f(e_0) &= t_0, \\ f(e_i) &= t_i - t_0 \text{ (linear) for } i > 0, \\ f(a_i) &= s_{i,0}(a) \text{ (linear) for } i \geq 0, \end{aligned}$$

and the formula $s_i = f \cdot \sigma_i$ is immediate.

**Exercise 33** If $A, B$ are two addresses, sitting in ${}_R Mod$, ${}_S Mod$, respectively, we denote by $Id : {}_A\Delta_n \to {}_B\Delta_n$ the "identity" $\sigma_i \mapsto \sigma_i$. Consider base changes $\alpha : A \to B$. Show that the assignment $A \mapsto {}_A\Delta_n$ and $\alpha \mapsto Id/\alpha : {}_A\Delta_n \to {}_B\Delta_n$ is a functor $\Delta_n : {}_c\mathbf{Mod} \to ObLoc$ on the category ${}_c\mathbf{Mod}$ of modules over commutative rings and diaffine morphisms.

To define global "free" objects among the $A$-addressed objective compositions with finite charts, we shall represent the nerve $n(G^I)$ by an isomorphic *standard representative nerve* $n^\star$ induced by a covering of the natural interval $[0, m] = \{0, 1, 2, 3, \dots m = card(G) - 1\}$ of natural numbers. For $n^\star$, we define the *global standard composition* ${}_A\Delta_{n^\star}$ at address $A$ by the interpretation of the local standard composition ${}_A\Delta_m$ which is given by the present covering of $[0, m]$. We are also given a *standard atlas* of ${}_A\Delta_{n^\star}$. In fact, for any subset $q = \{t_0, \dots t_c\} \subset [0, m]$ of $c+1$ elements, indexed corresponding to increasing values, we have the canonical injection $i_q : {}_A\Delta_c \to {}_A\Delta_m$ via $\sigma_j \mapsto \sigma_{t_j}$. This defines the standard atlas.

The universal property of this global standard composition reads as follows. Take the category $Covens$ of coverings of sets[4], and consider the covariant functor

$${}_A Cov_{n^\star} : ObGlob_{@A} \to \mathbf{Sets} : G^I \mapsto Hom_{Covens}(n^\star, (G, n_0(G^I))). \tag{15.15}$$

Then we have this result:

[4]See appendix H.2.1, example 102.

**Proposition 15** *The functor ${}_ACov_{n^\star}$ is representable by the standard global composition ${}_A\Delta_{n^\star}$, i.e., we have a bijection*

$$Hom_{Covens}(n^\star, (G, n_0(G^I))) \xrightarrow{\sim} Hom_{ObGlob_{@A}}({}_A\Delta_{n^\star}, G^I)$$

*which is functorial in the A-addressed composition $G^I$.*

The proof is left as an exercise. In particular, if we take the standard nerve $n^\star = n^\star(G^I)$ of the nerve of $G^I$ and then the corresponding 'identity' morphism $Id : n^\star \xrightarrow{\sim} (G, n_o(G^I))$, we obtain a corresponding bijective[5] morphism

$$res_{G^I}/Id_A : \Delta_{G^I} \to G^I \tag{15.16}$$

with the notation $\Delta_{G^I} = {}_A\Delta_{n^\star(G^I)}$, this object and the morphism $res_{G^I}/Id_A$ being called *the resolution of* $G^I$. Clearly, the associated simplicial morphism $n(res_{G^I}/Id_A) : n^\star(\Delta_{G^I}) \to n^\star(G^I)$ is an isomorphism.

In particular, due to the universal property of the global standard compositions, every morphism $\overset{\iota}{f}/Id_A : G^I \to H^J$ can uniquely be lifted to a corresponding morphism $res_{\overset{\iota}{f}/Id_A}$ of resolutions to make the diagram

$$\begin{array}{ccc} \Delta_{G^I} & \xrightarrow{res_{\overset{\iota}{f}/Id_A}} & \Delta_{H^J} \\ {\scriptstyle res_{G^I}/Id_A}\downarrow & & \downarrow{\scriptstyle res_{H^J}/Id_A} \\ G^I & \xrightarrow{\overset{\iota}{f}/Id_A} & H^J \end{array} \tag{15.17}$$

commute. We therefore have a resolution functor

$$res_{@A} : ObGlob_{@A} \to ObGlob_{@A} \tag{15.18}$$

and a natural transformation

$$\delta_{@A} : res_{@A} \to Id_{ObGlob_{@A}}. \tag{15.19}$$

The following deals with the reconstruction of the category $ObGl_{@A}$ from its subcategory of free objects.

### 15.2.2 Compositions from Module Complexes

**Summary.** The projection of the standard composition onto its generating composition canonically induces a module complex of global affine functions on the standard composition. This complex is used to reconstruct the generating composition from the standard composition.

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[5] It is, however, not an isomorphism in general.

The resolution functor $res_{@A}$ and its associated natural transformation $\delta_{@A}$ give rise to a module complex of affine functions $\Delta n\Gamma(G^I) = n\Gamma(G^I)|res_{G^I}/Id_A$ in $n\Gamma({}_A\Delta_{G^I})$, for each global A-addressed composition $G^I$. Call this complex *the resolution complex of composition* $G^I$. Moreover, this assignment commutes with the morphism of the resolution functor, i.e., for a morphism $\overset{\iota}{f}/Id_A : G^I \to H^J$ we have a canonical inclusion

$$n\Gamma(H^J)|\overset{\iota}{f}/Id_A \subset n\Gamma(G^I) \tag{15.20}$$

of the retracted resolution complex of $H^J$ in the resolution complex of $G^I$. The next step deals with the reconstruction of $G^I$ from $n\Gamma(G^I)$ and the related question of classification of global compositions by use of the resolution complex which is suggested by the functorial relation (15.20).

The generic situation from the preceding constructions is that we are given a module complex $M \subset n\Gamma(G^I)$, containing the constants $C$, and that we would like to construct a kind of "quotient" composition whose affine functions are those of $M$. We first look at the local situation.

**Definition 56** *Let $S \subset A@_RU$ be an A-addressed objective local composition in the R-module $U$. For a submodule $L \subset \Gamma(S)$ of affine functions on $S$, the evaluation map $\dot{\cdot} : S \to A@_RL^\star$ into the A-valued points of the dual module $L^\star = Lin_R(L, R)$ of $L$ is defined by $\dot{s}(a)(l) = l(s)(a)$.*

The problem is that the evaluation is not a morphism of local compositions in general. We have to investigate sufficient conditions for the existence of a morphism. In the special case which is of interest, we have this guarantee: In fact, let $S = {}_A\Delta_n \subset A^{\sqcup n}$. Then, the dotted points $\dot{\sigma_i} : A \to L^\star$ define the universal map $H_L : A^{\sqcup n} \to L^\star$, and we have interpreted $\dot{\cdot} : {}_A\Delta_n \to A@_RL^\star$ as a morphism of local compositions.

Next, suppose we are given two local compositions $S \subset A@_RU, T \subset A@_RV$ and a morphism

$$f/Id_A : S \to T,$$

together with a module $L_T \subset \Gamma(T)$ whose retract $L_T|f/Id_A$ is included in a module $L_S \subset \Gamma(S)$. We then have a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\dot{\cdot}} & A@_RL_S^\star \\ {\scriptstyle f/Id_A}\downarrow & & \downarrow{\scriptstyle A@|f^\star} \\ T & \xrightarrow{\dot{\cdot}} & A@_RL_T^\star \end{array} \tag{15.21}$$

where $|f^\star$ is the $R$-dual of the canonical linear map $|f : L_T \to L_S$.

**Exercise 34** Give a proof of the commutativity of diagram (15.21).

This construction yields a morphism $\dot{f}/Id_A : \dot{S} \to \dot{T}$ of local compositions in the ambient spaces $L_S^\star, L_T^\star$, respectively. With this technique, we may associate a global composition with a module complex $N \subset n\Gamma({}_A\Delta_{n^\star})$ of affine functions in the standard composition ${}_A\Delta_{n^\star}$ of a standard covering $n^\star$.

To this end, suppose we are given such a module complex $N$ of affine functions, and suppose that the restriction transition morphisms $N(\tau) \to N(\sigma)$, $\tau \subset \sigma$, are all surjective. This is certainly the case for retracted function modules from resolution morphisms. For a simplex $\sigma$ of the nerve $n({}_A\Delta_{n^\star}) \tilde{\to} n^\star$ we note $c(\sigma) = card(\cap_{q\in\sigma} q) - 1$.

*In the following discussion of classification, we shall tacitly assume that all module complexes of affine functions have surjective transition morphisms.*

With this, if we apply the construction from diagram (15.21) to the situation where $S = {}_A\Delta_{c(\sigma)} \subset A@A^{\sqcup c(\sigma)}$, and $T = {}_A\Delta_{c(\tau)} \subset A@A^{\sqcup c(\tau)}$ for simplexes $\tau \subset \sigma$ of ${}_A\Delta_{n^\star}$, and with $L_S = N(\sigma), L_T = N(\tau)$, then we have injective vertical arrows in the corresponding commutative diagram

$$\begin{array}{ccc} {}_A\dot{\Delta}_{c(\sigma)} & \longrightarrow & A@_R N(\sigma)^\star \\ {\scriptstyle \text{inclusion}}\downarrow & & \downarrow{\scriptstyle A@res^\star} \\ {}_A\dot{\Delta}_{c(\tau)} & \longrightarrow & A@_R N(\tau)^\star \end{array} \tag{15.22}$$

where $res$ is the restriction map. We write $\cap\sigma = {}_A\dot{\Delta}_{c(\sigma)}$, and therefore get a surjective morphism of diagrams of local compositions ${}_A\Delta_{c(\sigma)} \to \cap\sigma$ over $n^\star$. Setting ${}_A\Delta_{n^\star}/N = colim_{n^\star} \cap \sigma$, we have a commutative diagram of sets

$$\begin{array}{ccc} {}_A\dot{\Delta}_{c(\sigma)} & \longrightarrow & \cap\sigma \\ \downarrow & & \downarrow \\ {}_A\Delta_{n^\star} & \xrightarrow{/N=colim} & {}_A\Delta_{n^\star}/N \end{array} \tag{15.23}$$

induced by the dot morphisms of local compositions.

**Definition 57** *Call a module complex $N \subset n\Gamma(G^I)$ separating, iff for every zero-simplex (chart) $\sigma \in n_0(G^I)$, the dot map $\sigma \to A@_R N(\sigma)^\star$ is injective.*

Intuitively, this means that for any pair of points on any chart, there is a function of $N$ on this chart which separates these points. So, if in the above situation $N \subset n\Gamma({}_A\Delta_{n^\star})$ is separating, the dot maps are all bijective onto the images since the vertical maps in diagram (15.22) are injective. Therefore, the colimit diagram (15.23) has bijective horizontal arrows, and the images $\cap\sigma$ are injected into the limit ${}_A\Delta_{n^\star}/N$. So these images cover the limit and the images of the zero-dimensional simplexes build a canonical atlas of a global $A$-addressed composition, i.e., the diagram (15.23) becomes a bijective morphism of $A$-addressed global compositions.

So, if $N$ is separating, we have constructed a canonical global composition and a bijective morphism from the free object to a global composition which is defined by the functions of $N$.

**Definition 58** *We call this composition ${}_A\Delta_{n^\star}/N$ the $N$-quotient of ${}_A\Delta_{n^\star}$. The morphism ${}_A\Delta_{n^\star} \to {}_A\Delta_{n^\star}/N$ from diagram (15.23) is denoted by $/N$.*

In particular, if $res_{G^I} : {}_A\Delta_{G^I} \to G^I$ is the resolution of the $A$-addressed composition $G^I$, we have the resolution complex $\Delta n\Gamma(G^I)$. Clearly, the resolution complex is separating iff the

complex of affine functions $n\Gamma(G^I)$ is. So there is a $\Delta n\Gamma(G^I)$-quotient if the complex of affine functions $n\Gamma(G^I)$ is separating, in which case we also say that $G^I$ *is separating*. Of course, this is a property which is invariant under bijective morphisms among $A$-addressed compositions.

**Proposition 16** *If the $A$-addressed composition $G^I$ admits a projective atlas, i.e., an atlas whose charts have projective $R$-modules, then $G^I$ is separating.*

Proof. In this case, the canonical bidual map $U \to U^{\star\star}$ is injective for each ambient space $U$ of a projective atlas, see appendix E.4.2. Now, if $s \neq t$ are two points in the chart $\sigma \subset A@_R U$, there is $a \in A$ with $s(a) \neq t(a)$, and we may take a linear form $h \in U^\star$ such that $h(s(a)) \neq h(t(a))$ which means that the induced affine form $l = A@h|\sigma$ separates $s$ from $t$. QED.

**Proposition 17** *Suppose that $G^I$ is separating. Then we are given a commutative triangle of* covering set *isomorphisms*

$$\begin{array}{ccc} & {}_A\Delta_{G^I} & \\ {\scriptstyle /\Delta n\Gamma(G^I)}\swarrow & & \searrow{\scriptstyle res_{G^I}} \\ {}_A\Delta_{G^I}/\Delta n\Gamma(G^I) & \xrightarrow[f]{} & G^I \end{array} \tag{15.24}$$

*which stem from morphisms of global compositions except—possibly—for $f$.*

To look for conditions when $f$ is a morphism, we may concentrate on the local situation. This means that we are given a separating local composition $S = \{s_0, \dots s_n\} \subset A@_R U$ of cardinality[6] $n+1$. For the module $\Gamma(S)$ of affine functions $g : S \to A@_R R$ on $S$, we have the corresponding commutative triangle of set bijections:

$$\begin{array}{ccc} & {}_A\Delta_n & \\ {\scriptstyle /\Delta\Gamma(S)}\swarrow & & \searrow{\scriptstyle \cdot} \\ {}_A\Delta_n/\Delta\Gamma(S) & \xrightarrow[f]{} & S \end{array} \tag{15.25}$$

which becomes a triangle of morphisms of local compositions if we can derive $f$ from an affine morphism $F : \Gamma(S)^\star \to U$ of ambient modules. On the level of ambient modules, we have the following commutative square of affine morphisms:

$$\begin{array}{ccc} A^{\sqcup n} & \xrightarrow{\sqcup \dot{S}} & \Gamma(S)^\star \\ {\scriptstyle \sqcup S}\downarrow & & \downarrow{\scriptstyle q^\star} \\ U & \xrightarrow{b} & (U@_R R)^\star \end{array} \tag{15.26}$$

[6]Recall that in this discussion, we are considering global compositions with finite charts.

where the $b(u)(t) = t(u)$ is the extension of the bidual, where $q : U@_R R \to \Gamma(S)$ is the canonical surjection, and where $\sqcup S, \sqcup \dot{S}$ are the universal morphism associated with $S, \dot{S}$, respectively.

**Exercise 35** Verify the commutativity of diagram (15.26) by use of the universal property of the $(n+1)$-fold coproduct $A^{\sqcup n}$.

If $U$ is finitely generated and projective, the bidual $U \to U^{\star\star}$ is an isomorphism (appendix E.4.2), and the dual $r : (U@_R R)^\star \to U^{\star\star}$ of the inclusion $U^\star \to U@_R R$ defines a right-inverse $i$ of$b$. Therefore, we are given the commutative triangle

$$\begin{array}{ccc} & A^{\sqcup n} & \\ {\scriptstyle \sqcup \dot{S}} \swarrow & & \searrow {\scriptstyle \sqcup S} \\ \Gamma(S)^\star & \xrightarrow[i \cdot q^\star]{} & U \end{array} \tag{15.27}$$

which yields the required morphism for $f$ in diagram (15.25). This means that

**Theorem 15** *If the composition $G^I$ has a finitely generated, projective atlas*[7]*, the factorization diagram of proposition 17 is a diagram of global compositions.*

Proof. In fact, by proposition 16, $G^I$ is separating, and by the local diagram (15.27), applied to the charts of this situation, the map $f$ in diagram (15.24) is a morphism of global compositions, QED.

The factorization morphism $f$ has a chance to become an isomorphism if we succeed in constructing an ambient space morphism in the other direction. A sufficient condition for such inverse morphism is that the surjection $q : U@_R R \to \Gamma(S)$ has a section, i.e., a left inverse $p : \Gamma(S) \to U@_R R$. This implies that on the duals, one has $Id_{\Gamma(S)^\star} = q^\star \cdot p^\star$, and we may take $p^\star \cdot b$ to go back from $U$ to $\Gamma(S)^\star$. This condition is certainly satisfied if $\Gamma(S)$ is projective. We say that the global composition *has projective functions* iff its complex of affine functions is projective on the zero simplexes. So we have this result:

**Theorem 16** *If $G^I$ has a finitely generated projective atlas and projective functions, then the horizontal arrow $f$ in the factorization diagram of proposition 17 is an isomorphism.*

This condition is evidently satisfied if $R$ is semi-simple, i.e., a finite product of commutative fields. This means that in this case, we are able to reconstruct $G^I$ from its retracted affine functions on the resolution. Moreover, in this case, the retracted module can also be recovered from the quotient composition, i.e.,

$$\Delta n\Gamma(G^I) = n\Gamma({}_A\Delta_{G^I}/\Delta n\Gamma(G^I))|/\Delta n\Gamma(G^I) \tag{15.28}$$

so that we are now left with the question of characterizing those module complexes of affine functions in ${}_A\Delta_{n^\star}$ which could give rise to compositions having this free object as their resolution.

[7] All charts have finitely generated projective modules.

Clearly, the first condition on such a module complex is that it is separating. Secondly, we may suppose that it is finitely generated projective, i.e., its modules on the zero-dimensional simplexes are so. The first requirement is in particular the case for the global compositions in theorem 16 above since the charts have finitely generated projective ambient modules. The second requirement is the one we had in theorem 16. Also, under these conditions, the quotient composition has charts $\Gamma(S)^\star$ which are again finitely generated projective and separating. The third and last requirement is obvious: $N$ should contain the constant functions. So we proceed to the analysis of module complexes $N \subset n\Gamma({}_A\Delta_n)$ which are separating, finitely generated projective, and contain the constant functions; call these complexes *representative*.

## 15.3 Orbits of Module Complexes Are Classifying

**Summary.** By means of the representation of a composition via its module complex of functions on the purely combinatorial standard composition, we obtain a classification frame for global compositions.

– Σ –

Let us first recall the overall situation of resolutions. For a global composition $G^I$, we have the functorially associated standard resolution $\Delta_{G^I} = {}_A\Delta_{n^\star}$ with the standard covering $n^\star$ of $G^I$, and the resolution bijection $res_{G^I} : \Delta_{G^I} \to G^I$, as described in section 15.2.1. If a global composition $H^J$ is isomorphic to $G^I$ via a morphism $f \overset{\iota}{/} Id_A : G^I \to H^J$, which we now abbreviate by $f$ if no confusion is likely, we have a commutative diagram with horizontal isomorphisms:

$$\begin{array}{ccc} {}_A\Delta_{n^\star} & \xrightarrow{res_f} & {}_A\Delta_{n^\star} \\ {\scriptstyle res_{G^I}}\downarrow & & \downarrow{\scriptstyle res_{H^J}} \\ G^I & \xrightarrow{f} & H^J \end{array} \tag{15.29}$$

By the universal property of the standard compositions, the automorphism group $S_{A,n^\star}$ of the standard composition ${}_A\Delta_{n^\star}$ identifies to a subgroup of the symmetric group $\mathfrak{S}_{m+1}$ of permutations of ${}_A\Delta_{n^\star}$ if the standard covering is defined on the integer interval $[0, m]$ as discussed above. By retraction (see (15.4)), this group acts from the right on the set $Rep_{A,n^\star}$ of representative module complexes on ${}_A\Delta_{n^\star}$

$$ret : Rep_{A,n^\star} \times S_{A,n^\star} \to Rep_{A,n^\star} : (N, g) \mapsto N|g. \tag{15.30}$$

So, if our isomorphic global compositions $G^I, H^J$ have a finitely generated projective atlas and projective functions, their retracted function complexes $N_{G^I}, N_{H^J}$ on the standard composition ${}_A\Delta_{n^\star}$ are representative, and we have $N_{G^I} = N_{H^J}|res_f$.

Conversely, if we are given a representative module complex on ${}_A\Delta_{n^\star}$, we have a bijective morphism

$$r : {}_A\Delta_{n^\star} \to {}_A\Delta_{n^\star}/N \tag{15.31}$$

(see definition 58). Since the charts of the quotient have ambient spaces $N(\sigma)^\star$, the quotient has a finitely generated projective atlas. What are the functions of the quotient? Since the

simplicial complexes are isomorphic, it is sufficient to consider the local situation on a chart. So we are given a representative module[8]of affine function $N \subset \Gamma({}_A\Delta_n)$ in a local standard composition. An affine function $f$ on the image $\Delta \subset A@_R N^\star$ is induced by an affine morphism $F : N^\star \to R$, and since $N$ is finitely generated projective, we may write $F = e^t \cdot n^\star$, with the bidual $n^\star$ of an element $n \in N$. So the retraction of the induced function $f$ on $\sigma_i$, and on an argument $a \in A$ evaluates to

$$f(\sigma_i)(a) = t + n(\sigma_i)(a) \tag{15.32}$$

and, since the constants are in $N$, we may absorb the constant $t$ in $n$, which means that $n\Gamma({}_A\Delta_{n^\star}/N)|r) = N$. Since this retraction is isomorphic to the function module of the quotient, the latter is also representative.

Conversely, if we are given two representative module complexes $M, N \in Rep_{A,n^\star}$, and an automorphism $g \in S_{A,n^\star}$ with $N = M|g$, then we have a commutative diagram with isomorphism on the horizontal arrows

$$\begin{array}{ccc} {}_A\Delta_{n^\star} & \xrightarrow{\ g\ } & {}_A\Delta_{n^\star} \\ {\scriptstyle r_N}\downarrow & & \downarrow{\scriptstyle r_M} \\ {}_A\Delta_{n^\star}/N & \longrightarrow & {}_A\Delta_{n^\star}/M \end{array} \tag{15.33}$$

by the factorization of the resolution map $g \cdot r_M$ through the quotient ${}_A\Delta_{n^\star}/N$. So conversely, isomorphic representative module complexes give rise to isomorphic quotients, and we have proven this:

**Theorem 17** *The orbit space $Rep_{A,n^\star}/S_{A,n^\star}$ is in bijection with the set of isomorphism classes of $A$-addressed global compositions with projective functions and finitely generated projective atlases which have a covering complex isomorphic to $n^\star$. This bijection is induced by the retraction of the function module complex to the resolution ${}_A\Delta_{n^\star}$, in one direction, and by the quotient composition on a given representative module complex on ${}_A\Delta_{n^\star}$, in the other.*

In particular, this classification result is valid for the global compositions having as their address a module $A$ over a semi-simple commutative ring $R$, i.e., a finite direct product of commutative fields (see appendix E.2.3, theorem 48).

### 15.3.1 Combinatorial Group Actions

**Summary.** We discuss the action of the automorphism group of the standard composition on module complexes since it induces isomorphism classes of global compositions.

– Σ –

By the above classification theorem 17, we are interested in a more explicit description of the action of the automorphism group $S_{A,n^\star}$ on the set $Rep_{A,n^\star}$ of representative module complexes on the standard composition ${}_A\Delta_{n^\star}$. To begin with, the module complex of affine functions on ${}_A\Delta_{n^\star}$ reads as follows: We recall from section 15.2.1 that the standard covering $n^\star$

[8]So $N$ is finitely generated, projective, contains the constants, and is separating.

may be given in form of a covering of the integer interval $[0, m]$ by subsets (the zero simplexes) $\sigma = \{t_0, \dots t_{c(\sigma)}\}$ of $c(\sigma) + 1$ elements, and therefore by the standard atlas injections $i_\sigma : {}_A\Delta_\sigma \to {}_A\Delta_m : \sigma_j \mapsto \sigma_{t_j}$. On each simplex $\sigma$ of this covering, we have a corresponding local standard composition ${}_A\Delta_{c(\sigma)}$, and its function module is

$$\Gamma({}_A\Delta_{c(\sigma)}) \xrightarrow{\sim} (A@_R R)^{c(\sigma)+1}$$

whereas the constants correspond to the diagonal submodule

$$\Delta(A@_R R) \subset (A@_R R)^{c(\sigma)+1},$$

and our module complex $N$ must be a module

$$\Delta(A@_R R) \subset N(\sigma) \subset (A@_R R)^{c(\sigma)+1}$$

for each simplex $\sigma$, finitely generated projective as well as separating in dimension zero. What does it mean that it is separating? It means that for any zero-dimensional simplex $\sigma$, and for any pair $i \neq j$ of indices between 0 and $c(\sigma)$, there is an element $n \in N(\sigma)$ such that its coordinates $n_i$ and $n_j$ differ. Moreover, for any two simplexes $\tau \subset \sigma$, we have a commutative diagram

$$\begin{array}{ccccc}
\Delta(A@_R R) & \hookrightarrow & N(\tau) & \hookrightarrow & (A@_R R)^{c(\tau)+1} \\
\downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle res_{\tau,\sigma}} & & \downarrow{\scriptstyle pr} \\
\Delta(A@_R R) & \hookrightarrow & N(\sigma) & \hookrightarrow & (A@_R R)^{c(\sigma)+1}
\end{array}$$

with surjective vertical arrows induced by the projection to the right side. Further, an automorphism of the underlying global standard composition boils down to a permutation of the interval $[0, m]$ which is compatible with the covering. In particular, this is a finite group action which, on every module $N(\sigma)$ acts by induction from the permutation of components on the supporting direct sum module $(A@_R R)^{c(\sigma)+1}$.

Recall the diagonal embedding and its factorization from formula (11.12) in section 11.3.2:

$$0 \rightarrowtail A@_R R \xrightarrow{\Delta} (A@_R R)^{c+1} \xrightarrow{d} (A@_R R)^c \twoheadrightarrow 0$$

which is equivariant for the given permutation group action. So we have a direct equivariant decomposition $(A@_R R)^{c+1} = \Delta(A@_R R) \oplus (A@_R R)^c$ which carries over to the submodules:

$$N(\sigma) = A@_R R \oplus N(\sigma)_{red}, \text{ with } N(\sigma)_{red} \subset (A@_R R)^c,$$

The action on the reduced factor has been described in formula (11.14) of section 11.3.2. So the above projection diagram reduces to

$$\begin{array}{ccc}
N(\tau)_{red} & \hookrightarrow & (A@_R R)^{c(\tau)} \\
\downarrow{\scriptstyle res_{\tau,\sigma}} & & \downarrow{\scriptstyle pr} \\
N(\sigma)_{red} & \hookrightarrow & (A@_R R)^{c(\sigma)}
\end{array}$$

It is known that $N(\sigma)$ is projective of finite type iff $A^\star$ and $N(\sigma)_{red}$ are so (see appendix E.4.2). So, if we make the general hypothesis that $A^\star$ is projective of finite type, we may concentrate on the reduced part of the module complex, i.e., on a finitely generated projective module complex $N_{red} = N/C$ and on the reduced group action.

### 15.3.2 Classifying Spaces

**Summary.** This section is devoted to the classification theorem: There is an algebraic scheme whose rational points represent certain isomorphism classes of global compositions. The scheme is a fine moduli space in the sense of David Mumford.

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To begin with, in the local situation of a local standard composition ${}_A\Delta_c$ of dimension $c$, we consider a (covariant) Grassmann functor $X_r^{A,c} : \mathbf{ComRings}|_R \to \mathbf{Sets}$ on all commutative $R$-algebras. Writing $W_{A,c} = (A@_R R)^c$, if $s : R \to S$ is an $R$-algebra, we have

$$\begin{aligned} X_r^{A,c}(s) &= Grass_{W_{A,c},r}(S) \\ &= \{V \subset S \otimes_R W_{A,c} |\ S \otimes_R W_{A,c}/V = \text{ locally free of rank } r\}. \end{aligned} \tag{15.34}$$

Since[9] $S \otimes_R W_{A,c} \xrightarrow{\sim} ((S \otimes_R A)@_S S)^c$, this functor parametrizes $S$-modules of affine functions on ${}_{S\otimes_R A}\Delta_c$ which have locally free[10] quotients of rank $r$.

Next, we have to deal with the separation property. We are in a similar situation as in the discussion of the local classification in section 11.3.2, where we had to deal with subfunctors of the Grassmannian (formula (11.21)) with respect to a selected point (formula (11.22)). Here, the $S$-valued ambient module is the scalar extension $S \otimes_R W_{A,c}$. For any pair $1 \le i \le j \le c$, consider the difference projections

$$p_{i,j} : S \otimes_R W_{A,c} \to (S \otimes_R A)@_S S = \begin{cases} pr_i & \text{if } i = j, \\ pr_j - pr_i & \text{if } i \neq j, \end{cases} \tag{15.35}$$

where $pr_i$ is the projection onto the $i$-th factor.

The separation property means that no difference projection $p_{i,j}$ vanishes on the module $V \in X_r^{A,c}(s)$. For two indices $1 \le i \neq j \le c$, let $\Delta_{A,i,j,c}$ be the submodule of $W_{A,c}$ whose $i, j$ entries are equal, while the other entries vanish. Define these submodules of $W_{A,c}$:

$$V_{i,j} = \begin{cases} (A@_R R)^{i-1} \oplus 0 \oplus (A@_R R)^{c-i} & \text{if } i = j, \\ \Delta_{A,i,j,c} \oplus (A@_R R)^{i-1} \oplus 0 \oplus (A@_R R)^{j-i} \oplus 0 \oplus A@_R R)^{c-j-1} & \text{if } i \neq j. \end{cases}$$

We have these quotients $S \otimes_R W_{A,c}/S \otimes_R V_{i,j} \xrightarrow{\sim} (S \otimes_R A)@_S S$. Let us assume henceforth that $A$ is locally free of rank $m$. Then we have $S \otimes_R V_{i,j} \in Grass_{W_{A,c},m+1}(S)$. The separation

[9] In fact, $S \otimes_R (A@_R R) \xrightarrow{\sim} S \otimes_R (A^\star \oplus R) \xrightarrow{\sim} (S \otimes_R A)^\star \oplus S \xrightarrow{\sim} (S \otimes_R A)@_S S$. See [63] for scalar extensions on duals.

[10] The property "locally free" is equivalent to "finitely generated projective", see appendix F.2, theorem 58.

condition for $V$ then means that we do not have $V \subset S \otimes_R V_{i,j}$ for all $1 \leq i \leq j \leq c$. The latter condition defines a subfunctor $X^{A,c}_{r,i,j}$ of $X^{A,c}_r$ which is represented by a closed subscheme $Grass^{A,c}_{r,i,j}$ of the scheme $Grass^{A,c}_r$ over $Spec(R)$. In fact, it is the closed subscheme of the flag scheme $Drap^{A,c}_{r,m+1}$ defined by the $m+1$-codimensional flag component being fixed to $S \otimes_R V_{i,j}$. It is a closed subscheme since a section of structural morphism of the Grassmannian is a closed immersion, see appendix F.5, lemma 86 for details.

In other words, we have to take $V$ in the (open) complement scheme $O^{A,c}_r$ of all closed subschemes $Grass^{A,c}_{r,i,j}$ of $Grass^{A,c}_r$. So locally, the separation property defines an open subscheme of the Grassmannian which is also invariant under all permutations of the indexes, an action which is induced by the automorphisms of the underlying standard composition.

So if the zero-dimensional simplexes of the global standard composition are ${}_A\Delta_{c_i}, i = 1, 2, \ldots k$, our module complex is given by an open subscheme $O^{A,c.}_{r.} = \prod_{i=1,2,\ldots k} O^{A,c_i}_{r_i}$ of the fiber product of Grassmannians over $Spec(R)$. Since all combinations of coranks are possible, we should also take the coproduct $O^{A,c.} = \coprod_{r.} O^{A,c.}_{r.}$ of all these open subschemes on which the permutation (automorphism) group of the covering acts. To obtain the effective candidates, we have to consider coincidence on restrictions. So if $\rho$ is any simplex whose vertexes are $\sigma_1, \sigma_2$, we need to know that $N(\sigma_1)|_\rho = N(\sigma_2)|_\rho$. This evidently is a closed, equivariant condition and we have shown that the representative module complexes over the $R$-algebra $S$ which are of any rank configuration on the respective simplexes define the $S$-valued points of a locally closed subscheme $C^{A,n^\star}$ of a projective scheme over $Spec(R)$, and since any finite number of points is contained in an open affine subscheme in this situation (projective schemes, see appendix F.6), we know from appendix F.6, theorem 59, that the quotient scheme of orbits of the finite group action of $S_{A,n^\star}$ in the sense of the difference cokernel of the group action $\mu$ and the first projection $pr_1$

$$C^{A,n^\star} \times_{Spec(R)} S_{A,n^\star} \underset{\mu}{\overset{pr_1}{\rightrightarrows}} C^{A,n^\star}$$

exists. This means that we have this result:

**Theorem 18** *For an address $A$ which is locally free of rank $m$ over the commutative ring $R$, there is a subscheme $J^{n^\star}$ of a projective $Spec(R)$-scheme of finite type such that its $S$-valued points $J^{n^\star}(Spec(S))$ for an $R$-algebra $S$ are in bijection with the classifying orbits of module complexes $N$ in ${}_{S\otimes_R A}\Delta_{n^\star}$ which are locally free of defined co-ranks on the zero-simplexes of $n^\star$.*

In particular, if the ground ring $R$ is semi-simple, this theorem gives the classification of any global composition which is addressed in a finitely generated $R$-module $A$.

# Chapter 16

# Classifying Interpretations

*Weil die mathematische Methode die Wissenschaft ist,*
*die zur Zeit die am weitesten entwickelte Methodologie besitzt,*
*war mir daran gelegen,*
*sie zum Vorbild zu nehmen,*
*das uns helfen kann,*
*unsere gegenwärtigen Schwachstellen*
*zu beheben.*
Pierre Boulez [60, II, p.71]

**Summary.** This chapter exposes criteria for characterizing interpretable compositions in terms of classifying spaces. This is a central issue since interpretability yields access to instrumental parameters for the physical 'rendering' of a compositional structure. In contrast to general classification, *interpretable* molecular structures are difficult to classify. We also review global enumeration theory as well as global American Set Theory.

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We are not going to give further musicological comments and interpretations of the preceding and the following classification results. This will be dealt with in the next chapter. However, it should be kept in mind that this subject is far from pure mathematical exposition; the meaning of the classification techniques for musicology cannot be underestimated. It is the turning point between thinking and making music, between mental construction and physical realization. This is why affine functions are so important: they yield the entire potential of parametrizing music in acoustic realms.

The interpretation of isomorphism classes in terms of points of a scheme also lead to a comparative theory of global compositions: We can now deal with the problem of which compositions are more generic than others, so this entire approach from algebraic geometry opens the subject of germinal vs. derived ideas in musical composition.

In particular, the question of those compositions which can effectively be played on ensembles of musical instruments is embedded in the perspective of algebro-geometric specialization, and this is the view we shall deal with in this chapter.

## 16.1 Characterization of Interpretable Compositions

**Summary.** This section presents a condition for a composition to be interpretable. This condition regards the restriction behavior of affine functions which relates to *flasque* sheafs.

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Pursuing the path of the last chapter, we again assume that the address $A$ is a module over a commutative ring $R$, and we consider the category $ComGlob_A$ of $A$-adressed commutative global compositions. As we were able to classify those commutative global compositions in $ComGlob_A$ with $A$ locally free of a defined rank, and function modules which are locally free of defined coranks on the zero simplexes of their resolutions, by the orbits of the retracted function module complexes, it should also be possible to characterize the interpretable compositions by their function module complexes.

**Definition 59** *A module complex on a global composition is called* flasque *iff every section on a zero-simplex stems from a global section.*

For example, if a global composition $G^I$ is interpretable by the covering $J$ of a local composition $K \subset A@_R M$, for an $R$-module $M$, then we have a bijective morphism $f/Id_A : G^I \to K$, and the (global) functions of $K$, by retraction yield all the functions of $G^I$ on their charts (=zero-simplexes), this means:

**Lemma 28** *Interpretable global compositions have flasque function module complexes.*

Moreover, if $G^I$ is interpretable by a projective module $M$, the global functions are separating, this follows from proposition 16 of section 15.2.2. If a local composition has ambient module $M$ which is not projective, quite pathological things may happen.

**Example 30** Let us study such an example. Take the ring $R = \mathbb{Z}$ of integers, a free module $I = R^d$ of defined rank $d$, and take a non-zero prime ideal $m = (p)$ in $R$. Consider the $R$-module $M = I \oplus R/m$, and any address $A$. Clearly, we have $M@_R R \xrightarrow{\sim} e^R \cdot I^\star$ since all linear forms on $R/m$ vanish. Consider a local composition $L = \{x, y\} \subset A@_R M$ with $x = e^\xi \cdot u$, and $y = e^\eta \cdot u$, where $\xi, \eta \in R/m$ are two different elements, and where $u : A \to I$ is a surjection. Then no affine function on $M$ can separate $x$ from $y$ since their difference is annihilated by any affine function. Moreover, if we consider the interpretation of $L$ by the two singleton charts $\{x\}, \{y\}$, we have free function modules $\Gamma(\{x\}), \Gamma(\{y\}) \xrightarrow{\sim} R \oplus I^\star$ of rank $d + 1$. We therefore have an interpretation with locally free function modules and charts whereas the global functions are not separating, essentially because the ambient module is not projective.

But we have the following:

**Proposition 18** *Take $G^I$ in $ComGlob_A$ with $A$ locally free of rank $m$, and function modules which are locally free of defined co-ranks on the zero simplexes of their resolutions. Suppose that its function module is flasque, and that the global functions are separating. Then $G^I$ is interpretable.*

Proof. We know from the quotient construction of global compositions from function modules which backs the classification theorem 18 in section 15.3.2 that an atlas of $G^I$ is given by the duals $N(\sigma)^\star$ of the functions on the given charts $\sigma$ of $G^I$. So on the one hand, $G$ is injected into $A@_R\Gamma(G)^\star$ with ambient module $\Gamma(G)^\star$. On the other, we have a split[1] injection $i_\sigma : N(\sigma)^\star \to \Gamma(G)^\star$ into the dual of the global function module since the latter is flasque, and since by hypothesis, the function module $N(\sigma)$ and therefore its dual is locally free of defined rank. So the two charts $\sigma \subset A@_R\Gamma(G)^\star$, and $\sigma \subset A@_R N(\sigma)^\star$ are isomorphic, and we have reconstructed $G^I$ as an interpretation. QED.

In particular, if the ring $R$ is semi-simple, we obtain a complete characterization of interpretable global composition via their function modules in the resolution given in theorem 18 in section 15.3.2. But in the general case, a necessary and sufficient condition for module complex classes to yield interpretable global compositions is not known.

**Exercise 36** Let $A$ be the zero address, and $R$ be an infinite field. Take $\mu = max_\sigma(dim(R.\sigma))$ the maximum of dimensions of the modules associated with the charts $\sigma$ of $G^I$. Then an interpretation of $G^I$ which is guaranteed in proposition 18 can be constructed in an ambient vector space of dimension $\mu$.

Before terminating this classification discourse, we should add examples of non-interpretable global compositions. We shall discuss a further example of this type in chapter 17, but with a more semantic orientation. Here, we simply give the examples as facts of the classification discourse.

**Example 31** We discuss a non-interpretable global composition $G^I$ in the zero address, and over the field of real numbers, see figure 16.1. It has six points,

$$G = \{x_1, x_2, x_3, x_4, x_5, x_6\},$$

and three charts for its covering, i.e., $I = \{U_1, U_2, U_3\}$ with

$$\begin{aligned} U_1 &= \{x_1, x_2, x_3, x_4\}, \\ U_2 &= \{x_1, x_2, x_5, x_6\}, \\ U_3 &= \{x_3, x_4, x_5, x_6\}. \end{aligned}$$

The three covering charts are all in bijection with a 'square vertex' composition $S = \{a = (0,0), b = (1,0), c = (1,1), d = (0,1)\} \subset 0@_{\mathbb{R}}\mathbb{R}^2$ as follows:

$$\begin{aligned} U_1 &\tilde{\to} S : x_1 \mapsto a, x_2 \mapsto d, x_3 \mapsto b, x_4 \mapsto c, \\ U_2 &\tilde{\to} S : x_1 \mapsto a, x_2 \mapsto d, x_5 \mapsto c, x_6 \mapsto b, \\ U_3 &\tilde{\to} S : x_3 \mapsto b, x_4 \mapsto c, x_5 \mapsto a, x_6 \mapsto d, \end{aligned}$$

so that the intersections of any two of these charts yield an isomorphism of a two-point composition in $\mathbb{R}^2$. If we visualize this configuration by means of a subdivision of each chart as a union

[1] The image is a direct summand.

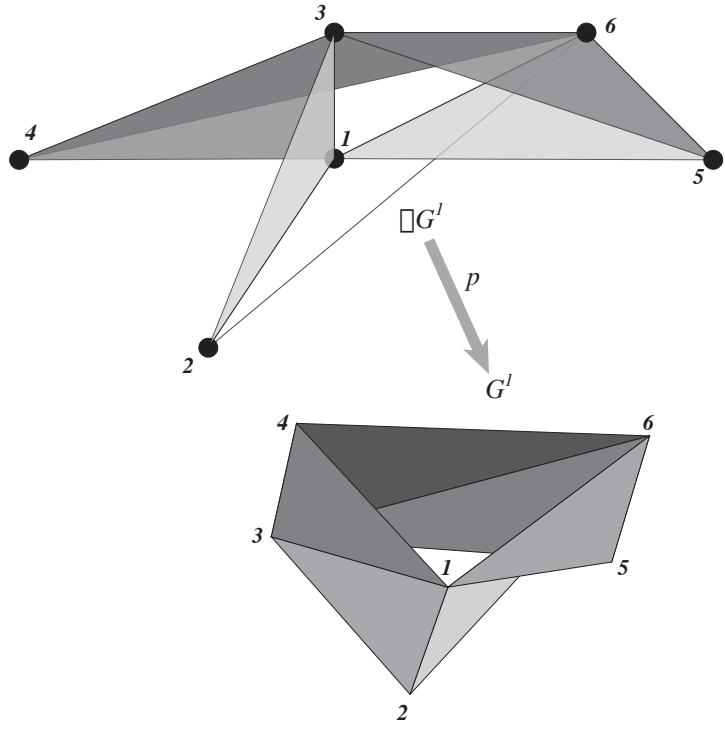


Figure 16.1: A non-interpretable composition and its resolution.

of two triangular surfaces, we obtain a Möbius strip, see figure 16.1. The resolution of $G^I$ in the same surface representation is also shown. Here, the charts are no longer plane compositions but tetrahedra in three space.

Let us show that $G^I$ is not interpretable. By the above results, it is sufficient to see that the global affine functions on this composition do not separate points. Let $f : G \to \mathbb{R}$ be an affine function (we suppress the zero address and just work in the respective ambient spaces), set $f(x_i) = f_i$ for $i = 1, 2, 3, 4, 5, 6$ and $f_{i,j} = f_i - f_j$. Since f is affine on each chart, we have on chart $U_1$: $f_{1,2} = f_{4,3}$, on chart $U_2$: $f_{1,2} = f_{5,6}$, on chart $U_3$: $f_{4,3} = f_{6,5}$. Therefore $f_{5,6} = f_{6,5}$, and $f_{5,6} + f_{6,5} = 0$, by definition, so $0 = f_{1,2} = f_{4,3} = f_{6,5}$, and no one of the pairs $(x_1, x_2), (x_3, x_4), (x_5, x_6)$ can be separated by $f$.

### 16.1.1 Automorphism Groups of Interpretable Compositions

**Summary.** We show that every finitely generated abelian group can be represented by the automorphism group of an interpretable composition.

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**Theorem 19** *Let $H$ be a finitely generated abelian group. Then there is a zero-addressed interpretation $G^I$ over $R = \mathbb{R}$ with $H \xrightarrow{\sim} Aut(G^I)$.*

The proof idea runs as follows. Suppose that we have $H \xrightarrow{\sim} \mathbb{Z}^r \times \mathbb{Z}_{m_1} \times \ldots \mathbb{Z}_{m_k}$. We shall construct a disjoint union $M^r \sqcup M_{m_1} \sqcup \ldots M_{m_k}$ of mutually non-isomorphic connected interpretations $M^r, M_{m_i}$ with $Aut(M^r) \xrightarrow{\sim} \mathbb{Z}^r$, and $Aut(M_{m_i}) \xrightarrow{\sim} \mathbb{Z}_{m_i}$.

The cell of our interpretations is a zero-addressed local composition $L_{\alpha,\beta} = \{0, \alpha, \beta, 1\} \subset 0@_{\mathbb{R}}\mathbb{R}$ for any pair $0 < \alpha < \beta < 1$ of real numbers such that $\alpha + \beta \neq 1$. Clearly, $L_{\alpha,\beta}$ is rigid, i.e., $Aut(L_{\alpha,\beta}) = Id$. Moreover, any two $L_{\alpha,\beta}, L_{\gamma,\delta}$ are isomorphic iff $\{\alpha, \beta\} = \{\gamma, \delta\}$.

Given a directed graph $D$ without zero-loops[2], take the global composition $Ex(D, L_{\alpha,\beta})$, *the extension of $L_{\alpha,\beta}$ by $D$*, which has the following structure: Its charts are all isomorphic to $L_{\alpha,\beta}$, and correspond one-to-one to the arrows of the graph $D$. Map the elements 0 in the copies of $L_{\alpha,\beta}$ to the graph vertexes corresponding to the arrow tails, whereas the elements 1 in the copies of $L_{\alpha,\beta}$ are mapped to the graph vertexes corresponding to the arrow heads. Identify these elements iff their images under this map coincide. Intuitively, this means that we replace each arrow by a copy of $L_{\alpha,\beta}$, with the 0 on the tail and the 1 on the head. The graph $D$ canonically identifies to the global subcomposition of $Ex(D, L_{\alpha,\beta})$ defined by the arrow tails and heads:

$$D \rightarrowtail Ex(D, L_{\alpha,\beta}) \tag{16.1}$$

What are the automorphisms of such an extension? They are related to an underlying automorphism of the arrow set of $D$. And on each chart $L_{\alpha,\beta}$ for a given arrow, the chart morphism must be the identity since the charts are rigid. So the automorphism must also conserve heads and tails of arrows, and therefore we have this fact:

**Lemma 29** *The automorphism group of $Ex(D, L_{\alpha,\beta})$ is isomorphic to the automorphism group of the directed graph $D$. No two extensions are isomorphic if their cells aren't.*

**Exercise 37** Give a proof of this lemma.

Suppose that the graph $D$ has no multiple arrows (independent of their direction). Then such an extension can be realized as an interpretation in $\mathbb{R}^3$. We are now ready to define the interpretation with the required automorphism group. Take the group $\mathbb{Z}^r \times \mathbb{Z}_{m_1} \times \ldots \mathbb{Z}_{m_k}$, and fix $r+k$ mutually different pairs $\alpha_i, \beta_i$ as indexes of mutually non-isomorphic cells $L_i = L_{\alpha_i,\beta_i}$. Consider the directed graph $D_\infty$ with vertexes $v_i, i \in \mathbb{Z}$ and arrows $a_i : v_i \to v_{i+1}, i \in \mathbb{Z}$. We have $Aut(D_\infty) \xrightarrow{\sim} \mathbb{Z}$, the group of translations of $\mathbb{Z}$. For a positive integer $m$, consider the regular polygon graph $D_m$ with vertexes $v_i, i = 0, \ldots m-1$ and arrows $a_i : v_i \to v_{i+1}, i = 0, \ldots m-1$, where we close the polygon by defining $v_m = v_0$. Clearly, $Aut(D_m) \xrightarrow{\sim} \mathbb{Z}_m$. We then consider the extensions $E_i = Ex(D_\infty, L_i), i = 1, \ldots r$, and $E_i = Ex(D_{m_{i-r}}, L_i), i = r+1, \ldots r+k$. No two of these interpretable compositions are isomorphic and we have $Aut(E_i) \xrightarrow{\sim} \mathbb{Z}, i \leq r$, as well as $Aut(E_i) \xrightarrow{\sim} \mathbb{Z}_{m_i}, r < i \leq r+k$. It is now immediate that the disjoint union

$$G^I = \coprod_{i=1,\ldots r+k} E_i \tag{16.2}$$

does the job, i.e., $Aut(G^I) \xrightarrow{\sim} \mathbb{Z}^r \times \mathbb{Z}_{m_1} \times \ldots \mathbb{Z}_{m_k}$. QED.

[2]No arrows with identical domain and codomain.

### 16.1.2 A Cohomological Criterion

**Summary.** Module complexes of define cochain complexes and their cohomology modules. We derive a necessary condition for the vanishing of the first cohomology and for the complex of the module complex of affine functions to be flasque, and hence the interpretability of the composition.

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If $G^I$ is a global composition at address $A$, everything over the commutative ring $R$, we may define the cochain complex $C^\star(G^I)$ as follows, and in congruence with the general cohomology theory of coefficient systems (see appendix H.3.1). Denote by $S_k n(G^I)$ the set of singular $k$-simplexes of the nerve $n(G^I)$. For each $k \geq 0$, we have the module

$$C^k(G^I) = \bigoplus_{s \in S_k n(G^I)} n\Gamma(s) \tag{16.3}$$

of $k$-cochains of functions. We have a linear differential coboundary map for each $k$:

$$d_k : C^k(G^I) \to C^{k+1}(G^I) \tag{16.4}$$

where for any singular $k+1$-simplex $s = (i_0, i_1, \ldots i_{k+1})$, we set

$$d_k(f)(s) = \sum_{j=0,1,\ldots k+1} (-1)^j f(s_j) \tag{16.5}$$

where $s_j$ is the face of $s$ after omitting vertex $i_j$. It is well known that $d_{i+1} \cdot d_i = 0$, and therefore, we have the usual cohomology modules

$$H^k(G^I) = Z^d/B^d \text{ with } Z^d = Ker(d_k), B^d = Im(d_{k-1})$$

for positive $k$, and $H^0(G^I) = Ker(d_0)$, which evidently identifies to the module $\Gamma(G^I)$ of global sections.

**Proposition 19** *Suppose that the $A$-addressed global composition $G^I$ has a nerve $n(G^I)$ which is a finite acyclic graph. Then $H^1(G^I) = 0$, and $n\Gamma(G^I)$ is flasque.*

Proof. The second statement is obvious by induction on the cardinality of the nerve: There are leaves, i.e., vertexes which are connected to (at most) one other vertex. Take such a leaf, omit it, and extend a function on the rest to the omitted leaf.

For the vanishing of first cohomology, we may suppose that the nerve is connected and take a cochain $f = (f(i,j))_{(i,j) \in S_1 n(G^I)}$ in $Z^1$. The vanishing of $df$ means that we have

$$f(i_1, i_2) - f(i_0, i_2) + f(i_0, i_1)|_{i_0 \cap i_1 \cap i_2} = 0$$

for any singular simplex $i_0, i_1, i_2$. In particular, we have $f(i,i) = 0$, and therefore, $f(i,j) + f(j,i) = 0$ for any singular 1-simplex $(i,j)$. We may suppose that there is a tree (a directed acyclic graph) $T$ such that its undirected image $|T|$ is isomorphic to $n(G^I)$. Denote by $A_T$ the arrow set and by $V_T$ the vertex set of $T$. We then conclude the existence of a 0-chain $g = (g(i))_{i \in I} \in C^0(G^I)$ with $dg = f$ by the following lemma (QED).

**Lemma 30** *Let* $f \in \bigoplus_{(i,j)\in A_T} \Gamma(i\cap j)$*, then there exists* $g \in \bigoplus_{i\in V_T} \Gamma(i)$ $(= C^0(G^I))$ *such that* $g(j) - g(i)|_{i\cap j} = f(i,j)$ *for all* $(i,j) \in A_T$*.*

Proof. Induction on $card(A_T) \geq 1$. For $card(A_T) = 1$, suppose that $f$ resides on $i, j$. Extend $f(i,j)$ to $h$ on $i$, and choose $g(j) = h, g(i) = 0$.

For the general case, take a $i_0 \in V_T$ which is the head of a non-empty set $A'$ of arrows, and where no arrow tails exist. Consider the subgraph $T'$ of $T$ obtained after removing $i_0$ and $A'$. It is a not necessarily connected graph, in fact a forest with possibly several maximal subtrees, i.e., connected components. By induction, there is a cochain $g' \in \bigoplus_{i\in V_{B'}} \Gamma(i)$ such that $g'(j) - g'(i)|_{i\cap j} = f(i,j)$ for all $(i,j) \in A_{T'}$. Since $T$ is a tree, every arrow $(i, i_0) \in A'$ defines exactly one connected component $Z_i$ of $T'$ which contains $i$.

We may add any global section $d$ to the system $(g'_i)_{i\in A_{Z_i}}$ without altering the differences which yield $g'(i) - g'(j)|_{i\cap j}$. Since $n\Gamma(G^I)$ is flasque, we may therefore choose $g'(i)$ freely; set $g'_i = f(i_0, i)$. The final definition of $g$ runs as follows: Take $g(i)) = g'(i)$ for all vertexes except $i_0$, and $g(i_0) = 0$. This does the job. QED.

The following examples (which are also exercises) show that we are far from understanding the connection of cohomology of global compositions to the problem of interpretability.

**Example 32** Consider the interpretation $G^I$ of the zero-addressed local composition $G \subset \mathbb{R}^2$ by three charts $G_1, G_2, G_3$ as shown in figure 16.2 Since every intersection $G_i \cap G_j$ is generating,

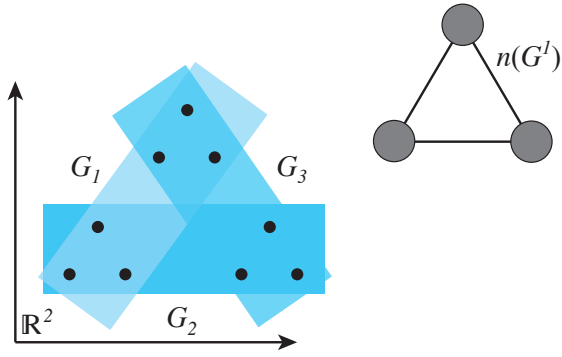


Figure 16.2: An interpretation with non-vanishing first cohomology.

we have $\Gamma(G^I) \tilde{\to} \Gamma(G_i) \tilde{\to} \Gamma(G_i \cap G_j)$. We have $C^0(G^I) \tilde{\to} (\mathbb{R}^2@\mathbb{R})^3$, $H^0(G^I) \tilde{\to} \mathbb{R}^3$, therefore $B^1(G^I) \tilde{\to} \mathbb{R}^6$, and[3] $Z^1(G^I) \tilde{\to} (\mathbb{R}^2@\mathbb{R})^3 \tilde{\to} \mathbb{R}^9$, whence $H^1(G^I) \tilde{\to} \mathbb{R}^3$.

**Example 33** This example is the "Möbius" strip composition introduced in example 31 above. It is not interpretable since the functions are not flasque, however, we have the global functions $\Gamma(G^I) \tilde{\to} H^0(G^I) \tilde{\to} \mathbb{R}^3$, and $H^1(G^I) = 0$.

[3] Please, check!

**Example 34** Take the zero-addressed global composition over the finite field $R = \mathbb{Z}_3$, defined as a gluing of two copies of $\mathbb{Z}_3 = 0@_{\mathbb{Z}_3}\mathbb{Z}_3$ along two common points, see figure 16.3 The

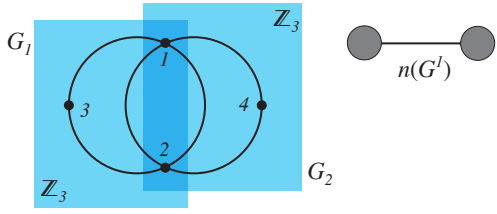


Figure 16.3: A non-interpretable composition with acyclic nerve and vanishing first cohomology.

composition cannot be interpreted since the three points in each card are colinear, but it is not possible to maintain this property in a common ambient $\mathbb{Z}_3$-vector space while distinguishing the two non-common points of the charts!

## 16.2 Global Enumeration Theory

**Summary.** Enumeration of interpretations in special cases, such as tesselations, or canons, has a strong root in the American tradition. Contributions by George Halsey, Edwin Hewitt, David Lewin, Dan Tudor Vuza, and Harald Fripertinger are discussed.

– Σ –

Enumeration theory deals with the calculation of isomorphism classes of interpretations of local compositions in finite abelian groups, i.e., in finite $\mathbb{Z}$-modules. Since these are never projective, the techniques of global classification presented in chapter 15 do not apply here. Further, to this date, only very special non-trivial coverings have been classified: Global enumeration theory is mostly restricted to zero-dimensional nerves, Fripertinger's recent classification of canons are an exception.

### 16.2.1 Tesselation

**Summary.** We review the work by George Halsey and Edwin Hewitt [204] on the tesselation of finite $\mathbb{Z}$-modules (abelian groups).

– Σ –

A finite $\mathbb{Z}$-module $M$ is a zero-addressed local composition, and we may consider its interpretations $M^I$ with mutually disjoint (non-empty) charts, i.e., with discrete nerves. Recall from sorite 6 that any isomorphism $f : U \xrightarrow{\sim} V$ among two such charts may be extended to an affine automorphism of $M$. Fix a subgroup $G \subset Aut(M)$. If these charts (U,M) are all isomorphic under $G$, we call $M^I$ *a $G$-isotypic tesselation of $M$*. So this is an interpretation by mutually isomorphic charts which can even be transformed into each other by a transformation in $G$, in other words, the class nerve $CN(M^I)$ (see section 14.5) is discrete and has constant

weights *under* $G$. We have the canonical bijective *tesselation morphism* $M^I \to M$ of zero-addressed compositions. The classifications in [204] deals with these tesselation morphisms: A $G$-isomorphism between such tesselation morphisms $M^I \to M$, and $M^J \to M$ is an element $g \in G$ which defines this commutative diagram

$$\begin{array}{ccc} M^I & \longrightarrow & M \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ M^J & \longrightarrow & M \end{array} \tag{16.6}$$

of global compositions[4]. In [204] a *tesselating chord* is defined as an isomorphism class of tesselation morphisms on $M$ under the group $G = e^M$ of translations on $M$. For any natural number $j$, we denote by $\Gamma_t(M, j)$ the number of tesselating chords in $M$ which represent tesselations by charts of cardinality $j$. We have two formulas for special cases: one for $j = 2$, and one for $j = 3$ if the 3-Sylow group[5] is cyclic.

**Theorem 20** [204, Satz 11.10] *Let $m(2)$ be the number of direct cyclic summands in the 2-Sylow group of $M$, and let $c$ be the number of elements of odd order in $M$. Then we have*

$$\Gamma_t(M, 2) = \frac{1}{2}(card(M) + 2^{m(2)} - c - 1).$$

**Theorem 21** [204, Satz 11.15] *If the 3-Sylow $S_3$ group of $M$ is cyclic, i.e., $S_3 \xrightarrow{\sim} \mathbb{Z}_{3^\lambda}$, and if $card(M/S_3 = d)$, we have*

$$\Gamma_t(M, 3) = 1 + \frac{1}{3}(d^2 - 1) + \frac{3}{8}d^2(3^{2(\lambda-1)} - 1).$$

In principle, the number $\Gamma_t(M, j)$ can be calculated by Sands' algorithm for all finite $\mathbb{Z}$-module $M$ which are Hajós groups (this property is related to Hajós' solution of Minkowski's problem; see [204, §12] for further references). For a local composition $(W, M)$, denote $Trans(W)$ for the translation symmetries of $W$. Then

**Definition 60** *A finite abelian group $M$ is called a* Hajós group *if either $M$ is trivial, or for every tesselation $M = \coprod_{v \in V} e^v(U)$ by translates of a subset $U$, either $Trans(U)$ or $Trans(V)$ is not trivial.*

All finite cyclic Hajós groups have been classified ([204, §12]). They are the groups of order $n = 1, n = p^\alpha, n = p^\alpha q$ for positive natural $\alpha$, $n = p^2q^2$, and $n = p^2qr, n = pqr, n = pqrs$, for any distinct primes $p, q, r, s$. In particular, $\mathbb{Z}_{72}$ is the smallest cyclic non-Hajós group.

Based on an algorithm similar to Sands' algorithm, the numbers of tesselation chords have been calculated in [204, §12] for all cyclic groups of order $\leq 24$ (they are automatically Hajós by the above). Their list is given in table (12.13) in [204].

[4] Instead, we could equivalently consider the interpretation which adds the chart $M$ to the atlas $I$ and then just look for $G$-isomorphisms of such interpretations of $M$.

[5] See appendix C.3.4.2.

### 16.2.2 Mosaics

**Summary.** We review the work by Harald Fripertinger [172] on the enumeration of mosaics, i.e., zero-addressed discrete interpretations of finite cyclic groups.

– Σ –

This classification deals with discrete interpretations $\mathbb{Z}_n^I$ of $\mathbb{Z}_n$ by $card(I) = k$ non-empty charts of arbitrary size. So the ambient module is more special than in section 16.6, but the partition is not necessarily isotypical. Also, in this case the calculation of isomorphism classes is not restricted to translations. Again, we are selecting a subgroup $G \subset Aut(\mathbb{Z}_n) \xrightarrow{\sim} e^{\mathbb{Z}_n} \cdot \mathbb{Z}_n^\times$, and we say that a *k-partition* is the canonical morphism $Part(I) = \mathbb{Z}_n^I \to \mathbb{Z}_n$ of a discrete interpretation $\mathbb{Z}_n^I$ with $card(I) = k$. Isomorphisms among partitions $Part(I)$, $Part(J)$ are defined by commutative diagrams as for tesselations. We ask for $g \in G$ that the diagram

$$\begin{array}{ccc} \mathbb{Z}_n^I & \longrightarrow & \mathbb{Z}_n \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{Z}_n^J & \longrightarrow & \mathbb{Z}_n \end{array} \tag{16.7}$$

commutes. With this, an orbit of partitions is called *a G-mosaic*.

According to the general combinatorial Pólya methodology applied by Fripertinger, the calculation of mosaic numbers requires the action of a finite group on a set of objects which represent the partitions. To this end, a partition $P$ is identified to an orbit of a function $p : \mathbb{Z}_n \to [1, n]$ under the left action of the symmetric group $\mathfrak{S}_n$, i.e., $P$ corresponds to an orbit $\mathfrak{S}_n \cdot p$. The reason is that the fibers of the function $p$ define a partition of $\mathbb{Z}_n$, and that we are only interested in the fibers, no matter which values they stem from, whence the action of the symmetric group on the value domain. On the other hand, the isomorphism $g \in G$ acts from the right on $p$, and we may identify the mosaics with the orbits of the action of the direct product $\mathfrak{S}_{[1,n]} \times G^{opp}$ on the function set $[1, n]^{\mathbb{Z}_n}$. Of course, the set $Mos_n^G$ of $G$-mosaics in $\mathbb{Z}_n$ is partitioned into the subsets $Mos_{n,k}^G$ of $k$-element partition mosaics for $k = 1, 2, \ldots n$. These subsets correspond to the function sets of functions $p : \mathbb{Z}_n \to [1, n]$ with $card(Im(p)) = k$.

As usual in Pólya theory, we need to calculate the cycle index $Z(G) = Z(G, P)$ as defined in section 11.4.1, formula (11.42). The notation is related to the second variable $P = \mathbb{Z}_n$, as in that context, and $G \subset \mathfrak{S}_P$ which we omitted in the previous notation of $Z$ since $P$ was fixed. With this index, a rational polynomial in the indeterminates $X_1, \ldots X_n$, with $n = card(P)$, if each variable $X_i$ in $Z(G, P)$ is replaced by a value $\xi_i$, we write $Z(G, P|X_i = \xi_i)$. Then we have the following enumeration theorem. Write $[\frac{n}{i}]$ for the greatest integer $\leq n/i$, and $s_i = i(X_1 + \ldots X_{[\frac{n}{i}]})$.

**Theorem 22** [172, Theorem 1] *With this notation, we put*

$$M_k = Z(G, \mathbb{Z}_n|X_i = \frac{\partial}{\partial X_i})Z(\mathfrak{S}_k, [1, k]|X_i = e^{s_i})|X_i = 0 \tag{16.8}$$

*for $1 \leq k \leq n$ (differential operators acting on the second factor, and then being evaluated at zero). Put $M_0 = 0$. Then we have*

$$card(Mos_{n,k}^G) = M_k - M_{k-1} \tag{16.9}$$

*and therefore*

$$card(Mos_n^G) = M_n. \tag{16.10}$$

A calculation by use of the computer program SYMMETRICA in [172] yields this table of mosaics for three important groups:

| $k =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{12}$ | 1 | 179 | 7254 | 51075 | 115100 | 110462 | 52376 | 13299 | 1873 | 147 | 6 | 1 |
| $TI_{12}$ | 1 | 121 | 3838 | 26148 | 58400 | 56079 | 26696 | 6907 | 1014 | 96 | 6 | 1 |
| $\overrightarrow{GL}(\mathbb{Z}_{12})$ | 1 | 87 | 2155 | 13730 | 30121 | 28867 | 13835 | 3667 | 571 | 63 | 5 | 1 |

This yields a total of 351773 $T_{12}$-mosaics, 179307 $TI_{12}$-mosaics, and 93103 $\overrightarrow{GL}(\mathbb{Z}_{12})$-mosaics.

This method can be refined to yield finer classifications of mosaics. We summarize one such refinement. If we are given a partition $Part(I)$ of $\mathbb{Z}_n$, not only the cardinality $card(I) = k$ is invariant under isomorphism of the group $G$, but also the *block type*, i.e., the sequence $\lambda(Part(I)) = \lambda = (\lambda_1, \lambda_2, \ldots \lambda_n)$ where $\lambda_i$ is the number of charts in $I$ with cardinality $i$. So we clearly have two linear conditions $\sum_i \lambda_i = k, \sum_i i\lambda_i = n$ for $\lambda$.

We again replace sets by functions. Given a type $\lambda$, take the set $Bi(\mathbb{Z}_n, [1,n])$ of bijections $f : \mathbb{Z}_n \to [1,n]$, and fix once and for all a partition $\Lambda$ of $[1,n]$ which is of type $\lambda$. Then the inverse image $f_\Lambda = f^{-1}\Lambda$ of $\Lambda$ under a bijection $f$ defines a partition of $\mathbb{Z}_n$, and two such inverse images $f_\Lambda, g_\Lambda$ define the same partition iff $g^{-1} \cdot f$ stabilizes $\Lambda$. Let $H_\Lambda \subset \mathfrak{S}_{[1,n]}$ be the stabilizer of $\Lambda$. Then the orbit set $H_\Lambda \setminus Bi(\mathbb{Z}_n, [1,n]$ of left action

$$H_\Lambda \times Bi(\mathbb{Z}_n, [1,n]) \to Bi(\mathbb{Z}_n, [1,n]) : (g, f) \mapsto g \cdot f$$

identifies to the set of partitions of type $\lambda$. To get the mosaics of this type, we add the well-known right action of $G$, and we obtain the canonical identification

$$Mos_{n,\lambda}^G \xrightarrow{\sim} H_\Lambda \times G^{opp} \setminus Bi(\mathbb{Z}_n, [1,n])$$

of the set $Mos_{n,\lambda}^G$ of $G$-mosaics of type $\lambda$ with the orbits of the two-sided action of $G$ and $H_\Lambda$. Using this identification, we have

**Theorem 23** [172, Theorem 2] *The number of $G$-mosaics of type $\lambda$ is given by the formula (due to de Bruijn)*

$$card(Mos_{n,\lambda}^G) = Z(G, \mathbb{Z}_n | X_i = \frac{\partial}{\partial X_i}) Z(H_\Lambda, [1,n] | X_i = iX_i) | X_i = 0. \tag{16.11}$$

Since the cycle index of a wreath product[6] of groups can be deduced from the indexes of its factors, and since the stabilizer is isomorphic to

$$\prod_i \mathfrak{S}_{[1,i]} \wr \mathfrak{S}_{[1,\lambda_i]},$$

the formula is controllable. The numbers of $T_{12}$-mosaics of all types have been calculated by the SYMMETRICA program and yield this table:

[6] See appendix C.3.2.

| $\lambda$ | | $\lambda$ | | $\lambda$ | | $\lambda$ | |
|---|---|---|---|---|---|---|---|
| $(12)$ | 1 | $(1,11)$ | 1 | $(2,10)$ | 6 | $(1^2,10)$ | 6 |
| $(3,9)$ | 12 | $(1,2,9)$ | 30 | $(1^3,9)$ | 12 | $(4,8)$ | 29 |
| $(1,3,8)$ | 85 | $(2^2,8)$ | 84 | $(1^2,2,8)$ | 140 | $(1^4,8)$ | 29 |
| $(5,7)$ | 38 | $(1,4,7)$ | 170 | $(2,3,7)$ | 340 | $(1^2,3,7)$ | 340 |
| $(1,2^2,7)$ | 510 | $(1^3,2,7)$ | 340 | $(1^5,7)$ | 38 | $(6^2)$ | 35 |
| $(1,5,6)$ | 236 | $(2,4,6)$ | 610 | $(1^2,4,6)$ | 610 | $(3^2,6)$ | 424 |
| $(1,2,3,6)$ | 2320 | $(1^3,3,6)$ | 781 | $(2^3,6)$ | 645 | $(1^2,2^2,6)$ | 1820 |
| $(1^4,2,6)$ | 610 | $(1^6,6)$ | 50 | $(2,5^2)$ | 386 | $(1^2,5^2)$ | 386 |
| $(3,4,5)$ | 1170 | $(1,2,4,5)$ | 3480 | $(1^3,4,5)$ | 1170 | $(1,3^2,5)$ | 2330 |
| $(2^2,3,5)$ | 3510 | $(1^2,2,3,5)$ | 6960 | $(1^4,3,5)$ | 1170 | $(1,2^3,5)$ | 3500 |
| $(1^3,2^2,5)$ | 3510 | $(1^5,2,5)$ | 708 | $(1^7,5)$ | 38 | $(4^3)$ | 297 |
| $(1,3,4^2)$ | 2915 | $(2^2,4^2)$ | 2347 | $(1^2,2,4^2)$ | 4470 | $(1^4,4^2)$ | 792 |
| $(2,3^2,4)$ | 5890 | $(1^2,3^2,4)$ | 5890 | $(1,2^2,3,4)$ | 17370 | $(1^3,2,3,4)$ | 11580 |
| $(1^5,3,4)$ | 1170 | $(2^4,4)$ | 2325 | $(1^2,2^3,4)$ | 8860 | $(1^4,2^2,4)$ | 4463 |
| $(1^6,2,4)$ | 610 | $(1^8,4)$ | 29 | $(3^4)$ | 713 | $(1,2,3^3)$ | 7740 |
| $(1^3,3^3)$ | 2610 | $(2^3,3^2)$ | 6005 | $(1^2,2^2,3^2)$ | 17630 | $(1^4,2,3^2)$ | 5890 |
| $(1^6,3^2)$ | 424 | $(1,2^4,3)$ | 8725 | $(1^3,2^3,3)$ | 11623 | $(1^5,2^2,3)$ | 3510 |
| $(1^7,2,3)$ | 340 | $(1^9,3)$ | 12 | $(2^6)$ | 554 | $(1^2,2^5)$ | 2792 |
| $(1^4,2^4)$ | 2325 | $(1^6,2^3)$ | 645 | $(1^8,2^2)$ | 84 | $(1^{10},2)$ | 6 |
| $(1^{12})$ | 1 | | | | | | |

### 16.2.3 Classifying Rational Rhythms and Canons

**Summary.** We give a short comment on the classification of rhythms and canons on rational onsets.

– Σ –

Following Vuza's context [552], we consider rhythms without any further parameters, except the onsets which we also restrict to the rationals. Formally, we work with zero-addressed objective local and global compositions on the space $Onset|_{\mathbb{Q}}$ of rational onsets. In Vuza's theory, a *(periodic) rhythm* is a zero-addressed local rhythm $R = e^{[-\infty,\infty]p}G$ over a finite germ $G \subset 0@Onset|_{\mathbb{Q}}$, and positive period $p$. Equivalently, $R$ is a zero-addressed objective local composition in $Onset|_{\mathbb{Q}}$ which has a non-zero translation in its automorphism, and which is locally finite. Call *Vuza rhythm* such a local rhythm, and we suppose that it is non-empty since otherwise everything is known. Therefore, the translation automorphisms of $R$ form the group $e^{p\mathbb{Z}}$, $0 < p$ being called *the period* $Per(R)$ of $R$. The group $Trans$ of rational translations acts on the set $R_{Vuza}$ of Vuza rhythms, and defines the Vuza classes as being the translation isomorphism classes or orbits in $Trans \setminus R_{Vuza}$ of Vuza rhythms. In other words, we are working on the additive group of rationals, i.e., on the $\mathbb{Z}$-module $\mathbb{Q}_{\mathbb{Z}}$.

The determination of Vuza classes runs as follows. Given a Vuza rhythm $R$, we write $R_u = \frac{1}{Per(R)}e^{-r}R$ the *unified* rhythm with period one which is obtained by contraction after a

shift by the minimal non-negative element $r$ of $R$. We have $Per(R_u) = 1$, and $R_u$ is embedded. Clearly, two Vuza rhythms of fixed period $p$ have the same class iff their unified rhythms do so.

Hence we may concentrate on classifying embedded rhythms with period one modulo translations. Since the $\mathbb{Z}$-module $\mathbb{Z}.R \subset \mathbb{Q}_{\mathbb{Z}}$ is finitely generated, it is monogeneous, and as it contains $\mathbb{Z}$, it must be of shape $\frac{1}{n_R}\mathbb{Z}$, with uniquely determined positive integer $n_R$. After a dilatation by the invariant $n_R$, we are left with the classification of zero-addressed objective local compositions $(R, \mathbb{Z})$ which

1. are generating: $\mathbb{Z}.R = \mathbb{Z}$,
2. have period $n_R$

under the group of integer translations. But this classification is clearly equivalent to that of the objects in $ObLoc_0^{gen,\mathbb{Z}_n}$ which have trivial translation automorphisms (they are "maximal" in the terminology of Halsey and Hewitt [204]). Since these classes are the classes of $ObLoc_0^{gen,\mathbb{Z}_n}$, minus those which have non-trivial translation groups $H \subset T_n$, and the latter are recursively determined as the classes stemming from $ObLoc_0^{gen,\mathbb{Z}_n/H}$, we have a recursive procedure to enumerate the classes in question. In [204, §6], this has been explicated. The classification itself can be performed along the lines of our exposition for the finite case in section 11.3.3. We leave it as an exercise to go through that procedure with the translation group instead of the full automorphism group.

Based on this local situation, Vuza [555] considers "unending rhythmic canons" which are defined as finite, non-empty subsets of a Vuza class, i.e., a non-empty finite set of Vuza rhythms of the same translation class. Equivalently, a Vuza canon is the interpretation $C^I$ of a zero-addressed objective local composition $C \subset 0@Onset|_{\mathbb{Q}}$ by a $Trans$-isotypic covering set $I$ consisting of Vuza rhythms, also called the voices of the canon, although no further parameters are considered; we subtend that the Vuza context must be interpreted as a projection of a Para-rhythm as discussed in section 13.4.3. Vuza [555] calls the common chart class the *ground class* $groundclass(C^I)$ *of the canon* $C^I$, whereas the Vuza class of the union $C$ is called the *resultant class* $resclass(C^I)$ *of the canon*. Two Vuza canons $C^I, D^J$ are called equivalent iff the resulting rhythms are translation equivalent under a translation $e^t$ which is compatible with the respective atlases, i.e., we have a commutative diagram

$$\begin{array}{ccc} C^I & \longrightarrow & C \\ {\scriptstyle e^t}\downarrow & & \downarrow{\scriptstyle e^t} \\ D^J & \longrightarrow & D \end{array} \tag{16.12}$$

as above for tesselations. So this situation is a generalization to non-discrete isotypic coverings, and not necessarily covering the entire ambient module, but this module is $\mathbb{Q}_{\mathbb{Z}}$ instead of a finite abelian group, as discussed by Halsey, Hewitt, and Fripertinger.

As with the local situation of one Vuza rhythm, the *period of a Vuza canon* is a well-defined positive rational number given by $Per(C^I) = Per(groundclass(C^I))$. Two Vuza canons $C^I, D^J$ are equivalent iff they have the same period $p$ and the contracted canons $\frac{1}{p}C^I, \frac{1}{p}D^J$ are equivalent canons of period one, so we may restrict our discussion to $C^I, D^J$ having this special period. Moreover, we may also suppose that the resulting classes are embedded, i.e.,

$C \subset \mathbb{Z}.C, D \subset \mathbb{Z}.D$. Since these classes are translation-equivalent, we obtain the same module $\mathbb{Z}.C = \mathbb{Z}.D = \frac{1}{n}\mathbb{Z}$, with the invariant *resulting divisor* $n_C = n_D = n$ as already discussed above in the local case. So, if we dilatate this situation by the resulting divisor, we have the generating local compositions $n.C \subset \mathbb{Z}, n.D \subset \mathbb{Z}$ with a period which divides n, and which are covered by local chart compositions which are all (strictly) $n$-periodic. So modulo this period, we have to look at interpretations $c^\iota, d^\kappa$ of local compositions $c, d \subset \mathbb{Z}_n$ by equipollent translation-isotypic atlases $\iota, \kappa$ which consist of $card(\iota) = card(I)$ (!) not necessarily generating local chart compositions which do not have non-trivial translations in $\mathbb{Z}_n$, i.e., are "translation rigid".

So our classification problem reduces to the classification of "canons" in the cyclic residue groups $\mathbb{Z}_n$. By definition, these are interpretations of generating, zero-addressed objective compositions by translation-isotypic translation rigid charts, but the charts are not necessarily generating. And the classification goes by translations on the ambient space $\mathbb{Z}_n$ which carry over to the interpretations. The classification of such interpretations of local compositions in $\mathbb{Z}_n$ cannot be settled by the known resolution theorems from chapter 15 since the involved function modules are not projective. However, we have a number of numerical invariants, such as the (translation) class nerve and the cohomology groups. We do not know how far these invariants are away from being classifying. In a recent work, Fripertinger has also classified such canons, but see [173].

A special family of Vuza canons are the *regular complementary canons of maximal category* (for short: RCMC-canons). In [557], Vuza presents an algorithm which enables the calculation of any tesselation $M = \coprod_{v \in V} e^v(U)$ of a group $M$ by translates of a subset $U$, where both $Trans((U)$ and $Trans(V)$ are trivial (a non-Hajós group, see definition 60). In particular, he proves that six is the minimal number of voices of an RCMC-canon and $n_R = 72$ is the shortest period. For a detailed discussion on Vuza's algorithm in the perspective of the theory of non-Hajós groups and the Minkowski conjecture, see [16]. The algorithm has been implemented in OpenMusic by Carlos Agon and Moreno Andreatta, see [17] for a complete list of solutions.

## 16.3 Global American Set Theory

**Summary.** American Set Theory has developed a number of "combinatorial" structures which relate to interpretations of pitch class sets. We give a short account to this sector, as it was developed by Forte, Lewin, Morris, and Rahn.

– Σ –

As mentioned in section 11.5.2.2, part of what we called the American tradition focused on global instead of local musical properties. The local/global dichotomy is already present in Allen Forte's book [159], in which the second part is concerned with the so-called Pitch-Class Set Complexes, i.e., sets of sets associated by virtue of the inclusion relation. Inclusion relations are basically of two type: the K and the Kh relations. By definition, given a pc set class $X$, called the *nexus*, a pc set class $Y$ is a member of the set complex about $X$ iff $Y$ can contain $X$ or can be contained in $X$ (or the corresponding for the complement of $X$), with some preliminary conditions on the cardinality of $X$ and $Y$ which are:

1. the inequalities $2 < card(X) < 10$ and $2 < card(Y) < 10$,

2. the inequalities $card(Y) \neq card(X)$ and $card(Y) \neq card(-X)$, where $-X$ means the complement of $X$.

Two sets that belong to the set complex about a given pc set class $X$ are said to be in the K relation. As pointed out by Forte, "the rule of set-complex membership yields aggregates of considerable size" [159, p.96]. A stronger condition, called the Kh relation, enables one to reduce drastically the number of sets which are in relation of a given set. The new family of sets is called the subcomplex of a given pc set class $X$. By definition it is the collection of all sets $Y$ such that $Y$ can contain or can be contained in $X$ *and* can contain or can be contained in the complement of $X$. A preliminary remark consists of the distinction between literal and abstract relations. Literal relations are among unordered collections of pitch classes, whereas abstract relations are among collections of pc sets related by some equivalence relation (usually transposition and/or inversion). In the case of the abstract inclusion, it may happen for example that a pc set class may be included in its complement, a statement that would be absurd in the case of literal inclusion. Forte's K and Kh relations are examples of abstract relations.

We agree with Robert Morris that "aside from reviews of Forte's book, there has been scant theoretical elaboration on the K and Kh relation in the literature" [382, p.175]. In order to give a new interpretation of Forte's set complexes, Morris suggests to represent K and Kh relations as lists of set classes displayed in complementary pairs which are called SC-comp lists [382, p.284]. Before describing the lattice representation of the K and Kh complexes, we follow Morris's discussion of a preliminary relation that he calls the KI relation. By definition two set classes $A$, $B$ are said to be in the KI relation iff $A \subseteq B$. The main difference compared to Forte's relations consists of the fact that K relates two pairs of set classes whereas KI simply relates two set classes. The KI relation is a partial ordering which can be displayed in a lattice representation called the "KI-inclusion lattice". The following example in figure 16.4 shows the KI-inclusion lattice about the octatonic collection 8-28 considered as the set class of the pc set $\{0, 1, 3, 4, 6, 7, 9, 10\}$.

Notice that the abstract inclusion relation has been independently theorized by Rumanian composer Anatol Vieru by means of the concept of "modal structure". By definition, a modal structure is an equivalence class of a pc set under simple transposition and without taking inversion into account. In [545, ch.3] Vieru explains how to generalize the concept of inclusion in the case of modal structures. The greater power of selectivity of Forte's Kh relation compared to the KI relation is well explained by the concept of SC-comp list we mentioned before. In this representation, each class is listed together with its abstract complement (the two set classes are separated by a slash). In the case of a self-complementary hexachord, e.g., the set class 6-5 corresponding to the pc set $\{0, 1, 2, 3, 6, 7\}$, this hexachord is listed alone. Forte's Kh relation may be reformulated now in terms of SC-comp lists. A couple of (abstract) complementary set classes $A/-A$ is said to be the Kh nexus of the SC-comp list $X$ if for all couples of complementary set classes $Y/-Y$ of $X$ the couple $A/-A$ is in the Kh relation with $Y/-Y$. Note that this new definition drops Forte's original condition by assuming that a couple $Y/-Y$ may be equal to the Kh nexus $A/-A$. The following figure 16.5 shows the great power of abstraction of the Kh complex about the octatonic collection 8-28/4-28.

For a discussion of some difficulties arising in the analytical application of Morris K and Kh relations as well as of a possible generalization of the SC-list concept in relation to Forte's concepts, see [382, pp.285-288 and pp.299-304]. As mentioned in section 11.5.2.3, the main

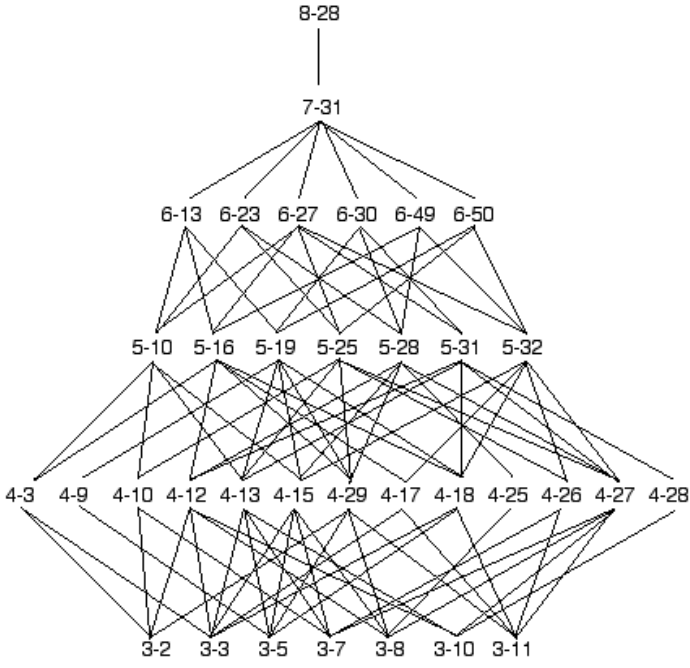


Figure 16.4: Abstract inclusion lattice for the "octatonic scale".

properties concerning set complexes have been implemented in Ircam's visual programming language OpenMusic.

We conclude this section by briefly discussing a different aspect of global American theory, which is usually called transformational theory. The theoretical basis of this approach is contained in the second part of David Lewin's book [300] as an alternative approach to the Generalized Interval System (GIS). As suggested by Lewin "Instead of starting with a GIS and deriving certain characteristic transformations therefrom, it is possible to start with a family of characteristic transformations on a musical space and derive a GIS structure therefrom" ([300, xiii]). Transformations produce a global network whose morphology is crucially dependent on the type of transformations which are modeled. Lewin discuss the case of Riemannian networks (i.e., networks involving so-called "Riemann transformations"[7]) and networks making use of serial and inversional transformation. As observed by Vuza ([553, p.277]), from a mathematical point of view, the GIS structure cannot be replaced by the concept of a space together with a simply transitive group of operations on it. Nevertheless, the interest in a transformational

[7] As mentioned by Richard Cohn in his recent survey on neo-Riemannian theory [93], Riemann transformations have been introduced in the music theory discipline by David Lewin in [298] and largely discussed in Lewin [300] as something acting on a consonant triad (i.e., a "Klang" after Riemann) in order to produce another "Klang". Examples of Riemann transformations are Lewin's MED function (i.e., the operation by which the transformed triad becomes the mediant of its MED-transform), the DOM operation (i.e., the transposition by 5 semitones) and some so-called "contextual inversions" as the operations taking any Klang into its relative or its parallel major/minor (see also our discussion on harmony in chapter 25)

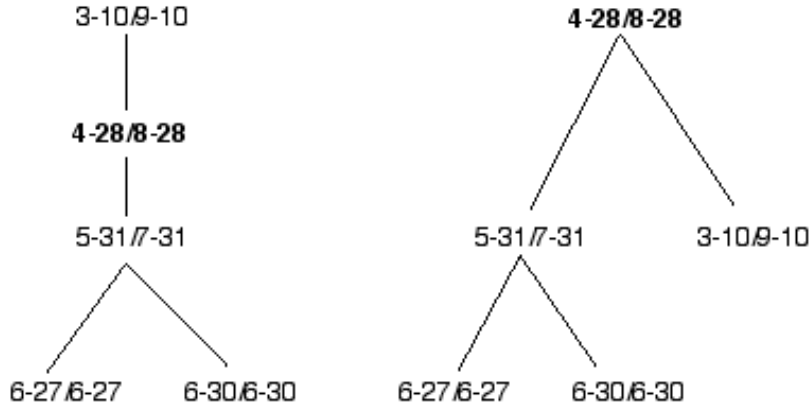


Figure 16.5: The $Kh$ complex around chord classes 4-28/8-28 in Morris' notation.

oriented music analysis has become more and more evident, particularly in relationships with the concept of 'contextual transformations', i.e., transformations which are sensitive to a particular aspect of a given musical context. This approach has been extensively discussed by Lewin in the four analytical essays of *Musical Form and Transformation* [301]. For an example of the musicological relevance of the contextuality of the transformational network approach, see Lewin's analysis of Karlheinz Stockhausen's *Klavierstück III* (pp.16–67), a piece which appears to be generated via a transformational network around the pc-set $\{0, 1, 2, 3, 6\}$. In some sense, as suggested by Robert Morris in his review of Lewin's book, the discovery of various transformational relationships in a piece can be regarded as building an abstract space in which a given piece lives. For a more detailed discussion of the concept of abstract space in composition and improvisation see [381].

## 16.4 Interpretable "Molecules"

**Summary.** The question of interpretability of molecular structures is very difficult. We make the problem clear and explain its musical aspect.

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The classification of global molecules in the sense of section 15.1.3 is evidently based on the classification of global compositions over the real numbers. So the important question concerning interpretable molecules resides in the interpretability of the underlying global compositions, a question which is settled by the criteria of section 16.1, using the module of affine functions. In order to interpret a global molecule, we need to know how to construct a global space where an underlying local molecule could subsist. So we know that there is a big space where an

underlying local composition exists. And we have a bilinear form and an orientation on each chart. At present, we do not know of criteria to decide whether these local Euclidean structures extend to an overall Euclidean space.

For the composer, passing from local Euclidean to global Euclidean structures in his global compositions means that he deals with the question of not only playing a global composition in a space of instrumental parameters, but also of playing the local distances between sound events in this instrumental parameter space. So if the property of interpretability means that an abstract global composition 'idea' can be played at all, the property of interpretability of the molecular Euclidean structures means that the local event configuration shape given by distances and angles (such as pitch or time distances, sound color distances, and similar metric categories) of the abstract 'idea' can also be played in a singular big instrumental parameter space. We are far from having settled this problem in theory or in practice.

# Chapter 17

# Esthetics and Classification

*La musique est la corporification de*
*l'intelligence qui est dans les sons.*
Józef Marja Hoene-Wroński (1778–1853)

**Summary.** Contrary to the seemingly bookkeping character of the concept of classification, the subject is deeply tied to esthetics. This is explicated and illustrated with a detailed example.

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This final chapter on global classification is a more practical and philosophical one, and it thereby clarifies the theoretical background of a demanding subject which will be dealt with later: performance theory. It stresses the basics of the intriguing problem of performance, viz, the obligation to shape a composition beyond the given score data. Why is this necessary? Isn't the score enough? What is added, why, and how?

In this chapter, we shall not answer all these questions but only one: What is added? We shall try to make this clear on a very abstract level: resolutions of global compositions. But this offers a powerful perspective on the deeper structural rationales for shaping performance.

We hope that we can convey the essential connection between performance and classification: both deal with understanding, the former more on the communicative level, the latter more on the semiotic level. So they are indebted to each other as communication has to control its contents whereas contents should be put in evidence in the making of music.

## 17.1 Understanding by Resolution: An Illustrative Example

**Summary.** Before concluding this chapter on classification with a more programmatic and philosophical outlook, the theory and its esthetic implications are illustrated by the detailed analysis of a small two-part composition.

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Our example deals with two easy two-voice compositions, and we want to analyze what it means to understand these pieces, each in itself and comparing them to each other: What is their commonality, what is the difference? Are they isomorphic? Are they two expressions of one and the same compositional idea?

To put everything into a completely familiar context without cumbersome mathematical abstracta, we select the ring $\mathbb{R}$ of real numbers and modules over $\mathbb{R}$, i.e., real vector spaces and their affine morphisms. This eases the geometry and gives us an occasion to visualize the classification ideas in ordinary three-space. For the music this means that we may consider real-valued onsets, pitch, duration, loudness, etc., and that we do not consider pitch classes but pitch as such, including microtonal intervals and infinitely subdivided time. Intuitively this means that very 'elastic' parameter conditions are admitted.

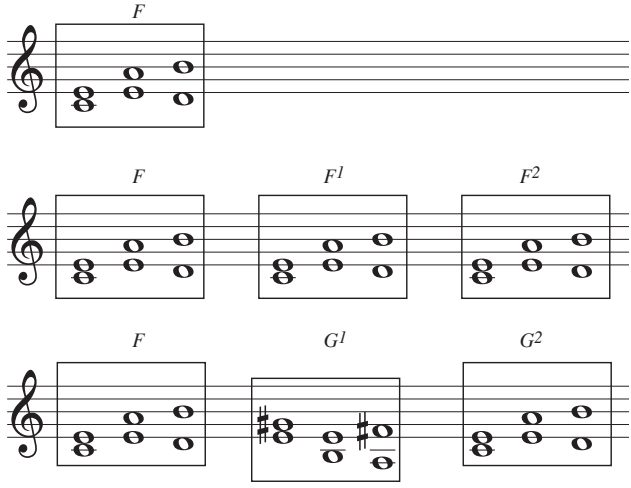


Figure 17.1: The compositional germ $F$ (top system) gives rise to two compositions $X$ (middle system), and $Y$ (bottom system), by use of contrapuntal replicates: translation and inversion.

The example—composed by a fictitious composer—where we start from, is a two-voice zero-addressed local composition $F \subset 0@OnPi$ from elementary note-against-note counterpoint, see figure 17.1, top system. Duration and other parameters are of no relevance now and will be added later. Suppose that our composer now extends this germ by two procedures, see figure 17.1, second and third system. In the first, he unites $F^0 = F$ to successive translates in time: Consider $F^1 = e^{(3,0)}F, F^2 = e^{(3,0)}F^1, \dots F^{k+1} = e^{(3,0)}F^k$, and then the union $X = \bigcup_{k=0,1,\dots r} F^k$. In the second, he unites $G^0 = F$ to successive translates in time, but every new translate is also inverted in pitch by $I$, the inversion at middle $e$: $G^1 = e^{(3,0)} \cdot IG^0, G^2 = e^{(3,0)} \cdot IG^1, \dots G^{k+1} = e^{(3,0)} \cdot IG^k$, and we get the union $Y = \bigcup_{k=0,1,\dots r} G^k$. The composer would like to understand the essential difference between $X$ and $Y$.

According to the Yoneda philosophy, we first examine the discrete interpretations of $X, Y$ and cannot find any difference since both just enumerate to $6(r+1)$ points. The next interpretation by use of vertical interval 'slices' does not show essential differences since the corresponding interval events in $X$ and $Y$ are just either the same or inverted copies, so these interpretations are isomorphic with each other. In the next refinement, $X^C, Y^D$, the composer lays the charts of couples of successive contrapuntal intervals, each, see figure 17.2. So each of the $r$ charts has four events. In $X^C$ as well as in $Y^D$, the first two charts are the same, they have these

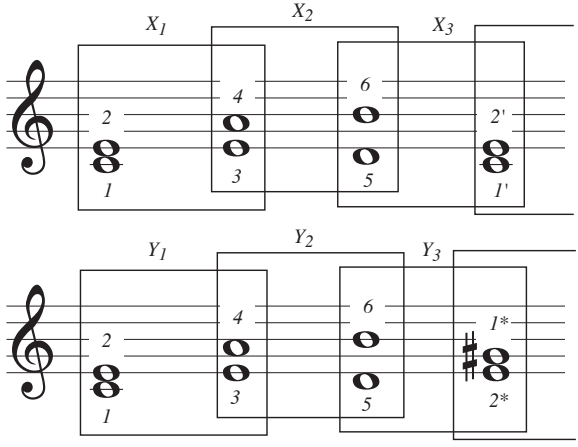


Figure 17.2: The interpretations shown here refer to charts which represent the contrapuntal sequence of two successive intervals per chart.

event numbers $X_1 = Y_1 = \{1, 2, 3, 4\}, X_2 = Y_2 = \{3, 4, 5, 6\}$. But the third charts are different: $X_3 = \{5, 6, 1', 2'\} \neq Y_3 = \{5, 6, 1^\star = I(1'), 2^\star = I(2')\}$. After that, things get repeated: $Y_4, Y_5$ are the inversions of $X_4, X_5$, respectively, etc.

In the next step, the composer tries to boil down these compositions to their common 'germ' $F$. In order to do so, he considers two point sets

$$F_X = \{1_X, 2_X, 3_X, 4_X, 5_X, 6_X\}, F_Y = \{1_Y, 2_Y, 3_Y, 4_Y, 5_Y, 6_Y\}$$

consisting of six points each. According to the above contrapuntal interpretations, the composer constructs two global compositions $F_X^\star, F_Y^\star$ on these sets by the following charts for $F_X^\star$ (see left part of figure 17.3):

$$\begin{aligned} X_1 &\rightarrow F_{X,1} = \{1_X, 2_X, 3_X, 4_X\} : i \mapsto i_X \\ X_2 &\rightarrow F_{X,2} = \{3_X, 4_X, 5_X, 6_X\} : i \mapsto i_X \\ X_3 &\rightarrow F_{X,3} = \{5_X, 6_X, 1_X, 2_X\} : 5 \mapsto 5_X, 6 \mapsto 6_X, 1' \mapsto 1_X, 2' \mapsto 2_X \end{aligned}$$

which are taken from the interpretation $X^C$. So this is a global composition which is isomorphic to the interpretation $F^A$ of $F$ by the atlas

$$A = \{\{1, 2, 3, 4\}, \{3, 4, 5, 6\}, \{5, 6, 1'2'\}\}.$$

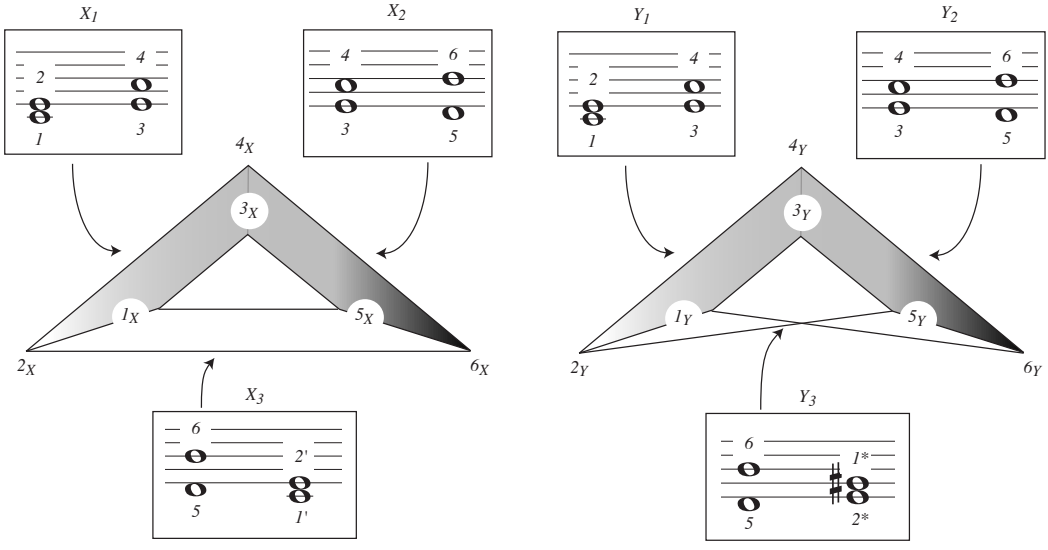


Figure 17.3: These two global compositions are abstract models of the composition principles which give rise to the contrapuntal interpretations as shown in figure 17.2. The left one is interpretable whereas the right one is not.

On the other hand, we construct a global composition $F_Y^\star$ by this atlas (see right part of figure 17.3):

$$\begin{aligned} Y_1 &\rightarrow F_{Y,1} = \{1_Y, 2_Y, 3_Y, 4_Y\} : i \mapsto i_Y \\ Y_2 &\rightarrow F_{Y,2} = \{3_Y, 4_Y, 5_Y, 6_Y\} : i \mapsto i_Y \\ Y_3 &\rightarrow F_{Y,3} = \{5_Y, 6_Y, 1_Y, 2_Y\} : 5 \mapsto 5_Y, 6 \mapsto 6_Y, 1^\star \mapsto 1_Y, 2^\star \mapsto 2_Y \end{aligned}$$

where the first two charts are defined as before whereas the third is different: The points $1^\star, 2^\star$ are identified via inversion to the points $1, 2$, therefore we have set $1^\star \mapsto 1_Y, 2^\star \mapsto 2_Y$ on the third chart.

These global compositions should be viewed as being formalizations of the compositional ideas in $X$ and in $Y$. Now, although the class nerves $CN(F_X^\star)$ and $CN(F_Y^\star)$ are equal, these compositions are essentially different. Whereas the first is interpretable, the second is not, it is in fact the composition which we had already discussed in example 31. So these compositions

are not isomorphic, and the compositional ideas are essentially different, though subtly differentiated from each other. The first composition can be played in its integrity on one space of musical coordinates, even in two-dimensional space, by construction. The other can never be played in this way. It is a somehow abstract object.

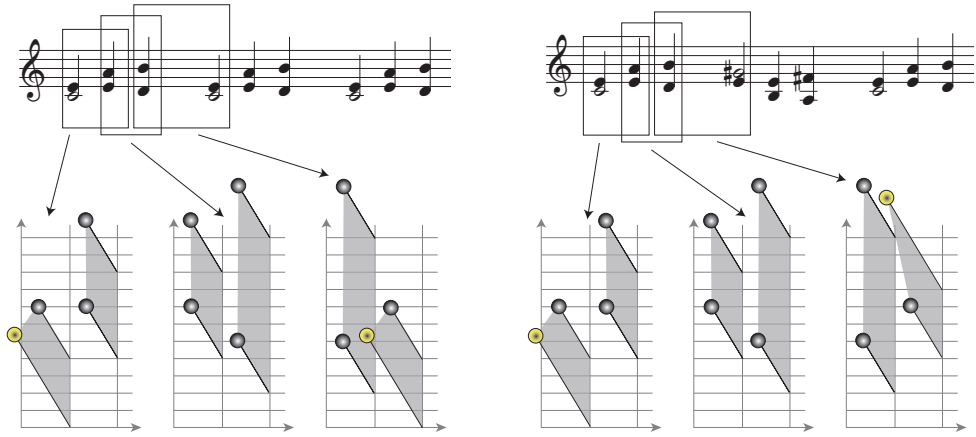

Figure 17.4: By means of differentiation of durations, the non-isomorphic abstract compositions become isomorphic.

It is a good exercise to ask for the subtle reasons why the second composition cannot be interpreted. The calculation of the function modules which was carried out in example 31 showed that the impossibility for separating functions is the parallelism of the lines drawn through couples of points in the charts. We have parallelisms $(1_Y, 2_Y) \parallel (3_Y, 4_Y)$, $(3_Y, 4_Y) \parallel (5_Y, 6_Y)$, and $(5_Y, 6_Y) \parallel (1_Y, 2_Y)$. This entails ample dependencies of functional values. The argument to blow up this flat world is to shift points into mutually more independent positions. Geometrically, this means that one should try to displace the four points on a chart to a general position, i.e., such that they span a non-degenerate tetrahedron.

But how should our composer accomplish this geometric task? One solution is to add more parameters to the given two-dimensional situation. This means, for example, to add duration, and to view the given compositions as living in a hyperplane of the product space $OnPi \oplus Duration$ with duration as third component, the hyperplane being defined by a constant duration, a whole note, say. This is in fact what we were drawing in the above figures.

To obtain a more general position of the chart events, we just have to alter the relative duration of one event: all notes being set to quarters, the first middle $c$ can be given duration 1/2. We show this change in figure 17.4. Now, the three charts are in fact isomorphic to each other: the first to the first, the second to the second, and the third to the third, because any two standard compositions are isomorphic!

The trick simply is to add enough parameters in order to accomplish the goal of being in a state to play the piece in a single big parameter space. But clearly, the new compositions are no longer isomorphic to the old ones, they do, however, project onto the given original samples.

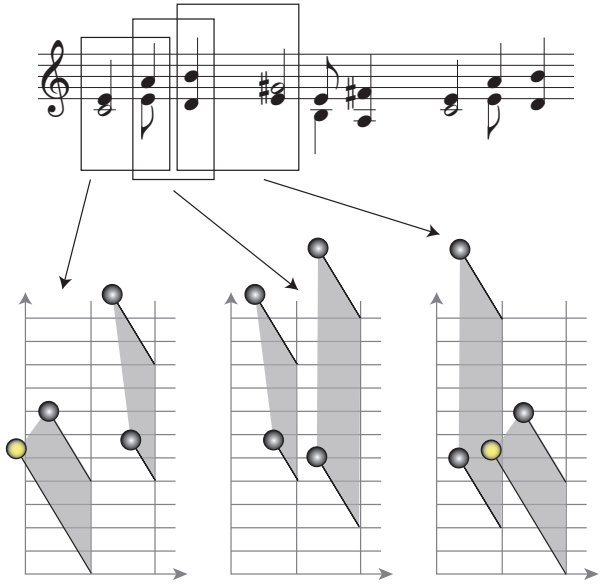

Figure 17.5: The resolution of the non-interpretable abstract model $F_Y^\star$ is obtained by further differentiation of durations such that each chart becomes a configuration of events in general position: In fact, every chart shows a non-degenerate tetrahedron of points.

More systematically speaking, one could have put any chart configuration into general position, i.e., the second chart, too, by just making an eight note from the second middle $e$, this is shown in figure 17.5. This figure of the resolution $\Delta F_Y^\star$ of $F_Y^\star$ shows that we have an evident bijection of note events $\Delta F_Y^\star \to F_Y^\star$ as given in the general theory of resolutions, but the positions of the events in the different charts project from a general to a special position. But where has the special position been hidden in the resolution structure? It is packed in the retract of the function module of the original composition. This retraced module tells the composer, whether and how global coordinate functions for instrumental rendering are at his disposition. We shall shortly come back to this point in the discussion of Varèse's approach to composition.

## 17.2 Varèse's Program and Yoneda's Lemma

**Summary.** Implicitly, the part of Yoneda's lemma dealing with variations of perspectives is of primordial importance in compositional concepts. In Edgar Varèse's programmatic writings, a thoroughly geometric approach to the Yoneda philosophy is sketched. It accomplishes the classical variational principle in composition; we give an overview to this central connection between modern mathematics and music.

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To begin with, let us recapitulate the impact of the resolution of a (commutative) global composition on the esthetics of music. The resolution $\Delta G^I$ of a global composition $G^I$ yields points which are in general position in every chart of the atlas. Moreover, the resolution is interpretable, and $G^I$ can be reconstructed from the retracted function module in the resolution.

Since the resolution's nerve is isomorphic to the original nerve, and since the resolution projects bijectively onto the original composition, no note event and no overlapping relation of charts is destroyed in the resolution. So we essentially have the same set of notes, except that they were enriched by a number of parameters which allow us to place these events in optimal relative position. So the resolution can—in principle—be played by physical instruments, and it can also—in principle—be played such that the old idea can be heard since old parameters can be preserved.

However, the freedom of choice for an optimal realization of the resolution (which after all is only determined up to isomorphism) is also mandatory since a good performance on the physical level has to cope with a number of additional conditions of human cognition. In fact, the auditory system, the instrumental skill of an artist, the material possibilities, the time frame at disposition, etc., all these conditions impose serious boundary values on the possible parameter values which can be accepted in a good realization. For example, these parameters have to reflect the syntactical structure of the composition, and not only the resolution's general position context. So time must be given a delicate role in the parametrization of events. And the distinction of events must also be optimized when the unfolding of a performance in time is to be a good communication stream. We shall come back to these subtleties later in the context of performance theory. Nonetheless, the resolution classification technique yields very important necessary conditions for a comprehensible parameter setting in performance.

What could now be, after all, the program of classification? Its core value is that it deals with understanding musical works. And we should stress that our concept of a musical work is not the narrow one which restricts to those individual opera which—at least in Europe—started to emanate in the Renaissance. It includes as well general musical corpora such as scales, systems, everything that can be represented by means of global compositions, and—in the limit—any denotator if we admit the most general topos of this theory.

From the precise parametric description of a work and of its ambiguities, this work appears as a point configuration in a more or less complex space (or form, if you prefer the denotator terminology). However this configuration is already a determinate perspective which shows a multitude of relations among its ingredients. It is the composer's perspective (now including an abstract 'composer' or creator of a general musical structure like a scale). For example, the choice of tonality, instrumentation, tempo, etc., are points of view which may or may not pertain to the composition, this is a question of the epoch of creation. But their character can undoubtedly be subject to variation. Among others, here we do address the question of historical instrumentation for early music.

In order to understand the relations among different parts of a composition, and even to simply recognize them, a change of the given perspective is mandatory. If a never seen object must be inspected, what should we do? You walk around it. This is the most common version of Yoneda's lemma. The analogy to cartography is straightforward: The natural perspective of the landscape in which we live does not coincide with the perspective which best meets our need for orientation. To reach this goal, we preferably build maps which show the landscape from an infinitely far vertical point.

The same happens to music. You play a piece in slow motion 'from very near', in a zoomed optics, a complex chord is arpeggiated, i.e., viewed from a skew angle, and so forth. This idea of variation of the perspective has also been integrated in the compositional thinking of the $20^{th}$ century. We want to illustrate this remarkable fact by a citation from Edgar Varèse's comments on his composition "Intégrales" [542, p.67]:

> *Die Intégrales wurden für eine räumliche Projektion entworfen. (...) Während wir in unserem musikalischen System Klänge anordnen, deren Werte festgelet sind, suchte ich eine Verwirklichung, bei der die Werte fortwährend im Verhältnis zu einer Konsanten verändert werden. (...)*
>
> *Um dies besser zu begreifen, übertragen wir, da das Auge viel schneller und geübter ist als das Ohr, diese Vorstellungen ins Optische und betrachten die wechselnde Projektion einer geometrischen Figur auf eine Fläche, wobei Figur und Fläche sich beide im Raum bewegen, aber jede nach ihren eigenen Geschwindigkeiten, die veränderlich und verschieden sind, die sich verschieben und rotieren. Die augenblickliche Form der Projektion ist durch die Relation zwischen Figur und Fläche in diesem Augenblick bestimmt. Aber wenn man erlaubt, daß die Figur und die Fläche ihre eigenen Bewegungen haben, ist es möglich, mit der Projektion ein äußerst komplexes und scheinbar unvorhersehbares Bild zu erhalten. Diese Qualitäten können noch vermehrt werden, wenn man die Form der geometrischen Figur ebenso wie ihre Geschwindigkeiten variiert. (...) Ich hoffe, innerhalb kurzer Zeit einen Apparat zur Verfügung zu haben, der es erlauben wird, ein räumliches Relief zu geben. Nur des Beweises wegen würde ich daran interessiert sein, die Intégrales eimal so zu realisieren, wie sie ursprünglich konzipiert worden sind.*

In 1960 Maurizio Kagel transferred these principles on paper strips and discs of the score for "Transición II" for piano, percussion, and two tapes.

Varese's idea basically is a remake of the *classical variation principle*. Bach's Goldberg Variations (BWV988), Beethoven's Diabelli Variations (op.120), or Webern's Variationen für Klavier (op.27) are compositions in this spirit. The subject is always an artistically interwoven change of perspectives of a theme: the variation in the parts of the theme and their relations.

**Principle 7** *Variation as a principle of musical shaping is nothing else than the identification of an idea—such as the theme—as a sum of its perspectives.*

Especially for Webern a composition is a cellular organism, a connected manifold (in the naive sense) of transformations, of ever changing perspectives, of metamorphoses of a single cellular germ (in the sense of Goethe), which in fact is Schönberg's dodecaphonic series.

In front of this historical background the classification problem of global compositions—together with its central process of resolution—appears as a *canonical program*. In particular, the nerve of a resolution, a concept related to that of a "cell complex" from algebraic topology, reminds us of the cellular organism alluded to by Webern. And the projections which Varèse describes in a visionary fashion show a surprisingly similar geometry to the projections of a resolution onto the original composition, projections which are distinguished in that they project a general position onto specializations. Finally, the variation of these projections corresponds to the variation of the modules of affine functions, i.e., the variation of the compositions which

are distinguished from each other via their retracted function modules on one and the same resolution.

But the variational principle is not only a compositional strategy, it equally, or even more dramatically, applies to the performance level. Performance—we shall discuss the issue in depth later—deals with a transformation from the mental score space to the physical space of the acoustic realization. But this transformation locally is a deformation of the "rigid" parameter values set out on the score. Why should the artist deform a perfect opus? Wouldn't this be blasphemy or at least a tremendous lack of respect? No, the added value of such a deformation is not a destruction of given structure, it is a subtle change of parametric perspectives which let the auditory still recognize the written relations, but on top of that puts configurations into general position such that their generic, or better: resolved, structure becomes "visible" on the auditory level—to restate it in the wording of Varèse.

**Principle 8** *So the* structural *rationale of performance is a strategy of small changes of the composer's perspective to make the resolution of the composition audible and thereby ease understanding of the underlying composition class (in the strict sense of classification).*

# Chapter 18

# Predicates

*Die Welt ist alles, was der Fall ist.*
Ludwig Wittgenstein [580]

**Summary.** Denotators are purely mathematical structures which do not specify "what is the case" and what is not. This chapter deals with the existence problem of music-related objects in contrast to mathematical 'fiction'. This amounts to loading mathematical constructs with an additional semiotic signification process in order to express "which denotators are the case". These existence specifications instantiate an interface between mental potentiality and historical actuality. It reveals two fundamentally different existentialities, termed "textual" and "paratextual" signification, respectively. The former involves predicates defined by classical extension over specific denotators, whereas the latter transcends pure extensionality and thus points into domains of open semiosis. Both, textual and paratextual predicates are essential enrichments of mathematical constructs: The platonic ontology is thereby supplemented by a differentiation which cannot be reduced to pure "mentality". The variety of textual predicates follows certain construction rules of logical and geometric nature and is founded in a triply typed set of "atomic" predicates of (a) mathematical, (b) musical, and (c) deictic types.

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## 18.1 What Is the Case: The Existence Problem

**Summary.** Music and musicology cannot refrain from distinguishing fiction against facticity. There are three reasons for this constraint: (1) historically, works, interpretations, and performances are in a substantial interaction which deals with what is the case; (2) in view of interactive discourse on music, actual perspectives of involved instances are not contingent but proper to the results; (3) the documentation of facts related to music are part of their accessibility *not less* than physical bodies do share and—by Einstein's gravitation theory—even *shape* a site in the universe of space-time.

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In mathematics, once a domain of objects has been consistently defined, it is no question that they are all available or 'exist' without further differentiation, respectively. Once prime numbers are defined, their instances are just there, no question of distinguishing explicitly those we have already dealt with from the others[1].

The fundamental difference between mathematical set or category theory and denotator theory is that the 'existence' of denotators with respect to music or musicology is *not* equivalent to their purely mathematical existence. Rather must we consider a specific type of allocation or instantiation, be it on a computer's memory or in an intellectual framework, such as the composer's mind or a given composition.

The point is that, within a fixed discourse, we do not permit automatic access to denotators. This is a rigorous discipline about what is given and what is only possibly given, a fundamental feature in musical thought or musicological analysis.

For example, in the diachronic evolution of the music system or within a specific material allocation within an information system[2], this may be relevant. In particular, if one deals with musicological analysis, it is essential to make precise the universe of objects one deals with, be it for classification or for ad hoc reference.

**Principle 9** *We do not deal with a priori limitation of the available 'material', but with a 'declaration duty' of what we are allowed to refer to.*

Denotators share a kind of mathematical existence. Whether we view them in a set or category-theoretical perspective, they participate in a layer of abstract existence. In the context of music semiotics, it becomes relevant to couple semiotic specification in the sense of Hjelmslev's glossematic [227] with existentiality: Denotation and connotation are ontologically sensitive concepts. They not only reflect ontology but are possibly responsible for its very production. This contrasts musicology with physics: The latter deals with a fairly objective subject to be described by mathematics, whereas the subject of musicology is far more human nature and as such does not only exist but is essentially *created*. Therefore this ontological enrichment has to be dealt with explicitly and in a differentiated way.

In order to differentiate denotator ontology from the musicological one, the denotator system is viewed as the denotative layer of a supersystem whose signs are called (musical) predicates. Here, the denotators play the role of significants, whereas the significates are instances of 'specifically' musicological meaning. This is the model to be developed in the sequel[3].

### 18.1.1 Merging Systematic and Historical Musicology

**Summary.** Predicates prepare the field for a reconciliation of systematic and historical musicology.

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[1] However, for computer mathematics, the prime numbers which have been dealt with are definitely more concrete than the others.

[2] A database, for example.

[3] It is based upon the insight of music semiotics [361] that musicological meaning is a multilayered fact to be successively constructed via Hjelmslev chains of denotator/connotator systems.

We cannot refrain from stressing how profoundly the distinction between denotators and predicates hits the very structure of musicology. Following a strong tradition [103], this is commonly divided into systematic and historical musicology. While historical musicology seems to deal with history and historicity of music, systematic musicology seems to deal with musical systems. However, it is well known that history is also history of somewhat, of some systems, whereas systems are also historic in nature. The contrast to physics, for example, is that history of physics is history of the knowledge of one and the same subject: nature. So the physical systems evolve as knowledge bases about one and the same thing. Their coherence is a priori guaranteed by the identity of nature and by the invariance of physical laws in time.

In contrast—and this is the argument which is usually given—music's identity is itself a historic variable since music is a constantly evolving human creation. The argument is however erroneous since music's identity has never been defined or even understood as a time-independent entity. In other words, the diachronic axis of music is essential, but not as a disconnecting instance, on the contrary, coherence of the historical development is part of the overall system. Like language, music is a semiotic system with diachronic and synchronic coordinates and laws which pertain specifically to dia- and synchronic locations. But the study of time-dependent systems is not essentially different from the study of time-independent systems such as physics[4]. In other words: The fact that laws of music are susceptible to be time-dependent is no reason to believe that there are no laws, that there is no all-embracing system beyond a valid chain of stories.

There are several methods to turn this pseudo-dichotomy of historic and systematic musicology into science. One way is to seek for time-invariant laws which model historic development as such in specific areas. Such an approach is undertaken in the theory of increasing paradigmatic groups as a significant parameter for historicity, see section 11.7.2. This approach resembles the second law of thermodynamics which—for a determined set of physical systems—defines a time arrow towards the future by postulated increase of entropy. This first approach is somewhat naive since it may happen that the dynamics of knowledge acquisition requires a conceptual reengineering. For example, the concept of reference frame in physics had to be revised because of the relativization of time in special relativity. The relativistic time concept had to be attached as a proper attribute to each reference frame, and was no longer an all-embracing, divine quality in the sense of Newton.

Whereas in physics, epistemological events of this size are rather rare, in fact appear as veritable scientific revolutions, *music is in a state of incessant big bang*: Creativity constantly adds to the fundamental data and knowledge. In other words, dynamical concept and theory handling is much more vital to musicology than to physics. Therefore a second method consisting of dynamic concept frameworks and explicit handling of the incessant creative impact is required. Dynamic concept frameworks have been developed in the previous theory of forms and denotators. The impact of incessant big bang creativity is handled by the theory of predicates.

More generally speaking, *historical sciences are likely to be absorbed by dynamical knowledge management* in the following sense. Understanding history means conceptual and data control of the world's diachronic dynamics[5]. Professional data control is no longer feasible

[4]Even physical laws do not persist eternally, it is argued that in the very beginning of the big bang, physical laws were rightly established; in fact at physical zero time, all laws break down.

[5]Wittgenstein's dictum heading this chapter should then be refined diachronically: "Die Welt ist alles, was mit der Zeit der Fall ist."

without powerful data base management systems, and conceptual control cannot survive if it is not part of a comprehensive concept management system. Otherwise, historic science runs out of self-control or downsizes to well-known variants of ideologies and "weltanschauungen". In this sense, *innovative musicology could play a role of a prototype for future historical science.*

## 18.2 Textual and Paratextual Semiosis

**Summary.** Review of Kofi Agawu's and Roman Jakobson's introversive and extroversive semiosis. Truth values and meaning. The problem of open semiosis. Classifying open semiosis: processuality, synchronic and diachronic pointers of competence and tradition. Introducing the formal system of a *predicative semiology.*

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To get off the ground with the discourse on musicological meaning, we review Agawu's work on music semiology [8] which builds on the tradition of Jakobson's [245, 246] research in modern poetology. Agawu follows Jakobson in distinguishing introversive vs. extroversive semiosis. Introversive semiosis is production of meaning on the basis of intratextual signs. Agawu calls them the universe of structure. Examples: Schenker's "Ursatz" (beginning/middle/ending), Ratner's model of harmonic functions, and, of course, all elementary signs for metric, rhythmical, motivic, harmonical, etc. structures. Introversive semiosis can be said to be production of textual meaning because the text is the relevant reference level for introversive semiosis.

Extroversive semiosis involves signs which transcend the system of musical signs in the narrow sense of the word. Agawu calls them "the universe of topics". Topics are signs which have a significate beyond the text. The author gives 27 examples within the form of his analysis (reaching from 1770 to 1830, i.e., embracing the first Viennese school):

> *alla breve, alla zoppa, amoroso, aria, bourrée, brilliant style, cadenza, Empfindsamkeit, fanfar, French overture, gavotte, hunt style, learned style, Mannheim rocket, march, minuet, musette, ombra, opera buffa, pastoral, recitative, sarabande, Seufzermotiv, singing style, Sturm und Drang, Turkish music.*

These signs have a surface which has a regular textual meaning involving particular groups of sound events with a particular structure. But the deeper meaning, in some sense a connotative significate, reaches beyond the text, and understanding it requires historical and/or music(ologic)al competence. It can only be adequately realized by the competence of the listener/musician and his/her idiomatic expertise.

Agawu's access to musical meaning is characterized by a dichotomy between precise textual and denotative semiosis and some kind of black box semiosis referring to an exterior context. This music(ologic)al context is an open system and should be treated as such. With regard to it, there is open ended semiosis dealing with what we call paratextual meaning.

The distinction between textual and paratextual meaning is motivated by the fact that meaning is never formally closed and must therefore be treated in an open way.

When building the predicate concept, one has to make the semiosis that takes place on the denotator level more open with respect to

- diachronicity: the set of predicates may change with time (cultural epochs);
- synchronicity: the set of predicates depends on the spatial (cultural) context;
- incompleteness: meaning may be incomplete, provisional or overtly undetermined.

Let us give an illustration: The diachronic extension of the repertoire of compositions determines the experimental material upon which music analysis has to be executed, tested and developed. It is not time-invariant. For instance, the Tristan chord is far more than just one item within an abstract list of chords. The synchronic analysis of ethnomusicological data is heavily dependent upon the cultural region from where they are applied. The extroversive meaning of data (such as "fermata") may be incomplete (from the point of view of performance) and one has to find a way of handling it formally. We do not yet know 'everything' a sign is supposed to convey, but we have to handle it within its context. In the life of musicians and musicologists, incomplete semiosis is the rule, not the exception!

To control this variety of sign processes it is necessary to set up an adequate system of signification mechanisms.

### 18.2.1 Textual and Paratextual Signification

**Summary.** Truth values and meaning; extension vs. intension.

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In order to distinguish potential from actual instances of denotators in the spirit of Agawu's universe of structure, it is necessary to be in a position to tell which instances "are the case" and which are not. This is why we stipulate that predicates are related to sets of denotator instances which do exist. For example, if we describe piano notes by special types of denotators, the predicate "piano notes of concerto XY" would cover all these denotators for the notes of the score XY attributed to the piano. We may say that this predicate creates meaning by extension.

Meaning by extension can be distinguished from meaning by intension. This is what Agawu alludes to when introducing topics. It turns out that intensional meaning involves a much richer type of semiosis. The following list enumerates some types of open semiosis (without claim of completeness):

- *Semiosis as a process*—To begin with, semiosis is not a state but a process so that richness of meaning increases or decreases as a function of system time. What could be an intensional meaning at a given moment can be transformed into an explicit extensional meaning after additional information was added, see [122].
- *Synchronic pointers: competence*—Meaning of the intensional type may consist in the reference to another instance which 'knows' more about the music in question. This is a pointer to a competence exterior to the structural data. For example: The expert teacher in piano music knows how a specific articulation sign has to be realized on a grand piano.
- *Diachronic pointers: tradition and progress*—Another pointer type of intensional character is directed towards historical topoi or towards paradigms of progress. As an example, meaning may be anchored in historical style knowledge.

## 18.3 Textuality

**Summary.** The category of denotators. The textual predicate system. Predicate expressions as significants. Classification of textual predicates by three dichotomies: arbitrary/motivated, punctual/relational, and objective/morphic. Classification of motivation mechanisms: logic and geometry.

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### 18.3.1 The Category of Denotators

**Summary.** We first have to introduce a category structure on the set of all denotators.

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In section 8.2, we briefly alluded to an extension of morphisms between local compositions to denotators of more general types. For the following discussion of textual predicates, we need such a framework which admits morphisms between any denotators.

In order to define morphism for denotators of any type, i.e., for the category *Den* of denotators, we start by the following axiom:

**Axiom 2** *If $x, y$ are denotators of different form type, the set $Den(x, y)$ of morphisms in Den is empty.*

This means that only typed morphisms are allowed here, in other words: The category of denotators is the coproduct

$$Den = Den_{\mathbf{Simple}} \sqcup Den_{\mathbf{Syn}} \sqcup Den_{\mathbf{Power}} \sqcup Den_{\mathbf{Limit}} \sqcup Den_{\mathbf{Colimit}} \tag{18.1}$$

of the subcategories of types as notated in the indexes. Let us now look at the different types. The type where we already have a good theory is the category $Den_{\mathbf{Power}} = Loc$ of local compositions. Observe that here, the morphisms do not presuppose that the form morphisms which induce the natural transformations $f$ of the respective subfunctors restrict to specific form types, i.e., the coordinator forms may be of any type. This means that if we want to associate a denotator $x : A \rightsquigarrow F(\xi)$ of form type $t_x$ with a denotator $y : B \rightsquigarrow G(\eta)$ of form type $t_y$, by a form morphism $h : x \mapsto y.\alpha$, say, then we can do so under their "wrapping" as singleton local compositions, i.e., we have $f/\alpha : \{x\} \to \{y\}$ for $f = h|_{\{x\}}$, as described in section 8.2. So

**Principle 10** *The only way to change type by morphisms is to wrap form morphisms into local compositions.*

Let us now define the subcategory $Den_{\mathbf{Simple}}$. Give two simple denotators $x : A \rightsquigarrow F(\xi), y : B \rightsquigarrow G(\eta)$, with simple forms $F \underset{Id}{\longrightarrow} \mathbf{Simple}(M), G \underset{Id}{\longrightarrow} \mathbf{Simple}(N)$, we set

$$Den(x, y) = \{(x, y, \alpha), \alpha \in A@B | \text{ there is } h \in M@N \text{ with } h.x = y.\alpha\} \tag{18.2}$$

and we denote such a morphism by $!/\alpha : x \to y$. This definition should be read properly with respect to the underlying identifiers. We have $\xi : A \to Fun(F), \eta : B \to Fun(G)$, a notation

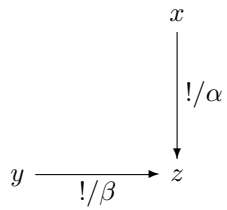
which we may use in view of Yoneda's lemma, by identification of $A@Fun$ with $Hom(@A, Fun)$. The existence of $h$ means that we have a form (i.e., a functor) morphism $\bar{h} : Fun(F) \to Fun(G)$ which is induced by $@h$. Then the equation $h.x = y.\alpha$ means that $\bar{h}.\xi = \eta.\alpha$. The rest is obvious.

**Proposition 20** *The subcategory* $Den_{\mathbf{Simple}}$ *has arbitrary fiber products.*

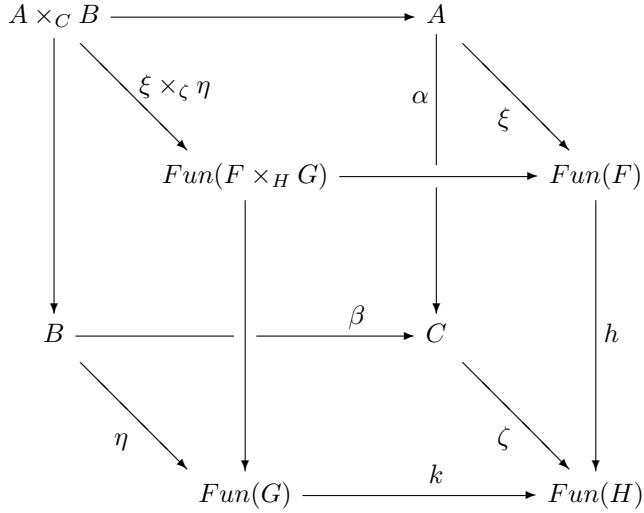
Proof. Suppose we are given three simple forms

$$F \underset{Id}{\longrightarrow} \mathbf{Simple}(M), G \underset{Id}{\longrightarrow} \mathbf{Simple}(N), H \underset{Id}{\longrightarrow} \mathbf{Simple}(L),$$

three respective denotators $x : A \rightsquigarrow F(\xi), y : B \rightsquigarrow G(\eta), z : C \rightsquigarrow G(\zeta)$ and two morphisms




of simple denotators. Suppose that $!/\alpha$ is induced by a morphism $h : M \to L, k : N \to L$. Then we have a fiber product form $F \times_H G : Id \rightsquigarrow \mathbf{Simple}(M \times_L N)$ with identifier equal to the fiber product of the identifiers of $F, G, H$ under the given form morphisms. Since these form morphisms are induced by two module morphisms $h, k$, the projections $F \times_H G \to F, F \times_H G \to G$ are also induced by the projections $M \times_L N \to M, M \times_L N \to N$. With this in mind, there is a denotator $x \times_z y : A \times_C B \rightsquigarrow F \times_H G(\xi \times_\zeta \eta)$ which is universally defined to make this diagram




commute, and we are done. QED.

For synonymy, define the *reference denotator* $x^\sim : A \rightsquigarrow F^\sim(\xi^\sim)$ of a denotator $x : A \rightsquigarrow F(\xi)$ with $F \longrightarrow \mathbf{Syn}(F^\sim)$ and $\xi^\sim = Id(\xi) \in A@Fun(F^\sim)$. This is essentially the same as $x$, albeit

leaving aside the synonymous 'disguise'. Suppose by recursion that for two synonymy denotators $x : A \rightsquigarrow F(\xi)$, $y : A \rightsquigarrow F(\eta)$, we have already defined and symbolized morphisms of referenced denotators by the fractional notation $f/\alpha : x^\sim \to y^\sim$ with address change $\alpha : A \to B$. Then we identify the morphisms $Den(x, y)$ with the set of triples $\{(x, y, f/\alpha) |\ f/\alpha \in Den(x^\sim, y^\sim)\}$; and we denote such a morphism by the evident symbol $f/\alpha : x \to y$. The composition of such morphisms is evident, and this (sub)category is defined. This subcategory has fiber products iff its referenced counterpieces have. Observe that this is a problematic requirement for circular denotators; in a mathematical discussion, the situation should receive a special treatment.

For the limit type, give two $x : A \rightsquigarrow F(\xi), y : B \rightsquigarrow G(\eta)$, with forms $F \underset{Id}{\longrightarrow} \mathbf{Limit}(\mathbb{D}), G \underset{Id}{\longrightarrow} \mathbf{Limit}(\mathbb{E})$. By definition, we have $\xi = (\xi_d)_{d \in |\mathbb{D}|}, \eta = (\eta_e)_{e \in |\mathbb{E}|}$ where $|\mathbb{D}|, |\mathbb{E}|$ denote the corresponding diagram vertex sets. In this case, a morphism $f/\alpha : x \to y$ means the following:

1. an address change $\alpha \in A@B$,
2. the symbol $f/\alpha$ for a family of morphisms $(f_d/\alpha : x_d \to y_{\phi(d)})$ with a map $\phi : |\mathbb{D}| \to |\mathbb{E}|$, and where the denotators $x_d, y_{\phi(d)}$ are reference denotators as above, i.e., $x_d : A \rightsquigarrow F_d(\xi_d)$ for the $d$-th form $F_d$ of the diagram.

In particular, we suppose that the types of the factor domains and codomains are identical. The composition of two such morphisms is again evident. In this subcategory, fiber products can be constructed as follows: Give a diagram of three limit denotators

$$\begin{array}{ccc} & & x \\ & & \downarrow{\scriptstyle f/\alpha} \\ y & \xrightarrow[g/\beta]{} & z \end{array}$$

with morphisms $f/\alpha : x \to z = (f_d/\alpha : \xi_d \to \zeta_{\phi(d)})$ for a map $\Phi : |\mathbb{D}| \to |\mathbb{F}|$ and $g/\beta : y \to z = (g_e/\beta : \eta_e \to \zeta_{\psi(e)})$ for a map $\Psi : |\mathbb{E}| \to |\mathbb{F}|$. The fiber product has a diagram scheme which is the fiber product set $\mathbb{P} = |\mathbb{D}| \times_{|\mathbb{F}|} |\mathbb{E}|$ (no arrows), together with its canonical projections into the factor diagrams. On a pair $(d, e)$ over the vertex $h$ of $\mathbb{F}$, the diagram evaluates to a denotator $p_{d,e}$ at the address $A \times_C B$ which is given by the cartesian square

$$\begin{array}{ccc} p_{d,e} & \longrightarrow & x_d \\ \downarrow & & \downarrow \\ y_e & \longrightarrow & z_h \end{array} \tag{18.3}$$

and whose existence is supposed by circular or regular recursion. It is easy to verify that this defines a fiber product on $Den_{\mathbf{Limit}}$. For the colimit type, give two $x : A \rightsquigarrow F(\xi), y : B \rightsquigarrow G(\eta)$, with forms $F \underset{Id}{\longrightarrow} \mathbf{Colimit}(\mathbb{D}), G \underset{Id}{\longrightarrow} \mathbf{Colimit}(\mathbb{E})$. Here, we have either $x = y$ where we just take the identity as a morphism, or else it is supposed that the diagrams are both discrete, in which case both, $x, y$ live in coproduct spaces, $x \sim x_d \in A@d, y \sim y_e \in B@e$, say. Then we identify $Den(x, y)$ with the morphism set $Den(x_d, y_e)$ as above. In all other cases, no morphisms

are provided in the colimit type. We leave it as an exercise to verify the existence of fiber products for the subcategory $Den_{\mathbf{Colimit}}$.

We stress once more that these are recursive definitions which require a special treatment in case the denotators pertain to circular forms. Observe that, for example, if we have a denotator $x$ in a product form of two simple factors with modules $M, N$, this form is isomorphic to the simple form associated with the direct sum $M \oplus N$. However, the denotator is not isomorphic to its corresponding denotator $x'$ in the simple form of the direct sum! It only becomes isomorphic if wrapped as a singleton, i.e., $\{x\} \tilde{\to} \{x'\}$. One could also see this technique as a "type casting" procedure, known from (strongly) typed computer programming languages.

#### 18.3.1.1 Morphisms as Denotators

**Summary.** For formal programming reasons, it might be useful to transform morphisms into denotators. We look for such techniques.

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If $\alpha \in A@_R B$ is an address change over a fixed ring $R$, and $x : A \rightsquigarrow F(\xi), y : B \rightsquigarrow G(\eta)$ are two denotators, the triple $(x, y, \alpha)$ can be seen as a denotator as follows: We have these sections and projections of addresses: $i_A : A \rightarrowtail A \oplus B, p_A : A \oplus B \twoheadrightarrow A$ and $i_B : A \rightarrowtail A \oplus B, p_B : A \oplus B \twoheadrightarrow B$. We also have the injection $p_B@B : A@B \rightarrowtail A \oplus B@B$. We refer to these maps in the following discussion.

Take the two injections $p_A@F : A@F \rightarrowtail A \oplus B@F$, and $p_B@G : B@G \rightarrowtail A \oplus B@G$. Consider the form $F \times G \times B \underset{Id}{\longrightarrow} \mathbf{Limit}(F, G, @B)$. Then the triple $(x, y, \alpha)$ is identified with a denotator $(x, y, \alpha) : A \oplus B \rightsquigarrow F \times G \times B(\xi, \eta, \alpha)$. This settles the description of simple denotators.

For the local compositions, take the usual morphism situation:

$$\begin{array}{ccc} x & \longrightarrow & @A \times F \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle @\alpha \times h} \\ y & \longrightarrow & @B \times G \end{array} \tag{18.4}$$

which represents the morphism $f/\alpha : x \to y$. We may take the graph $\Gamma_f \subset @A \times F \times @B \times G \tilde{\to} @A \oplus B \times F \times G$. This is a $A \oplus B$-addressed local composition in the form $F \times G$. Taking the form $exp(F, G) \underset{Id}{\longrightarrow} \mathbf{Power}(F \times G)$, we have this denotator $\Gamma_{f/\alpha} : A \oplus B \rightsquigarrow exp(F, G)(\gamma)$ and we can couple it with the equally addressed denotator $(x, y, \alpha)$ from above to get a denotator $(f/\alpha)$ in form $exp(F, G) \times F \times G \times B \underset{Id}{\longrightarrow} \mathbf{Limit}(exp(F, G), F \times G \times B)$:

$$(f/\alpha) : A \oplus B \rightsquigarrow exp(F, G) \times F \times G \times B(\Gamma_{f/\alpha}, (x, y, \alpha))$$

for the given morphism of local compositions.

The case of synonymy is trivial since the morphisms are just "synonymy casts" of already given morphisms.

The limit and colimit situations essentially reduce to a finite list of morphisms or one selected morphism, respectively, and this is settled by the above techniques; we leave it as an exercise to work out the details.

We should add two remarks:

- Morphisms of forms can be restated as new forms. In fact, if $h : F \to G$ is any morphism of forms, look at the form

$$\Gamma_h \underset{\Gamma_h : Fun(F) \rightarrowtail Fun(F) \times Fun(G)}{\longrightarrow} \mathbf{Limit}(F, G).$$

  The morphism is represented by the identifier of this form. This can be used to represent general limit data by products: If we have a diagram $\mathbb{D} : D \to \mathbf{Mod}^{@}$, each diagram arrow $f_{d,e,k} : d \to e$ can be wrapped in its graph form $\Gamma_{f_{d,e,k}}$ of limit type (in fact a product of two forms). Of course, one should not believe that wrapping a morphism in an identifier is an essential simplification of real problems, but it shows that morphisms are controlled on the level of identifiers within forms, and that, conversely, controlling forms qua objects means controlling the whole category of forms.

- If two denotators of same address and form have same coordinate $\xi$ but different names "$d_1$", "$d_2$", we can shift this difference into the coordinate by adding a simple factor of character strings to the given form, and restating the coordinates as $(d_1, \xi), (d_2, \xi)$ such that now, the coordinates are different and not only the total denotator data.

  This discussion makes clear that we may either look at denotators or at the category of denotators, the information is essentially the same: Each morphism yields an object, so that the knowledge about all objects is sufficient to describe knowledge about the whole category *Den*. This is why we shall in fact use the whole category and not only its objects in the following discourse about predicates.

### 18.3.2 Textual Semiosis

**Summary.** We introduce the formal system of a textual semiosis, including the question of switching between predicates and denotators.

$$-\ \Sigma\ -$$

The textual predicate system is a formal system of semiosis which manages the difference between mathematically possible denotators and denotators which are the case in some technical or more informal data base. The typical application is the storage of denotators on computer data bases for music research including retrieval, communication, and extension of knowledge.

A *textual semiosis (over the category Den)* is a map

$$sig_{Den} : Tex \to Texig(Den)_I \tag{18.5}$$

with a domain $Tex \subset Ex =< UNICODE >$ in the monoid of all $UNICODE$ strings[6], which we call *expressions* here.

To define its codomain, let $Den_\infty = \coprod_{1 \leq n} Den^n$ be the union of all powers of the arrow set ${}_1Den$, including the identities which represent the denotators as objects. Moreover, we are

[6] $UNICODE$ is an extension of the $ASCII$ set to non-European letters and symbols.

given a module $I$ of 'truth values' whose role will be discussed in the sequel[7]. This module will only be evoked via its role within a simple form

$$Val(I) \underset{Id}{\longrightarrow} \mathbf{Simple}(I) \tag{18.6}$$

of truth values[8]. This in turn gives rise to what is really needed, i.e., the form

$$TRUTH(I) \underset{Id}{\longrightarrow} \mathbf{Power}(Val(I)) \tag{18.7}$$

whose set of $A$-addressed *truth denotators* (any address admitted) is denoted by $\mathbb{T}_I^A$, whereas the union of all such sets over the totality of addresses is denoted by $\mathbb{T}_I$ and called the set of truth denotators. In other words, we have a 'fibration' $add : \mathbb{T}_I \to \mathbf{Mod}$ whose fiber $add^{-1}(A)$ at address $A$ is $\mathbb{T}_I^A$.

The codomain $Texig(Den)_I$ is the set $\mathbb{T}_I^{Den_\infty}$ of all *characteristic functions* on the set of all finite length tuples of denotators or morphisms of denotators, and with values in $\mathbb{T}_I$. If $\chi \in Texig(Den)_I$, and if $x \in Den_\infty$, the address of $\chi(x)$ is often related to the addresses which are involved in $x$. For example, if all these addressed coincide and are equal to $A$, then $\chi(x)$ should also live in $\mathbb{T}_I^A$, but this is not mandatory and depends upon the construction of $\chi$ from previously defined functions.

To understand the truth denotator set, let us first discuss the completely reduced case of zero address and zero truth module, i.e., $A = I = 0_0 = 0$. This is the standard situation of truth values in topos theory: They are just the set of morphisms $1@\Omega$; in our situation of the presheaves over the module category **Mod**, the final object is representable: $1 = @0$, and by Yoneda, we have $1@\Omega \tilde{\to} \Omega(0) = Sub(@0)$. Warning: This topos $\mathbf{Mod}^{@}$ is not Boolean. In fact, a topos of presheaves $\mathbf{Sets}^{\mathbf{C}}$ is Boolean iff $\mathbf{C}$ is a groupoid, i.e., iff all arrows are isomorphisms (see [314, exercise VI.2,p.343]). In general, we have $A@TRUTH(I) = A@\Omega^{@I} \tilde{\to} Sub(@(A\times I))$, therefore for the zero situation, a denotator $D \in \mathbb{T}_0^0$ is $D : 0 \rightsquigarrow TRUTH(0)(d)$ and has coordinate $d \subset @0$, a sieve in the zero module over the zero ring. To describe all sieves in the zero module, consider a category $\mathbf{C}$, and the equivalence relation

$$R \sim S \text{ iff } \mathbf{C}(R,S) \text{ and } \mathbf{C}(S,R) \text{ are both non-empty}$$

on $\mathbf{C}$ whose classes are denoted by $[R]$. On $\mathbf{C}/\sim$, we have a directed tree graph[9] structure $B_{\mathbf{C}}$ where an arrow $[R] \to [S]$ is defined iff there is a morphism $f : R \to S$ for representatives $R, S$. On any directed tree $T$, we can build the duplication tree $Dup(T)$ as follows: Take two disjoint copies $T_0, T_1$ of $T$ and draw one arrow $d_x : x_0 \to x_1$ for each couple $(x_0, x_1)$ of points corresponding to the original point $x$ of the tree $T$.

Call a subset of a directed graph $G$ *open* iff it contains the domains of all arrows whose codomains it contains. The set $Open(G)$ of all open sets in $G$ defines a topology. In fact, the empty set and $G$ are open, and any union/intersection of opens is open. Therefore, the topological space $Open(G)$ is a Heyting algebra in the usual sense (with intersection, union and implication, see appendix G.5.1 and [314, p.51]). With this in mind we have the following description of all truth denotators modulo names:

[7] We refrain from introducing non-representable truth value objects here and restrain to modules as there are merely speculative reasons to generalize.

[8] More generally, a truth form may be any form $F$, and not only a simple one.

[9] This is a directed graph whose undirected associated graph (forget about arrow directions) has no cycles.

**Theorem 24** *The Heyting algebra $Open(Dup(B_{\mathbf{Rings}}))$ of open sets of the duplicated tree $Dup(B_{\mathbf{Rings}})$ and the Heyting algebra $Sub(@0)$ of (zero) truth coordinates are canonically isomorphic.*

Proof. Let $X \subset @0$ be a sieve, and $A$ any address. Then either $A@X = \{!_A\}$ (the unique arrow to the final module) or $A@X = \emptyset$. So such a sieve is defined by the subset $supp(X)$ of addresses where $A@X = \{!_A\}$. Conversely, if any subset $S$ of addresses is given, this corresponds to the support set of a sieve iff it is closed under address changes, i.e., iff, whenever $A \in S$ and there is an address change $B \to A$, then $B \in S$.

Therefore, the support set $S$ of a sieve must be a union of module classes in the above sense, i.e., elements of $B_{\mathbf{Mod}}$. We therefore view $S$ as a subset of vertexes in $B_{\mathbf{Mod}}$ which is open in $B_{\mathbf{Mod}}$.

Now, every non-empty $R$-module $M$ is equivalent to the zero-module $0_R$ over the same ring. And for two such modules, there is a morphism iff there is an underlying ring homomorphism. Therefore, the restriction of the module class tree to the non-empty modules is isomorphic to the class tree of the ring category $B_{\mathbf{Rings}}$. Moreover, the empty modules are also classified according to their coefficient rings, and therefore, the class tree of these modules is also isomorphic to the class tree $B_{\mathbf{Rings}}$ of rings. Finally, there is a morphism $\emptyset_R \to 0_R$ whereas there is no morphism in the other direction. Therefore, the tree $B_{\mathbf{Mod}}$ is isomorphic to $Dup(B_{\mathbf{Rings}})$. Moreover, a sieve support set $S$ of module classes in $B_{\mathbf{Mod}}$ must be open, and we are done. QED.

**Exercise 38** Observe that there are open sets in $Dup(B_{\mathbf{Rings}})$ which are not defined by a discrete set of leaves. For example, take the direct product ring $R = \prod_{i=1,2,3,\ldots} \mathbb{Z}_{p_i}$ over all prime fields for increasing primes $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$. Consider the infinite increasing sequence of ideals $I_k = \prod_{k \leq i} \mathbb{Z}_{p_i}$ and the corresponding chain of projections

$$R \to R/I_1 \to R/I_2 \ldots R/I_k \to R/I_{k+1} \ldots$$

which defines non-equivalent rings and whose opening (the smallest open set which contains this sequence) is not defined by a leaf.

**Example 35** A first example of a characteristic function is the attribution of one of the two universal values $\top, \bot : 1 \to \Omega$ with $\top = @0, \bot = \neg\top = \emptyset$ for each tuple $f$ of denotators. This may and *will* happen in a completely uncontrollable, mathematically non-foreseeable way, see section 18.3.3.

**Proposition 21** *The coordinate Heyting algebra $0_{\mathbb{Z}}@\Omega$ of truth denotators for address $A = 0_{\mathbb{Z}}$ and truth module $0_0$ is isomorphic to the Heyting algebra of sieves $S \subset \circledcirc\mathbb{Z}$ in the category* **Rings**.

The proof is left as an exercise.

For any truth value module $I$ and address $A$, and for any local composition $t \subset A@I$, we have the canonical functorialization $\hat{t}$ (see section 6.2.3), in particular, for every value $\tau \in A@I$, we have $\hat{\tau} = \{\tau\}\hat{}$. So every 'set' $t \subset A@I$ of $A$-addressed 'truth values' in $I$ gives rise to a truth denotator in $\mathbb{T}_I^A$.

**Example 36** Take, for example, the value module $S_1 = \mathbb{R}/\mathbb{Z}$, i.e., the real circle group (a $\mathbb{Z}$-module). For each number $\phi \in [0,1]$, we have the local, zero-addressed truth denotator (coordinate) $\phi\hat{} = \widehat{[0,\phi[} \in \mathbb{T}_{S_1}^{0_\mathbb{Z}}$. This means that we have traced a (naive) fuzzy logic on the unit interval by the interval truth denotators $\phi\hat{}$, included in the special values $0\hat{} = \emptyset \subset \phi\hat{} \subset 1\hat{} = \widehat{S_1}$. The upper limit is the largest objective local composition in $@(0_\mathbb{Z} \times S_1)$.

**Principle 11** *Philosophically, this means that fuzzy logic (and also other variants according to the given value module) is interpreted as a logic of intervallic local compositions in the unit group. This is quite the spirit which we shall now evoke: to view logical values as special local compositions, and to rephrase statements about local compositions as if they were generalized truth values. In music, there is no deeper reason to restrict truth to the final power $\Omega^1$ of the subobject classifier instead of extending the discourse to general powers $\Omega^I$.*

**Exercise 39** For every morphism of truth value modules $h : I \to J$, we have a canonical form morphism

$$TRUTH(h) : TRUTH(I) \to TRUTH(J)$$

defined by images of subfunctors, and a corresponding map

$$\mathbb{T}(h) : \mathbb{T}_I \to \mathbb{T}_J : d : A \rightsquigarrow TRUTH(I)(\delta) \mapsto h.d : A \rightsquigarrow TRUTH(J)(Im(\delta))$$

on the truth denotators. So we have a corresponding map

$$Texig(Den)(h) : Texig(Den)_I \to Texig(Den)_J$$

which canonically extends to a change of textual semioses, i.e., if

$$sig_{Den} : Tex \to Texig(Den)_I$$

is a given textual semiosis, the composition

$$h.sig_{Den} = Texig(Den)(h) \cdot sig_{Den}$$

defines a new textual semiosis.

More generally, given two textual semioses

$$sig^1_{Den} : Tex^1 \to Texig(Den)_I \text{ and } sig^2_{Den} : Tex^2 \to Texig(Den)_J$$

a morphism $sig^1_{Den} \to sig^2_{Den}$ is a pair $(u, h)$ with a set map $u : Tex^1 \to Tex^2$ on the expressions and a morphism $h : I \to J$ on the truth modules, such that the diagram

$$\begin{array}{ccc} Tex^1 & \xrightarrow{sig^1_{Den}} & Texig(Den)_I \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle Texig(Den)(h)} \\ Tex^2 & \xrightarrow{sig^2_{Den}} & Texig(Den)_J \end{array} \tag{18.8}$$

commutes. This defines the *category of textual semioses* on *Den*. Verify the details.

In particular, we may now compare different textual semioses according to variations on the expressive as well as on the truth value module levels.

The construction of the category of textual semioses in exercise 39 will be used when we have to combine characteristic functions which evaluate to different truth domains. This is essential in the whole construction process of musicologically relevant functions and predicates (see below, definition 61). The problem is not a formal one: As soon as one has opened the truth domains from naive logic of "Yes" and "No", combinations of logical statements must deal with a comparison paradigm, otherwise the choice of logic may terminate in a failure of communication between different logical territories.

Very often, a characteristic function is not really relevant except to a small subset of $Den_\infty$, e.g., the zero-addressed local compositions. For the remaining arguments, the function takes value 'false', more precisely, the value is the empty sieve $\perp_A \in A@\Omega^I \tilde{\to} (A \times I)@\Omega$ at address $A$, write also $\perp$ if the address is clear or irrelevant. Define the *support* $supp(\chi)$ *of a characteristic function* $\chi \in Texig(Den)_I$ as the set of tuples $f \in Den_\infty$ such that $\chi(f) \neq \perp$. Whenever the value of a characteristic function is not specified, it is supposed to be 'false' (the address of this $\perp_A$ still being undetermined).

**Definition 61** *A* predicate *for the category Den is a pair* $(E, sig_{Den})$ *of a textual semiosis* $sig_{Den}$ *over Den, together with an expression* $E$ *in* $sig_{Den}$*. The characteristic function* $sig_{Den}(E)$ *is called the* content *of the predicate, whereas the mapping (the functional relation) of* $sig_{Den}$ *is called its* signification*. The predicate's* extension $ext(E)$ *is the content's support* $supp(sig_{Den}(E))$*.*

*In a more sloppy language we shall identify a predicate with its expression if the rest (the semiosis in its very meaning!) is clear. Also we will refer to the predicate's textual semiosis when referring to the predicate's truth module etc.*

We now introduce the concept of a predicative object (a pre-object), morphism (a pre-morphism), or more generally a tuple of morphisms. The denotators are the fictitious (mathematical) objects and we need those denotators which are supported by predicates. So let $E$ be a predicate of a given semiosis. Then a *predicative object, morphism, tuple* (for this predicate) is a pair, denoted by $x/E$, where $x$ is a denotator, a morphism or a tuple of morphisms such that $E(x) = sig_{Den}(E)(x) \neq \perp$, i.e., $x \in ext(E)$. In general, pre-objects and pre-morphisms do not build categories since predicates are completely 'orthogonal' to the underlying mathematical structures. It will however be a special task to delimit classes of predicates which relate to the underlying mathematics. Essentially, pre-objects are introduced to grasp those entities which are the case in the sense of topos-theoretic, and more or less fuzzy sense as explained above. And to guarantee the most unsystematic approach to facticity as it happens to pervade the mud of the humanities.

Summarizing, if we denote the collection of all pre-objects, pre-morphisms and pre-tuples by $Den_\infty/sig$, we have a 'forgetful' map

$$sig/ : Den_\infty/sig \to Den_\infty : x/E \mapsto x$$

whose fiber $sig/^{-1}(x)$ represents the *full facticity* of $x$ with respect to the given semiosis. Setting ${}_0Den_\infty = \prod_{1 \leq i} ({}_0Den)^i$ we complete the general terminology with this definition:

**Definition 62** *A predicate* $E$ *in* $sig$ *is called*

- *objective iff* $supp(E) \subset {}_0Den_\infty$*,*

- *morphic iff it is not objective,*
- *punctual iff* $supp(E) \subset Den$*,*
- *relational iff it is not punctual.*

The above observation that a morphism of denotators may be restated as denotator means that *we may restate morphic and relational predicates as objective and punctual ones.*

Although time is not mentioned explicitly in this construction, the concrete situation is that the entire textual semiosis is not time-independent. The point is that the construction of new pre-denotators from old ones is not automatic but has to be performed by someone. This is not a serious drawback for the theory, but we should be aware that the setup changes every time a new item is added or removed! The details of this dynamics will become more clear in the following construction discourse.

A final point concerning reality and fiction: finiteness conditions. The realistic situation is that every predicate should only have a finite number of supporting denotators, a computer hard disk can not have infinitely many items. This is a delicate condition if we start producing new pre-denotators from given ones, since evident logical or geometric procedures yield infinite support if they are applied carelessly. So the *axiom of finite facticity support* for predicates is an omnipresent boundary condition to be observed in all the following construction.

#### 18.3.2.1 Predicates as Denotators

**Summary.** Predicates should be convertible into denotators for the sake of management in information technology, but also because this is an essential method for generating new denotators from given ones. We discuss such techniques.

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We suppose that we are given a predicate $E$, and that it is objective and punctual, a restriction which is not substantial—modulo some restatement acrobatics as described above. By the technique from section 18.3.1.1, we may further suppose that all the denotators of its support have the same address $A$, viz the direct product of all the addresses of its denotators. And we may suppose that different denotator names also have different coordinates. The set $supp(E)$ is of course not a denotator, but there is a canonical way to make one: Suppose that we have the list $FormSupport(E) = (F_i)_{i=1,\dots n}$ of forms pertaining to denotators of $support(E)$. Define this operator[10] on forms:

$$SupportForm(E) \underset{Id}{\longrightarrow} \mathbf{Limit}((P(F_i))_{i=1,\dots n}) \tag{18.9}$$

with the factor forms

$$P(F_i) \underset{Id}{\longrightarrow} \mathbf{Power}(F_i).$$

This defines a canonical restatement of $supp(E)$, in fact, take $supp(E)_i = supp(E) \cap supp(F_i?)$, where we view $F_i?$ as the predicate with support exactly the denotators of form $F_i$. Set $Coord_i =$

[10] The index limit $n$ could be a finite of infinite number, but as supports will be finite in every practical situation, the finite case is the default case.

set of coordinates of $supp(E)_i$. Then we have the denotator

$$den(E) : A \rightsquigarrow SupportForm(E)((Coord_i)_{i=1,\dots n}). \tag{18.10}$$

As we suppose that the coordinates distinguish denotators, we get a one-to-one representation of the predicate $E$ by a product of objective, $A$-addressed local compositions.

The usage of this kind of denotators is not harmless! In fact it is important since we want to build denotators by this technique, but we do not want to produce contradictions by circularity, as, for example by Russel's antinomy! However, since the construction involves already given denotators, the new one is just a further construction which uses given supports.

But then, we have to pay attention to avoid conflicts with history! The point is that the construction of $den(E)$ involves the actual status of the predicate $E$. So, as time passes, $E$ may change its signification (sic!) and we should update the associated denotator. For example, if $E$ is already constructed from other predicates, this may be a cascade of updating duties. So the updating process may even involve circular constructs (referring to $E$), we will not produce contradictions since the referred to instances of $E$ will be the last one, as with all replacement processes of type "$x = x + 1$" in programming.

We insist on this conflict potential since in the increasingly accelerated cycle of production, documentation, dissemination, and reflection of knowledge, a dramatic interaction between successive cycles may cause such strong intercyclic distortion forces that *knowledge crashes*, since the known and the new may overlap to the degree where the older 'version' cannot even stabilize its contents before the new 'version' intervenes and puts into question its predecessor.

### 18.3.3 Atomic Predicates

**Summary.** The criteria for atomicity. Classification of atomic textual predicates: mathematics, primavista, and deixis (shifters).

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As announced earlier, predicates are not just invented in the air but germinate from a well-delayed arsenal of elementary or "atomic" predicates. The different types of atomicity are first treated before we deal with the combinatorial chemistry of compound predicates in section 18.3.4. Evidently, this concept reflects the overall topography of construction in music and its theory. Basically, we distinguish three approaches to music related predicates: 1) those which are given by a fundamentally mathematical reasoning without further relation to music or to other rationales, 2) those which are defined by a purely music(ologic)al reasoning, and 3. those which are just the user's decisions beyond any further foundation, i.e., deictic morphemes in the sense of semiotics (see [361, 2.5.1] for a more semiotic discussion of the present topic). They will be discussed in the sequel.

#### 18.3.3.1 Mathematical Predicates

**Summary.** Why do we need mathematical predicates since denotators already share mathematical existence? Examples of mathematical predicates for the working musician and musicologist.

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We insist on distinguishing pure mathematical existence from musical relevance (musical existence) of mathematical properties in the lexical foundation for arbitrary predicates. This may—at first sight—seem unnecessarily restrictive. Why should a mathematical criterion not be unrestrictedly accessible within music and its theory? The reason is that, besides some historical attempts to identify music with mathematics (e.g., the Pythagorean school), mathematical objects are not automatically relevant to music. For example, a Fibonacci sequence does not automatically have musical meaning. Only when it is introduced as a composer's or analyzer's criterion will it gain the status of a predicative instance. Musical meaning consists of predicates which are not free from mathematical criteria, but have to be given a semantic status beyond mathematics. One of the reasons for this duty is that cognitive aspects of music must be examined with respect to their relevance within a given context. For example, the discussion of consonance and dissonance perception should be sensitive towards the mathematical procedures involved in the recognition of sonance classes. As soon as cognitive performance has to be traced in the cortical or subcortical tissue, the question of modules for mathematical tasks becomes primordial.

So mathematical predicates have extensions of denotators defined by mathematical criteria. One looks at mathematically defined properties of denotators, such as onset quantities or Boolean specifications, and then selects those denotators which fulfill these properties. Here fulfillment means attributing to denotators truth values in some set $\mathbb{T}_I^A$ of truth denotators. Let us start with four elementary examples which may be extended and completed in many ways.

1. *Chords.* We want to specify denotators which comprehend chords as sets of simultaneous notes. To this end, sets of notes, for piano say, are defined by the form $NoteGroup \xrightarrow[Id]{}$ **Power**$(Piano\text{-}Note)$, using the well-known coordinator $Piano - Note$. Then, by definition, the expression "$PianoChord$" will take truth value $\top \in \mathbb{T}_0^0$ precisely on the denotators
$$Ch : A \rightsquigarrow NoteGroup(Note_1, \ldots Note_n)$$
such that all its substance points $Note_1, \ldots Note_n$ have one and the same $E$-formed onset coordinate $E_1 = E_2 = ...E_n$. It takes the false value $\perp$ in all other cases (we take $I = 0_0$ here). Recall that this pre-denotator is denoted by $Ch/PianoChord$.

2. *Melodic motifs.* We again start with the form $NoteGroup$ in the above example, but require now that the onset coordinates $E_i$ be pairwise distinct and define the content of the expression "$Motif$" to evaluate to $\top_A \in \mathbb{T}_0^A$ those denotators
$$Mt : A \rightsquigarrow NoteGroup(Note_1, \ldots Note_n)$$
which verify this condition: $Mt/Motif$, $\perp_A \in \mathbb{T}_0^A$ for all other denotator morphisms with codomain $A$, and $\perp \in \mathbb{T}_0^0$ in all other cases. So these values also take care of the address for objects and morphisms in *Den*.

3. *$N$-element groups of notes.* We fix a natural number $N$ and want to select those sets of piano notes which have at most $N$ elements. Denote this predicate by the expression $< N$, thus defining the corresponding pre-denotators by $D/ < N$, the truth values being again taken from $\mathbb{T}_0^0$. This last example shows that expressions may contain variables from the mathematical concept framework, such as $N$ in this case.

4. *Form positions.* Each denotator refers to its form, address, and coordinate. Hence we are also interested in the information which comes from this data. For example, we may look for the form's type and define a characteristic function with values[11] in $\mathbb{T}^{A_{[\mathbb{Z}]}}_{\mathbb{Z}_5}$, with this prescription: We codify the forms by $i(\mathbf{Simple}) = 0, i(\mathbf{Syn}) = 1, i(\mathbf{Limit}) = 2, i(\mathbf{Colimit}) = 3, i(\mathbf{Power}) = 4$, and then attribute the value $t(d) \in \mathbb{T}^{A_{[\mathbb{Z}]}}_{\mathbb{Z}_5}$ defined by the constant sieve $i(type(form(d)))^\frown$ for denotators $d$ or codomains $d$ of morphisms, and the false value $\perp \in \mathbb{T}^{A_{[\mathbb{Z}]}}_{\mathbb{Z}_5}$ else.

These four examples suggest that there may be a legitimate musical or musicological interest to introduce certain mathematical properties for denotators. Nevertheless, these properties are of a purely mathematical nature and cannot be deduced from other properties by formal arguments; this is why they are arbitrary to the predicate system.

#### 18.3.3.2 Primavista Predicates

**Summary.** Primavista (= PV) predicates are the main source for score-related predicates. We give a list and discuss some representative PV predicates for classical European notation; further examples of non-European PV predicates.

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PV predicates are predicates related to scores. The concept of score is however not restricted to the classical European music culture, even if we shall only give a rather complete list of primavista predicates for classical European music. From this list, some selected predicates will be discussed in more detail in order to illustrate the general procedure and to present templates for other predicates.

*List of typical primavista predicates from classical European music scores:*
Staves, Braces and Systems, Ledger Lines and Octave Signs, Clefs, Stems, Flags and Beams, Rests and Pauses, Ties, Key Signatures, Time Signatures, Accidents, Bar-lines and Repeat Signs, Slurs, Dynamic Marks, Articulation, Ornamentation Signs, Tempo Indications, Arpeggio, Composer, Name of the Composition, Expression, Instruments, Lyrics, Comments, Jazz-Harmony, Gestures, Number Sheets (see [435]). For non-European predicates see [103, Vols. 8 & 9].

**Absolute Tempo.** The absolute tempo indication can be either a numerical Mälzel sign or by verbal indications. To grasp this information, we take the forms

$$Maelzel \underset{Id}{\longrightarrow} \mathbf{Limit}(Rate, Onset) \tag{18.11}$$

$$Rate \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}) \tag{18.12}$$

$$VerbAbsTpo \underset{Id}{\longrightarrow} Simple(\mathbb{Z} < ASCII >) \tag{18.13}$$

$$AbsTpo \underset{Id}{\longrightarrow} \mathbf{Colimit}(Maelzel, VerbAbsTpo) \tag{18.14}$$

[11] Recall that the notation $A_{[\mathbb{Z}]}$ means restriction of scalars to the integers, i.e., reduction to the underlying abelian group, see appendix E.1.1.

where a denotator $m : 0_{\mathbb{Z}} \rightsquigarrow Maelzel(x, y)$ denotes that the Mälzel sign $m$ is read as "Play $x$ units of duration $y$!"; the meaning of the $VerbAbsTpo$ is clear, and denotes indications such as "Play *'Lento assai'*!". The total form codifies the alternative between Mälzel and verbal indications.

The next step is the embedding of the absolute tempo information in a named composition and at a precise onset position. We have

$$AbsTpoInComp \underset{Id}{\longrightarrow} \mathbf{Limit}(AbsTpoEvt, NComp) \tag{18.15}$$

$$AbsTpoEvt \underset{Id}{\longrightarrow} \mathbf{Limit}(AbsTpo, Onset) \tag{18.16}$$

$$NComp \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z} < ASCII >) \tag{18.17}$$

and understand that the absolute tempo event denoted under the form $AbsTpoEvt$ is referred to a composition which is represented by a name. Of course, this could be refined, but the idea is clear from this.

Until now, we have just defined forms which yield the spaces where predicates could play a role. The predicate which is expressed by "Absolute Tempo" then is more than the abstract set of denotators. It should tell us which absolute tempo indications are the case from the point of view of a score reading expert. The trace of such a judgment is the predicate's content. If it is not always very clear from the philological point of view whether there is a determined absolute tempo, we could take the truth value of the characteristic function in $\mathbb{T}_{S_1}^{O_{\mathbb{Z}}}$ and thereby model the fuzzy approach as explained above. Therefore the value for a denotator $atp : O_{\mathbb{Z}} \rightsquigarrow AbsTpoInComp(x)$ is set to the fuzzy value $\phi\widehat{\ }$ attributed to the expert judgment that $atp$ is the case with "certainty $0 \leq \phi \leq 1$", and $\perp$ in all other cases.

**Jazz CD Reviews.** If a jazz researcher wants to investigate the role of certain jazz directions in the historical line, it may be important to refer to a judgment of CD releases in professional jazz media. Let us therefore introduce a PV predicate about the importance of a CD in a determined review. The idea is that the predicate is set up by a jazz expert who judges the importance: Is the CD review saying that the product is important? Yes? No? "Rather yes, but, well, I changed my mind about the contents of that review.", etc. So we want a fuzzy, time-dependent predicate $Importance$ on denotators which parametrize jazz CD reviews. Here are the forms:

$$CD \underset{Id}{\longrightarrow} \mathbf{Limit}(Label, Number, Year, Title) \tag{18.18}$$

$$Label \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z} < ASCII >) \tag{18.19}$$

$$Number \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}) \tag{18.20}$$

$$Year \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}) \tag{18.21}$$

$$Title \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z} < ASCII >) \tag{18.22}$$

$$Review \underset{Id}{\longrightarrow} \mathbf{Limit}(CD, Media, Year, Number) \tag{18.23}$$

$$Media \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z} < ASCII >) \tag{18.24}$$

Under the *Importance* predicate, a review denotator *rev* may take truth values in $\mathbb{T}^{O_{\mathbb{Z}}}_{\mathbb{Z}^3 \times S_1}$. This means that we attribute the fuzzy value in the module $S_1$ as above, but we also denote the date (on $\mathbb{Z}^3$) to fix the time-dependence of the judgment which may change even if we concentrate on an individual expert. The rest is clear. Why is this time parameter inserted in the predicate and not in the denotator? Because it is a quality of the predication and not of the review. Of course, one could also have introduced the time parameter in the review form, but then, the predication would be hidden in the denotators instead of the truth values.

**Fushi in Noh's Utai Chant** The Noh tradition (see [269] for the following facts) includes a sophisticated chant formalism which is titled "Utai", a triply articulated structure. We distinguish between melodic, dynamic, and speech articulation in the Utai. In this example, we shall concentrate on the melodic Utai which is described by a combination of melodic units, the *"fushi"*, see figure 18.1. Each fushi (or combination of such) is a

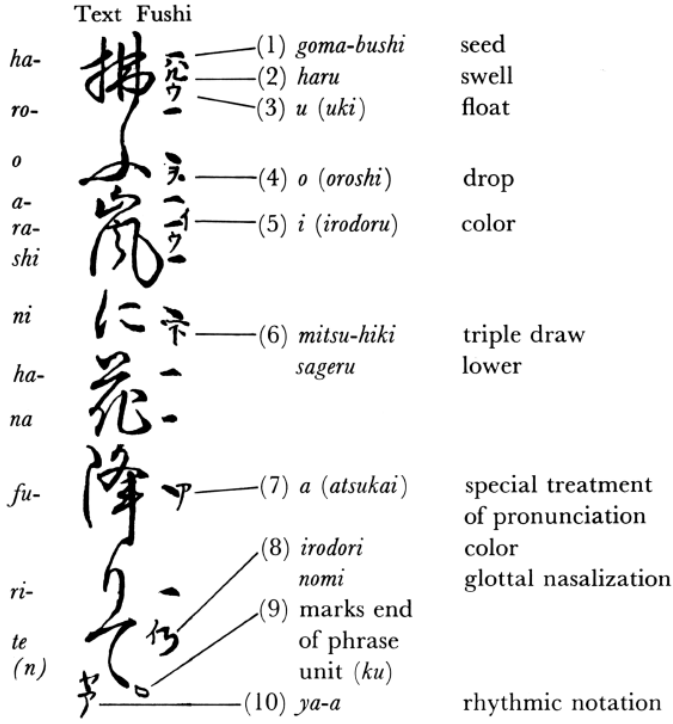


Figure 18.1: An example of a Noh Utai song text (left column) with the fushi sequence (column to its right).

prescription to shape the melodic content of one text syllable in duration, pitch, and color. So fushis are paired with syllables which we codify by the 16-BIT UNICODE codification (the Japanese-Chinese-Korean (JCK) character set, in this case). This is encoded in the

form

$$JCK \underset{Id}{\longrightarrow} Simple(\mathbb{Z} < JCK >) \tag{18.25}$$

where we use the free $\mathbb{Z}$-monoid algebra over the JCK part of the UNICODE alphabet to encode text.

The proper fushi part is built from the three fushi articulations into basic, special, and pitch-change segments. As it is not guaranteed that the special fushi signs are integrated in UNICODE, we better give a plain visual representation of those signs by a sufficiently fine binary encoding, on a $k + l$-Bit basis, say, i.e., in the forms

$$FushiPic \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}_2^k \times \mathbb{Z}_2^l) \tag{18.26}$$

with $k$ Bits for the pixel color encoding and $l$ Bits for the pixel number. This enables us to add a name to each fushi sign. However, the fushi part is not just one of these denotators but it has a recursive compound structure since fushis may be enriched by auxiliary sequences (strings or lists) of fushis for pitch and other specifications. For any form $F$ we may consider the circular list form where the entries are of form $F$:

$$List_F \underset{Id}{\longrightarrow} \mathbf{Limit}(F, ListEntry_F) \tag{18.27}$$

$$ListEntry_F \underset{Id}{\longrightarrow} \mathbf{Colimit}(List_F, Count) \tag{18.28}$$

$$Count \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}) \tag{18.29}$$

Let us then codify the fushi sign strings in form

$$FushiSTRG \underset{Id}{\longrightarrow} \mathbf{Syn}(List_{FushiPic})$$

and then construct the fushi concept as a circular form as follows:

$$FSH \underset{Id}{\longrightarrow} Limit(FushiSTRG, FushiOrnament) \tag{18.30}$$

$$FushiOrnament \underset{Fin}{\longrightarrow} \mathbf{Power}(FSH) \tag{18.31}$$

with the usual finite set identifier. We therefore have a fushi structure which is essentially the same as the $MakroEvent$ structure discussed in 6.7. With this, we may finally complete the fushi structure associated with a syllable:

$$Fushi \underset{Id}{\longrightarrow} \mathbf{Limit}(Basic, Special, PitchChange) \tag{18.32}$$

$$Basic \underset{Id}{\longrightarrow} \mathbf{Syn}(FSH) \tag{18.33}$$

$$Special \underset{Id}{\longrightarrow} \mathbf{Syn}(FSH) \tag{18.34}$$

$$PitchChange \underset{Id}{\longrightarrow} \mathbf{Syn}(FSH) \tag{18.35}$$

where the three synonymous forms refer to the basic fushi (usually named *sugu, hiki, mawashi*), the special fushi (two types, *hashiri, yari*, the latter being a class of recursively

ornamented trill fushi), and thirdly: pitch-change fushis. Our construction terminates with the form of combination

$$JCKFU \underset{Id}{\longrightarrow} \mathbf{Limit}(JCK, Fushi)$$

which yields the units of the list form $List_{JCKFU}$ fo the full fushized Utai text, and we may pose

$$Utai \underset{Id}{\longrightarrow} \mathbf{Syn}(List_{JCKFU}) \tag{18.36}$$

whose denotators are the melodically articulated Utai texts. It is clear how to complete this construction in order to add all the philologically necessary information from the Noh tradition.

In this context, we may define what are the existing melodic Utai songs by a PV predicate $MelodicUtaiSequence$ which takes its values in some fuzzy (uncertainty of identification) $\mathbb{T}^{0_{\mathbb{Z}}}_{S_1}$ to be $\phi\widehat{\ }$ for $0 \leq \phi \leq 1$ in the same sense as with the above example on absolute tempo.

#### 18.3.3.3 Shifter Predicates

**Summary.** Organizing the non-lexical resources: the relevance of shifter predicates in production and reception of music and its science.

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Until now, atomic predicates were drawn from lexical data, be they of notational primavista nature related to scores or of mathematical nature. There is another major source for the creation of predicates: deictic criteria. For such a predicate the extension may be different, depending on its user. Whereas in language, lexical signs dominate and determine a fixed ontology, the basic means of semiosis in music (and to some degree also in musicology) are shifters: being and becoming a musical object may be a result of individual decisions. Once a predicate has been introduced, it may be used just like the others; however, its mode of coming into existence will remain important. In accordance with communicative coordinates, shifters are differentiated relative to poietic and esthesic perspectives. The neutral level is omitted here since it excludes the deictic dimension by definition.

*Poietic shifters* typically come up when the composer decides to specify a determined set of denotator instances which will become the objects of the composition. For example, the choice of a composition's motivic germ $MyMotif : A \rightsquigarrow NoteGroup(N_1, \ldots N_k)$ by a composer does not favor just any possible predicate loaded with the property of being a motif, i.e., verifying a predicate $MyMotif/Motif$; on the contrary, it is this particular motif that was chosen objectively by the composer and it is this choice that gives it a special position among the denotators of form $NoteGroup$ and among the predicates encompassed within the predicate expressed by $Motif$. The expression corresponding to this very individuum could be denoted by $ThisMotivicGerm$, an expression that is evaluated to $\top$ if and only if we deal with this composer's concrete and unique choice of the germ. The predicate $MyMotif/ThisMotivicGerm$ has exactly one affirmative answer and no possibility to be generated within a genuinely lexical environment.

The *esthesic counterpart* of creational shifters is the set of extensions considered by an analytical interest. Often, there is no lexical reason to consider a particular composition or a special chord, there is just scientific curiosity, or interest. In this context, a composition which we may identify by a simple character string denotator $S$ for the composition's name, is marked by !, i.e., we write $S!$ for the fact that there is a special interest in $S$.

Let us terminate this subject by two remarks and a principle statement:

- Comparison of different truth value domains is eased by the concept of a morphism of textual semioses which is induced by morphisms between corresponding truth value modules. So at least in principle we have a canonical technique for comparing truths as they are modeled in specific and adequate contexts.
- Truth in the topos-theoretic sense is not meant to model some formal languages but to add contents to the poor truth value of absolute logic in order to target towards a possible fusion of logic and beauty—after the marriage of logic and geometry initiated by topos theory.

**Principle 12** *After all, it would be wonderful if one day, we could say that something is true* because *it is beautiful. The feeling of such an insight has been around for a long time—let us finally face its substance.*

### 18.3.4 Logical and Geometric Motivation

**Summary.** Analysis of the motivation mechanism for textual predicates. Besides the classical logical "recombination" of predicates, music preconizes more geometric methods in order to produce new predicates from given ones. Motivation is the technical counterpart of productive navigation in the EncycloSpace.

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We announced that predicates are mainly generated by certain constructions from arbitrary basics such as mathematical, PV, and shifter predicates. Recall that semiotically, constructing the signification of signs by a determined mechanism from given significations is called motivation (see 2.3.4). So we now have to deal with motivation from the three named (arbitrary) basics.

One must, however, point out a major difference between predicative existence and mathematical fiction in the motivation process. The construction (motivation) of new denotators from given ones is a purely mathematical routine process: it simply has to fit in the generic mathematical framework of soundness. But with predicates, generating new denotators which "are the case" from already instantiated facticity is not only an abstract affair: these objects have to be the case for us, they have to be available, on a storage media, in a determined reservoir of shared information, in other words, predicates are the elements of knowledge in the sense of ordered access to information. So accessibility of the predicates' extensions is a serious requirement. Let us look at this condition in the different arbitrary basics.

Start with the arbitrary PV predicates. They mean music(ologic)al facticity. They have extensions somewhere, and this somewhere can be accessed without restrictions, it is not only as if it existed, it is within our reach. So the existing sonatas of Beethoven, the existing Noh

Utai, the existing jazz compositions of Duke Ellington are a knowledge base which is strictly antagonistic to possible, but never conceived music works.

**Principle 13** *If we are going to extend this level of facticity, we should meet the requirement that the knowledge ontology is conserved and not evaporated into fiction of whatever nature.*

The mathematical predicates are a bit more delicate. We contended above that these are introduced to indicate that there is a mathematical predication which is music(ologic)ally relevant. Nonetheless, such a predicate may have an infinite extent. So it is a fact of music but not necessarily a finite one. The point is that this differentiation from purely mathematical existence has to be combined with the requirement of principle 13. More precisely:

**Principle 14** *If a mathematical predicate is conceived, it must be set up to cope with the accessibility requirement of knowledge. This accessibility is guaranteed by an explicit declaration of its characteristic function, and not necessarily by the predicate's extension which could be infinite or simply too large for any human storage device. If such mathematical predicates are used to generate extensions, they must meet the specific storage conditions of the medium, be it a technological or a human one (memory).*

In the case of deictic predicates, the facticity is more a question of declaration discipline: We do not want to deal with uncertainty in the choice of predicate extension instances. Rather would it be important to say that you have to attribute to a determined denotator a fuzzy truth value instead of saying that you do not know whether you want to consider that denotator or not. So this means

**Principle 15** *Deictic predicates must be understood as the duty to declare facticity versus the freedom of leaving your decision unsettled. The latter is not part of the predicate system but of the psychology of the decision process in its making. This is not a proper part of the predicate theory which is a precise theory of fuzziness in music and not a properly fuzzy theory.*

Summarizing, we want to construct new predicates from already given ones in the sense of a consistent extension of knowledge ontology:

**Principle 16** *The explicit and unrestricted access to information must be declared on the germinal level of arbitrary predicates and conserved in each motivation process.*

After these preliminary reflections on knowledge propagation, we may proceed to the description of motivation mechanisms. Basically, two levels of construction can be distinguished: the logical and the geometric motivation.

*Logical* motivation means that new predicates are constructed from given ones by use of the *codomains* of characteristic function. In fact, the topos-theoretic truth domains $\mathbb{T}_I^A$ are provided with logical functionality of the associated Heyting algebras, such as conjunction, disjunction, implication, and negation. This type of logical operation extends to universal and existence quantifiers.

*Geometric* motivation acts on the *domain* or characteristic functions. This means that the topos-theoretic constructs such as fiber products, fiber coproducts, power objects and other

universal objects are used to build new denotator objects in $Den_\infty$ on which truth values are applied.

Of course, nothing can prevent us from applying these methods in a not so topos-theoretic manner or to introduce still other motivation methods. This is no problem, the scope of these delimitations is rather to put into evidence the richness of the purely topos-theoretic vein of motivation.

Before giving further comments on these motivation methods, we should present some easy typical examples.

**Logics: conjunction, disjunction, implication, and negation.** This type is completely straightforward. We suppose given two predicates (named) $P, Q$ which have their truth values in the same domain of truth denotators $\mathbb{T}_I^A$. Then the logical combinations $P \wedge Q$ evaluates as follows. Let $P(f) = p : A \rightsquigarrow TRUTH(I)(\pi)$ and $Q(f) = q : A \rightsquigarrow TRUTH(I)(\kappa)$ for an object $f$ of $Den_\infty$. Then we set

$$P \wedge Q(f) = p \wedge q : A \rightsquigarrow TRUTH(I)(\pi \wedge \kappa)$$

where $\pi \wedge \kappa$ is calculated in the Heyting algebra of the functor $A@Fun(TRUTH(I))$. The analogous construction works for the other logical operations. (If we deal with different truth denotator domains, we have to be provided with a common domain, for example by use of colimits such as fiber sums.)

Despite the evident procedure there is a delicate point in these construction. If we look for the extensions of predicates and suppose that the extensions of the given predicates are traced on some human or man-made data storage or memory devices, it is not clear whether the same may also be the case for the resulting logical combinations. For disjunction and conjunction this works since the extensions are—roughly speaking—unions or intersections of the given extensions. For negation, however, the resulting predicate may obtain an infinite, not controllable extension. In other words, negation of facticity is not necessarily the same type of facticity. In fact, what can be conserved in facticity is not the extension but the operation which takes a decision on every proposed tuple of denotator morphisms.

**Logics: Existence and universal quantifiers.** In the existence and universal quantifier situation, typically involving a predicate $P(x, y)$ with two free variables $x, y$, these two logical operators, $\exists_x P(x, y)$, and $\forall_x P(x, y)$, invoke a variable, $x$, say, which must be run through and checked for truth of the given predicate. In our situation, this check will not be a dichotomic one between true and false, but an 'integration' over all values of $P(x, y)$ as $x$ varies. The statement $\exists_x P(x, y)$ then reads as the maximal possible truth value of all $P(x, y)$ as $x$ varies, so if there is at least one truth value $\top$, then we assert that $\exists_x P(x, y)$ has this value. Analogously, the statement $\forall_x P(x, y)$ reads as the minimal possible value among all $P(x, y)$ as $x$ varies.

To put it more formally, suppose that a predicate $P$ of a textual semiotic evaluates on every $n$-tuple $f = (f_1, \ldots f_k)$ of denotators to a truth denotator $P(f) \in \mathbb{T}_I^A$. Fix an index $1 \leq i$ of tuples. Then the quantifiers are defined with respect to the denotator variable at this index. To this end, if $X \subset \mathbb{T}_I^A$, denote by $lub(X)$ its least upper bound whereas $glb(X)$ denotes the greatest lower bound of $X$ in the complete Heyting algebra

$\mathbb{T}^A_I$. Further, if $d \in Den, 1 \leq i \leq k+1$, and if $f = (f_1, \dots f_k) \in Den^k$ is a $k$-tuple, write $f_{>i}d = (f_1, \dots f_{i-1}, d, f_i, \dots f_k)$. Then the existence quantifier is the predicate $\exists_i P$ which at the $k$-tuple $f$ evaluates to

$$\exists_i P(f) = \begin{cases} P(f) & \text{if } k+1 < i, \\ lub(\{P(f_{>i}d) | d \in Den\}) \text{ if } 1 \leq i \leq k+1, \end{cases} \tag{18.37}$$

whereas the universal quantifier at the $k$-tuple $f$ evaluates to

$$\forall_i P(f) = \begin{cases} P(f) & \text{if } k+1 < i, \\ glb(\{P(f_{>i}d) | d \in Den\}) \text{ if } 1 \leq i \leq k+1. \end{cases} \tag{18.38}$$

Observe that the calculation of the *lub* and the *glb* needs only the set $supp(P)$. So if this one is so small that a computer or human memory can control it, the quantifier predicates can be calculated at a comparably small expense. If this support is not finite, or if it is too large to be stuck on a memory device, the range of the bound $i$th variable $d$ must be restricted to a controllable predicate or else the predicate $P$ must be joined to a predicate $Q$ such that $supp(P \wedge Q)$ becomes controllable by the available memory. In this situation, we write $\exists_{i,Q} P, \forall_{i,Q} P$.

We cannot solve the details of this problem of facticity here, but we feel obliged to insist on the relevance of its investigation because abstract mathematical reality is not sufficient for predicates to be handled by human technology and culture. So one should in any case point out the delicate turning points.

**Geometry: fiber products.** Geometric motivation implies predicate construction methods which are built on the domain $Den_\infty$ of characteristic functions. We want to illustrate this for fiber products on denotators, a universal construction which—modulo some circularity traps—is known to exist for all five typed denotator subcategories (see section 18.3.1).

We take two mathematical predicates having their support on ${}_0Den$ (see section 18.3.3.1):

$$\text{Chords } Ch/PianoChord, \tag{18.39}$$

$$\text{Melodic motifs } Mt/Motif, \tag{18.40}$$

and a PV predicate $OnBeetSon$ which takes value $\top \in \mathbb{T}^0_0$ exactly for each zero-addressed local composition $C$ consisting exactly of all onsets of the onset-bearing events of one of the 32 Beethoven sonatas (and $\bot$ else), so we are looking for the predicative support objects

$$C/OnBeetSon. \tag{18.41}$$

To relate these three predicates, we consider a mathematical predicate $OnsetProj$ supported on $\mathbb{T}^0_0$ with $\top$ by morphisms of local compositions $f : X \to Y$ such that $X$ has form $NoteGroup$, $Y$ has form $Onset$, and $f$ is the projection morphism retaining the note events' onsets.

To relate the different truth denotator domains, we have to reduce the motif domain $\mathbb{T}_I^A$ to $\mathbb{T}_0^0$, but this is straightforward by the unique projection $A \to 0$, what yields $\mathbb{T}_I^A \to \mathbb{T}_0^0$, and this is what we carry over from the truth values for motifs.

Finally, the geometrically motivated predicate $BeetMotChordFiber$ has its $\top$ support on octuples $f = (X_1, X_2, X_3, X_4, f_1, f_2, f_3, f_4)$ where the first four objects are veritable objects in $Den$ (identity morphisms), and the second four are morphisms of $Den$. The value $\top$ in $\mathbb{T}_0^0$ is taken on all octuples of *cartesian* diagrams (in $ObLoc$)

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & X_2 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_3} \\ X_3 & \xrightarrow{f_4} & X_4 \end{array} \tag{18.42}$$

such that

$$X_2/Motif, X_3/PianoChord, X_4/OnBeetSon, \tag{18.43}$$
$$f_3/OnsetProj, f_4/OnsetProj \tag{18.44}$$

are all true.

This construction can be decomposed as follows: The predicate $BeetMotChordFiber$ on octuples is built as a logical conjunction of several projection predicates. A projection predicate is a geometric motivation as follows: Given an increasing sequence of positive natural numbers $j_\cdot = j_1, \dots j_s$, denote by $\pi_{j_\cdot}$ the natural transformation $Den_\infty \to Den_\infty$ which on a $k$-tuple $f = (f_1, \dots f_k)$ evaluates to

$$\pi_{j_\cdot}(f) = (f_{j_1}, f_{j_2}, f_{j_s}). \tag{18.45}$$

Then the projection $j$th predicate $P.\pi_{j_\cdot}$ deduced from a given predicate $P$ is defined by

$$P.\pi_{j_\cdot}(f) = P(\pi_{j_\cdot}(f)).$$

This is an easy geometric motivation built upon this projection. We may then consider the mathematical predicate $Cart$ of cartesian diagrams which gives $\top$ on octuples $f = (X_1, X_2, X_3, X_4, f_1, f_2, f_3, f_4)$ if they yield a cartesian square 18.42. So our above predicate reads as a logical motivation from projection motivation via

$$\begin{aligned} BeetMotChordFiber(f) = \quad & Cart \wedge Motif.\pi_2 \wedge PianoChord.\pi_3 \\ & \wedge OnBeetSon.\pi_4 \wedge OnsetProj.\pi_7 \\ & \wedge OnsetProj.\pi_8(f) \end{aligned}$$

which means that we have built this predicate by a (logical) conjunctive motivation and a (geometric) projection motivation from mathematical and PV predicates. Strictly speaking, this second construction of $BeetMotChordFiber$ is a synonymous predicate: It should have another expression, but its content is the same as the first one's content; in fact, synonymy of predicates is a common feature as in every semiotic system.

Now, the predicate *BeetMotChordFiber* is not quite what we have in mind: We would like to speak about the fiber product as a local composition, and not about the whole bunch of auxiliary structures. So the final construction would be this:

$$\begin{aligned} BeetMotChordFiberObject(X) = \quad & \top \text{ iff there is an octuple } f \\ & \text{with } \pi_1(f) = X \\ & \text{and } BeetMotChordFiber(f) = \top. \end{aligned}$$

**Exercise 40** Try to unify the above existence condition with the logical motivation via the existence quantifier as two special cases of a more general logical existence construction. (Hint: use more general projections.)

So we finally get a predicate which selects local compositions (in fact chords) consisting of piano notes which share the onset of a note of one motif and of a chord within the onsets of one of Beethoven's sonatas.—But we should, once again, be aware that the verification of the cartesian predicate is a mathematical operation of complex nature. It involves mathematical existence and universal quantifiers for the verification of a universal property. The operationalization of such predicates on the basis of computer implementation can be quite tedious since it always must be solved on a constructivist basis.

## 18.4 Paratextuality

**Summary.** Beyond textual semiosis, a type of predicate semiosis can occur which is not limited to extensions over denotators.

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Besides textual semiosis, expressions of predicates can also (or exclusively) invoke historical and stylistic competence, for example, and thus point at exterior strata of signification. However, this does not enforce physical reality; it can still live on the mental level of signification. Classification of paratextual signification includes reproductive and productive behavior. Reproductive paratextual predicates include performance, lyrics, expressive, choreographic, and musicological aspects. Productive paratextual predicates can be of lexical nature, such as ornaments or improvisational patterns—or else of shifting character, such as creational and unintentional predicates. For a more systematic treatment of this subject, we have to refer to [361, 2.6].

It is however important to give a short account of the germ of paratextuality as it is preconized by the very nature of truth denotators in the textual semiosis. In fact, we have already learned that the classical logic of extensionality is canonically embedded in the truth denotator domains $\mathbb{T}_I^A$ which means that extensionality and truth evaluation is only a very 'poor' signification codomain. It is natural to view this situation as a special case of a more general validation operator $\omega$ process where a validation just gives us back a denotator for every given tuple, i.e., we have a signification process

$$\omega : Den_\infty \to Den$$

which attributes some values to tuples, values which share a proper extratextual semantics, such as ‘truth’ or ‘falsity’.

For example, such an operator is given for the evaluation of certain PV predicates, such as “Fermata”: An operator $\omega$ would turn this abstract denotator $myFermata$ (which essentially has the coordinates onset and duration) into a tempo curve $\omega(myFermata)$ which may be described by a parameter set for a spline, or by a string which represents a mathematical expression for that tempo curve, etc. This operator process does however not yet produce physical contents, but it gives an explicit representation of such objects in physical terms. So the generalized textuality which evaluates to not necessarily truth-oriented denotators yields a connotation stratification for deeper semantics beyond text of whatever abstraction.

This successive absorption of semantics by denotators is not only a methodological mechanism, it is already a tendency in music history where contents of musical signs have always been progressively split into connotational substrata which in turn have been absorbed by conceptualization and music theory [122].

# Chapter 19

# Topoi of Music

*Das ist wohl schon die Mathematik des "Neuen Zeitalters".*
Alexander Grothendieck [200] on "Geometrie der Töne" [340]

**Summary.** This chapter is a conceptual synthesis of the previous achievements. We show that the overall structure of the category *Glob* of global compositions carries a Grothendieck pretopology via finite covering families. It is well known that such a pretopology generates a Grothendieck topology $J$ and therefore a Lawvere–Tierny topology $j$ on the presheaf topos $Glob^{\mathbb{Q}}$. We discuss the associated instances, such as the subobject classifier sheaf $\Omega$, and the subtopos $\mathrm{Sh}(J, Glob)$ of sheaves.

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In this chapter we tacitly make use of the topos theory of Grothendieck topoi, refer to appendix G.4.

## 19.1 The Grothendieck Topology

**Summary.** This section introduces the Grothendieck topology on *Glob* via covering families. Musical motivation is given.

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The finiteness of common local and global compositions is not favorable to intuition from standard topology, but we have known the application of traditional methods from combinatorial topology and corresponding Čech cohomology for classification purposes of musical compositions in chapter 15. However, this is not the full potential of topological methodology: This one has been developed with great success by Grothendieck in algebraic geometry, and applied—among others—by Pierre Deligne in his celebrated solution of the Weil conjectures, see also [209].

The idea is now fully absorbed by topos theory in the context of sheaf construction from presheaf categories. Here, the yoga of topology boils down to the gluing catechism which gives

insights to compatibility of local data without having to deal with the original intuitive neighborhood connotations from Euclidean geometry. Therefore it is a natural idea to integrate the Grothendieck topos construction which generalizes the Čech canon for the analysis of musical composition categories, or even of denotator categories provided they admit fiber products, a mandatory condition for the definition of Grothendieck topologies via covering families (see appendix G.4). To begin with, we have the following theorem:

**Theorem 25** *The category Glob of global compositions is finitely complete.*

Proof. We have to show[1] that *Glob* has fiber products and a final object. The latter is already known from section 8.3.3 in the local case, but for global compositions, the same object does the job. The former claim is known for local compositions, and the fiber product of the underlying functors is known to exist. So the proof reduces to a control of the atlas index maps and the gluing of local fiber products. The fiber product of the index sets and their maps defines a system of subfunctors of the fiber product of the functors which underlie the global compositions. The union of these subfunctors is therefore covered by charts which are isomorphic to fiber products of local compositions and which clearly glue together as desired. QED.

**Exercise 41** Show that the fiber product of interpretable compositions is interpretable.

Despite this positive result, this is not the right perspective to introduce Grothendieck topologies. In fact, consider the fiber product of two charts $U_1, U_2$ in a global composition $G^I$. Since their index singleton injects into different indexes of the covering set, the fiber product $U_1 \times_G U_2$ is empty! The reason for this strange fact is that we had—and still have—serious musicological reasons for specifying the index map $\iota$ as well as the associated family $\alpha$ of address changes of a morphism $\overset{\iota}{f/\alpha}$. So this is a strong dichotomy between musicological setting of global compositions and the traditional setting with mathematical manifolds, where the atlas is only an existence and not a uniqueness requirement. If we stick to the musicological categories, such as *Loc*, *Glob*, we cannot apply those powerful tools of Grothendieck pretopologies as we would like to do: even the intersection of two charts would be intractable qua fiber product. So the approach for this conceptual environment must be slightly weakened: it is reasonable to forget about the index and address change maps and to retain only the transformation $f$ between sets or functors, respectively. In other words, we consider the categories ${}_\mu Loc$, ${}_\mu Glob$, ${}_\mu ObLoc$, ${}_\mu ObGlob$ as follows ($\mu$ for "mathematical"). The objects are those of the old categories *Loc*, *Glob*, *ObLoc*, *ObGlob*. The morphisms are just the maps of the underlying functors (or sets in the objective case). So we have, by definition, surjective "forgetful" functors

$$\mu : \quad Glob \to {}_\mu Glob \tag{19.1}$$

$$\mu : \quad Loc \to {}_\mu Loc \tag{19.2}$$

$$\mu : \quad ObGlob \to {}_\mu ObGlob \tag{19.3}$$

$$\mu : \quad ObLoc \to {}_\mu ObLoc \tag{19.4}$$

(and the other known subcategories) which are all defined by the action $\mu(\overset{\iota}{f/\alpha} : G^I \to H^J) = (G^I, H^J, f : G \to H)$ which we shall also write as $f : G^I \to H^J$, and which are all the

[1] See appendix G, proposition 99.

identity on the objects. We shall denote the image of $Hom(G^I, H^J)$ by $G^I @ H^J$, the set of mathematical morphisms, as already introduced in definition 50 of section 14.4, to make the reference to the manifold mathematics evident. So intuitively, a mathematical morphism is just a morphism of compositions where we forget about the underlying chart index and address change information—it is just known that they are there, as in manifold theory.

If we restrict to the subcategories $Loc_A$ over a fixed address and without address changes, then evidently, we have an isomorphism $\mu : Loc_A \xrightarrow{\sim} {}_\mu Loc_A$, whereas the restriction to the global compositions $GlCo_A$ with fixed address $A$ gives us as a fiber of a morphism $f$ all index maps which give rise to $f$. More precisely:

**Proposition 22** *Let $f \overset{\iota}{/} \alpha : G^I \to H^J$ be a morphism of global composition. Let $\kappa : I \to J$ any set map such that the $H$-chart $V_{\kappa(i)}$ contains $im(U_i)$, for every $i \in I$. Then $f \overset{\kappa}{/} \alpha$ is a morphism.*

Proof. If $im(U_i) \subset V_{\kappa(i)} \cap V_{\iota(i)}$, then we may apply the transition isomorphism between the intersection of the two charts $V_{\kappa(i)}, V_{\iota(i)}$ to induce a morphism of local compositions $U_i \to V_{\kappa(i)}$ with the same address change. QED.

Observe that the general transformation $\mu$ does not reflect isomorphisms. In fact, if $G^I \to G$ is a refinement map for a local composition $G$, as discussed in example 23, section 14.2, then the identity $Id_G : G^I \to G$ of the underlying functor $G$ is an isomorphism in ${}_\mu Glob$, but generically the objects are evidently not isomorphic qua global compositions if $card(I) > 1$.

Before introducing a Grothendieck topology on ${}_\mu Glob$, we have to guarantee the existence of fiber products. Here is the theorem:

**Theorem 26** *The category ${}_\mu Glob$ of global compositions and mathematical morphisms is finitely complete.*

Proof. Evidently, the final object of $Glob$ does also the job for ${}_\mu Glob$. So we are left with the proof that ${}_\mu Glob$ has fiber products. We are given a pair of mathematical morphisms $f : G^I \to H^J, g : L^K \to H^J$. As these are natural transformations we may consider the fiber product $G \times_H L$ of functors with its canonical projections to $G, L$. This functor can be covered by the non-empty fiber products $P_{i,k} = G_i \times_H L_k$ of the chart subfunctors $G_i, L_k$ in $I \times K$. Let $(G_i, L_k) \in I \times K$ be such that $G_i \times_H L_k$ is not empty. Take $J$-charts $H_{\iota(i)}, H_{\kappa(j)}$ such that $f$ on $G_i$ factorizes through $H_{\iota(i)}$, and such that $g$ on $L_k$ factorizes through $H_{\kappa(k)}$. Then evidently, the image of $P_{i,k}$ in $H$ is contained in $H_{i,k} = H_{\iota(i)} \cap H_{\kappa(j)}$, and we have $G_i \times_H L_k \xrightarrow{\sim} G'_i \times_{H_{i,k}} L'_k$, where $G'_i, L'_k$ are the inverse images of the intersection $H_{i,k}$ in $G_i, L_k$, respectively. By the existence of fiber products of local compositions, these fiber products are local compositions, and we obtain an evident atlas together with two mathematical morphisms of the fiber product functor (which is a global composition) into the factors. The details are left as an exercise for the reader. QED.

To make the ideas precise, the present approach to Grothendieck topology will be focused[2] on the category ${}_\mu Glob$. We consider the following system of covering families: For a global

[2]Generalizations to other denotator categories are possible but we do not have concrete concepts and leave this subject to future developments.

composition[3] $G$, the covering basis $K(G)$ consists of all surjective finite families $(f_i : G_i \to G)_i$, i.e., their functorial images generate the functor $G$. Clearly, isomorphisms are such families, fiber products of such families are such families, and the composition of such families are again finite and surjective. Therefore we have a basis (originally called a pretopology) for a Grothendieck topology. Call the Grothendieck topology $J$ of these covering families the *finite cover topology* on ${}_{\mu}Glob$.

**Example 37** The interpretation $G^I$ of a local composition $G$ by a finite covering $I$ of subcompositions is in $K(G)$.

**Example 38** The family of local subcompositions of a defining finite atlas of global composition is a covering family.

**Example 39** More generally, if we add all finite intersection compositions as considered in the theory of affine functions on the compositions nerve, we obtain another covering family.

So musicologically, the finite cover topology is the "least common denominator" of all our previous local-global constructs. In other words, all hitherto known concepts and techniques for gluing together finite assemblies of local data in music and music theory are special cases of the overall approach by the finite cover topology. This also applies to the Čech cohomology of the nerve which we have shortly discussed in section 16.1.2.

**Remark 4** The finite cover topology is not subcanonical (this would mean that every representable presheaf $\textcircled{\mu}X$ over ${}_{\mu}Glob$ is a sheaf). The point is that the exactness condition for sheaves only yields a natural transformation from a family of compatible mathematical morphisms, but not (in general) a mathematical morphism. Possibly, a narrower definition of covering families would yield subcanonical topologies.

**Example 40** Suppose we are given a predicate $P$ with values in $\mathbb{T}_I^A$ and support on the local compositions, to fix the ideas. Consider this data as a function $t$ on $Loc$. We would like to be able to extend this function to the global compositions. To this end, suppose that this predicate is invariant under isomorphisms of local compositions. As the truth denotator domain is a complete Heyting algebra, we may attribute to a global composition $G$ with charts $G_i$ the truth values $t(G) = sup_i(t(G_i))$ or else $t(G) = inf_i(t(G_i))$ which are well defined by our invariance assumption and do extend the values on local compositions.

### 19.1.1 Cohomology

**Summary.** We sketch the topos-theoretic Čech cohomology associated with the finite cover topology $J$ on ${}_{\mu}Glob$.

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[3] Pay attention to the different morphism set of mathematical morphisms whereas the objects are unchanged, even if we often omit the covering atlas to ease notation!

Following the ideas of Verdier's exposition [22, exposé V], every covering family $f. = (f_i : G_i \to G)_{i\in I}$ induces the following simplicial diagram. Let $n(f.)$ be the simplicial complex of all subsets $S \subset I$ with non-empty fiber products $\prod_G^{i\in S} G_i$. For a natural number $0 \leq k$, denote by $S(f.)_k$ the set of singular $k$-simplexes, i.e., maps $s : \Delta_k \to I$ on the abstract standard $k$-simplex $\Delta_k = \{0, 1, 2, \ldots k\}$ such that $im(s) \in n(f.)$. For such a singular $k$-simplex $s$, we write $\prod_s(f.) = \prod_G^{i=0,\ldots k} G_{s(i)}$ and $\Sigma_k(f.) = \coprod_{s\in S(f.)_k} \prod_s(f.)$.

Every set map $q : \Delta_k \to \Delta_l$ induces a universal map

$$\Sigma(q) : \Sigma_l(f.) \to \Sigma_k(f.) \tag{19.5}$$

which on a cofactor $\prod_s(f.)$ of $\Sigma_l(f.)$ at singular $l$-simplex $s$ is the arrow

$$\Sigma(q) : \prod_s(f.) \xrightarrow{x} \prod_{s.q}(f.) \longrightarrow \Sigma_k(f.)$$

where $x$ is the universal arrow defined by the respective projections. In particular, we have the arrows $\sigma_i = \Sigma(q_i)$ with $q_i : \Delta_k \to \Delta_{k+1}$ the omission of the $i$th entry for $0 \leq i \leq k+1$, i.e., $q_i(j) = j$ if $j < i$, and $q_i(j) = j+1$ else. Finally, we also consider the augmentation map $\sigma_0 : \Sigma_0(f.) \to G$ given by the coproduct of the original family $f.$ of morphisms into $G$.

Summarizing, we get the simplicial diagram $\Sigma(f.) =$

$$\ldots \quad \begin{array}{c} \xrightarrow{\sigma_0} \\ \xrightarrow{\sigma_1} \\ \xrightarrow{\sigma_2} \\ \xrightarrow{\sigma_3} \end{array} \quad \Sigma_2(f.) \quad \begin{array}{c} \xrightarrow{\sigma_0} \\ \xrightarrow{\sigma_1} \\ \xrightarrow{\sigma_2} \end{array} \quad \Sigma_1(f.) \quad \begin{array}{c} \xrightarrow{\sigma_0} \\ \xrightarrow{\sigma_1} \end{array} \quad \Sigma_0(f.) \xrightarrow{\sigma_0} G$$

of global compositions which is the basis of the Čech cohomology if we succeed in applying contravariant functors with values in abelian groups to this diagram.

Suppose that we are given such a functor $F$, defined in a category which comprises the objects and arrows of the simplicial diagram $\Sigma(f.)$. Then we have the cochain complex $C^\star(f., F) =$

$$\ldots \xleftarrow{d_2} F(\Sigma_2(f.)) \xleftarrow{d_1} F(\Sigma_1(f.)) \xleftarrow{d_0} F(\Sigma_0(f.)) \xleftarrow{d_{-1}} 0 \tag{19.6}$$

with the alternate sums

$$d_k = \Sigma_{i=0,\ldots k+1}(-1)^i F(\sigma_i)$$

of abelian group homomorphisms for indexes $0 \leq k$ and $d_{-1} = 0$. This cochain complex has its usual cohomology groups $H^k(f., F) = Ker(d_k)/Im(d_{k-1})$ for $0 \leq k$.

**Example 41** *Affine functions.* In section 15.1.1, we have dealt with the category $ComGlob_A$ of commutative global compositions where the address $A$ is a module over a commutative ring $R$. We now want to look at affine functions on functorial global compositions. More precisely, we work in the following category ${}^R Glob$: instead of $\mathbf{Mod}^{@}$, we work in the category $\mathbf{Mod}_R^{@}$ of presheaves over $R$-modules over the commutative ring $R$. And ${}^R Glob$ is the category of global compositions over this context. Clearly, ${}^R Glob$ is finitely complete, and so are ${}^R Glob_A$, ${}^R_\mu Glob$, and ${}^R_\mu Glob_A$.

To define affine functions on a global composition $G^I$ in ${}^R_\mu Glob_A$, consider the objective composition $\hat{A} = \widehat{A@_R R}$. We claim that the set $Hom_A(G^I, \hat{A})$ of morphisms which leave the address fixed, is canonically provided with an $R$-module structure. For any commutative local composition $K \subset A@_R M$ with ambient module $M$, which is an $R$-module, i.e., $K = R.K$, each $f$-slice $f@_R\hat{K}$ for $f : B \to A$ is an $R$-module when identified with its second factor $K.f$. Now, given two morphisms $u/Id_A, v/Id_A : G^I \to \hat{K}$, we can add them or apply a scalar multiplication when restricted to each $f$-slice:$f@_R G^I \to f@_R\hat{K}$. This defines an $R$-module structure on $Hom_A(G^I, \hat{K})$. In particular, for $M = R$ and $K = A@_R R$, we get the required module structure. We call the module $Hom_A(G^I, \hat{A})$ the *module of global affine functions on* $G^I$ and denote it by $\Gamma(G^I)$.

We therefore have a representable contravariant functor (15.6)

$$\Gamma : {}^R Glob_A \to {}_R\mathbf{Mod}$$

which associates the $R$-module $\Gamma(G^I)$ of global affine functions with a given global composition $G^I$. It is easily seen that this functor is well defined on the mathematical category ${}^R_\mu Glob_A$ and gives rise to a synonymous functor

$$\Gamma : {}^R_\mu Glob_A \to {}_R\mathbf{Mod}$$

entailing cohomology $R$-modules $H^i(f., \Gamma)$ for any covering family $f.$ in ${}_\mu ObLoc_A$.

- For the default covering family $\iota.$ of charts which define $G^I$, this gives back the cohomology modules $H^i(\iota., \Gamma) = H^i(\Gamma)$ discussed in 16.1.2.
- Consider next the (mathematical) resolution morphism (15.16)

  $$res_{G^I} : \Delta_{G^I} \to G^I$$

  and its restriction

  $$res_i : {}_A\Delta_i \rightarrowtail \Delta_{G^I} \to G^I \tag{19.7}$$

  to the resolution's standard composition chart ${}_A\Delta_i$ for chart $i \in I$ of $G^I$. This family $res.$ is covering for $G^I$ and we therefore obtain the *resolution cohomology modules* $H^i_\Delta(G^I) = H^i(res., \Gamma)$ for the affine function functor.

  Clearly, we have an epimorphism of simplicial diagrams

  $$\Sigma_{res} : \Sigma(res.) \twoheadrightarrow \Sigma(\iota.) \tag{19.8}$$

  which is induced by the canonical morphisms

  $${}_A\Delta_{i_1} \times_G \ldots \times_G {}_A\Delta_{i_k} \twoheadrightarrow U_{i_1} \cap \ldots \cap U_{i_k}$$

  for the $G^I$-charts $U_{i_j}$. Therefore, the corresponding function complexes are related by an exact sequence

  $$0 \longrightarrow C^\star(\iota., \Gamma) \longrightarrow C^\star(res., \Gamma) \longrightarrow C^\star(res., \Gamma)/C^\star(\iota., \Gamma) \longrightarrow 0 \tag{19.9}$$

of differential $R$-module complex homomorphisms. So by the standard procedures for exact sequences of differential complexes, we have a long exact cohomology sequence

$$\ldots \longrightarrow H^i(G^I) \longrightarrow H^i_\Delta(G^I) \longrightarrow H^i(\Delta/G^I) \xrightarrow{\delta} H^{i+1}(G^I) \longrightarrow \ldots$$

with the cohomology modules $H^i(\Delta/G^I)$ defined by the quotient differential complex $C^\star(res.,\Gamma)/C^\star(\iota.,\Gamma)$.

**Example 42** *Linear forms.* For this example, we work over a fixed $R$-module $A$ as address, $R$ being a commutative ring. Recall from the discussion of module complexes in section 15.1 that for a commutative (objective) global composition $G^I$ in $ComGlob_A$ and an $R$-module $V$ (in particular the already known module $A@_R R$ which yields the classical forms), we have the module complex ${}_VG^I = Hom_R(G^I, V)$ of $V$-forms on $G^I$. We want to make this complex into a contravariant functor ${}_\mu Glob_A \to \mathbf{Mod}_R$. On the category $ComLoc_A$, it is the well-known dual functor to the associated module functor $R : ComLoc_A \to \mathbf{Mod}_R$. It sends a local composition $K \subset A@M$ to the module ${}_VK = Hom_R(R.K, V)$. For each global composition $G^I$ in $ComGlob_A$, select an atlas $(K_i \subset A@M_i)_{i\in I}$ and restrictions $(K_\sigma \subset A@M_{i(\sigma)})_{\sigma\in n(G)}$ where $i(\sigma)$ is a selected vertex index of the simplex $\sigma$. When taking the limit $\Gamma({}_VG^I) = lim_{\sigma\in n(G^I)}\, {}_VK_\sigma$ for global sections, this is clearly already the limit on the double index restrictions. So a global section $x \in \Gamma({}_VG^I)$ is just a family $(x_i)_I$, $x_i : R.K_i \to V$ such that the restrictions $x_i|R.K_j$ and $x_j|R.K_i$ coincide.

If we are given a morphism $f\overset{\iota}{/}Id_A : G^I \to H^J$, and if we select an atlas $(L_j \subset A@N_j)_{j\in J}$, the global sections are mapped into each other by the universal property of limits via

$$\Gamma({}_VH^J) \to {}_VL_{\iota(i)} \to {}_VK_i$$

for every $i \in I$. Moreover, this linear map is invariant under a change of the index map $\iota$. In fact, if we have a second index map $\kappa : I \to J$, the map

$${}_V\Gamma(H^J) \to {}_VL_{\kappa(i)} \to {}_VK_i$$

factorizes through the intersection of charts:

$${}_V\Gamma(H^J) \to {}_VL_{\iota(i),\kappa(i)} \to {}_VK_i$$

and we are done. Moreover,the functor ${}_V\Gamma(G^I)$ is also natural in the module $V$, i.e., any module homomorphism $u : V \to W$ entails the evident natural homomorphism ${}_u\Gamma : {}_V\Gamma(G^I) \to {}_W\Gamma(G^I)$. Therefore

**Proposition 23** *Given an $R$-module $V$, we have a contravariant functor of $V$-forms*

$${}_V\Gamma : {}_\mu Glob_A \to \mathbf{Mod}_R \tag{19.10}$$

*which associates the $R$-module ${}_V\Gamma(G^I) = \Gamma({}_VG^I)$ of global sections of $V$-valued linear forms to each global composition $G^I$ in the mathematical category over the fixed address $A$. The functor ${}_V\Gamma$ is natural in $V$.*

In particular, for the integers as ground ring, and for any covering family $f.$ of a global composition $G^I$ which is addressed in a $\mathbb{Z}$-module $A$, if we look at the projective $\mathbb{Z}$-module system $\mathbb{Z}/\mathbb{Z}_{l^r}$ and its projective limit, the $l$-adic integers[4] $\mathbb{Z}_l$, we first have cohomology groups

[4]With the usual unusual confusing index notation...

$H^\star(f., {}_{\mathbb{Z}/\mathbb{Z}_{l^r}}\Gamma)$ and then their limit, the *l-adic form cohomology*

$$H_l^\star(G^I) = lim_r H^\star(f., {}_{\mathbb{Z}/\mathbb{Z}_{l^r}}\Gamma) \tag{19.11}$$

on $G^I$.

## 19.1.2 Marginalia on Presheaves

**Summary.** We briefly discuss function presheaves, and the subobject classifier.

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This section is a sketch of what should be investigated more carefully and systematically.

### 19.1.2.1 Function Presheaves

**Summary.** Function sheaves are a basic instance for global music theory, we review the corresponding discussion of the resolution covering.

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The resolution covering is a remarkable example for the finite cover topology since it is not just a covering by subcompositions, it is somewhat analogous to étale topology, the set-theoretic map is even bijective, only the relative position of points has changed. We feel that this presheaf type is crucial for understanding music. Let us make this more explicit—even beyond the resolution covering.

We recall that the resolution was a procedure to lift non-interpretable global compositions to interpretable ones built on point configurations in general position. Now, this approach may be interpreted from still another angle. Whereas classification uses resolutions to calculate the 'minimal perspective of a global composition' which tells you everything of its structural contents, it could also happen that a composer does not want to classify a composition but to view it as a germ of a real composition, a concentrated abstract version of something which will be used only on a different level. So the germinal composition could be a global composition $G^I$ which will only be used via some covering morphism family $f_i : G_i^{I_i} \to G^I$ which the composer uses to grasp particular local aspects of the germ. From this construction, she could also deduce function modules to realize her music in determined parameter spaces. A study of such poietic aspects and applications of general covering families of the finite cover topology is still outstanding, but surely will furnish powerful compositional tools.

### 19.1.2.2 The Subobject Classifier Ω

**Summary.** The subobject classifier is a key object in a topos. We have a closer look at its structure and function in the topoi ${}_\mu Glob^{\mathbb{Q}}$ and Sh($J$, ${}_\mu Glob$).

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Recall the general relation between presheaves and sheaves. Given the finite cover topology $J$ on ${}_{\mu}Glob^{@}$, we have the subcategory $\mathrm{Sh}(J, {}_{\mu}Glob)$ of $J$-sheaves and its embedding $i : \mathrm{Sh}(J, {}_{\mu}Glob) \rightarrowtail {}_{\mu}Glob^{@}$. Whereas the subobject classifier $\Omega$ of ${}_{\mu}Glob^{@}$ gives the set

$$\Omega(G^I) = \{ \text{ sieves } S \subset @G^I \}$$

of $G^I$-sieves, the subobject classifier $\Omega_{Sh}$ of $\mathrm{Sh}(J, {}_{\mu}Glob)$ evaluates to the subset

$$\Omega_{Sh}(G^I) = \{\text{closed sieves } S \subset @G^I\}$$

of closed $G^I$-sieves, i.e., the sieves $S$ such that any (mathematical) morphisms $f : H^T \to G^I$ covering[5] $S$ are contained in $S$. In terms of covering families this means that whenever there is a covering family $f_i : H_i^{T_i} \to H^T$ such that every composite morphism $f.f_i \in S$, then $f \in S$. From a predicative point of view, a closed sieve $S \subset @G^I$ corresponds to a predicate $P_S$ whose support is $S$, i.e., $P_S(f) = \top$ iff $f \in S$, and $P_S(f) = \bot$ else. The predicate $P_S$ is characterized by

- $P_S(f)$ is invariant among mathematically equivalent arrows,
- invariance under right multiplication, i.e., if $P_S(f) = \top$, then $P_S(f.g) = P_S(f)$ whenever the product makes sense;
- $P_S(f) = \top$ if there exists a covering family $f.$ such that $P_S(f.f_i) = \top$.

There are different approaches to view the sheaves within the presheaves. First, we have the adjointness property: There is a left adjoint $a : {}_{\mu}Glob^{@} \to \mathrm{Sh}(J, {}_{\mu}Glob)$ for the embedding $i$, called the associated sheaf functor, see appendix G.4.1: For any presheaf X and sheaf Y, we have a functorial isomorphism $Hom_{{}_{\mu}Glob^{@}}(X, i(Y)) \xrightarrow{\sim} Hom_{\mathrm{Sh}(J, {}_{\mu}Glob)}(a(X), Y)$. It resides on the sheafification operator $P \mapsto P^+$, see appendix G.4.1 which yields the associated sheaf $\mathbf{a}P = (P^+)^+$. To this property corresponds the construction of the Lawvere–Tierny topology $j : \Omega \to \Omega$ which associates with every sieve $S \subset @G^I$ its closure $\overline{S}$ consisting of all morphisms which cover $S$. In terms of predicates, this amounts to adding to a predicate all morphisms which fulfill the third property above.

## 19.2 The Topos of Music: An Overview

**Summary.** This section summarizes the overall categorization of music with regard to the topos-oriented structures worked out from denotators to Grothendieck topologies.

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The overall theory of forms and denotators is a concept framework based on the topos of presheaves over the category **Mod** of modules and of selected subcategories. Special types of denotators: the local compositions (power type) are naturally provided with a local-global paradigm. In fact, they are essentially subobjects of special functors, together with induced morphisms, quite similarly to elementary approaches to affine algebraic varieties and to local

[5] i.e., such that $f^{\star}S$ covers $J(H^T)$.

Figure 19.1: Alexander Grothendieck.

differential geometry. This part of the denotator theory—which stresses rather the conceptual construction than the geometric manifold approach—is embedded in the global theory which can be seen as a passage to inductive limits, i.e., 'patchworks' of local objects. This latter construction is essentially different from the algebraic and differential geometry in that the atlas covering is not variable but an intrinsic data from the musicological point of view. The inductive limit over variable atlases is not the point for musicology, it would be a too coarse approach.

However, the patchwork of local structures can be boiled down to a more mathematical manifold theory: the categories of global compositions, together with mathematical morphisms. In this framework, it is possible to introduce natural Grothendieck topology, for example the finite cover topology considered above. This one is also the Grothendieck topology which englobes all the local global approaches which we have performed in interpretation and classification theory.

Undoubtedly, larger denotator subcategories should be accessible by either globalization processes or—at least—by Grothendieck topologies which extend the one sketched above. This is a priori feasible since we have seen that fiber products are also given for a large class of denotators as soon as they are not pathologically circular.

The cohomology theory for Grothendieck topoi, as sketched by Verdier in the legendary SGA seminar, is naturally given via function presheaves, for example. We do not know whether the predicate sheaves can also reveal cohomological spin-offs since it is not evident how truth denotators should be given structures of abelian groups. They rather live in Heyting algebras, and one should first adapt these structures to become candidates for cohomology theory. However, if truth denotators are just singletons in truth modules, they evidently add up to groups. But this perspective is neither within the reach of our experience with concrete examples nor

do we have theoretical results which would enforce such a development.

Summarizing, the topos of music first centers around the concept architecture of music objects in the general denotator theory, and therein around the presheaf topos over modules, and secondly, it evolves to a universe of local-global perspectives which are readily described by Alexander Grothendieck (see figure 19.1) with his awe inspiring topologies and their functors and cohomology theories.

The following development of this book will now descend to more concrete investigations, regarding those fields which are more in the tradition of the composers, music theorists, and performers. This does not mean that we can forget about the general framework, on the contrary, it will often not be possible to understand the technical and conceptual procedures without knowing about the topoi which back the more down-to-earth scenery.

# Chapter 20

# Visualization Principles

*For they shall see eye to eye, when the Lord shall bring again Zion.*
ISAIAH 8

**Summary.** As a compensation to the abstract nature of general topos theory, some principles for visualizing such abstract objects are mandatory, in particular in view of implementations on computerized knowledge bases. We give an account of such principles as they are being applied in graphical interface design.

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## 20.1 Problems

Visual navigation on general databases is a difficult task for three reasons:

- The data structure is not a priori in a geometric shape.
- The geometric shape, if it occurs, is not a priori adapted to human 3D vision.
- Objects may be composed of other objects which in turn are composed, and so on in a recursive way. Visualization then should take care of a recursive architecture.

The first obstacle may happen if we are given a bunch of textual objects. How should we arrange them as if they were points in a geometric space? Or else, if we are given a collection of chords in a score, is it possible to deal with such objects as if they were nicely distributed as points or spheres or whatever geometric objects in an adequate geometric space?

The second obstacle may occur when we have a high-dimensional geometry, for example six-dimensional representation of tones (onset, pitch, loudness, duration, glissando, crescendo). How could we grasp all these dimensions without forgetting some of them by plain projection to 2D or 3D partial representations, such as 2D for onset and pitch?

The third obstacle is a fundamental design problem. If we are able to visualize 2D situations, for onset and pitch, say, this only works if the visualization of these coordinates is

already settled. But in general, the coordinate references are not elementary decimal or integer numbers, so how is it possible to set up a recursively stable visualization strategy?

Apart from this agenda, visualization has to make knowledge accessible, i.e., visualization must provide a distinguished knowledge navigation tool. We have already discussed knowledge navigation in chapter 5. From that discussion, one could draw the following condensed definition of knowledge:

**Definition 63** *Knowledge is ordered access to information.*

Accordingly, navigation is intimately related to knowledge. In fact, knowledge involves the access activity. And it obeys ordering principles, this access is not a random walk as it is the case in present internet surfing environments. So the instantiation of knowledge involves navigation on ordered spaces, not just blind date behavior. For example, in a traditional encyclopedia, the retrieval of a keyword starts with a linear movement along the alphabetic order axis, but then, when one delves into the text, presents a completely chaotic navigation environment: Only some rare linkage arrows tell us where to navigate, the rest must be constructed by semantic lecture, interpretation and then rebuilding of new keywords to deepen the searched concept's understanding. The textual abstraction is produced by an extreme knowledge hiding, whose disclosure requires considerable efforts in supplementary knowledge navigation. Let us put this insight as follows:

**Principle 17** *Visual navigation must be built on orderings*[1].

We may now recall the treatment of denotator orderings in section 6.1.3 and—more technically—in section 6.8. We have shown there that for quite general addresses, there is a generic recursion process to define linear orderings on denotator systems of any form space. Let us rephrase those results in an informal way:

**Fact 9** *Any system of denotators can be linearly ordered according to a generic recursive procedure which is based on linear orderings of address modules. These module orderings can be constructed from canonical orderings of elementary modules, such as ground rings, and basic quotient modules, in a natural way.*

This fact lends itself to visualization: To begin with, in all relevant cases, we only deal with finite, and mostly comprehensible, collections of denotators, of size, say, below $10^6$. This means that we can deal with a number of points on the real line. But this is not sufficient, since the abstract order relation is not ideal for representing the underlying 'distances'. If, for example, a number of pitches is represented by pure order relations, this reduces to an equidistant arrangement of points of the real axis and destroys the pitch distances. Of course, it is not wrong to recur to this equidistributed arrangement which we may call the *generic linear visualization*. One must be able to invoke this one if no further information is required. But we have to ask for this:

[1] Observe that visualization means using the ordered field $\mathbb{R}$, and this means writing—typically—decimal numbers, which in turn amounts to building on the ordering of the natural numbers $0 < 1 < 2 < \ldots 9$.

**Principle 18** *A generic construction method for natural distances among successive items in the generic visualization must be developed to generate intuitive visual distance perception.*

Call the visual representation on the basis of principle 18 the *metrical linear visualization.* The ongoing implementation of denotator visualization[2] has already realized this principle on the modules which occur in 'real life'.

The next problem to be settled is to deal with high-dimensionality, more precisely: more than three dimensions in product type denotators. Now, the principle of metrical linear visualization basically solves this problem, but it is too flat, we are able to visualize in 3D and should make use of this human feature. So we have to respect the following *packing principle*:

**Principle 19** *In a product type space, the linear visualization should be distributed among a number of packages of coordinate spaces such that at most three spatial visualization axes are needed.*

So, for example, we should pack three real dimensions in the linear visualization of a first axis, another four dimensions in the linear visualization of a second axis. The remaining dimensions should be packed into some different visualization level. Even if we apply the linear visualization principle to these latter dimensions, we cannot, by construction, add more spatial visualization axes. We therefore propose the following *object visualization principle*:

**Principle 20** *In the product (more generally speaking: limit) type, the non-spatial—possibly linearly packed groups of—linear dimensions must be attributed to object parameters of multimedia objects.*

Such objects can be specified by geometric shapes, such as spheres, cubes, faces, by surface texture, by sonic parameters such as sound, earcons[3], and by gestural parameters, such as handling behavior, like drag and drop response, of more or less physical nature (elasticity, inertia, gravity, magnetism).

Summarizing, we obtain a distribution of high-dimensional denotator visualization among space and object visualization. With this in mind, how does navigation look? You may specify the space/object distribution and associated linear (metrical) visualization according to the previous principles, including selection of adequate multimedia objects. You may move around within such visual spaces, zoom in and out, and interact with the multimedia objects, like listening to tones, feel the response of a piano key, etc.

But other navigation methods are at our reach, let us just name some evident variants:

1. Pick a specific selection of factors or cofactors in products or coproducts; this is the classical "coordinate projection" or "book selection" method.

2. Greeking[4] of deeper recursive information. This means that in the recursive denotator space ramification, one decides to collapse all values and to forget about existing differences. This is very important for first approaches to complex databases.

[2] PhD Work of Stefan Göller at Zürich University.

[3] Earcons are icons for the ears.

[4] An operation known from text applications, meaning that a structure is blurred, recalling the English idiom "that sounds Greek to me".

3. Differentiating between more or less generic visualizations. In a coproduct, one may concentrate on the 3D visualization of one cofactor, leaving the adjacent and more distant cofactors in a decreasingly detailed state of visualization.

## 20.2 Folding Dimensions

According to the *packing principle* (Principle 19) we need a method to fold dimensions. Taking principles 17 and 18 into account, this method needs to fulfill the following properties:

- order preserving
- injective

Fair enough, it is not *homeomorphic*, as it can't be *injective* **and** *homeomorphic*. Obviously there are many ways to define such a mapping; we choose the following generally valid algorithm. Denote by $p_i : \mathbb{R}^n \to \mathbb{R}$ the $i^{th}$ projection, and by $p_{i\ldots i+k} : \mathbb{R}^n \to \mathbb{R}^{k+1}$ the projection onto the coordinates $i$ through $i+k$.

### 20.2.1 $\mathbb{R}^2 \to \mathbb{R}$

We are given a set $\mathcal{P}$ of points in $\mathbb{R}^2$.

1. Partition $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \ldots \cup \mathcal{P}_{n-1}$ with $\forall i \in \{0, 1, \ldots n-1\}\ \forall x \in \mathcal{P}_i : p_1(x) = t_i = const.$, and $t_i \neq t_j$ if $i \neq j$.
2. Subdivide the x-axis by giving every $\mathcal{P}_i$ a lower and upper bound $u_i$ and $o_i$:

$$u_i = \begin{cases} 2t_i - o_i & : \quad (i = 0) \wedge (n > 1), \\ \frac{t_i + t_{i-1}}{2} & : \quad (i \geq 1) \wedge (n > 1), \\ t_i - \frac{1}{2} & : \quad else; \end{cases}$$

$$o_i = \begin{cases} 2t_i - u_i & : \quad (i = n-1) \wedge (n > 1), \\ \frac{t_{i+1} + t_i}{2} & : \quad (i < n-1) \wedge (n > 1), \\ t_i + \frac{1}{2} & : \quad else, \end{cases}$$

3. Map every $\mathcal{P}_i$ into $]u_i, o_i[$ using a bijective strictly increasing map $\mu_i$ on $\mathbb{R}$ (such as of arctan shape, see figure 20.1).
4. The maps $\mu_i : \mathcal{P}_i \to ]u_i, o_i[$ define the map $\mu : \mathcal{P} \to ]u_0, o_{n-1}[$.

**Fact 10** *Note that the first step in the algorithm needs the set $\mathcal{P}$ to be finite in the first coordinate; therefore $\mu$ is* not total *on $\mathbb{R}^2$. This is not a real limitation, as we won't try to visualize infinite sets.*

**Lemma 31** *If we impose the lexicographic ordering on $\mathbb{R}^2$, $\mu$ is order-preserving and therefore injective.*

Proof. There are two possible cases: If $p_1(x_1) < p_1(x_2)$, then $\mu(x_1) \in ]u_i, o_i[$ and $\mu(x_2) \in ]u_j, o_j[$ with $i < j$. But then, by construction of the intervals $]u_i, o_i[, ]u_j, o_j[$, $]u_i, o_i[ < ]u_j, o_j[$, hence $\mu(x_1) < \mu(x_2)$. Else, if $p_1(x_1) = p_1(x_2)$ and $p_2(x_1) < p_2(x_2)$, then on these points, $\mu = \mu_i$ for an index $i$, and $\mu_i(x_1) < \mu_i(x_2)$ since $\mu_i$ is strictly increasing, QED.

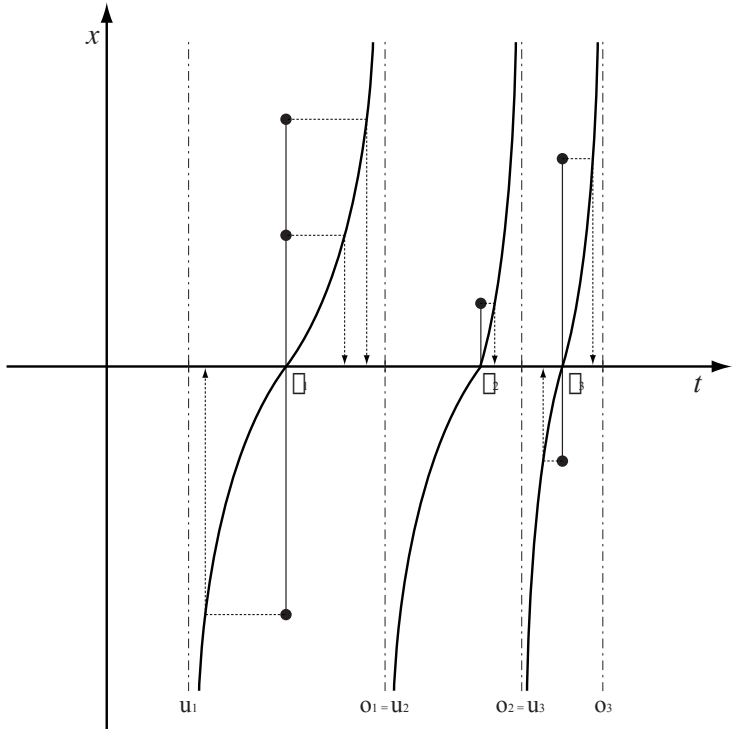



Figure 20.1: The two-dimensional folding algorithm $\mu$.

### 20.2.2 $\mathbb{R}^n \to \mathbb{R}$

With $\mu : \mathbb{R}^2 \to \mathbb{R}$ we can define $\hat{\mu} : \mathbb{R}^n \to \mathbb{R}$ inductively:

$$\hat{\mu}(x) = \begin{cases} \mu(x) & : \quad n = 2, \\ \mu(p_1(x)\ ,\ \hat{\mu}(p_{2\ldots n}(x)) & : \quad n > 2. \end{cases}$$

**Lemma 32** *$\hat{\mu}$ is order-preserving and hence injective.*

Proof. The map $x \mapsto (p_1(x), \hat{\mu}(p_{2\ldots n}(x))$ is order-preserving by induction, and $\mu$ is so by lemma 31, QED.
Note that $p_1(\mathcal{P})$ becomes the coarse factor, whereas $p_{2\ldots n}(\mathcal{P}))$ the fine one.

### 20.2.3 An Explicit Construction of $\mu$ with Special Values.

What remains undefined so far is the *bijective proper increasing mapping* from step 3 in 20.2.1. We define this map with two special real values *low* and *high* in mind, which will be explained in the following section 20.3. We define the following (partial) maps $\tau$ and $\phi$ on $\mathbb{R}^5, \mathbb{R}^{10}$, respectively, see figure 20.2.

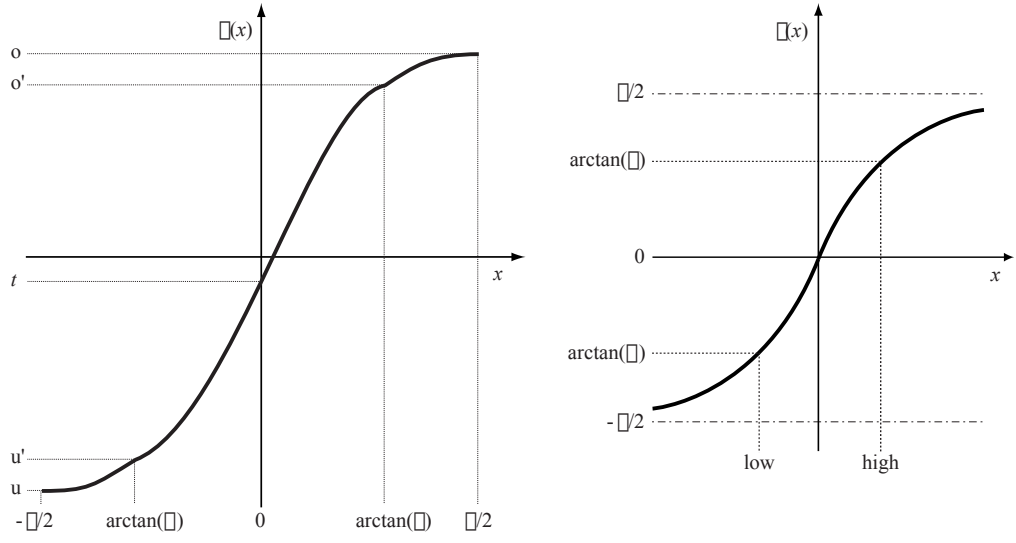



Figure 20.2: The maps $\tau$ and $\phi$.

$$\tau(\alpha, \beta, h, l, x) = \tan^{-1}\left(\frac{\beta - \alpha}{h - l}(x - l) + \alpha\right),$$

$$\phi(o, o', u, u', t, \alpha, \beta, h, l, x) = \begin{cases} \frac{u'-u}{\frac{\pi}{2}+\tan^{-1}(\alpha)}\, x + \frac{\pi}{2}) & +u \quad : \quad -\frac{\pi}{2} < x \leq \tan^{-1}(\alpha) \\ \frac{t-u'}{-\tan^{-1}(\alpha)}\, x - \tan^{-1}(\alpha)) & +u' \quad : \quad \tan^{-1}(\alpha) < x \leq 0 \\ \frac{o'-t}{\tan^{-1}(\beta)}\, x & +t \quad : \quad 0 < x \leq \tan^{-1}(\beta) \\ \frac{o-o'}{\frac{\pi}{2}-\tan^{-1}(\beta)}\, x - \tan^{-1}(\beta)) & +o' \quad : \quad \tan^{-1}(\beta) < x \leq \frac{\pi}{2} \end{cases}$$

The composed map on $x \in \mathbb{R}^2$

$$\phi_\tau(o_i, o_i', u_i, u_i', \alpha, \beta, high, low)(x) =$$
$$\phi(o_i, o_i', u_i, u_i', p_1(x), \alpha, \beta, high, low, \tau(\alpha, \beta, high, low, p_2(x)))$$

maps

$$\begin{array}{lcl} p_2(x) = low & \mapsto & u_i' \\ p_2(x) = high & \mapsto & o_i' \\ p_2(x) = -\infty & \mapsto & u_i \\ p_2(x) = +\infty & \mapsto & o_i \end{array}$$

which is useful for our visualization. Note that in the variable $p_2(x)$, $\phi_\tau$ is *continuous* at $\tan^{-1}(\alpha)$, 0, $\tan^{-1}(\beta)$ but not *differentiable*.

## 20.3 Folding Denotators

With $\hat{\mu}$ defined, we can visualize finite sets in real vector spaces of any dimension in 3D. But what we really want, is to visualize *denotators*. For the time being, we exclusively address *zero-addressed* denotators. Recall from chapter 6 that a denotator can be either simple—with its coordinate in a module—or compound, such as a limit, a colimit or a powerset of 'lower-level' denotators, to speak in the naive denotator terminology. A typical example is shown in figure 20.3, where A to L are again denotators. If we provide every simple denotator with the ability

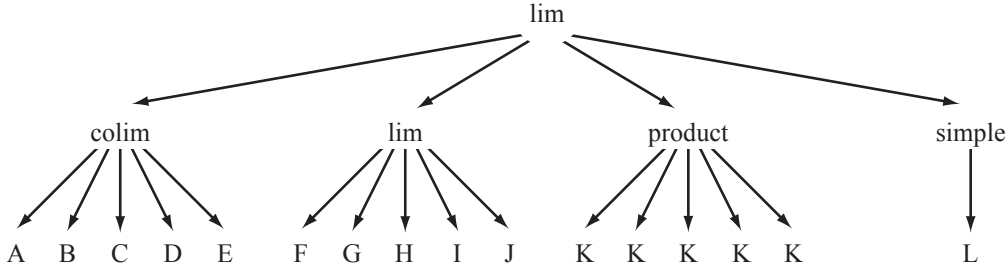



Figure 20.3: A typical compound denotator tree.

to produce a real-valued representation of its coordinate, we can map every denotator to $\mathbb{R}$ by applying $\hat{\mu}$ all the way up the denotator tree, as shown in figure 20.4. The simple denotator gets this real value from the underlying module; the module also provides us with two more values *low* and *high*, representing *natural* borders such as lowest and highest note on the piano, which is useful for the subsequent visualization. But how can we apply $\hat{\mu}$ to a limit- or colimit-denotator?

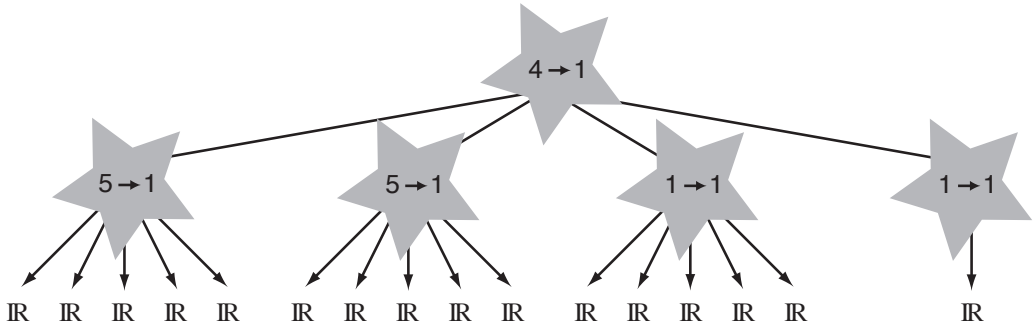



Figure 20.4: Mapping every denotator to $\mathbb{R}$ by applying $\hat{\mu}$ all the way up the denotator tree

### 20.3.1 Folding Limits

When we forget about the limit's diagram, a limit is nothing but a product. Assuming that we already know (by induction) how to fold its factors, we just apply $\hat{\mu}$ to the limit as if it were of type $\mathbb{R}^n$. Later in the visualization we can add the diagram's information by use of color, transparency, etc. By this we visualize a denotator as shown in figure 20.5 as a set of points in the ordinary n-space as shown in figure 20.6.

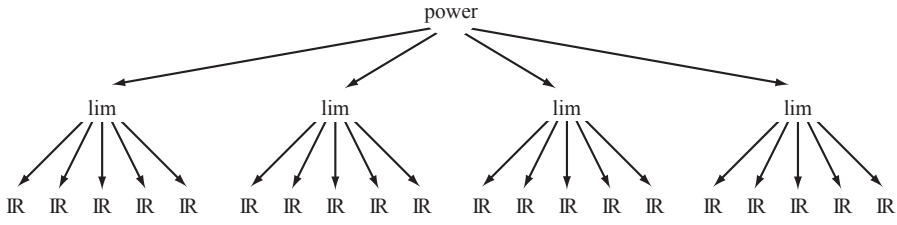



Figure 20.5: Folding limit denotators.

### 20.3.2 Folding Colimits

The folding of colimits is something different. While the semantic of a limit is some kind of a vector space with some additional constraints, the notion of a colimit is the one of a library. A point in a colimit space is like a book or a set of books from a library while the cofactors mean different types of books and the diagram controls which books you may take out together. Therefore we decided to fold a set of colimit denotators (like in figure 20.7) in the following way:

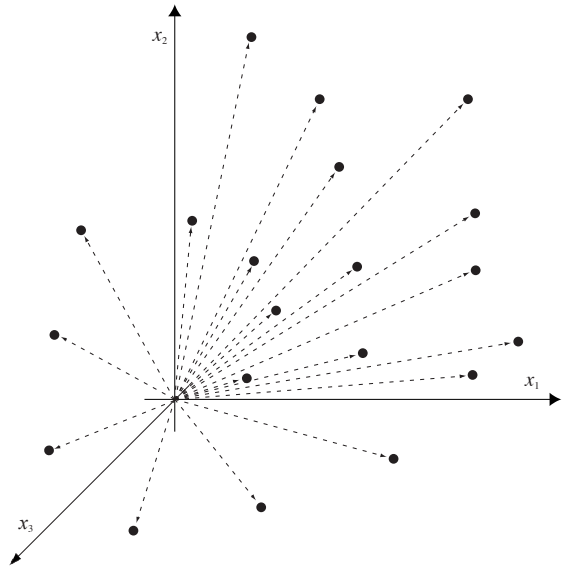
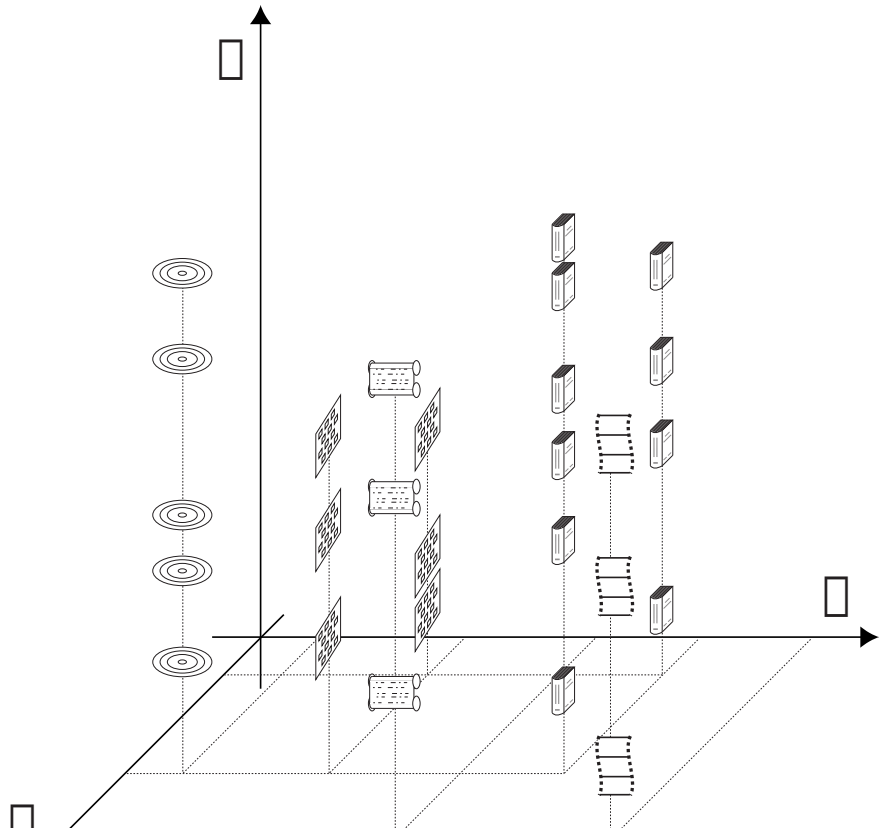



Figure 20.6: Visualization of a denotator as shown in figure 20.5 by a set of points in an ordinary n-space.



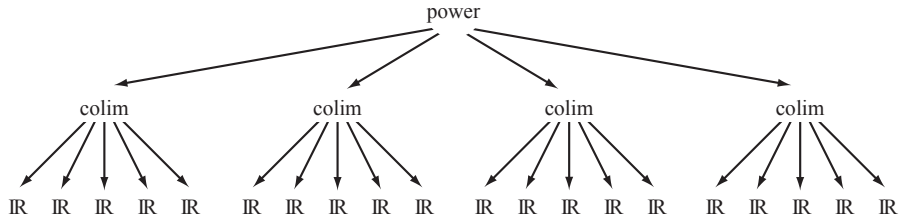



Figure 20.7: A set of colimit denotators.

Firstly we choose which cofactors we bundle together; this information goes to the first axis. On the second we chose to map the index of the cofactor in question and the actual data is folded in the third axis. By this way we can get different kinds books or CDs, etc. in rows of shelves like in figure 20.8.

### 20.3.3 Folding Powersets

The most natural way to deal with a powerset denotator is to view it as the collection or 'container' of its elements and proceed by folding one element after the other recursively with respect to their type. (Note that all elements of the powerset are of the same type.) We call this *powerset disclosure*. The preceding folding of limit and colimit denotator collections ('powerset of limit', 'powerset of colimit') were special cases of this general situation. However, the 'powerset of powerset' case is somewhat special: Here we represent the elements in a $2 \times n$ array, we use their position in their canonically ordered arrangement as first and a their *barycenter*-value as second coordinate. This arrangement is then folded with the $1 \times n$ folding matrix.

Logically there is also an *undisclosed* way of folding a powerset denotator. In this case the barycenters of the $n$ elements are folded with the standard $1 \times n$ folding matrix. This *undisclosed* folding is the default case if a powerset denotator is itself a (co-)factor or element of another denotator.

### 20.3.4 Folding Circular Denotators

In sections 6.5 and 6.7, the problems of circular denotators where already discussed. When actually working with those denotators, one has to take care not to run into infinite loops. Therefore we decided to run a three-level strategy: Down to level $\gamma$ the usual folding takes place, between $\gamma$ and $\tau$ all the real values from the foldings are generalized with the $index(x)$ function[5], and at level $\tau$ the folding process stops completely.

[5]See appendix D.5.1.

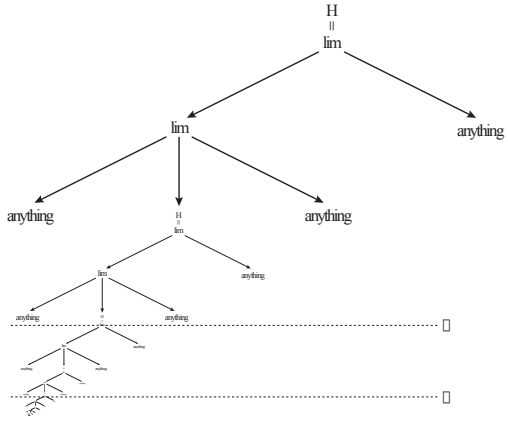



Figure 20.9: Down to level $\gamma$ the usual folding takes place, between $\gamma$ and $\tau$ all the real values from the foldings are generalized with the $index(x)$ function, and at level $\tau$ the folding process stops completely.

## 20.4 Compound Parametrized Objects

Even if we are now in the position to be able to transform any denotator into a 3-dimensional vector we often prefer not to fold all those $n$ dimension down to 3 but to keep some more and map them on any graphical, acoustical, haptic, etc. property of our display object. Therefore we need lots of different objects—let's just call them *multimedia objects*—with many different parameters like color, transparency, pitch, stiffness, etc.

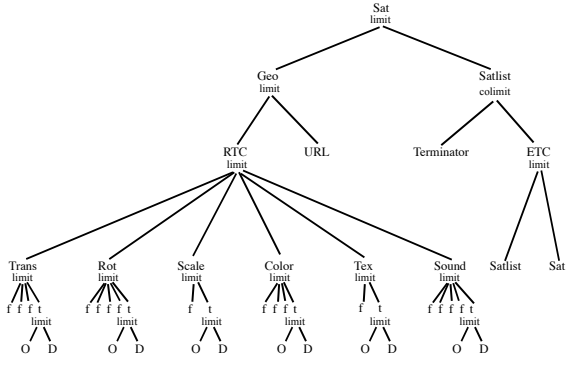



Figure 20.10: Description of a compound multimedia object as a *circular denotator*.

As it is not recommended to actually program each object again and again, we decided to describe a compound multimedia object as a *circular denotator space* (form) (see figure 20.10). The *Sat* form (*Satellite*) consists of a list of (sub-)satellites and the actual geometric, acoustical and time-critical information in the *RTC* denotator space, which actually is a limit

space according to the above information types.

We can for example construct a Pinocchio puppet (see figure 20.11) by defining a denotator for the body$^0$, one for the neck$^1$, the head$^2$, the hat$^3$, the arm$^4$, the foot$^5$, and the leg$^6$. Reusing these denotators we can define the whole Pinocchio. The $f$-denotator coordinates in the $RTC$-denotators provide the parameter functions to steer the concerning properties respectively. By this construction every part of the Pinocchio can be moved, rotated, scaled, etc. as shown in figure 20.11. The $Sat$ denotator space is recursively stable, which means that the parameters still work if the Pinocchio itself becomes a subpart of a more complex multimedia object.

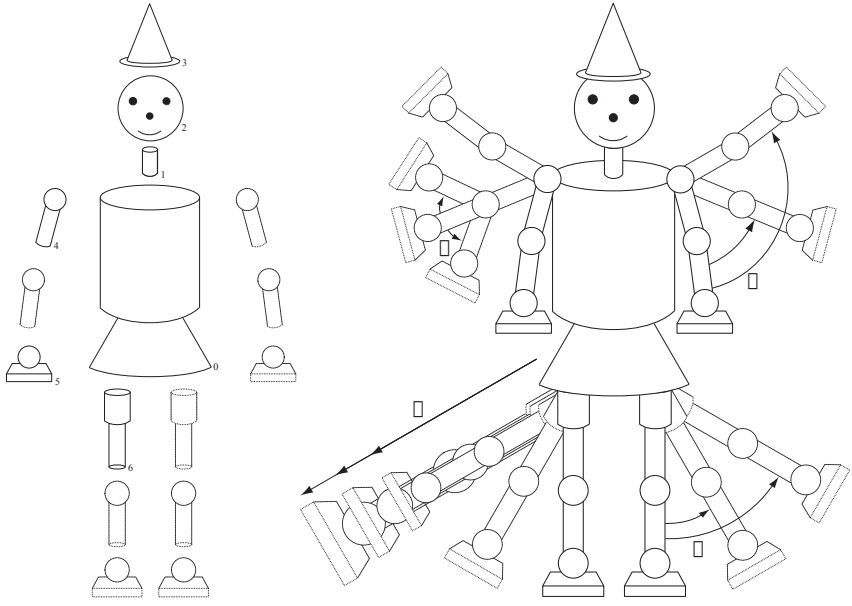



Figure 20.11: Construction of a Pinocchio puppet by defining a denotator for the body$^0$, one for the neck$^1$, the head$^2$, the hat$^3$, the arm$^4$, the foot$^5$, and the leg$^6$. Reusing these denotators we can define the whole Pinocchio.

## 20.5 Examples

The following example shows a denotator like the one in figure 20.5; the corresponding limit is 19-dimensional and was folded to seven dimensions to fit the Pinocchio object, which provides four object parameters additionally to the x-y-z-position.

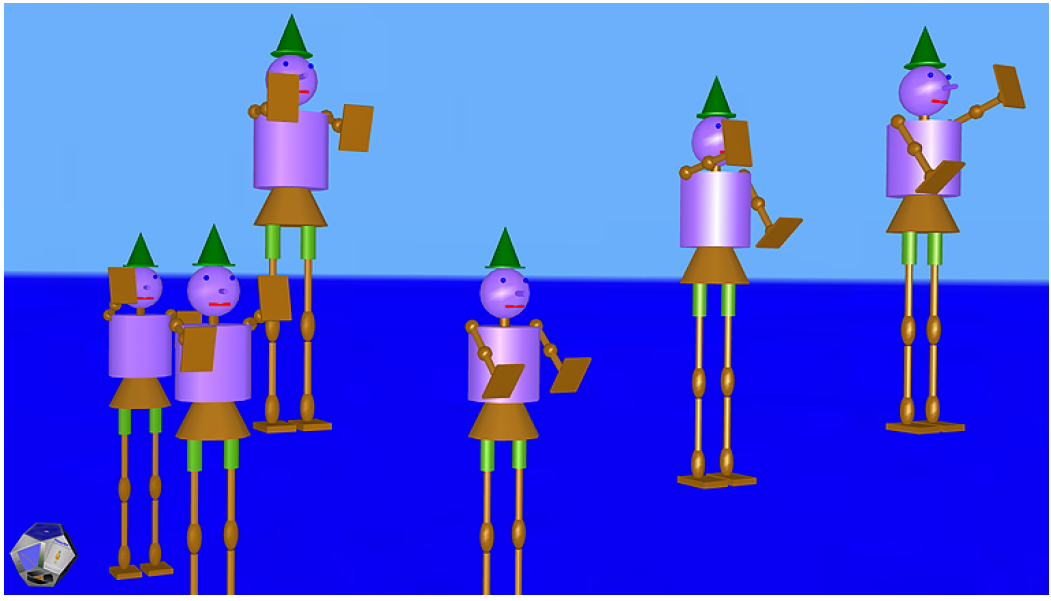


Figure 20.12: In this figure some objects are shown, with relevant coordinates mapped to the leg length and the body color.

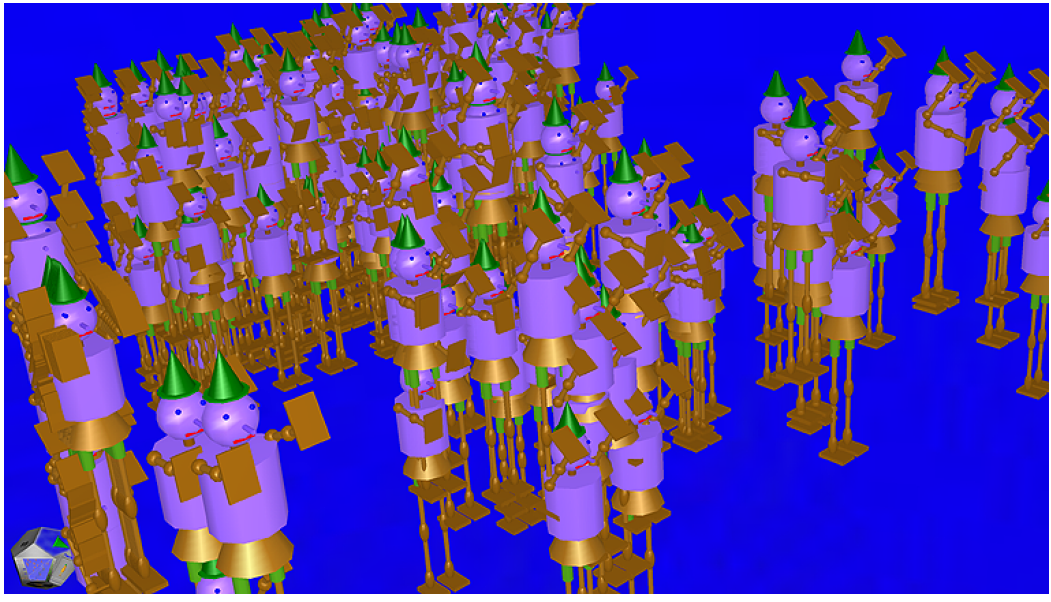


Figure 20.13: In this figure some 400 objects are visible with different quite random parameter mapping.

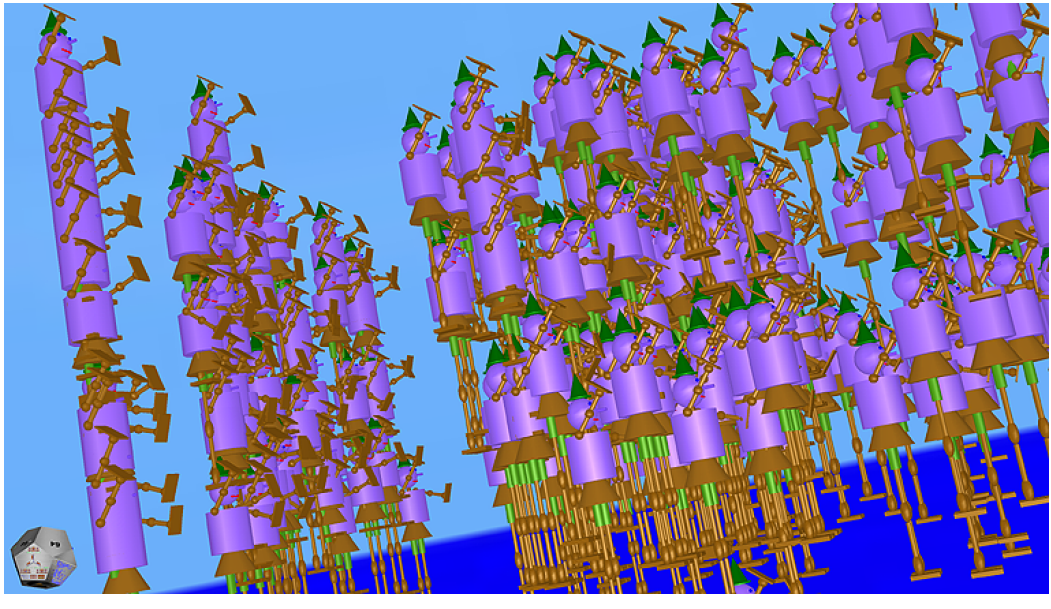


Figure 20.14: This figure shows the same data but with a more sensible parameter mapping. One immediately sees more of the structure; colinear and complanar subsets arise.

# Part V

# Topologies for Rhythm and Motives

# Chapter 21

# Metrics and Rhythmics

*. . . wir operieren mit einem zweiten Grundbegriffe,*
*dessen Feststellung wir uns nun zuzuwenden haben,*
*demjenigen des verschiedenen Gewichtes der Zeiten,*
*der* ***metrischen Qualität.***
Hugo Riemann [453, p.8]

**Summary.** Metrics and rhythmics are an excellent elementary test-case for global structures in music. We shall critically review two commonly known approaches: the Riemann and the Jackendoff–Lerdahl theories. We then develop the concepts of global time structures and their topologies, including associated weight functions.

– Σ –

In this chapter, we refer to the introductory discussions of meter and rhythm in section 7.2.2 (local case) and section 13.4.3 (global case). We shall above all concentrate on the metrical aspect since it yields a simple and formally transparent situation from which the rhythmical complexification can easily be unfolded.

## 21.1 Review of Riemann and Jackendoff–Lerdahl Theories

**Summary.** We review these theories and trace back the concepts of metrical weight and metrical hierarchies.

– Σ –

The reader should however be aware that we cannot include a philologically valid analysis of these works here, our scope is a modest one: to present the tie between these historically important outlets and our own approach *as it really happened*. The interest of such a tie is to start reflecting about the incredibly fuzzy state of conceptualization in music which Riemann also had to suffer ([453, p.VIII]):

*Meine musikalische "Dynamik und Agogik" (1884) arbeitet sich noch mühselig durch den Wust falscher Definitionen, die, wie ein undurchdringliches Gewirr der Wurzeln von Sumpfpflanzen unter dem anscheinend klaren mit Blumen geschmückten Wasserspiegel, sich jedem Vordringen in den Weg stellen.*

### 21.1.1 Riemann's Weights

**Summary.** Hugo Riemann's approach in [453] is discussed, stressing his conceptual specifications of meter and rhythm, in particular the weight idea.

– Σ –

Riemann's concept of rhythm deals with a regular division of time ([453, p.2]):

*Freilich ist aber doch wohl kein Zweifel, daß in der* erkennbaren Gleichmäßigkeit *der Zeitteilung wirklich ein besonderes lustgebendes Moment liegt und zwar eben gerade dasjenige, welches man Rhythmus nennt...*

The temporal period defining a rhythm is not an abstract one, it is related to the concrete music events of the given composition ([453, p.8]):

*Welches in jedem Einzelfalle das Grundmaß [des Rhythmus, G.M.] ist, ergibt sich wie bereits betont, nicht aus dem abstrakten absoluten Mittelmaß, sondern aus dem der konkreten Melodie selbst.* Die Zählzeiten *(Schlagzeiten, rhythmischen Grundzeiten)* gewinnen unter allen Umständen erst reale Existenz durch ihre Inhalte.

For Riemann, rhythm is a result of grouping musical events to equitemporally distributed units, and to relate the grouped events to the units as relative instances (*Relativität der rhythmischen Qualität* [453, p.7]). This is why we have chosen the definition of a rhythm as described in section 13.4.3. This is also stressed in the explication of rhythmical unity ([453, p.9]):

*... als Inhalt einer den Pulsschlag des Rhythmus bildenden Einheit ist prinzipiell eine Mehrheit von wahrnehmbaren Erscheinungen* (also für die Musik: Tongebungen) *anzunehmen.*

However, a detailed study of Riemann's theory is not our scope and should be taken up in the framework of mathematical music theory as it has been taken up in motif theory with Rudolph Reti's approach by Chantal Buteau [73].

In contrast to rhythmical grouping, Riemann views metrical quality as a differentiation among onsets (of notes) which comes from their roles and relations. This quality is stated as a weight ([453, p.8]):

*... wir operieren mit einem zweiten Grundbegriffe, dessen Feststellung wir uns nun zuzuwenden haben, demjenigen des* verschiedenen Gewichtes der Zeiten*, der* **metrischen Qualität**.

The weight is meant to be a metaphor for the degree of importance:

*... die Unterscheidung der einander folgenden Zeiteinheiten in wichtige und minder wichtige oder wie man zu sagen pflegt:* schwere *und* leichte ...

### 21.1.2 Jackendoff–Lerdahl: Intrinsic Versus Extrinsic Time Structures

**Summary.** The time domain is a prototype of a highly ambiguous dimension where intrinsic structures, such as notes, are mixed with extrinsic structures, such as bar-lines. The former are part of the musical material, the latter are part of the abstract time framework. We discuss the difference by the example of Ray Jackendoff's and Fred Lerdahl's approach in [243].

– Σ –

In this section we shall compare the metrical analysis of the Jackendoff–Lerdahl theory (GTTM, Generative Theory of Tonal Music) with what is proposed by the present mathematical music theory and its implementations in the RUBATO® platform's MetroRUBETTE® for metrical analysis. For a more generic discussion of the GTTM method, see [366].

In order to perform a metrical/rhythmical analysis or interpretation of a musical composition, the basis of such an investigation must be questioned. This one is linked to the "neutral" work data on one hand, and to the analytical approach on the other. Every investigation of the neutral data presupposes a choice of relevant aspects. This choice occurs very early in the propaedeutic evaluation.

For example, in the first Viennese school, bar-lines, period limits, notes, and pauses are recognized. In a metrical/rhythmical analysis, the question of the bar-lines' relevance would be set forth: Will this type of events strongly influence our analysis? As is shown in the MetroRUBETTE® concept, the answer is not necessarily "Yes" or "No", a fuzzy decision between these two extremal values is possible and reasonable.

In this situation, a normative analysis would imply that we decide to include bar-lines as external formal limits of metrical/rhythmical analysis. This bed of Procrustes is what GTTM prescribes in its metrical structure analysis: Bar-lines are always instances of the metrical structure, this is implicit in the fourth rule[1]. of metrical structure[2]. Here are the four MWFRs:

- *MWFR 1. Every attack point must be associated with a beat at the smallest metrical level present at that point in the piece.*
- *MWFR 2. Every beat at a given level must also be a beat at all smaller levels present at that point in the piece.*
- *MWFR 3. At each metrical level, strong beats are spaced either two or three beats apart. (This rule is recognized as idiom-specific in GTTM.)*
- *MWFR 4. The tactus and immediately larger metrical levels must consist of beats equally spaced through the piece. At subtactus levels, weak beats must be equally spaced between strong beats.*

[1] There are two types of GTTM rules: Well-formedness rules (WFR), and preference rules (PR). The rule system has four parts: G: grouping structure, M: metrical structure, TSR: time-span reduction, and PR: prolongational reduction. Accordingly, the two rule types are symbolized by the part prefix and the type postfix. For example, metrical well-formedness rules are MWFR. The complete index of GTTM rules is given in [243, p.345ff.]

[2] See also [243, p.71]: *Yet metrical intuitions about music clearly include at least one specially designated metrical level, which we are calling the* tactus.)

Worse than that: even non-existent onsets are inserted in determined levels in order to enforce periodicity where the text does not show such. This is a consequence of (for example) the axiom MWFR3 which asks that successive metrical levels have duration (period) ratios 1:2 or 1:3. All other ratios are forbidden! The analysis of eight bars from Mozart's *Jupiter* Symphony are a typical example, see figure 21.1.

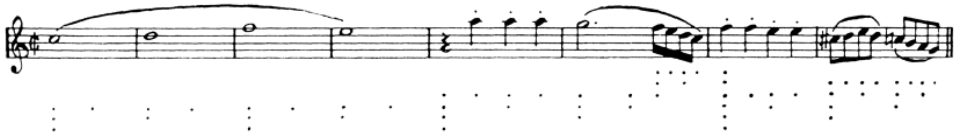



Figure 21.1: The metrical hierarchy in the finale's eight bar theme of Mozart's *Jupiter* Symphony, according to the GTTM, [243, p.73].

It shows that the whole notes in the first four bars have to be enriched by half-note points in order to cope with the succeeding onset structure. It also shows that the bar-lines are points even when there is no such note-onset. Such a procedure is not only rigid, it is also dangerous since the complexity of meter/rhythm is destroyed by secondary instances such as bar-lines, and by a spurious subdivision dogma (1:2, 1:3). And it forces a norm which only cements bad mass taste instead of representing possible compositional intensions.

But even in tonal music, also in modern tonal music, rhythm and metrics are much less tied to bar-lines than the formal score notation suggests. For instance, in contemporary jazz[3], we recognize extremely complex poly- and microrhythms which escape bar-lines and would be eliminated by the said normative standardization as being non-metrical or non-rhythmical music.

Moreover, the GTTM casting of metrical structure makes evident a remarkable drawback of non-computerized musicology[4]: The effective usage of the following metrical analysis method (21.2) would have been fictitious without computers because of its combinatorial complexity. We in fact argue that sometimes, a normative approach is only propagated because operational control of analysis would break down without such a "terrible simplification", so the simple and wrong is preferred to reality which would possibly force one to wait until the power of tools has reached an adequate level. During the design period of the workstation RUBATO® it was therefore mandatory to ask for the following condition:

**Axiom 3** *We need concrete tools which are adequate to the analyzed work, and which yield an immanent analysis, not merely abstract or brutally normative principles.*

Despite our critical review of the GTTM method, the idea to cover the onsets of a given composition by "levels" has been a fruitful input for our global metrical perspective. It gave us the motivation to introduce global meters from the concept of a local meter, which were introduced in [340, pp.32-34], and from the insight that simultaneous presence of several local meters in a piece of music is the rule and not the exception (see the example [340, Bild 11,

[3]Listen, e.g., to Steve Coleman's CD *The Sonic Language of Myth*, BMG(RVA Victor 1998).
[4]It is well known that the authors of GTTM do not preconize computer-aided music analysis.

p.34]). The crucial difference between the global meter theory and the GTTM metrical structure theory is—besides our non-normative attitude—the construction of level functions from global data (in fact the nerve structure) and not their a priori definition, see the level function defined in section 13.4.3.

## 21.2 Topologies of Global Meters and Associated Weights

**Summary.** The hierarchic structure of global meters gives rise to topologies and associated weights, i.e., numerical functions which contain information about the local connection of onsets within their topological neighborhoods. Metrical topologies are the theoretical background for the MetroRUBETTE® module for metric and rhythmic analysis in the RUBATO® software discussed in section 41.1.

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To make the discussion more transparent, we only deal with the zero address and meters in the *Onset* domain and leave it to the reader to add the extensions to proper *Para*-meters as defined in section 13.4.3. Let us first recall the maximal meter topology on a local composition $X$. This one is associated to the interpretation $X^{MaxMet}$ whose charts are all finite intersections of the maximal local meters in $X$. We know that every closed irreducible set is the closure of any of its maximal points with respect to the dominance relation. So if $x, y$ have the same closure $\overline{\{x\}} = \overline{\{y\}}$, then they are mutually dominating, or equivalently, their simplexes coincide, $Sp(x) = Sp(y)$, or equivalently, their minimal open neighborhoods $U(x), U(y)$ coincide (lemma 20 in section 13.4.3). Recall from appendix F.2.1 the left adjoint construction of the continuous map $i : X \to X^s$ of a topological space $X$ into its associated sober space $X^s$ which sends $x$ to the point $\overline{\{x\}}$. This map is a quasi-homeomorphism (see lemma 85, appendix F.2.1), so the sober topology is coarser than the quotient topology. But $i$ is also surjective, so any open set of the quotient topology lives in the sober topology, i.e., the sober topology and the quotient topology coincide. Let $Sp(X) = \{Sp(x) \in n(X) |\ x \in X\}$. From this information we deduce this result:

**Lemma 33** *The sober quasi-homeomorphism $i : X \to X^s$ for the maximal meter topology is open, surjective, and (therefore) the topology of $X^s$ is the quotient topology. We have a bijection*

$$is_X : X^s \xrightarrow{\sim} Sp(X) : i(x) \mapsto Sp(x)$$

*of sets.*

We may rephrase the sober topology definition to generate a topology on the nerve $n(X^{MaxMet})$ which extends the topology on $Sp(X)$ which is induced by the bijection $is_X$. By definition, open sets in $X^s$ are given by

$$V^s = \{\overline{\{x\}} |\ x \in X,\ \text{there is } y \in V, Sp(y) \subset Sp(x)\}$$

for any open sets $V$ in $X$. If we use the transition bijection, we have the corresponding open sets

$$is_X(V^s) = \{Sp(x) |\ x \in X,\ \text{there is } y \in V, Sp(y) \subset Sp(x)\}$$

and this can be rephrased to extend to the entire nerve by

$$V^{sp} = \{\sigma |\ \sigma \in n(X^{MaxMet}),\ \text{there is } y \in V, Sp(y) \subset \sigma\}.$$

We have $V^{sp} \cap Sp(X) = is_X(V^s)$. Let $n(X^{MaxMet})^s$ be the *maximal meter nerve topology* which is generated by the sets $V^{sp}$. Observe that we have $\bigcup_i V_i^{sp} = (\bigcup_i V_i)^{sp}$ whereas the intersection $V_1^{sp} \cap V_2^{sp}$ need not be a generating open set. However, the restriction of the maximal meter nerve topology to the subspace $Sp(X)$ is the induced topology. So we have this:

**Proposition 24** *With the maximal meter and the maximal meter nerve topologies as defined above, the map $Sp : X \to n(X^{MaxMet})^s$ becomes a continuous map whose image (as a subspace with the relative topology) identifies to the associated sober space $X^s$.*

This justifies the following definition:

**Definition 64** *An $\mathbb{R}$-valued* sober weight *on the zero-addressed local composition $X \subset Onset$ is a function $W : X \to \mathbb{R}$ which factorizes through the nerve map $Sp$. A factor $nW : n(X^{MaxMet}) \to \mathbb{R}$ is called a* nerve weight *for $W$. A nerve weight $nW$ is said to be* induced *iff its values are determined by the values on the vertexes, i.e., by the restriction $n_0W = nW|_{n_0(X^{MaxMet})}$.*

**Example 43** If $n_0W$ is any $\mathbb{R}$-valued function, we may take the sum $nW(\sigma) = \sum_{M \in \sigma} n_0W(M)$ of vertex values. A typical vertex function will be discussed for the MetroRUBETTE® (section 41.1): Fix a real exponent $prof$, weight's "profile", and two natural numbers $min \leq max$. The function associates with the maximal local meter $M \in n_0(X^{MaxMet})$ the number

$$n_0W(M) = \begin{cases} l(M)^{prof} \text{ if } min \leq l(M) \leq max, \\ 0 \text{ else.} \end{cases} \tag{21.1}$$

So musically, a sober weight is a valuation of onset events in $X$ by their simplexes, i.e., their position within the covering by maximal local meters which are particular points in the topological space $(X^{MaxMet})^s$. No systematic treatment of the real vector space of sober weights or, more generally, the cohomology of such weights, has been undertaken to date.

**Exercise 42** In this context, one may consider the covering $MaxMet$ of $X$ and then restrict everything to the integers as a ground ring. Given a prime number $l$, the functors of $\mathbb{Z}_{l^r}$-forms yields the $l$-adic cohomology $H_l^\star(X^{MaxMet})$ introduced in equation (19.11). We argue that this information should be important for the classification of global meters, and also, for defining refined sober weights. Elaborate the form functor on the maximal meters and its cohomology.

The method of induced weights can be generalized to interpretations in the onset domain. Suppose that $X \subset Onset$ is interpreted to yield the global composition $X^I$. This may happen if we generate this interpretation from a knowledge about the origin of $X$. For example, it could happen that $X = pr_{Onset}(Y^J)$, the projection of a given global composition into the onset domain, and that we have the charts $Y_j \subset Y$ of a covering $J$. Then we obtain an interpretation $X^I$ of $X$ by the projections $pr_{Onset}(Y_j)$ of these charts. We shall learn from section 41.1 that

this is a very common situation in music research, for instance when collecting different kinds of rhythmically significant objects, such as notes, bar-lines, pauses, etc.

We may then consider the maximal meter topology on each chart $X_i$ of $X^I$. Suppose we are given a sober weight $W_i : X_i \to \mathbb{R}$ for each chart. Then we may construct an induced weight from the nerve $n(X^I)$. For $x \in X$, let $Sp_I(x)$ be the simplex of $x$ with respect to the covering $I$. Then we define $W(x) = \sum_{X_i \in Sp_I(x)} W_i(x)$. This enables us to create mixed weights for points in an interpretation and thusly build a refined weighting of onsets with respect to different predicative specifications. In particular, we shall see in 41.1 that the Jackendoff–Lerdahl approach can be simulated (and this in a refined way) by reintroducing the bar-lines with a specific weight to add up to the total weight of an onset.

## 21.3 Macro-Events in the Time Domain

**Summary.** This section reviews macro-events in the light of topologies of global meters and rhythms and the associated weights.

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The previous discussion of temporal denotators was centered around topological relations, such as dominance, of onset events. It is evident that topological relations yield a tool for grouping onset events according to their metrical role. To this end, we review the definitions of local rhythms from section 13.4.3. In that context, a ($A$-addressed) rhythmic germ is a local composition $G \subset A@Rhythm(Para)$.

We shall deal with germ denotators $G$ which have the shape

$$G : A \rightsquigarrow Para(g_1, \ldots g_m)$$

where

$$Para \underset{Fin(Rhythm(Para)) \rightarrowtail \Omega^F}{\longrightarrow} \mathbf{Power}(Rhythm(Para))$$

with $F = Fun(Rhythm(Para))$. So the form $Para$ is circular and reappears as a parameter space associated with the onset coordinator. We know from section 6.5 that $Para$ exists. Such a denotator is a $Para$-rhythmic germ as well as a macro denotator as defined in formula (6.116). The point is that the germ parameter space coincides with the germ space. Intuitively speaking, such a germ is a local composition of onsets which are parametrized by germs of onsets, etc., recursively, until—in a 'tame' case—the parametrizing germs reduce to the empty set.

This being true, we want to associate such a macro germ with each maximal meter topology on a (zero-addressed) local composition $X \subset Onset$. To this end, consider the Hasse diagram[5] $Dom(X)$ for the partial ordering of the dominance relation on $X$ under the maximal order topology. This defines a macro germ as follows. We take the level function[6] $lev(x), x \in X$, and define by level recursion: $Dom(x) = (x, \emptyset)$ if $lev(x) = 0$, and

$$Dom(x) = (x, \{(Dom(y)_1 - x, Dom(y)_2) |\ y \in pre(x)\}) \tag{21.2}$$

[5] See appendix C.2, definition 121.
[6] See appendix C.2.

in the other cases with the two components $Dom(y)_1, Dom(y)_2$ of the macro $Dom(y)$. Then we put $Dom(X) = \{Dom(x)|\ x \in Max(X)\}$.

**Example 44** Let us make an example, see figure 21.2

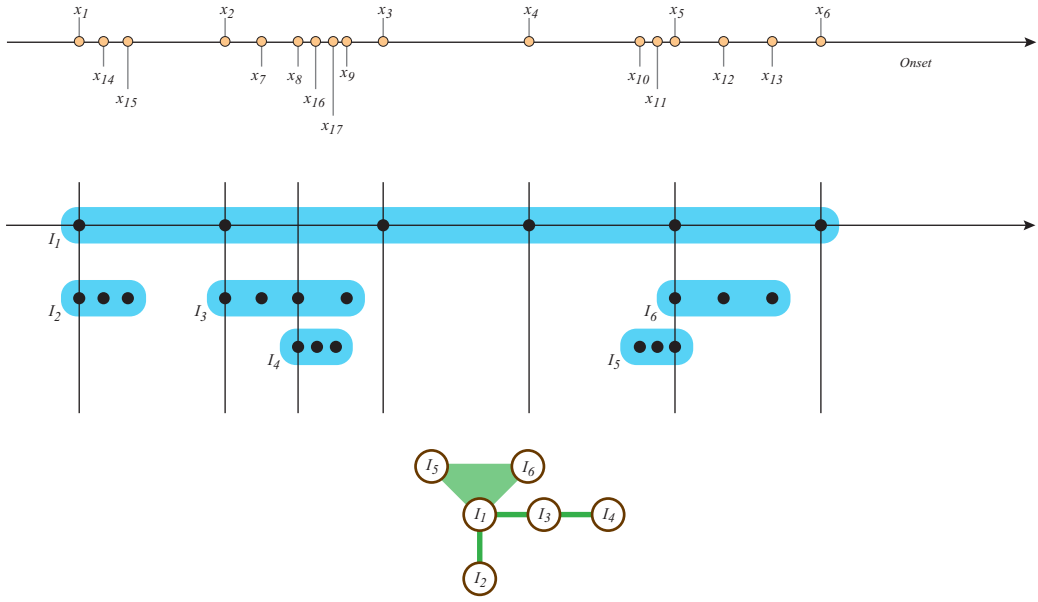


Figure 21.2: An irregular metrical structure $X$ and its nerve. The associated macro germ is described in the text.

Denoting simplexes by $\{I_1, I_5, I_6\} = \Delta(1,5,6), \{I_1, I_5\} = \Delta(1,5)$, etc., we have these $Sp$ function values:

$$
\begin{aligned}
Sp(x1) = \Delta(1,2), Sp(x2) = \Delta(1,3),\\
Sp(x3) = Sp(x4) = Sp(x6) = \Delta(1), Sp(x5) = \Delta(1,5,6),\\
Sp(x7) = Sp(x9) = \Delta(3), Sp(x8) = \Delta(3,4),\\
Sp(x10) = Sp(x11) = \Delta(5), Sp(x12) = Sp(x13) = \Delta(6),\\
Sp(x14) = Sp(x15) = \Delta(2), Sp(x16) = Sp(x17) = \Delta(4).
\end{aligned}
$$

So the points $x1, x2, x5, x8$ are generic, and we have

$$\begin{aligned}
Dom(x1) &= (x1, \{(x3 - x1, \emptyset), (x4 - x1, \emptyset), (x6 - x1, \emptyset), (x14 - x1, \emptyset), \\
&\qquad (x15 - x1, \emptyset)\}), \\
Dom(x2) &= (x2, \{(x3 - x2, \emptyset), (x4 - x2, \emptyset), (x6 - x2, \emptyset), (x7 - x2, \emptyset), \\
&\qquad (x9 - x2, \emptyset)\}), \\
Dom(x5) &= (x5, \{(x3 - x5, \emptyset), (x4 - x5, \emptyset), (x6 - x5, \emptyset), (x10 - x5, \emptyset), \\
&\qquad (x11 - x5, \emptyset), (x12 - x5, \emptyset), (x13 - x5, \emptyset)\}), \\
Dom(x8) &= (x8, \{(x7 - x8, \emptyset), (x9 - x8, \emptyset), (x16 - x8, \emptyset), (x17 - x8, \emptyset)\}).
\end{aligned}$$

We see that the same point, e.g., $x4$ appears in three different macro ramifications since it is a specialization of three different generic points. But flattening this macro-event eliminates this multiplicity. We then have

$$Dom(X) = \{Dom(x1), Dom(x2), Dom(x5), Dom(x8)\}$$

which gives a complete picture of the dominance hierarchy; make a figure of this dominance hierarchy.

If we are given a macro germ, it is possible to disregard its ramification structure which is deeper than a given limit, this operation of "greeking"[7] can be defined by replacing the macro satellite sets by the empty set from a given depth on.

**Exercise 43** Give a rigorous definition of greeking macro germs.

[7] An operation known from text applications, meaning that a structure is blurred, recalling the English idiom "that sounds Greek to me".

# Chapter 22

# Motif Gestalts

*In the organic sphere one cell is different from all the others.*
*By a magic interplay between these identical yet different cells,*
*the higher forms of life come into existence.*
*In an astoundingly analogous way one musical motif,*
*one theme releases another as*
*an expression of its own innermost idea,*
*yet the latter is a being*
*entirely different from the first.*
Rudolph Reti [444, p.359]

**Summary.** This chapter is not only a good test-case for the mathematization of elementary music concepts, it is above all a refined study of turning fuzzy concept sketches of the humanities into precise and consistent frameworks—without the expected side-effect of "terrible mathematical simplification". In the present case of motives, our topic is—within the general task of grasping motivic phenomena—the construction of Rudolph Reti's immanent motif analysis.

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Understanding the motivic, melodic, and thematic organism of a musical composition is one of the most delicate tasks of musicology. It has three principal components which characterize its problematic status: semantic depth, formal complexity, and ill-defined aspects.

Semantic depth is the phenomenon that motives, melodies, and themes are not only formal constructs, i.e., denotator structures with little or no paratextual and easy textual predication. A composition's motif is a germ of a structural hierarchy, unfolding into its most diversified ramifications, variations, and fragmentations. This hierarchy is an expression of meaning, of multiply layered semantic depth which may transcend pure textuality and point at more philosophical, ethical or at least esthetical programs.

Formal complexity appears when one tries to grasp von Ehrenfels' easy-going definition of a gestalt. One recognizes that its characteristics: super-summativity, and transformational invariance, are very complex requirements if rendered in a precise concept framework. We have already discussed some of these issues in chapter 12. This problem makes clear that the

construction of semantic layers from motivic, melodic, or thematic surface is a task which involves some structural complexity.

Ill-defined aspects of motivic, melodic, and thematic analysis show up if one asks for abstraction principles when defining gestalt qualities. It is commonly accepted in motif theory (if that name were really adequate) that gestalt is connected to simplification and class building from the given material. It virtually never happens that the given note information is taken literally to build gestalts. But it is completely unclear which aspects of the given material should be retained, and in what form, or even in what variety of particular forms.

In what follows, we have attempted to give a number of solutions or to make proposals for such, and to discuss the relations between our approaches and the fuzzy but sensitive ideas from traditional musicology.

The application of the following topological methods in motif theory to programming software for motif analysis is important and will be dealt with in chapter 41.2.

## 22.1 Motivic Interpretation

**Summary.** The first step in the present motive analysis framework is the insight that motives cannot be processed without preliminary abstraction. We present musical motivations and case studies.

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To begin with, recall the definition of a motif from section 7.2.3: If $A$ is an address, a (A-addressed) motif $M$ is an objective local composition $M \subset A@F$ whose ambient space $F = Onset \oplus Pitch \times Para$, for any Form $Para$ of additional "parameters", and such that for any pair of different elements in $M$, their projections to the onset space are different. We shall write $M_{OP}$ for the projection of $M$ onto its $OP$ component, i.e., the local composition $M_{OP} \subset A@(Onset \oplus Pitch)$, and analogously $M_O \subset A@Onset$. We therefore have bijective projection morphisms

$$M \xrightarrow{pr_{OP}} M_{OP} \xrightarrow{pr_O} M_O$$

for motives.

Before entering into the technical discussion, let us have a look at the auxiliary space $Para$. As with meters, we can also use this space to realize macro constructs in the motivic context. More precisely, in this case the motif $M$ is a denotator of Form $Para \underset{Fin(F) \rightarrowtail \Omega^{Fun(F)}}{\longrightarrow} \mathbf{Power}(F)$. The flattening and greeking operations discussed in chapter 21 can be applied mutatis mutandis. So the parameter space may be an external space regarding duration, sound colors, etc., or a macro refinement construct which englobes hierarchies of onset plus pitch events. We denote by $A@MOT_F$ the set[1] of $A$-addressed motives of space $F$, and we write $MOT_F$ for the set zero-addressed motives, and $A@MOT$ ($MOT$) for the set of $A$-addressed (zero-addressed) motives in the space $F = Onset \oplus Pitch$ (the parameter space is trivial). For a positive natural number $n$, we denote by $A@MOT_{F,n}, MOT_{F,n}, MOT_n$, respectively the corresponding sets of motives with cardinality $n$, we therefore have $A@MOT_F = \coprod_n A@MOT_{F,n}$, etc.

[1] Pay attention to the fact that this is not a functor since distinction of points by their onsets is not functorial!

Our first abstraction is a strong one: We shall forget about all parameters from the *Para* component and concentrate on the $OP$-projection $M_{OP}$, i.e., on $A@pr_{OP} : A@MOT_F \to A@MOT$. Although our analyses will deal with the real motives, their properties will be 'filtered' through $pr_{OP}$. This is not mandatory, any extension of the forthcoming theory to the other parameters is feasible, for instance inclusion of sound colors, loudness or duration in order to describe similarity of motives. So our exposition is just a prototypical one concentrating on the core parameters. We shall describe a compensatory action to this preliminary reduction below.

**Definition 65** *A (finite)* motivic interpretation *of a local composition $X \subset A@F$ is an interpretation by a (finite) subset $\mu \subset A@MOT_F$. The motives of $\mu$ are called the* covering motives *of the motivic interpretation $X^\mu$, and $\mu$ is called the* motif space. *We say that a motivic interpretation (or its motif space)* satisfies the submotif existence axiom (SEA) *iff every submotif of a covering motif is also covering whenever its cardinality equals at least the minimal cardinality of covering motives. The nerve of a motivic interpretation is called the* motivic nerve.

This extends the definition of a melody given in definition 42, sec section 13.4.4. Motivic interpretations occur for local compositions $X$ in score spaces $F$ with events having onset and pitch coordinates. Typically, the covering motives are selected by upper and lower limits $min, max$ of admitted motif cardinality and by the maximal admitted difference *span* of onsets within a motif. We shall deal with this kind of covering satisfying the SEA in the discussion of the MeloRUBETTE® in section 41.2. A generalization of this selection method is given by a preliminary interpretation $X^I$ of $X$, for example by the charts of the voices in a polyphonic composition. We may then consider all motives within one of these charts, and subjected to the above limits (here the lower limit should be set to 1 in order to guarantee a covering of the given composition). Of course, there is already a strong semantics in the definition of a motivic interpretation. This one may be generated essentially by mathematical rules, by prima vista predicates (such as slurs, beams), by shifter predicates, or by combinations thereof, see section 18.3.3 for this differentiation where we refer the reader to anchor corresponding semantic questions.

Evidently, the intersections $\bigcap\sigma$ of the simplexes $\sigma$ of a motivic nerve are also motives, and we obtain a base for a topology on $X$. In contrast to the maximal meter topology, this one is not interesting since often, the intersection $\bigcap\sigma$ is a singleton, and the topology becomes discrete. We call the simplex $Sp_\mu(x)$ of a point $x \in X$ the *motivic simplex of $x$* (with respect to $\mu$). As with the metrical theory, the simplex of a point should give information about its role in the zoo of selected motives. The following study of topologies on motif spaces aims at providing us with that information.

This possibility to select motives by use of charts and their predicative background is a certain compensation for the preliminary reduction to onset and pitch. On the other hand, the analytical, cognitive or compositional perspective still does not work with the full information produced by the onset and pitch data. For example, in counterpoint one is interested in the categorization of interval sequences: the cantus firmus and discantus may move in different directions: in parallel, oblique, or contrary motion. And this enforces a very coarse classification of motivic gestalts. In American contour theory [381], pitch is only regarded as a number in an abstract discrete pitch space whereas onset is abstracted to a pure ordering index.

Therefore, a general abstraction concept framework is needed to grasp the variety of perspectives which contribute to the motivic gestalt concept. This is what we want to expose in the following section. It will help us in the construction of topological relations on a motif space as described above.

Although the motif concept works for any address, all the abstractions which are known from musicological/cognitive contexts germinated on zero-addressed motives. In this setting, they all make use of the linear ordering on a motif $M$ which is induced by the bijection with its onset projection. Now, this projection can be given a linear ordering if the address $A$ is such that $A@Onset$ is linearly ordered. For example, if we have $\mathbb{Z}$-addressed motives, we know from section 7.2.3 that the onset coordinates of their points $D$ are represented by arrows $ON_D \Rightarrow OFF_D$. This data bears a canonical linear lexicographic ordering: first order the onset times, and then the offset times. Moreover, since the pitch coordinates are represented by arrows $PIFF_D \Rightarrow POFF_D$, the same construction works for pitch. This elementary situation can be used to generalize zero-addressed abstractions to $\mathbb{Z}$-addressed generalizations, but there is no intuition or method to do so for general addresses.

The following abstraction examples will therefore be of rather selective nature, restricting to zero and integer addresses, but the theory will nonetheless be carried out on a general address level, starting from the general axioms which we can learn from the given examples.

## 22.2 Shape Types

**Summary.** Several abstraction methods are in usage in musicology. We establish the formal concept of *shape types* and present a list of common constructs.

– Σ –

We get off by the formal definition of a shape type and then give representative examples thereof:

**Definition 66** *Given a parameter space $F$ and an address $A$, a* shape type *$t$ is a family $(\Gamma_{t,n})_{n\in\mathbb{N}_+}$ of non-empty sets, together with a set map*

$$t : A@MOT_F \to \Gamma_t \tag{22.1}$$

*with codomain $\Gamma_t = \bigcup_n \Gamma_{t,n}$ such that $t(A@MOT_{F,n}) \subset \Gamma_{t,n}$, for each index $n$. Equivalently, we may give the $n$-components*

$$t_n : A@MOT_{F,n} \to \Gamma_{t,n} \tag{22.2}$$

*with the direct sum $t = \coprod t_n$. The spaces $\Gamma_{t,n}$ are called the* spaces of (abstract $n$-motives of type) *$t$ and their elements are called* abstract $n$-motives of type $t$*; the space $\Gamma_t$ is called the* space of $t$-abstract motives*. For an abstract motif $m \in \Gamma_t$ we write $abcard_t(m)$ for the minimal index $n$ such that $m \in \Gamma_{t,n}$, this number is called the* abstract cardinality of $m$*. If $M \in A@MOT_F$, we write $abcard_t(M) = abcard_t(t(M))$ and call this number the* $t$-abstract cardinality of $M$*.*

We shall use the following notation: $\Gamma_t|_k$ denotes the subset of abstract motives of abstract cardinality $k$, and we set $A@MOT_F|_k = t^{-1}(\Gamma_t|_k)$, whence $\Gamma_t = \coprod_k \Gamma_t|_k$, and $A@MOT_F = \coprod_k A@MOT_F|_k$.

**Assumption 1** *In the sequel we shall always make the assumption that the abstract cardinality is compatible with inclusions, i.e., $M \subset N$ implies $abcard_t(M) \leq abcard_t(N)$.*

All examples which we shall present satisfy this property.

In principle it is possible and reasonable to view the abstract motives as denotators, but we will refrain from this formalism since the denotator theory is not (yet?) primordial here and would only burden the symbolism.

## 22.2.1 Examples of Shape Types

**Summary.** We present the rigid, diastematic, elastic, and toroidal shape types.

– Σ –

All the following types will undergo the announced generic abstraction from parameters outside onset and pitch.

### 22.2.1.1 Rigid Types

This abstraction is a generic one with respect to the others, in fact, it is a one-to-one restatement of the $OP$-projection $M_{OP}$ of a motif $M$ in terms of the ordering which is induced by the onset projection if this is given at the selected address $A$. For $A = \mathbb{Z}^r$, we have the well-known lexicographic ordering of $\mathbb{Z}^r@Onset$ denotators $o = (o_0, o_1, \dots o_r), o_i \in \mathbb{R}$. In this case, the onset ordering is transported to the points of $M_{OP}$, i.e., the rigid shape type $t = Rg$ is defined by $Rg(M) = (m_0, m_1, \dots m_{c-1})$ where the sequence $(m_0, m_1, \dots m_{c-1}) \in \Gamma_{Rg,c} = \mathbb{R}^{2(r+1)\times c}$ is the sequence of all elements of $M_{OP} = \{m_0, m_1, \dots m_{c-1}\}$ with $c = card(M)$ in their onset ordering. In this setup, we have $abcard_{Rg}(M) = card(M)$, $\Gamma_{Rg} = \coprod_n \Gamma_{Rg,n}$, $\Gamma_{Rg}|_n = \Gamma_{Rg,n}$, and $\mathbb{Z}^r@MOT_F|_n = \mathbb{Z}^r@MOT_{F,n}$. Concretely, $Rg(M) = (m_0, m_1, \dots m_{c-1})$ is denoted by a $2(r+1) \times c$-matrix

$$Rg(M) = \begin{pmatrix} o_{0,0} & \dots & o_{0,i} & \dots & o_{0,c-1} \\ \dots & & & & \\ o_{r,0} & \dots & o_{r,i} & \dots & o_{r,c-1} \\ p_{0,0} & \dots & p_{0,i} & \dots & p_{0,c-1} \\ \dots & & & & \\ p_{r,0} & \dots & p_{r,i} & \dots & p_{r,c-1} \end{pmatrix} \text{ with } m_i = \begin{pmatrix} o_{0,i} \\ \dots \\ o_{r,i} \\ p_{0,i} \\ \dots \\ p_{r,i} \end{pmatrix} \tag{22.3}$$

and where the upper half $Rg_o(M)$ refers to the onsets whereas the lower half $Rg_p(M)$ refers to the pitch coordinates.

A second shape type $\Delta Rg$, the *rigid difference shape type* is directly derived from $Rg$: We take $\Gamma_{\Delta Rg,c} = \mathbb{R}^{2(r+1)\times(c-1)c/2}$ and define

$$\begin{aligned} &\Delta Rg(M) = (m_j - m_i)_{0\leq i<j\leq c-1} \text{ if } 1 < c, \\ &\Delta Rg(M) = 0 \text{ if } c = 1. \end{aligned} \tag{22.4}$$

We shall usually represent the $i, j$-indexed entities in the Delta rigid difference vector as coefficients $\Delta Rg(M)_{i,j}$ of an upper triangular $c \times c$-matrix (starting at the upper codiagonal).

As a specialization of the rigid difference shape type, we may consider a selected $k$th codiagonal with index difference $j - i = k$. This *rigid k-difference shape type* will be denoted by $\Delta_k Rg$, we have $\Gamma_{\Delta_k Rg,c} = \mathbb{R}^{2(r+1)\times(c-k)}$, and

$$\Delta_k Rg(M) = (m_{i+k} - m_i)_{0\leq i<c-k} \text{ if } 1 < c,$$
$$\Delta_k Rg(M) = 0 \text{ if } c = 1. \tag{22.5}$$

#### 22.2.1.2 Diastematic Types

The diastematic type $t = Dia$ comes from classical musicology where one is often only interested in the direction of a melodic movement. The *diastematic shape type* is a 'child' of the rigid difference shape type. It resides on the index function[2] index(x) when applied to the coefficients of the rigid difference shapes. We take $\Gamma_{Dia,c} = \mathbb{Z}^{(r+1)\times(c-1)c/2}$ and define

$$Dia(M)_{i,j} = (index(\Delta Rg(M)_{i,j,s}))_{r+1<s\leq 2(r+1)} \tag{22.6}$$

with the $s$-index being the row number of the matrix. And we may evidently carry over the codiagonal procedure to the diastematic situation in defining $\Gamma_{Dia_k,c} = \mathbb{Z}^{(r+1)\times(c-k)}$, and

$$Dia_k(M) = (Dia(M)_{i,i+k})_{0\leq i<c-k} \text{ if } 1 < c,$$
$$Dia_k(M) = 0 \text{ if } c = 1. \tag{22.7}$$

This is the *diastematic k-difference shape type*, the special case of $k = 1$ is also called *diastematic index shape type*. For the special case $r = 0$, the diastematic shape type includes the COM matrix used in AST contour concepts, see [74, 380].

As a function of this data, we may shape contrapuntal step types as follows. Take the address $A = \mathbb{Z}$, a case which we have already illustrated in section 7.2.3 as being a common contrapuntal situation. We have the values $Dia_{i,j} = (u, v)$ which means that the cantus firmus movement from interval $i$ to interval $j$ has index $u$ whereas the discant movement has index $v$. Therefore, the product $u.v$ is positive iff we have parallel motion, it is zero iff we have oblique motion, and it is negative iff we have contrary motion. More generally, we may define the *contrapuntal motion shape type* $t = Contra$ as a 'child' of the diastematic shape type as follows: We take $\Gamma_{Contra,c} = \mathbb{Z}^{r\times(c-1)c/2}$ and put

$$Contra(M)_{i,j} = (Dia(M)_{i,j,s}.Dia(M)_{i,j,r+2})_{r+2<s\leq 2(r+1)},$$
$$Contra(M) = 0 \text{ if } c = 1. \tag{22.8}$$

In the case $r = 1$ we get the above contrapuntal example. As above, we may also project to the $k$-th codiagonal and consider the corresponding *contrapuntal k-motion shape type k-Contra*, the details are left to the reader. The special case of the 1-*Contra* shape type really gives us an abstraction which codifies exactly the successive contrapuntal motion type picture.

**Exercise 44** Try to define a *reduced diastematic shape type* by use of the above lexicographic order (end of section 22.1) on the pitch coordinate vectors and the diastematic values $-1, 0, +1$ if pitch decreases, is unchanged, or increases. Describe the shape spaces $\Gamma_{RedIndia,c}$ for the codiagonal projection of this reduced diastematic type.

[2] See appendix D.5.1.

#### 22.2.1.3 Elastic Type

From the cognitive point of view, it might be reasonable to concentrate on geometric configurations up to dilatations and translations. This type $t = Elast$ is codified as a 'child' of the rigid difference shape type modulo dilatations, i.e., the diagonal action of the multiplicative group $\mathbb{R}_+^\times$ on the coefficients of the matrix $\Delta Rg(M)$. The orbits of the matrices can be parametrized by special homogeneous coordinates. We observe that each matrix $\Delta Rg(M)$ has in its onset part (upper half) its positive first coefficient $o_{a(i,j),i,j}$, for each matrix position $(i,j) = (0,1),\dots(c-2,c-1)$ in (22.4). Take the unique matrix dilatation such that $\sum_j o_{a(i,j),i,j} = 1$, then we have a unique representation of that $\mathbb{R}_+^\times$-orbit. So we first fix a function $a : \{(0,1),\dots(c-2,c-1)\} \to \{0,1,\dots r\}$ and denote $\Sigma a = \sum_{i,j} a(i,j)$. We then define the smallest non-vanishing (positive) coefficients $o_{a(i,j),j}, (i,j) = (0,1),\dots(c-2,c-1)$ of the $(i,j)$ position, together with the unity sum condition. The other coefficients are all free. So the abstract motif space looks as follows. For a fixed function $a$, we have the vector of initial values $(o_{a(i,j),i,j})_j \in |\Delta_{c(c-1)/2-1}|^o$, the interior of the affine $c(c-1)/2-1$-simplex. The other free values are distributed below the initial values in the rigid difference matrix (22.4). Each position $(i,j)$ has $2(r+1) - (a(i,j)+1) = 2r - a(i,j) + 1$ free values, which add up to $\sum_{i,j} 2r - a(i,j) + 1 = (2r+1)(c-1) - \Sigma a$ and are therefore in $\mathbb{R}^{(2r+1)(c-1)-\Sigma a}$. This means that we have the shape spaces $\Gamma_{Elast,c} = |\Delta_{c(c-1)/2-1}|^o \times \coprod_a \mathbb{R}^{(2r+1)(c-1)-\Sigma a}$.

#### 22.2.1.4 Toroidal Type

For the toroidal shape type $t = Toroid_\gamma^{m,l}$, we restrict to the zero address. We are given two positive natural numbers $m, l$, and an *affine grid basis* $\gamma = (o, x, y) \in (Onset \oplus Pitch)^3$, consisting of an origin $o$, and a linear grid basis $x, y$ which define the grid[3] $G(\gamma) = o + \mathbb{Z}.x + \mathbb{Z}.y \subset Onset \oplus Pitch$. With respect to $\gamma$, every point $p \in Onset \oplus Pitch$ has a unique representation $p = \gamma(p) + \xi.x + \eta.y$ with $0 \le \xi, \eta < 1$ and $\gamma(p) \in \gamma$. We then associate with $p$ the grid point $p_\gamma = \gamma(p) + \xi.round(x) + \eta.round(y)$ (see definition 134 in appendix D.5.1 for the rounding function). Let $p_\gamma = o + a.x + b.y$ be the representation of $p_\gamma$ in the grid. Then we have the element $p_{\gamma,m,l} = (a \bmod m, b \bmod l) \in OnPiMod_{m,l}$ whose coordinates are in the discrete torus $\mathbb{Z}_m \oplus \mathbb{Z}_l$ as discussed in formula (6.43) of section 6.4.1. If a motif $M \in MOT_{F,n}$, this defines a local composition $Toroid_\gamma^{m,l}(M) = \{p_{\gamma,m,l} |\ p \in M_{OP}\}$. It is contained in the set $\Gamma_{Toroid,n} = ObLoc_0^{\le n, OnPiMod_{m,l}}$ of local, zero-addressed compositions in $OnPiMod_{m,l}$ with cardinality at most $n$, and we have a shape map

$$Toroid_\gamma^{m,l} : MOT_{F,n} \to \Gamma_{Toroid,n} \tag{22.9}$$

on the given cardinality level $n$.

**Exercise 45** Observe that in this case, the shape spaces are not disjoint, but the inclusion assumption 1, section 22.2.1, is verified.

By use of the rigid type, we can also define a *toroidal sequence shape type* $TorSeq$ as follows. Again, we fix an affine grid $\gamma$ and a pair $m, l$ of positive integers. Take the (zero-addressed) rigid abstract motif $Rg(M) = (m_0, \dots m_{c-1})$, then the sequence $TorSeq(M) = (m_{0,\gamma,m,l}, \dots m_{c-1,\gamma,m,l}) \in \Gamma_{TorSeq,c} = 0@(OnPiMod_{m,l})^c \tilde{\to} (\mathbb{Z}_m \oplus \mathbb{Z}_l)^c$.

[3] Recall that the underlying module of $Onset \oplus Pitch$ is $\mathbb{R}^2$.

## 22.3 Metrical Similarity

**Summary.** This section deals with the metrical similarity of abstract motives. This concept is the base of motivic topologies used to understand Reti's thinking.

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The abstraction process which yields the shape space $\Gamma_t$ is only the first identification step. In fact, after this shape abstraction, we need to deal with similarity of motives. This relation will be defined on abstract motives and then retracted to the original motives in the following sense:

**Definition 67** *Given a shape type* $t : A@MOT_F \to \Gamma_t$, *a* distance function *is a sequence* $d = (d_n)_{1\leq n}$ *of pseudo-metrics* $d_n$ *which are defined on* $\Gamma_t|_n$. *A* $t$-distance function *is the sequence* $d_t = (d_{t,n} = d_n|A@MOT_F|_n)_{1\leq n}$ *of pseudo-metrics which are induced on the spaces* $A@MOT_F|_n$ *by the distance function* $d$, *i.e., if* $M, N \in A@MOT_F|_n$, *then* $d_{t,n}(M, N) = d_n(t(M), t(N))$. *If no confusion is possible, we omit the index* $n$ *and just write* $d_t(M, N)$.

### 22.3.1 Examples of Distance Functions

Let us now discuss distance functions for shape types which have been introduced in section 22.2.1.

#### 22.3.1.1 Distances for Rigid Types

The shape spaces for rigid types (subsubsection 22.2.1.1) are real matrix spaces $\Gamma_{Rg,n} = \mathbb{R}^{2(r+1)\times c}$ and $\Gamma_{\Delta Rg,n} = \mathbb{R}^{2(r+1)\times(c-1)c/2}$. On such a space, we have the usual Euclidean metric which defines the Euclidean distance function $Ed$ and the corresponding Euclidean $Rg$- or $\Delta Rg$-distance function $Ed_{Rg}, Ed_{\Delta Rg}$.

We may also consider the abstract cardinality and define the *relative Euclidean distance function*: Let $abcard(x) = abcard(y) = n$, for two abstract motives, then we set $REd(x, y) = Ed(x, y)/n$ for rigid type $Rg$, and $REd(x, y) = Ed(x, y)/(n - 1)$ for shape type $\Delta Rg$. Such a relative distance takes into account that on the *cognitive level*, we would like to say that the distance is a kind of mean value of all the point distances between the first and first, second and second, etc., last and last points of two motives. This reason also holds for the other relative distances to be defined in the sequel.

#### 22.3.1.2 Distances for Diastematic Types

For the diastematic types (subsubsection 22.2.1.2), the abstract motives also lie in matrix spaces, this time with integer coordinates, and we may also take the Euclidean distance function induced from the surrounding real matrix spaces.

A relative Euclidean distance function on $\Gamma_{Dia}$ which relates to the number of components in the triangular matrix is given by $REd(x, y) = 2Ed(x, y)/(r + 1)c(c - 1)$. For the codiagonal shape types $Dia_k$, we set $REd(x, y) = Ed(x, y)/(r+1)(c-k)$. For the contrapuntal shape type $Contra$, we set $REd(x, y) = 2Ed(x, y)/rc(c - 1)$.

#### 22.3.1.3 Distances for Elastic Type

As in the previous situations, we can apply the Euclidean distance function to the elastic shape type (subsubsection 22.2.1.3) since its abstract motives are also embedded in a real matrix space. In this case, it may be more adequate to take the metric which is deduced[4] from the 1-norm on n-space since in this case, the component of initial values in the affine simplex has just length 1. Since it is well-known[5] that the topology on n-space is independent of the defining norm, this will have no influence on the following topological considerations.

A relative Euclidean or 1-norm distance can be defined via a division factor $2(r+1)(c-1)$ which takes into account the number of motif points and the dimension of the address-related columns.

#### 22.3.1.4 Distances for Toroidal Types

On the discrete torus $\mathbb{Z}_m \oplus \mathbb{Z}_l$, one defines a metric $steps(x, y)$ by the minimal number of grid steps on all possible walks from $x$ to $y$ which is a kind of geodesic distance on this discrete torus. On the shape space $\Gamma_{TorSeq,c} \xrightarrow{\sim} (\mathbb{Z}_m \oplus \mathbb{Z}_l)^c$, we may take the norm, Euclidean, or uniform distance construction. Given two sequences $x = (x_i), y = (y_i) \in \Gamma_{TorSeq,c}$, we accordingly set

$$d_{1,c}(x, y) = \sum_i steps(x_i, y_i), \tag{22.10}$$

$$d_{2,c}(x, y) = \sqrt{\sum_i steps(x_i, y_i)^2}, \tag{22.11}$$

$$d_{\infty,c}(x, y) = max\{steps(x_i, y_i) |\ i = 0, \ldots c-1\} \tag{22.12}$$

and get a corresponding distance on toroidal sequences. This construction is also used in counterpoint theory, see chapter 30.

For the *Toroid* shape type, one can define a distance via the lexicographic ordering among torus points, thereby ordering a local composition lexicographically and then applying the above distance functions for the lexicographic sequences of abstract motives ([73]).

**Exercise 46** Fill in the details of the *Toroid* distance function and verify that it is a pseudometric.

## 22.4 Paradigmatic Groups

**Summary.** Together with abstraction mechanisms, groups of symmetries act on spaces of abstract motives and—in several cases—on the proper motif spaces. These groups are termed *paradigmatic* groups since they relate to Jean-Jacques Nattiez' paradigmatic theme, as discussed in 11.7.1. An important case are those actions which are *equivariant* with respect to the shape maps. The formal definitions of *gestalts* are associated with specific abstractions and group actions. This demonstrates that the concept of gestalt is the result of a multilayered concept hierarchy involving several non-automatic parameter choices.

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[4] See appendix I.1.1.
[5] See appendix I.1.2.

As a matter of fact, comparison of motives is not restricted to abstraction and similarity, we know from the general discussion of paradigmatic concepts in chapter 10 that the topological similarity which has been induced (though not executed, this is the scope of the topological theory of motives in section 22.6) must be complemented by the transformational similarity. The latter deals with group actions on shape types in the following sense:

**Definition 68** *A* paradigmatic group $P$ for a shape type $t : A@MOT_F \to \Gamma_t$ *is a left action* $\pi : P \times \Gamma_t \to \Gamma_t$ *on the shape space which leaves the components* $\Gamma_{t,n}$ *invariant.*

*An* equivariant paradigmatic group $P$ for a shape type $t : A@MOT_F \to \Gamma_t$ *is a pair of group actions*

$$\gamma : P \times A@MOT_F \to A@MOT_F, \pi : P \times \Gamma_t \to \Gamma_t$$

*which is equivariant*[6] *with respect to the shape map* $t$ *and such that* $P$ *is a paradigmatic group for* $t$*.*

We also write

$$\begin{array}{ccc} A@MOT_F & \xrightarrow{t} & \Gamma_t \\ \circlearrowleft_{P:\gamma} & & \circlearrowleft_{P:\pi} \end{array}$$

or more concisely (if the actions must not be specified)

$$A@MOT_F \xrightarrow[\circlearrowleft_P]{t} \Gamma_t$$

for this situation.

**Lemma 34** *If* $P$ *is a paradigmatic group for type* $t$*, then the abstract cardinality is an invariant for abstract motives, i.e., the group action leaves invariant the components* $\Gamma_t|_n$*.*

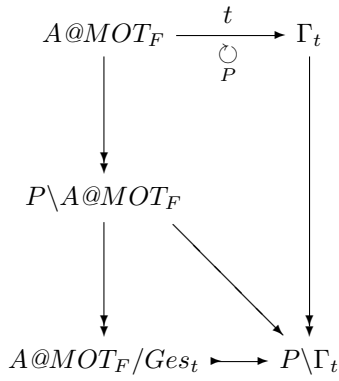
The proof is an easy exercise. We are now ready to define the gestalt concept.

**Definition 69** *The orbits of a paradigmatic group* $P$ *action on* $\Gamma_t$ *are called the* abstract gestalts of shape type $t$*. The inverse images* $t^{-1}G$ *of* abstract gestalts $G$ of $t$ *are called the* $t$-gestalts*. If* $M \in A@MOT_F$*, then its* $t$*-gestalt* $t^{-1}(P.t(M))$ *is denoted by* $Ges_t^P(M)$ *or* $Ges_t(M)$ *if* $P$ *is clear.*

**Sorite 7** *Let* $P$ *be a paradigmatic group for shape type* $t$*. Then every* $t$*-gestalt* $Ges_t(M)$ *is contained in* $\Gamma_t|_{abcard_t(M)}$*. If the paradigmatic group is equivariant, then the orbit of a motif* $M$ *is contained in the gestalt:* $P.M \subset Ges_t(M)$*, i.e., a gestalt is a disjoint union of orbits. We*

[6] See appendix C.3.1, example 70.

*therefore have the commutative diagram*




This follows immediately from the definitions and lemma 34.

**Definition 70** *The* cardinality of a gestalt $Ges_t(M)$ *is defined as* $abcard_t(N)$ *for any* $N \in ges_t(M)$.

## 22.4.1 Examples of Paradigmatic Groups

Here are some group actions which are of practical use. Preliminary remark: For any group $P \subset \overrightarrow{GL}(\mathbb{R}^2)$ of affine transformations on $Onset \oplus Pitch$, one has the canonical induced action $\gamma$ on the onset and pitch coordinates of motives in $A@MOT_{F,n}$. Many paradigmatic actions are defined as equivariant actions which issue from this canonical action on the proper motives, i.e., one first transports the motivic action to a motivic shape action and then tries to generalize the latter to any abstract motif.

The most prominent group is the *affine counterpoint group* $CP = T_{2,\mathbb{R}} \rtimes LCP$, the group which is generated by the normal subgroup $T_{2,\mathbb{R}} = e^{\mathbb{R}^2}$ of translations and the linear counterpoint group $LCP = \langle U, K \rangle \tilde{\to} K_4$ which is a Klein group[7].

### 22.4.1.1 Paradigmatic Groups for Rigid Types

For the rigid shape type $Rg : \mathbb{Z}^r@MOT_F \to \Gamma_{Rg}$, we have the above canonical action of $p \in CP$ on a motif $M$: $p(M) = \{p(m) |\ m \in M\}$. Evidently, the retrograde motion reverses the time order of elements in $M$. Therefore we can define a $CP$-action $\pi$ on $\Gamma_{Rg,c}$ by the rule

$$p(x_0, \dots x_{c-1}) = \begin{cases} (p(x_0), p(x_1), \dots p(x_{c-1})) & \text{if } p \text{ does not have the retrograde component,} \\ (p(x_{c-1}), p(x_{c-2}), \dots p(x_0)) & \text{(reversed order) else.} \end{cases} \tag{22.13}$$

[7] The four-element group of plane symmetries generated by inversion $U$ (reflection at zero pitch) and retrograde $K$ (reflection at zero onset), see also section 8.1.1.

Clearly, this defines an equivariant action

$$\underset{\circlearrowleft}{\mathbb{Z}^r@MOT_F}_{CP:\gamma} \xrightarrow{Rg} \underset{\circlearrowleft}{\Gamma_{Rg}}_{CP:\pi}$$

for rigid shape type.

For the rigid difference shape type, the affine action of $CP$ reduces to the linear action, i.e., we have the quotient action $\Delta\pi$ via an element $q \in LCP \xrightarrow{\sim} CP/T_{2,\mathbb{R}}$ on $\Gamma_{\Delta Rg,c}$ by the following rule. If we write $d = (d_o, d_p)$ for the onset-pitch decomposition, and $U(d) = (d_o, -d_p)$ we have

$$q((d_{i,j})_{i,j}) = \begin{cases} (U(d_{i,j}))_{i,j} & \text{if } q = U, \\ (d_{c-1-j,c-1-i})_{i,j} & \text{if } q = K. \end{cases} \tag{22.14}$$

This gives the equivariant action

$$\underset{\circlearrowleft}{\mathbb{Z}^r@MOT_F}_{CP:\gamma} \xrightarrow{\Delta Rg} \underset{\circlearrowleft}{\Gamma_{\Delta Rg}}_{CP:\Delta\pi}$$

for rigid difference shape type.

#### 22.4.1.2 Paradigmatic Groups for Diastematic Types

The canonical action of the affine counterpoint group on the motives also carries over to an equivariant action on the diastematic shape types. Since the latter are children of the rigid difference shape type, we have to see if we can have an equivariant action

$$\underset{\circlearrowleft}{\Gamma_{\Delta Rg}}_{LCP:\Delta\pi} \xrightarrow{index} \underset{\circlearrowleft}{\Gamma_{Dia}}_{LCP:Dia\pi}$$

between rigid difference and diastematic shape types (in this direction!). Again, we may define the action $Dia\pi$ on the generators and set

$$q((ind_{i,j})_{i,j}) = \begin{cases} ((-ind_{i,j}))_{i,j} & \text{if } q = U, \\ ((-ind_{c-1-j,c-1-i}))_{i,j} & \text{if } q = K \end{cases} \tag{22.15}$$

for a diastematic index family $(ind_{i,j})_{i,j}$. Evidently, this action makes the *index* map equivariant. So as above, the quotient $CP \to LCP$ defines the equivariant action

$$\underset{\circlearrowleft}{\mathbb{Z}^r@MOT_F}_{CP:\gamma} \xrightarrow{Dia} \underset{\circlearrowleft}{\Gamma_{Dia}}_{CP:Dia\pi}$$

of the affine counterpoint group in the diastematic case. This clearly carries over to the different codiagonal projections since the codiagonals are invariant under the retrograde action.

**Exercise 47** Define an equivariant action of the affine counterpoint group on the contrapuntal motion shape type by use of the diastematic action.

#### 22.4.1.3 Paradigmatic Groups for Elastic Type

The paradigmatic counterpoint group action is a bit more delicate for the elastic type since we have to take care of the initial value functions $a$ and the change of these function under the counterpoint group. As the elastic type is a child of the rigid difference type, we again want to define an equivariant action of the linear counterpoint group $LCP$ for the elastic map

$$\begin{array}{ccc} \Gamma_{\Delta Rg} & \xrightarrow{elast} & \Gamma_{Elast} \\ \circlearrowleft_{LCP:\Delta\pi} & & \circlearrowleft_{LCP:Elast\pi} \end{array}$$

To begin with, the inversion $U$ leaves the initial value function $a$ invariant, whereas the retrograde $K$ maps $a$ to $K.a$ with $K.a(i,j) = a(c-1-j, c-1-i)$. Since the multiplicative action of $\mathbb{R}_+^\times$ commutes with any linear action, and since the condition $\sum_j a(j) = 1$ is invariant under the linear counterpoint group, we can take the rigid difference action on the homogenous coordinates in the elastic shape spaces $\Gamma_{elast}$, together with the above initial value function transformation. The $K$-action permutes the summands of the disjoint union $\coprod_a \mathbb{R}^{(2r+1)(c-1)-\Sigma a}$ according to the rigid difference action and the operation $K.a$ on the disjoint union "index" $a$.

**Exercise 48** Fill in the details of the previous elastic action.

#### 22.4.1.4 Paradigmatic Groups for Toroidal Types

This case is a bit different. There is no analogous equivariant action of the affine counterpoint group for toroidal types since the rounding function in the affine frame quantization is not compatible with the inversion and retrograde operations (check this!). We however have an action of the affine group (or any subgroup) $G_{m,l} = \overrightarrow{GL}(\mathbb{Z}_m \oplus \mathbb{Z}_l)$ of the torus $\mathbb{Z}_m \oplus \mathbb{Z}_l$ on the abstract motives. In fact, if $Q \in \Gamma_{Toroid}$, the pointwise action $\pi_{m,l}$ of $g \in G_{m,l}$, $g.Q = \{g.q |\ q \in Q\}$ gives the action on abstract toroidal motives. So here the abstract gestalts are the isomorphism classes of local compositions.

For the toroidal sequence shape type, we have the usual coordinatewise action $\pi_{m,l}$ of $g \in G$ on a sequence $x. = (x_i)_i$ via $g.x. = (g.x_i)_i$.

## 22.5 Pseudo-metrics on Orbits

**Summary.** Metrical similarity between abstract motives is extended to gestalts which essentially refer to orbits of abstract motives under paradigmatic groups.

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It is not sufficient to have metrical relations among the abstract and real motives, we also want to tell what it means that two gestalts are metrically neighboring. To this end, we need a preliminary knowledge about actions of groups of isometries. We know from appendix I.1.1, lemma 98, that if a pseudo-metric $d$ and an action of a group $G$ by isometries[8] are given on a space $V$, then the orbit space $G\backslash V$ is naturally provided with a pseudo-metric $d^*$.

[8] See appendix I.1.1, definition 173.

So we only have to look for isometric actions in our examples, and we obtain a pseudo-metric on the abstract gestalt spaces. Let us now check where this is the case.

1. **Rigid Types**.

   For the rigid shape type, the distance between two matrices is the Euclidean norm of their difference. On the other hand, the inversion and retrograde are isometries on the onset-pitch space, therefore, the counterpoint group $CP$ acts by isometries.

   The same is valid for the rigid difference shape type which has only its triangular matrix entries permuted and is a priori invariant under translations; the same holds for the codiagonal projections and relative distances.

2. **Diastematic Types**.

   The counterpoint group acts via its linear projection $LCP = \langle U, K \rangle$ on the diastematic shape type. Thereby, only signs are altered and indexes of coefficients are permuted (equation (22.15)). This leaves the Euclidean distance invariant. Same result for the codiagonal projections and relative distances.

3. **Elastic Type**.

   Since the action of $LCP$ on the elastic type is the action of this group on the rigid difference shape type with the corresponding homogeneous coordinates, $LCP$ also acts by isometries on the elastic type.

4. **Toroidal Types**.

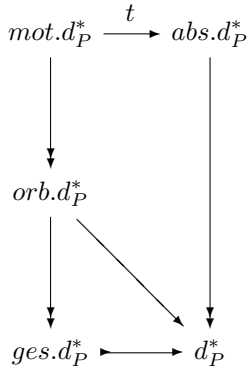
   Here, we have to observe the step number between two points on the torus $\mathbb{Z}_m \oplus \mathbb{Z}_l$ when an affine automorphism is applied. In general, this not an invariant. For instance, we have the fourth multiplication on $\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$ (diagonal matrix with 5 in its pitch direction) which sends a fifth (minimal number of 5 steps) to the minor second (one step). However, if the numbers $m, l$ are relatively prime and the Euler functions[9] are $\phi(m), \phi(l) \leq 2$, then the minimal step number is conserved. This function value is the case for a very small number of cases: $m, l = 2, 3, 4$, and the combinations $(m, l) = (2, 3), (3, 2), (3, 4), (4, 3)$.

   In this shape type, we should therefore concentrate on selected subgroups of the general affine group in order to preserve distances. One such subgroup is the $LCP$ which acts in the canonical way on the factors.

After these case studies we can define what is an orbit pseudo-metric for shape types. We say that a paradigmatic group $P$ for a shape type $t$ with distance $d$ *consists of isometries* if $P$ acts as group of isometries on each $d_n$-pseudo-metric shape space $\Gamma_t|_n$. If this is the case, we have *a pseudo-metric* $d^*_{P,n}$ *on each orbit space* $P\backslash\Gamma_t|_n$. In a more sloppy language we shall also say that we have a pseudo-metric $d^*_P$ on the abstract gestalt space $P\backslash\Gamma_t$ when we think of the sequence of pseudo-metrics $(d^*_{P,n})_n$. We shall also use this convention for all pseudo-metrics induced on the following spaces: Reconsider the diagram just before section 22.4.1. By retraction of the abstract gestalt pseudo-metric $d^*_P$ to all the codomains, we obtain a system of pseudo-metrics which we denote as indicated in the corresponding diagram which now automatically

[9] See appendix C.3.4.1, proposition 72.

turns into a diagram of (sequences of) pseudo-metrics




where we have omitted the underlying (real or abstract) motif sets.

## 22.6 Topologies on Gestalts

**Summary.** The overall comparison of gestalts (of different cardinalities) is conceived. It is built upon the *inheritance property* which essentially captures the psychological fact that recognition of motives often amounts to recognition of their submotives.

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We are now ready to introduce topologies on motif and gestalt spaces. Except for the toroidal type, this approach combines the elements of gestalt and metrical similarity while comparing motives. In the toroidal situation, we shall however introduce a more algebro-geometric idea to topology which deals with specialization instead of metrics.

### 22.6.1 The Inheritance Property

**Summary.** This section is devoted to the formal discussion of the inheritance property and to the study of different shape types with respect to this property.

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Though metrical properties suggest that we may have a common topology on $A@MOT_F$ defined by use of pseudo-metrics, it turns out that there is an important restriction to this program, a restriction which is of an evident cognitive semantic. We postpone this rationale and first present the formal aspect.

**Definition 71** *Given a shape type $t : A@MOT_F \to \Gamma_t$, together with a distance function*[10] *$d$ on $\Gamma_t$, we say that this data* has the inheritance property *iff for each motif $M \in A@MOT_F|_n$, any submotif $M^* \subset M$, and any real number $\epsilon > 0$, there is a real number $\delta > 0$ such that if $N \in A@MOT_F|_n$ is such that $d_t(M, N) < \delta$, then there is a submotif $N^* \subset N$ such that $abcard(M^*) = abcard(N^*)$ and $d_t(M^*, N^*) < \epsilon$.*

[10] Recall from definition 67 that this induces a $t$-distance function $d_t$ on $A@MOT_F$.

Intuitively speaking, *neighboring motives have neighboring submotives*. An equivalent definition states that a particular function on the motives is continuous: Suppose that for two motives, $M^*, N$, $abcard(M^*) \leq abcard(N) = n$. We define

$$Sub_{M^*}(N) = \{N^* \subset N |\ abcard(N^*) = abcard(M^*)\},$$

the set of submotives of $N$ with fixed abstract $t$-cardinality $abcard(M^*)$. Consider the function

$$d_{t,M^*,n} : A@MOT_F|_n \to \mathbb{R} \tag{22.16}$$

defined by $d_{t,M^*,n}(N) = Min_{N^* \in Sub_{M^*}(N)}(d_t(M^*, N^*))$. Then the inheritance property reads

**Definition 72** *Given a shape type $t : A@MOT_F \to \Gamma_t$, together with a distance function $d$, we say that this data* has the inheritance property *iff for any motif $M^*$ and any $abcard(M^*) \leq n$, the function $d_{t,M^*,n}$ is continuous in every supermotif $M$ of $M^*$ (i.e., $M^* \subset M$ of abstract cardinality $n$ with respect to the pseudo-metric topology of $d_n$).*

In fact, suppose that the second version of the inheritance property holds. Then $d_{t,M^*,n}$ is continuous in any supermotif $M$ of $M^*$ of abstract cardinality $n$. But then, $d_{t,M^*,n}(M) = 0$, and for a given positive $\epsilon$ there is $\delta$ such that $d_t(M, N) < \delta$ implies $|d_{t,M^*,n}(N) - d_{t,M^*,n}(M)| = d_{t,M^*,n}(N) < \epsilon$, and this is the claim of the first version of the inheritance property. The converse is evident.

This property is not automatic. An illustrative 'pathology' is the diastematic $k$-difference shape type $t_{Dia_k}$. Let us take the codiagonal difference $k = 1$, i.e., the diastematic index shape type, see figure 22.1.

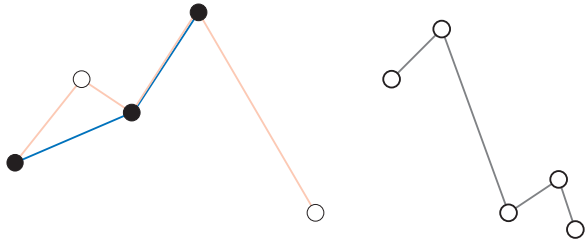

Figure 22.1: Two motives $M, N$ which are at diastematic distance zero but $N$ has no submotif at this distance which corresponds to the submotif $M^*$ drawn with black points.

Here, we work in $MOT_n$, and we look at the motif $M \in MOT_5$ with $Dia_1(M) = (1, -1, 1, -1)$, and the submotif $M^* \subset M$ with $abcard(M) = 3$ and $Dia_1(M^*) = (1, 1)$. Take the motif $N \in MOT_5$ with $Dia_1(N) = Dia_1(M)$. The distance is $d_t(M, N) = 0$, but $N$ has no submotif $N^*$ with $Dia_1(N^*) = Dia_1(M^*)$, and so the inheritance property cannot be fulfilled. For rigid and elastic types we have affirmative results:

**Proposition 25** *Rigid shape types $Rg, \Delta Rg$, and corresponding Euclidean distance functions $Ed_{Rg}, Ed_{\Delta Rg}$ have the inheritance property.*

Proof. This is evident for the $Rg$-type since the passage to a nearby submotif of $N$ is reflected by a projection onto a submatrix of the rigid image $Rg(N)$ which is defined by selection of those columns which are defined by the submotif $M^*$. And this projection is continuous. On the $\Delta Rg$-type, we have to observe that the matrix columns of the $\Delta Rg$-abstractions of submotives are given by summing up the $Rg$-difference vectors of their supermotives. But this is a continuous operation, and we are done. QED.

Observe that this property and the following are independent of the chosen metric on the rigid or elastic shape spaces, since the continuity is only a topological property, and we know that the topology on the real $n$-space is independent of the defining norm.

**Proposition 26** *Elastic shape type and Euclidean distance has the inheritance property.*

Proof. The elastic shape type is derived from the rigid difference type by normalization of the initial value function $a$ to $\Sigma a = 1$. This means that we just divide the rigid difference matrix by $\Sigma a$, and this is a continuous function on the interior $|\Delta_{c(c-1)/2-1}|^o$ of the affine $c(c-1)/2-1$-simplex which defines the shape space, see 22.2.1.3. QED.

**Proposition 27** *Diastematic shape type Dia and Euclidean distance has the inheritance property.*

**Exercise 49** Give a proof of proposition 27.

**Proposition 28** *If we have an equivariant paradigmatic group action*

$$A@MOT_F \xrightarrow[\overset{\curvearrowright}{P}]{t} \Gamma_t$$

*which consists of isometries with respect to the distance function $d$, and if the shape type $t$ and $d$ have the inheritance property, then the gestalt distance $mot.d_P^*$ and shape type $t$ also has the inheritance property.*

Proof. Clearly, we have $(mot.d_P^*)_{t,M^*,n}(N) \leq d_{t,M^*,n}(N)$, and then, continuity of $d_{t,M^*,n}$ in supermotives of $M^*$, where the function value is zero, implies continuity of $(mot.d_P^*)_{t,M^*,n}$ in supermotives of $M^*$. QED.

### 22.6.2 Cognitive Aspects of Inheritance

**Summary.** This section makes the inheritance property plausible from the psychological and cognitive point of view.

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Before we proceed to the construction of motivic topologies, we should briefly inspect the cognitive relevance of the basic inheritance property. Recognition of similarity of motives is a complex task of gestalt recognition. Little if anything is known about recognition of auditory gestalts, in particular, it is not known which shape type could be a cognitively relevant one,

and whether there is a kind of combination of different shape types introduced above which could represent good cognitive information.

A particular problem of auditory gestalt recognition is the transitory nature of auditory gestalts: in the common situation of a piece being presented to a listener, motives pass by very fast and comparison of such transient objects must be an economic process, typically taking a time interval around 0.5 *sec*. Now, similarity of longer motives, consisting of eight tones, say, is not likely to be perceived as an irreducible entity. Fast processing must rely on elementary gestalt aspects of such motives. Mathematically speaking, it is plausible that gestalt perception breaks down to the perception of an interpretation of such motives, i.e., to the perception of motivic interpretations, i.e., melodies in the technical sense of global motivic compositions.

If we assume that we are given melodies $M^I$ which are covered by motives $M_i$ of a limited small cardinality $c$, then similarity of motives $M$ can only be mediated via similarity of their charts $M_i$. So the present hypothesis is that similarity perception of large motives $M$ breaks down into two subtasks: similarity perception of their charts $M_i$, and perception of the nerve $n(M^I)$. In this framework, the chart similarity check is related to the inheritance property. In fact, the latter means that similarity of charts is a necessary condition for similarity of the big motives. So if our cognitive performance is not strong enough to check similarity among big motives, it can at least make the chart check and thereby learn whether a necessary condition for similarity of big motives holds.

Now, this necessary condition takes place iff the inheritance property holds: This is the cognitive interpretation of this topological property.

### 22.6.3 Epsilon Topologies

**Summary.** Topologies on motif and gestalt spaces are introduced in case we are provided with a pseudo-metric and the inheritance property is fulfilled.

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To begin with, we define the 'open discs' which will yield a base for a motivic topology:

**Definition 73** *If we have an equivariant paradigmatic group action*

$$A@MOT_F \xrightarrow[\overset{\curvearrowright}{P}]{t} \Gamma_t$$

*which consists of isometries for the distance function d, if* $M \in A@MOT_F$*, and if* $\epsilon$ *is a positive real number, the* $\epsilon$-neighborhood of $M$ *is the set*

$$D_\epsilon(M) = \{N \in A@MOT_F |\ (mot.d_P^*)_{t,M}(N) < \epsilon\} \tag{22.17}$$

*which implicitly means that the candidates* $N$ *have abstract cardinality at least equal to* $abcard(M)$*.*

**Proposition 29** *If we have an equivariant paradigmatic group action*

$$A@MOT_F \xrightarrow[\overset{\curvearrowright}{P}]{t} \Gamma_t$$

*which consists of isometries for the distance function d and such that the inheritance property is fulfilled, then the system of the $\epsilon$-neighborhoods $D_\epsilon(M)$, all $\epsilon, M$, is a base of a topology $\mathfrak{T}_{t,P,d}$ on $A@MOT_F$. This topology is called the* epsilon topology *(for the data $t, P, d$).*

Proof. Let $D_{\epsilon_1}(M_1), D_{\epsilon_2}(M_2)$ be two $\epsilon$-neighborhoods and take a motif $O \in D_{\epsilon_1}(M_1) \cap D_{\epsilon_2}(M_2)$. We must look for an $\epsilon_3 > 0$ such that $D_{\epsilon_3}(O) \subset D_{\epsilon_1}(M_1)$. A similar argument yields an $\epsilon_4 > 0$ which does the job for $M_2$, and the smaller of the two epsilons solves the problem.

By construction, we have $(mot.d_P^*)_{t,M_1}(O) = q < \epsilon_1$. Take a submotif $O_1 \subset O$ with $q = mot.d_P^*(M_1, O_1)$. By the inheritance property, we find a positive $\epsilon_3$ such that $mot.d_P^*(O, Q) < \epsilon_3$ implies $(mot.d_P^*)_{t,O_1}(Q) < \epsilon_1 - q$. Then clearly, $D_{\epsilon_3}(O) \subset D_{\epsilon_1}(M_1)$. QED.

The next concepts relate gestalts and their representatives to motif inclusion:

**Definition 74** *If $P$ is a paradigmatic group for shape type $t$, we say that* gestalts behave well for $t$ *iff for every submotif pair $M^* \subset M$, and motif $M_1 \in Ges_t(M)$, there is a submotif $M_1^* \subset M_1$ with $M_1^* \in Ges_t(M^*)$.*

**Definition 75** *Let $G^*, G$ be two gestalts for shape type $t$. Then we say that $G^*$* is a small gestalt in $G$ *and write $G^* \sqsubset G$ iff there are motives $M^* \in G^*, M \in G$ such that $M^* \subset M$.*

The proofs of the following two propositions are left as an exercise for the reader:

**Proposition 30** [73, Prop.3] *If $P$ is a paradigmatic group for shape type $t$ which behaves well, then the small gestalt relation $\sqsubset$ is reflexive, transitive and antisymmetric*[11]*.*

**Proposition 31** [73, Prop.8] *Suppose that we are given an equivariant paradigmatic group action*

$$A@MOT_F \overset{t}{\underset{P}{\circlearrowright}} \Gamma_t$$

*which consists of isometries for the distance function d, such that the inheritance property is fulfilled, and such that the pseudo-metric $ges.d_P^*$ on $A@MOT_F/Ges_t$ is a metric. Then gestalts behave well for $t$.*

**Exercise 50** Show that the action of the affine counterpoint group $CP$, or more generally the equivariant action of any group $P$ with finite orbits in the shape space, defines a metric $ges.d_P^*$ on $A@MOT_F/Ges_t$. In particular, the described actions of $CP$ on rigid difference, diastematic, and elastic types have this property.

**Corollary 7** [73, Prop.9] *Given the epsilon topology $\mathfrak{T}_{t,P,d}$ on $A@MOT_F$, consider the quotient topology $\mathfrak{T}_{t,P,d}/Ges_t$ on the gestalt space $A@MOT_F/Ges_t$ for the gestalt mapping $Ges_t : A@MOT_F \twoheadrightarrow A@MOT_F/Ges_t$. Suppose that $ges.d_P^*$ is a metric. Then $Ges_t$ is open and the system*

$$D_\epsilon(H) = \{G \in A@MOT_F/Ges_t |\ \exists G^* \sqsubset G, ges.d_P^*(G^*, H) < \epsilon\}, H \in A@MOT_F/Ges_t, 0 < \epsilon,$$

[11] See also appendix C.2.

*forms a base for* $\mathfrak{T}_{t,P,d}/Ges_t$. *More precisely, this follows from the formulas*

$$Ges_t(D_\epsilon(M)) = D_\epsilon(Ges_t(M)), \tag{22.18}$$

$$Ges_t^{-1}(Ges_t(D_\epsilon(M))) = D_\epsilon(M). \tag{22.19}$$

The topology $\mathfrak{T}_{t,P,d}/Ges_t$ is called the *epsilon gestalt topology*. The following is immediate from the previous constructions:

**Corollary 8** *For the epsilon topology* $\mathfrak{T}_{t,P,d}$ *on* $A@MOT_F$ *and the epsilon gestalt topology* $\mathfrak{T}_{t,P,d}/Ges_t$ *on* $A@MOT_F/Ges_t$, *and for a positive integer* $n$, *the relative topology*[12] $\mathfrak{T}_{t,P,d}|_n$ *on* $A@MOT_F|_n$ *is the pseudo-metric topology of* $mot.d_P^*|_n$, *whereas the relative topology on* $\mathfrak{T}_{t,P,d}/Ges_t|_n$ *on* $A@MOT_F|_n/Ges_t$ *is the pseudo-metric topology of* $ges.d_P^*|_n$.

**Exercise 51** Use assumption 1, section 22.2.1, to show that $N \in D_\epsilon(M)$ implies $abcard(M) \leq abcard(N)$. It follows that $\bigcup_{n\leq i} A@MOT_F|_i$ is open for any positive index $n$.

**Exercise 52** Show that $N \in D_\epsilon(M)$ implies $Ges_t(N) \subset D_\epsilon(M)$.

**Exercise 53** Show that in the epsilon topology, if we have a submotif $M^* \subset M$ and $0 < \epsilon$, then there is $0 < \delta$ such that $D_\delta(M) \subset D_\epsilon(M^*)$.

## 22.7 First Properties of the Epsilon Topologies

**Summary.** For the total motif space $A@MOT_F$ as well as for the motif space $\mu$ associated with a motivic interpretation $X^\mu$ of a finite local composition $X$, some elementary properties of motivic topologies are discussed.

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Suppose that we are given the motivic topology $\mathfrak{T}_{t,P,d}$ as introduced above. Whenever a motivic interpretation $X^\mu$ of a finite local composition $X$ is given according to definition 65, section 22.1, the motif space $\mu \subset A@MOT_F$ will be given the relative topology $\mathfrak{T}_{t,P,d}|\mu$ which we also call the *epsilon topology*. Moreover, the image $\mu/Ges_t$ of $\mu$ under the gestalt map $Ges_t$ is also given the relative topology $\mathfrak{T}_{t,P,d}/Ges_t|\mu/Ges_t$. To ease notation, we shall henceforth write $\mathfrak{T} = \mathfrak{T}_{t,P,d}, \mathfrak{T}_\mu = \mathfrak{T}_{t,P,d}|\mu$, and $\mathfrak{T}/Ges = \mathfrak{T}_{t,P,d}/Ges_t, \mathfrak{T}_{\mu/Ges} = \mathfrak{T}_{t,P,d}/Ges_t|\mu/Ges_t$ if the background assumptions are unambiguous.

**Assumption 2** *In this section we always assume that we are given a type* $t$, *a pseudo-metric* $d$ *on the space of* $t$, *an equivariant paradigmatic group* $P$ *for* $t$, *which acts by isometries. We further assume that the inheritance property is verified and such that the gestalt distance* $ges.d_P^*$ *is in fact a metric.*

**Proposition 32** [73, Prop.11] *The epsilon topologies* $\mathfrak{T}/Ges$ *and* $\mathfrak{T}_{\mu/Ges}$ *are* $T_0$[13].

[12] See appendix H.1.2.
[13] See appendix H.1.4, definition 167.

**Corollary 9** *Irreducible subsets in the epsilon topologies $\mathfrak{T}/Ges$ or $\mathfrak{T}_{\mu/Ges}$ have at most one generic*[14] *point.*

**Lemma 35** *If the group $T_{Onset}$ of all onset translations is contained in the paradigmatic group $P$, the topological spaces $(A@MOT_F, \mathfrak{T})$ and $(A@MOT_F/Ges, \mathfrak{T}/Ges)$ are irreducible.*

However, the motif space of $\mathfrak{T}$ is not sober in this case since it is irreducible, but no motif can be the generic point of the space since such a motif is a proper specialization of all its proper supermotives.

**Lemma 36** *Given an epsilon topology $\mathfrak{T}$, if $M, N \in A@MOT_F$, the following statements are equivalent:*

(i) *There is $\epsilon > 0$ with $N \notin D_\epsilon(M)$.*

(ii) *$Ges_t(N) \notin D_\epsilon(Ges_t(M))$.*

(iii) *$Ges_t(M) \not\sqsubset Ges_t(N)$.*

Proof. If $abcard(N) < abcard(M)$, the above exercise 51 tells us that both, $N \notin D_{\epsilon(M)}$, and $Ges_t(M) \sqsubset Ges_t(N)$ are the case for any $\epsilon > 0$. Suppose $abcard(N) \leq abcard(M)$. If $Ges_t(M) \not\sqsubset Ges_t(N)$, then $\epsilon = (ges.d^*_P)_N(M) > 0$ because the gestalt distance is a metric, and therefore $N \notin D_\epsilon(M)$. Conversely, if there is a $\epsilon > 0$ with $N \notin D_\epsilon(M)$, then by definition of disk neighborhoods, $\epsilon = (ges.d^*_P)_N(M) > 0$, so no submotif of $M$ can be in the gestalt of $N$. From corollary 7 we know that $N \notin D_\epsilon(M)$ iff $Ges_t(N) \notin D_\epsilon(Ges_t(M))$, and we are done.

**Sorite 8** *Given an epsilon topology $\mathfrak{T}$, if $M, N \in A@MOT_F$, the following three statements are equivalent:*

(i) *$M$ dominates $N$.*

(ii) *$Ges_t(M)$ dominates $Ges_t(N)$.*

(iii) *$Ges_t(N) \sqsubset Ges_t(M)$.*

*Moreover these statements are equivalent:*

(i) *$\{M\}^- = \{N\}^-$.*

(ii) *$\{Ges_t(M)\}^- = \{Ges_t(N)\}^-$.*

(iii) *$Ges_t(M) = Ges_t(N)$.*

*The following two statements are equivalent:*

(i) *$\{M\}^- = Ges_t(M)$.*

(ii) *$Ges_t(M) \in A@MOT_F/Ges_t$ is closed.*

[14] See appendix F.2.1.

*Moreover, if $M$ is closed, then $abcard(M) = 1$, and conversely, if the spaces $\Gamma_{t,n}$ are mutually disjoint. We have*

$$\{M\}^- = \{N \in A@MOT_F |\ Ges_t(N) \sqsubset Ges_t(M)\}$$

*and*

$$\{Ges_t(M)\}^- = Ges_t(\{M\}^-) = \{Ges_t(N) \in A@MOT_F |\ Ges_t(N) \sqsubset Ges_t(M)\}.$$

*In particular:*

$$Ges_t(M) = \{N |\ M \text{ dominates } N\} \cap \{N |\ N \text{ dominates } M\}. \tag{22.20}$$

The only point to observe here is that a motif $M$ with $abcard(M) > 1$ has its singleton subsets as submotives of abstract cardinality 1, and therefore their gestalts are different from $Ges(M)$ and $M$ cannot be closed. The converse is evident.

The last property (22.20) is a purely topological characterization of a motif's gestalt. This means that talking about "gestalts" in our technical sense is absorbed by a purely topological fact, i.e.:

**Fact 11** *The entire "gestalt" concept has been absorbed by a topological approach.*

**Proposition 33** *If $A@MOT_F|_1/Ges$ is finite, then $(A@MOT_F, \mathfrak{T})$ is quasi-compact*[15].

Proof. Let $C$ be an open cover of $A@MOT_F$, then

$$A@MOT_F|_1 = \coprod_{i=1,\dots k} Ges(M_i) = \bigcup_{i=1,\dots k} \{M_i\}^-$$

is a finite union of its gestalts and of the closures of its member representatives $M_i$. So there is a finite subset $C' = \{U_1, \dots U_l\}$ of $C$ that covers all representatives $M_i$. Since $MOT_{F,1} \subset A@MOT_F|_1$, each singleton motif is in one of $C'$'s open sets $U_j$. Therefore any motif $M$ is in the union of the open sets in $C'$, and $C'$ is a finite subcovering of $C$, QED.

### 22.7.0.1 Relative Topologies

In this general framework we consider the relative topologies $\mathfrak{T}_\mu, \mathfrak{T}_\mu/Ges$ when $\mu$ is a motif space for an interpretation $X^\mu$ of a finite local composition $X$. For the relative topology on $\mu$, we have these notations:

- $D^\mu_\epsilon(M) = D_\epsilon(M) \cap \mu$.
- $Ges^\mu_t(M) = Ges_t(M) \cap \mu$.

We have seen in corollary 9 that irreducible closed sets in the epsilon topologies $\mathfrak{T}_\mu, \mathfrak{T}_\mu/Ges$ have at most one generic point. Now, since $\mu$ is a finite set, an irreducible closed subset $W \subset \mu$ has a maximal index $n(W)$ such that $W \cap (\bigcup_{n(W) \leq i} A@MOT_F|_i) \neq \emptyset$. But by the metric hypothesis on gestalts, this intersection must consist of one single gestalt since this intersection is irreducible and the metric implies $T_2$ on the abstract cardinality level $n(W)$. So this gestalt is the generic gestalt of $W$, and we have

[15] See appendix H.1.4, definition 169.

**Proposition 34** *On a finite space $X$, the gestalt topology $\mathfrak{T}_\mu/Ges$ is sober.*

The above procedure also yields [73, prop.21] an algorithm for constructing the irreducible components of $\mu/Ges$. First, start with the gestalts $Ges(M_i), i = 1, \ldots r$ of maximal abstract cardinality in $\mu$. These points define irreducible components. In $\mu/Ges - \bigcup_{i=1,\ldots r}\{Ges(M_i)\}^-$, take the maximal abstract cardinality gestalts $Ges(N_1), \ldots Ges(N_s)$ which define irreducible components $\{Ges(N_j)\}^-$ of $\mu/Ges$, and so on, until the remainder is empty. Now, if any irreducible component $C$ is given, it must be covered by at least two of such irreducible components both of which have proper intersection with $C$. And it easily follows that $C$ must be in fact one of them.

**Proposition 35** *On a finite space $X$, the gestalt space $\mu/Ges$ (with the gestalt topology $\mathfrak{T}_\mu/Ges$) is homeomorphic to the associated sober space*[16] *$\mu^s$.*

Proof. Let $W \subset \mu$ be irreducible and closed. We know that it is a union of relative gestalts $Ges^\mu_t(M)$. Since $\mathfrak{T}_\mu/Ges$ is the quotient topology, the image $Ges(W)$ is irreducible and closed. Since $\mu/Ges$ is sober, $Ges(W)$ has a unique generic point $Ges(N)$. Then by sorite 8, $W = \{N\}^-$. Therefore, the canonical continuous map $q : \mu \to \mu^s$ is surjective. By sorite 8, it has the same fibers as the gestalt map. So the canonical[17] continuous map $p_\mu : \mu^s \to \mu/Ges$ is a bijection. Let $V \subset \mu^s$ be an open set. By definition of the sober topology, it is associated to an open set $U \subset \mu, V = U^s$, and we have $q^{-1}(V) = U$. So the image $p_\mu(V) = Ges(U)$ is open, and we are done. QED.

**Exercise 54** If instead, we take the total motif space $A@MOT_F$ with the epsilon topology $\mathfrak{T}$, we have a canonical continuous surjection $p : A@MOT^s_F \to A@MOT_F/Ges_t$ which induces a homeomorphism on the image $q(A@MOT_F)$ of the canonical map $q : A@MOT_F \to A@MOT^s_F$.

**Proposition 36** *Let $X^\mu$ satisfy the SEA property, $n_{min}$ being the minimal abstract cardinality of members of $\mu$, and $\mu_{min} = \{M \in \mu |\ abcard(M) = n_{min}\}$. Then the closed points are contained in $\mu_{min}$. If the shape spaces $\Gamma_{t,n}$ are mutually disjoint, then the set of closed points is exactly $\mu_{min}$.*

Proof: Exercise.

### 22.7.1 Toroidal Topologies

**Summary.** The special abstraction type of pitch and onset class formation leads to a similarity concept of gestalts which cannot be dealt with by use of (pseudo-)metrics. It is related to the topological specialization type discussed in section 12.2.2.

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[16] See appendix F.2.1.
[17] See appendix F.2.1, proposition 93.

In this case we shall consider a completely different kind of topologies which stems from algebraic geometry rather than from Euclidean geometry. Omitted proofs in this section can be found in [73, ch.4]. To fix the ideas, we take the toroidal shape type with shape spaces

$$\Gamma_{Toroid,n} = ObLoc_0^{\leq n, OnPiMod_{m,l}}$$

of local, zero-addressed compositions in $OnPiMod_{m,l}$ with cardinality at most $n$ and shape map $Toroid = Toroid_\gamma^{m,l}$ as introduced in formula (22.9). Here, the abstract cardinality is the usual cardinality of a local composition. Also we take the full affine group $P = \overrightarrow{GL}(\mathbb{Z}_m \oplus \mathbb{Z}_l)$ as a paradigmatic group introduced in 22.4.1.4.

#### 22.7.1.1 Dominance Topology

The analogy to the pseudo-metric topologies on the shape spaces $\Gamma_{Toroid}|_n$ is played by the dominance topology introduced in formula (12.4) of section 12.2.2. This time, we essentially work in $\mathbb{Z}_m \oplus \mathbb{Z}_l$ and consider abstract motives $x, y \subset \mathbb{Z}_m \oplus \mathbb{Z}_l$ of abstract cardinality $abcard(x) = abcard(y) = n$. We say that $y$ *is an abstract specialization of* $x$ (in symbols: $x \twoheadrightarrow y$) iff there is a bijective morphism of local compositions $s : x \to y$; so until now, no topology is involved, and the terminology seems a bit ambiguous, but the reason for this confusion will be given soon. Given two motives $M, N \in A@MOT_F|_n$, we also say that $N$ *is a specialization of* $M$ (in symbols: $M \twoheadrightarrow N$) iff $Toroid(N)$ is an abstract specialization of $Toroid(M)$.

The following lemma makes clear what it means that two abstract motives are mutually abstract specializations.

**Lemma 37** *For the toroidal type $Toroid$, if $x, y$ are two abstract motives, the following statements are equivalent:*

(i) *The abstract motives $x, y$ are mutually abstract specializations $x \twoheadrightarrow y, y \twoheadrightarrow x$.*

(ii) *The local compositions $x, y$ are isomorphic.*

(iii) *We have $Ges_{Toroid}(x) = Ges_{Toroid}(y)$.*

Proof. The equivalence of (ii) and (iii) follows from sorite 6, section 8.3.5. Clearly (ii) implies (i); for the converse, suppose we have two bijective morphisms $f : x \to y, f : y \to x$. Then their compositions $g \circ f, f \circ g$ are bijective and there is a positive power $n$ such that $(g \circ f)^n = Id_x, (f \circ g)^n = Id_y$, whence $(g \circ f)^{n-1} \circ g$ is the right and left inverse of $g$, and $x, y$ are isomorphic. QED.

This implies that the following settings are well defined.

**Definition 76** *If $M, N$ and $x, y$ are two motives and abstract motives, respectively, then we say that* the gestalt $Ges(N)$ is a specialization of gestalt $Ges(M)$ *(in symbols $Ges(M) \twoheadrightarrow Ges(N)$) and that* the abstract gestalt $P.y$ is an abstract specialization of the abstract gestalt $P.x$ *(in symbols $P.x \twoheadrightarrow P.y$), respectively, iff $M \twoheadrightarrow N$ and $x \twoheadrightarrow y$, respectively.*

**Corollary 10** *Two abstract gestalts (two gestalts) are mutually abstract specializations (mutually specializations) iff they are equal.*

For a fixed abstract cardinality $k$, we have the following Kuratowski closure operator[18] on $A@MOT_F|_k/Ges$. If $X \subset A@MOT_F|_k/Ges$, then we set

$$\bar{X} = \{y \in A@MOT_F|_k/Ges| \ \exists x \in X \text{ such that } x \twoheadrightarrow y\}.$$

Clearly, in this dominance topology, the sets $[x) = \{y| \ y \twoheadrightarrow x\}$ are the smallest open neighborhoods of $x$ and define a base for the dominance topology on the gestalt space. On the inverse image $A@MOT_F|_k$ of motives having abstract cardinality $k$, we take the inverse image of the dominance topology, i.e., the base $[M) = \{N| \ Ges(N) \twoheadrightarrow Ges(M)\} = \{N| \ N \twoheadrightarrow M\}$, the latter equality being the case because by lemma 37, dominance is invariant modulo gestalts. Clearly, the gestalt map $A@MOT_F|_k \to A@MOT_F|_k/Ges$ is open and the dominance topology is the quotient topology.

#### 22.7.1.2 Specialization Inheritance and Specialization Topology

In the next step of our construction, we look at inheritance properties for the dominance topology.

**Definition 77** *If for any triple $M, M^*, N \in A@MOT_F$ with $M^* \subset M$ and $M \twoheadrightarrow N$, there is $N^* \subset N$ such that $M^* \twoheadrightarrow N^*$, we say that* specialization is inherited.

*If for any triple $M, N^*, N \in A@MOT_F$ with $N^* \subset N$ and $M \twoheadrightarrow N$, there is $M^* \subset M$ such that $M^* \twoheadrightarrow N^*$, we say that* specialization is co-inherited.

**Proposition 37** [73, prop.27,cor.5] *Every toroidal shape type $Toroid^{m,l}_{\gamma}$ is inherited and co-inherited.*

This property guarantees that we may deduce another topology in the spirit of the Epsilon topologies, but now starting from dominance topology:

**Proposition 38** [73, def.19,prop.28,cor.6] *The system*

$$El(M) = \{N \in A@MOT_F| \ \exists N^* \subset N \text{ such that } Ges(N^*) \in [Ges(M))\}$$

*of* elementary neighborhoods *of motives $A@MOT_F$ is a base for a topology on $A@MOT_F$, which is called the* specialization topology $\mathfrak{T}^{m,l,sp}_{Toroid,\gamma} = \mathfrak{T}^{sp}_{Toroid}$. *Elementary neighborhoods are invariant under gestalts.*

*The quotient topology $\mathfrak{T}^{sp}_{Toroid}/Ges$ on the gestalt space $A@MOT_F/Ges_{Toroid}$ is also called* specialization topology. *Under these topologies, the gestalt map is an open continuous map.*

*The specialization topology $\mathfrak{T}^{sp}_{Toroid}/Ges$ has a base which consists of all elementary neighborhoods (in fact images of the elementary neighborhoods of the motif space) of shape*

$$El(G) = \{H \in A@MOT_F/Ges| \ \exists H^* \sqsubset H \text{ such that } H^* \in [G)\}$$

*for $G \in A@MOT_F/Ges$.*

[18] See appendix H.1. Kuratowski closure operators are used to define topologies.

In the specialization topology, we have analogous results as in the epsilon topology, regarding the closure of a point:

**Proposition 39** [73, prop.31,prop.39,prop.41,prop.42] *Ff $G \in A@MOT_F/Ges$ for the specialization topology $\mathfrak{T}^{sp}_{Toroid}/Ges$, then*

(i) $El(G) = \{H \in A@MOT_F/Ges |\ H \text{ dominates } G\}$.

(ii) $\{G\} = \bar{G} \cap El(G)$.

*For the specialization topology $\mathfrak{T}^{sp}_{Toroid}$, if $M \in A@MOT_F$ is a motif, then*

(i) $El(M) = \{N \in A@MOT_F |\ N \text{ dominates } M\}$.

(ii) $Ges(M) = \bar{M} \cap El(M)$.

*The specialization topology $\mathfrak{T}^{sp}_{Toroid}/Ges$ is $T_0$, irreducible, and quasi-compact.*

Many of the properties for the specialization topologies $\mathfrak{T}^{sp}_{Toroid,\mu}$ on a motif space $\mu$ of a finite local composition $X$, and $\mathfrak{T}^{sp}_{Toroid,\mu/Ges}$ on $\mu/Ges$ are analogous to those which we have discussed for the epsilon topologies. We refer to Buteau's work [73, ch.4] for the details. In particular, the irreducible components for $\mathfrak{T}^{sp}_{Toroid,\mu}$ can be calculated by a recursive procedure which is similar to the one discussed in section 22.7.0.1.

**Exercise 55** Show that the specialization topology $\mathfrak{T}^{sp}_{Toroid,\mu}$ still homeomorphic to the associated sober space of the specialization topology on gestalts.

Remark. In [73, ch.4], it is shown that specialization relations, together with different subgestalt relations can be used to define a *gestalt specialization category* $\mathfrak{Ge}(A@MOT_F/Ges_{Toroid})$.

**Principle 21** *As an overall result from the construction of motivic topologies, we can now consider* generic points *and related dominance structures from the theory of sober spaces. This is the basis for a topological realization of Reti's ideas in the sense that the* semantic charge *connoted with the very concept of a motif turns out to be a precise instance of the genericity properties of the topological motif space which is associated with a given score. In other words: Cognitive and structural motivic semantics is related to and reduced from the topological dominance configurations in the motivic topologies.*

We come back in section 22.9 to the more detailed technique of motif weights to make motivic semantics explicit on a quantitative level.

## 22.8 Rudolph Reti's Motivic Analysis Revisited

**Summary.** After the precise construction of immanent motivic "organisms" associated with determined scores, we conclude this chapter with a critical review of Rudolph Reti's classical texts in the light of the presented theory. For more details, see [73, ch.5.2].

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### 22.8.1 Review of Concepts

Complete works about motivic analysis are practically non-existent, therefore Reti's book [444] is still a good reference to the methodology of traditional musicological motif theory. Despite his accurate overview on the evolution of a motif through a composition, Reti does not use rigorous terminology, i.e., he uses the same word for different concepts or different words for the same concept. It seems very difficult to grasp its contents and substance. This vague terminology produces hazy deductions, not to say contradictions. However, behind his words, Reti's conception of a first sketch for a thematic theory of music is considerable.

Reti is not even concerned with reliable terminology:

> [444, p.12] *In general, the author does not believe in the possibility or even desirability of enforcing strict musical definitions.*

But it is undeniable that before any attempt to interpret Reti's motivic analysis within our mathematical model, we must take a closer look at his terminology whose nuances are essential for bringing consistency into Reti's motivic score analysis.

Let us get off ground with Reti's definition of a motif:

> [444, p.2-3] *... any musical element, be it a melodic phrase or fragment or even only a rhythmical or dynamical feature which, by being constantly repeated and varied throughout a work or a section, assumes a role in the compositional design somewhat similar to that of a motif in the fine arts.*

Since Reti asserts that he handles the words "motif" and "theme" without categorical distinction, we are aware that in some of the following quotations or examples, the word *theme* can be, for our purposes, replaced by *motif*. All the same, Reti points out a difference between the two concepts by defining a theme as

> [444, p.3] *... a fuller (compared to a motif) group or "period" which acquires a "motivic" function in a composition's course.*

According to Reti, both a "motif" and a "theme" have the same function and therefore are reduced to almost the same concept. A first distinction between these two notions is length. Then, Reti adds that a theme is composed of few motives. Moreover, the difference of the length has consequences on how and where motives and themes can occur in the composition. A motif being short, it can be easily repeated throughout the composition. It can also be modified and inserted in parts having different moods. These frequent occurrences create a certain unity within the piece. A theme has a similar function but cannot, as easily as the motif, be modified or/and inserted in the composition.

We now state a crucial idea in Reti's book. Reti asserts that abstractions have to be made in order to compare themes and motives:

> [444, p.243] *Similar* liberties *were taken in our analysis on several other occasions...by* rearranging *the design..., by exchanging the octaves in which some phrases were notated in the score.*

In a more general way, when talking about classical symphonies, Reti claims that

> [444, p.13-14] *He (the classical composer) strives toward homogeneity in the inner essence... toward variety in the outer appearance. Therefore he changes the surface but maintains the substance of his shapes... Tempo, rhythm, melodic detail, in fact the whole character and mood are altered and adjusted to the form in which the composer conceived them fitting to the new movement.*

Reti clearly affirms that we have to make abstraction from some qualities of a motif in order to make the connection with its transformations in other parts of the piece. As a matter of fact, Reti always makes abstraction when comparing sequences of notes. For example, it can be the simple abolition of note loudness or more complex procedures such as the exchange of octave positions inside a given sequence of notes.

We should remark that Reti never uses the word "gestalt"[19]. Instead, he is concerned with comparisons of "shapes". The word "shape" is not chosen at random and is met throughout the book:

> [444, p.205] *... a theme is that shape around which...*

The importance of the shape concept is that it englobes the idea of abstraction. Nevertheless, Reti does not have a well-defined usage of this concept. In similar situations, such as in examples [444, 328,p.210] and [444, 334,p.213], both words "motif" and "shape" are used with the same meaning. The reason for this lack of rigor is simply that when comparing motives, natural abstractions like the abolition of loudness are somehow unconsciously operated. For Reti, starting a comparison with shapes seems trivial.

According to Reti, the shape of a motif is incessantly heard, literally or modified, through the piece. The shape is imitated, varied or transformed:

> [444, p.240]
>
> 1. imitation*, that is literal repetition of shapes, either directly or by inversion, reversion, and so forth;*
> 2. varying*, that is, changing of shapes in a slight, well traceable manner;*
> 3. transformation*, that is, creating essentially new shapes, though preserving the original substance.*

In other words, a motif has a "smooth shape", i.e., its shape can be imitated or smoothly modified and remains the same, or as Reti asserts, it remains "identical"[20]. However the accepted transformations of shapes can reach a limit:

> [444, p.355] *...his (Beethoven's) most impressive thematic constructions lie on the border line between being matchless master strokes of transformation or utterances wherein the thematic bond has almost dissolved in that very transformation process by which they were created.*

[19] Nonetheless, the gestalt concept is omnipresent. We shall see that Reti's implicit definition of a gestalt is first the choice of an abstraction perspective, and second all the immalleable transformations (the imitations) of the motif's shape in the chosen perspective.

[20] The words "identity" and "identical" are used over and over in the book and their meaning is not the one that one is accustomed to from logic! For example, see [444, pp.13,21,38,102,167,243,...].

Reti states that a shape can not be endlessly transformed and still remain identical with its initial shape: identity has a 'limit of transitivity'. This is by itself a contradiction since any accepted concept of "identity" is transitive. However, there are subtle identification problems which, in case the analyst is not precise enough, i.e., does more identification than the situation requires, cause contradictions while the essential points drop out. Unfortunately, Reti did not realize the implications that bears his use of the concept of "identity".

Reti tries to avoid this contradiction by constant reference to the first appearance of a motif in the score. Reti usually affirms that a shape is a transformation, imitation or variation of a motif by, most of the time, comparing the shape of the first motif appearance with the considered transformation. In a sense, this "identity" is an order relation, the first appearance $A$ being "greater" than all the later appearances, $B, C, \ldots : A > B, A > C, \ldots$, but transitivity is not relevant here, because, besides $A > B, A > C \ldots$, no relation $B > \ldots, C > \ldots$ etc is considered. So the transitivity question $A > B \;\&\; B > C \stackrel{?}{\Rightarrow} A > C$ is of no interest. But again, we are confronted with a non-sense definition of the identity: it is not even symmetric! In any case the common concept of identity is violated. In the next section, we shall clarify how to use "identity of motives" in a consistent way.

Although Reti was imprecise he was not irreversibly wrong. The abuse of the term "identity" is somewhat clarified in the concept of a motif, unfortunately hidden by a fluffy terminology. For Reti, a motif conceives

> [73, p.108] *the first appearance of a certain group of notes with shape abstraction such that, throughout the whole composition, there are "many" note groups having respective shapes (i.e., shape abstractions) which are imitation, variation or transformation of the first shape.*

To distinguish the different concept levels in our discussion, we denote by *Reti-motif* the above "definition" of a motif.

## 22.8.2 Reconstruction

We now proceed in reconstructing Reti's concept framework from the point of view of the present motif theory.

### 22.8.2.1 Choice of Parameters

We first fix the parameter space $F = Onset \oplus Pitch \oplus Loudness \oplus Duration$ since for the context of Reti's book, no other relevant tone parameters are of use. To be clear, this framework does not include pauses or bar-lines as relevant tone objects, only 'sounding' notes are considered. Here are the detailed settings (see also appendix A for these standard values):

- The first note has *Onset* value 0.
- Duration values are taken by the prescription that 1.0 in the *Onset* coordinate corresponds to the literal mathematical value of 4/4 duration.
- For the *Pitch* values, we select the usual gauge with middle $C = 0$, and the chromatic pitch set being parametrized by the integers, i.e., $C\sharp = D\flat = 1$, $D = 2$, etc.

- *Loudness* is gauged by the usual setting where $mf \sim 0$, $f \sim 1$, $ff \sim 2$, etc., and $p \sim -1$, $pp \sim -2$, etc.

- By *toroidal type* we intend $Toroid_{\gamma}^{12\times 12}$ with a default grid[21] $o = (0,0), x = (1,0), y = (0,1)$.

Furthermore, as might be expected, we always deal with the zero-address in Reti's context.

#### 22.8.2.2 Shapes, Imitations and Transformations

Let $X \subset 0@F$ be the local composition consisting of all notes within the given composition, and with respect to the above gauge of the four parameters. Then let $\mu^*$ be the set of all motives within $X$.

Reti's implicit first step in his motivic analysis of a composition is the choice of a determined set $\mu \subset \mu^*$ of (mathematical) motives, i.e., "sequences of notes" which he wants to compare with each other in order to find imitations, variations and transformations of the shape of a certain sequence of notes which he will then call a motif (a Reti-motif in our terminology). Call our mathematical motives $M \in MOT_F$ *Math-motives* in order to distinguish them from Reti's more involved Reti-motives. In fact, it is remarkable that most of the Math-motives in $\mu$ are no way qualified for becoming contributors to Reti-motives. They are just a kind of motivic 'raw material'. Furthermore, Reti naturally allows himself to compare parts (subsequences) of the chosen Math-motives (sequences). He normally does not regress to the single notes, thus he gives himself a lower limit of motif cardinality. This all means that in the mathematical model the SEA property must hold with respect to the lower limit for $\mu$. Most of the time Reti applies the limit of the intervals, i.e., $n_{min} = 2$ in SEA. Let us now reconstruct Reti's motif definition

> *...the first appearance of a certain group of notes with shape abstraction such that, throughout the whole composition, there are "many" note groups having respective shapes (i.e., shape abstractions) which are imitation, variation or transformation of the first shape.*

within the mathematical model:
**"... with shape abstraction..."** As seen in the previous part of this section, the choice of an abstraction of the sequences of notes is the starting point for comparison. To 'simulate' Reti's abstraction procedure by our model, we select a shape type type $t : MOT_F \to \Gamma_t$, and we model the shape abstraction of a Math-motif $M \in \mu$ by its associated abstract motif $t(M) \in \Gamma_t$.
**"... which are imitation..."** We must emphasize that declaring that a shape is an "imitation of another" suggests the choice of a collection of admissible imitation transformations, a choice which we associate with a choice of a paradigmatic group $P$ in the mathematical model. For example, most of the time strict repetitions (translations in time) are admissible, and it is clear that the choice depends on the selected shape abstractions. The implicitly conceived gestalt of a sequence of notes in Reti's book perfectly corresponds to a gestalt of a Math-motif in a motif space.

[21] This grid is by no means mandatory. According to the concrete situations, we shall define better adapted quantization data.

According to Reti's analysis, three examples [444, example 35, p.33] (first scene of Schumann's Kinderszenen, shape I), [444, example 20b, p.21] (second version of Beethoven's Ode an die Freude shape), and [444, example 1, p.11] (descending third in measure 19 of the first movement of Beethoven's Ninth Symphony) of imitations seems to coincide with the counterpoint group.

The choice of abstraction (the choice of the types) greatly influences our view on a composition, and this is a strong point of this analysis since it takes into consideration different qualities and characters of the composition.

We now introduce the distance $ges.d_P^*$ (defined in section 22.5) of Math-motives into our model of Reti's motivic analysis. In a sense the distance $ges.d_P^*$ between motives, i.e., two sequences of notes such that their shapes have the same number of "elements" (same abstract cardinality in the selected shape type), gives a measure for their diversity. For Reti, two sequences of notes with shape having the same number of elements are identical (imitated), closely alike (varied or transformed) or different (not related). In the mathematical model, this kind of relation is more subtle. The shapes of two sequences of notes "being identical" means having distance 0, "being closely alike" requires to know "how close" they are from each other and being different requires to know "how far" they are from each other. There is no longer only three rough categories but instead, we are given subtly nuanced relations between sequences of notes.

**Remark 5** The $ges.d_P^*$ distance is a priori defined on $MOT_F$ and not on $\mu$. This means that the calculation of $ges.d_P^*$ for two sequences of notes *within* a given composition $X$ is determined by first looking at their shapes (following a given abstraction), and then, given a paradigmatic group $P$, by comparing not only all their imitations within the composition but also all the a priori imaginable imitations. According to rigid type, we easily see that a similar distance defined *only* on $\mu$ would yield absurd results. Moreover, these 'imaginary' shapes are naturally, but probably also unconsciously, used by Reti.

**"...variation or transformation..."** Variations and transformations of a sequence of notes are more delicate to translate into the mathematical model. Since the difference between variation and transformation is simply a question regarding the level of change between shapes, we merely consider transformations, i.e., variations are viewed as special cases of transformations. There seem to be no disadvantageous consequences to this identification.

We recall that, according to the analyst Reti, there is a limit to a transformation of a shape such that it still relates to the original shape, i.e., such that it remains "closely alike". In mathematical terminology, there is a maximal admissible distance $\epsilon_{max}$ between the Math-motives. In the motivic topology, this limit corresponds to a fixed positive real number $\epsilon_{max}$. The latter is a border line such that if two shapes with the same abstract cardinality are less than $\epsilon_{max}$ distant, then they are considered as being "closely alike". However, transformations do not necessarily maintain the number of elements. In the mathematical model, transformations of shapes can be translated to three different situations corresponding to the change of number of elements of the shapes:

1. If two shapes have the same number of elements and are closely alike, then, in the motivic space, their Math-motives $M$ and $N$ are in the other's $\epsilon_{max}$-neighborhood, respectively, which is $M \in D_{\epsilon_{max}}(N)$ and $N \in D_{\epsilon_{max}}(M)$. Example: [444, examples 85,86, p.62]

(Palestrina's Missa Marcelli, comparison of beginnings of the two first sections in Kyrie Eleison and the Christe Eleison).

2. If in the terminology of Reti, a shape $A$ is "closely alike" inside the shape $B$, then, in the mathematical model, for a given shape type, this means that a Math-motif $M$ with shape $t(M) = B$ is in the $\epsilon_{max}$-neighborhood of the Math-motif $N$ with $t(N) = A$: $M \in D^{\mu}_{\epsilon_{max}}(N)$. Intuitively speaking, this means that the shape of $N$ is closely alike to the shape of a shape of a submotif of $M$. Example: [444, example 118, p.80] (first movement of Brahms' Second Symphony).

3. If a part $A^*$ of a shape $A$ is closely alike inside shape $B$ (and conversely in Reti's terminology), then, in the motivic space, there is a submotif $M^*$ of $M$ with shape $A^*$ and $A$, respectively, and such that $N$ with shape $B$ is in the $\epsilon$-neighborhood of $M^*$: $N \in D^{\mu}_{\epsilon_{max}}(M^*)$. Example: [444, example 218, p.143] (Brahms' First Rhapsody).

If the relation between a shape and its transformation is musically too hidden inside the transformation, in the sense that it is hardly heard or even not at all heard, then this is, what Reti calls, an "indirect affinity". This phenomena is similarly translated into motif space terminology as a transformation with possibly larger $\epsilon_{max}$.

#### 22.8.2.3 Reti's Identity Relation Revisited

In the previous section we mentioned the inconsistent use by Reti of the concept of "identity" in the sense of imitations, variations and transformations. However in the mathematical model, identity of gestalt can be compared with Reti's use of the concept of Identity. In fact, we know from sorite 8, that, under certain conditions, $Ges_t(M) = Ges_t(N)$ for two Math-motives $M$ and $N$ iff $M \in D^{\mu}_{\epsilon}(N)$ and $N \in D^{\mu}_{\epsilon}(M)$, for all $\epsilon > 0$.

This gives us the possibility to review the gestalt identity concept under a purely topological perspective. We can see now that Reti's violation of the strict concept of identity is related to violation of one or several parts of the above neighborhood condition for identity of gestalt. It is undeniable that the identity of motives should be limited to imitations, i.e., the identification of motives with their gestalts yields consistent statements. The identification with transformations creates a system where every Math-motif can be identified to any other Math-motif! This is certainly not what Reti wanted to state unrestrictedly. A reconstruction of Reti's theory (and other fuzzy attempts, such as [428]) should be undertaken on the basis of topological concepts as modeled above.

## 22.9 Motivic Weights

**Summary.** Motivic weight functions are a means for understanding motivic topologies by use of real 'coordinate functions'. We describe the technique and its meaning for semantic purposes, in particular, regarding abstraction from motif parameters.

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In the analysis of metrical structure of section 21.2, definition 64, we have introduced sober weights. This can be carried over to motivic structure without any further modification; recall (see section 21.2) that for the motivic interpretation $X^\mu$, we have the nerve map $Sp : X \to n(X^\mu)$. Also recall that an $\mathbb{R}$-valued sober (motif) weight on the $A$-addressed local composition $X \subset A@F$ is a function $W : X \to \mathbb{R}$ which factorizes through the nerve map $Sp$. A factor $nW : n(X^\mu) \to \mathbb{R}$ is called a nerve weight for $W$. A nerve weight $nW$ is said to be induced iff its values are determined by the values on the vertexes, i.e., by the restriction $n_0W = nW|_{n_0(X^\mu)}$.

In the implementation of motif theory on the RUBATO®'s MeloRUBETTE®, such induced sober motif weights have been used for performance oriented motivic analysis. We shall discuss this approach in chapter 41.2. In the present discussion, we want to concentrate on the semantics of such weights. To calculate the weight $W(x)$ means that we give the tone event $x$ a value which stems from the weight values of motives $n_0W(M), x \in M$, including some additional configuration functions which take care of the simplex structure $Sp(x)$. Basically, this means that we validate $x$ in its motivic environment: "What is the motivic position of $x$ within the given motif space $\mu$ of $X$?" So we give each tone event $x \in X$ a 'motivic coordinate'. Isn't this too coarse? The point is that for performance, complex coordinates are not useful since the instrumentalist really needs numeric information of how strongly that key should be stroked. And that may be ok for performance, but it is not the yoga of motivic analysis. There is a more faithful approach to motivic coordinates if we step over to the nerve weight $nW$, which we suppose to be induced, to make the ideas more transparent.

Restating the precedent question, a nerve weight fixes the motivic coordinate $nW(M)$ of any motif $M \in \mu$. Why are we—beyond applications to performance theory—interested in such coordinates? The point is that the motivic epsilon or toroidal topologies are not immediately understandable in terms of classical Euclidean coordinates, in fact, these topologies have more of an algebro-geometric flavor, including generic points, specializations, and related 'far-out' structures. It has been shown in the discussion of Reti's theory that these tools are adequate. But the prize was a strong deviation from numerical, quantitative representation. The immediate semantical potential of motivic topologies is not a numeric one. On this basis, the introduction of nerve weights should provide us with a reconstruction of quantitative aspects of topological perspectives, possibly with a bunch of numerical invariants which could in the ideal case approach a complete set of invariants.

One approach to this goal has been realized by the MeloRUBETTE®. Let us just have a look at that methodology. Fix a shape type $t$, an equivariant paradigmatic group $P$ of isometries, the standard situation of our motif theory. Given a positive limit number $\epsilon$, we look at the $\mathfrak{T}$ neighborhood configuration on $\mu$ as seen from a fixed motif $M$. What does this object 'know' from its topological position? We could be interested in knowing how many motives are in the disk neighborhood $D^\mu_\epsilon(M)$. This would yield a kind of 'presence' $pr_{\mu,\epsilon}(M)$ of $M$ within larger motives, up to some tolerance $\epsilon$ of shape similarity. It could also be interested in knowing how much motives $M$ participate in their 'presence', thereby yielding a kind of 'content' $ct_{\mu,\epsilon}(M)$ of $M$. This data can be used to give the motif a measure for its relevance or weight in the present motif space $\mu$, a weight function

$$nW_\epsilon(M) = \omega(pr_{\mu,\epsilon}(M), ct_{\mu,\epsilon}(M)), \tag{22.21}$$

for example with $\omega(x, y) = x.y$. This is only one approach to grasp topological information via numerical values, but we just want to give a concrete idea of how to proceed in general. Given

these weights $nW_\epsilon : \mu \to \mathbb{R}, \epsilon > 0$, we obtain a map

$$nW : \mu \to \mathbb{R}^{\mathbb{R}_+} \tag{22.22}$$

on the vertexes of the nerve $n(X^\mu)$, being extended to the entire nerve $nW : n(X^\mu) \to \mathbb{R}^{\mathbb{R}_+}$ via a given induction function, i.e., $nW(\sigma) = f(nW(M), M \in \sigma)$. Since the motif space is finite, we may restrict to a finite sequence $\epsilon_\cdot = \epsilon_1 < \epsilon_2 \ldots < \epsilon_k$ of limit numbers to grasp all changes within these weight functions. This means that we consider the weight $nW_{\epsilon_\cdot} : \mu \to \mathbb{R}^k$, and its induced extension $nW_{\epsilon_\cdot} : n(X^\mu) \to \mathbb{R}^k$. It is a major task in motif theory to construct weight functions following the scheme of $nW_{\epsilon_\cdot}$ in order to investigate the abstract motivic topologies via their images in real $n$-space.

**Remark 6** The AST-ideas on similarity, as developed by Morris, Lewin, Rahn, and others may be looked up in Morris' work [380, p.103ff.].

# Part VI

# Harmony

# Chapter 23

# Critical Preliminaries

*Aber atonal wird man irgend ein Verhältnis von Tönen*
*sowenig nennen können, als man ein Verhältnis von Farben*
*aspektral oder akomplementär bezeichnen dürfte.*
*Diesen Gegensatz gibt es nicht.*
Arnold Schönberg [478, p.240]

**Summary.** The present part on harmony is a traditionally dominant and extended portion of music theory. Therefore, it is adequate to review some of the important approaches to harmony. This chapter is however far from a complete synthesis of harmony and its history. We have selected three representative approaches which are systematically elaborate and theoretically founded: H. Riemann, P. Hindemith, and H. Schenker. The following overview concentrates on the divergence between claim and realization, and it does, once again, lay bare the enormous difficulty to set up a precise discourse about music *without*—hélas—the power of mathematical language. Also this critique is not thought to be a preliminary to something which in the subsequent chapters of this part will be perfectly solved by mathematical music theory. The discourse simply tries to persuade music theorists that a) the commonly cultivated status quo of the subject is scientifically unacceptable, and b) that mathematically sharpened concepts, constructs, and models can show ways to more in-depth and precise understanding of harmony—without banning it to history and "atonal" negation. Generic harmony is a universal perspective of music, and it is unscientific as well as near-sighted if not anti-musical to abandon harmonic paradigms *instead of* embedding them into a diachronically and synchronically open, unified, and universal concept framework. To be clear, the *main question* is not to defend or instantiate *any* ideology of harmony—this is the unhappy business of Pythagorean fundamentalists—but to investigate its possible semiotic functions in musical works and their communicative explication, to develop an adequate language, and to propose consistent and sound models of harmonic processes.

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The theory of harmony as a theory of chords and chord progressions comprises a wide range of different approaches, methods and questions. Thereby the historic context is only one factor

among others to explain these differences. Harmonic research is situated in the intersection of various fields of interests, which correspond to different approaches and questions. For instance Paul Hindemith's interest in *Unterweisung im Tonsatz* [224] is that of a composer—so his focus is to explain, in how far and why diverse combinations of tones as chords differ in their esthetic effects. Arnold Schönberg's *Harmonielehre* [478] aims at students of composition as well. In comparison to this perspective Schenker and his adepts analyze the harmonic structure of pieces of music in order to explain the impression of "inner coherence" in this piece which includes a description of different hierarchic levels. This approach results in a completely distinct definition of what is called a "chord" compared to, for instance, Hindemith's theory. In the following preliminaries to the mathematical theory of harmony, we don't want to give a complete summary of all these different investigations, but to sketch some of the different ways of classifying chords and thereby shed light at least on some typical questions concerning chords and chord progressions.

## 23.1 Hugo Riemann

**Summary.** This section briefly reviews Hugo Riemann's concept of triad-based tonality as it was developed in his writings, such as *Musikalische Logik* [449] and *Musikalische Syntaxis* [450], from Rameau's early adumbration. Riemann's construction of tonality from a chord progression is described. It is based on the concept of *thesis* (German "These"); the tonal coherence of a composition is constituted by a succession of theses. To conclude, we discuss Riemann's unfinished program to assign a harmonic function to strictly *every* possible chord, given a specific tonality.

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Rameau was one of the first theorists to deal with the classification of the variety of possible combinations of tones as chords in order to find systematic laws of combination. He classified chords on the basis of thirds, which means to understand the combinations of thirds which build chords. Thus a fourth-sixth-chord can be traced back to a triad by an inversion, i.e., by adding an octave to the lowest tone. The triad is the starting point not only of the horizontal, but also of the vertical organization of the tones in a piece of music. The system of tonal functions is adumbrated, but not fully exposed.

This exposition can be found in Riemann's *Musikalische Logik* [449]. It involves a classification of chords into the three categories *tonic, dominant, subdominant* and the discussion of the cadence ("Kadenz") as a specific progression of chords when subsumed under these categories. His system includes a classification of chords on the basis of the minor and the major triad. This method classifies every chord by tracing it back to a triad (or a combination of triads) it was derived from (e.g., by altering the chord's single tones). Consequently, it involves the problem that the preimage from which a chord is derived is often far from unequivocal. The classification of all chords into the three categories tonic, dominant, and subdominant, is ambiguous as well.

Another important question Riemann deals with in *Musikalische Syntaxis* [450] is that of tonality. How does the phenomenon of tonality arise within a specific chord progression such that one chord is classified as a tonic? Riemann discusses examples of such chord progressions,

but in some cases there is more than one possible triadic solution for the suspected tonality—a fact which Riemann does not reflect. So the question, how to find criteria for deciding upon possible tonalities in non-trivial ambiguous situations, remains open.

## 23.2 Paul Hindemith

**Summary.** This section briefly reviews Paul Hindemith's harmony as developed in *Unterweisung im Tonsatz* [224]. In order to explain the tension/relaxation within a chord progression, he constructs a hierarchy of intervals which is driven by their consonance character. The chords are then classified according to the hierarchical position of their constituting intervals.

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Hindemith's classification of chords starts with a discussion of the intervals with respect to their stability and consonant or dissonant character. He critizes the approach to alter all chords into triadically composed chords. Chords are classified according to the intervals they consist of in such a way that stable or consonant intervals generate stable chords. Among all chords, the triads are the most stable ones. Chord progressions are analyzed in terms of different stabilities of the consecutive chords. For instance, if a given chord consists of "more consonant" intervals and the next one of "less consonant" intervals, an effect of tension is produced. By this definition, Hindemith creates curves of tension within long chord successions. The concept of tension is an instrument to explain possible effects or impressions when listening to such a chord successions.

## 23.3 Heinrich Schenker and Friedrich Salzer

**Summary.** This section reviews Friedrich Salzer's *Structural Hearing* [470] as an updated version of tonal theory following Heinrich Schenker's ideas, as exposed in [474] and [475]. This theory considers a hierarchy of structural layers from the background ("Ursatz des Hintergrundes"), traversing the transformational middleground ("Mittelgrund der Verwandlungsschichten"), and developing into concrete music on the foreground ("Vordergrund"). Structural chords on a higher hierarchical level are complemented by (relative) prolongational chords on a lower level, expressing harmonic "motion" between structural instances. We discuss the question of ambiguities and indeterminacies in the case-specific definition of such a hierarchy.

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Salzer (as representative of Schenkerian analyses) critizes those harmonic approaches, which structure a given piece of music into a progression of chords and assign functions to them by analyzing the specific chord gestalts. Instead, he proposes the classification of chords into two groups—structural and prolongational chords—in order to explain phenomena like 'musical motion' or 'coherence'. Structural chords support the harmony, while harmonic meaning is not assigned to prolongational chords—they mediate between structural chords and support the 'motion' from one harmonic function to the next one. This approach starts with a grouping of the piece into periods and themes, followed by the specification of the structural chords within

these periods. There are however no general arguments or criteria for the process which decides about the chords belonging to structural or prolongational categories. Also, the problem of which tones form a chord remains unsolved as well.

**Remark 7** We should mention Sigfrid Karg-Elert's harmony [259] which is based on a strictly polar symmetry between major and (aeolian) minor modes, as well as Arthur von Oettingen's symmetry-based modulation theory [406]. A thorough discussion and understanding of their works is still unsettled.

# Chapter 24

# Harmonic Topology

*Tyger! Tyger! burning bright*
*In the forests of the night,*
*What immortal hand or eye*
*Could frame thy fearful symmetry?*
William Blake (1757–1827)

**Summary.** This chapter introduces a systematic correspondence between chords and symmetries. It lays the morphological fundament for a semantic theory of harmonic functions which will be exposed in the following chapter. Essentially, the idea of such a correspondence is to carry over the structural discourse on harmony to richer objects by an intermediate switch to "richer" addresses.

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In sections 11.3.6 and 11.3.7, we have presented a preliminary study of symmetries of chords, and of the classification of self-addressed chords in simple ambient spaces of modules with finite length. In this chapter, the idea of switching between chords and symmetries will be developed a bit more systematically and then applied to the construction of harmonic semantics as suggested by Noll [400].

The basic idea is that to understand the full morphological potential of chords, it is not sufficient to look at their points, rather one should include the information given by their inner symmetries. Now, there are two ways to deal with this idea. The first is an ontological switch. It resembles the concept from quantum mechanics, stating that the measurable physical reality is a trace—in fact a bunch of eigenvalues—of more abstract hermitian (linear) operators: the observables. The higher reality of these operators is the semantic source of microscopic physics whose seemingly contradictory phenomenology of wave and particle are just projections of the consistent higher reality of observables [150]. In this vein, chords would be replaced by endomorphisms of the underlying form space.

The drawback of this approach is that it enforces a new reality which is not that of addressed points but resides in abstract morphisms between form spaces. But music should not have its reality split into what has meaning and what is audible. It should avoid doing so as

much as physics should. We cannot speak for the physical ontology, but for music, there is an elegant solution which has been prepared via the very approach to musical objects which is based on addressed denotator substance. We contend that the richness of ontological reference which is offered on each address is essential and sufficient to condense the morphic flavor of symmetries on the level of address ontology (see also our discussion in 8.3.4, in particular principle 3)—provided that "generic" addresses can be found for a given form space.

There is a further advantage of the 'address condensation' of morphic structures: With this tool it becomes possible to consider global compositions on the basis of local compositions on a generic address, a construction which Noll calls "Morphemfeld" in [403], and which he uses to (re)build harmonic semantics in the spirit of Hugo Riemann.

We are aware that the following approach is not evaluated at all its implications, in fact, the harmonic topology is but a first application of a more general topological interpretation of denotator morphology. We nevertheless feel obliged to sketch this general perspective to make the stream of ideas evident and their coherence plausible.

## 24.1 Chord Perspectives

**Summary.** This section deals with chords in Euler and pitch class spaces, together with a comparative study while applying the enharmonic projection of self-addressed Euler spaces onto pitch class spaces.

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We start our discourse from concrete situations in harmony which arise from chord descriptions in different ambient spaces for pitch and pitch classes. Noll's term "perspective" for pitch denotators at different addresses seems adequate: it emphasizes the ontological perspective offered on each address, as well as the change of perspective induced by a change of address. These examples also will prepare the reader for the semantic discussion of chord morphology in the next chapter.

### 24.1.1 Euler Perspectives

**Summary.** This section discusses zero-addressed and self-addressed chords in the context of pitch classes in just tuning.

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Recall from section 7.2.1 that the coordinators of $EulerModule$ and $EulerPlane$ are $\mathbb{Q}^3$ and $\mathbb{Q}^2$, with the canonical decompositions $\mathbb{Q}^3 = \mathbb{Q} \cdot o \oplus \mathbb{Q} \cdot q \oplus \mathbb{Q} \cdot t$ and $\mathbb{Q} \cdot q \oplus \mathbb{Q} \cdot t$, respectively. Within these spaces, one identifies the subspaces $EulerZModule = EulerModule|_{\mathbb{Z}}$ and $EulerZPlane = EulerPlane|_{\mathbb{Z}}$ which are defined by integer coordinates, i.e., the lattices generated by the canonical bases $o, p, q$ and $p, q$, respectively. We shall mainly start from these lattices when talking about harmony in just tuning.

For example, the C-chromatic scale of Vogel, from which one octave is shown in the lower half of figure 7.6, is a zero-addressed chord $\chi$ in the $EulerZModule$. In order to have a better

representation of this $\chi$, we transform it under the linear space automorphism

$$S = \begin{pmatrix} 1 & 2 & 2 \\ -2 & -3 & -4 \\ 0 & 0 & 1 \end{pmatrix} \tag{24.1}$$

and get the more transparent configuration $\chi^\star = S.\chi$ as shown in figure 24.1. To understand this

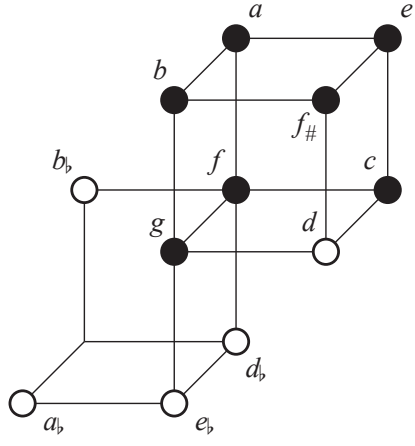


Figure 24.1: The transformed chromatic scale makes its inner symmetries evident.

chord better, observe that it contains (the heads and tails of) four $\mathbb{Z}$-addressed points $g \Rightarrow e_\flat$, $g \Rightarrow b$, $g \Rightarrow f_\sharp$, $g \Rightarrow a_\flat$ with common tail $g$. Then $g \Rightarrow e_\flat + g \Rightarrow b$ is a zero length arrow, in fact a major third 'coupling', and $g \Rightarrow f_\sharp + g \Rightarrow a_\flat$ is a zero length arrow, a minor second coupling. A similar situation holds for exactly one other set of four $\mathbb{Z}$-addressed points, i.e., $f \Rightarrow a$, $f \Rightarrow d_\flat$, $f \Rightarrow c$, $f \Rightarrow b_\flat$ with common tail $f$. Again, we have these relations: $f \Rightarrow a + f \Rightarrow d_\flat$ is a zero length arrow, a minor third coupling, and $f \Rightarrow b_\flat + f \Rightarrow c$ is a zero length arrow, a fourth coupling. These two four-point configurations are each contained in one of two parallel planes. Besides these two four-element $\mathbb{Z}$-addressed chords, there is one $\mathbb{Z}$-addressed point $g \Rightarrow d$ in the first plane, and one $\mathbb{Z}$-addressed point $f \Rightarrow e$ in the second plane.

Now, if $u/1 : \chi^\star \xrightarrow{\sim} \chi^\star$ is an automorphism, it must either exchange the two $\mathbb{Z}$-addressed chords or leave them invariant, and it must also exchange the two additional points or leave them fixed. It is easily seen that an automorphism which leaves the planes fixed must be the identity since it must leave the above couplings fixed. But there is an automorphism which exchanges the two planes, and by the previous remark, this will be the unique non-trivial automorphism.

The automorphism has this effect:

$$u/1 : b \rightsquigarrow d_\flat$$
$$u/1 : d \rightsquigarrow e$$
$$u/1 : e_\flat \rightsquigarrow a$$
$$u/1 : f_\sharp \rightsquigarrow c$$
$$u/1 : a_\flat \rightsquigarrow b_\flat$$
$$u/1 : g \rightsquigarrow f$$

and on the original chromatic scale $\chi$ (!) is given by the affine $EulerZModule$ automorphism $U = e^s.U_0$ with translation $s = (-5, 2, 1)$ and matrix

$$U_0 = \begin{pmatrix} 11 & 18 & 24 \\ -4 & -7 & -8 \\ -2 & -3 & -5 \end{pmatrix} \tag{24.2}$$

which is automatically an involution since it exchanges the planes of the two chords which are centered around $g$ and $f$. Also observe that the $\mathbb{Z}$-addressed point $g \Rightarrow f$ is reversed to $f \Rightarrow g$ under $u/1$. The interesting thing about this involution is that it yields an involution $e^6.5$ of $OnMod_{12}$ if we transport the tone names as usual, i.e., $c \sim 0, d_\flat \sim 1, \ldots b \sim 11$.

Moreover, if we consider the translation $\chi$ to $\Xi = e^{-h}\chi$ by the vector $h = (-3, 1, 1)$, the automorphism of this scale chord is the conjugate $V = e^{-h}.U.e^h$. But the elements of $\Xi$ are the just interval quantities of the chromatic chord $\chi$ when viewed from $h$. The above dichotomy which was given by the two parallel planes now corresponds to the two following six-by-six element dichotomy of $\Xi$:

$$\Xi_K = \{(0,0,0), (0,1,-1), (4,0,-2), (-2,1,0), (6,-1,-2), (2,0,-1)\},$$
$$\Xi_D = \{(7,-2,-2), (1,-1,0), (3,-2,0), (3,-1,-1), (7,-3,-1), (5,-2,-1)\},$$

and its interpretation in terms of differences of numbers in $\mathbb{Z}_{12}$ as above, e.g., $b - b \sim 11 - 11 = 0, d - b \sim 2 - 11 = 3, e_\flat - b \sim 3 - 11 = 4$, etc. yields the consonance-dissonance dichotomy $K = \{0, 3, 4, 7, 8, 9\}, D = \{2, 5, 10, 1, 6, 11\}$ of classical counterpoint. The literally transformed automorphism is the involution $e^2.5$ of $OnMod_{12}$, the (unique) *autocomplementarity function* of the counterpoint model from which the rules of elementary counterpoint will be deduced in part VII. We come back in section 24.1.3 to this phenomenon.

Instead of looking at a chord of $\mathbb{Z}$-addressed points with tail in $g$, one can also take a $\mathbb{Z}^3$-addressed 'dominant seventh' point $D^7$ with tail in $g$, and three arrowheads in $b, d, f$. The automorphism $U$ is completely determined by this self-addressed point since it defines an affine base of the ambient space. So $U$ is uniquely determined by $U(D^7)$. The same is true for the 'tonic seventh' point $T^7$ with tail $g$ and arrowheads in $c, e, b_\flat$; same for the 'subdominant seventh' point $S^7$ with tail $f$ and arrowheads $a, c, e_\flat$ (observe that the major seventh would not work in any of these points!). So the three previous self-addressed seventh points $X^7$ are all sufficient to describe the entire automorphism, i.e., $U(X^7)$ determines $U$.

Consider the canonical pitch class projection

$$mod_o : EulerModule \to EulerPlane$$

defined in 7.11. This induces the projection of the corresponding chord spaces $mod_o : EulerChord \to o\text{-}ClassChord$. Since $mod_o$ is a projection with integer matrix, we have a corresponding integer situation $mod_o : EulerZModule \to EulerZPlane$ and

$$mod_o : EulerZChords \to o\text{-}ClassZChord$$

with $EulerZChord = EulerChord|_{\mathbb{Z}}, o\text{-}ClassZChord = o\text{-}ClassChord|_{\mathbb{Z}}$, the restrictions to $\mathbb{Z}$-valued points of chords.

Two $A$-addressed chords $C_1, C_2$ in $EulerModule$ are called *an inversion of each other* iff $mod_o(C_1) = mod_0(C_2)$. Inversion equivalence classes of chords, i.e., pitch class chords, are central objects in harmony. The present just pitch class space is spanned by the fifth $q$ and by the third class $t$. Vogel's zero-addressed scale $C\text{-}chromatic$ which is the projection $mod_o(\chi)$ of the chromatic chord $\chi$ discussed before, gives rise to an infinity of zero-addressed scales

$$X\text{-}chromatic = e^X.C\text{-}chromatic$$

in $EulerZPlane$ for any zero-addressed class $X$ in $EulerZPlane$.

**Proposition 40** *Consider the lattice* $L = \mathbb{Z}.(1,0,0) + \mathbb{Z}.(0,-4,1) + \mathbb{Z}.(0,0,3)$. *Then the* $L$*-translates of* $\chi$ *define a tiling of* $EulerZModule$.

Proof. The $mod_o(L)$ translates of $C\text{-}chromatic$ define a tiling of $EulerZPlane$, and the octave translates fill up the space in octave direction. Observe, that the vector $(0,-4,1)$ corresponds to the fifth comma defined in 6.33, whereas the third generator of the lattice is $Kq+19.o+3.Kt$, so *the lattice is generated by the octave and the two commata.* We shall discuss this lattice in a moment.

The zero-addressed major scale $C\text{-}major$ and natural minor scale $C\text{-}minor$ give rise to corresponding shifted scales

$$X\text{-}major = e^X.C\text{-}major, X\text{-}minor = e^X.C\text{-}minor.$$

In the same way, the just zero-addressed harmonic minor scale $X\text{-}harm$ stems from the harmonic minor scale $C\text{-}harm$ which differs from $C\text{-}minor$ by a non-flattened $b$, whereas the zero-addressed melodic minor scale $X\text{-}mel$ stems from the melodic minor scale $C\text{-}mel$, and this one differs from $C\text{-}minor$ by non-flattened $a, b$.

For the time being, we shall use the same tone names for pitch classes as for their original pitches in $EulerZModule$. As with the $EulerZModule$, we have prominent $\mathbb{Z}^2$-addressed, i.e., self-addressed points in $EulerZPlane$ which determine any endomorphism $V : EulerZPlane \to EulerZPlane$ because their tail and arrowheads are an affine base. Here are such candidates: the 'major triad' point $C^5$ with tail $C^5(0) = c$ and heads $C^5(e_1) = g(= q)$, $C^5(e_2) = e(= t)$, the 'minor triad' point $c^5$, same as $C^5$, except $c^5(e_2) = e_\flat(= q - t)$.

#### 24.1.1.1 Just Mutation

**Summary.** On the $EulerZModule$ a Lie algebra structure can be introduced in order to give a "processual" interpretation of the enharmonic projection (see 24.1.3).

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For the following Lie algebra theory, see appendix E.4.4. The $EulerZModule$ is charged with a Lie algebra structure in the following way. We represent the module of $EulerZModule$ in the basis $o, q, r = Kt - q$ ($Kt = t - 4q - 4o$ being the syntonic comma, see 6.33) and induce the Lie ($\mathbb{Z}$-)algebra structure from the module isomorphism

$$sl_2 : \mathbb{Z}.o \oplus \mathbb{Z}.q \oplus \mathbb{Z}.r \xrightarrow{\sim} \mathfrak{sl}(2, \mathbb{Z})$$

defined[1] by

$$sl_2(o) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, sl_2(q) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, sl_2(r) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

The Lie bracket $[x, y]$, i.e., Noll's *mutation* ([400]), on this basis reads as follows: $[o, q] = 2q, [o, r] = -2r, [q, r] = o$, the other values are mandatory by skew symmetry.

With this Lie bracket, the adjoint endomorphisms $ad(q), ad(r)$ are nilpotent, by definition, and we have the Lie algebra automorphisms

$$Q = \quad exp(ad(q)) = Id + ad(q) + ad(q)^2/2, \tag{24.3}$$
$$R = \quad exp(ad(r)) = Id + ad(r) + ad(r)^2/2. \tag{24.4}$$

It is well known that on the level of $\mathfrak{sl}(2, \mathbb{Z})$, these automorphisms $Q, R$ identify to the conjugation by

$$exp(sl_2(q)) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, exp(sl_2(r)) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

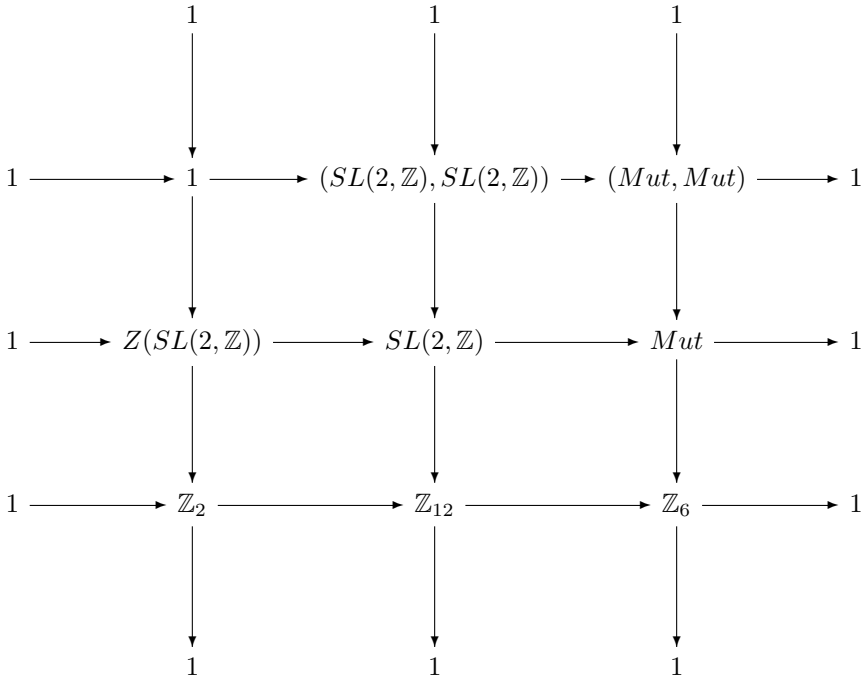
respectively, a generator set of $SL(2, \mathbb{Z})$. Consider the group

$$Mut = \langle Q, R \rangle \subset Aut(EulerZModule)$$

generated by these operators and the center $Z(SL(2, \mathbb{Z})) = \{\pm Id\}$. Then we have this result:

[1] See appendix E.4.4, example 80, for the definition of $\mathfrak{sl}(2, \mathbb{Z})$.

**Theorem 27** [400, III.4.3, Theorem 1] *We have a group diagram*




*with exact columns and rows and the canonical homomorphisms that are related to the conjugacy operation of the special linear group on* $\mathfrak{sl}(2,\mathbb{Z})$.

In fact, in [400, III.4.3], it is proven that $SL(2,\mathbb{Z})/(SL(2,\mathbb{Z}),SL(2,\mathbb{Z})) \tilde{\to} \mathbb{Z}_{12}$, but the center generator $-Id$ is not in the commutator, and therefore projects to the element 6 of order 2 in $\mathbb{Z}_{12}$, so we are done.

Furthermore, we may define a set map

$$Ex : EulerZModule \to SL(2,\mathbb{Z})$$

by $Ex(a.o+b.q+c.r) = exp(sl_2(q)^b.exp(sl_2(r)^c$ ($o$ goes to the identity). By the above theorem, this map induces a surjective group homomorphism

$$ENH : EulerZModule \to \mathbb{Z}_{12}.$$

Consider the subform $CommaZModule$ of $EulerZModule$ which is spanned by the Pythagorean and Syntonic commata $Kq, Kt$, and by the octave $o$. Then we have

**Theorem 28** [400, III.4.3, Theorem 2]
*The kernel $Ker(ENH)$ equals the subform $CommaZModule$.*

We shall discuss this result in section 24.1.3 in the context of enharmonic identification methods. However, the point of the preceding construction is that it essentially introduces a

group of space automorphisms on $EulerZModule$ which reside on an "exponentiation lifting" from zero-addressed points $o, q, r$ to self-addressed points, i.e., $Id, Q, R$, related to a linear Lie algebra interpretation of the underlying module. So it not only lifts zero-addressed points to constant self-addressed points via address change projection, but gives them a non-trivial lifting. At present, no musicological justification of this Lie algebra construction is known—except that it does give a new interpretation to enharmonic identification. It is in particular not known whether there is a more general and/or not arbitrary, viz., canonical construction behind this idea. It is also not clear why the special linear group should be invoked as a lifting of the mutation operator group $Mut$, and a musicological interpretation of the mutation operators $Q, R$ is not known.

### 24.1.2 12-tempered Perspectives

**Summary.** This section discusses constant and self-addressed chords in the context of pitch classes in 12-tempered tuning.

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This situation stems from the 12-tempered part of the $EulerModule$, i.e., the subform 12-$Temp$ of $EulerModule$ of the one-dimensional $\mathbb{Z}$-submodule $\mathbb{Z}.\frac{1}{12}.o \subset \mathbb{Q}.o \oplus \mathbb{Q}.q \oplus \mathbb{Q}.t$, together with its projection

$$mod : 12\text{-}Temp \to 12\text{-}PitchClass \xrightarrow{\sim} PiMod_{12}$$

of octave identification. So we have to keep in mind that this construction is completely separated from the just tuning spaces. We are going to discuss the relation between these two approaches in section 24.1.3 below.

Beyond the general Yoneda principle, and to our knowledge of mathematical music theory, the historically[2] first passage from zero-addressed tones in $PiMod_{12}$ to self-addressed tones, i.e., tones in $\mathbb{Z}_{12}@PiMod_{12}$, was suggested by the question in [340, section 4.2.3.2.2] concerning the deduction of important chords of music theory from basic principles of the underlying pitch spaces. In the foundations of harmony, they try to deduce the structure of important chords, above all the major and minor triad, from the acoustic phenomenon of partials in Fourier analysis, for example with Rameau [433], Schönberg [478], or Hindemith [224]. A slightly different deduction is offered by Oettingen [406] or Vogel [547] by use of the $EulerZPlane$ where fundamental, third, and fifth define a special generating zero-addressed local composition.

"Deduction" means that it is attempted to justify the eminent role of those chords from an ontological point of view, as an a priori condition that precedes their role as structural fundamentals of harmony. In any case, this foundation is based on just tuning and contradicts the practice of well-tempered tuning. Mostly, the argument in favor of this contradiction is that we are living a compromise, a neat refusal of the tempered tuning being the exception, such as with Vogel who even ran a political initiative to banish tempered instruments from public schools.

Our concern is however not a dogmatic one, we rather would offer an approach to fundamental harmony which also works for tempered tuning and—at the same time—evokes the

[2] See Noll's motivation in [400, I.1.4].

rationales in just tuning ideology. In his approach, Schönberg views the major triad as an "imitation of its fundamental tone by partials". Starting from $c$, he denotes the sequence of partials $c', g', c'', e'', g'', b_\flat'', c''', \ldots$ where the first three octave pitch classes are $c, g, e$ (in this order!), i.e., an inversion equivalent of the major triad. There is no immanent reason to stop at this stage, except that one step further, the catastrophe happens: the "natural seventh" (German: "Naturseptime"), with tonality $b_\flat$, where after 2,3,5 the new prime 7 appears in the frequency ratio, and we have to extend the $EulerModule$. So the triad is not really given as a closed unit, but as a first segment of an infinite chain.

For the 12-tempered tuning, the 'partial' argument does not work, and we have to look for other mechanisms. Whereas the partial frequencies are positive multiples of the fundamental frequency, the 12-tempered tuning offers endomorphisms on 12-$Temp$ and on $PiMod_{12}$. Let us look at the pitch classes in $PiMod_{12}$. If $k \in \mathbb{Z}_{12}$ is a zero-addressed pitch class, and if $f = e^x.n$ is an endomorphism, its effect on $k$ is $f(k) = e^{k.n}x$, the $k$-fold translation of $n$ from $x$. So the action of $f$ on all pitch classes $k = 0, 1, 2, 3, \ldots 11$ is just the sequence

$$x, x + n, x + 2.n, \ldots x + 11.n$$

which in music theory is called the *$n$-circle from $x$* (the number $n$ corresponding to multiplication in $\mathbb{Z}_{12}$).

**Exercise 56** An instructive example is the "circle of fourths" and the "circle of fifths" (just run through the circle of fourths in the opposite direction). Major scales are usually arranged in this order:

$$\text{ascending: } C, F, B_\flat, E_\flat, A_\flat, D_\flat, \quad \text{descending: } C, G, D, A, E, B, F_\sharp$$

where $F = e^5.C$, etc., and $G = e^{-5}.C$, etc. It is well-known that the alteration signs increase one-by-one on the fourth circle, whereas they increase one-by-one on the fifth circle when we transgress the scales in this order. Why? Since this question only regards the white and black keys, we may really work in tempered terminology.

The musician's common argument is that we may go from $C$ to $F$ by alteration $b \mapsto b_\flat$, and than proceed analogously. But the black keys do not shift along the transition from $C$ to $F$. The exact reason is that we may multiply everything by 5 and then work in the chain of fourths. Thereby, $C$ switches to $5.C = \{b, e, a, d, g, c, f\}$, an uninterrupted sequence of fourths. And the shifted scale $F = e^5.C$ switches to $5.F = 5.e^5.C = e^1.(5.C)$, the shift of the switched scale $5.C$ by one unit. This shifting operation is repeated in every fourth step of scales, and therefore, each new scale adds one black key, i.e., one flattening of a white key. Check the details of this argument. Try to convince musicians that the argument does not work for other white-black-key distribution such as those which would define "white = melodic C minor, black = remaining keys".

The 12-tempered simulation of the partial argument is this: We start from the fundamental tone $c = 0 \in \mathbb{Z}_{12}$ and take an endomorphism $S = e^7.3$ of $PiMod_{12}$. We consider the monoid $\langle S \rangle$ generated by $S$ and let it act on the fundamental. We have this orbit:

$$\langle S \rangle(c) = \{c, S(c) = g, S^2(c) = e, S^3(c) = g, \ldots\} = I_C$$

which is precisely the major $C$ triad. Observe that the order of images is the same as with partials: $c, g, e$, except that it is completed on this triad, no further tones must be cut in order to obtain the triad.

**Exercise 57** Show that the minor triad $I_{c_m}$ is a similar orbit $\langle T\rangle(g)$. Show that there are exactly two endomorphisms $W$ such that $\langle W\rangle(x) = I_C$ for a selected tone $x$.

This suggests the question of whether this method produces other prominent chords and whether there are chords which look essentially different from those defined by this orbit method.

The second question is easily answered: Yes, take the dominant seventh chord $V_C^7 = \{g, b, d, f\}$. This one cannot be defined as an orbit of a cyclic monoid $\langle S\rangle$ which acts on a single tone. One needs at least two generators for such an orbit, e.g., $R = e^8.3, T = e^2.0$. We then have

$$\begin{gathered}\langle R\rangle(g) = \{g, R(g) = f, R^2(g) = b, R^3(g) = f, \ldots\},\\ \langle T\rangle(g) = \{g, T(g) = const = d \ldots\},\\ R(d) = d, \text{ and therefore}\\ \langle R, T\rangle(g) = V_C^7.\end{gathered}$$

Possibly, this bigeneric monoid simulates what Oettingen called "Bissonanz", however here this one is not given by two unrelated triads (as with Oettingen) but by two symmetries acting on one fundamental tone.

Before we set forth a systematic discussion of this generative approach to chord theory, let us answer the first of the above questions. We call a zero-addressed chord of shape $Ch = \langle S\rangle(x)$ a *circle chord* (generated by $S$ and $x$). Any chord which is isomorphic to a circle chord is also a circle chord. In fact, if $h : PiMod_{12} \xrightarrow{\sim} PiMod_{12}$ is an automorphism inducing an isomorphism $h : \langle S\rangle(x) \xrightarrow{\sim} X$, then we have $X = \langle h.S.h^{-1}\rangle(h(x))$. So we are left with the problem to describe all circle chords for representatives of conjugacy classes of symmetries of $PiMod_{12}$. The table of these sixteen chord classes in figure 24.2 shows that—except one class number 33 (see the classification of chords in appendix L.1)—all classes are *common chords*.

**Fact 12** *Circle chords really yield a generative fundament for basic chords in harmony, and one may refrain from evoking just temperament and tuning compromises when laying this basis to harmony.*

### 24.1.3 Enharmonic Projection

**Summary.** There is an *enharmonic* projection from Euler pitch classes to 12-tempered pitch classes which relates constructions in the two spaces. These relations are discussed.

– Σ –

Just tuning and well-tempered tuning are fundamentally different choices of pitch spaces within the $EulerSpace$. The first is a free $\mathbb{Z}$-submodule of rank one generated by $\frac{1}{12}o$, whereas the second is free of rank three, i.e., the $EulerZModule$ and intersects the well-tempered module

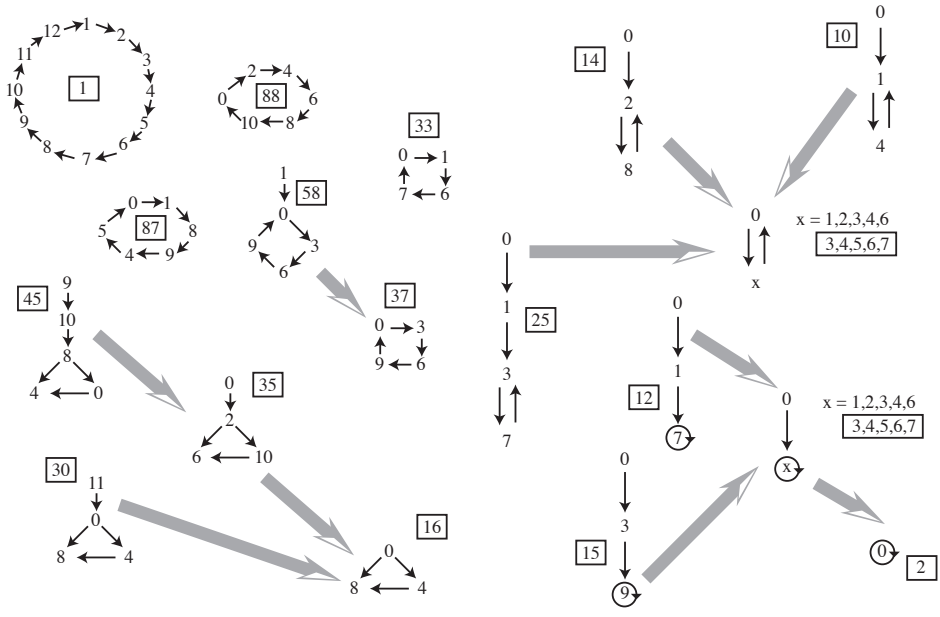


Figure 24.2: The 16 circle chords in $PiMod_{12}$, together with their generating symmetry action (arrows). The numbers refer to the classification of chords. A large arrow from one chord to another means that the target chord is isomorphic to a subchord of the start chord, and that the subchord is generated by a suborbit of the same symmetry action as for the superchord.

only in the octave-submodule $\mathbb{Z}.o$. The usual practical argument for switching between the two tunings is that the pitch differences between the corresponding Euler points are relatively small.

The theoretical counterpart of this switch resides on the so-called enharmonic identification under the comma module $CommaZModule$. Classically, two points in the $EulerZModule$ are called *enharmonic (with each other)* iff their difference lies in the submodule $\mathbb{Z}.Kt \oplus \mathbb{Z}.Kq \oplus \mathbb{Z}.o$ of the $CommaZModule$. We shall more generally say that they are enharmonic iff their difference is in the $EulerZModule$, i.e., they are identified by the enharmonic projection

$$enh : EulerZModule \to EulerZModule/CommaZModule \tilde{\to} \mathbb{Z}_{12}$$

(see section 24.1.1.1, theorem 28) which besides the fifth and third commata annihilates the octave. Enharmonic identification is a construction which resides on the cognitive rationale of commata, i.e., their role for auditory identification, but it has nothing to do with tempered tuning. *The enharmonic projection is not a tuning, but an abstraction.* So making music theory on this abstraction is one business, whereas realizing this abstraction on the 12-tempered tuning is another.

In his processual interpretation of enharmonic identification via theorem 28, Noll charges the enharmonic projection by a cognitive dimension in the following sense. Look at the com-

mutative diagram issued from section 24.1.1.1:

$$\begin{array}{ccc} EulerZModule & \xrightarrow{Ex} & SL(2,\mathbb{Z}) \\ \downarrow{\scriptstyle enh} & \searrow{\scriptstyle ENH} & \downarrow \\ \dfrac{EulerZModule}{ComaZModule} & \xrightarrow{\sim} & \dfrac{SL(2,\mathbb{Z})}{(SL(2,\mathbb{Z}),SL(2,\mathbb{Z}))} \end{array}$$

where the upper horizontal map $Ex$ is the exponential map, and the lower isomorphism identifies both quotient groups to $\mathbb{Z}_{12}$. The lifting map $Ex$ interprets the enharmonic homomorphism $enh$ as follows: A point in the $EulerZModule$ is lifted to a product of two-by-two matrices, starting from the lifting of the fifth to the upper triangular generator, whereas the third is lifted to the lower triangular generator. The general Euler point is lifted to a product of such matrices. This cognitively means that the cognitive identification of an Euler point is given by a sequence of steps of either fifth or third generator type. The non-commutativity of the concatenation of fifth and third generators is resolved by identification of commutators $x.y.x^{-1}.y^{-1}$ with the identity. And this precisely yields the modular group $\mathbb{Z}_{12}$. If instead, we stay in the modular group $SL(2,\mathbb{Z})$, the non-commutativity of the fifth and third has been interpreted by Noll [400] as a rationale of the enharmonic identification in the same sense that non-commutativity of operators gives rise to the uncertainty relation in quantum mechanics: enharmonicity resembles uncertainty. As we already stated, the nature of the modular lifting—up to now—has neither formal nor musicological or cognitive justifications; not every mathematical structure yielding $\mathbb{Z}_{12}$ in some way is automatically musicologically justified.

However, the enharmonic projection helps understand the above remark concerning the automorphism $U$ of the chromatic scale $\chi$ (see the linear part $U_0$ of $U$ in formula 24.2) when it is transferred to pitch classes via "literal transformation".

**Definition 78** *An endomorphism $\Phi$ of the $EulerZModule$ is called* enharmonic *iff it commutes with the enharmonic projection, i.e., iff there is a (necessarily unique) endomorphism $\phi$ of $PitchMod_{12}$ such that the diagram*

$$\begin{array}{ccc} EulerZModule & \xrightarrow{\Phi} & EulerZModule \\ \downarrow{\scriptstyle enh} & & \downarrow{\scriptstyle enh} \\ PitchMod_{12} & \xrightarrow{\phi} & PitchMod_{12} \end{array} \tag{24.5}$$

*commutes.*

Here is a characterization of enharmonic endomorphisms:

**Proposition 41** (i) *Every translation $\phi = e^s$ is enharmonic.*

(ii) *An endomorphism $\Phi$ of the $EulerZModule$ is enharmonic iff its linear part $\Phi_0$ is, i.e., iff $\Phi_0(CommaZModule) \subset CommaZModule$.*

(iii) *The enharmonic endomorphisms define a multiplicative monoid*

$$End_{enh}(EulerZModule) \subset End(EulerZModule).$$

*The invertible elements among the enharmonic endomorphisms define a subgroup, the* enharmonic group $\overrightarrow{GL}_{enh}(EulerZModule)$.

(iv) *Rewriting the Euler points in the basis* $o, Kt, q$, *and* $\Phi_0$ *as a matrix*

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

*in this basis,* $\Phi$ *is enharmonic iff* $a_{21} \equiv a_{23} \equiv 0 \mod 12$.

Proof: Exercise for the reader, except for the statement concerning the enharmonic group $\overrightarrow{GL}_{enh}(EulerZModule)$: Observe that an invertible linear endomorphism which leaves the $CommaZModule$ invariant automatically induces an automorphism of $CommaZModule$ since its index in $EulerZModule$ is finite. This statement also follows directly from the calculation of the inverse of a matrix and the criterion (iv) of the proposition.

**Corollary 11** ([400, III.4.1]) *The autocomplementarity involution* $U$ *of the just chromatic chord* $\chi$ *and its conjugate* $V$, *the autocomplementarity involution of* $\Xi$, *are both enharmonic.*

In fact, the first follows from the matrix criterion above, and the second follows from the fact that translations, and therefore conjugation with translations, are enharmonic.

**Corollary 12** ([400, III.4.1]) *The chromatic chord* $\chi$ *and its translate* $\Xi$ *define two sections* $OnPiMod_{12} \to EulerZModule$ *to the enharmonic projection. The enharmonic autocomplementarity involutions* $U$ *and* $V$ *of* $\chi$ *and* $\Xi$, *respectively, induce the autocomplementarity involutions* $e^6.5$ *and* $e^2.5$ *on* $\chi$ *and* $\Xi$, *respectively.*

This follows from corollary 11 and from proposition 40.—So the literal translation is in fact induced by the enharmonic projection, and we easily recognize that the tiling lattice in proposition 40 is the $CommaZModule$. So the autocomplementarity function $U$ permutes the tiling lattice and therefore also the tiling as such. Conversely, modulo $CommaZModule$-translations, any affine permutation of the chromatic tiling identifies to the autocomplementarity involution.

**Theorem 29** *The group of affine automorphisms of the* $\chi$*-chromatic tiling of the* $EulerZModule$ *is the semi-direct product* $e^{CommaZModule} \rtimes \langle U \rangle$, *of the autocomplementarity involution* $U$ *with the* $CommaZModule$*-translations. The same holds with, mutatis mutandis,* $\chi$ *being replaced by* $\Xi$, *and* $U$ *by the* $\Xi$*-involution* $V$.

**Fact 13** *Musicologically, this means that the autocomplementarity of consonant and dissonant intervals (see chapter 30 for this subject) is induced by a unique inner symmetry on the level of just tuning and its (discrete) interpretation by Vogel's chromatic chord.*

The moral of this discussion is that in both cases, 12-tempered and just tuning, certain endomorphisms of the underlying pitch space are crucial structures for understanding zero-addressed harmonic, and—as will be shown in the counterpoint theory—contrapuntal phenomena. Intuitively, the above example of the automorphism groups of chromatic interpretations (by Vogel's $\chi$ or by its 'intervallic shift' $\Xi$) suggests that this order two group is composed of two agents: the identity for the consonant, and the autocomplementarity involution for the dissonant 'character'. In the same spirit, the preceding generators of zero-addressed circle chords in 12-tempered pitch class space suggested that these agents should be viewed as if they were tones of a more general nature. In the sequel we shall set up a more systematic context for such agents and their relations.

## 24.2 Chord Topologies

**Summary.** Chord topologies are built upon chord sets which are invariant under collections of symmetries. These sets define a topological basis.

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Although this section deals with and stems from investigations of harmonic structures, it turns out that some fundamental buildings can be carried over to completely general local compositions. We prepend this setup before specializing to proper harmony.

### 24.2.1 Extension and Intension

**Summary.** The general technical setup for intension-extension topologies (to be presented in the next section) in terms of local compositions is presented.

– $\Sigma$ –

Let $F$ be any form space (which we identify with its functor if no confusion is likely), $A$ an address, $a \subset @A \times F$ an $A$-addressed local composition in $F$, and $f : F \to F$ an endomorphism. We shall say in this context that *$f$ is an endomorphism of $a$* if $f|a/1 : a \to a$ is a morphism, i.e., if $Im(a) \subset a$. Denote by $SemiEnd(F), MonEnd(F)$, respectively, the set of subsemigroups, submonoids of $End(F)$ (with the identity $Id_F$ as neutral element), respectively.

**Definition 79** *Let $M \subset End(F)$ a set of endomorphisms of $F$ and $A$ an address. The set*

$$Ext_A(M) = \{a \in A@\Omega^F |\ f \text{ is an endomorphism of } a \text{ for all } f \in M\} \tag{24.6}$$

*is called a* basic extension of $F$ at address $A$. *The sets*

$$Int(M) = \{m \in SemiEnd(F) |\ M \subset m\}, \tag{24.7}$$

$$Int^e(M) = Int(M) \cap MonEnd(F), \tag{24.8}$$

*are called* basic (semigroup, monoid, respectively) intensions of $F$.

*For a set $E \subset A@\Omega^F$, the monoid*

$$Int(E) = \{f |\ f \in End(F), f \text{ is an endomorphism of every } a \in E\} \tag{24.9}$$

*is called* the intension[3] of $E$.

**Sorite 9** *With the above notation, we have the following facts:*

(i) *For $M \subset N \subset End(F)$, we have $Ext_A(N) \subset Ext_A(M)$, $Int(N) \subset Int(M)$, and $Int^e(N) \subset Int^e(M)$.*

(ii) *For $M \subset End(F)$, $Ext_A(M) = Ext_A(\langle M\rangle) = Ext_A(\langle M\rangle_e)$[4], $Int(M) = Int(\langle M\rangle)$, and $Int^e(M) = Int^e(\langle M\rangle) = Int^e(\langle M\rangle_e)$.*

(iii) *For $M, N \subset End(F)$,*

$$Ext_A(M) \cap Ext_A(N) = Ext_A(M \cup N) = Ext_A(\langle M, N\rangle) = Ext_A(\langle M, N\rangle_e),$$

*and*

$$\begin{aligned} Int(M) \cap Int(N) &= Int(M \cup N) = Int(\langle M, N\rangle), \\ Int^e(M) \cap Int^e(N) &= Int^e(M \cup N) = Int^e(\langle M, N\rangle) = Int^e(\langle M, N\rangle_e). \end{aligned}$$

(iv) *For a set $E \subset A@\Omega^F$, we have $E \subset Ext_A(Int(E))$, and for a set $M \subset End(F)$, we have $M \subset Int(Ext_A(M))$.*

(v) *For an $A$-addressed local composition $a \subset @A \times F$, we set $Int(a) = Int(\{a\})$ and call it* the intension of $a$. *Then we have*

$$Ext_A(Int(a)) = \{b|\ f/1 : a \to a \text{ implies } f/1 : b \to b\}.$$

(vi) *For a set $M \subset End(F)$ and an address change $\alpha : A \to B$, we have*

$$Ext_B(M).\alpha \subset Ext_A(M).$$

*If $\alpha$ is a retraction, i.e., there is a right inverse $\beta : B \to A$, $\alpha.\beta = Id_B$, then, for any local composition $a \subset @B \times F$, $Int(a.\alpha) = Int(a)$, whence*

$$\alpha^{-1}(Ext_A(M)) = Ext_B(M).$$

(vii) *For a set $M \subset End(F)$, and $a, b \in Ext_A(M)$, we have $a \cap b, a \cup b \in Ext_A(M)$.*

(viii) *If $a \subset A@F$ is an objective local composition, the set of endomorphisms $f/1 : a \to a$ qua objective local composition coincides with the intension of its functorial counterpart $\hat{a}$.*

Proof. All statements except (vi), (vii) and (viii) are straightforward. Statement (viii) follows from the fact (see proposition 3 in chapter 8) that objective local compositions build a full subcategory of the functorial local compositions, and that underlying form endomorphisms

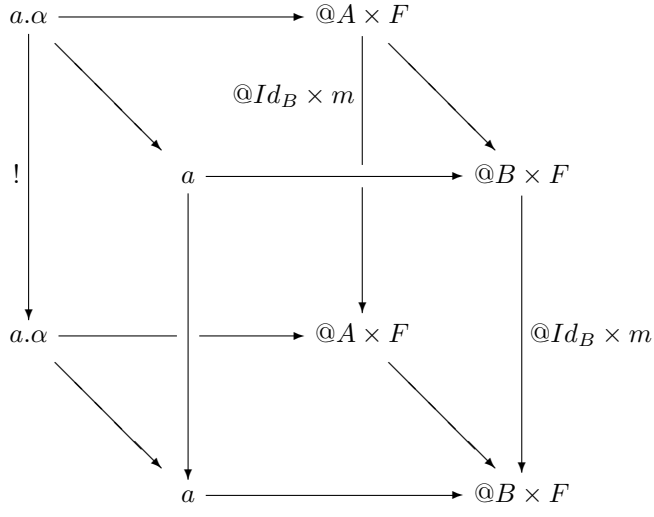
[3] Terminology introduced by Noll [403] in the case of self-addressed chords of pitch classes and to be justified below in chapter 25.

[4] $\langle M\rangle$ is the semigroup generated by $M$, whereas $\langle M\rangle_e$ is the monoid generated by $M$, see appendix C.2.3.

of the functorial versions carry over to the objective versions. To prove statement (vi), let $a \in Ext_B(M)$. Then, by definition, we have a cartesian diagram

$$\begin{array}{ccc} a.\alpha & \longrightarrow & a \\ \downarrow & & \downarrow \\ @A \times F & \xrightarrow{@\alpha \times Id_F} & @B \times F \end{array} \tag{24.10}$$

and therefore, for any $m \in M$, we have a unique vertical arrow ! making the diagram




commute. This shows that $a.\alpha \in Ext_A(M)$. To prove the second part of (vi), consider $a.\alpha \in Ext_A(M)$. Since under the hypothesis on $\alpha$, the transformation $@\alpha \times Id_F$ is epi, and since the upper horizontal square is cartesian, the morphism $a.\alpha \to a$ is also epi. Therefore, there is a vertical arrow $a \to a$ making the entire cube commute, and we conclude $a \in Ext_B(M)$ and therefore, $Int(a.\alpha) = Int(a)$. By the first part of (vi), we conclude that $\alpha^{-1}(Ext_A(M)) = Ext_B(M)$. QED.

### 24.2.2 Extension and Intension Topologies

**Summary.** Intension and extension topologies on local compositions are presented.

– Σ –

With the above technicalities, we may now introduce different topological structures.

**Definition 80** *The collection of basic semigroup, monoid, respectively, intensions of $F$ define bases for two topologies on $SemiEnd(F), MonEnd(F)$, respectively, the* (semigroup, monoid, respectively) intension topologies

$$SemInTop(F), InTop(F)$$

of $F$, *respectively.*

**Lemma 38** *The projection*

$$?_e : SemiEnd(F) \to MonEnd(F)$$

*(see appendix C.2.3, exercise 84) is continuous and open onto the (basic) open subspace* $MonEnd(F) = Int(Id_F)$ *of* $SemiEnd(F)$.

Proof. Clearly, $MonEnd(F) = Int(Id_F) \subset SemiEnd(F)$. To see that $?_e$ is continuous, take a basic monoid intension $Int^e(M)$. Its inverse image under $?_e$ is $\{s|\ M \subset s_e\}$. Suppose first that $Id_F \notin M$. Then $M \subset s_e = s \cup \{Id_F\}$ implies $M \subset s$, and conversely. So $?_e^{-1}(Int^e(M)) = Int(M)$. If $Id_F \in M$, then $s \in ?_e^{-1}(Int^e(M))$ iff $M \setminus \{Id_F\} \subset s_e \setminus \{Id_F\}$. This implies that $?_e^{-1}(Int^e(M)) = Int(M \setminus \{Id_F\})$, so the map is continuous. It is open since we have $?_e(Int(M)) = Int^e(M)$, and we are done.

**Sorite 10** *The intension topologies have these properties:*

(i) *With the preceding notation, if* $s \in SemiEnd(F), m \in MonEnd(F)$, *respectively, then*

$$Int(s) = \{t \in SemiEnd(F), s \subset t\}, Int^e(m) = \{n \in MonEnd(F), m \subset n\}$$

*are the smallest open neighborhoods of* $s, m$ *in* $SemInTop(F), InTop(F)$, *respectively.*

(ii) *A subset* $X \subset SemiEnd(F), X \subset MonEnd(F)$, *respectively is closed iff for a subsemigroup* $t$, *submonoid* $n$, *respectively,* $t \subset s, s \in X$ *or* $n \subset m, m \in X$ *implies* $t \in X, n \in X$, *respectively.*

(iii) *The closure of a point* $s \in SemiEnd(F), m \in MonEnd(F)$, *respectively, is* $\bar{s} = \{t \in SemiEnd(F)|\ t \subset s\}, \bar{m} = \{n \in MonEnd(F)|\ n \subset m\}$, *respectively.*

(iv) *A closed subset* $X \subset SemiEnd(F), X \subset MonEnd(F)$, *respectively, is irreducible iff for* $m, n \in X$, $\langle m, n\rangle \in X, \langle m, n\rangle_e \in X$, *respectively.*

(v) *If every monoid in* $MonEnd(F)$ *is finitely generated (for example, if End(F) is finite), then* $SemInTop(F)$ *and* $InTop(F)$ *are sober.*

Proof. We prove the monoid case, the other is completely analogous. The first three statements are straightforward. As to the fourth, the irreducibility means that for any two $m, n \in X$, their smallest neighborhoods $Int(m), Int(n)$ intersect in $X$, i.e., there is $r \in X, n, m \subset r$, therefore the monoid $\langle m, n\rangle_e$ is in $X$ since $X$ is closed. As to the last statement, if $X$ is closed and irreducible, there is at most one generic point. In fact, $\bar{x} = X = \bar{y}$ means $x \subset y \subset x$. To find the generic point, consider the monoid $U = \langle m, m \in X\rangle_e$. Since $U$ is finitely generated, there are finitely many $m_1, \ldots m_k$ such that $U = \langle m_1, \ldots m_k\rangle_e$. But by the one but last statement, this is a member of $X$, and we have the generic point. QED.

**Definition 81** *The collection of basic extensions of* $F$ *at address* $A$ *defines a base for a topology on the set* $A@\Omega^F$ *of* $A$*-addressed local compositions in* $F$, *the* extension topology $ExTop_A(F)$ of $F$ at address $A$.

**Corollary 13** *For a local composition $a \in ExTop_A(F)$, the set*

$$\begin{aligned} Ext_A(Int(a)) &= \{b|\ f/1 : a \to a \text{ implies } f/1 : b \to b\} \\ &= \{b|\ Int(a) \subset Int(b)\} \end{aligned}$$

*is the smallest neighborhood of $a$, whereas the topological closure of $a$ is*

$$\begin{aligned} \bar{a} &= \{b|\ f/1 : b \to b \text{ implies } f/1 : a \to a\} \\ &= \{b|\ Int(b) \subset Int(a)\}. \end{aligned}$$

This follows from sorite 9, statement (v).

**Corollary 14** *The three topologies $ExTop_A(F), SemInTop(F), InTop(F)$ are irreducible with generic points $A@F\hat{}, End(F), End(F)$, respectively (non-unique). The topology $ExTop_A(F)$ is the inverse image topology of $InTop(F)$ under the map*

$$Int_A : ExTop_A(F) \to InTop(F) : a \mapsto Int(a) \tag{24.11}$$

*which therefore is automatically continuous. The fibers $Int_A^{-1}Int_A(a)$ consist of the $A$-addressed local compositions $b$ in $F$ with $\bar{b} = \bar{a}$.*

Proof. The irreducibility claims are clear. For a set $M \subset End(F)$, we have $a \in Ext_A(M)$ iff $M \subset Int(a)$ iff $Int(a) \in Int^e(M)$, in other words, the inverse image $Int_A^{-1}Int^e(M)$ equals $Ext_A(M)$.

**Corollary 15** *For an address change $\alpha : B \to A$ which is a retraction, the natural map $.\alpha : ExTop_A(F) \to ExTop_B(F)$ is continuous,*

$$\begin{array}{ccc} ExTop_A(F) & \xrightarrow{.\alpha} & ExTop_B(F) \\ & {\scriptstyle Int_A} \searrow \quad \swarrow {\scriptstyle Int_B} & \\ & InTop(F) & \end{array} \tag{24.12}$$

*commutes.*

This follows from sorite 9, statement (vi), in particular, we have $Int(a.\alpha) = Int(a)$ in this case.

**Proposition 42** *The induced topology on the image*

$$Int_A(ExTop_A(F)) \subset InTop(F)$$

*is homeomorphic to the induced topology of the canonical image*

$$i(ExTop_A(F)) \subset ExTop_A(F)^s$$

*in the sober topology of $ExTop_A(F)$. If all submonoids in $MonEnd(F)$ are finitely generated, the map $Int_A$ factorizes through $ExTop_A(F)^s$.*

Proof. We already know that the fibers of $Int_A$ and of $i$ coincide. A base for the topology of $Int_A(ExTop_A(F))$ is the collection of sets $I(M) = \{Int(a)|\ M \subset Int(a)\}$, $M \subset End(F)$. A base for the induced topology on $i(ExTop_A(F))$ in the sober topology is given by $E(M) = \{\bar{a}|\ Ext_A(M) \cap \bar{a} \neq \emptyset\}$. But this set, by corollaries 13, 14, equals the set $\{\bar{a}|\ \exists b, Int(b) \subset Int(a),$ and $M \subset Int(b)\}$, and this equals $I(M)$. If all submonoids are finitely generated, we know from sorite 10, statement (v), that the topology $InTop(F)$ is sober, and the statement follows from the universal property of the associated sober space. QED.

**Remark 8** The fact from proposition 42 which relates the closure of an $A$-addressed local composition $a$ in the extension topology, or, equivalently: the associated point in the sober topology, with its intension $Int(a)$ justifies the terminology of Noll in [400] where $Int(a)$ is called "Hüllakkord", in English "chord closure" or "closure chord". The difference to our point of view is, of course, that the intension of a chord is a set of endomorphisms, and not a chord. But this will be clarified in the following section.

### 24.2.3 Faithful Addresses

**Summary.** This section discusses the relation between endomorphisms and their "trace" on local compositions which live at addresses that capture enough information to produce a faithful image of endomorphisms. We deduce topological properties of such addresses.

– Σ –

**Definition 82** *Let $A$ be an address, $F$ a form space, and $z \in A@F$. The point $z$ is called* faithful, full, fully faithful, *respectively, if the map $.z : End(F) \to A@F : f \mapsto f.z$ is injective, surjective, bijective, respectively. We also say that the* address $A$ is faithful, etc., *if there is a point $z$ at that address which has the corresponding property.*

The classical example of a fully faithful point is the case of a representable space $F = @M$ for a module $M$, and $z = Id_M$.

**Definition 83** *Let $A$ be an address, $F$ a form space, and $z \in A@F$. Then we have a map*

$$ex_z : SemInTop(F) \to ExTop_A : m \mapsto m.z\widehat{\ }$$

*which associates with a semigroup $m$ the functorial local composition of the $m$-orbit of $z$; we set $monex_z = ex_z|InTop(F)$ for the restriction of $ex_z$ to the monoids.*

**Proposition 43** *If the point $z \in A@F$ is faithful, then the map $monex_z$ is a continuous section of $Int_A$, i.e.,*

$$Int_A \cdot monex_z = Id_{InTop},$$

*and the induced topology on $Im(monex_z)$ identifies to $InTop(F)$.*

Proof. If $M \subset End(F)$, then the inverse image of $Ext(M)$ under $monex_z$ is the set of monoids $m$ such that $M \subset End(ex_z(m))$. But the latter is $m$. In fact, by sorite 9, statement (viii), the endomorphisms of $ex_z(m)$ coincide with those of the objective trace $m.z$. But if $f$ is

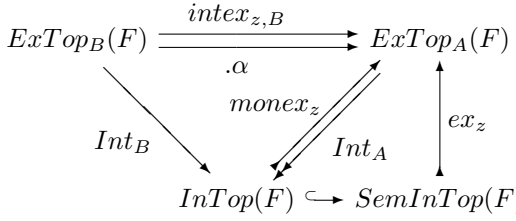
such an endomorphism, we have $f.Id_F.z = \mu.z, \mu \in m$. Since $z$ is faithful, $f = \mu$. Since trivially, $m \subset End(ex_z(m))$, the claim follows, and therefore, $M \subset End(ex_z(m))$ means $M \subset m$, i.e., the inverse image of $Ext(M)$ is $Int(M)$. Moreover, by the above, we also have $Int_A \cdot ex_z(m) = End(ex_z(m)) = m$ QED.

This means that the topological space $InTop(F)$ of monoids of $F$ can be viewed as a retraction of the topological space $ExTop_A$ of local compositions at a faithful address $A$ (as a function of the chosen faithful point). In particular, if $B$ is any address, and if $z$ is a faithful point at address $A$, we have the composed continuous map

$$intex_{z,B} = ex_z \cdot Int_B : ExTop_B \to ExTop_A \tag{24.13}$$

which associates with any $B$-addressed local composition the "objective trace" of its endomorphism monoid at the faithful address $A$. This means that the closure of a local composition $a$ can be reinterpreted in terms of the objective local composition $intex_{z,B}(a)$, the *$z$-objective (or, less precisely, $A$-objective) closure of $a$*. This is, what in the elementary case of $F = @\mathbb{Z}_{12}, B = 0, z = Id_{\mathbb{Z}_{12}}$ was studied by Noll in [400]. Here, we have a fully faithful point and therefore a complete identification of endomorphisms and points. We should, however keep in mind the ontological difference of endomorphisms and points. But mathematically, this very special case of pitch class theory is indifferent to ontological or more general mathematical perspectives.

So far, if $z \in A@F$ is faithful and $\alpha : A \to B$ is a retraction, we have a diagram with these continuous maps




where a local composition $a$ in $ExTop_B$ is mapped to its natural image $a.\alpha$ and to its objective closure $intex_{z,B}(a)$. In this general setup, the two images have nothing in common. We shall now discuss a context where these images are intimately related.

To this end, we suppose that first, $z$ is fully faithful. This means that $.z$ is bijective, call $/z$ its inverse. We further suppose that the image $B@F\hat{}.\alpha = B@F.\alpha\hat{}$ of the maximal $B$-addressed objective local composition in $F$ corresponds to a set of mutually right-absorbing endomorphisms of $F$ under $/z$, i.e., for all $u, v \in B@F$,

$$u.\alpha/z \cdot v.\alpha/z = u.\alpha/z.$$

Again, the classical example with $F = @M, A = M, B = 0, z = Id_M$, and $\alpha : M \to 0$ the zero-homomorphism, yields the embedding of zero-addressed local compositions as self-addressed local compositions with points $m \in M$ corresponding to the constant points $e^m.0$. These are evidently mutually right-absorbing.

A point $z$ that is fully faithful and induces a set of mutually right-absorbing endomorphisms of $F$ on $B$-addressed points of $F$ is called *$B$-absorbing*[5]. In the following calculations, we shall use the Yoneda notation $x : X \to F$ for the point $x \in X@F$ corresponding to $x : @X \to F$.

**Lemma 39** *Let $z$ be a $B$-absorbing point and $\alpha : A \to B$ a retraction. If $g \in B@F$, then $g_\alpha := g.\alpha/z$ is constant on $B$-valued points of $F$, i.e., $g_\alpha.h = g$ for all $h : B \to F$.*

Proof. Since $\alpha$ is an epimorphism, it suffices to show $g_\alpha.h.\alpha = g.\alpha$. But $g_\alpha.h.\alpha = g_\alpha.h_\alpha.z = g_\alpha.z = g.\alpha$, QED.

**Corollary 16** *Let $z$ be a $B$-absorbing point, $\alpha : A \to B$ a retraction, $b = b^\diamond \subset @B \times F$ a non-empty objective local composition at $B$, and $x : B \to F$. Then $x_\alpha$ is an endomorphism of $b$ (or, equivalently $\check{b}$), iff $x \in \check{b}$.*

**Corollary 17** *Let $z$ be a $B$-absorbing point, $\alpha : A \to B$ a retraction, and $m \in SemInTop(F)$ a semigroup. Denote by $|m|_\alpha$ the inverse image $(\alpha^{-1}(ex_z(m)^\vee))^\frown$ in $B@F$, and call it* the $B$-constant part of $m$ *(to be precise, we would have to specify the absorbing point and the retraction). Then we have*

$$m \subset Int(|m|_\alpha).$$

Proof. Let $g \in m, x \in |m|_\alpha$, and denote $y = g.x \in B@F$. Taking over the notation of lemma 39, $y.\alpha = g.x.\alpha = g.x_\alpha.z = g_x.z, g_x \in F@F$. By hypothesis, $x_\alpha \in m$, and so is $g_x$, i.e., $y.\alpha = g_x.z$, whence $y \in |m|_\alpha$. QED.

**Definition 84** *For an address change $\alpha : A \to B$, form space $F$, and a local composition $a \subset @A \times F$, we denote*

$$a|\alpha = a \cap B@F^\frown.\alpha$$

*and call it the* $\alpha$-restriction of $a$.

Observe that even for objective local compositions, their restriction need not be objective!

**Proposition 44** *Let $z$ be a $B$-absorbing point, $\alpha : A \to B$ a retraction, and $b = b^\diamond \subset @B \times F$ a non-empty objective local composition at $B$. Then we have*

$$(intex_{z,B}(b)|\alpha)^\vee = \check{b}.\alpha. \tag{24.14}$$

*In particular, for two objective, non-empty $B$-addressed local compositions $b_1, b_2$, we have*

$$intex_{z,B}(b_1)|\alpha = intex_{z,B}(b_2)|\alpha$$

*iff $b_1 = b_2$.*

[5] More precisely: "$\alpha$-absorbing".

Proof. We have

$$\begin{aligned}(intex_{z,B}(b)|\alpha)^\vee &= (intex_{z,B}(b))^\vee \cap B@F.\alpha \\ &= End(\check{b}).z \cap B@F.\alpha\end{aligned}$$

and the claim follows from corollary 16, QED.

Therefore, if $z$ is an absorbing point and $\alpha : A \to B$ a retraction, any objective $B$-addressed local composition $b$ in $F$ fulfills $b.\alpha \subset intex_{z,B}(b)$. This turns the terminology of an "objective closure" into an operator that adds new elements to the given original local composition. The closure gives us back the original local composition by restriction to the "$B$-constant" points, i.e., the natural image of $B@F$ under the address change $\alpha$.

Compared to the original approach [400], this presentation is independent of the specific properties of the basic form space $F$. If restricted to the objective local compositions, we have this fact:

**Fact 14** *Let $ObExTop_B(F)$ the subspace of $ExTop_B$ consisting of objective local compositions. Under the conditions of proposition 44, we have a continuous injection*

$$obintex_{z,B} : ObExTop_B(F) \to ObExTop_A(F), \tag{24.15}$$

*and we get the original objects back via $\alpha$-restriction.*

### 24.2.4 The Saturation Sheaf

**Summary.** Sheaves of monoids and monoid algebras are associated with each topological space $ExTop_A(F)$.

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For an address $A$ and space form $F$, if $M \subset End(F)$, the basic open set $Ext_A(M)$ identifies to $Ext_A(\widehat{M})$ if we define

$$\widehat{M} = \bigcap_{a \in Ext_A(M)} Int_A(a), \tag{24.16}$$

the *$A$-saturation of $M$*. In fact, $a \in Ext_A(M)$ iff $M \subset Int_A(a)$, and therefore, $M \subset \widehat{M}$, whence $Ext_A(\widehat{M}) \subset Ext_A(M)$, the converse is clear. The saturation of $M$ is the largest monoid defining the given basic open set. This gives rise to a sheaf of monoids as follows.

**Proposition 45** *Select an address $B$ and space form $F$. On an open set $U \subset ExTop_B(F)$ the presheaf $Sat_B^F$ defined by*

$$Sat_B^F(U) = \begin{cases} \bigcap_{u \in U} Int(u) \text{ if } U \neq \emptyset, \\ 1 \text{ if } U = \emptyset, \end{cases}$$

*together with the transition inclusion maps $i_V^U : Sat_B^F(U) \hookrightarrow Sat_B^F(V)$ for open sets $V \subset U$, is a sheaf of monoids, the* saturation sheaf of $F$ at address $B$.

*If $\alpha : A \to B$ is a retraction, the continuous map $.\alpha : ExTop_B(F) \to ExTop_A(F)$ canonically extends to a map of sheaves $.\alpha : Sat_B^F \to Sat_A^F$.*

Proof. This presheaf is a sheaf because the restrictions are inclusions (everything within the big monoid $End(F)$), and because any two open sets intersect by the irreducibility of $ExTop_B(F)$. Moreover, in the case of a retraction $\alpha$, we know from sorite 9, statement (vi), that for any local composition $a \in ExTop_B(F)$, we have $Int(a) = Int(a.\alpha)$. To define the sheaf morphism, we can restrict to basic open sets $Ext_A(M)$. We have to construct a morphism of monoids $Sat_A^F(Ext_A(M)) \to Sat_B^F(\alpha^{-1}Ext_A(M))$. By sorite 9, statement (vi), we also know that $\alpha^{-1}Ext_A(M) = Ext_B(M)$. So we need $Sat_A^F(Ext_A(M)) \to Sat_B^F(Ext_B(M))$. But we have

$$Sat_A^F(Ext_A(M)) = \bigcap_{a \in Ext_A(M)} Int(a) \subset \bigcap_{b \in Ext_B(M)} Int(b.\alpha),$$

and

$$\bigcap_{b \in Ext_B(M)} Int(b.\alpha) = \bigcap_{b \in Ext_B(M)} Int(b) = Sat_B^F(Ext_B(M)),$$

so the inclusion map of monoids $Sat_A^F(Ext_A(M)) \hookrightarrow Sat_B^F(Ext_B(M))$ does the job. QED.

**Remark 9** Given any commutative ring $R$, the above constructions canonically extend to sheaves ${}_RSat_A^F$ and morphisms of $R$-algebras via the monoid algebras ${}_RSat_A^F(U) = R\langle Sat_A^F(U)\rangle$. We leave the details to the reader.

The saturation sheaf will be used in the discussion in chapter 25 of the tonal function concept as it was developed in [400, 404].

**Exercise 58** The stalk of the saturation sheaf $Sat_B^F$ at a point $a \in ExTop_B(F)$ is the monoid $Int(a)$.

# Chapter 25

# Harmonic Semantics

*"Logik" ist in der Funktionstheorie*
*ein fundamentaler, aber dunkler Begriff.*
Carl Dahlhaus [100, p.95]

**Summary.** This chapter is about "understanding" aggregates of pitches in their combination on the syntagmatic/paradigmatic axes. This requires constructions of targets of such an understanding, i.e., harmonic *semantics*. The present theory succeeds in a (re)construction of function-theoretic semantics which is based on the paradigms associated with classical tonal functions. We discuss different approaches, among others the morphological theory of Noll and the approach of Mazzola based on chains of triads.

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We have already seen in the previous discussion of metrical, rhythmical, and motivic issues that the question concerning the position of a compositional item leads to topological considerations. This will also hold for the harmonic perspective. The topology comes in by the fact that the syntagmatic network of a composition is filtered by a paradigmatic assimilation of units such that this simplified structure can be read as a logical one. It is not the scope of this book to establish dogmatic theories of how this logic should look. We rather want to make clear what general processes are involved when logical constructions are elaborated from the crude pitch configurations.

There are two general differentiations of such logical methodologies: First, the harmonic functions must be related to the basic address where harmonic facts take place, i.e., the address of the local and/or global compositions which we investigate. This will in fact turn out to generate fundamentally different harmonic ontologies, such as it has been made clear by Noll's dissertation [400]. Second, harmonic functions germinate from prototypical harmonic units, such as major or minor triads, chains of thirds and the like, where one starts on a firm harmonic (pre)judice and then wants to extend the commonly accepted to less common chords. The point here is the extension approach from common germs to general situations. But it seems that the topological flavor of all the known harmonies is the common denominator, and that, beyond this, the basic constructors of harmony have very little to do with the special pitch domains.

We have therefore anticipated the chapter on harmonic topologies, in fact, as already stressed in that introduction, a chapter which is only accidentally directed towards harmony.

We are not yet in the position to claim that all these topological approaches converge in the common topos-theoretic logic which of course specializes to the logic of open sets of topologies. But it is nevertheless important to open the path to a purely topological interpretation of musical paradigmatics, be it via combinatorial topology or via more or less exotic (but nevertheless canonically defined) sober topologies.

The topos logic of predicates as described in section 18.3.2 does in fact suggest the following reinterpretation of harmonic topologies. In the truth form defined in formula (18.7)

$$TRUTH(I) \underset{Id}{\longrightarrow} \mathbf{Power}(Val(I)), \tag{25.1}$$

we took a special simple form $Val(I)$ associated with a truth module $I$. Instead, we could have taken any form $F$ and considered the $F$-truth form

$$TRUTH(F) \underset{Id}{\longrightarrow} \mathbf{Power}(F), \tag{25.2}$$

whose values stem from $F$ instead of $Val(I)$. This means that the set $\mathbb{T}_F^A$ of truth denotators at address $A$ are the $A$-addressed local compositions $a \subset @A \times F$. We have the usual Heyting algebra structure of subobjects in $\mathbb{T}_F^A$, together with its logic, as discussed in 18.3.2. But now, we have more: the extension topology $ExTop_A(F)$ with its natural Heyting logic of open sets (see appendix C.2.2 and [314, p.51]). What we should keep in mind in the following discourse is this:

**Principle 22** *The extension topologies are a natural topos-theoretic framework for predicate logic.*

One the one hand, this principle preconizes a seamless transition from logic to geometry as already indicated in principle 12 in section 18.3.3.3, but on the other, it does also confirm the fuzzy character of harmonic logic in so far as the extension topology is quite contrary to a common geometric space. Rather could it be associated with the irreducible ring spectra in algebraic geometry.

## 25.1 Harmonic Signs—Overview

**Summary.** This section gives an overview on levels of harmonic semantics.

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Generally speaking, the semantics of harmonic signs has always been built via ontological "superspaces", such as the metaphysical level of the Pythagorean tetractys: Greek musicology was based upon transcendence, and the core activity of Pythagorean scholars was the introspection of this mystic unit via instrumental practice. It seems that—at least in European tradition—semantics of musical harmony has been attached to a transcendence of one or another type.

In modern times, especially with Hugo Riemann's never accomplished program after Rameau's incipit, harmonic semantics has been focused on the attribution of tonic, dominant, and subdominant meaning to chords. This semantics is a very mysterious one since it is quite formalistic on one hand, but carries with it a certain amount of romantic connotation, in the spirit of Moritz Hauptmann's dialectic overloading of octave, fifth, and third [548], say.

We shall refrain from such transcendental semantics and restrict our analysis to the structural framework which can be recognized as a machinery for attributing to given chord configurations their harmonic functionalities.

The basic setup is always this (concrete examples follow below):

1. A pitch space form $F$ is chosen.
2. At a specific address $A$, a set $StandardChords$ of local compositions in $F$ is selected, playing the role of *standard chords* where the hoped-for semantics will be based, and where one knows from experience/history/practice what to expect.
3. A set of interpretations $S^I$ of local "scale" compositions $S$ at $A$ by determined subsets of $I \subset StandardChords$ is defined which will define the admitted *tonalities* in the specific setup.

On this background, the harmonic semantics of any chord, i.e., any local composition $c$ at selected (envisaged by the specific approach) addresses $B$ in $F$, will be constructed by one of these two strategies:

**Degree Theory** This approach relates $c$ to the given standard chords and/or tonalities by coverings, i.e., $c$ is included in a specific selection of standard chords and/or tonalities. This means that a chord is viewed as either a part of a standard structure, or else a part of a covering by standard structures, e.g., the seventh chord $G^7 = V_C^7$ is covered by two triadic degrees $V_C, VII_C$. *The moral of this approach is that the semantical process is a completion of a partial sign to fit in a standard object.*

**Function Theory** This approach relates $c$ to the given standard chords and/or tonalities by topological proximity to a specific selection of standard chords and/or tonalities. Such a procedure presupposes a topological environment, in fact, we shall see that the extension topologies are the formal realization of this task. *The moral of this approach is that the semantical process uses similarity to standard objects without necessarily being part of them as in degree theory, i.e., commonality instead of completion.*

Evidently, both approaches imply degrees of fuzziness: Completion, because this can be achieved in different ways. For example, an interval $\{c, g\}$ may be completed to a major third degree $C = \{c, e, g\}$ or else a minor third degree $Cm = \{c, e_\flat, g\}$, whereas this interval may represent the tonic function in $C$ or the subdominant function in $G$, for example. But this is the very nature of the harmonic approach which is a genuinely paradigmatic one. The problem of harmonic semantics is not its polyvalence, but the formally clear theory of such polyvalence, because only the precise analysis of polyvalence can qualify or disqualify a harmonic theory.

We should stress that the degree-theoretic and function-theoretic approaches do not imply that harmonic functions can only be generated by the second approach. The degree-theoretic

approach may also be used to attribute functional values, but these values will then be calculated from the complete standard chords instead of functions which are calculated from similarity to standard objects.

**Remark 10** We have to note another source of fuzziness which is often applied to both strategies: Instead of the given chord, one admits its deformation by a shift $c'$ of $c$ which is induced by an alteration (see section 7.5). As we have already discussed this deformation process, we shall restrict ourselves here to non-alterated harmonic semantics. The decisive point here is that one should keep in mind the alteration theorem 1 in section 7.5.1 which states that—in its negative interpretation—any two chords may be shifts of each other if the alteration interval is large enough. In contrast to this not very elegant deformation rhetorics stands the empirical use of alterations in existing musical styles which sometimes is quite subtle. In jazz harmony, for example, the fifth of a dominant seventh chord can be altered without problems (Domiant7/-5 or Dominant7/+5), but the seventh not (an alteration of the minor seventh invariably results in a loss of the dominant function). In a better Approach to alterations, one should not develop harmony on the given form, but on a "tangent form $F[\epsilon]$" (corresponding to the tangent module $M[\epsilon]$ in the representable case). Presently, no further theory is known in this direction.

## 25.2 Degree Theory

**Summary.** We give three short accounts of computer programs for degree theory.

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### 25.2.1 Chains of Thirds

**Summary.** Software for analysis by chains of thirds: Mazzola's *presto*® chord analyzer.

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The *presto*® software[1] analyzes zero-addressed chords in $PiMod_{12}$. The chords in *presto*® are abstracted from any local compositions in the four-dimensional space $OPLD_{\mathbb{Z}} = Onset \oplus Pitch \oplus Loudness \oplus Duration|_{\mathbb{Z}}$ via projection to $PiMod_{12}$. The program also retains the local composition's lowest pitch, see figure 25.1. The user may then select a reference pitch class; by default the class of the retained lowest pitch is chosen. This means that in fact, the chord is not taken as a local composition, but as a global one with two charts: the given chord, and an additional chart, the singleton of the reference pitch, the latter being a representative of the so-called "fundamental" pitch in classical harmony. This is an interpretative activity and cannot be justified without a deeper analysis of the context of the given chord. So a priori, the reference pitch is a variable which has to be integrated within a context-sensitive calculus. The projected local composition $S$ is then confronted with all chains of thirds (see section 13.4.2) which start from the reference pitch and have a minimal cardinality such that they contain $S$. For example, the chord $\{0, 1, 2\}$ has 23 minimal third chains when calculated from the reference 1. The list of all possible chains of thirds is shown in appendix L.2. Among these third chains

[1] See chapter 49 for details.

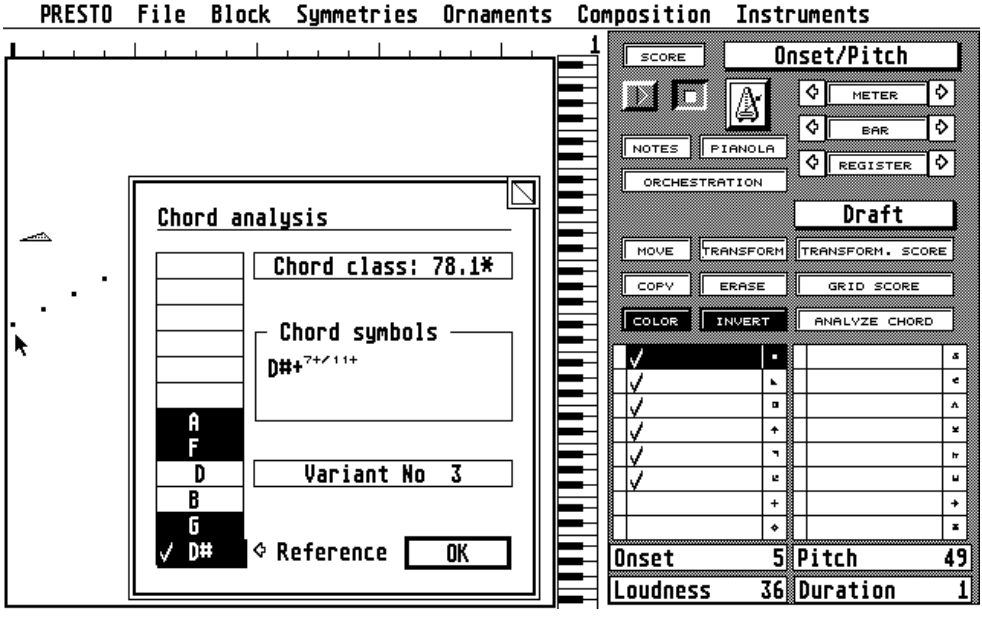


Figure 25.1: The sound-event set of ascending small filled squares to the left is analyzed by the software's chordmaster routine (middle small window entitled "Chord analysis"). This one shows the chord's pitch classes in third chain sequence from below. The black notes are those of the chord, the white ones in between are the added pitches from the embedding third chain. The isomorphism class of this chain (78.1*) is indicated, as well as the standard denotation of the chain in jazz chord symbols. The variant number (No 3 here) is the number of the chain out of the total number of possible chains. The reference pitch is the lowest pitch, it can be changed by the user to any other pitch class (even outside the chord).

the chains with absolute minimal cardinality can be found, it is the set $3Chain(S)$ defined in 13.4.2. This set is independent of the reference assumptions.

Although it does not determine the isomorphism class of $S$ in general, it gives a lot of information for the harmonic semantics of the chord. The absolutely minimal third chains of the chord are not yet function-theoretic constructs, but they can and will be used to build function-theoretic values, see section 25.3.3.

The complete list of third chains in appendix L.2 also contains the symbols from the American jazz notation. It is well known that there are many and not really optimal variants of this notation. The *presto*® implementation of our list has been charged with one variant of the jazz lead-sheet notation that can be justified by the total number of third chains (including 12-chains, not as with traditional (and not the new!) jazz practice, where chords only have a restricted number of notes). Basically the procedure is this. We have chosen one of the four 12-tone chains to build the reference interval step succession. It is reasonable to choose the one which is lexicographically first (most packed to the left) with respect to the "white keys", i.e., to the diatonic scale from $C$. This maximal chain of thirds has its initial subchains which all

are named with the usual numbers. So the first chain from $C$ is $C = \{0, 4, 7\}$, the second is $C7 = \{0, 4, 7, 10\}$, etc., until the last and total chain

$$C23 = \{0, 4, 7, 10, 2, 5, 8, 11, 3, 6, 9, 1\} \quad (25.3)$$
$$= \{C, E, G, B_\flat, D, F, A_\flat, B, E_\flat, G_\flat, A, D_\flat\}. \quad (25.4)$$

All degree names will be related to this reference chain, e.g., if we have a "13" in a symbol, this relates to $A_\flat$ with respect to the $C$ chain, and to the corresponding transposition in the general case. For the triadic chains, we use the traditional four types of major $C$, minor $Cm = \{0, 3, 7\}$, diminished $C0 = \{0, 3, 6\}$ and augmented $C+ = \{0, 4, 8\}$ chains. The larger chains are built upon these basic cases, and the added symbols refer to the above degree codification. For example, the symbol $Cm7 + /11$ means that we add the major seventh 11 and go until the 11-chain, including the none ("9" adding pitch class 2) and the undecime ("11" adding pitch class 5). So the deviations from the normal chain are indicated by $m, 0, +$ for the triadic initial piece, and then by the alterations $x+, x-$ for the other deviations from the norm. In some cases, the single alterations are not sufficient, for example, the chain nr. 46: $\{0, 4, 8, 11, 3, 7\}$ would have the notation $C+7+/9+/11++$. Alternatively, we can write $C+7+/9+/13+/\widehat{11}$ to indicate that 11 has been omitted and instead, 13+ was added. This notation (well known to mathematicians) can also be used to describe precisely the pitch classes of the given chord: we add a hat for every omitted ingredient, and we write this information after the standard symbol. For example, the fifth $\{C, G\}$ would read as $C/\widehat{3}$ or $Cm/\widehat{3}-$.

The set of third chains $3Chain(S)$ of a non-empty chord $S$ evidently is a simplex in the nerve of the third chain interpretation of $PiMod_{12}$. We shall see in section 25.3.2 that harmonic functions starting from degree theory can be viewed as fuzzy truth functions on this nerve, i.e., functions defined on each vertex and then extended to the entire chain in the affine way to the geometric realization of the nerve. This type of functions was already encountered in the metric and melodic analysis under the label of "weight functions". So the topological flavor of the degree theory reduces to the combinatorial nerve of the standard chords (the third chains in this case). The question of the function values of standard chords with respect to specific tonalities will be discussed in section 25.3.2.

### 25.2.2 American Jazz Theory

**Summary.** John Amuedo's MAX programs CHORD-CLASSIFIER, SCALE-FINDER, and SCALE-MONITOR for analysis of AST-based jazz harmonies.

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In his music theory thesis [15], Amuedo has programmed a bunch of LISP routines for AST and jazz-theoretic degree classification of chords. We have already mentioned the AST part of this work in section 11.5.2.3 and section 16.3. This software-part is used in the three harmonic analysis routines. The 76 standard ("fundamental" in Amuedo's wording) chords as well as the 18 standard scales of this approach are retrieved from relevant American jazz literature[2]. Like *presto*®'s chord analyzer, the present chord-analysis works on pitch classes (*pc*s) in $PiMod_{12}$,

[2]Above all, it is [67], for further literature, see [15, Bibliography: Jazz harmony]

however, the class of the lowest pitch in the original integer pitch space is retained to specify the chord and scale names.

The chord names or symbols are derived from seven groups of "foundation chords". In the table below is their list (see the list in section 11.5.2.3 for the definition of the different "norm" symbols).

This list is complemented by Amuedo's "Shortest List" of 18 "essential jazz scales" shown below. The CHORD-CLASSIFIER routine takes the input chord with integer coefficients (also, like *presto*®, not necessarily as a 'vertical' chord), retains the *pc*s and the lowest note's *pc*. The output is a list of seven entries. Apart the general AST-classification routine SET-SLAVE, this information is deduced from the standard chord list and from the essential scale list. The last four entries are just different labelings in terms of AST norms, such as DNR, DNF, DPF, INF, we refrain from this accounting overhead. The first entry: "RootPos." exhibits the lowest pitch class, together with a standard chord from the above list, possibly extended to higher third-chain notes. If this task fails, MAX' LISP-routine inserts the chord's DNR on this position. The second entry "Inversion:" exhibits other possible chords from the standard list which could describe the given chord by use of another fundamental note when applying inversions. The third entry "PC-Aliases:" lists all possible chord names irrespectively of the given DNF.—The SCALE-FINDER routine finds scales which are "compatible" with a given chord in the sense of *pc*-subset relations. This routine outputs all scale types, together with possible transpositions such that they contain the chord's *pc*s.—The third routine SCALE-MONITOR is more pedagogical and just for exercises in selected scales.

On this basis, a harmonic analysis is reduced to a set of degree symbols (lead-sheet notation) for chords (together with fundamental notes) which are grouped according to typical initial triad, as used in *presto*® and other degree notation. The enrichment with respect to *presto*®'s notation consists in a more function-theoretic flavor of the names, in the case of "dominant" chord types, for example. This information is—however—also present in *presto*®'s fundamental note selection option, such that the distinction between $Am7$ and $C6$ is managed by the distinction between the fundamental $A$ and $C$ in the two cases, respectively, and also with the consequence that the minimal third chains in these cases are not the same.

In both cases, we end up with a determined list of standard chords (including standard scales in the shape of "large chords") which describe the given chord and reduce its study to the harmonic semantics of these empirically or systematically retrieved standard objects.

| Lead-sheet symbol | DNR | ZNF | Representative |
|---|---|---|---|
| Major Type (Ma) | | | |
| Ma | 317.0 | 0,4,7 | C,E,G |
| Ma6 | 437.3 | 0,3,5,8 | C,E,G,A |
| Ma7 | 416.1 | 0,1,5,8 | C,E,G,B |
| Dominant Type (Do) | | | |
| Do | 439.3 | 0,3,6,8 | C,E,G,Bb |
| Dob5 | 430.3 | 0,2,6,8 | C,E,Gb,Bb |
| Do5# | 425.2 | 0,2,4,8 | C,E,Ab,Bb |
| Augmented Type (Au) | | | |
| Au | 318.0 | 0,4,8 | C,E,Ab |
| Au7 | 434.2 | 0,3,4,8 | C,E,Ab,B |
| Minor Type (mi) | | | |
| mi | 314.0 | 0,3,7 | C,Eb,G |
| mi6 | 428.2 | 0,2,5,8 | C,Eb,G,A |
| mi7 | 437.2 | 0,3,5,8 | C,Eb,G, Bb |
| miM7 | 413.1 | 0,1,4,8 | C,Eb,G, B |
| Diminished Type (di) | | | |
| di | 313.0 | 0,3,6 | C,Eb,Gb |
| di6 | 440.0 | 0,3,6,9 | C,Eb,Gb,A |
| di7 | 428.1 | 0,2,5,8 | C,Eb,Gb,Bb |
| diM7 | 412.1 | 0,1,4,7 | C,Eb,Gb,B |
| Major Flat Five Type (Mb5) | | | |
| Mb5 | 316.0 | 0,4,6 | C,E,Gb |
| Mb5M7 | 415.1 | 0,1,5,7 | C,E,Gb,B |
| Sustained Four Type (Su) | | | |
| Su2 | 308.0 | 0,2,5 | C,D,F |
| Su5 | 310.2 | 0,2,7 | C,F,G |
| Su7 | 310.1 | 0,2,7 | C,F,Bb |

Added to the common foundation chords by Amuedo

| Sustained Root Type (R) | | | |
|---|---|---|---|
| R25 | 310.0 | 0,2,7 | C,D,G |
| R27 | 307.1 | 0,2,4 | C,D,Bb |
| Stacked Perfect-Fourth/Perfect-Fifth (P4, P5) | | | |
| P4*3 | 427.1 | 0,2,5,7 | C,F,Bb,Eb |
| P5*3 | 427.2 | 0,2,5,7 | C,G,D,A |

| Name | DNF | Representative |
|---|---|---|
| Alt_Dominant+5 | 834 | C,Db,Eb,E,F#,G,Ab,Bb |
| Augmented | 646 | C,Eb,E,G,Ab,B |
| Bebop_Major | 832 | C,D,E,F,G,G#,A,B |
| Bebop_Seventh | 817 | C,D,E,F,G,A,Bb,B |
| Blues_(6-note) | 656 | C,Eb,F,Gb,G,Bb |
| Blues_(7-note) | 708 | C,Eb,E,F,Gb,G,Bb |
| Diminished | 835 | C,D,Eb,F,Gb,Ab,A,B |
| Harmonic_Minor | 738 | C,D,Eb,F,G,Ab,B |
| Hung/Gypsy/Byz | 729 | C,D,Eb,Gb,G,Ab,B |
| Locrian+n9 | 815 | C,Db,D,Eb,F,Gb,Ab,Bb |
| Major | 743 | C,D,E,F,G,A,B |
| Major_b6 | 742 | C,D,E,F,G,Ab,B |
| Melodic_Minor | 739 | C,D,Eb,F,G,B |
| Penta_Dom | 549 | C,D,E,G,Bb |
| Penta_Lydian | 550 | C,E,F#,Ab,B |
| Penta_Major | 5501 | C,D,E,G,A |
| Penta_Minor | 555 | C,D,Eb,G,A |
| Whole_Tone | 674 | C,D,E,Gb,Ab,Bb |

### 25.2.3 Hans Straub: General Degrees in General Scales

**Summary.** Hans Straub has initiated a general harmony, including modulation theory, as a generalization of Mazzola's original approach [327]. Here we just give a summary of Hans Straub's approach to scales and degrees.

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In the author's original approach to harmony in [327], the diatonic scale $X$ was covered by triadic degree chords to yield the triadic interpretation $X^{(3)}$ discussed in section 13.4.2. The corresponding cadence and modulation theory was subsequently generalized by Daniel Muzzulini [390], and this one still more generalized by Straub [514].

In Straub's general approach, a scale is not given as a starting structure, but as a result, in fact the union, of a set $S$ of "degree" zero-addressed chords $s \subset 0@PiMod_{12}$. No further requirement upon similarity or structure of the chosen chords is imposed. For example, the "Hora Cero" interpretation consists of the set of seven three-element chords

$$S_{\text{Hora Cero}} = \{C, Dm, Em, F, G, Am, B_\flat\}$$

with a nerve that looks like a belt which has been pinched to a point (the $G$ vertex) in a position. Another interpretation is called "Dorico Flamenco" because of its Flamenco origin,

its seven chord atlas is

$$S_{\text{Dorico Flamenco}} = \{Dm, C, B0, Am, G, F, E\}$$

both generating scales of type 17.1 in the chord classification appendix L.1.

The spirit in Straub's approach is not so much the corresponding generality of cadence and modulation theory (which will be taken up in the following chapters), but the insight that much of existing music traditions, in particular popular music or American jazz, as we have seen in section 25.2.2, is defined by a grown chord vocabulary that is not subjected to a scale, on the contrary, it generates 'scales' that issue from the chord vocabulary, however not scales that will play a melodic role, only harmonically driven background material.

This approach is however a purely degree-theoretic one, even in a very radical vein: No approach to theoretical background structures to allowed chords is imminent. You are just allowed to play a number of chords per "tonality" and to use cadences and modulations according to determined rules, but no further function theory is addressed.

The question of harmonic semantics is indeed not explicit here. Although the jazz idiom has a strong affinity to European function semantics, it is not true that this and other ethnological/cultural contexts are or can be cast in this kind of abstraction. Some ideosyncratic usages of chord vocabularies happen to look like absolute pitch perception: No comparison between different pitches leads to identification, each pitch is an individual. In harmony, this would mean that some usages of chords relates to a kind of "absolute harmonic position", every used chord is irreducible, in fact a *semantic atom* of harmony.

## 25.3 Function Theory

**Summary.** We discuss the topological similarity approach to harmonic functions, as it has been prepared in the previous chapter 24.

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In this section, we consider two addresses, $A, B$, a retraction $\alpha : A \to B$, a form $F$, and a $B$-absorbing point $z : A \to F$.

In traditional harmony, for example in Schönberg's treatise [478], the harmonic band between two chords is the set of common notes. We have seen in section 13.4.2 that the Möbius strip shaped nerve of the triadic interpretation of a diatonic scale represents the information about harmonic bands among certain sets of triadic degrees. In that naive setup, the harmonic relation was given by common notes. Following the empirical insight 11.3.6, the deeper semantics of a (zero-addressed) chord may be recognized from its symmetries [400], or, to put it in the wording of harmonic topology, from the self-addressed objective closure (24.2.3). Putting this together with our empirical insight, one may consider the harmonic bands between the objective closures of chords, instead of the original, usually zero-addressed objects.

This motivates Noll's definition of allomorphic chords for a given semigroup of symmetries ([400]). Here is our rephrasing of that definition (Noll does not refer to topological structures) in the general terms of harmonic topologies.

**Definition 85** *Let $m$ be a monoid in $InTop(F)$. Then the* $B$-allomorphic extension of $m$ *is the open set*

$$AllExt_B(m) = \bigcup_{n \prec m, n \neq 1} Ext_B(n)$$

*of all points in $ExTop_B(F)$ which contain some non-trivial submonoid $n$ of $m$, i.e., some non-closed monoid $n$ which is topologically dominated by $m$. The elements of $AllExt_B(m)$ are called the* $B$-allomorphs of $m$.

Evidently, $Ext_B(m) \subset AllExt_B(m)$. In order to make the common endomorphisms precise, we have locally closed[3] subsets

$$AllExt_B(n, m) = Ext_B(n) - \bigcup_{n \prec l \prec m, n \neq l} Ext_B(l)$$

for all non-trivial dominance relations $n \prec l \prec m$ for monoids $l$ strictly larger than $n$. This gives a partition

$$AllExt_B(m) = \coprod_{n \prec m, n \neq 1} AllExt_B(n, m) \tag{25.5}$$

of an allomorphic extension into locally closed subsets. Such a subset $AllExt_B(n, m)$ is called the *strict $B$-extension of $n$ in $m$*. Except the open part $AllExt_B(m, m) = Ext_B(m)$, all strict extensions have empty interiors. Observe that in general, the intersection of two allomorphic extensions is *not* an allomorphic extension, so these open sets do not define a basis of a topology.

**Example 45** We take the classical situation $B = 0, A = \mathbb{Z}_{12}, F = @A$, however with the usual identification of elements of $F$ as multiples of fifths. Then Noll's "second dominant morpheme in C major" is $D^\star = \langle \alpha^\star = e^8.3, \beta = e^9.8 \rangle_e$, a bigeneric monoid.

Allomorphic extensions are a means for defining harmonic functions via extensions, this is the idea in [400, I.3.1, Definition 4]. Given specific monoids $M_{m,n}$ which "signify" certain harmonic functions (see below for details), the representation of these functions extends to the (zero-addressed) chords in $AllExt_0(M_{m,n})$. This approach reveals the thoroughly intuitionistic logic of topological spaces [314, I.7/8]. In fact, by the irreducibility of $ExTop_B(F)$, extensions—and a fortiori allomorphic extensions—are never disjoint. So there is little chance to have a unique functional representation for a given chord. Even on the level of strict extensions, we may have non-empty intersections $AllExt_B(n, m) \cap AllExt_B(n', m')$ for different pairs $(n, m)$, $(n', m')$. We have this lemma on empty strict extensions:

**Lemma 40** *With the above notation, if the defining monoid $m$ is saturated, i.e., equal to its saturation $\hat{m}$, and its non-trivial submonoid $n$ is not, then $AllExt_B(n, m) = \emptyset$.*

Proof. From the construction of the saturation sheaf it is clear that a monoid inclusion $n \subset m$ implies $\hat{n} \subset \hat{m}$. Therefore, if $m = \hat{m}$ and $n$ is not saturated, $x \in Ext_B(n)$ implies $x \in Ext_B(\hat{n})$ and therefore $AllExt_B(n, m) = \emptyset$.

[3] Intersections of open and closed sets, see appendix H.1.4, definition 168.

**Remark 11** In general, it is difficult to give these functional specifications a quantitative weight, in terms of percentages, say. The only evident measure of the role of a chord $X$ in $AllExt_B(m)$ is the position of its strict extension—the unique non-trivial submonoid $n \subset m$ such that $X \in AllExt_B(n, m)$—in the lattice $\bar{m}$ (the closure of $m$) of submonoids. If the entire descent from monoid to submonoids is finite, one may for example take the relative depth of descent from top to define a numerical weight of a submonoid.

We still do not know very much about the relations between the saturation sheaf and the topological properties of the extension topology. But here are some exercises for straightforward connections:

**Exercise 59** With the above notation, prove these statements:

1. If $U \subset ExTop_B(F)$ is open, then the basic extension of the monoid of sections $Sat_B^F(U)$ satisfies $U \subset Ext_B(Sat_B^F(U))$.
2. If for two monoids $n \prec m$, then $AllExt_B(n) \subset AllExt_B(m)$.
3. We have $Sat_B^F(AllExt_B(m)) = \bigcap_{n \prec m, \text{ saturated and non-trivial}} n$. If $m$ verifies the descending chain condition[4] then

$$Sat_B^F(AllExt_B(m)) = \bigcap_{\text{minimal among saturated and non-trivial } n \prec m} n.$$

### 25.3.1 Canonical Morphemes for European Harmony

**Summary.** We discuss the construction principle for bigeneric morphemes, the crucial objects for Noll's reconstruction of traditional European harmony.

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We have so far prepared the semantic field via extension and allomorphic extension concepts. So we are ready to look for monoids which may recover the harmonic semantics known from classical function theory.

As above in example 45, we are now working in the situation $B = 0, A = \mathbb{Z}_{12}, F = @A$, the case studied in [400] with the elements of $F$ being interpreted as multiples of fifths. Following the suggestion from section 24.1.2 concerning the structure of certain common chords, in particular circle chords, the construction of special monoids (or their objective trace under an absorbing point $z = Id_{\mathbb{Z}_{12}}$ in the present case) has been set up following a bigeneric approach. In fact, the circle chords are the zero-constant traces of self-addressed chords $monex_z(\langle f, e^c.0\rangle_e)$ associated with a bigeneric monoid, whereas the example $V_C^7$ is the zero-objective trace of $monex_z(\langle e^8.3, e^2.0, e^7.0, \rangle_e)$ which is associated with a trigeneric monoid.

The bigeneric approach then takes two generators $\alpha_d = e^d.3, \beta_a = e^a.8$ ($d$ for "diminished", $a$ for "augmented", will be justified below) and considers the "morpheme" monoids[5]

$$M_{d,a} = \langle \alpha_d, \beta_a \rangle_e \tag{25.6}$$

[4] Every descending chain of submonoids is stationary, for example for a finite endomorphism monoid $End(F)$.
[5] In [400], only the generated semigroups are considered, but the results are the same.

generated by $\alpha_d, \beta_a$ for any coefficients $a, d \in \mathbb{Z}_{12}$. The experimental evidence of the crucial role of these monoids is given by the following proposition. It uses the affine Lie bracket $[xy] = x.y - y.x$ which is defined on any monoid of endomorphisms $X@X$ of a module $X$. Further, we need to encode the translation classes of some zero-addressed chords in $\mathbb{Z}_{12}$. We need the class numbers 2,5,6,7,10, and 14, see appendix L.1 for the class numbers. Class 2 contains two translation classes, 2.0 (=2 in the class table), and 2.1. Classes 5,6,7 are translation classes. Class 10 splits into four translation classes, two for each subclass 10, 10.1 of the table. We denote them again by 10.1, 10.2, 10.3, 10.4 according to the lexicographic order. Class 14 splits into two translation classes 14.1, 14.2.

**Proposition 46** [400, III.3.3, Lemma 2] *Given two coefficients $a, d \in \mathbb{Z}_{12}$, the translation class of the zero-constant part $|M_{d,a}|_0$ of the morpheme monoid $M_{d,a}$ is a function of the Lie bracket $[\alpha_d \beta_a] = e^{5d+2a}$, and we have these translation classes:*

| *Translation Class of constant part of $M_{d,a}$* | *Lie invariant $5d + 2a$* |
|---|---|
| *2.0* | *0* |
| *5* | *3 or 9* |
| *6* | *4 or 8* |
| *7* | *6* |
| *10.1* | *1* |
| *10.2* | *11* |
| *10.3* | *5* |
| *10.4* | *7* |
| *14.1* | *10* |
| *14.2* | *2* |

The point of this proposition is that these classes are very common. In fact, the Lie invariant $5d + 2a$ yields these chord types for the constant parts of the above morphemes:

1. $5d + 2a = 1$ gives a major chord such as $\{c, e, g\}$,
2. $5d + 2a = -1$ gives a minor chord such as $\{c, e_\flat, g\}$,
3. $5d + 2a = -2$ gives a (minor) seventh chord without fifth, such as $\{g, b, f\}$,
4. $5d + 2a = 2$ gives a (minor) sixth chord without tonic, such as $\{e_\flat, g, a\}$.

This means that we get very classical chords for the Lie invariants $\pm 1, \pm 2$. These Lie invariants are taken to define the functional semantics of tonic, dominant, and subdominant values in [400].

Before we step over to the tonal function calculus in 25.3.2, we should understand the general principle which governs the construction of bigeneric morphemes. In [400], the situation

was uniquely $F = @\mathbb{Z}_{12}$, and no systematic account was given for the "bi" in the bigeneric approach. It was also not clear which should be the relation between the two generators. This is what we want to complete now. To this end, recall from appendix C.2.3, definition 125, the idempotent component $\sqrt{x}$ of an idempotent element $x \in End(F)$ of the endomorphism monoid of a form $F$. Recall also from appendix C.2.3, proposition 66, that for a representable form $F = @Q$, the idempotent components are all conjugate under translations to idempotent components of linear idempotents $x_{U,V}$ which correspond to direct decompositions $Q = U \oplus V$.

In $\mathbb{Z}_{12}$, we have the four direct decompositions

$$\begin{aligned}\mathbb{Z}_{12} &= \mathbb{Z}_{12} \oplus (0),\\ \mathbb{Z}_{12} &= (0) \oplus \mathbb{Z}_{12},\\ \mathbb{Z}_{12} = 3.\mathbb{Z}_{12} \oplus 4.\mathbb{Z}_{12} &\xrightarrow{\sim} \mathbb{Z}_4 \oplus \mathbb{Z}_3,\\ \mathbb{Z}_{12} &= 4.\mathbb{Z}_{12} \oplus 3.\mathbb{Z}_{12}\end{aligned}$$

which are pairwise coupled by the exchange of factors, corresponding to the exchange $x \mapsto 1-x$ of idempotents. The second group of idempotents is the Sylow decomposition of $\mathbb{Z}_{12}$, the only non-trivial direct decomposition. The above selection of generators $\alpha_d, \beta_a$ corresponds exactly to the Sylow decomposition; we take $\alpha_d \in \sqrt{e^x.pr_{\mathbb{Z}_4}}, \beta_a \in \sqrt{e^y.pr_{\mathbb{Z}_3}}$. More precisely,

$$\alpha_d = e^d.\pi_{\mathbb{Z}_4}, \pi_{\mathbb{Z}_4} = -pr_{\mathbb{Z}_4} = (3),$$

and

$$\beta_a = e^a.\pi_{\mathbb{Z}_3}, \pi_{\mathbb{Z}_3} = -pr_{\mathbb{Z}_3} = (8),$$

twice the negative projections as linear factors, and any shift coefficients $a, d$, which in fact can be arbitrary because of the choice of the linear part!

So the recipe is this: Take any non-trivial direct decomposition of the underlying module $Q = U \oplus V$ and the associated idempotents $p_U, p_V$, take two elements $\alpha \in \sqrt{e^x.p_U}, \beta \in \sqrt{e^y.p_V}$ in idempotent components which are conjugate to the components $\sqrt{p_U}, \sqrt{p_V}$, and build the bigeneric monoid $\langle \alpha, \beta \rangle_e$. So the *bi*generic approach corresponds to the direct decomposition into two factors, the relation among these generators is the complementary role of the direct factors in the idempotent relation $x \mapsto 1 - x$.

Let us apply this generalization to the just tuning space $EulerZModule = \mathbb{Z}.q \oplus \mathbb{Z}.t$. This rank-two $\mathbb{Z}$-module has infinitely many direct decompositions into two rank-one factors, for example the above one, defining the projections $p_1, p_2 = 1 - p_1$ onto the factors $\mathbb{Z}.q, \mathbb{Z}.t$, respectively, or the decomposition $\mathbb{Z}.q \oplus \mathbb{Z}.Kt$ via syntonic comma $Kt$. Let us look at two canonical generators for the idempotents $p_1, p_2$. Set $\pi_1 = -p_1, \pi_2 = -p_2$, and then

$$\alpha_{a,\xi} = e^{\binom{a}{\xi}}.\pi_1, \beta_{\eta,b} = e^{\binom{\eta}{b}}.\pi_2.$$

This entails

$$\alpha_{a,\xi}^2 = e^{\binom{0}{\xi}}.p_1, \beta_{\eta,b}^2 = e^{\binom{\eta}{0}}.p_2,$$

and $\alpha_{a,\xi}^3 = \alpha_{a,\xi}, \beta_{\eta,b}^3 = \beta_{\eta,b}$. Then we have this zero-constant chord:

$$|\langle \alpha_{a,\xi}, \beta_{\eta,b} \rangle_e|_0 = \{\binom{a-\eta}{\xi}, \binom{\eta}{b-\xi}, \binom{\eta}{\xi}\},$$

which yields the major triad $I_C = \{\binom{0}{0}, \binom{1}{0}, \binom{0}{1}\}$ for $\eta = \xi = 0, a = b = 1$. So the direct decomposition approach not only yields standard chords in 12-tempered space, but also in just tuning. No further developments have been investigated to date in the just tuning space.

So if ever the form is representable, $F = @Q$ for a module $Q$, if $Q = U \oplus V$ is a direct decomposition with corresponding projections $p_U, p_V$, and if $x, y \in Q$, we have the generators of a bigeneric monoid $\alpha_x^U = e^x. - p_U, \beta_y^V = e^y. - p_V$ which live in the idempotent components $\sqrt{e^{x-p_U(x)}.p_U}, \sqrt{e^{y-p_V(y)}.p_V}$, respectively. If $s = e^t.s_0 \in \overrightarrow{GL}(Q)$ is any invertible endomorphism of $Q$, then the conjugate generators are

$$s.\alpha_x^U.s^{-1} = \alpha_{s_0(x)}^{s(U)},$$
$$s.\beta_y^V.s^{-1} = \beta_{s_0(y)}^{s(V)},$$

and we have the conjugate monoid

$$s.\langle \alpha_x^U, \beta_y^V \rangle_e.s^{-1} = \langle \alpha_{s_0(x)}^{s(U)}, \beta_{s_0(y)}^{s(V)} \rangle_e$$

whose zero-constant part is

$$s(|\langle \alpha_x^U, \beta_y^V \rangle_e|_0).$$

This means:

**Proposition 47** *The transformations via $s \in \overrightarrow{GL}(Q)$ of all zero-constant chords in $@Q$ derived from bigeneric morphemes are the zero-constant chords derived from $s$-conjugate generators. In particular, all translates of zero-constant chords derived from bigeneric morphemes are again of this type.*

In particular, the full translation classes of common zero-addressed chords in $@\mathbb{Z}_{12}$ derived from bigeneric morphemes are derived in this way, this is the situation encountered in [400].

Generalizations of this approach are obvious: One may step over to direct decompositions into three, four, etc., factors and corresponding multigeneric monoids, one may also take into account more general bigeneric chords with not so common zero-constant parts, and, last, but not least, one may reconsider the whole theory for more general addresses $B$, not just the zero address, and for more general forms (representable or not).

**Remark 12** At present, we do not know how far the canonical transitive $GL(2, \mathbb{Z})$-action on the set of direct, non-trivial decompositions $\mathbb{Z}^2 = U \oplus V$ could be connected to the idea of the abelian group $SL(2, \mathbb{Z})/(SL(2, \mathbb{Z}), SL(2, \mathbb{Z}))$ being identified with the set $\mathbb{Z}_{12}$ of pitch classes (see section 24.1.1.1).

### 25.3.2 Riemann Matrices

**Summary.** The Riemann program suggests a special matrix associated with every possible chord. The coefficients of this *Riemann matrix* are accounts of the role of the chord within the coordinates of tonalities, modes, and function values. We give the formal definition of the Riemann matrix and its construction principles.

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We are now ready for considerations of tonal functions built upon the preceding mechanisms from degree and function theory[6]. The essence is this:

**Principle 23** *Any function theory should aim at defining tonal functions for any existing chord[7], i.e., relations between instances of the concept of a tonality and "semantic pointers", the harmonic function values, which provide the syntactic structure with means for understanding harmonic coherence.*

This statement is not meant as a cryptic sentence to cheat humanities, rather do we delimit our subject from levels of understanding which are not the scope of exact science, viz. textual predication in the sense of section 18.2. In this vein, functional values are just names that are attributed to chords by means of precise evaluation modalities. So the only intrinsic meaning we can offer is extensionality, the determination of fibers lying over specific functional value symbols. The names are not really relevant, except that they may evoke some paratextual mechanism. Our approach yields textual predicates which give every chord a truth value in some truth module, and such that this evaluation is built upon the structure theory thus far developed.

The collection of tonal functions is defined by three domains: A name set TON of "tonalities", a name set VAL of "tonal function values", and a form space $TRUTH(I)$ for of "truth denotators" in a specific truth module (or, more generally, a truth form) $I$ and at a specific address $B$ for a form $F$.

Formally speaking (see also formula (18.7)), a tonal function $TF_{f,t}(f,t) \in \mathrm{VAL} \times \mathrm{TON}$, is a parametrized set of objective textual predicates which are encoded by maps

$$TF_{f,t} : ExTop_B(F) \to \mathbb{T}_I^B$$

on the supporting domain $ExTop_B(F)$. For such a collection of predicates, the *Riemann matrix* is the matrix $TF = (TF_{f,t})_{(f,t)\in \mathrm{VAL}\times\mathrm{TON}}$ of predicates. Given such a matrix, for any chord $a \in ExTop_B(F)$, its Riemann matrix is by definition the matrix[8] of all values, i.e.,

$$TF(a) = (TF_{f,t}(a))_{(f,t)\in \mathrm{VAL}\times\mathrm{TON}} \in \mathbb{M}_{\mathrm{VAL},\mathrm{TON}}(\mathbb{T}_I^B), \tag{25.7}$$

and therefore, we have the Riemann matrix map

$$TF : ExTop_B(F) \to \mathbb{M}_{\mathrm{VAL},\mathrm{TON}}(\mathbb{T}_I^B).$$

Here, a tonality symbol $t \in \mathrm{TON}$ is codified by its value vector $TF_t(a) = (TF_{f,t}(a))_{f\in\mathrm{VAL}})$ on all chords $a \in ExTop_B(F)$. This is a fuzzy evaluation of tonal roles of chords with respect to the function symbols in VAL.

[6] Recall that we are still given two addresses $A, B$, a form $F$, and an absorbing retraction $\alpha : A \to B$.

[7] See also Dahlhaus' paper [100] on Riemann theory.

[8] The set of all matrices with rows in a set $m$, columns in a set $n$, and coefficients in a space $X$, is denoted by $\mathbb{M}_{m,n}(X)$.

### 25.3.3 Chains of Thirds

**Summary.** This section gives an elementary construction of a Riemann matrix, based on chains of thirds and minimal embeddings of chords in such chains.

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This setup works in the zero-address and form $F$ with $Fun(F) = @\mathbb{Z}_{12}$. The tonality set is the set of "tonics" with just one tonic per pitch class

$$\text{TON} = \{C, D_\flat, D, E_\flat, E, F, G_\flat, G, A_\flat, A, B_\flat, B\}.$$

The value set is the classical six-element set for tonic, dominant, subdominant, each in major and minor mode:

$$\text{VAL} = \{T, D, S, t, d, s\},$$

with the minor problem of symbol confusion ($D$ is doubly coded, but this is the tradition). The truth module is the already discussed fuzzy group $I = S_1 = \mathbb{R}/\mathbb{Z}$ (see example 36), and the values will be just the denotators associated with the half open interval subsets $\phi\hat{} = [0, \phi[$ of $S_1$.

From section 25.2.1 we learned that this approach deals with the third chain simplex $3Chain(c)$ of a chord $c$. Given a tonality symbol $f$ in our context, and a third chain or singleton $a$, we shall define the function values $TF_{f,t}(a)$ and then extend this evaluation to any chord $c$ by the formula

$$TF_{f,t}(c) = at(\frac{1}{card(3Chain(c))}\Sigma_{a\in 3Chain(c)}at^{-1}(TF_{f,t}(a))) \tag{25.8}$$

which includes the case of third chains and singletons as zero simplexes of the third chain interpretation. Here, the auxiliary function

$$at : [0, \infty[ \to [0, 1[$$

is supposed to be a strictly increasing bijection[9], for example $at(x) = \frac{2}{\pi}\arctan(x)$. The reader may generalize this formula by more sophisticated contributions of the simplex vertexes, we just show the principle here. The principle is that a chord's tonal function is some kind of average of the tonal function taken over all minimal third chains, i.e., the "standard" chords covering the—possibly exotic—given chord.

We are left with the definition of the tonal function for singletons and non-trivial third chains. Again, this is a prototypical formalism, to be adopted by the reader for specific needs. Suppose that a third chain $a$ is given by the sequence $a_1, a_2, \ldots a_k$, then we suppose given the non-negative values

$$TF_{f,t}(a_i)$$

and then set

$$TF_{f,t}(a) = at(3CH(at^{-1}(TF_{f,t}(a_1)), at^{-1}(TF_{f,t}(a_2)), \ldots at^{-1}(TF_{f,t}(a_k)))$$

[9]This normalization is used to cast the raw values in the unit interval of the circle group.

with a non-negative function $3CH$ on the singleton arguments, e.g., the sum of their second powers. One may in particular implement singleton values which yield good truth values for major, minor and similar common triads.

The formulas for the singletons $x$ should be such that the values for basic tonality symbol $f = C$ yield the values for general tonality symbols, according to the formula

$$TF_{f,t}(x) = TF_{C,t}(e^{-t}(x)) \tag{25.9}$$

if the symbol $f$ corresponds to the tonic $t$. So this method is based on an elementary truth evaluation on the singletons *relative to one reference tonality* and then uses different constructors via third chain evaluation and then minimal covering families of third chains. This method has been implemented in RUBATO®'s HarmoRUBETTE®, see section 41.3 for details.

The above approach seems to be numerically oriented. It is however still a logical approach in the fuzzy logic module $S_1$. Nonetheless, such a specific truth value assignment is not mandatory, one may think of much more abstract truth modules $J$ and then, after having defined a method that works on $J$, transform such abstract values to more 'practical' values via appropriate truth module morphisms $J \to I$.

### 25.3.4 Tonal Functions from Absorbing Addresses

**Summary.** This section discusses the tonal functions of chords associated with absorbing addresses, generalizing the idea of self-addressed morphemes in [400].

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This approach does not take values of a numerical nature, but gets off on a purely topological level of truth levels. The philosophy is that the evaluation is not so much information about truth or falsity, but about the esthetic position within a 'neighborhood system of harmonic centers'. Harmonic evaluation in this spirit follows a 'logic' where the location is more relevant than the decision between dichotomic alternatives. It is in fact an open question whether Riemann's harmonic logic ever was meant to be a restatement of classical binary logic, and whether its role was ever limited to yield decisions between correct and false musical syntax.

We start on the truth value form functor $I = ExTop(F)$ for a pitch form $F$ (e.g., $F = PiMod_{12}, EulerZModule$), defined on the subcategory of addresses with retractions as morphisms. This generates the truth form $TRUTH(I)$ and, by definition, its functor values $B@TRUTH(I) = B@\Omega^{ExTop(F)}$ at address $B$. So a subset $U \subset ExTop_B(F)$, i.e., an element $U \in B@2^{ExTop(F)}$, gives rise to a truth value $U\hat{} \in B@TRUTH(I)$, and we have integrated topologically deduced subsets of the $B$-addressed extension topology of form $F$.

On this basis, the evaluation $TF_{f,t}(c)$ of a chord $c$ in $ExTop_B(F)$ follows this scheme: The tonality symbol $t$ is associated with an interpretation $S_t^{M_t}$ of an $A$-addressed scale $S_t \subset ExTop_A(F)$. The elements of $M_t$ are images $U_m = monex_z(m)$ of monoids[10] $m \in InTop(F)$. The monoids are deduced from the interpretation $S_{t_0}^{M_{t_0}}$ for a basic tonality $t_0$ by conjugation with the translation[11] that corresponds to the tonic shift from $t_0$ to $t$, or another transformation for more involved collections of tonal symbols.

[10] The "harmonic morphemes" according to the terminology of Noll [403].

[11] See the calculations of morpheme transformations for bigeneric morphemes in section 25.3.1.

This method's function symbols relate to the basic interpretation $S_{t_0}^{M_{t_0}}$ and its charts. If $m$ is such a chart's monoid, some specific specialization monoids, i.e., non-trivial submonoids $n_{m,i}, i = 1, \ldots q(m)$ are chosen to define the function symbols $f_{m,i}$. For a general tonality symbol $t$, the function symbols are associated with the correspondingly transformed monoids and submonoids.

This being true, the function value $TF_{f,t_0}(c)$ for a symbol $f$ corresponding to the specialization pair $n_{m,i} \prec m$ is the locally closed allomorphic extension set $AllExt_B(n_{m,i}, m)$ iff $c$ is an element of this set, and $\emptyset$ else.

In what follows, we want to give some examples of tonal functions from absorbing addresses. They are all focused on the classical context of pitch form $F = PiMod_{12}$ with fifth identifier, addresses $A = \mathbb{Z}_{12}, B = 0$, and absorbing address retraction $z : A \to 0$.

**Example 46** Bigeneric Major Tonality [400]. We take up the bigeneric morpheme monoids $M_{d,a}$ from equation (25.6) and their associated self-addressedchords $M_{d,a}^z = monex_z(M_{d,a})$. We are given the usual twelve tonality symbols

$$\text{TON} = \{C, D_\flat, D, E_\flat, F, G_\flat, G, A_\flat, A, B_\flat, B\}$$

and calculate the pitch class $p$ corresponding to tonality symbol $X$ in multiples of fifths, e.g., $G$ corresponds to $p = 1$, we write $G = e^1C$, to be short. The $X = e^pC$-major tonality is defined by the twelve morpheme charts with the respective functional symbols as follows: We choose $d = 1$ and set

| Function Symbol | $T$ | $D$ | $S$ | $t$ | $d$ | $s$ |
|---|---|---|---|---|---|---|
| Consonant Mode | $M_{1,p+4}^z$ | $M_{11,p+9}^z$ | $M_{3,p+11}^z$ | $M_{9,p+1}^z$ | $M_{7,p+6}^z$ | $M_{11,p+8}^z$ |
| Dissonant Mode | $M_{10,p+4}^z$ | $M_{8,p+9}^z$ | $M_{0,p+11}^z$ | $M_{0,p+1}^z$ | $M_{10,p+6}^z$ | $M_{2,p+8}^z$ |

The corresponding predicates are related to the extensions (the unique open allomorph) $Ext_0(M_{d,a}^z)$ and take symbols $T, D, S, t, d, s$ for the "consonant mode", and $T^\star, D^\star, S^\star, t^\star, d^\star, s^\star$ for the "dissonant mode"[12], i.e.,

$$\text{VAL} = \{T, D, S, t, d, s, T^\star, D^\star, S^\star, t^\star, d^\star, s^\star\},$$

and the value

$$F_{e^pC,v}(c) = \begin{cases} Ext_0(M_{a(p,v),d(p,v)}) \text{ if } c \in Ext_0(M_{a(p,v),d(p,v)}), \\ \emptyset \text{ else,} \end{cases}$$

with $v \in \text{VAL}$, and

$$\begin{aligned} (a(p,T), d(p,T)) &= (1, p+4), \\ (a(p,D), d(p,D)) &= (11, p+9), \\ &\ldots \\ (a(p,s^\star), d(p,s^\star)) &= (2, p+8), \end{aligned}$$

[12] Terminology from [400].

according to the above table. Recall that the zero-constant parts of these morpheme charts are classical triads, as discussed in proposition 46 of section 25.3.1 and the remarks following that proposition. We also see that the terminology of a "consonant" or "dissonant" mode is now plausible by the triads in the dissonant mode being related to seventh and tritonus intervals.

This example can be enriched by adding selected allomorphs to the above open allomorphs. A priori, we would preconize the full allomorphic extension of each of the above morphemes, and then build a harmonic function theory upon the complete 'allomorphic spectrum'.

**Exercise 60** Calculate and draw the nerve of the twelve-chart interpretation by the semigroup morphemes[13] $ex_z(\langle \alpha_d, \beta_a \rangle)$ and the sub-nerve of its zero-constant trace by the bigeneric morphemes in example 46. The nerve of the semigroup submorphemes tells more about the intersection configurations because the identity of a monoid 'collapses' all information about other common points.

**Example 47** Third Degree Tonality. We have already defined the triadic interpretation $X^{(3)}$ of a diatonic scale $X$ in section 13.4.2. This was a zero-addressed situation which we now 'blow up' in the self-addressed context[14]. Again, we have the twelve pitch class related tonalities

$$\mathrm{TON} = \{C, D_\flat, D, E_\flat, E, F, G_\flat, G, A_\flat, A, B_\flat, B\}.$$

We first stick to the basic $X = C$ and start on seven monoid morphemes, $M_I = Int(I_C), M_{II} = Int(II_C), \ldots M_{VI} = Int(VI_C), M_{VII} = Int(VII_C)$. We have $Int(I_C) = M_{1,4}$, i.e., the generators $\alpha, \beta$ generate the entire monoid $Int(I_C)$, which is the semigroup generated by $\alpha, \beta$, united with the identity[15]. The analog is true for the other monoids $Int(II_C), \ldots Int(VI_C)$ since degrees $II$ to $VI$ are isomorphic to degree $I$. However, $Int(VII_C)$ is more complicated. The symmetry group of $VII_C$ is a Klein group and the sub-semigroup of non-invertible elements is not bigeneric in the sense discussed above. Here it is:

$$\begin{aligned} Int(VII_C) - Sym(VII_C) \quad = \quad & \{e^1 10, e^2 0, e^5 0, \\ & e^5 6, e^1 16, \\ & e^8 3, e^8 9, \\ & e^1 2, e^7 2, e^3 10, e^9 10, e^1 8, e^7 8, e^{10} 8, \\ & e^3 4, e^6 4, e^9 4\} \end{aligned}$$

[13] These can also be defined as being the intersection of the monoid morphemes with the semi-group of all non-invertible elements of $End(F)$.

[14] Recall that we work in pitch space $PiMod_{12}$ with the fifth identifier.

[15] We know from appendix C.2.3 that

$$\begin{aligned} \mathbb{Z}_{12} @ \mathbb{Z}_{12} \quad = \quad & \sqrt{Idempot(\mathbb{Z}_{12} @ \mathbb{Z}_{12})} \\ = \quad & \sqrt{1} \sqcup \\ & \sqrt{4} \sqcup \sqrt{e^3 4} \sqcup \sqrt{e^6 4} \sqcup \sqrt{e^9 4} \sqcup \\ & \sqrt{3} \sqcup \sqrt{e^4 3} \sqcup \sqrt{e^8 3} \sqcup \\ & \sqrt{0} \sqcup \sqrt{e^1 0} \sqcup \sqrt{e^2 0} \sqcup \sqrt{e^3 0} \ldots \sqcup \sqrt{e^{11} 0} \end{aligned}$$

is the idempotent decomposition of $\mathbb{Z}_{12} @ \mathbb{Z}_{12}$ into invertible endomorphisms, those above linear factor 4, those above linear factor 3, and those above linear factor zero. Since these linear factors commute, the components, which are conjugate with each other above a fixed linear idempotent (appendix C.2.3, proposition 66), are all semigroups.

We denote the monoid generated by these non-invertible elements of $Int(VII_C)$ by $Int_0(VII_C)$. The charts of our self-addressed triadic interpretation are the six charts $M_I = intex_z(I_C), M_{II} = intex_z(II_C), \ldots M_{VI} = intex_z(VI_C)$, and the degree seven chart $M_{VII} = monex_z Int_0(VII_C)$. The function values are symbolized according to all possible allomorphs of the seven chart monoids, which yield the truth values as in the preceding example. We leave the details to the reader.

The intersection configuration is remarkable. The nerve of the objective traces of the $ex_z$-charts from the sub-semigroups of non-invertible elements[16] in the monoids

$$Int(I_C), \ldots Int(VI_C), Int_0(VII_C)$$

is a priori a simplicial complex containing the nerve of the zero-constant part. In this case, no new intersection configuration intervenes: *As in the zero-constant case, the nerve of the self-addressed extension of the triadic interpretation of a diatonic scale is again a Möbius strip.* Figure 25.2 shows the intersection configuration in detail.

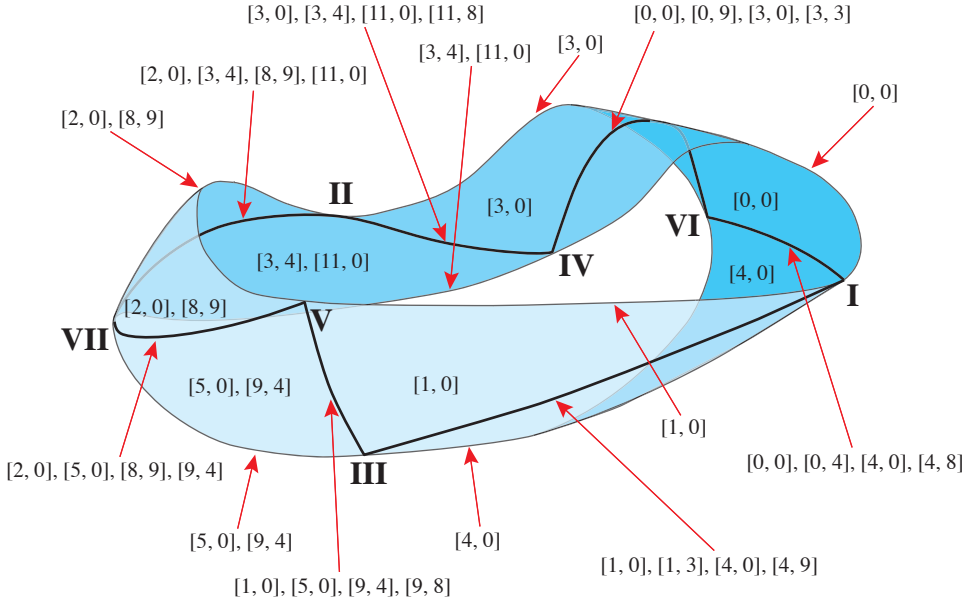


Figure 25.2: The non-invertible elements of the self-addressed charts built upon the endomorphisms of classical triads define a nerve which is identical to the nerve from the constant parts of the self-addressed triads! The figure shows the self-addressed chart intersections for $C$ tonality, everything calculated in the fifth-identifier form $PiMod_{12}$.

**Remark 13** The passage from zero-addressed to self-addressed degrees maintains the nerve, but if we take one more step from the self-addressed to the functorial setup, the Möbius strip is

[16] These are all elements except the identity.

enriched by new simplexes. More concretely, we take the self-addressed degrees $X \subset \mathbb{Z}_{12}@\mathbb{Z}_{12}$ and consider the associated functorial local compositions $\hat{X} \subset (\mathbb{Z}_{12}@\mathbb{Z}_{12})\hat{}$ and the global composition $Z^I$ defined by this covering $I$. In this setup, the nerve of the self-addressed configuration is of course part of the nerve of the functorial configuration $Z^I$, but we encounter new simplexes. Specifically:

**Exercise 61** Show that the functorial degrees $I\hat{}$ and $II\hat{}$ have a non-empty intersection and thereby extend the Möbius strip of the self-addressed triadic covering. Hint: Find a morphism $f : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ which yields an $f$-slice $f@I\hat{} \cap f@II\hat{} \neq \emptyset$.

# Chapter 26

# Cadence

*Was ist C-Dur?*
*Wahrlich, Herr Mazzola, eine gute Frage!*
Rudolf Wille [575]

**Summary.** Cadences are shorthand representations of tonalities. (We do not discuss the other meanings—e.g., the solo cadence in the sense of a concert climax—of this typically homonymic term.) There is a variety of approaches to realize such a representation. We give an explicit definition of the concept of a cadence with respect to varied addresses and ambient spaces. In particular, we present the very classical cadences, those related to self-addressed function theory, and more exotic self-addressed cadences which relate to symmetries rather than to tones or sets of such objects.

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The concept of a cadence as a final cadence or a cadential formula to consolidate a tonality goes back to Italian theorists of the $16^{th}$ century. In this chapter, we shall discuss this meaning of the word and not that of the concert solo cadence.

In this sense, a cadence is a good example for the construction of sign structures in music. The signifier expression is fixed in a cadential formula, such as $I$-$IV$-$I$-$V$-$I$, which is supposed to obtain its signification in a specific context, and thereby points to its signified content: the "uniquely determined tonality" or the "composition's conclusion", respectively. We shall only deal with the former meaning because this one is also the structural substrate of the latter: the conclusion is expected because a specific, uniquely determined tonality is recognized from a cadential formula.

If this tonality (whatever the technical meaning of this word may be) is the significate of the given cadence formula, a mathematical function of type "cadence formula=function(tonality)" is given. The characteristic of this function is that it is *injective*. Two tonalities are associated with different cadence formulas: $I_C$-$IV_C$-$I_C$-$V_C$-$I_C$ differs from $I_D$-$IV_D$-$I_D$-$V_D$-$I_D$.

Implicitly, the existence of a cadence function depends upon two conditions which usually are not noticed by theorists because they seem straightforward:

- The first deals with the very concept of a tonality. Making this concept precise is a condition for a unique determination of its instances whatsoever. The recognition of a tonality from its cadence presumes that you are able to distinguish it from musical objects that are not tonalities or are so, but share a different flavor which you would eliminate in the present context. For example, the recognition of the European $C$-major tonality from the classical Rameau cadence $I_C$-$IV_C$-$I_C$-$V_C$-$I_C$ presupposes that we would not search in a set of Turkish or other non-European tonalities. So we may suppose being given a set $X$ of global compositions which define the extension of the concept of a tonality.

- The second condition relates to the range of the possible cadence formulas. In fact, searching for a formula presupposes that one will find it in a delimited domain of musical cadence expressions. In other words, one should know what is a cadence formula in the present context and what is not. Call $P$ the set of admitted cadence formulas, or "cadence parameters" as the elements of $P$ parametrize the domain of formulas. We shall see that for all known situations, $P$ can be chosen to be a set of global compositions.

## 26.1 Making the Concept Precise

**Summary.** This section describes the concept of a *cadence*.

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**Definition 86** *Given two sets $X, P$ of tonalities and cadence parameters, respectively, as introduced above, and a map $\kappa : X \to P$, we say that $\kappa$ is* cadential *in a tonality $x \in X$ if the fiber $\kappa^{-1}\kappa(x)$ is the singleton $\{x\}$, i.e., $\kappa$ is injective in $x$. The map $\kappa$ is called* cadential *or* a cadence*, iff it is cadential in every tonality $x \in X$.*

We should however be aware that the general nature of a cadence map cannot be restrained to mathematical or even functorial properties. A cadence is a relation which can issue from any motivation, be it historical, systematic or arbitrary. Essentially, the above definition is the one used in [327, 328, 340]. In the present theory language, we would prefer restating that definition in terms of textual predicates. Disregarding the fact that global compositions are not denotators, we would have to extend the cadence map to a 'global predicate' $Cad$ outside $X$ by the value $\kappa(y) = y/Cad = \emptyset$ for $y \notin X$. Inside $X$, we then would have the values $x/Cad$ in $Glob_F^A$, the set of $A$-addressed global compositions with charts living in $\mathbb{T}_F^A$ for a specific *cadence* truth form $F$. Whenever we use this wording for cadences, keep in mind that no extension of the category *Den* of denotators to global objects has been defined so far, except for powerset denotators, i.e., local compositions. Nonetheless, we shall use this sloppy language because the environments are evident.

Before discussing the examples, we should emphasize the dramatic ambiguity of these concepts between truth and beauty. The truth values for a cadence are global compositions, i.e., objects of musical esthetics rather than poor truth denotators. A cadence gives you a parametrization of musical objects—tonalities in this case—in an auxiliary space of musical objects in order to obtain a shorthand representation of these objects. Rather than evaluating these objects by truth or falsity, the cadence positions them in an esthetic parameter space.

Observe also that we did not give a specific delimitation of the tonality concept, it is just a word which will be filled with meaning in the following examples. Nonetheless, we should recall the discussion of function theory in chapter 25.3. There, "tonality" was introduced as a symbol in an abstract set $TON$. Its usage was the construction of a bunch of textual predicates $TF_{f,t}$ which give each chord a tonal function (in this case: an allomorphic set in the topological extension space). These predicates may be used to define cadences, but the effort is considerable, we leave this as an exercise[1]. The link between the function predicates and the cadence predicates is that the function predicates are evaluated by use of global compositions (such as the triadic interpretation of a scale). These global compositions correspond to the tonalities in the set $X$ for cadences. Notice that the scope of tonal functions is not the identification of tonalities, but the attribution of a tonal value for any given chord, and this is much more than mere identification.

## 26.2 Classical Cadences Relating to 12-tempered Intonation

**Summary.** This first example reviews the classical cadences of European tonal music in relation to pitch classes in $PiMod_{12}$.

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The first group of cadences relates to zero-addressed (commutative) compositions, more precisely, to compositions stemming from the pitch class ambient spaces $\mathbb{Z}_{12}$ for 12-tempered tuning and $\mathbb{Z}^2$ for just temperament. These cases are also prominent since we have extensive modulation theorems for them.

These approaches are also prototypical in so far as the used atlases consist of quite special charts (typically triadic coverings) which could be generalized to richer and more realistic coverings (and partly have been by Straub's investigations). But it is just this simplistic perspective which shows that even under elementary conditions, interesting and experimentally significant structures may emerge. The experimental investigations will be explicated in chapter 27 dedicated to modulation phenomena in classical scores from European literature.

### 26.2.1 Cadences in Triadic Interpretations of Diatonic Scales

**Summary.** This section describes the cadences related to triadic degrees.

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We start with the classical example. Let $X = Dia^{(3)}$ be the twelve-element set of zero-addressed triadic interpretations of all major scales in $\mathbb{Z}_{12}$. Take the cadence parameter set $P = ObGlob_0^{\mathbb{Z}_{12}}$ of global, zero-addressed compositions with charts in $\mathbb{Z}_{12}$. For a given sequence $J = 1 \leq j_1 < j_2 < \ldots j_k \leq 7$ of indices for triadic charts, we define the map $\kappa_J(S^{(3)}) = \{J_{j_1}, J_{j_2}, \ldots J_{j_k}\}$ with image the sub-interpretation of the covering defined by the charts from the subsequence $J$. For example, if $J = 1, 4$, we get the value $\kappa_J(E^{(3)}) = \{I_E, IV_E\}$, a two-chart-interpretation of the six-element pitch class set $\{e, g_\sharp, b, a, c_\sharp\}$. The extension of such a

[1] Speaking of a cadence in the context of tonal function predicates, this cadence would be formalized by the function vector $TF_t$ discussed in sec section 25.3.2.

function to other triadic interpretations, for example of melodic minor scales, would evidently destroy cadentiality. On the other hand, ignorance of the precise formula (first and fourth triadic degrees) would make it impossible to retrace the argument. For example, if we had also at our hands the first and fifth triadic degrees, it could not be decided whether the interpretation is the image $\{I_E, IV_E\}$ or $\{I_A, V_A\}$.

Although the function $\kappa_J$ is a cadence for the given tonality domain $Dia^{(3)}$, the usual selection method for 'good' sequences asks for more. The idea is that you only know that the associated interpretation is defined by some $J$, but one is not sure about which one has been applied. So the requirement is that the tonality should be uniquely determined whatever $J$ may be the defining selection parameter. This leads to the concept of a cadential set of triadic degrees.

**Definition 87** *Given a set $X$ of interpretations of objective, local $A$-addressed compositions of form space $F$, a set $C \subset I$ of charts of an element $S^I \in X$ is called* cadential in $S^I$ *iff $S^I$ is the only interpretation $T^J$ in $X$ such that $C \subset J$. The cadential set $C$ is called* minimal cadential in $S^I$ *iff it has no proper subset which is also cadential in $S^I$.*

*A family $(C_x)_{x \in X}$ of (minimal) cadential sets $C_x$ for each interpretation $x \in X$ is called a* (minimal) cadential family *for $X$.*

The classical example $X = Dia^{(3)}$ yields five index sequences for minimal cadential families:

$$J_1 = (II, III), J_2 = (II, V), J_3 = (III, IV), J_4 = (IV, V), J_5 = (VII).$$

This means that the family $J_1$, for example, defines its set $\{II_{S^{(3)}}, III_{S^{(3)}}\}$ for the triadic interpretation $S^{(3)}$. The fourth one is the reminiscence to Rameau's cadence *I-IV-I-V-I*, except that the first degree triad is superfluous since no tonic is stressed in this triadic interpretation.

**Remark 14** Observe that in the present definition of cadential sets, no syntactical position is defined, so the sequence in the Rameau cadence *I-IV-I-V-I* is not reflected, only the set $\{I, IV, I, V, I\}$ is of interest in this context. A further specification of syntactical relations is necessary for the concrete applications, but this is another subject which we do not open up here.

In [390], Daniel Muzzulini calculated all minimal cadential sets for the following tonality sets $X$ (by computer). We take the translation orbits $X = e^{\mathbb{Z}_{12}}.S^{(3)}$ of triadic interpretations of seven-element scales $S \subset \mathbb{Z}_{12}$ by the seven degrees consisting of those three-element subsets such that every second pitch class in the clock order is selected. The number of cadential sets varies between 5 and 21, the first number being taken by the triadic major scale interpretations $Dia^{(3)}$ while the maximum of 21 cadence sets is taken by the orbit of harmonic minor scale interpretations $Har^{(3)} = e^{\mathbb{Z}_{12}}.c_h^{(3)}$, in fact, here every pair $(X, Y)$ of two different degree indexes defines a minimal cadence set, and we have $\binom{7}{2} = 21$ sets.

A more modern definition of cadential sets runs as follows. We consider a set $X$ of $A$-addressed interpretations

$$x = S^I \to S \to @A \times F$$

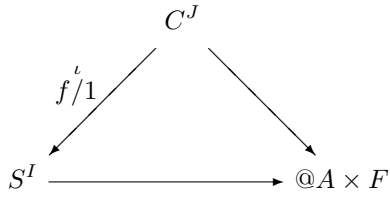
in form space $F$. A cadential interpretation for $S^I$ is an interpretation

$$C^J \to C \to @A \times F$$

which factorizes through $S^I$, i.e., there is a morphism

$$\overset{\iota}{f}/1 : C^J \to S^I$$

such that the triangle




(26.1)

commutes. This implies that each chart of $C^J$ is associated with a chart of $S^I$, and that this is done in such a way that the points of the charts are mapped into each other in accordance with the underlying embeddings of the local support compositions $S, C$. And we ask that there is no factorization through an interpretation $y \neq x$ in the set $X$. Morally (which in this book reduces to topos theory), such a cadential interpretation is a point in the full sieve $@x$ which characterizes $x$ within the collection $X$. So cadential sets relate to selecting 'separating' points $c_x$ in full sieves $@x, x \in X$. A general theory of such constructs has not been developed to date.

### 26.2.2 Cadences in More General Interpretations

**Summary.** This section describes the cadences related to generalized degrees in general scales, as investigated by Hans Straub.

– Σ –

Starting from the generalization of minimal cadence sets for the seven-element scales studied in [390], Hans Straub [514] has undertaken a vast generalization of cadence set types on seven-element scales. A (minimal) cadential set is the same as above, but it need not cover the scale, it is a more abstract representation not of the single scale notes, but of the degree representation of the scale. For example, one may consider as degrees all the triadic chords of minor, major, diminished, or augmented types, which are contained in a given scale. Complete computer-aided (here, computers become definitively mandatory) calculations have been carried out for minimal cadence sets consisting of a number of three-element degrees of any type, no classical triadic chords are preferred.

If the degrees are all the triadic chords of minor, major, diminished, or augmented types in a scale, the nerve dimensions may vary from zero to seven, and not every scale is covered by this degree system. But this degree system has the advantage that all degree chords are classical, which is not the case in Muzzulini's generalization. However, in Straub's general concept framework, one finds degree systems consisting of seven triadic chords which have no cadence sets at all, for example for the scales of type 60 and 62 (in our classification table L.1).

It turns out that 21 is the absolute maximum of possible minimal cadence sets for systems of seven three-element chords in seven-element scales, and that it is achieved by the majority of scales.

In the very general setup, Straub recognizes that a particular type of degree systems is interesting. According to the degree-theoretic approach which includes coverings by standard structures (as sketched in section 25.1), Straub therefore considers *irreducible* degree systems $DEG$ in the sense that no element of $DEG$ can be written as a union of other elements of $DEG$. Evidently, systems consisting of elements of equal cardinality, such as triadic systems, are irreducible. Also, a system is always built by a unique irreducible subsystem. A classification of all irreducible degree systems has not been accomplished to date. It is however important to embed the classical European degree concepts in this variational environment to understand how special or singular the classical choice is. According to Straub's investigations, the distinguished position of the classical diatonic, harmonic, and melodic scales does not emerge on the general level of degree systems and the associated cadence systems, but only with respect to characteristic properties in modulation theory (see chapter 27 for this discussion). This latter aspect has however not been investigated in depth.

**Exercise 62** Show that a degree system of a finite scale is always built by a unique irreducible subsystem.

## 26.3 Cadences in Self-addressed Tonalities of Morphology

**Summary.** Relating to interpretation of self-addressed tonalities, minimal cadential sets have been calculated by Thomas Noll, we present and discuss the results.

– Σ –

In [403], the basic form is the classical 12-tempered pitch class space $F = PiMod_{12}$, with fifth identifier. In the vein of our example 47 in chapter 25, major and minor tonalities are constructed as interpretations of $\mathbb{Z}_{12}$-addressed local compositions. Elements $e^a.b$ are denoted by $[a, b]$. We restrict to the description of $C$-major and $C$-minor, the other 22 tonalities being defined via transposition.

The basic local composition (the 'scale') has 23 elements, eight of them being constant, i.e., stemming from zero-addressed tones:

$$\begin{aligned} C\text{-major} = \{ \quad & [0,0],[1,0],[2,0],[3,0],[4,0],[5,0],[8,0],[11,0], \\ & [1,3],[2,3],[3,3],[8,3],[11,3], \\ & [0,4],[3,4],[9,4], \\ & [4,8],[8,8],[9,8],[11,8], \\ & [0,9],[4,9],[8,9]\}, \end{aligned}$$

whereas the interpretation is given by 10 charts which are related to the function symbols $T, D, S, D^*, s, s^*$ in major and $t, d, s, s^*, D, D^*$ in minor. Each of these main function symbols is associated with a "Gegenklang" and a "Parallelklang" symbol $g, p$, respectively. We have these

major tonality charts[2]:

$$
\begin{array}{ll}
Tg = [1,3],[4,9],[1,0],[4,0], & Tp = [4,8],[0,4],[0,0],[4,0] \\
Sg = [3,3],[0,9],[0,0],[3,0], & Sp = [11,8],[3,4],[11,0],[3,0] \\
Dg = [11,3],[8,9],[2,0],[5,0], & Dg^* = [8,3],[8,9],[11,0],[5,0] \\
Dp = [9,8],[9,4],[1,0],[5,0] & \\
sG = [11,3],[8,9],[11,0],[8,0], & sG^* = [2,3],[8,9],[2,0],[8,0] \\
sP = [8,8],[0,4],[0,0],[8,0] &
\end{array}
$$

which are motivated by the allomorphs to the bigeneric morpheme construction from section 25.3.1 (see also [403]).

**Exercise 63** Draw the nerve of this interpretation, and the nerve of the constant subcharts. Compare them to the nerve (harmonic strip) of the triadic interpretation from example 47 in chapter 25.

With these settings, the set $X$ of tonalities is defined by the 24-element set of 12 major tonalities $C$-major, $D_\flat$-major,... $B$-major, and 12 minor tonalities $C$-minor, $D_\flat$-minor,... $B$-minor. The cadence maps are given by 56 minimal cadential sets. Half this number defines the minimal cadential sets and are applied to major tonalities only (!), the other half operates on minor tonalities. This means that the the 28 candidates for major tonalities characterize any major tonality within $X$, whereas the other half does the job for the minor tonalities. The following list from [403] shows the candidates for major tonalities. With each chart $Z$, the number in brackets indicates the number of zero-addressed chords with up to four tones, which is contained in the extension $Ext_0(Z)$.

Observe that in general, replacing the chart by its zero-addressed extension is not cadential. The point is that in a concrete score, only zero-addressed chords will appear, at least without any additional constructions. So the question of how to represent a self-addressed chord on the zero-addressed level arises. To look for zero-addressed extensions means that one switches to the topological framework and seeks zero-addressed chords in the extension of a chart $Z$. A refined method would consist in the consideration of allomorphs as discussed in the function-theoretic logic above. This method should however be worked out in more detail. In fact, it is possible that a zero-addressed chord is contained in several extensions, and, worse than that, in extensions stemming from different cadential sets. So the overall information is that a zero-addressed chord is contained in some extensions for some cadential sets. The problem is this: How can one decide which tonality is the best candidate for the placement of the given chord?—Such a question becomes important if one needs recognition algorithms in a systematic tonality analysis of given scores, on a software platform such as RUBATO®, for example.

Still more generally set, a zero-addressed chord need not be viewed as a local object, but it may be interpreted by subchords which live in some extension sets, much like the degree concepts in non-irreducible chords of Straub's above approach. No systematic research has been carried out to date.

[2]The symbol construction in [403] is not thoroughly logical.

Minimal cadence sets in major self-addressed tonalities.

| | | |
|---|---|---|
| $Dg^*$ (26) | $Tg$ (12) | |
| $Dg^*$ (26) | $Tp$ (32) | |
| $Sg$ (12) | $Dg^*$ (26) | |
| $Sg$ (12) | $Dg$ (12) | |
| $Sg$ (12) | $Dp$ (32) | |
| $Sg$ (12) | $sG^*$ (26) | |
| $sG^*$ (26) | $Tg$ (12) | |
| $sG^*$ (26) | $Tp$ (32) | |
| $Sp$ (32) | $Dg^*$ (26) | |
| $Sp$ (32) | $Dp$ (32) | |
| $sP$ (32) | $Dg$ (12) | |
| $sP$ (32) | $Sg^*$ (26) | |
| $Sg$ (12) | $sG$ (12) | $Tg$ (12) |
| $Sg$ (12) | $sG$ (12) | $Tp$ (32) |
| $Sg$ (12) | $sP$ (32) | $Tg$ (12) |
| $Sg$ (12) | $sP$ (32) | $Tp$ (32) |
| $sG$ (12) | $Dg$ (12) | $Tg$ (12) |
| $sG$ (12) | $Dg$ (12) | $Tp$ (32) |
| $sG$ (12) | $Dp$ (32) | $Tg$ (12) |
| $sG$ (12) | $Dp$ (32) | $Tp$ (32) |
| $sP$ (32) | $Dg$ (12) | $Tg$ (12) |
| $sP$ (32) | $Dg$ (12) | $Tp$ (32) |
| $sP$ (32) | $Dp$ (32) | $Tg$ (12) |
| $sP$ (32) | $Dp$ (32) | $Tp$ (32) |
| $Sp$ (32) | $sG$ (12) | $Tg$ (12) |
| $Sp$ (32) | $sG$ (12) | $Tp$ (32) |
| $Sp$ (32) | $sP$ (32) | $Tg$ (12) |
| $Sp$ (32) | $sP$ (32) | $Tp$ (32) |

The corresponding minor list is deduced from the preceding one by these exchanges: $Tg \to tG, Tp \to tP, Sg \to dG, Sp \to dP, Dg \leftrightarrow sG, D^*g \leftrightarrow s^*G, Dp \leftrightarrow sP$.

## 26.4 Self-addressed Cadences by Symmetries and Morphisms

**Summary.** More *exotic* cadences can be introduced by use of symmetries instead of chords. This example will be applied in the analysis of L. van Beethoven's "Hammerklavier"-Sonata in

section 28.2.

– Σ –

The preceding examples have dealt with different addresses, but they had in common that the addresses of the tonality sets and of the cadence parameter sets were the same in each case. One can also try to associate tonalities and cadences on different addresses. One example of such a procedure, which we shall use in the discussion of the modulation architecture in Beethoven's op. 106 (section 28.2), will be discussed hereafter.

We start on the space $ExTop(PiMod_{12})$ of zero-addressed local compositions on pitch classes. To each such local composition $Ch$, the continuous injection in formula (24.13)

$$intex_{z,B} : ExTop_0(PiMod_{12}) \to ExTop_{\mathbb{Z}_{12}}(PiMod_{12})$$

associates the 'self-addressed chord' $intex_{z,B}(Ch)$ of all endomorphisms. The idea is that one may associate to this fractal chord a self-addressed chord defined by a specific sub-monoid of endomorphisms. Since $intex_{z,B}$ is injective, it is a cadence map on any set of 'tonalities' $X \subset ExTop_0(PiMod_{12})$. So if $X$ is any set of zero-addressed local compositions of pitch classes, playing the role of scales, we have a first example of a cadence via $intex_{z,B}|_X$. The example we are aiming at is the selection of a specific group of automorphisms in $intex_{z,B}(Ch)$. For example, we may consider a group $G$ of automorphisms of $PiMod_{\mathbb{Z}12}$ and then consider the intersection (in the naive sense, not as functors) $intex_{z,B,G}(Ch) = monex_z(G) \cap intex_{z,B}(Ch)$. For example, take the group $G = TI_0$ of inversions and transpositions on $\mathbb{Z}_{12}$ and then take the intersection $intex_{z,B,TI_0}(Ch)$, i.e., the group of 'counterpoint pitch symmetries' on $PiMod_{\mathbb{Z}12}$ which induce automorphisms of $Ch$. Or else take the full automorphism group $Aut(PiMod_{\mathbb{Z}12}) \xrightarrow{\sim} \overrightarrow{GL}_1(\mathbb{Z}_{12})$.

Clearly, such a map is not injective on all of $ExTop_0(PiMod_{12})$ since the automorphisms of a zero-addressed chord are the same as those of its complement. If we do however restrict the cadence domain to the domain $X_6$, those objective zero-addressed local compositions with up to six elements, then the map

$$intex_{z,B,\overrightarrow{GL}_1(\mathbb{Z}_{12})} : X_6 \to ExTop_{\mathbb{Z}_{12}}(PiMod_{12})$$

has a nice property of cadentiality:

**Lemma 41** *The map $intex_{z,B,\overrightarrow{GL}_1(\mathbb{Z}_{12})}$ on the domain $X_6$ is cadential exactly in the three diminished seventh chords (of class number 37, pitch classes in semitone steps):*

$$\begin{aligned} \mathfrak{M}_0 &= \{1,4,7,10\}, \\ \mathfrak{M}_1 &= e^1.\mathfrak{M}_0 = \{2,5,8,11\}, \\ \mathfrak{M}_2 &= e^2.\mathfrak{M}_0 = \{0,3,6,9\}. \end{aligned}$$

A proof results from the contemplation of the symmetry groups of chord classes in the class list of Appendix L.1.

This means that the diminished seventh chords are the only ones that can be recovered from their symmetries! As with the self-addressed tonalities above, these maps have the drawback that their codomain values are somewhat above naive reality of score objects. But it is possible to give a more down-to-earth restatement of this map in terms of orbits!

To this end, recall that the set $Dia^{(3)}$ of triadic interpretations of diatonic scales is the orbit of the counterpoint pitch group $TI_0$. Any subgroup $G \subset TI_0$ generates the partition $orb(G)$ of $G$-orbits on $Dia^{(3)}$, an element of the set $Part(Dia^{(3)})$ of partitions of $Dia^{(3)}$. Let this set be the new parameter set $P = Part(Dia^{(3)})$ of our map, and take again $X = X_6$ as domain, i.e., $\kappa_{orb} : X_6 \to Part(Dia^{(3)})$. We define

$$\kappa_{orb}(Ch) = orb(intex_{z,B,TI_0}(Ch))$$

with the obvious restatement of $\mathbb{Z}_{12}$-addressed elements as automorphisms. Then:

**Lemma 42** *The map $\kappa_{orb}$ is cadential exactly in the three diminished seventh chords*

$$\mathfrak{M}_0, \mathfrak{M}_1, \mathfrak{M}_2.$$

We leave the proof as an exercise for the reader, but observe that there are exactly two orbits in the three cadential chords: We have

$$\kappa_{orb}(\mathfrak{M}_0) =$$
$$\{W = \{B_\flat^{(3)}, D_\flat^{(3)}, E^{(3)}, G^{(3)}, A^{(3)}, C^{(3)}, E_\flat^{(3)}, G_\flat^{(3)}\}, \quad W^* = \{B^{(3)}, D^{(3)}, F^{(3)}, A_\flat^{(3)}\}\},$$

the others are transpositions thereof. These two orbits, the large $W$ ("world") and the small $W^*$ ("antiworld") will play a crucial role in the analysis of the modulation architecture of Beethoven's op.106.

## 26.5 Cadences for Just Intonation

**Summary.** For use in just modulation theory as it has been worked out by Hildegard Radl [429], we present minimal cadence sets in third-fifth and Pythagorean tuning.

– Σ –

We recall the context from just and "justest" tuning introduced in section 13.4.2.2. We are working in the *EulerPlane* space spanned by fifth and third axes, as well as in the Pythagorean subspace *PythagorasLine* spanned by the fifth axis alone, and related to the *EulerPlane* by the projection modulo third comma.

In all the following cases, we shall only discuss the triadic interpretations of the $C$ scale since the other scale interpretations are deduced from this one by transpositions, and the corresponding cadential sets are the transposed sets, so the names of degrees in cadential sets just vary by the scale symbol. We shall therefore omit this specification in order to keep everything readable.

### 26.5.1 Tonalities in Third-Fifth Intonation

**Summary.** The minimal cadential sets of degrees are described for major, (natural) minor, harmonic minor, and melodic minor tonalities.

– Σ –

The just triadic *major* scale interpretation $C^{(3)}$ has the seven degrees[3]

$$I, II, III, IV, V, VI, VII$$

described in figure 13.12. The minimal cadential sets are these:

$$J_1 = (II), J_2 = (III, VI), J_3 = (III, IV), J_4 = (IV, V), J_5 = (VII), J_6 = (V, VI). \quad (26.2)$$

The just triadic *natural minor* tonality $c^{(3)}$ is the isomorphic image of $C^{(3)}$ under the 180 degree rotation around the middle between tones $c$ and $g$, see also figure 7.5 which shows the scale and the seven degrees. The minimal cadential sets follow from the transformation of the above major cadence sets 26.2 and are these[4]:

$$J_1 = (VII), J_2 = (III, VI), J_3 = (V, VI), J_4 = (IV, V), J_5 = (II), J_6 = (III, IV). \quad (26.3)$$

The just triadic *harmonic minor* scale interpretation $c_h^{(3)}$ works as follows: The harmonic minor scale and its degrees are shown in figure **??**. The scale $c_h$ is selected out from the chromatic scale shown in figure 7.5, the degrees are uniquely determined by the tone names given in the chromatic scale.

The minimal cadential sets are these:

$$J_1 = (III), J_2 = (II), J_3 = (VII), J_4 = (I, IV), \quad (26.4)$$

$$J_5 = (I, V), J_6 = (I, VI), J_7 = (IV, V), J_8 = (IV, VI), J_9 = (V, VI). \quad (26.5)$$

The just triadic *melodic minor* scale interpretation $c_m^{(3)}$ works as follows: The melodic minor scale and its degrees are shown in figure **??**. The scale $c_m$ is selected out from the chromatic scale shown in figure 7.5, the degrees are uniquely determined by the tone names given in the chromatic scale.

The minimal cadential sets are these:

$$J_1 = (I), J_2 = (II), J_3 = (III), J_4 = (IV, V), J_5 = (VI), J_6 = (VII). \quad (26.6)$$

### 26.5.2 Tonalities in Pythagorean Intonation

**Summary.** The minimal cadential sets of degrees are described for Pythagorean major, (just) minor, harmonic minor, and melodic minor tonalities.

– Σ –

The Pythagorean scale interpretations are defined by the comma projection $EulerPlane \to PythagorasLine$, applied to the synonymous just tonalities discussed above.

**Exercise 64** Draw all the above major and minor scales in the Pythagorean tuning.

[3] We do not consider the alternative degrees $II^*$ and $VII^*$ from figure 13.12.

[4] The rotation transforms major degrees into minor degrees as follows: $I \mapsto I, II \mapsto VII, III \mapsto VI, IV \mapsto V, V \mapsto IV, VI \mapsto III, VII \mapsto II$.

Here are the minimal cadential sets:
*Major*:

$$J_1 = (VII), J_2 = (II, III), J_3 = (II, V), J_4 = (III, IV), J_5 = (IV, V).$$

*Natural Minor*, here, the degrees are coupled by a reflection on $PythagoreanLine$[5] and accordingly yield these minimal cadential sets:

$$J_1 = (II), J_2 = (VII, VI), J_3 = (VII, IV), J_4 = (VI, V).$$

*Harmonic minor*:

$$J_1 = (III) \text{and all pairs of any two different degrees} \neq III.$$

*Melodic minor*:

$$J_1 = (III), J_2 = (I, II), J_3 = (I, V), J_4 = (I, VII), \tag{26.7}$$
$$J_5 = (II, IV), J_6 = (II, VI), J_7 = (IV, V), J_8 = (IV, VII), \tag{26.8}$$
$$J_9 = (V, VI), J_9 = (VI, VII). \tag{26.9}$$

[5]The symmetry $e^1.(-1)$ on the line transforms major degrees into minor degrees as follows: $I \mapsto I, II \mapsto VII, III \mapsto VI, IV \mapsto V, V \mapsto IV, VI \mapsto III, VII \mapsto II$.

# Chapter 27

# Modulation

*Denn das Wesentliche an einer Modulation*
*ist nicht das Ziel, sondern der Weg.*
Arnold Schönberg [478]

**Summary.** This chapter deals with the central issue of modulation between two given tonalities. It involves explicit models of tonalities, of cadences and—even more crucial—of the transition process from one tonality to its successor. The present model involves the analogy to elementary particle physics: Modulation is viewed as a 'force interaction' between two 'tonality particles' which is mediated by a 'modulation quantum'. The model allows for a complete *calculation of fundamental degrees of modulation* in congruence with A. Schönbergs harmony [478]. The model is realized for diatonic tonalities in 12-tempered and just tuning. It has been extended to all 7-element scales in 12-tempered tuning and to a number of scales in just tuning. The 12-tempered extension reveals a privileged position of the diatonic scale with regard to this modulation theory. We conclude the chapter with a discussion of the basic role of modulation models and their application to optimize harmonic paths in the sense of section 27.2.

– Σ –

In the historical development of mathematical music theory, the modulation models were the initial major topic. The first results were traced in [326, 327]. The original problem was to understand the role of the diminished seventh chord in the modulation strategy of Beethoven's "Hammerklavier" sonata op.106. It turned out that the inner symmetries of that chord seemed to govern the possible modulatory transitions and their specific stamp. This entailed the idea that modulation could be described by use of 'modulation symmetries' between start and target tonality, and that the pivotal chords (those which enforce the switch between the involved tonalities) could be extracted from a set of 'modulatory tones' which admits such modulation symmetries as *inner* symmetries. The first (not so satisfactory) attempts in fact focused on the diminished seventh chords, but then, the definition of such modulatory sets turned out to work perfectly when coupled to cadential sets for the target tonalities instead of the rather artificially chosen diminished seventh chord.

Interestingly enough, no essential extensions of those early methods have been proposed to date. Which does not mean that the results have not been greatly extended to many general scales and to just tuning. But the general setup of the involved concepts is still the old one.

This is why we shall start this subject with a general view on the modulation methodology thus far developed in the following section 27.1.3. This will also reveal a fascinating perspective towards a "grand unification" of harmony and counterpoint as it has been suggested by Noll's definition of Riemann consonances and dissonances [400].

It is important to review the crucial role of the very concept of modulation in music. The medieval concept of "modulatio", as it is encountered in Augustinus' famous early definition *musica est ars bene modulandi*, has the quite general meaning of rational organization of tones and rhythms or, in the later medieval tradition, and more generally, musical composition. The modern meaning of a tonal modulation is a specialization of the original concept and was only introduced in the 18th century (see article "modulatio" in [457]).

The following approach essentially relates to tonal modulation, but the mathematical formalism and related canonical generalizations suggest a broader conceptualization. This means that the idea of a tonal modulation is in fact a special case of a potential theory of transitions between instances of a particular musical structure. Transition between tonalities can be paralleled by transitions between motives, rhythms, global objects, and by transitions between more sophisticated representations of tonalities (such as the self-addressed setup), motives, rhythms, and global objects. So the theory would have to describe transition mechanisms and their expressions in terms of compositional instances of practical use.

In this spirit, and it is the mathematical universality which preconizes the development, modulation should be rephrased in the understanding of medieval tradition, i.e., as a rational construction of music in all its structural streams. Unfortunately, this latter perspective is by no means a contemporary vein of musicology or even computer music science. But it turns out that any future system of music composition technology should deal with the general problem of transitional structures in all musical layers, such as harmony (in its most general setup), motivic, and rhythmical layout—including their interactions on local and global levels.

## 27.1 Modeling Modulation by Particle Interaction

**Summary.** This section introduces the motivation for developing modulation models. These models are designed to convey a harmonic transformation process. We discuss the 'particle interaction' model which allows us to simulate A. Schönberg's fundamental modulation degrees *wherever* he described direct modulations. The model is thoroughly formalized for 12-tempered and just tuning. The corresponding modulation theorems are proven and commented.

– Σ –

To begin with, this typical case of a model in musicology should be understood with care: It is *not* a dogmatic prescription of what a modulation should be, and a fortiori not a declaration of what kind of tonalities, cadences and modulation mechanisms should be englobed. Everyone can modulate as she likes. Modulation is even and—hélas quite often in popular music—processed without any reflection: In the commercial shortcuts, after a song unit (following the four-period blues scheme, for example) a song is suddenly transposed by a number of semitone

steps and restarts without any preparation or transitory passage on a new shifted tonality. In our approach, we evidently are more interested in the description of possible relations between adjacent tonalities in musical compositions, and in the question of how such relations should be expressed explicitly on the material level of harmonic syntax.

### 27.1.1 Models and the Anthropic Principle

**Summary.** We sketch the role of mathematical models as a means to investigate the anthropic principle of music history, i.e., the historical selection of distinguished solutions of theoretical and practical approaches to musical rule systems within a variety of fictitious alternatives.

– Σ –

The following modulation model is a first, and a very simple construction which however yields very good results and has the typical property of mathematical models in music: To enable a quasi-automatic generalization to situations where the classical music theory for which the model was constructed has no answer. In the case of modulation which originally was modeled for major scales, the generalization extends to arbitrary scales.

The property of extensibility of a mathematical model relocates the existing music theory (which it models) in a field of potential, fictitious theories. This puts the historically grown facticity into a relation with the potential 'worlds of music'. The purely historic justification of existing modulation rules, for example, does not give us reasons for this choice, and this makes the purely historical approach a poor knowledge basis: We know that something is the case, but not why, and why other possibilities are not. In contrast, the mathematical approach gives us a field of potential theories wherein the actual one can be asked for its possible special properties with respect to non-existing variants. This differentia specifica is a remarkable advantage of mathematical methodology against the historical approach of musicology which cannot, in its poor intellectual performance, embed the facts in a variety of fictions and thereby understand the selection of what is against what is not.

This evokes Leibnitz idea that the existing world is the best of all possible worlds: Is the existing music theory the best possible choice? Or is it at least a distinguished one? In cosmology, this idea has been restated under the title of the "anthropic principle" [40]. It says that the physical laws are the best possible for the existence of humans, more precisely (and less radically), it is the theorem stating that a slight variation of the fundamental constants, such as the gravitational constant, or the electric charge of electrons and protons, would make any higher molecular complexity as is necessary for carbon-based biochemistry impossible.

### 27.1.2 Classical Motivation and Heuristics

**Summary.** We describe Schönberg's tripartite modulation scheme and the task of formalization.

– Σ –

In his classical text on harmony [478], Schönberg has described tonal modulation from one tonality to another as a three-part process of neutralization, turning point to the new tonality, and cadential confirmation, see figure 27.1. The central statement is the list of modulation

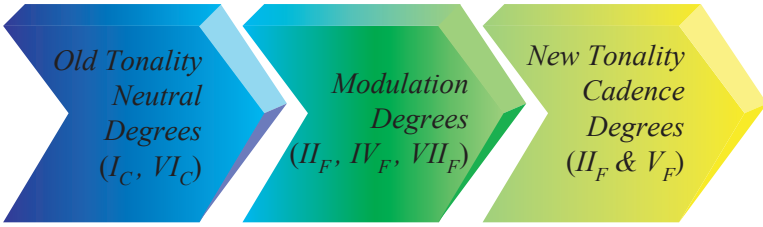



Figure 27.1: The three parts of Schönbergs modulation scheme in the case of a modulation from $C$-major to $F$-major.

degrees in the middle turning point part. A mathematical model must start with a precise restatement of the musicological terms and facts in a rigid mathematical terminology. So we have to answer the questions

1. What is the set of admitted tonalities?
2. What is a degree?
3. What is a cadence?
4. What is the underlying modulation mechanism?
5. How can the modulation degrees be deduced from a solution of the preceding questions?

To begin with, we want to work in the context of well-tempered tuning, and we shall only look for pitch classes, i.e., for for zero-addressed points in $PiMod_{12}$. So our pitch class set identifies to the cyclic group $\mathbb{Z}_{12}$. We consider twelve possible diatonic scales, $C = \{0, 2, 4, 5, 7, 9, 11\}$ and its eleven translates $F, B_\flat, E_\flat, ..., E, D, G$, in the cycle of fourth.

To define tonalities, we then consider the set

$$Dia^{(3)} = C^{(3)}, F^{(3)}, B_\flat^{(3)}, E_\flat^{(3)}, ..., E^{(3)}, D^{(3)}, G^{(3)}$$

of triadic interpretations. The degrees in tonality $T^{(3)}$ are identified with the triadic charts $X_T, X = I, II, \dots VII$, everything being visualized by means of the nerve, i.e., the harmonic strip of a triadic interpretation.

This means that a seventh chord, e.g., $\{d, f, a, c\}$ in $C^{(3)}$, is not simply $II_C$, but must be 'interpreted' as a union $II_C \cup IV_C$ of degrees in $C$. And the chord $\{d, f_\sharp, a\}$ is not a second degree in $C$ with "sharpened third"; degrees are strictly triadic degrees within the respective tonalities!—We should recall the Yoneda philosophy from section 9.3 in this context. Accordingly, an object $X$ is identified by the functor of its 'perspectives', i.e., by the system of all morphisms $f : Y \to X$ in the given category. In the category *Glob* of global compositions, where we are presently working, our perspective is the canonical morphism $X^{(3)} \to X$ onto the basic scale $X$. Another perspective, such as a modal interpretation $X_f \to X$ (see section 13.4.2), or a combined perspective of triads and modal tone $X_f^{(3)} \to X$, would yield a variant of the present approach.

**Remark 15** For the traditionally educated in harmony, this self-control is hard to understand. It seems that one can no longer understand everything. But understanding everything amounts to understanding nothing. We had already seen in section 13.4.2.1 that Riemann's attempt to define tonal functions on all chords had to fail. And the technique of alterations is questionable in harmony since it does not lead to contradictions like function theory, but produces meaningless statements. We know from Mason's theorem in section 7.5.1 that two sharps and three flats suffice to interpret any chord in a seven-note scale as an alteration of a chord in any other seven-note scale! Mason's result tells us that with a sufficiently high number of alteration signs, nothing is in fact said if one interprets a chord in this way as an alteration of a chord in a determined scale.

With these definitions, and with the concept of a cadence being related to minimal sets of degrees (see definition 87 in section 26.2.1) defining uniquely the surrounding tonality, the mechanism of modulation from tonality $S^{(3)}$ to tonality $T^{(3)}$ is defined by a symmetry which transforms $S^{(3)}$ into $T^{(3)}$. As in particle physics, we will materialize such a transformation force by a modulation quantum $M$. By definition, this is a zero-addressed local composition in $PiMod_{12}$, together with an interpretation $M^{(3)}$ by triadic chords such that the modulation symmetry is an inner symmetry of $M$, plus two more technical properties. The point of this modulation quantum is that it is 'musical matter'—much like the physical modulation quanta are materialized forces—and therefore preconizes a set of degrees to be played in the middle part of Schönberg's process shown in figure 27.1.

As to the technical properties of $M^{(3)}$, the first one states that the intersection $M \cap T$ is rigid, i.e., the group of symmetries $Sym(M \cap T)$ is trivial. It guarantees that the modulation symmetry $S^{(3)} \to T^{(3)}$ is uniquely determined. The second property requires that the modulation quantum be a minimal set of pitch classes which satisfies the other properties. This is an economical condition designed to exclude superfluous tones. The precise definitions will be given below in section 27.1.4.

The central statement of this modulation model is that such modulation quanta exist for all pairs of tonalities, and that they yield exactly the modulation degrees described by Schönberg [478]. Figure 27.2 shows such a quantum $M^{(3)}$ for the modulation from $C$ to $E_\flat$. We recognize the two harmonic strips (light) as well as the connecting quantum (dark) in its nerve representation. The modulation degrees are exactly those vertexes of $M^{(3)}$ which lie in the harmonic band (four degrees in the example of figure 27.2) of the target tonality. The statements of this model and its background mechanism, viz. the inherent symmetry transformation, will also be tested on the repertoire of classical music, such as Beethovens op. 106, for example (see section 28.2). For the latter, it is possible to obtain a deeper understanding of its well-known intricate and complex modulation plan, switching back and forth between a "world" around the main tonality $B_\flat$-major, and the "anti-world" around tonality $B$-minor (theses of Erwin Ratz and Jürgen Uhde, see section 28.2.2).

The above modulation model illustrates perfectly the "anthropic principle" subject which emerges from mathematical modeling. The model will be extended to any seven-tone scale and corresponding harmonic bands, see section 27.1.5, and one can thereby locate the case of the major scale in a world of potential variants. The main result in this variational context is that the major scale occupies a singular position in that its modulation arsenal turns out to be the minimal possible variety among all scales which admit modulations between any two

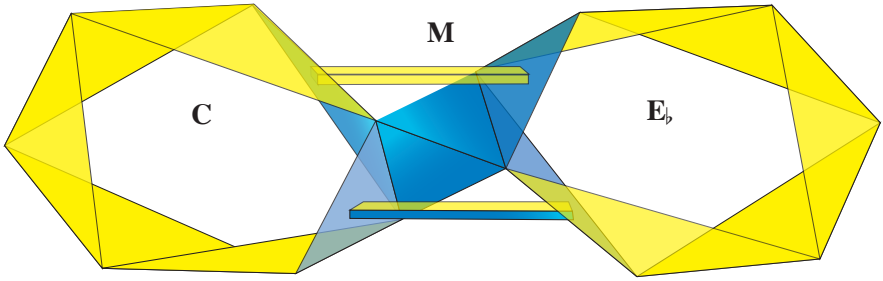



Figure 27.2: Visualization of a modulation quantum $M^{(3)}$ for the transition from $C$ to $E_\flat$. The nerve representation of $M^{(3)}$ shows the connecting middle structure in dark color, in particular it contains two five-dimensional simplexes (symbolized by horizontal bars) above and below a three-dimensional simplex (the tetrahedron).

tonalities. So the historical dominance of the major scale receives a supplementary justification in a perfectly systematic framework, and one which the purely historical approach can a priori not provide.

### 27.1.3 The General Background

**Summary.** The general situation is sketched in order to establish a connection to the topos-theoretic perspective and grand unification options.

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In the preceding classical model of modulation, the space form was $PiMod_{12}$, and we were concerned with a collection $X = Dia^{(3)}$ of triadic interpretations $S^{(3)}$ of scales $S$, i.e., zero-addressed local compositions in $PiMod_{12}$. Moreover, these interpretations were all isomorphic with each other under symmetries which are given on the space $PiMod_{12}$; more precisely, we considered a full orbit $X = G.C^{(3)}$ under the action of a group $G$ of symmetries on the space form. Further, the cadence was chosen to be the one with minimal cadence sets (according to definition 87). Finally, the modulation quantum $M^{(3)}$ was a triadic interpretation of a zero-addressed local composition in $PiMod_{12}$. We have not yet proven *that* such quanta exist, and *when* they do so in the specific setups. We shall discuss this topic in detail later in sections 27.1.4, 27.1.6. For the time being, we are only interested in an adequate generalization of this conceptual setup in terms of the theory of global compositions. We should again stress that this generalization has not yet been completed by corresponding theorems as they exist in some classical cases.

The evident drawback of the classical model is the restriction to isomorphic interpretations. In general, transitions between instances of a type of musical structures cannot be

restricted to isomorphic instances, proper deformations and specializations should be possible. In particular, tonal modulation should also work between non-isomorphic tonalities, modulations between non-isomorphic motives should be the common case, and modulations between completely different rhythms are just what any interesting composition should englobe after a century of jazz. We shall come back to rhythmical modulations between isomorphic rhythms in section 28.3.2, this example uses the present theory in the onset dimension.

It is remarkable that modulation between non-isomorphic tonalities has not yet been treated in a systematic way in traditional music theory. The reason may be that in traditional music cultures, the selection of tonalities is a quite stable preference which identifies the traditional music culture and therefore would contradict its own message while being allowed to change its isomorphism class.

Let us now set up a more general approach to modulation. We first have to identify the basic objects. The obvious generalization for tonalities and modulation quanta is that they are interpretations of local compositions at an address $A$ in a form space $F$. We first have to consider a set $X$ of interpretations. Such an interpretation $S^I$ maps epi onto its local support $S$ which means that we have a composed arrow (not a mono, in general!)

$$S^I \to S \rightarrowtail @A \times F \tag{27.1}$$

of global and local, not necessarily objective, compositions. We next give a morphism between such global compositions

$$S^I \to T^J \tag{27.2}$$

which should generalize the isomorphism in the classical case. But we cannot consider 'abstract' global morphisms since we also want to apply these morphisms to the modulation quantum. Therefore, we are given a morphism of the underlying local compositions $f/1 : S \to T$ which is, by definition, induced from a "symmetry" $h : F \to F$ of the form space. Notice that it is not necessary to suppose $h$ being an isomorphism. So we are looking for global morphisms $\overset{\iota}{f}/1 : S^I \to T^J$ which extend the local morphism $f/1$ to the chart configuration, i.e., the diagram

$$\begin{array}{ccccc} S^I & \longrightarrow & S & \longrightarrow & @A \times F \\ {\scriptstyle \overset{\iota}{f}/1}\downarrow & & {\scriptstyle f/1}\downarrow & & \downarrow{\scriptstyle 1\times h} \\ T^J & \longrightarrow & T & \longrightarrow & @A \times F \end{array} \tag{27.3}$$

with the evident horizontal arrows commutes. Next, we suppose that we are given an interpreted cadence composition, also $A$-addressed in $F$, for each tonality $S^I$:

$$C^T_{S^I} \to C_{S^I} \rightarrowtail @A \times F \tag{27.4}$$

which lives in $S^I$, i.e., we have a factorization of interpretations of local compositions in $F$:

$$\begin{array}{ccccc} C^K_{S^I} & \longrightarrow & C_{S^I} & \longrightarrow & @A \times F \\ {\scriptstyle 1\overset{\tau}{/}1}\downarrow & & 1/1\downarrow & & \downarrow{\scriptstyle 1} \\ S^I & \longrightarrow & S & \longrightarrow & @A \times F \end{array} \tag{27.5}$$

and this means that not only is the support $C_{S^I}$ a subcomposition of $S$, but the degrees of the cadence are also associated with degrees of this tonality via the map $\tau$. In the above classical case, this means that the triadic degrees of the cadence set must be triadic degrees in the englobing tonality. Cadentiality in this situation means that whenever we have two such factorizations of $C^K_{S^I}$ through $S^I, T^J$ in the tonality collection $X$, then we must have $S^I = T^J$.

The last of our objects is the modulation quantum. This is again an $A$-addressed interpretation

$$M^L \to M \rightarrowtail @A \times F \tag{27.6}$$

and we ask that the modulation morphism be an endomorphism of this quantum, more precisely, the diagram

$$\begin{array}{ccccc} M^L & \longrightarrow & M & \longrightarrow & @A \times F \\ {\scriptstyle \overset{\lambda}{m/1}}\downarrow & & {\scriptstyle m/1}\downarrow & & \downarrow{\scriptstyle 1\times h} \\ M^L & \longrightarrow & M & \longrightarrow & @A \times F \end{array} \tag{27.7}$$

commutes under a suitable morphism $\overset{\lambda}{m/1}$, a condition which reduces to the condition that we find a chart map $\lambda$ which is compatible with the restriction endomorphism $m$ of the ambient 'symmetry' transformation $1 \times h$. So in the setup of the harmonic topologies, we just ask that $M \in Ext_A(h)$, and that this local condition be compatible with the charts of $M$'s interpretation.

The classical condition that the target tonality $T^J$ as well as the modulation quantum should contain the cadence set now means that we have a commutative diagram

$$\begin{array}{ccc} C^K_{T^J} & \longrightarrow & M^L \\ \downarrow & & \downarrow \\ T^J & \longrightarrow & T^J \cap M^L \end{array} \tag{27.8}$$

where the intersection[1] $T^J \cap M^L$ is the fiber product $T^J \times_{@A\times F} M^L$. The rigidity condition now states that the only invertible symmetry $k : F \to F$ which induces an endomorphism of the intersection $T^J \cap M^L$ is the identity. This condition may be enriched by the additional conditions, for example, that $k$ must also induce an isomorphism which transforms interpretations within $X$ into each other.

In the classical definition of the modulation quantum $M^{(3)}$, there is a last requirement, i.e., that the intersection $T \cap M$ be covered by the degrees of $T^{(3)}$ which are contained in $M$. In the general setup this reads as follows: For each subset $J^*$ of the given atlas $J$ of $T^J$, we have a sub-composition $T^{J^*} \rightarrowtail T^J$ and therefore a corresponding fiber product diagram

$$\begin{array}{ccc} T^{J^*} \cap M^L & \longrightarrow & M^L \\ \downarrow & & \downarrow \\ T^{J^*} & \longrightarrow & @A \times F \end{array} \tag{27.9}$$

Take the unique maximal sub-composition $T^{J_{max}}$ such that the projection $T^{J^*} \cap M^L \to T^{J^*}$ has a section $\sigma : T^{J_{max}} \to T^{J^*} \cap M^L$. (The empty sub-composition evidently has this property,

[1] Observe that even when the factors are objective, their intersection will not be so in general!

so existence is evident.) Intuitively, this means that we take all charts in $T$ which also live in $M$. Then we ask that the canonical composed map

$$T^{J_{max}} \xrightarrow{\sigma} T^{J^*} \cap M^L \to T \cap M$$

is an *epimorphism*.

The philosophy of this approach is that a modulation looks like a perspective from the old tonality into the new one, and this is quite the same as the approach of Noll to Riemann's concept of consonance and dissonance (see section 30.2.1) which views this qualification as an expression of the morphisms between two chords, i.e., as a perspective view of relative consonance/dissonance of one chord with respect to another. The point in modulation theory is that we do not consider the perspectives (morphisms) as an ultimate structure, but want them to be an expression of a materialization (the quantum) in order to grasp the material trace of 'modulation forces'.

### 27.1.4 The Well-Tempered Case

**Summary.** The 12-tempered modulation theorem is proven for diatonic scales. We visualize the nerves of the modulation quanta and compare to A. Schönberg's lists in [478].

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This technical section follows the general modulation theorem as exposed in Muzzulini's paper [390]. This modulation theorem generalizes the earlier approaches in [327, 328, 340].

Muzzulini considers a seven-element scale $S$ in $PiMod_{12}$ and its interpretation $S^{(3)}$ by seven three-element degrees $X_S \subset S$ defined by taking all subsets of form $X_S = \{s_i, s_{i+2}, s_{i+4}\}$ with indexes modulo 7 and referring to any indexation $S = \{s_0, s_1, \dots s_6\}$ of the $S$-elements along the circle line. The naming of these degrees is not relevant for the interpretation, however, one usually enumerates the degrees as with ordinary diatonic scales, i.e.,

$$I_S = \{s_1, s_3, s_5\}, II_S = \{s_2, s_4, s_6\}, \dots VII_S = \{s_6, s_1, s_3\},$$

although the naming is a function of the indexation of $S$. By construction any such triadic interpretation has a Möbius strip as its geometric nerve. From the classification table appendix L.1, we see that there are 66 translation classes, 38 translation-inversion classes, and 25 affine classes of seven-element scales. The first two groups also conserve interpretations and therefore are also orbits of triadic interpretations. All minimal cadential sets have been calculated by computer for the 66 translation classes (this can be done on the basis of the 38 translation-inversion classes). The number of these minimal cadential sets are indicated in the third column of table appendix N.1.1. We already know the five minimal cadential sets for major scales (No. 38.1 in the classification appendix L.1) from the cadence discussion in section 26.2.1. The maximal number of cadential sets appears for harmonic minor scales (and only here): Here, each pair of different triadic degrees is a minimal cadential set, so we have 21 such sets. Besides the major diatonic scale, there are two other scale orbits with this minimal cadence set number: No.52 and No. 62. The melodic minor scale has 15 minimal cadence sets.

Call a scale $S$ is *rigid* iff its symmetry group $Sym(S)$ is trivial. If an inner symmetry of $S$ induces an automorphism of the triadic interpretation $S^{(3)}$, we say that it is an inner symmetry

of $S^{(3)}$. If the group of inner symmetries of $S^{(3)}$ is trivial, $S^{(3)}$ is also called rigid. The following lemma from [390] is clear:

**Lemma 43** *No translation of a seven-element scale is an inner symmetry, and any inversion in $Sym(S)$ is an inner symmetry of $S^{(3)}$. The only possible non-trivial inner symmetries of $S^{(3)}$ are inversions. In particular, $S^{(3)}$ is rigid iff $Sym(S)$ contains no inversions.*

Given a pair $S^{(3)}, T^{(3)}$ of different triadic interpretations in the same translation orbit $T_{12}.X^{(3)}$, a *modulator* is a symmetry $g$ which induces an isomorphism $g : S^{(3)} \xrightarrow{\sim} T^{(3)}$. A modulator must be of type $g = e^t.h$, where $h$ is an inner symmetry of $S^{(3)}$. By the preceding lemma, $h$ must be an inversion or the identity. *Hence a modulator must be a translation or an inversion.*

**Definition 88** *Given a pair $S^{(3)}, T^{(3)}$ of different triadic interpretations in the same translation orbit $T_{12}.X^{(3)}$, a* modulation from $S^{(3)}$ to $T^{(3)}$ *is a pair $(g, \mu)$ consisting of a modulator for $S^{(3)}, T^{(3)}$, together with a minimal cadential set $\mu$ for the target tonality $T^{(3)}$.*

Given a modulation $(g, \mu)$ from $S^{(3)}$ to $T^{(3)}$, the axiomatic properties of a modulation quantum $Q \subset PiMod_{12}$ are these:

**Property 1**

$$(\pounds_\mu)\begin{cases}(1) & \textit{The modulator } g \textit{ is an element of } Sym(Q).\\ (2)_\mu & \textit{All triads in } \mu \textit{ are subsets of } Q.\\ (3) & \textit{The intersection } TI_{12} \cap Sym(T \cap Q) \textit{ is trivial, and}\\ & T \cap Q \textit{ is covered by triads of } T^{(3)}.\\ (4) & \textit{The quantum } Q \textit{ is a minimal set with properties } (1) \textit{ and } (2)_\mu.\end{cases}$$

The main problem of this construction is the existence of a modulation quantum. A modulation is said to be *quantized* iff it has at least one modulation quantum. The modulation theorem below will give an exhaustive answer to this problem. If a modulation $(g, \mu)$ has a quantum $Q$, denote by $(Q \cap T)^{(3)}$ the interpretation of the trace of $Q$ in $T$ by the triads of $T$ which cover $T \cap Q$. The interpretation of $Q$ by triads of $S^{(3)}$ and $T^{(3)}$ is denoted by $Q^{(3)}$. The degrees of $(Q \cap T)^{(3)}$ are called the *pivots* of this quantized modulation. Observe that, by property $(\pounds_\mu), (3)$, the modulator is uniquely determined by the quantum.

In the following modulation theorem [390], we distinguish the 38 $TI_{12}$-orbits of seven-element scales.

**Theorem 30** *In the following, let $0 \neq p \in \mathbb{Z}_{12}$, and $T^{(3)} = e^p.S^{(3)}$.*

- *Scales with rigid triadic interpretation: For each scale $S$ of the 28 $TI_{12}$-orbits with rigid interpretations, and for arbitrary $p$, there exists at least one quantized modulation from $S^{(3)}$ to $T^{(3)}$.—The maximum of 226 quantized modulations occurs for the orbit of scale type No.54.1 (the harmonic minor scale), while the minimum of 53 quantized modulations occurs for the orbit of scale type No.41.1.*

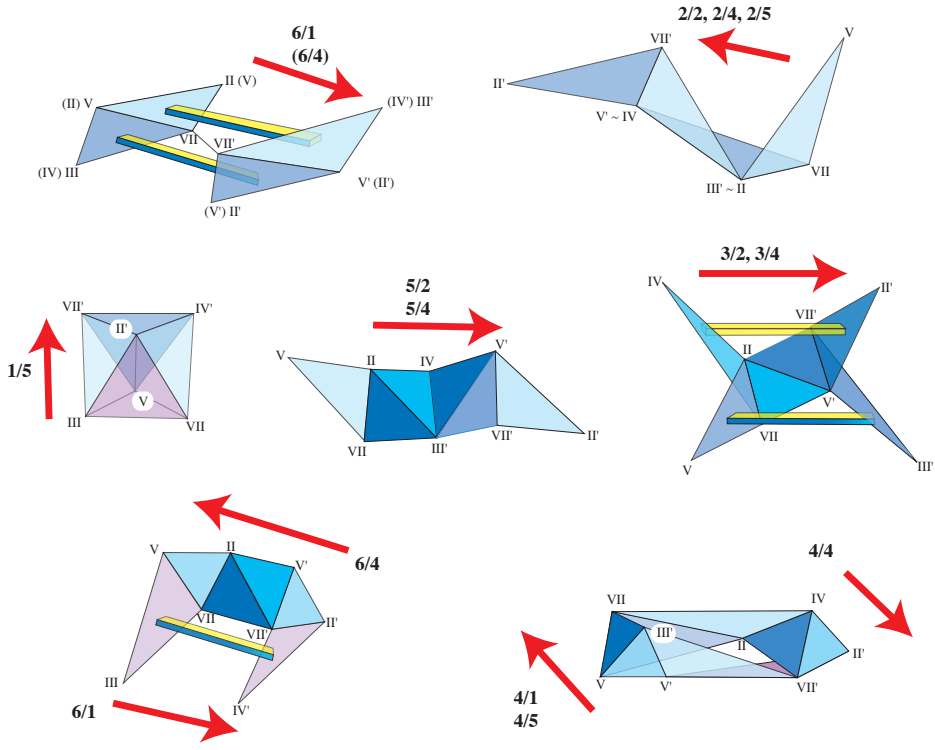


Figure 27.3: Visualization of all nerves of interpreted modulation quanta $Q^{(3)}$ for modulations on $Dia^{(3)}$. Certain nerves are higher than three-dimensional. There, we have symbolized the higher-dimensional simplexes by bars which connect all vertexes of the related lower-dimensional simplexes. For example, the bars of the quantum for 3/2, 3/4 symbolize a five-dimensional simplex connecting the vertexes of two triangular surfaces, each.

- *Scales with non-rigid triadic interpretations: For orbit of scale type No.52 and No.55, there exist quantized modulations except for $p = 1$ and $p = 11$. For the orbit of scale type No.38 and No.62, there exist quantized modulations except for $p = 5$ and $p = 7$. The six remaining orbits have at least one quantized modulation for each p.—The maximum of 114 quantized modulations occurs at the orbit of scale type No.47.1 (melodic minor scale). Among the scales with quantized modulations for each p, the minimum of 26 occurs at the orbit of scale type No.38.1 (diatonic major scale).*

Idea of proof. The proof [390] used a computer program to calculate quanta and pivots as follows: For a given pair $S^{(3)}, T^{(3)} = e^p.S^{(3)}$, choose a corresponding modulation $(g, \mu)$. Then there is exactly one candidate $Q$ for a quantum which fulfills properties $(1), (2)_\mu$, and (4) of $\pounds_\mu$. In fact, $Q$ is the orbit of the tones of $\mu$ under the group of symmetries generated by the modulator. This candidate is rejected if property (3) does not hold, and we may proceed with a new modulation.

Figure 27.3 shows the nerves of all quanta for modulations in the orbit $Dia^{(3)}$ of diatonic major scales.

For lists of modulation data, such as quanta, cadence sets, pivot sets, for different scale classes, please consult the tables N.1.1, N.1.2, N.1.3, and N.1.4 in appendix N.1.

### 27.1.5 Reconstructing the Diatonic Scale from Modulation

**Summary.** We discuss and comment the modulation theorem 30 for 12-tempered diatonic 7-tone scales. The overall behavior of the classical diatonic and melodic or harmonic minor scales for the present modulation model turn out to be in a distinguished position among all 7-tone scales. They can thus be reconstructed from their role in modulation processes; this gives modulation theory a new perspective within historical development of scales for musical composition which exemplifies the anthropic principle of mathematical models in music theory.

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The guarantee of the existence of quantized modulations is the alias of the historically grown rule canon within the mathematical model. This theorem in fact guarantees quantized modulations for all couples in $Dia^{(3)}$, and the pivotal degrees coincide with the pivotal degrees in Schönberg's treatise [478] wherever he considers *direct* modulations.

But the mathematical model also applies to the non-diatonic scales of seven tones and to any translation pair of triadic interpretations. So the modulation model immerses the classical case $Dia^{(3)}$ in a variety of modulation scenarios which have never been dealt with in historical contexts. This power of variation is a characteristic feature of the systematic approach in mathematical music theory. Let us briefly review the historical approach in order to understand this point. It cannot reduce to a pure tracing of what is the case in a diachronic development. This was also remarked by Dahlhaus [103, Bd.10,pp.104-105]:

> *Ist demnach sogar unter Historikern das Ausmaß, in dem die Chronologie als "substantiell" — als "wesentlich" für den "Zusammenhang" von Phänonemen — gelten soll (...), zumindest umstritten, so sollte es vollends einem Systematiker erlaubt sein, ohne übertriebene chronologische Zaghaftigkeit Ideen, Institutionen und Praktiken aufeinander zu beziehen und zu "Strukturen" zusammenzufassen, deren Enstehungszeiten um Jahrzehnte differieren, deren innerer Konnex aber keiner strikten äußeren Gleichzeitigkeit bedarf, um den Status einer wissenschaftlichen, nicht einer bloß spekulativen Hypothese zu erhalten. Und so wenig der konstruktive Zug des Verfahrens zu leugnen ist, so unbestreitbar dürfte es sein, daß ohne ihn weder eine Systematik noch eine Historiographie, die sich nicht in schierer Annalistik erschöpft, überhaupt möglich wären.*
>
> *(...)*
>
> *Erst die systematische Konstruktion öffnet den Blick dafür, welche Tatsachen einer Geschichte angehören, die zu erzählen lohnend erscheint.*

In the terminology of Saussurean semiology, Dahlhaus recognizes that the diachronic dimension of music must also be understood as an expression and extension of the system of music. And that this extension only becomes worth being considered qua instance of an overall system.

Clearly, the surface character of the diachronic axis with respect to the systematic background is paralleled by the surface character of the synchronic axis. In fact, this axis refers to the ethnological dimension of music, i.e., its cross-cultural variability. It would be as annoying as pure historiography to repertorize ethnological diversification if the fundamental question of systematic connections over the cultural localities could not steer ethnological musicology. In other words, the spatio-temporal extension and diversification of music could not be understood without its characteristic nature qua system. A semiotic system cannot be reduced to a bunch of records over time and space.—This is the basic axiom of Saussure's original linguistic approach—even if it turns out to be hard to recognize the system's laws.

So why is the modulation theorem in its present form (theorem 30) necessary to the comprehension of the music system? Wouldn't it suffice to have the original theorem for $Dia^{(3)}$? In the synchronic axis it would in fact: It gives the different transposition distances a common basis from which pivotal degrees can be deduced. The quantum model of modulation is the system, and the different transposition distances are synchronic places in a certain time slice of music history.

But that does not explain why the European corpus of classical compositions could grow to such a prominent and voluminous status, a status which is strongly based upon some standard scales, the diatonic major, the melodic, and the harmonic minor scales, and an associated harmony whose top performance crystallizes in tonal modulation.

We do not claim that the modulation theorem explains all reasons for the prominent status of these scales, but we claim that it gives the prominent scales a prominent position in modulation processes. From this, we deduce that the historical success of these scales could relate to their modulatory excellence.

The position of the harmonic minor scale is such that its number of 226 quantized modulations is maximal among all (seven-tone) scales with rigid triadic interpretation. Among the scales with non-rigid triadic interpretation and quantized modulations for each translation, the maximal number of 114 quantized modulations occurs for the melodic minor scale. So these minor scales have a maximal freedom of modulatory actions. Among the scales with non-rigid triadic interpretation and quantized modulations for each translation, the diatonic major scale is charged with the minimum of 26 quantized modulations. The scale No. 41.1 with a minimum of 53 quantized modulations has no interpretation in the historical scale repertory, to our knowledge.

To understand these extremal positions, observe that the modulation theorem does not prefer any of these scales in its concept framework or hypotheses. So it is a priori not expectable that the three scales of diatonic major, melodic, and harmonic minor could play any role in their variety of quantized modulations. The scale No. 41.1 is also in a distinguished position in this context, but we do not have any interpretation of this fact.

However, the historic "choice" of scales does coincide with three of four scales that are exhibited by this generic modulation model. So the two final questions remain:

- How can we recognize the modulation model in the historic traces?
- If one accepts the model's presence in the systematic background of the historic development, to what degree has the historic development been guided by system parameters which are related to the given modulation model?

The first point will be attacked in the experimental chapter 28 on applications of the theorem to classical compositions of the European tradition. The second question is hard to tackle since the question relates to the innermost reflections or mere intuitions which may have guided composers and—to a lesser degree—theorists.

Independently of its role as an explanatory mechanism for historical phenomena, the present model can very well be used to produce modulations in new contexts. For instance, one may try to write compositions in exotic scales and to use modulation rules from this model. Or one may also apply this model to other parameter spaces. In fact, the modulation model does not preconize the pitch domain in its structural premises! One may also apply the modulation rules to time, loudness or other musical parameters in order to cope with Augustinus' philosophy that music is the art of good 'modulation'. This latter approach has been tested in a composition *Synthesis* [339] for piano, percussion, and base. In the first movement of that composition, modulation was applied to the time domain, i.e., to rhythmical modulation instead of tonal modulation. We shall also discuss this experiment in section 28.3.2.

The table in appendix N.1.5 shows examples of direct schematic and short modulations (as usual in the modulation texts) for every fourth translation $p = 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 11$, starting from $C$-major. The degrees, which are separated by commata, are meant to be played in time order, one after the other.

### 27.1.6 The Case of Just Tuning

**Summary.** This section is devoted to the proof of the modulation theorem for just tuning, presented in [331, 340] for major scales, and elaborated by Hildegard Radl [429] to harmonic, melodic minor, and other scales. The theorem sheds some light on the particular chromatic scale in just tuning, as considered by Martin Vogel [547].

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#### 27.1.6.1 Just Scales and their Triadic Interpretations

We shall first of all deal with just scales in the *EulerPlane* as introduced in section 7.2.1.2. The scales $S$ which we shall encounter in the sequel will all be generating, i.e., $\mathbb{Z}S = \mathbb{Z}^2$, the ambient module whose first axis encodes fifths while the second encodes thirds. Therefore, it will be equivalent to speak about automorphisms and about inner symmetries. For any zero-addressed local composition $C$ in the *EulerPlane*, we say that it is *rigid* iff $Aut(C) = Id$.

We are going to discuss these scale types (refer to section 7.2.1.2 for just scales and section13.4.2.2 for triadic interpretations):

- The set $Maj$ of major scales, i.e., the translates $X = e^t C$ of just $C$-major $C$, together with the set $Maj^{(3)}$ of triadic interpretations by the seven degrees $I, II, \ldots VII$, excluding the variants $II^*, VII^*$.

- The set $NatMin$ of natural minor scales, i.e., the translates $X = e^t c_{nat}$ of natural $C$-minor $c_{nat}$, together with the set $NatMin^{(3)}$ of triadic interpretations.

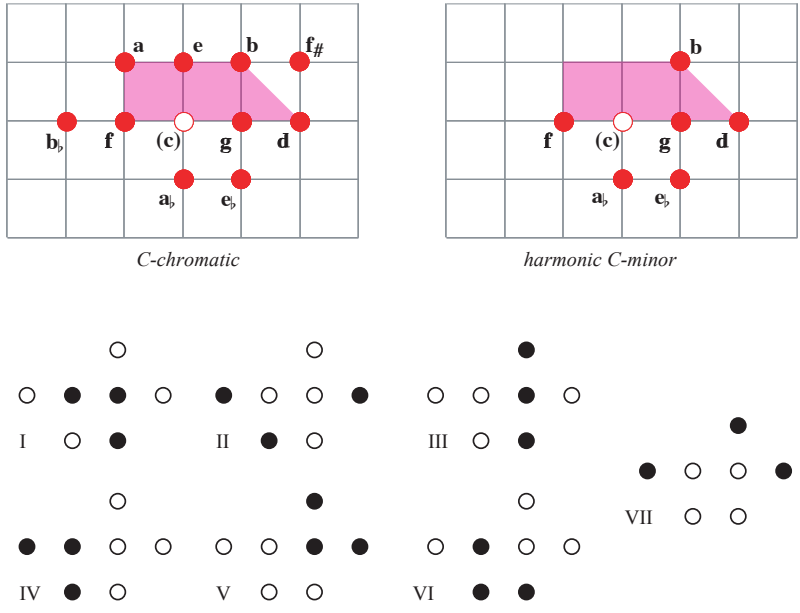


Figure 27.4: The harmonic minor scale and its triadic interpretation.

- The set $HarMin$ of harmonic minor scales, i.e., the translates $X = e^t c_{har}$ of harmonic $C$-minor $c_{har}$, see figure 27.4, together with the set $HarMin^{(3)}$ of triadic interpretations.
- The set $MelMin$ of melodic minor scales, i.e., the translates $X = e^t c_{mel}$ of melodic $C$-minor $c_{mel}$, see figure 27.5, together with the set $MelMin^{(3)}$ of triadic interpretations.
- The 32 translation orbits $AltMaj(i)$ of seven-element scales $C(i)$ which are defined as follows, see figure 27.6: Take the fifth pairing $c, g$. Take the a priori *$C$-scale frame* shown on top of figure 27.6. Then, choose any additional five tones which are either the major tones $f, d, a, e, b$ or one of their alterations, whence we obtain $2^5 = 32$ variants. Of course, these known special cases are included: $C = C(1)$, $c_{nat} = C(5)$, $c_{mel} = C(9)$, $c_{har} = C(11)$. And the corresponding triadic interpretation sets $AltMaj(i)^{(3)}$.

On the one hand, we shall discuss modulations within the orbits

$$Maj^{(3)}, NatMin^{(3)}, HarMin^{(3)}, MelMin^{(3)}, AltMaj(i)^{(3)},$$

on the other, we shall also switch between major and natural minor scales, this latter case meaning that we take the larger orbit under translations and 180°-rotations.

#### 27.1.6.2 Modulations and Quanta

We now have to adapt the crucial definitions from 12-tempered modulation to just modulation.

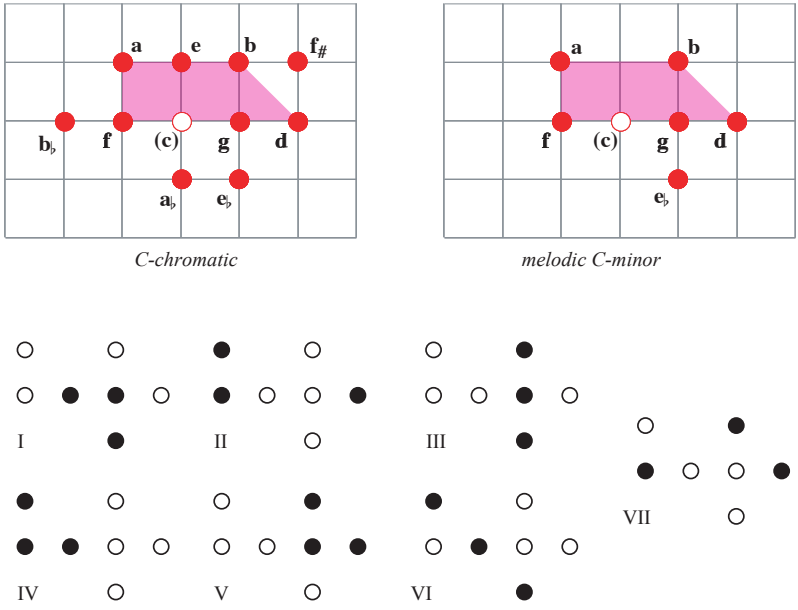


Figure 27.5: The melodic minor scale and its triadic interpretation.

**Definition 89** *Fix a subgroup* $G \subset \overrightarrow{GL}(\mathbb{Z}^2)$. *Given a pair* $S^{(3)}, T^{(3)}$ *of different triadic interpretations of one of the above scale types which live in the same orbit* $G.X^{(3)}$, *a* modulation from $S^{(3)}$ to $T^{(3)}$ *is a pair* $(g, \mu)$ *consisting of a modulator* $g \in G$ *for* $S^{(3)}, T^{(3)}$, *together with a minimal cadential set* $\mu$ *for the target tonality* $T^{(3)}$.

Given a modulation $(g, \mu)$ from $S^{(3)}$ to $T^{(3)}$, the axiomatic properties of a modulation quantum $Q \subset EulerPlane$ are these:

**Property 2**

$$(\pounds_{\mu}^{just}) \begin{cases} (1) & \textit{The modulator } g \textit{ is an element of } Sym(Q). \\ (2)_{\mu} & \textit{All triads in } \mu \textit{ are subsets of } Q. \\ (3) & \textit{The intersection } T \cap Q \textit{ is rigid, and} \\ & T \cap Q \textit{ is covered by triads of } T^{(3)}. \\ (4) & \textit{The quantum } Q \textit{ is a minimal set with properties } (1) \textit{ and } (2)_{\mu}. \end{cases}$$

#### 27.1.6.3 Automorphisms of Triadic Interpretations of Seven-Element Scales

Let $S^{(3)}$ be the triadic interpretation of a generating seven-element scale

$$S \subset EulerPlane,$$

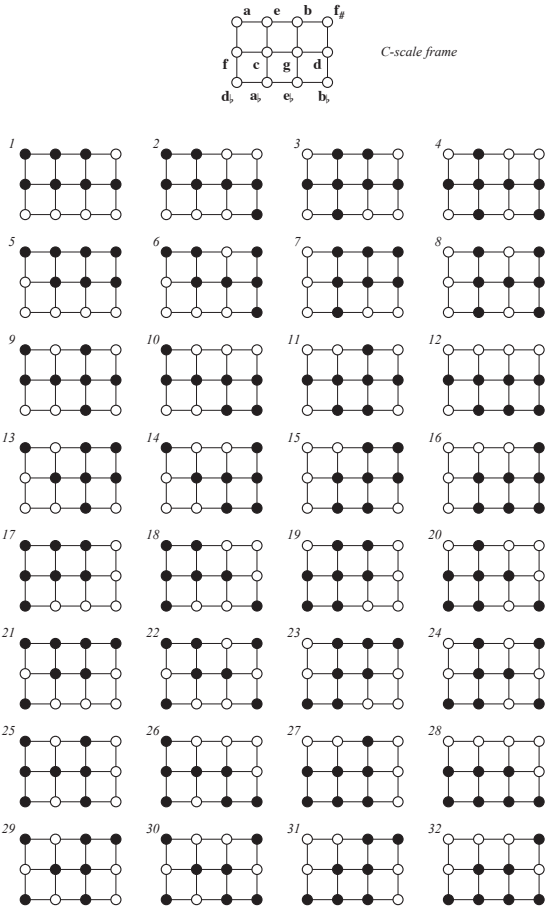


Figure 27.6: The 32 altered scales are derived from the $C$-major frame on top.

for example the above types in section 27.1.6.1. Let $\sigma \in Sym(S)$ be an automorphism which induces an automorphism $\sigma \in Sym(S^{(3)})$. Since this also induces an automorphism of the nerve $n(S^{(3)})$ which is a Möbius strip, we have combinatorial information on the possible permutations induced by $\sigma$. Clearly, $\sigma$ is of finite order. Since the boundary of the nerve is a circle, and since the adjacency relations of the seven vertexes on this circle must be preserved, there is only the dihedral group of the regular heptangle, i.e., the identity, the order-two reflections, and the order-seven translations on the vertexes, and hence, since the elements of the scale are recovered by the two-simplexes, the same is true for the scale.

On the other hand, if we consider any finite subgroup $H \subset GL_2(\mathbb{Z})$, then Minkowski's theorem (see appendix C.3.5, theorem 43) tells us that the composed projection

$$H \rightarrowtail GL_2(\mathbb{Z}) \rightarrow GL_2(\mathbb{Z}_3)$$

is injective. But we have $card(GL_2(\mathbb{Z}_p)) = (p-1)^2 p(p+1)$ for any prime $p$ (see appendix C.3.4)

which is 48 for $p = 3$. Hence, if we apply this fact to the linear part of our automorphism, order 7 is forbidden for our automorphisms, and we are left with orders 1 and 2. This means that if $\sigma = e^a.A$, then we have $A^2 = Id, A.a + a = 0$ for $A \in GL_2(\mathbb{Z}), a \in \mathbb{Z}^2$.

Suppose we are given a modulator $\phi : S^{(3)} \to T^{(3)}$ between two just tonalities, then it has the shape $\phi = \tau.\sigma$ with a translation $\tau = e^b$ and an automorphism $\sigma = e^a.R$ of $S^{(3)}$ as above. Since the modulation quantum must be a full orbit of the cadence $\mu$'s tones under the group $\langle\phi\rangle$ generated by $\phi$, it is important to know the shape of an orbit $\langle\phi\rangle(x)$ of a point $x \in \mathbb{Z}^2$ of the *EulerPlane* space. Here is the formula ([429, p.12]):

$$\langle\phi\rangle(x) = e^{\mathbb{Z}.(R+Id)b}x \cup e^{\mathbb{Z}.(R+Id)b}.\phi(x). \tag{27.10}$$

So we have one 'branch' of this orbit which is just the translation orbit of $x$ under translation $e^{(R+Id)b}$, and the other 'branch' which is the translation orbit of $\phi(x)$ under the same translation. If $\sigma = Id$, both branches are shifted against each other by $b$, and they are generated by a translation orbit under $e^{2b}$.

**Exercise 65** Draw an image of the orbit of $C$-major under $b = (9, 2)$ and $\sigma$ being the unique non-trivial automorphism $A$ of $C$-major, as described in formula 8.6.

#### 27.1.6.4 Finiteness of Modulation Domains

**Definition 90** *The* modulation domain *of a tonality $S^{(3)}$ is the set of shift vectors $b \in \mathbb{Z}^2$ such that there is a quantized modulation from $S^{(3)}$ to $T^{(3)} = e^b S^{(3)}$.*

In a good number of cases, the modulation domain of a tonality is finite. We investigate this situation.

In order to control the possible intersections between the target tonality and the orbit $\langle\phi\rangle(X)$ of a set $X$ of tones in the target tonality which stands for the potential modulation quantum, we should control the cases where the different transforms of a scale are disjoint from each other.

**Proposition 48** *With the above notation, if $C$ is the major scale in EulerPlane, then the transforms*

$$e^{z.(R+Id)b}C, e^{w.(R+Id)b}C, e^{z.(R+Id)b}.\phi(C), e^{w.(R+Id)b}.\phi(C)z, w \in \mathbb{Z},$$

*are mutually disjoint except for a finite number of shift vectors $b$.*

Proof. The translation vector $b = (b_1, b_2)$ induces a translation vector $(R + Id)b = b_2.(-1, 2)$. So clearly, the two branches $e^{\mathbb{Z}.(R+Id)b}C$ and $e^{\mathbb{Z}.(R+Id)b}.\phi(C)$ are disjoint if $|b_1|$ is sufficiently large. On the other hand, the vertical shift component by $b_2$ of the scale $C$ to the second branch $e^{\mathbb{Z}.(R+Id)b}.\phi(C)$ under $\phi$ cannot hit the first branch for $1 < |b_2|$ since the vertical component of $b_2.(-1, 2)$ is twice the vertical shift component $b_2$, and thus, the branches cannot intersect in the vertical progression. QED.

**Corollary 18** *For modulations among just major tonalities, the modulation domain is finite.*

Proof. Consider the minimal cadence sets $J_1, \ldots J_6$ as defined in formula (26.2). It is easily seen that the union of every cadence set, except $J_1 = \{II\}, J_5 = \{VII\}$, has no rigid superset in the scale. So only $J_1 = \{II\}, J_5 = \{VII\}$ can be candidates for cadence sets in the quantum construction ($\pounds_\mu^{just}$). But these two degrees are not rigid. Now, the quantum, if it exists, must be the orbit of either of these degrees under the group $\langle\phi\rangle$. But this is just one of these two degrees in every transformed instance of the scale. So, if the transformed instances are all disjoint from each other, the trace of the quantum in the target scale is either $II$ or $VII$, so it is not rigid. Now, by the above proposition, this is the case for all but a finite number of shift values $b$, and we are done. QED.

### 27.1.7 Quantized Modulations and Modulation Domains for Selected Scales

**Summary.** We present the quantized modulations for modulations between major scales, between natural minor scales, between major and natural minor scales, between harmonic minor scales, and between melodic minor scales. We finally discuss Radl's results concerning the 32 altered scales introduced in 27.1.6.1 and their modulation domains.

– Σ –

In the following sections, we present the modulation domains and the pivotal degrees for a number of standard scales. The proofs are omitted since they are a tedious and lengthy, but not very illuminating work of case distinction. The details are found in Radl's work [429] which in its algorithmic part should be incorporated in a computer-aided algorithm.

**Theorem 31** *This variety of modulation results, i.e., the existence of modulation quanta is not written in the closed form of a theorem. Nevertheless, it should be read as a highly segmented existence theorem of just modulation theory. We shall refer to this list of results when referring to the modulation theorem for just tuning.*

The main ideas are these:

- to use the finiteness statement of corollary 18 and to digress on the different cases in this finite set of possibilities;
- to use the fact that a modulation by a shift $b$ can be used to calculate the modulation in the negative direction $-b$.

**Exercise 66** Calculate the $-b$-case from the results for the $b$-case.

We should also mention that Radl has equally calculated some modulation results for other situations in the Euler context, such as scales in the Pythagorean tuning, and in the three-dimensional EulerSpace from third, fifth and seventh components. We withdraw from a discussion of these ramifications.

#### 27.1.7.1 Modulation Between Major Tonalities

Suppose that we have to modulate from $S^{(3)}$ to $T^{(3)} = e^b S^{(3)}$. We have two modulators in this case:

$$\Phi_1 = e^b, \tag{27.11}$$
$$\Phi_2 = A.e^b. \tag{27.12}$$

The modulations are listed in detail in table appendix N.2.1. Here, we just present the modulation domain in figure 27.7. We see that this domain covers all the tones of the major scale (except

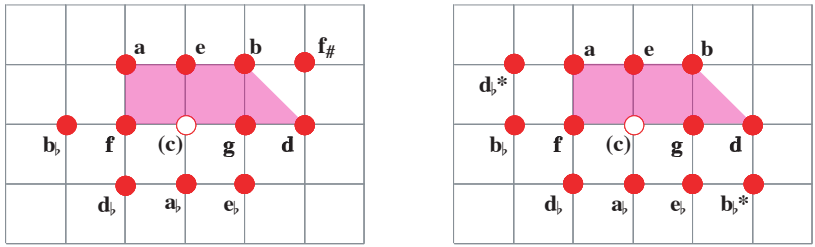


Figure 27.7: The modulation domain from just $C$-major.

the tonic itself which is excluded by definition) as well as six external tones. The modulation domain is very similar to Vogel's chromatic. From this compositional point of view, Vogel's chromatic is optimally chosen as a modulatory domain. Other chromatics, such as Michel's [374] or Roederer's [462] are less adapted to this modulation domain.

#### 27.1.7.2 Modulation Between Natural Minor Tonalities

We have the two modulators

$$\Phi_1 = e^b,$$
$$\Phi_2 = e^b.A.$$

The modulations are listed in detail in table appendix N.2.2. We just present the modulation domain in figure 27.8.

As may be expected from the symmetry relation between major and natural minor type, this domain is the same as that of major tonalities, however only absolutely. Relative to the scale, the domain is a rotated one.

#### 27.1.7.3 Modulation From Natural Minor to Major Tonalities

We have the two modulators

$$\Phi_1 = e^b.A,$$
$$\Phi_2 = e^b.B, B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

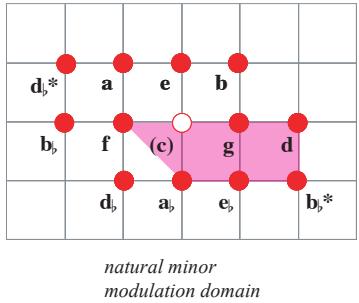


Figure 27.8: The natural minor modulation domain from natural $C$-minor.

The modulations are listed in detail in table appendix N.2.3. We just present the modulation domain in figure 27.9.

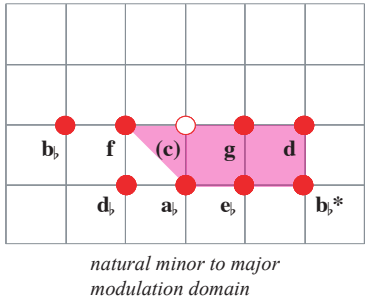


Figure 27.9: The modulation domain from natural $C$-minor to major tonalities.

This is a remarkable result since in the well-tempered situation, there is no distinction of the scales for major and natural minor, it is just the tonic which differs. So here, we have a modulation between different translation classes.

#### 27.1.7.4 Modulation Steps From Major to Natural Minor Scales

We have the two modulators (same as above).

$$\begin{aligned}\Phi_1 &= e^b.A,\\ \Phi_2 &= e^b.B, B = \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}.\end{aligned}$$

The modulations are listed in detail in table appendix N.2.4. We just present the modulation domain in figure 27.10.

This is the symmetric result to the above one.

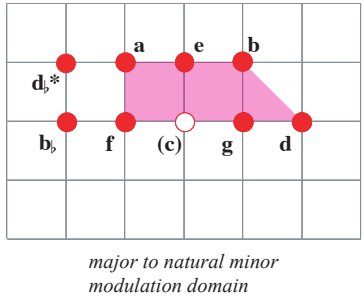


Figure 27.10: The modulation domain from $C$-major to natural minor tonalities.

#### 27.1.7.5 Modulation Steps Between Harmonic Minor Scales

We have the unique translation modulator

$$\Phi = e^b.$$

The modulations are listed in detail in table appendix N.2.5. We just present the modulation domain in figure 27.11.

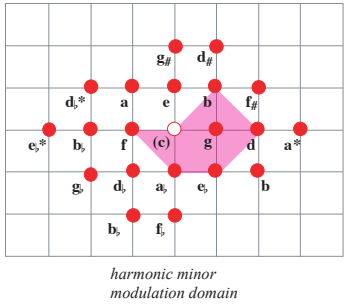


Figure 27.11: The modulation domain from $C$ harmonic minor tonalities.

This modulation domain is extremely large compared to the others. Perhaps this is a reason why harmonic minor is so prominent in classical European harmony, and also, to some degree, in composition.

#### 27.1.7.6 Modulation Steps Between Melodic Minor Scales

We have the two modulators

$$\begin{aligned} \Phi_1 &= e^b, \\ \Phi_2 &= e^b.A. \end{aligned}$$

The modulations are listed in detail in table appendix N.2.6. We just present the modulation domain in figure 27.12.

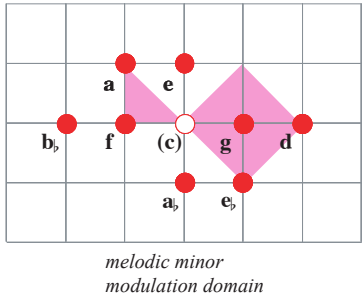


Figure 27.12: The modulation domain from $C$ melodic minor tonalities.

#### 27.1.7.7 General Modulation Behavior for 32 Altered Scales

This investigation in Radl's work was undertaken to get a global view of possible scale behavior in modulation when deriving scales from the major scale by determined alterations as described above. From the table N.2.7, we can distinguish these cases.

- *has no modulations* if its modulation domain is empty (always excluding the start tonality!),
- *has infinite modulations* if its modulation domain is infinite,
- *has modulations* if its modulation domain is not empty (always excluding the start tonality!), *has limited modulations* if the transitive closure (all tonics which can be reached by successive modulations from relative modulation domains) of its modulation domain is not total space.

The cases (15, 19) with no modulation are pathological. The cases with infinite modulations (21, 22, 24, 29, 30, 32) are too indeterminate since we can modulate in a set of tonalities which cannot be distinguished from their perceivable pitch structure, see also the discussion of valence theory in appendix B.2. The cases (8, 25) of limited modulations means that modulations cannot be concatenated to yield any target tonality. The remaining cases are the reasonable ones. There, we have reduction to modulations with just four target tonalities of the cases 2, 7, 13, 14, 23, same for the cases 10, 16, 17, 18, 20, 26, 27, 31. So these cases have very small domains against the remaining cases which all reduce to one of the special tonality types of major, natural minor, harmonic minor or melodic minor which have relatively large modulation domains (eight and more targets against four in the previous cases 2,7,...). The only case which remains is number 6, and this is a variant of the melodic scale. So we may summarize that the known scales have a much larger number of target domains than the other scales in the frame of the reasonable scales. *This is also a confirmation of Muzzulini's results on the distinguished position of major, melodic minor, and harmonic minor in the well-tempered case.*

## 27.2 Harmonic Tension

**Summary.** Harmonic tension regards the valuation of syntagmatic chord complexes. This concept adds to the isolated semantics of chords a semantic charge of complex syntagmatic arrangements. We discuss a distance-based concept of harmonic tension and related mathematical structures, such as the Riemann tensor algebra. The chapter concludes with a technical discussion of optimal harmonic paths, including algorithms for optimal path search.

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The harmonic evaluation of isolated chords does not give the full information about a preferable choice of harmonic evaluation in the syntagmatic context. Classical harmony has not dealt with this problem except in the rudimentary concept of cadences. We shall here set up a formalism to grasp harmonic context structures from given Riemann matrix data. This is an important subject since it is very difficult to define a global syntagmatic equilibrium on the harmonic level. The Riemann tensor algebra will give a first approach to the formal treatment of the problem of local and global harmonic tension. This approach has been used in the HarmoRUBETTE®and will be discussed on a more practical level in section 41.3.

### 27.2.1 The Riemann Algebra

**Summary.** We introduce the Riemann algebra as a quiver algebra that is built on the Riemann matrix.

– Σ –

We take up the formalism of Riemann matrices introduced in section 25.3.2, including the basic data of that context, i.e., a name set TON of "tonalities", a name set VAL of "tonal function values", and a form space $TRUTH(I)$ for of "truth denotators" in a specific truth module (or, more generally, a truth form) $I$ and at a specific address $B$ for a form $F$. To make the ideas concrete, and also related to the HarmoRUBETTE®module of RUBATO®(see section 41.3 for this tool) we shall work in the truth module $I = \mathbb{R}$. The form $F$ can be any form which englobes the parameters for harmony (this is however not a formal condition in the following discussion). The address $B$ will be any real vector space.

We first need a mathematical construction: the Riemann algebra.

**Definition 91** *Given value and tonality sets* VAL, TON*, a space form* $F$ *with address* $B$ *(a real vector space), the* Riemann algebra *(over the real numbers) is the quiver algebra*[2] $Rie(\mathrm{VAL}, \mathrm{TON}, F, B)\rangle$ *over the reals for the complete quiver*[3] *(the* Riemann quiver*) over the set* $\mathrm{VAL} \times \mathrm{TON} \times ExTop_B(F)$*).*

The definition is motivated by the usual situation in harmony: When one analyzes a piece of music, there is a sequence of chords which are given certain function values with respect to certain tonalities. Such sequences are formally restated as being paths in the Riemann quiver. The Riemann algebra is needed in order to make calculations on different harmonic paths in

[2]See appendix D.1.1.1 for this concept.
[3]See appendix C.2.2, definition 65.

the Riemann quiver, and in order to evaluate the best ones among a given selection of such paths. This is needed in order to decide which path should be taken when we have a certain way of giving weights to special sub-paths according to specific approaches to harmony.

The basic data in such a calculation are real-valued weights which we assign to certain paths. For example, one may exhibit a classical cadence path of shape $I_X \to IV_X \to V_X \to I_X$ and give it a high positive weight in order to express that such sequences are good paths. Or one may like to express that a passage $V_X \to I_X$ is a good relaxation movement, where as the other direction $I_X \to V_X$ is not and receives negative values, say. So these weight assignments regard more or less long paths in the Riemann quiver and should help in determining the quality, i.e., weight of a given path through a piece of music.

Before delving into details, observe that this approach is still a relative one in the following sense. Suppose we are given a sequence of chords $a_0, a_1, a_2, \dots a_n$, and suppose that we have two harmonic interpretations of them in terms of two Riemann paths

$$(v_0, f_0, a_0) \to (v_1, f_1, a_1) \to \dots (v_n, f_n, a_n),$$
$$(v_0^*, f_0^*, a_0) \to (v_1^*, f_1^*, a_1) \to \dots (v_n^*, f_n^*, a_n)$$

which we want to compare: Which one is harmonically preferable? One may therefore calculate "weights" for these paths and decide upon these weights. This is what we shall achieve. But there is also a question concerning absolute weights: What is the harmonic weight of the last chord $a_n$, compared to the first one $a_0$? Is it legitimate to say that this is just the weight of a selected Riemann path? It is certainly not because the local values of the weights could add up to some path weight which always gives the last chord a higher weight than the first one.

This question is in fact the question about the global development of harmonic tension. Up to date, no musicological theory has ever dealt with this problem: Is it possible to calculate global harmonic tension from local knowledge? It is certainly not true that every piece of music should have a global slope of zero, i.e., the beginning tension being equal to the ending tension. For example, in a Scriabin sonata, such as op. 72 *Vers la flamme*, the final tension is expected to be much higher than the beginning one. So how should one calculate global 'absolute' tension? We shall come back to this issue in the discussion of the HarmoRUBETTE® software module in section 41.3.

### 27.2.2 Weights on the Riemann Algebra

**Summary.** Weights on the Riemann algebra are predicates which measure the quality of a harmonic development path.

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So let us now work on the weight functions on harmonic paths. Suppose that we are given a harmonic, real-valued *default weight function* $w$ on all paths $(v_0, f_0, a_0) \to (v_1, f_1, a_1) \to \dots (v_i, f_i, a_i)$ of lengths $i = i_1, i_2, \dots i_k$. Then we may choose

- a set of coefficients $\mu_i, i = i_1, i_2, \dots i_k$, and,
- for each such $i$, a system $\lambda^i$ of coefficients in the sub-path operators $\sqsubset_i^{\lambda^i}$,

and define this linear form on the Riemann algebra:

$$\Omega(p) = \sum_{i=i_1,i_2,\ldots i_k} \mu_i.\omega(\sqsubset_i^{\lambda^i}\ (p)) \tag{27.13}$$

where the linear form $\omega$ extends the original $w$ on the spaces spanned by the paths in the images under the endomorphisms $\sqsubset_i^{\lambda^i}$. The idea of this definition is that we suppose that certain paths are already weighted (in fact by $\omega$). We then extend the weights to all paths by use of the sub-path operators. These operators take the given weights into account, each with its weight coefficient from the systems $\sqsubset_i^{\lambda^i}$. These systems give a sub-path its specific weight according to its position within the argument $p$. This is reasonable since we will in general give a sub-path a different harmonic meaning in a path according to whether it stays in the beginning, the middle, or the end. After this, the different contributions of sub-paths of specified lengths $i = i_1, i_2, \ldots i_k$ are weighted against each other according to the role of the sub-paths of these lengths in the present harmonic analysis. For example, if the harmonic concept gives more importance to short sub-paths of length 2, against lengths above 5, say, then the coefficient $\mu_2$ should be significantly larger than the coefficients above $\mu_5$. It may even happen that certain lengths play a completely different role than others, as we shall see in a moment.

So let us make a concrete example concerning the default weight function. The address is $B = 0$, and we are working in the function-theoretic context discussed in section 25.3.3. This means

$$\begin{aligned} Fun(F) &= @\mathbb{Z}_{12}, \\ \text{TON} &= \{C, D_\flat, D, E_\flat, F, G_\flat, G, A_\flat, A, B_\flat, B\}, \\ \text{VAL} &= \{T, D, S, t, d, s\} \end{aligned}$$

and the circle group truth module $I = S_1$ for the function theory, assigning a half open interval $TF_{f,t}(a) = \phi(a)\hat{} = [0, \phi(a)[$ of $S_1$ to each chord $a \in ExTop_0(F)$, function value $f \in$ VAL, and tonality $t \in$ TON.

Based on this information, we want to define the linear form $\omega$ on all paths of the Riemann quiver. By definition, the present form will vanish on all paths of length larger than 1. So this tension theory is very local, just limited to contributions from isolated chords and from transitions between two successive chords.

Let us first deal with the vertexes, i.e., the isolated chords. We have to define the values $\omega(f, t, a)$ for function values $f \in$ VAL, tonalities $t \in$ TON, and chords $a \in ExTop_0(F)$. Suppose that $TF_{f,t}(a) = \phi(a)\hat{}$ with $\phi(a) \neq 0$. Then we set $\omega(f, t, a) = \ln(\phi(a))$. If $\phi(a) = 0$, the corresponding predicate is $\phi(a)\hat{} = \emptyset$. This means that in a given harmonic path, we encounter a chord which has "False" as its harmonic truth value. Such a path should fail to be a competitor in the selection of a best harmonic path. More generally, we would like to eliminate paths containing chords with a value $\phi(a) \leq \phi_{min}, 0 \leq \phi_{min}$. So we replace $\omega(f, t, a) = \ln(\phi(a))$ by $\omega(f, t, a) = -\infty$ if $\phi(a) \leq \phi_{min}$. Of course, this infinity value is incompatible with the definition of a common linear form. But if we read the linear form codomain $\mathbb{R}$ as the set of intervals $]-\infty, r[, r \in \mathbb{R}$, together with the transported structure of $\mathbb{R}$-module, we can extend the set by the empty interval $\emptyset$ and add the rules $s.\emptyset = \emptyset, s \in \mathbb{R}$ and $\emptyset + ]-\infty, r[ = ]-\infty, r[ + \emptyset = \emptyset$. This structure is still a semigroup under addition and has a bilinear scalar multiplication. The empty interval is however an absorbing element: Every linear expression involving the empty

set is absorbed by this one and reduces to the empty set. So the values of our "linear form" live in the truth space $\mathbb{T}^0_{\mathbb{R}}$. We have in fact defined the first piece of a predicate on tuples of chords, given by the predicates $TF_{f,t}$ and the limiting value $\phi_{min}$.

The second piece of our predicate uses the linear form on paths of length 1. This contribution deals with the tension between two positions in the Riemann matrix. We are given a path $(f_1, t_1, a_1) \to (f_2, t_2, a_2)$. We would like to measure the harmonic tension of this transition as such, i.e., independently of the involved chords. The value $\omega((f_1, t_1, a_1) \to (f_2, t_2, a_2))$ will therefore be just a function of the arrow $(f_1, t_1) \to (f_2, t_2)$ in the complete quiver over the Riemann matrix index set VAL × TON, call this quiver the *Riemann index quiver*. Two points of this quiver share two components: the function values and the tonality symbols. In our case, the function values are also split into a set $\mathrm{VAL}_{type} = \{T, D, S\}$ of function value types and a set $\mathrm{VAL}_{mode} = \{min, maj\}$ of major or minor mode. So we shall restate the index quiver as the complete quiver over the triple product $\mathrm{VAL}_{type} \times \mathrm{VAL}_{mode} \times \mathrm{TON}$.

Given a path $(f_1^{type}, f_1^{mode}, t_1) \to (f_2^{type}, f_2^{mode}, t_2)$, we have to take into account the component steps in each coordinate. For each of these three coordinates, we set up a real-valued matrix to express the respective tension component. For the function type, the $3 \times 3$-matrix is

$$T_{\mathrm{VAL}_{type}} = (t^{type}_{XY})_{X,Y \in \mathrm{VAL}_{type}}.$$

This means that in a 1-path where the type changes from $X$ to $Y$ enforces a tension of quantity $t^{type}_{XY}$. The corresponding meaning for mode changes is quantized by the $2 \times 2$-matrix

$$T_{\mathrm{VAL}_{mode}} = (t^{mode}_{mn})_{m,n \in \mathrm{VAL}_{mode}}.$$

Finally, the change of tonality is codified by the $12 \times 12$-matrix

$$T_{\mathrm{TON}} = (t^{\mathrm{TON}}_{st})_{s,t \in \mathrm{TON}}.$$

Observe that neither of these matrices will be symmetric or even distance like, i.e., having zero diagonal etc. For instance, the change from major to minor is supposed to be lower valued than the converse change in classical European music.

With these component contributions, we may finally define the transition default values as follows:

$$\begin{aligned}&\omega((type_1, mode_1, tonality_1) \to (type_2, mode_2, tonality_2))\\ &= (t^{type}_{type_1 type_2})^2 + (t^{mode}_{mode_1 mode_2})^2 + (t^{\mathrm{TON}}_{tonality_1 tonality_2})^2.\end{aligned}$$

Of course this Euclidean approach can be replaced by any equivalent distance definition, this one is only one typical solution. So we never have minus infinity here, the only "False" contributions stem from the values on isolated chords as explained above.

To put everything together, the zero-length and one-length components are combined to yield the formula (27.13), i.e.,

$$\Omega(p) = \mu_0.\omega(\sqsubset_0^{\lambda^0}(p)) + \mu_1.\omega(\sqsubset_1^{\lambda^1}(p)) \tag{27.14}$$

where the $\mu$-coefficients give a relative weight to the path-length contributions. We shall see in section 41.3 that the HarmoRUBETTE® implements exactly this type of formula with the

special values $\lambda^0_{\cdot\cdot} = \lambda^1_{\cdot\cdot} = 1$, and zero else, except that the nominal values in that implementation are the exponential values $e^v$ of our values $v$.

This construction yields an objective tension predicate

$$Tension(\lambda^{\cdot}, \mu_{\cdot}, \omega, \phi_{min})$$

on $n$-tuples $(a_0, a_1, \ldots a_n)$ of zero-addressed chords of the given form in the truth space $\mathbb{T}^0_{\mathbb{R}}$. The predicate yields the empty set as soon as one chord in the sequence has truth value "False". This evaluation can be used to calculate best paths in the harmonic development.

### 27.2.3 Harmonic Tensions from Classical Harmony?

**Summary.** We discuss a particular "harmonic perspective" of harmonic tension proposed by Anja Fleischer in [154].

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The above construction of a tension predicate has the advantage that it works for any chord as proposed by Hugo Riemann. But it is not the usual approach in traditional harmony. In this latter context, it happens that most chords are not given any specific weight. We present one such approach as it was explicated in a diploma thesis by Anja Fleischer in [154]. In that approach, she took the classical binary logic and gave selected paths values of either "True" or "False" in the circle group truth module. Moreover, the value domains are chosen more in the flavor of Riemannian Harmony as it is taught in music schools.

The function list is this:

$$\text{VAL} = \{T, t, Tv, tv, D, d, Dv, dv, S, s, Sv, sv\}$$

which are major tonical, dominant, and subdominant values, together with their minor and substitute (German: "Vertreter") variants.

The tonality list is this:

$$\begin{aligned}\text{TON} \quad = \quad & \\ & \{DurC, DurCis, DurD, DurDis, DurE, DurF, DurG, DurGis, \\ & DurA, DurB, DurH, Mollc, Mollcis, Molld, Molldis, \\ & Molle, Mollf, Mollg, Mollgis, Molla, Mollb, Mollh\}.\end{aligned}$$

The 24 symbols relate to German major (=Dur), minor (=Moll), and the twelve tonality names. Here is the list of core chords plus values in the sense that these values are those where a chord takes the "True" value in the harmony predicate. The chord data in the first coordinate relate

to the same form $F = PitchMod_{12}$ as before.

$$
\begin{aligned}
True \sim \\
&\{(0,2,3,tv),(0,2,4,t),(0,2,5,s),(0,2,6,sv),(0,2,7,t),(0,2,7,tv)\\
&(0,2,9,s),(0,2,9,sv),(0,3,7,Tv),(0,4,5,s),(0,4,7,T),(0,4,9,t),\\
&(0,4,9,d),(0,4,10,t),(0,4,11,t),(0,5,7,s),(0,5,7,sv),(0,5,8,Sv)\\
&(0,5,9,S),(0,7,9,t),(0,7,10,tv),(0,7,11,t),(0,7,11,tv),(0,9,11,t),\\
&(0,9,11,d),(1,4,9,tv),(1,4,9,sv),(1,7,9,tv),(1,9,11,tv),(1,9,11,dv),\\
&(2,4,5,s),(2,4,6,sv),(2,4,7,t),(2,4,7,d),(2,4,8,tv),(2,4,8,dv),\\
&(2,4,9,s),(2,4,9,sv),(2,4,11,t),(2,4,11,tv),(2,4,11,d),(2,4,11,dv),\\
&(2,5,7,d),(2,5,7,dv),(2,5,9,s),(2,5,11,D),(4,5,7,t),(4,5,7,d),\\
&(4,5,8,tv),(4,5,8,dv),(4,5,8,sv),(4,5,9,s),(4,5,11,t),(4,5,11,tv),\\
&(4,5,11,d),(4,5,11,dv),(4,7,9,t),(4,7,9,tv),(4,7,9,d),(4,7,9,dv),\\
&(4,9,11,t),(4,9,11,tv),(4,9,11,d),(4,9,11,dv),(7,9,10,dv),(7,9,11,d)\}.
\end{aligned}
$$

The other chords of same interval structure are given the corresponding values relative to the related translation operations. This is a kind of historical "kernel" of harmonic knowledge. The question is whether one should just give "False" values to all chords that are not covered by this list. I would not, but it is difficult to establish a transparent rule set to evaluate the other chords (and these are most of the total chord system). The choice of falsity is the worst case scenario, but the present list does not give a good alternative, one just did not find general rules to be applied in the negative cases. We have included this example to make evident the dramatic gap between what is known and what is fuzzy and problematic in historical contexts.

### 27.2.4 Optimizing Harmonic Paths

**Summary.** This section deduces algorithms for valuations of entire paths under modulation constraints. The latter subject reveals the exorbitant complexity of harmonic analysis and throws new light on the question of complexity and depth in the humanities.

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In the preceding optimization procedure, changes of tonalities (for example in the distance matrix $T_{\mathrm{TON}}$) could be accounted for by numerical weights such that a tonality change is much more expensive than a conservation of the current tonality. Nonetheless, there is no finer tuning of the neighborhood of a harmonic development when a tonality change is recognized. The following algorithm is adequate for such a purpose. Let us reconsider the above case described in formula (27.14). Each chord of our chord sequence $(a_0, a_1, \ldots a_n)$ has its Riemann matrix evaluation, and the replacement by the "False" value for values below $\phi_{min}$. Suppose that every chord of the sequence has a Riemann matrix with at least one non-false coefficient. Call the sequence *coherent* iff there is at least one tonality symbol such that each Riemann matrix in the sequence has at least one non-false coefficient at this tonality. A coherent sequence allows a harmonic path without "false" chord values living in one and the same tonality. Suppose further that one is interested in getting a minimum of tonality changes. This requirement is

met as follows. Start with the maximal coherent subsequence $(a_0, a_1, \ldots a_{m_1})$ starting from the first chord (no gaps allowed). Then consider all maximal coherent subsequences (no gaps allowed) starting from $a_1, \ldots a_{m_1}, a_{m_1+1}$ (if possible). From these sequences, take the sequence with maximal last index: $a_{i_2}, \ldots a_{m_1}, a_{m_1+1}, \ldots a_{m_2}$, then repeat this procedure starting from index $i_2$ and so on. After a finite iteration, we end up with the covering of the whole sequence by coherent subsequences $(a_0, a_1, \ldots a_{m_1})$, $(a_{i_2}, \ldots a_{m_2})$, ... $(a_{i_k}, \ldots a_n)$. In each sequence, we may select a best path in a common tonality. This results in a spline-like configuration of paths $p_1, p_2, \ldots p_k$ with possibly overlapping parts. These parts can be taken as modulatory regions, more precisely: regions of neutralization which are followed by regions of pivotal chords and cadences. A deeper analysis of such splining regions with regard to the modulation theorems (subsections 27.1.4 ff. and 27.1.6 ff.) is pending.

# Chapter 28

# Applications

*Aus der Tatsache, daß sich die von den Theoretikern*
*geschaffenen Strukturen und Formen meist als*
*künstlich und zuweilen als abwegig herausgestellt haben,*
*folgt nicht, daß es überhaupt*
*keine allgemeine Struktur gibt, die nicht*
*eines Tages von einer*
*besseren Analyse der Musik, die*
*alle ihre Erscheinungsformen in Zeit und Raum*
*berücksichtigt, freigelegt werden könnte.*
Claude Lévi-Strauss [294]

**Summary.** This chapter deals with illustrations of the modulation model in chapter 27. It treats short and longer examples from Bach to modern jazz. This exercise should make clear the methodological background of these musicological experiments and illustrate the theory exposed in chapter 4.

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Before working out the concrete examples, we should emphasize that this approach is not meant to be a confirmation of any classical music theory. It is rather a discussion of a selection of empirical material in the light of the previous modulation theorems. The delicate point here is that the modulation theorems do not preconize any historical specification, they couldn't care less. These theorems give a list of pivotal degrees and not the answer to the question of historical adequacy. For such questions, one would have to look for other criteria.

A principal branch of such criteria comes from the communicative dimension of musical topography. When a work is analyzed, it is first taken in its neutral setup as a determined denotator of a more or less complex structure. As such, it has several pointers to poietic determinants, for example by the composer's name or/and the year of publication. When considering the poietic coordinate, one could ask for the adequacy of the modulation model in question with respect to the historical position of the composer and the composer's knowledge base. Questions as this one arise:"Did the composer (Beethoven, for instance) use the type of reflections as

described in the modulation model to construe the modulations in the given score (op.106. for example)?" Up to unknown and highly unlikely secret historical legacies, the answer is clearly "No!". But this is definitely not the discourse of our enterprise. We are not discussion conscious compositional strategies. Rather we are confronting the result of compositional efforts: the score (denotator) with the present modulation models. We are, so to speak, in the physicist's position to investigate nature's laws without ever knowing if and what and how a divine being could have built nature.

The default answer of a musicologist would be that we are not physicists in so far as we know quite a bit about Beethoven's approach to modulation or at least, we know that he was *not* in a state of considering those mathematical tools used in the mathematical model of modulation. And therefore—this the argumentation of the musicologist—any such modeling remains disconnected from the historical reality.

But it is rightly this latter concept of historical reality which we have to question in view of Dahlhaus' request for a systematic (re)construction of the dia- and synchronic system of music cited in section 27.1.5. Reality is not just a set of phenomena, but the totality of agents which shape the historical dynamics of the system. It is precisely the invisible and counter-intuitive strata of a system which usually are responsible for its unfolding. The absence of conscious traces of Beethoven's modulatory constructs does not explain these modulations and it does a fortiori not explain the global architecture of these modulations in the development of a large piece, such as Beethoven's op. 106. Moreover, the incredible organization and process of such masterworks cannot be explained by biographic factors, these things are of a much finer nature than the gross facts of a composer's life. And even if the composer had made statements about his poietic activity, this would just be one perspective upon the work's neutral (objective) data. There is no guarantee that this perspective must be the best possible. It is very likely that the intuitive and ingenious forces cover a strong majority of the work's characteristic and outstanding instances.

So what can we retain from an application of a mathematical model of modulation if not the historical adequacy? We can retain the explanatory power, i.e., the adequacy of model and instance, theory and experiment. How much of the given example can the model explain, and in which detail? This is what we shall explain in the following examples, and not, whether or not the composer would have thought in the same lines.

A last and personal remark about composers: From all composers that I have known, the great majority—also those who work with explicit constructive methods—was never in a state of explaining their own works to a satisfactory degree. Good works have always been far from algorithmic runs—even with composers of a knowledgeable mathematical flair or training, such as Tom Johnson or Jan Beran or myself. This is not a dogmatic statement, of course. It is the insight that good music is an enormously complex affair for which the present knowledge is not yet in a state of giving exhaustive construction algorithms.

## 28.1 First Examples

**Summary.** This section discusses short example from Johann Sebastian Bach, Wolfgang Amadeus Mozart, and Claude Debussy.

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In all subsequent examples, we shall refer to the modulation theorem in the well-tempered case. This seems legitimate in the historical context, starting from Bach. But it is not a priori clear which theorem should be applied in general.

### 28.1.1 Johann Sebastian Bach: Choral from "Himmelfahrtsoratorium"

**Summary.** The example exposes the first eight bars of the composition and shows a remarkable congruence of the model with a piece of "nature" which is far from being poietically conceived according to the model or its expression in music theories.

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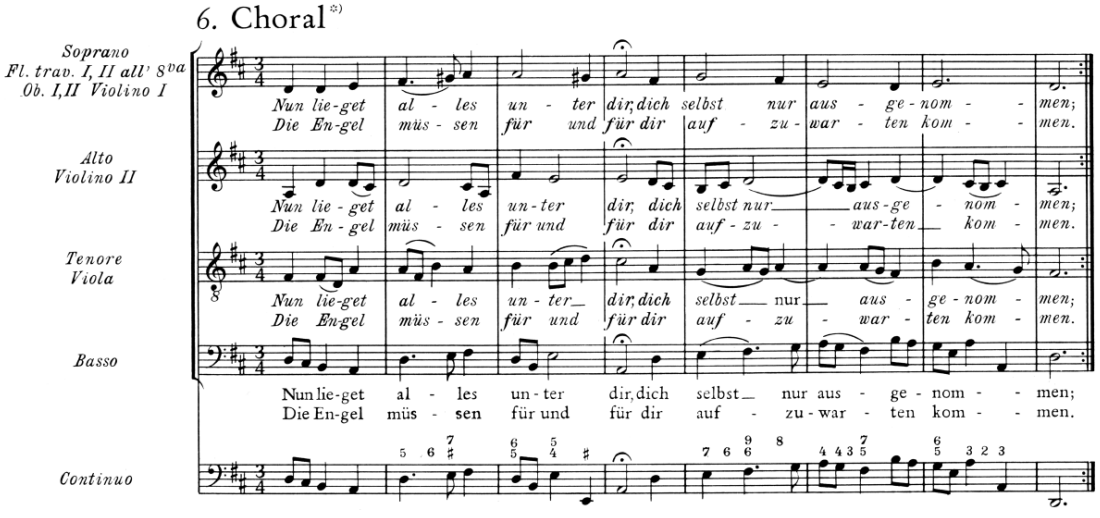


Figure 28.1: Johann Sebastian Bach: Choral from the *Himmelfahrtsoratorium* [33, p.41] (with kind permission of the Bärenreiter Publishers).

In this example, we consider bars 1 through 8 (see figure 28.1). For our analysis we strictly take the perspective of triadic degree interpretations, i.e., the global objects contained in $Dia^{(3)}$. Chords are always zero-addressed and are deduced from onsets in the sense that whenever there is a new onset, one considers the chord consisting of all pitch classes of events which start here or did not end at this onset or before.

According to our triadic degree perspective, such a chord will not remain uninterpreted, it will be covered by degrees of a triadic degree interpretation in the sense already explicated in section 27.1.2. For chords which are not triadic degrees, we choose an interpretation by a minimal number of triadic degrees. For example, the interval chord $\{d, a\}$ could be embedded in the major triad $\{d, f_\sharp, a\}$ or in the minor triad $\{d, f, a\}$. Since this embedding is not unique, we have listed our choices in figure 28.2. We shall view this process as a sequence of two modulations, the first from $D^{(3)}$ to $A^{(3)}$, the second back from $A^{(3)}$ to $D^{(3)}$. The scope is to compare the harmonic

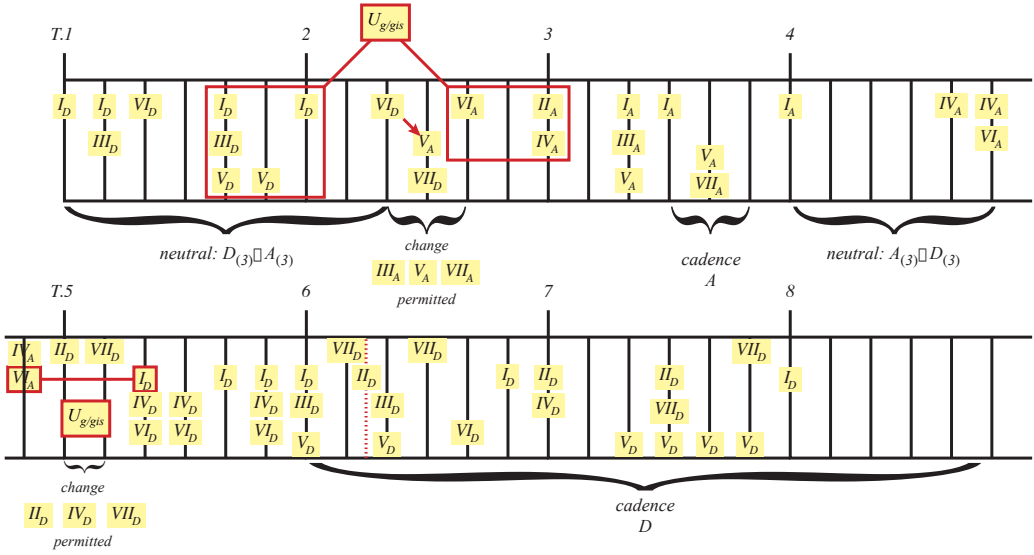


Figure 28.2: Two successive modulations with Bach from $D$-major to $A$-major, and back. In both cases, the pivotal degrees correspond to those predicted by the (well-tempered) modulation theorem. Moreover, within the symmetric degree neighborhood of the turning point of the first modulation, the modulator, i.e., the inversion $U_{g/g\sharp}$, becomes evident.

development with instances of the well-tempered[1] modulation theorem. The modulator is the inversion $U_{g/g\sharp}$ (recall that we denote $U_{g/g\sharp}$ for the inversion $U_3$, a reflection at the point between $g$ and $g\sharp$). The pivot degrees for the modulation $A^{(3)} \rightsquigarrow D^{(3)}$ are $III_A, V_A, VII_A$ whereas the modulation $D^{(3)} \rightsquigarrow A^{(3)}$ admits the pivotal degrees $II_D, IV_D, VII_D$ (see appendix N.1). The modulator symmetry $U_{g/g\sharp}$ gives rise to the following correspondences:

$$\begin{array}{lll} I_D \sim VI_A & VII_D \sim VII_A & VI_D \sim I_A \\ II_D \sim V_A & & V_D \sim II_A \\ III_D \sim IV_A & & IV_D \sim III_A. \end{array}$$

The first bar and the first two onsets of bar 2 are *neutral* since all appearing *neutral* degrees $I, III, V, VI$ of $D^{(3)}$ are also degrees of $A^{(3)}$. The third onset in bar 2 presents the pivots $V_A, VII_A$ according to table appendix N.1.2 and may therefore be viewed as the turning point of this modulation to $A^{(3)}$. The following onsets until the fermata at the beginning of bar 4 are a cadence of the target tonality $A^{(3)}$ with minimal cadence set $\{VII_A\}$, or, distributed among several onsets, with $\{II_A, V_A\}$. Beyond this it is interesting to observe the neighborhood of the turning point: it makes the modulator evident. The last four onsets before the turning point present $I \cup III \cup V, V, I$, and (less weighted as onset of the fifth to the delayed $V_A$) $VI_D$.

[1] We often use the sloppy wording "well-tempered" instead of "12-tempered", but are conscious that there are subtle differences in the terms.

After the turning point and before the cadence towards the end of bar 3, appear two onsets of $VI, IV \cup II$ of $A^{(3)}$. These three degrees of $A^{(3)}$ correspond (under the modulator) to the degrees $I, III, V$ of $D^{(3)}$ before the turning point.

The modulation in the reversed direction brings the neutral degrees in bar 4, then in the first two onsets of bar 5 the turning point with the admitted pivots $II_D$ and $VII_D$, and finally, in bar 6, follows the cadence according to the same pattern as observed in the first modulation.
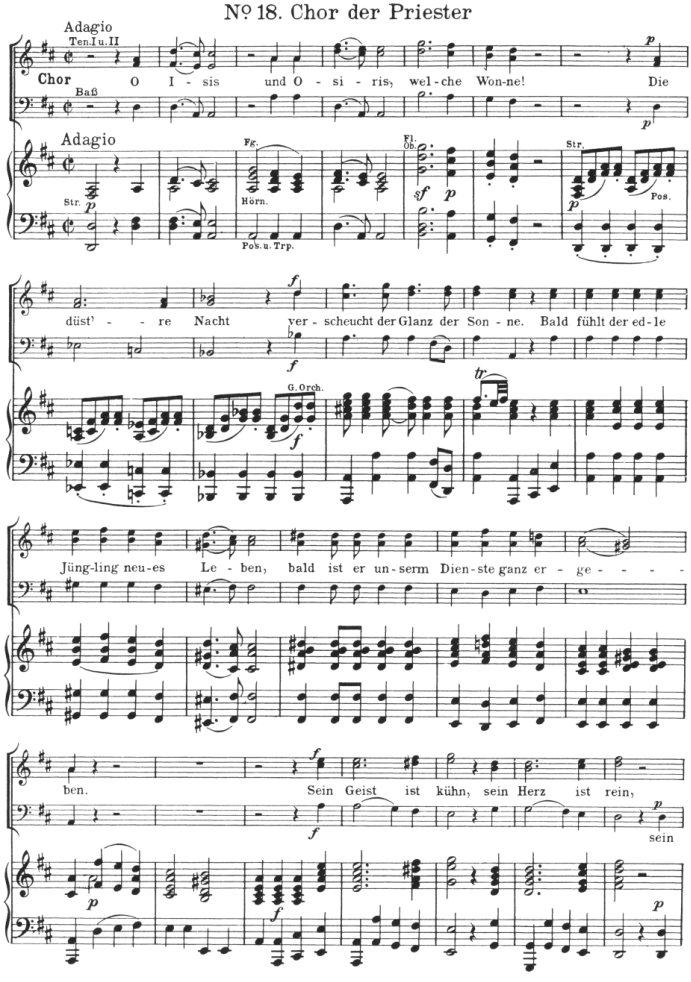
Figure 28.3: Wolfgang Amadeus Mozart: *Zauberflöte*, Pianoexcerpt (Soldan), second Scene No. 18, *Chor der Priester* [383, p.118] (with kind permission of the C.F. Peters Publishers, Frankfurt/Main).

### 28.1.2 Wolfgang Amadeus Mozart: “Zauberflöte”, Choir of Priests

**Summary.** This second example yields—apart from a provocative congruence—an interesting text-based *semantic* interpretation of the modulatory structure, an interpretation which seemingly did not appear from more traditional harmonic analyses.

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We are considering bars 1-18 of scene No. 18: “Chor der Priester” (see figure 28.3). The interpretation via chord onsets relating to triadic interpretations in $Dia^{(3)}$ is shown in figure 28.4. We however have to comment on this figure since three onsets in bar 8 are not contained in any triadic structure from $Dia^{(3)}$: the diminished seventh chord $\mathfrak{M}_2 = \{f_\sharp, a, c, e_\flat\}$. It is interpretable via degrees as a union of $VII_G, VII_{B_\flat}, VII_{D_\flat}$, and $VII_E$.

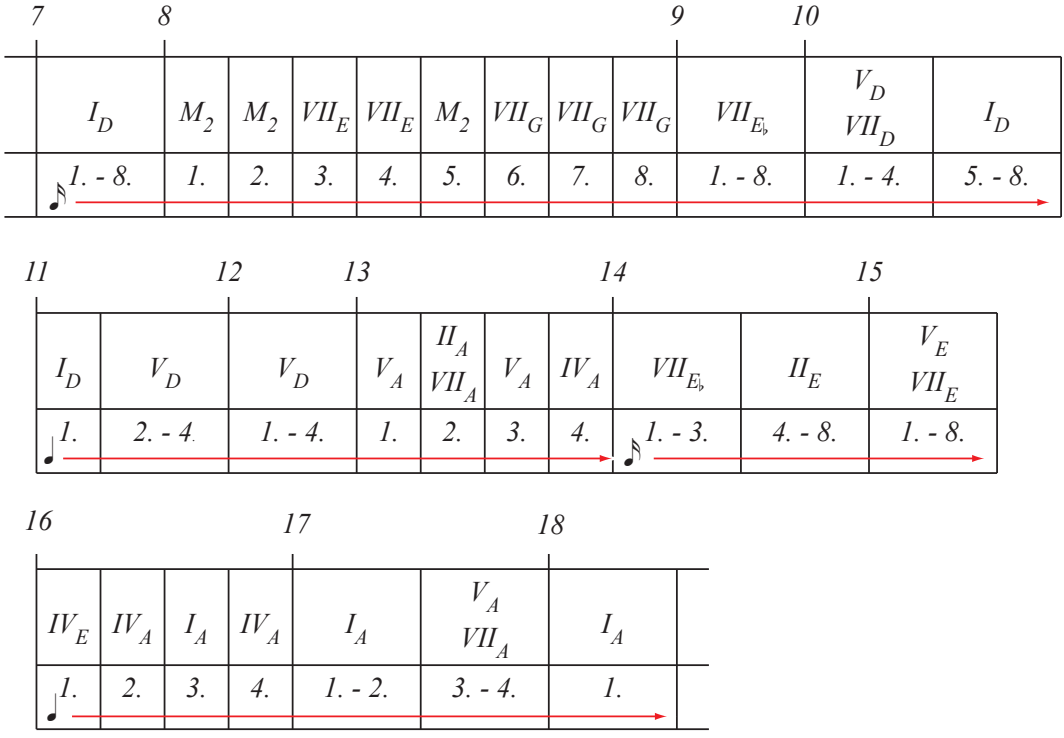


Figure 28.4: The configuration of triadic degrees as it is shown here from the perspective of modulation theory. In bar 8, the diminished seventh chord plays a special role since it is not interpretable within $Dia^{(3)}$.

To begin with, the score text poses a problem in bar 8 since we now leave $D$-major, where the degrees were positioned until bar 7. But we also leave $Dia^{(3)}$. In fact, bars 8 and 9 can be interpreted as these degrees in $g_h^{(3)}$ (see also 26.2.1): five times $II \cup VII$, then $VII$, and eight

times $I$. After that we return to $D$-major, and we have no more such conflicts until degree $VII_{E_\flat}$ at the beginning of bar 14, which can also be viewed as $VII_{E_\flat}$ or $II_C$ in harmonic minor tonalities. If we interpret the strict neighborhood of the diminished triad as lying in $A$-major, it appears as being isolated and cannot be understood easily. It is interesting that these two places are in a special position when related to the text semantics. The diminished seventh chord, when exiting the orbit $Dia^{(3)}$, is the moment of "düst'rer Nacht", the diminished triad $VII_{E_\flat}$ corresponds to the text which tells about the dark night, death, the polar concept against "life".

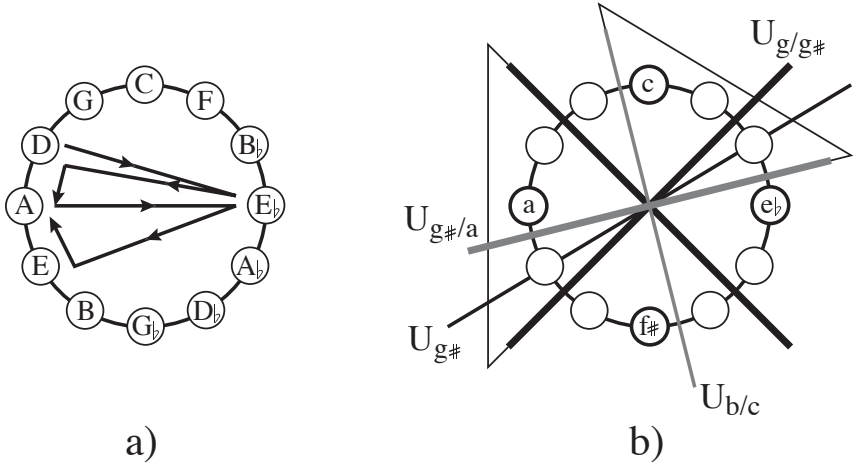


Figure 28.5: Part a) of this figure shows the modulation paths within the circle of fourths. Here, $E_\flat$ is the pole to $A$, and to its neighboring tonalities $D, E$ which are symmetrically positioned with respect to the polar axis $A, E_\flat$. The two semantic poles "night" (first $E_\flat$) and "life" (second $E_\flat$) are placed on distinguished tonality positions in front of the contextual tonalities $D, A, E$. Part b). The five modulators of the present modulations are partitioned into three groups: Two pairs of orthogonal axes, $U_{e/f}$ and $U_{g/g_\sharp}$, as well as $U_{b/c}$ and $U_{g_\sharp/a}$, and the axis $U_{g_\sharp}$ which lies symmetrically with respect to these two pairs.

Let us try to set up a connection between the text-semantic extra position of $\mathfrak{M}_2$ and of $VII_{E_\flat}$ within the modulatory configuration. To this end, we write $III_{E_\flat}$ in bar 9, and in bar 14, we write $II_E$ after $VII_{E_\flat}$. Bar 8, i.e., $\mathfrak{M}_2$, is not interpreted by degrees, but as a representation

of modulators. Here is our modulation scheme (the pivots are bold-faced):

| Modulator: | $U_{e/f}$ | $U_{e/f}$ | $U_{g/g_\sharp}$ | $U_{g_\sharp}$ | $U_{b/c}$ | $U_{g_\sharp/a}$ |
|---|---|---|---|---|---|---|
| Tonality: | $D \rightsquigarrow$ | $E_\flat \rightsquigarrow$ | $D \rightsquigarrow$ | $A \rightsquigarrow$ | $E_\flat \rightsquigarrow$ | $E \rightsquigarrow$ $A$ |
| Circle of fourths: | 5. | 7. | 11. | 6. | 5. | 1. |
| Pivots: | *II* | *II* | | *II* | **II** | *II* |
| | **III** | | *III* | *III* | *III* | |
| | | *IV* | | (*IV*) | | *II* |
| | *V* | **V** | **V** | *V* | *V* | **IV** |
| | *VII* | **VII** | *VII* | **VII** | *VII* | *VII* |
| Appearance: | *b.*9 | *b.*10 | *b.*13 | *b.*14 | *b.*14 | *b.*16 |

In figure 28.5, part a), the modulation paths are drawn on the circle of fourths. One recognizes the two segments in $E_\flat$ as a pole to $A$ an its symmetrically positioned neighbors $D, E$. The two semantic poles "night" (first $E_\flat$) and "life" (second $E_\flat$) are placed on distinguished tonality positions in front of the contextual tonalities $D, A, E$. Also, the modulators appear in a neat symmetric way: Part b) of figure 28.5 shows the axes of five different inversions and the diminished seventh chord $\mathfrak{M}_2$. These five modulators are divided into three groups: The pair of orthogonal axes (minor thirds!) associated with $U_{e/f}$ and $U_{g/g_\sharp}$ is contained in the symmetry group of $\mathfrak{M}_2$. It controls the first three modulations, i.e., the 'edges' $D - E_\flat$ and $D - A$ in figure 28.5, part a). Hence, $\mathfrak{M}_2$ appears as a 'support' of two modulators. These $\mathfrak{M}_2$-symmetries are in fact visible in bar 8. On the change $a \mapsto f_\sharp = U_{g/g_\sharp}(a)$, from the first to the second eighth onset, on the change $c \mapsto a = U_{e/f}(c)$ from the second to the third, on the change

$$\{a, e_\flat\} \mapsto \{f_\sharp, c\} = U_{g/g_\sharp}(\{a, e_\flat\}) = U_{e/f}(\{a, e_\flat\}) = e^3(\{a, e_\flat\})$$

from the fifth to the sixth, and on the change $c \mapsto a = U_{e/f}(c)$ from the sixth to the seventh eighth onset.

Symmetrically to this, the two 'edges' $E - E_\flat$ and $E - A$ in part b) of figure 28.5 are mediated by the equally orthogonal axes associated with $U_{b/c}$ and $U_{g_\sharp/a}$. And the middle modulation $A \rightsquigarrow E_\flat$ can be viewed as 'symmetry edge' on part b) of figure 28.5; its modulator $U_{g_\sharp}$ is equally inserted in a symmetric way between the double pair of orthogonal axes. This beautiful symmetry, which is semantically accompanied by the linguistic text, has the advantage that we can recognize an inner and semantic connectivity among the four tonalities $D, A, E, E_\flat$ instead of five tonalities $D, A, E, E_\flat$, and $g_h$, without any semantical connectivity.

### 28.1.3 Claude Debussy: "Préludes", Livre 1, No.4

**Summary.** This analysis again produces a possibly unexpected congruence with the model and succeeds in giving a very explicit interpretation to the title's "parfum".

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It is particularly important to recall from the introduction to this chapter that we do not make any statements about the poietic rationales of these compositions in our examples.

And that our analysis is nonetheless a scientifically legitimate and valid approach, much as human physical models of nature. We would strongly suggest to set up comparative discourses between traditional music analysis, jazz analysis, and the mathematical approach. For obscure reasons, this type of discourse has been applied very rarely[2] . We would however prefer to lead such a discourse on the level of neutral analysis, and not under the petitio principii perspective that knowing the composer's innermost agitations (who will ever know that?!) preconizes a distinguished analysis. We would rather like to see the objective advantages in the understanding of what is written down in the score.

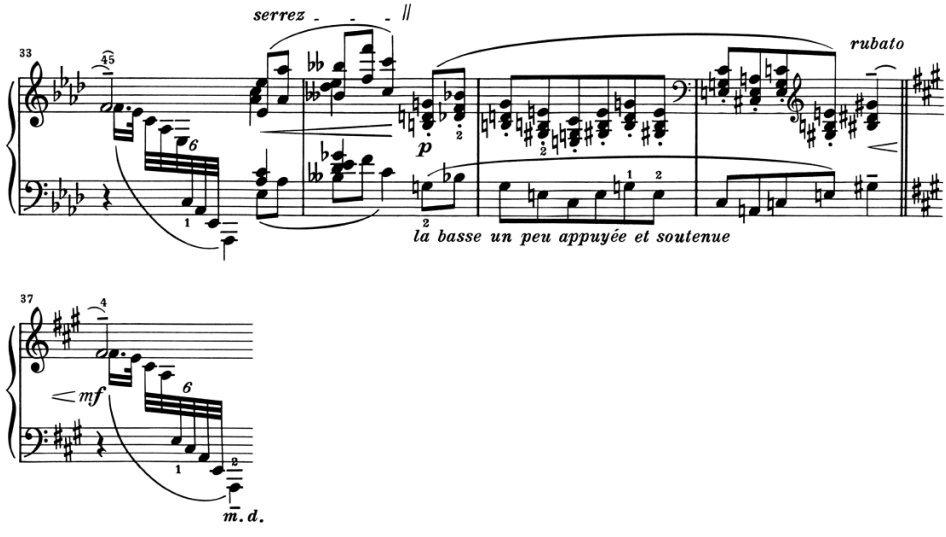


Figure 28.6: Claude Debussy: **Les sons et les parfums...** "Préludes", Livre 1, Nr. IV, [113] (with kind permission of the Henle Publishers).

We want to study a piece in Debussy's "Préludes", Livre 1. It is No. 4 "Les sons et les parfums tournent dans l'air du soir", a title which is taken from Baudelaire's "Harmonie du soir", third verse. We want to deal with the modulation from $A_\flat$-major to $A$-major (this is what the signature suggests), bars 33-37, see figure 28.6. The analysis starts on the last fourth (indication: *serrez...*) of bar 33 and terminates on the arpeggio at the beginning of bar 37.

The degree onsets are shown in figure 28.7. In contrast to the preceding example, there is no problem here to immerse chords within triadic interpretations. Rather do we encounter two problems: the mixed variety of tonalities and the ambiguity of most onsets with respect to tonal specification. It is particularly remarkable that a cadence of $A$-major does not occur. Moreover, the rhythmically simplistic sequence of mixed chords, which alters between $I_C$, $V_C$, and $V_A$, $I_A$ (bars 35 to 2/3 of bar 36, indication: *la basse un peu appuyée*) and are close to shocking in their strangeness to the context.

If we suppose that there is a modulation from $A_\flat$ to $A$, what could be the role of $C$? One can testify the presence of $C$ by the cadence $IV, V$, in fact the mentioned $I_C$-$V_C$ sequence, by

[2] An exceptional case is found in *Musiktheorie*, Vol. 4/2000.

the arpeggio of the chord $I_C \cup VI_C$ of the left hand, and by the melody line of the right hand. The role of $C$ cannot be that of an intermediate tonality since $C$ incessantly interplays with the target tonality. Also it is impossible that modulation targets to the intersection of $A$ and $C$ since the end clearly points away from $C$. The only possibility which remains is that $C$ does not play the role of a modulatory station, but of a modulator, if we stick to the present model.

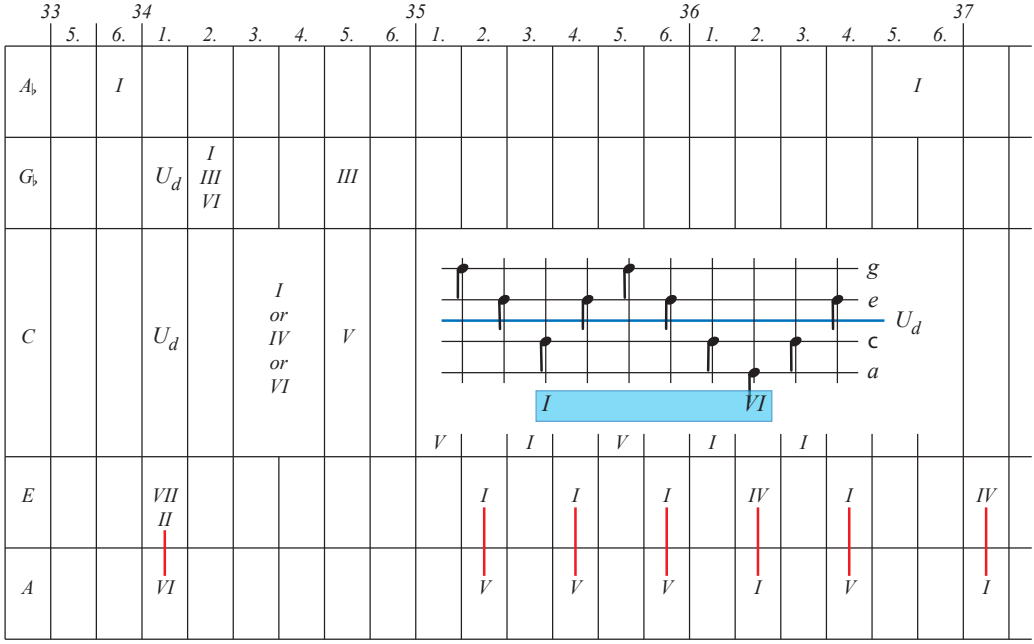


Figure 28.7: In the modulation from $A_\flat$ to $E$ (and $A$), the modulator, i.e., the inner symmetry of $C$ and $G_\flat$ is evidenced by chordical and motivic representation of this double scale material.

The triadic interpretation $C^{(3)}$ has exactly one non-trivial inner symmetry, namely the inversion $U_d$. There is a hint for this symmetry in the line of the left-hand voice in bars 35-36: Here, we have the mentioned arpeggio of $I_C \cup VI_C$, a chord which is symmetrical with respect to $U_d$, a fact which is felt as a to-and-fro movement within the chordically completed arpeggio (indication to play the bass!). Moreover, we observe three tones $d_\flat, e_\flat, f$ at the second onset of bar 34, which lie in $IV \cup V$ of $G_\flat^{(3)}$, and the last onset of the same bar can also be specified as $III$ of $G_\flat^{(3)}$. However, relating to symmetries, $G_\flat$ and $C$ are equivalent. Also, regarding inner symmetries, $G_\flat$ and $C$ are equivalent; i.e., the inversion $U_d$ is 'represented' by both, $C$ and $G_\flat$.

But if $U_d$ is the modulator, then it operates via $A_\flat^{(3)} \rightsquigarrow E^{(3)} = U_d(A_\flat^{(3)})$, and not to target $A^{(3)}$. This should happen with the pivots $II, III, V, VII$ or with $II, IV, V, VII$ (second cadence option). This is exactly what happens in the beginning of bar 34: $II_E \cup VII_E$ appear as unambiguous pivots towards $E^{(3)}$. The other degrees which do not pertain to $C$ or to $G_\flat$

are drawn in figure 28.7. They are all in $E$ as well as in $A$, except the last onset $I_{A\flat}$ in bar 36, which—together with the onsets in $G\flat$—concludes the tonality $A\flat$ in an atmospheric flavor the tonality $A\flat$ which is in symmetrical position to $E$.

This not only helps to understand the modulatory function of the tritonally related tonalities $C$ and $G\flat$, it also makes evident the ambiguity of the modulation—and thereby the charm of this passage: The piece modulated to $E$, but the cadence of $E$ did not take place, one remains within the intersection $E \cap A$, and the first degree of $A$ in bar 37 is a fact that can only be interpreted in an ambiguous way; it could also be $IV_E$. So tonality $E$ has been introduced in a defined way, but without cadencing, rather by neutralizing it with respect to $A$.

In this view, the title receives a structural reading. The tones are really turned within the modulation (via $U_d$), the "parfume" of $A\flat$ remains mixed with the "parfumes" of $A$ and $E$ until the end, it keeps returning like a memory.

## 28.2 Modulation in Beethoven's Sonata op.106, $1^{st}$ Movement

**Summary.** This longer example illustrates complex connections of modulatory processes within the Allegro movement of the "Hammerklavier" Sonata. We give an interpretation and precision of the famous theses by Erwin Ratz [434] and Jürgen Uhde [534] with respect to the overall architecture of the modulation processes.

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### 28.2.1 Introduction

**Summary.** This introduction sketches the key position of the "Hammerklavier" Sonata in the history of European piano music.

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In contrast to the preceding examples, this one will not deal with short modulatory passages but with *the overall organization of tonalities and architecture of modulatory passages.* The methods will be the same as in the preceding examples, but we shall only sketch the details—except for some punctual zooms. A detailed discussion, including motivic aspects ("motivisch-thematische Arbeit") and their connection to harmony can be found in [328]; some analysis of the motivic aspects have also been discussed in example 24 of section 14.2.

We want to investigate the Allegro movement in Beethoven's "Hammerklavier" sonata op.106 [46], a very famous and equally difficult late work which deaf Beethoven had composed 1817–1819. In his description of op.106 [257], Joachim Kaiser starts with the lapidary phrase: "Grösste Sonate der Musikgeschichte." The sonata's public premiere was executed in 1836 in Paris by Franz Liszt. But the "Hammerklavier" sonata is not only a technical and mnemotechnical challenge for the interpreter, it is also a mysterium for the sonata theory—developed by Adolf Bernhard Marx [320] in 1850s—which established the sonata form as an accepted norm.

This sonata form ("Sonatenhauptsatzform") is an architectural and processual scheme of a large musical form which is framed by a syntactic subdivision into exposition, development, recapitulation, and coda (see figure 28.8).

The comparison with the classical sonata form will only be a starting point of our analysis of the present *individual* composition. It is well known that the sonata scheme is not adequate for more than a statistical estimation of really existing sonata movements. However as a starting point, we can use the scheme in order to evaluate knowledge and limits of traditional musicology with the complexity of tonal and modulatory structures of the present sonata.

Scheme of the sonata form for the Allegro movement in Ludwig van Beethoven's op.106

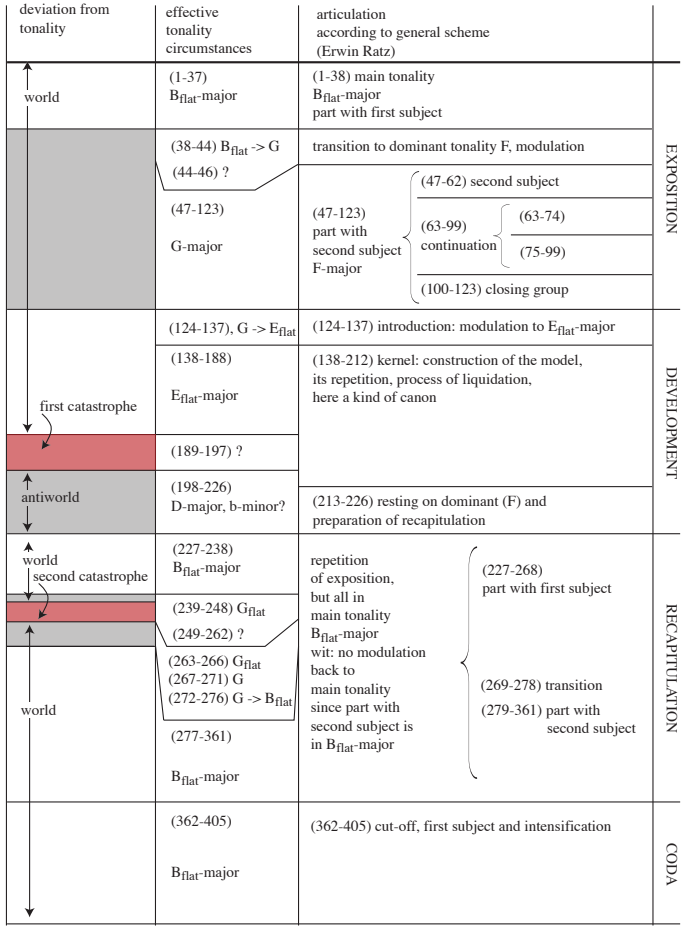


Figure 28.8: The sonata scheme of the first movement of the "Hammerklavier" sonata op.106, compared to the normed sonata scheme.

Our analysis is not thought of as a contradiction but as a confirmation and elaboration of existing analyses. In particular, we take over the fundamental *world-antiworld thesis* of Erwin Ratz [434] and the *"catastrophe theory"* of Jürgen Uhde [534] which relates to the specific modulatory role of the diminished seventh chords in the sonata. With regard to the formal ex-

plicitness of our model, the take-over will however include precision, completion, and unification of aforesaid world-antiworld thesis and catastrophe theory.

### 28.2.2 The Fundamental Theses of Erwin Ratz and Jrgen Uhde

**Summary.** This section presents the thesis of Ratz and its weltanschauung, as well as the thesis of Uhde. A restatement of these theses in group-theoretic terms is given. As its consequence, a particular modulation architecture is predicted. This will be verified in section 28.2.3.

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In [434], Ratz expressed the idea that the "empire of tonalities" in op.106 is polarized into a "world" around the pole $B_\flat$-major, the sonata's main tonality, and an "antiworld" around the counterpole $b$ minor. In [534], Uhde supplemented this thesis in so far as two so-called "catastrophes" occur in the Allegro movement when the "world" is left in order to enter the "antiworld". Catastrophes are dramatic modulatory processes which differ significantly from normal modulations (see figure 28.9). We discuss the following thesis:

**Thesis 4** *Let $\mathfrak{M}_0 = \{c_\sharp, e, g, b_\flat\} \subset PiMod_{12}$ the local composition of the diminished seventh chord. Then the modulation structure of op.106 is determined by the modulators in the sense of theorem 30 which are contained in $Aut(\mathfrak{M}_0)$.*

What is the relation of this thesis with Ratz' world/antiworld? To begin with, the group $Aut(\mathfrak{M}_0)$ solves the following purely group-theoretic problem: We look for a *maximal* subgroup $M$ of the group of inversions and translations $TI_{12}$ on $PiMod_{12}$ which acts as modulator group on $Dia^{(3)}$, but under the restriction that no group element transforms $B_\flat^{(3)}$ to $D^{(3)}$. Here, the interpretation $D^{(3)}$ stands for the minor tonality $b$ of ionian mode. Under the action of such a subgroup, the set $Dia^{(3)}$ is divided in at least two orbits (one for $B_\flat$, and one for $b$). A priori, it is not clear whether there are several such subgroups, and how many orbits such maximal subgroups will produce. But it turns out that the subgroup is uniquely determined, viz., $M = Sym(\mathfrak{M}_0) \cap TI_{12}$, these are the inversions and translations which leave $\mathfrak{M}_0$ invariant. We have the surjection $Sym(\mathfrak{M}_0) \to Aut(\mathfrak{M}_0)$, and it is easily seen that its restriction

$$M \to Aut(\mathfrak{M}_0)$$

is an isomorphism. So we may identify these groups. Moreover, there are exactly two orbits under this automorphism group, namely the world

$$W = Aut(\mathfrak{M}_0).B_\flat = \{B.D_\flat, E, G, A, C, E_\flat, G_\flat\},$$

and the antiworld

$$W^* = Aut(\mathfrak{M}_0).D = \{D, F, A_\flat, B\}.$$

We are now in state to make more precise Ratz' hypothesis: The tonalities of $W$ are those of Ratz' "world", whereas the complementary set $W^*$ defines Ratz' "antiworld". This casts our possibilities in two ways, we are given a precise dichotomy $W/W^*$ and the admitted modulators, i.e., the elements of the automorphism group $Aut(\mathfrak{M}_0)$.
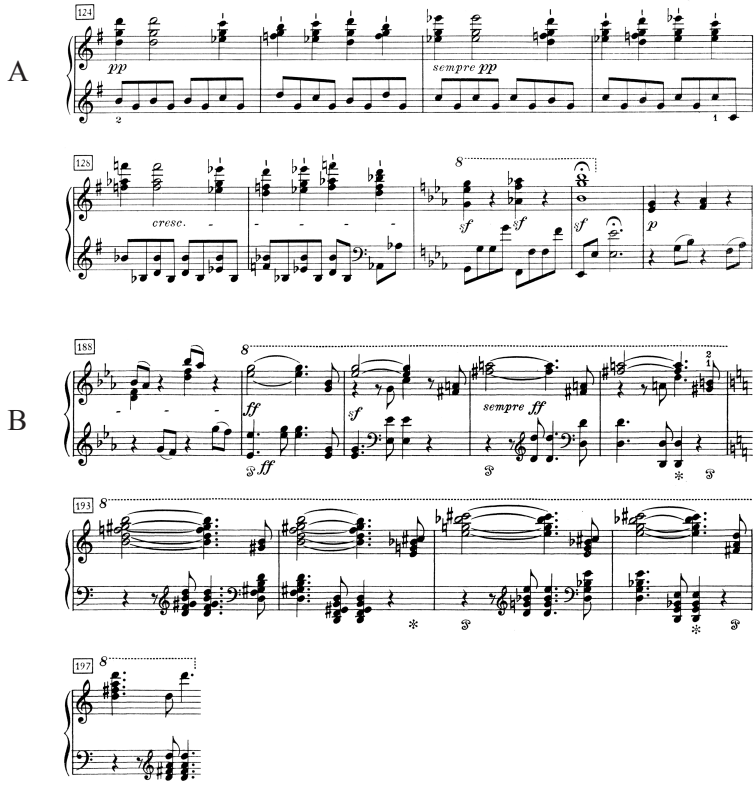
Figure 28.9: Ludwig van Beethoven: Two modulations in the first movement of op.106 [46] (with kind permission of the C.F. Peters Publishers, Frankfurt, Frankfurt/Main). The first (A), from $G$ to $E_\flat$, is a common one, whereas the second (B), from $E_\flat$ to $b$-minor, is a catastrophe modulation in the sense of Uhde.

The modulatory situation here is that of a *restriction of modulators* to the group $Aut(\mathfrak{M}_0)$. What is the meaning of this condition for the modulation model? As long as the modulator which is described by the modulation theorem is contained in $Aut(\mathfrak{M}_0)$, there is no problem to apply the theorem and to look for pivots. But a modulation to the third circle of fourths, like $C \rightsquigarrow E_\flat$ or $A \rightsquigarrow C$ only admits translations in our restricted framework. In this case we must refrain from rigidity condition (3) (the triviality of the intersection $TI_{12} \cap Sym(T \cap Q)$) in property 1 of section 27.1.4. In fact, this condition which guaranteed the uniqueness of the modulator is superfluous now: there is only this translation symmetry, no other candidate! But then, there is a corresponding theorem where the modulators are the translations $e^{\pm 3}$.

Therefore, one should modulate in the sense of restricted modulators and pivots within $W$ and within $W^*$, whereas modulation between two tonalities living in different worlds should yield a catastrophe in the sense of Uhde, whaich means in particular that in such a situation,

diminished seventh chords—which are the 'creators' of the catastrophe—should become visible at the surface of the score.

### 28.2.3 Overview of the Modulation Structure

**Summary.** We present the modulatory architecture and the modulations which split into "ordinary" cases and "catastrophes" in the sense of Uhde.

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We now have to test these postulates in the total plan of modulations of the Allegro movement. The modulation plan looks like this:

$$\overset{W}{B_\flat} \underset{e^{-3}}{\rightsquigarrow} \overset{W}{G} \underset{U_g}{\rightsquigarrow} \overset{W}{E_\flat} \underset{!}{\rightsquigarrow} \overset{W^*}{D/b} \underset{U_{d/d_\sharp}}{\rightsquigarrow} \overset{W^*}{B} \underset{!}{\rightsquigarrow} \overset{W}{B_\flat} \underset{U_{b_\flat}}{\rightsquigarrow} \overset{W}{G_\flat} \underset{U_{a_\flat/a}}{\rightsquigarrow} \overset{W}{G} \underset{e^3}{\rightsquigarrow} \overset{W}{B_\flat}.$$

In order to view these modulations on the circle of fourths and relative to the world/antiworld dichotomy, see figure 28.10. For each modulation, the modulators from $Aut(\mathfrak{M}_0)$ are indicated. Here are the single modulations in detail:

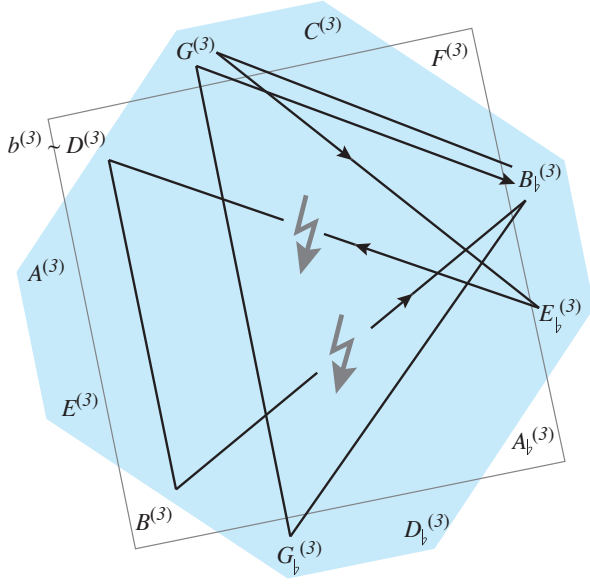


Figure 28.10: The graph shows the modulation plan in the Allegro movement of Beethoven's op.106 in the tonality system arranged on the circle of fourths. The start switches from $B_\flat$ to $G$. The inverse modulation occurs at the end, and both follow the same procedure, see sections 28.2.4 and 28.2.11. Except these initial and terminal movements, the modulation plan is perfectly symmetric around the symmetry axis between $B_\flat^{(3)}/E_\flat^{(3)}$ and $A^{(3)}/E^{(3)}$.

### 28.2.4 Modulation $B_\flat \rightsquigarrow G$ via $e^{-3}$ in $W$

The first modulation $B_\flat \rightsquigarrow G$ in the transition (bars 39-46) to the second subject could in principle be performed by use of a "pedal modulation" [478]. We do however not encounter this modulation, but 'merely' a sequence of $VII_G$-degrees whose top notes are shifted by minor thirds from each other, i.e., exactly the situation of the pivot $VII$ and the third translation, as predicted by the modulation under restricted modulators.

### 28.2.5 Modulation $G \rightsquigarrow E_\flat$ via $U_g$ in $W$

This modulation is bipartite (first part: bars 124-127, second part: bars 128-129). Before we encounter the pivots $VII - V - VII$ of $E_\flat$ according to modulation table appendix N.1 in part two, we hear tone $g$ as an octave interval: pedal and stationary voice. Here the inversion at $g$ is made evident (figure 28.11). On the one hand, this section is a cadence of $c_{mel}$ whose inner symmetry is $U_g$. One recognizes a bipartite contrapuntal motive structure of which the second part (bars 126-127) is the inversion of the first part (bars 124-125) in pitch $c_\sharp$. But modulo octave (in the pitch class space), this inversion is the same as $U_g$. Since $c_\sharp$ is not contained in the present scales, it is delegated to the octave in $g$. This means that the modulator $U_g$ is motivically evidenced in this first modulatory section preceding the pivots and the cadence.

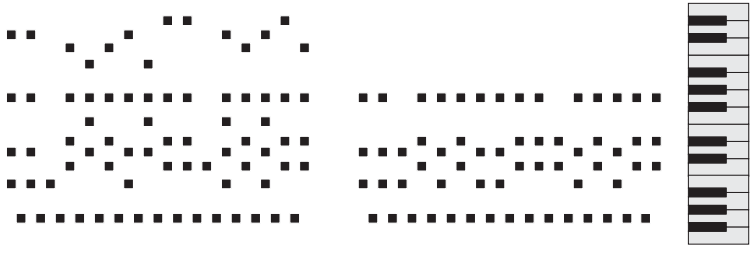

Figure 28.11: If we omit tone $f$ in bars 124-127 (it serves for the identification of $c_{mel}$) and transpose all pitches into one octave between the two $g$s (to the right), then we recognize the motivic inversion symmetry between bars 124-125 and bars 126-127.

### 28.2.6 Modulation $E_\flat \rightsquigarrow D/b$ from $W$ to $W^*$

This modulation is a catastrophe in the sense of Uhde since it leads to the antiworld $W^*$. As we may recognize already from the score (B) in figure 28.9, bars 189-197 are of a dramatic shape. Any elaborate motivic, rhythmic or harmonic effort is postponed in favor of a pertinent presentation of diminished seventh chords. An approach to modulation $VI_{E_\flat}$, $I_D$ (bars 189-192) fails, the resolution of all alteration signs indicates the exit from tonal space. We hear the "generator" of the catastrophe, the diminished seventh chord as such.

### 28.2.7 Modulation $D/b \rightsquigarrow B$ via $U_{d/d_\sharp} = U_{g_\sharp/a}$ within $W^*$

The process is resolved chordically here and impregnated by two simultaneous local meters (figure 28.12): in the left hand a 6/4 meter via triplets, in the right hand 8/8 meter. In both expressivo indications (bars 205, 209) appears (bar 207, and repeated in bar 211) in the triplets in the left hand the motif $f_\sharp - b - f_\sharp$ and $b - f_\sharp - b = U_{d/d_\sharp}(f_\sharp - b - f_\sharp)$. And the right hand plays $d$ and $d_\sharp = U_{d/d_\sharp}(d)$ in bars 205-209 as well as $e$ and $c_\sharp = U_{d/d_\sharp}(e)$. In bar 209 appears at the beginning degree $I_B$ and after that in the right hand a chromatic sequence which comprises the seven pitches $B^* = \{c_\sharp, d, d_\sharp, e, f, f_\sharp, g\}$. Now, $B = U_{d/d_\sharp}(D) = e^5.11(D)$. But by a concatenation with the fifth circle symmetry we produce $e^5.5 = U_{d/d_\sharp}.e^0.7$ out of $U_{d/d_\sharp}$. This result lies in $Sym(\mathfrak{M}_0)$ (!), and we exactly get

$$B^* = e^5.5(B),$$

so that we really are situated in $B$—up to an inner symmetry of $\mathfrak{M}_0$.

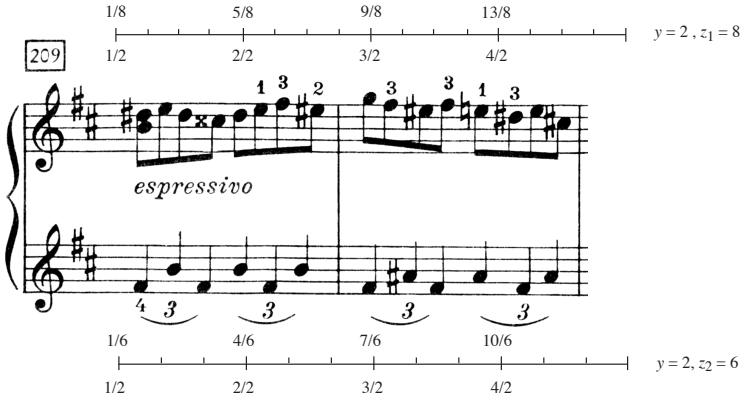


Figure 28.12: Example of two simultaneous local meters, corresponding to the left and right hand in bars 209-210 of the first movement of Beethoven's op.106.

### 28.2.8 Modulation $B \rightsquigarrow B_\flat$ from $W^*$ to $W$

The next modulation leads us back from the antiworld $W^*$ to the world $W$ (bars 214-226). Corresponding to the short stay in $W^*$ the return is an easy business. There follows a sequence of arpeggiated intervals which is strongly based upon $\mathfrak{M}_0$ and ends on $I_{B_\flat}$:

| | | |
|---|---|---|
| twice | $e, c_\sharp$ | within $\mathfrak{M}_0$ |
| six times | $g, e$ | within $\mathfrak{M}_0$ |
| four times | $g_\sharp, e$ | |
| four times | $b_\flat, e$ | within $\mathfrak{M}_0$ |
| four times | $b_\flat, f$ | $I_{B_\flat}$ |

### 28.2.9 Modulation $B_\flat \rightsquigarrow G_\flat$ via $U_{b_\flat}$ within $W$

For this modulation within the world $W$, we can apply the normal modulator. The modulation is a fast one (bars 238-239), although the involved tonalities $B_\flat$ and $G_\flat$ are separated by four fourths. At the end of bar 238, the neutral degree $I_{B_\flat}$ is followed by its inversion $VI_{G_\flat} = U_{b_\flat}(I_{B_\flat})$, so the modulator is again put into evidence. Immediately after that, at the beginning of bar 239, follows $\{b_\flat, f, a_\flat\}$ which lies in $III \cup V$ of $G_\flat$ and corresponds to the pivots.

### 28.2.10 Modulation $G_\flat \rightsquigarrow G$ via $U_{a_\flat/a}$ within $W$

This second Uhde catastrophe (bars 249-262) is highly dramatic. At the beginning (bars 249-250), we remain within the large orbit $W$: $I$ of $G_\flat$. As with the first catastrophe, we then encounter a pronounced series of diminished seventh chords. In bars 259-262, it terminates in the intersection $B \cap G_\flat$: we do not know whether the change or orbits has been successful or not. The decision is only taken in bar 263 with pitch $f$ which is not in $B$, but in $G_\flat$. So we actually did not really leave the world, at least not unambiguously. This 'delusion' is particularly refined since we hear the "fanfare" with $III_G$ which could also be viewed as $I$ of $b$-minor. But this third degree corresponds to a pivot of our model, the inversion of the bass in the moment of the forte onset at the end of bar 266: $b = U_{a_\flat/a}(f_\sharp)$ shows the same symmetry as for the above modulation $D/b \rightsquigarrow B$ within the antiworld after the first catastrophe. So we do not move from $B$ to $D$ as it is suggested by the change of signature in bar 267 and from the pitch material before the appearance of $f$ in bar 264. This superficial impression is an "allusion" to the antiworld situation, unveiled however as being an illusion and is resolved in a consistent way. We have moved from $G_{flat}$ to $G$.

### 28.2.11 Modulation $G \rightsquigarrow B_\flat$ via $e^3$ within $W$

This last modulation proceeds completely regularly according to the scheme that we already know from the inverse modulation in the transition to the second subject as discussed above.

## 28.3 Rhythmical Modulation in "Synthesis"

**Summary.** The modulation model does not restrict to time from its logical structure. This fact is exploited in a composition of rhythmical modulation for percussion ensemble and piano.

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The harmonic modulation model is based upon the pitch class construction modulo octave, i.e., the space $PiMod_{12}$ based on the module $\mathbb{Z}_{12}$ derived from the semi-tone pitch space based on the integers, modulo the submodule generated by the octave quantity 12. We have viewed the tonalities as being scales, together with their triadic interpretations by the seven well-known degrees. But there is nothing substantial to the choice of these spaces. The mathematical framework is completely insensitive to the forms and denotators which implement the underlying parameter and class spaces.

So why not apply the modulation theory to a concept framework which is mathematically the same as for harmony, but which is semantically different, more precisely: the new approach presents a space and its denotators in the context of onset time and its rhythms.

We shall develop this switch from pitch to time in the next section and apply this theory to a rhythmical modulation in movement No. 1 of Guerino Mazzola's jazz concert "Synthesis" [339] for piano, large percussion ensemble, and e-bass.

Before delving into technicalities, we should however observe a fundamental difference between two time qualities involved in musical composition. We have known time as a form related to onset and duration. These dimensions are part of what defines events in scores. They are comparable to other event parameters such as pitch, loudness etc. A local or global composition is a configuration of events which represent the composition's substance. This material data is without any logical or strategic specification. It does not include the composer's poietic construction plans or the analyst's structural evaluation.

But modulation is a structure that is not only defined on the level of the effective neutral, pivotal, and cadence degrees, it is rather built on the tripartite strategy Neutralization—Turning Point—Cadence as defined by Schönberg. This is a logical construction on the syntactical level: a sequence of three functional units in the syntagmatic string of musical development. Like abstract logical schemes, such as modus barbara ("a implies b" and "b implies c" implies "a imples c"), this is not a priori a syntagmatic string in the material musical time. It regards logical time rather than material time. In harmonic modulation, we have three logical stages in the pitch domain, and their unfolding on the material time line is only a representation of an abstract process, not the process as such.

Substantially, in harmonic modulation we have an excellent example for Augustinus' definition of music as an art of instantiation of good rational strategies. As already mentioned in the previous discussion of Augustinus' definition, modulation could also regard rhythmical strategies, for example. It is very important to distinguish logical time from material time in this case since here, the syntagmatic string which embodies the logical time is superposed to the material time of rhythmical structures.

### 28.3.1 Rhythmic Modes

**Summary.** This section describes the transfer of harmonic modality to rhythmic categories, and from this derives the modulations in the rhythm domain.

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The transfer of harmonic dimensions to rhythmic ones must deal with the semantic specifications of these dimensions. In fact, pitch is a space category which carries a strong connotation of sound quality, i.e., of instrumental realization. No abstract pitch quality has ever been used in harmony as soon as realistic music parameters are to be described. We could of course say that the abstraction from instrumental realities is only a question of habituation, and that abstract time could also be thought of in an abstract comprehension which already worked for harmony.

However, it is easy to imagine a more or less good orientation in the pitch space whereas it is difficult to imagine an orientation in an abstract time space, since the distinction between later and earlier time onsets is only relative and risks failure if no supplementary time-dependent

structures can be found. Moreover, the superposition of logical and material time layers asks for a clear marking of material time events.

This is why we would like to add supplementary, logically redundant markers of time such that one can use such redundancies to shape time in a rhythmical modulation theory. The idea is as follows: Suppose the onset time is parametrized by integers, $Onset_{\mathbb{Z}} \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z})$. Suppose further (in analogy with the octave period) that we have the translation $e^{12}$ in time on $Onset_{\mathbb{Z}}$ in the sense of a time period. This could be realized by a 12/8 meter, for example, where the eighth duration stands for the unit time in $Onset_{\mathbb{Z}}$. As a marker parameter, we select the form $Percussion \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z})$ whose zero-addressed denotators are percussion instruments as represented in a list with integer indexes. This could for example be a list of percussive sounds in an MIDI-environment where the numbers stand for the program changes (see below for more concrete setups). Our events are situated in the combined space

$$PerOns \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z} \oplus \mathbb{Z}) \xrightarrow{\sim} PO \underset{Id}{\longrightarrow} \mathbf{Limit}(Onset_{\mathbb{Z}}, Percussion)$$

and will be called *percussion events* in this context. But we want more, i.e., macro percussion events, in order to construct profiled markers. So we consider these macro percussion event spaces as they were introduced in section 13.4.3.1:

$$\begin{aligned} &Knot_{PerOns} \underset{Id}{\longrightarrow} \mathbf{Limit}(PerOns, Makro_{PerOns}) \\ &\text{with } Makro_{PerOns} \underset{f:F \xrightarrow{\sim} 2^{FK} \rightarrowtail \Omega^{FK}}{\longrightarrow} \mathbf{Power}(Knot_{PerOns}) \\ &\text{and } F = Fun(Makro_{PerOns}), FK = Fun(Knot_{PerOns}). \end{aligned}$$

A macro rhythmic germ $R^*$ is a finite local composition in the ambient space $Knot_{PerOns}$. And an infinite macro local Para-rhythm is paraphrased by the 12-periodic local composition $e^{[-\infty,\infty]12}R^*$. In the sequel, we shall only consider germs $R^*$ which are in bijection to their time projections $R^*_{Ons}$, and such that $R^*_{Ons}$ are contained in the period interval $[0, 12[$. This means that we may equivalently consider the rhythm classes modulo the time period 12. These objects are the residual classes of the macro rhythmic germs $R^*$ which we now identify with the germs because of our additional assumptions. So we are working in the macro class space:

$$\begin{aligned} &Knot_{PerOns_{12}} \underset{Id}{\longrightarrow} \mathbf{Limit}(PerOns_{12}, Makro_{PerOns_{12}}) \\ &\text{with } Makro_{PerOns_{12}} \underset{f:F \xrightarrow{\sim} 2^{FK} \rightarrowtail \Omega^{FK}}{\longrightarrow} \mathbf{Power}(Knot_{PerOns_{12}}) \\ &\text{and } F = Fun(Makro_{PerOns_{12}}), FK = Fun(Knot_{PerOns_{12}}), \end{aligned}$$

deduced from

$$PerOns_{12} \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}_{12} \oplus \mathbb{Z})$$

in an analogue construction to scales in pitch class spaces. Intuitively, these macro rhythmic germs in $Makro_{PerOns_{12}}$ are just time scales whose points are loaded with a rhythmic satellite object each.

As to modulation theory with such objects, we can apply the modulation model for 12-tempered tuning, but we have to take care of the satellites! There may be many rhythmic scales

with the same time projection! This means that the translation and time-inversion symmetries will have to carry over the (unaltered) satellites. This is as if the harmonic modulation would be carried out with "colored" pitch classes, whereas the inversion of pitch classes would preserve colors.

### 28.3.2 Composition for Percussion Ensemble

**Summary.** This final section discusses the concrete realization of the rhythmical modulation model, essentially by use of a large ensemble of percussive instruments.

– Σ –

The composition in question here is a rhythmical modulation in movement No. 1 of Mazzola's jazz concert "Synthesis" [339]. The modulation takes place after the exposition in the sonata scheme of this movement (3:18-5:48 on piece # 1 of [339]). This developmental start is written in 12/8 measure, in fact the measure which we need for the rhythmical modulation in a 12-periodic scheme as exposed above. The modulation is built upon the rhythmic scale (the time projection of the macro germ) corresponding to the complement $\widehat{62}$ of No. 62 the chord classification list in appendix L.1. The macro germ $G^*$ above $\widehat{62}$ has the following shape. For each onset $x \in \widehat{62}$, we have a corresponding denotator $((x, p_x), Sat_x)$ of form $Knot_{PerOns_{12}}$. The $p_x$-coordinate is an integer value for a percussion sound, whereas the satellite $Sat_x$ is a zero-addressed denotator of form $Makro_{PerOns_{12}}$. We choose for each satellite a zero-addressed three-element motif of form $PerOns_{12}$,

$$Sat_x = \{(0,0), (do_x^1, dp_x^1), (do_x^2, dp_x^2)\},$$

with first element the origin $(0,0) \in \mathbb{Z}_{12} \oplus \mathbb{Z}$ (see also figure 28.13). The choice of these three-element satellites is due to the general construction principle of the "Synthesis" concert, i.e., the use of the 26 isomorphism classes of three-element motives for all melodic and rhythmic structures. This principle was already encountered in section 11.6.3 where we discussed the third movement of the concert.

The modulation goes from the rhythmic macro germ $G^*$ to its symmetric image $H^* = R(G^*)$ under the retrograde motion $R$ which fixes the seventh tone (first tone of the seventh degree). The lower half of figure 28.13 shows the retrograde germ with the transported satellites. This is the germ for the "rhythmic target scale". According to the modulation theory (well-tempered case), there is a determined set of pivots related to a selected cadence in the target scale, and we can get off with the explicit construction.

In the "Synthesis" concert, we first set up six bars in order to define the start rhythm scale. In each bar 1-5, a new tone of the start rhythm scale is added, and two tones are added in bar 6, so we have the complete scale. Observe that this addition of tones means that each added tone can be repeated in the successive bars and thereby enriches the previous rhythms.

From bar 7 to bar 12, the tones of $G^*$ are successively removed so that we have a neutralization process here. The modulator is made evident by a rhythmic motif which is built around the above retrograde symmetry $R$, including its repetition during the whole second modulation segment presenting the pivots, from bar 13 to bar 23.

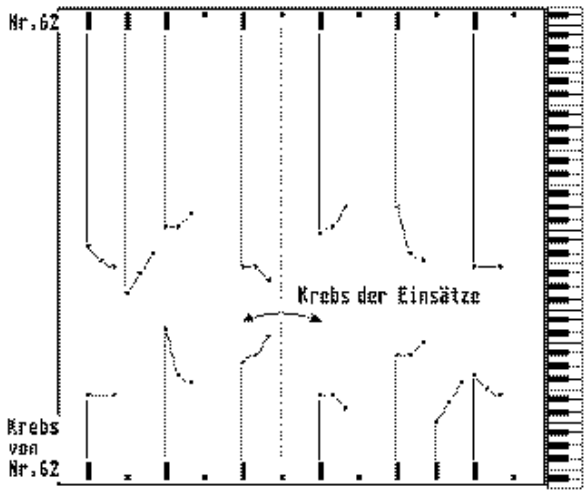


Figure 28.13: Above: The rhythmic macro germ $G^*$ of the rhythmical modulation in the first movement of the "Synthesis" concert [339]. Below: The retrograde of $G^*$ with the satellites being carried over to the transformed onset-percussion events.

Already in bar 21, the new tonic of $H^*$, which is (by definition) the image of the old tonic—with its characteristic rhythmic motif satellite—, is played and remains alive until the end of the modulatory process.

This means that the new first onset is no more the old one, but occurs on onset 9/8 (modulo 12/8) of the old bar onset. This change is realized by a new bar onset at onset 9/8 of bar 24. From this point on, we have 5 bars to confirm the target rhythm and to terminate the modulation.

Both cadences, that of the start and that of the target rhythm, are accompanied by a kind of regular falling drop sequence of two eights duration tones in order to stress the beginning of the respective bar units.

Recall from section 13.4.3.1 that the macro rhythms can be flattened down to the sets in the effective form $PerOns_{12}$ of rhythmic events. But the construction of the effective material had to be performed on a higher conceptual level of macro events. Probably this kind of conceptual upgrading of modulatory processes could also be used to understand existing and construct new harmonic modulations, or even melodic modulations if a modulation model of motivic structures (in $OnPiMod_{n,m}$, say) is available.

# Part VII

# Counterpoint

# Chapter 29

# Melodic Variation by Arrows

*punctus est cuius pars nulla est*
Euclid [215, book I]

**Summary.** The ideas of tangent objects described in section 7.5 are applied to define contrapuntal intervals as tangential "arrows" from the cantus firmus to the discantus tone. This formalism fits with the idea that the discantus is a kind of *melodic variation around the cantus firmus line*. The ring structure of the set of such arrows is discussed and motivated from the musical perspective.

– Σ –

The mathematical theory of counterpoint is an excellent subject to illustrate the idea of mathematical conceptualization of musical and musicological objects. It shows that mathematical subtleties are no formal overhead and can help to grasp music(ologic)al differences at the highest level. After the introduction of arrows as a formal restatement of the contrapuntal interval concept, we shall see in section 30.2.1 that this setup provides us with an astonishing relation between harmonic and contrapuntal objects, a relation which was never observed by musicological tradition.

This insight could have deep consequences for the understanding of the hitherto unresolved transition process from polyphony to homophony, from counterpoint to harmony as a basis of musical composition.

## 29.1 Arrows and Alterations

**Summary.** Arrows are a conceptual refinement of ordered intervals in the pitch domain. We compare these and related concepts and work out their specific differences.

– Σ –

Most of the common modules of simple space forms describe aspects of elementary music objects which can be related to an abstraction from physical events. However, musical reflection

is a mental and symbolic approach, and this is not only the case on the higher level of compound object interaction as it was developed in the theory of global compositions. We already observe such a symbolism on the most elementary understanding of what we mean when talking about a tone.

We are referring to the fact that in systematic musicology, some German specialists (see e.g., [205]) distinguish between "Ton" (tone) and "Tonort" (a tone's place), when discussion alterations. For example, in the 12-tempered system, the Tonort of $f_\sharp$ is the same as the Tonort of $g_\flat$, whereas these two tones are thought to be different since their origin, one $f$, one $g$, expresses a musical thinking that cannot be traced anymore on the level of the common Tonort.

This understanding is based on the concept of musical objects as being grown, dynamic entities and not merely static states. In this approach, a tone is not a point but a *variation* of a point, something that has a direction from past to present, in short: an 'arrow'. This unveils a fundamental stream of musical thinking: that of varied structures.

In this part, we want to discuss a central instance of this type of thinking, namely counterpoint. The objective is to interpret contrapuntal intervals as "arrows" and to deduce the basic rules of counterpoint. We cannot present a general theory of varied structures yet. Instead, we shall give a prototypical sketch of how such a theory could look.

In the following, we suppose that we are working in a form space $S_q$ of simple type, module[1] $M$ and identifier automorphism $q : M \xrightarrow{\sim} M$ (giving rise to the functorial identifier automorphism $@q$). For example: $S_{Id} = EulerModule$ with $M = \mathbb{Q}^3$, or $S_1 = PiMod_{12}$ with $M = \mathbb{Z}_{12}$, or $S_{(7)} = FiPiMod_{12} = PiMod_{12,(7)}$, the fifth identifier pitch class space, etc. In contrapuntal reasoning, one is not primarily interested in single pitches, but in "arrows" starting from a basic pitch $x$: the *cantus firmus* and ending in a target pitch $y$, the *discantus*. In order to catch this information, one could come back to the arrow approach exposed in section 7.2.3, i.e., to the $\mathbb{Z}$-addressed denotators $D : \mathbb{Z} \rightsquigarrow S_q(ON = x, OFF = y)$. For the following reason we refrain from this point of view[2]: The $\mathbb{Z}$-addressed denotators have too little canonical algebraic structure for our purposes. We shall see this in a moment.

The formalization we choose for counterpoint is the dual numbers approach described in section 7.5. The module $M$ is enriched to the dual numbers module $M[\varepsilon]$, and we also admit the presence of the identifier module automorphism $q : M \xrightarrow{\sim} M$ which induces the canonical automorphism $q[\varepsilon]$ of $M[\varepsilon]$; call the corresponding space $S_q[\varepsilon]$. For example we would have $EulerModule_q[\varepsilon] \underset{@q}{\longrightarrow} \mathbf{Simple}(\mathbb{Q}^3[\varepsilon])$, more generally, with the notations of appendix G.5.3:

$$S_q[\varepsilon] \underset{Id(S_q)[\varepsilon]=@(q[\varepsilon])}{\longrightarrow} \mathbf{Simple}(coord(S_q)[\varepsilon])$$

for an automorphism $q$ of the coordinator module $coord(S_q)$. In this setting, if $T$ is an address, a *$T$-addressed contrapuntal interval of space form $S_q$* is a $T$-addressed denotator $D : T \rightsquigarrow S_q[\varepsilon](x + \varepsilon.i)$, $x + \varepsilon.i \in T@M = T@coord(S_q)$. If $q = Id_M$, we also omit the $q$-index, and without any further address specification, we tacitly suppose the zero address. There are four canonical surjections [3]

$$p_{cf}, p_{int}, \alpha_+, \alpha_- : S_q[\varepsilon] \to S_q$$

[1] Always over a commutative ring $R$ in this context.

[2] This point of view is however the adequate for harmonic considerations as we have already learned in the self-addressed theory.

[3] Pay attention to the fact that these surjections are operating on the form functors and not on the frames.

associated with the first (cantus firmus) and second (interval) projections $M[\varepsilon] \to M$ and with the synonymous alterators of sweeping and hanging orientation $\alpha_+, \alpha_-$ introduced and motivated musicologically in section 7.5. In section 29.5 we shall also consider more general addresses (in particular self-addressed contrapuntal intervals), but for the time being, we stick to the zero-addressed contrapuntal intervals.

The enrichment of the dual number modules $M[\varepsilon]$ des not lie in the $R$-linear structure, it is based on the $R$-algebra structure of $R[\varepsilon]$ with $\varepsilon^2 = 0$. In other words, the well-known possibility of address killing

$$X@B\tilde{@}S_q = X \boxtimes B@S_q$$

described in section 11.2 could lead to an identification of $\mathbb{Z}$-addressed arrows in $S_q$ with zero-addressed arrows in $\mathbb{Z}@S_q$, but we need an additional algebraic structure on the latter form space.

## 29.2 The Contrapuntal Interval Concept

**Summary.** This section presents a formal definition of contrapuntal intervals or *arrows*, together with the two possible orientations of *sweeping* and *hanging* counterpoint.

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The point of departure of this theory is the fact that the known physical and mathematical concepts of consonance and dissonance are not adequate to the musical paradigm of an antagony of interval categories since they only conceptualize gradual changes in sonance. This unsatisfactory situation was also recognized by Hugo Riemann [456] in his critique of Euler and Helmholtz. Our problem is to turn this antagonistic paradigm into a mathematical concept framework in order to deduce the announced counterpoint model.

Our requirements would not be satisfied if we thought of an interval as being a set of two pitch events. This would in particular not do justice to the concept of voices. The cantus firmus could not be distinguished from the discantus, and the crossing of voice could not be conceived. It is adequate to contrapuntal reasoning to distinguish a basic cantus firmus tone from the dependent discantus tone, i.e., to consider arrows as defined above, and to indicate the orientation.

In the sense of this musical requirement, we shall now describe *an oriented contrapuntal interval* formally as a pair

$$(\alpha_\pm, D : 0 \rightsquigarrow S_q[\varepsilon](x + \varepsilon.i))$$

of a sweeping or hanging orientation plus a contrapuntal interval $D$ as defined above. When the rest is clear, we shall also reduce the description to the simple pair $(\alpha_\pm, x + \varepsilon.i)$ or even $x + \varepsilon.i$, if the orientation is clear. Whereas here, $x$ is the cantus firmus of the contrapuntal interval, the quantity $x \pm i$ is the discantus of the interval.

Within a given orientation, the two-part style "note against note" is now interpreted as a sequence

$$x_s + \varepsilon.i_s, s = 1, 2, 3, \ldots$$

of arrows. If a change of orientation happens, the indexes have to be split into regions of constant orientation. The historical development of the counterpoint of many parts is now reflected in

the historically variable characteristic types of geometric variations of the Gregorian chant of the cantus firmus' melodic line by arrows. Other tone attributes, such as duration or loudness, are neglected in this elementary motivic exposition. Except some minor remarks concerning just pitch spaces, we shall focus our discussion to the 12-tempered pitch class space $PiMod_{12,q}$ and its contrapuntal space $PiMod_{12,q}[\varepsilon]$ of pitch class arrows, $q$ being an automorphism , e.g., one of the four circle automorphisms $q = (1), (5), (7), (11)$, of $\mathbb{Z}_{12}$. This is quite congruent with the contrapuntal reasoning. For example, the octave extension of a perfect consonance is again perfect, idem for imperfect consonances and for dissonances [468]. So after this reduction we are left with 144 contrapuntal intervals for each orientation, the arrows in $PiMod_{12,q}[\varepsilon]$.

## 29.3 The Algebra of Intervals

**Summary.** The set of arrows is canonically provided with the structure of an algebra. The formal definition and first mathematical properties are exposed.

– Σ –

The module $\mathbb{Z}_{12}[\varepsilon]$ of $PiMod_{12,q}[\varepsilon]$ is not only a $\mathbb{Z}_{12}$-module but also a $\mathbb{Z}_{12}$-algebra, i.e., a module with a bilinear, associative and unitary multiplication defined by the nilpotent element $\varepsilon$ with $\varepsilon^2 = 0$ (see also example 76 in appendix D.1.1). The product of contrapuntal intervals $a + \varepsilon.b, c + \varepsilon.d$ is defined by

$$(a + \varepsilon.b).(c + \varepsilon.d) = ac + \varepsilon.bc + \varepsilon.ad + \varepsilon^2.bd = ac + \varepsilon.(bc + ad)$$

which yields a new contrapuntal interval. An invertible element of $\mathbb{Z}_{12}[\varepsilon]$ is an element $a + \varepsilon.b$ with $a = 1, 5, 7, 11$, and the inverse element is $(a + \varepsilon.b)^{-1} = a - \varepsilon.b$. Corresponding to the symmetries in $\mathbb{Z}_{12}$, the symmetries in $\mathbb{Z}_{12}[\varepsilon]$ are the affine endomorphisms (over the ring $\mathbb{Z}_{12}[\varepsilon]$)

$$e^{a+\varepsilon.b}(u + \varepsilon.v)$$

which is invertible iff $u = 1, 5, 7, 11$.

### 29.3.1 The Third Torus

**Summary.** The third torus is the geometric representation of $\mathbb{Z}_{12}$ associated with its Sylow decomposition[4] . We discuss the mathematical structure and its metrical properties.

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The module isomorphism $T : \mathbb{Z}_{12} \xrightarrow{\sim} \mathbb{Z}_3 \oplus \mathbb{Z}_4 : z \mapsto (z(mod 3), -z(mod 4))$ with its inverse $T^{-1}(z_3, z_4) = 4.z_3 + 3.z_4$ identify the cyclic module $\mathbb{Z}_{12}$ with a discrete torus. Under this isomorphism, we have an isomorphism of forms $T : PiMod_{12} \xrightarrow{\sim} PiThirds_{3,4}$ with the third class form $PiThirds_{3,4} \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z}_3 \oplus \mathbb{Z}_4)$. Hereby, the first component of a $T$-transformed pitch class is the number of its major thirds while the second component is the number of minor thirds that add to the given pitch when counted from zero. For example, $T(7) = (1, 1)$, i.e., "fifth = major third plus minor third", see figure 29.1.

[4] See theorem 41 in appendix C.3.

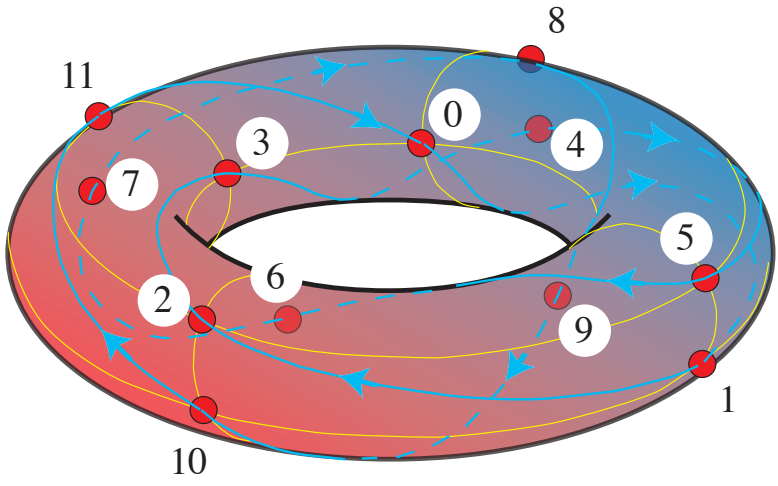


Figure 29.1: On the torus of thirds, the third relations among pitch classes and intervals are represented in a geometric way. The sequence of semi-tone steps appears as an entwined closed spiral.

To describe 'pure intervals' in pitch classes, starting from the form

$$IntMod_{12} \underset{Id}{\longrightarrow} \mathbf{Syn}(PiMod_{12})$$

for interval quantities in pitch classes, we also use the third torus structure as a synonym for interval quantities in terms of thirds, i.e.,

$$IntThirds_{3,4} \underset{Id}{\longrightarrow} \mathbf{Syn}(PiThirds_{3,4}).$$

As in differential geometry, one has the space of contrapuntal intervals $IntMod_{12,q}[\varepsilon]$ or the corresponding third tangent torus

$$IntThirds_{3,4,q}[\varepsilon]$$

in terms of a cantus firmus torus where twelve "tangent" tori are attached: The contrapuntal intervals are viewed as tangents to their cantus firmus points: A tangent torus $I_x = x + \varepsilon.\mathbb{Z}_{12}$ is attached at each of its points $x$, see figure 29.2.

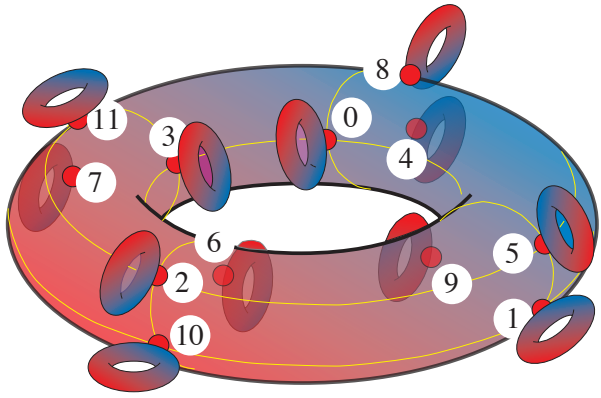


Figure 29.2: The contrapuntal intervals viewed as tangents to their cantus firmus points.

Let us review the affine automorphism group $\overrightarrow{GL}(\mathbb{Z}_{12})$ in terms of the Sylow representation of $\mathbb{Z}_{12}$ as a discrete torus. The group is evidently generated by these four symmetries:

$$\begin{aligned} c_1 &= e^0.11 \text{ multiplication by } -1;\ \text{inversion} \\ c_2 &= e^0.5 \text{ multiplication by } 5;\ \text{make fourth circle} \\ c_3 &= e^3.1 \text{ addition of constant } 3;\ \text{minor third chain} \\ c_4 &= e^4.1 \text{ addition of constant } 4;\ \text{major third chain} \end{aligned}$$

This system is not only musically meaningful and therefore turns the group $\overrightarrow{GL}(\mathbb{Z}_{12})$ into a musically meaningful group by the concatenation principle 2, it also shows that all automorphisms preserve the third distance in the following sense: Let $x, y \in \mathbb{Z}_3 \oplus \mathbb{Z}_4$. Define $d(x, y)$ as the minimal number of ascending or descending minor or major third steps on the third torus to move from $x$ to $y$; for example: $d(0, 9) = 1$ since $0 - 3 = 9$, $d(0, 1) = 2$ since $0 + 4 - 3 = 1$, etc. The distance is just the minimal number of edges on the discrete third torus that connects the two points $x$ and $y$. This distance, in fact a metric, is left invariant under each of the four generators $c_1$ to $c_4$, see figure 29.3 for the evidence.—Summarizing:

**Proposition 49** *The group $\overrightarrow{GL}(\mathbb{Z}_{12})$ leaves invariant the third distance, i.e., it is a group of isometries of the interval torus $IntThirds_{3,4}$.*

## 29.4 Musical Interpretation of the Interval Ring

**Summary.** This section deals with musical interpretations of the operations in the interval ring. We take this occasion to deepen the topic of mathematical "overhead" structures and their role in building musical concepts.

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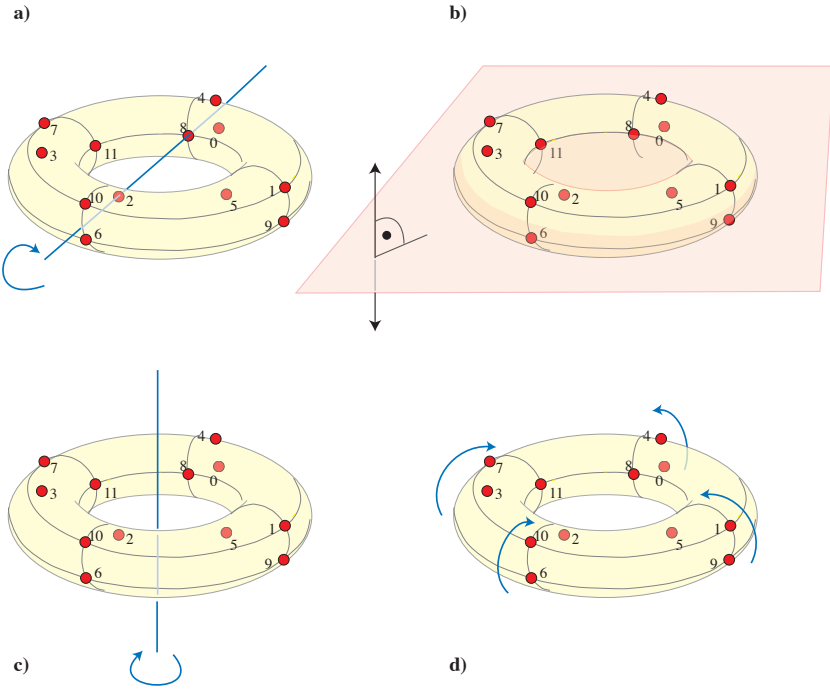


Figure 29.3: The elementary symmetries on the third torus. The generator $c_1$ is a 180°-rotation around the axis through 0 and 6; $c_2$ is a reflection at the torus' equatorial plane; $c_3$ is a 90°-rotation around the polar axis; $c_4$ is a tilting movement in by 120°. Each of these generators preserves the third distance on the torus.

*In this section we only deal with invertible symmetries.*

We want to understand what could be the musical meaning and interpretation of multiplication in $PiMod_{12}[\varepsilon]$, or, mathematically equivalent, in $IntMod_{12,q}[\varepsilon]$. This is indeed a crucial situation. A priori, music theory never considered such a multiplication of intervals, and a conservative attitude could very well prohibit such an extension as an inadmissible mathematical overhead. But this is a classical situation with mathematical objects in the sciences, be it in physics, chemistry or musicology: Mathematical structures have more properties than their application needs. So there is always a degree of mathematical overhead. The question is rather whether ingredients of that overhead can be enriched by meaning within the applying science. More precisely, we are not preconizing the power of a conceptual oracle with mathematical properties. We simply associate mathematical properties with musical properties (in our case) and try to profit from the semantical added value in favor of a richer mathematization of musical phenomena.

As already exposed in section 8.3, we shall again use the concatenation principle 2 from section 8.3 here. This means that understanding a symmetry in $e^t.g \in \overrightarrow{GL}(\mathbb{Z}_{12}[\varepsilon])$ means understanding its factors in a determined factorization. As in section 8.3, we shall present a system of 'elementary', musically understandable symmetries which generates all symmetries.

To begin with, we shall treat the translations $e^t = e^t.1$ and the multiplications $g$ separately. For a translation quantity $t = a+\varepsilon.b$, we have $e^t = e^{a+\varepsilon.b} = e^a.e^{\varepsilon.b}$. Furthermore, $g = u+\varepsilon.v = u.(1+\varepsilon.uv) = u.(a+\varepsilon.1)^{uv}$.

This yields the following system:

1. $s_1 = e^a$,
2. $s_2 = e^{\varepsilon.b}$,
3. $s_3 = u, u = 1, 5, 7, 11$,
4. $s_4 = 1 + \varepsilon.1$.

Here are the musical interpretations of the system's symmetries:

**1.** $s_1 = e^a$**.** Symmetry $s_1$ causes a transposition if the interval by $a$ since we have $s_1(x+\varepsilon.y) = (a+x+)+\varepsilon.y$.

**2.** $s_2 = e^{\varepsilon.b}$**.** Symmetry $s_2$ causes a transposition of the discant voice by $b$ while the cantus firmus remains unaltered, i.e., $s_2(x+\varepsilon.y) = a+\varepsilon.(y+b)$. this constant enlargement of the distance between cantus firmus and discantus corresponds to a traditional technique of double counterpoint.

An example of an interesting symmetry, a concatenation of types 1 and 2, is given for $t = a \pm \varepsilon.1$. For $t = a+\varepsilon.1$, minor intervals are transformed into corresponding major intervals under the transposition by $a$. Conversely, $t = a+\varepsilon.1$ transforms major intervals into corresponding minor ones under the transposition by $a$. This means that also not 'strict' parallels of thirds, such as

$$2+\varepsilon.3, 7+\varepsilon.4 = e^{5+\varepsilon.1}(2+\varepsilon.3), 11+\varepsilon.3 = e^{4-\varepsilon.1}(7+\varepsilon.4)$$

are viewed as translation symmetries.

**3.** $s_3 = u, u = 1, 5, 7, 11$**.** Symmetries of this type multiply the cantus firmus, and the distance between the voices, and hence also the discantus, by $u$. Since we cannot reduce this discussion to the involved pair of tones, and hence not to the basic pitch domain $\mathbb{Z}_{12}$, we must have a closer look at this situation.

**3.1** $u = 11$**.** The cantus firmus is reflected at the origin $c = 0$ whereas the voice distance $11.b$ means the octave complement to $b$. For example, the sweeping minor third $d$, $f(= 2+\varepsilon.3)$ is transformed into the sweeping major sixth $b_\flat$, $g(= 10+\varepsilon.9)$. Observe that, by definition, symmetries do not alter the underlying orientation.

**3.2** $u = 5$**.** Musically, the distance $b$ of the interval tones can be of equal interest as a multiple of minor seconds $(= 1)$ or of fourths $(= 5)$. The symmetry $u = 5$ connects intervals of equal values in these two perspectives. In a multiplication by 5, the *minor second distance* $b$ of an interval $a+\varepsilon.b$ is transformed into the *fourth distance* $5.b$.

**3.3** $u = 7$**.** For this value, we have an analogous argument as for the preceding case, with fifths instead of fourths. One could also invoke the concatenation principle and use the factorization $7 = 5.11$ to reduce this case to the cases 3.1 and 3.2.

**4** $s_4 = 1 + \varepsilon.1$**.** With this symmetry, the cantus firmus remains fixed whereas the voice distance increases by the distance of the cantus firmus from the origin $c = 0$. Thereby, the cantus firmus acquires a new function, i.e., to appear itself as a 'discantus' with respect to the origin. Thus its distance to the origin is added to the given discantus as a *fixed reference quantity*. If we repeat the application of this symmetry to the resulting interval, the same reference quantity is again added to the discantus, etc. Repeated application of $s_4$ therefore generates a circle on the discantus of the original interval, a circle whose step width is defined by the cantus firmus' distance to the origin. Musically speaking, the origin could be imagined as being a tonic. The appearance of the origin as a reference pitch is only justified by our choice of the generator $s_4$ in the given system. If we took $e^{-\varepsilon.b}.s_4$ instead, then $b$ would play the role of the reference pitch.

**Example 48**

$$\begin{aligned} 3 + \varepsilon.4 \mapsto\ & 3 + \varepsilon.7 = (1 + \varepsilon.1).(3 + \varepsilon.4) \mapsto \\ & 3 + \varepsilon.10 = (1 + \varepsilon.1).(3 + \varepsilon.7) \mapsto \\ & 3 + \varepsilon.1 = (1 + \varepsilon.1).(3 + \varepsilon.10) \mapsto \\ & 3 + \varepsilon.4 = (1 + \varepsilon.1).(3 + \varepsilon.1) \end{aligned}$$

**Example 49**

$$\begin{aligned} 6 + \varepsilon.3 \mapsto & 6 + \varepsilon.9 = (1 + \varepsilon.1).(6 + \varepsilon.3) \mapsto \\ & 6 + \varepsilon.3 = (1 + \varepsilon.1).(6 + \varepsilon.9) \end{aligned}$$

By the way, this symmetry type is the only one within our generator system which connects the cantus firmus and the discantus components in an irreducible manner. This symmetry type crystallizes an essential difference to the common reasoning in the pitch class space $PichMod_{12}$.

## 29.5 Self-addressed Arrows

**Summary.** Since arrows formally behave much like tones, self-addressed arrows can be introduced as a natural generalization of self-addressed tones and (ordinary) arrows via *address change* according to 8.3.4. This extension is described—together with a canonical projection which will play a major role in the theory of consonances and dissonances in section 30.2.1.

In this section, we want to concentrate on the algebraic relations between self-addressed tones and contrapuntal arrows. Fix an arbitrary commutative ring $R$ and consider the canonical $R$-linear injection $i : R \to R[\varepsilon]$. We also have the functorial $R$-linear injection $[\varepsilon] : R@_R R \to R[\varepsilon]@_{R[\varepsilon]}R[\varepsilon] : f \mapsto f[\varepsilon]$ defined in section 7.5. Furthermore, we have two $R$-linear address change injections $c : R \xrightarrow{\sim} 0_R@_R R \to R@_R R, c[\varepsilon] : R[\varepsilon] \xrightarrow{\sim} 0_R@_R R[\varepsilon] \to R[\varepsilon]@_{R[\varepsilon]}R[\varepsilon]$. This entails the following commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\ i\ } & R[\varepsilon] \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c[\varepsilon]} \\ R@_R R & \xrightarrow{\ [\varepsilon]\ } & R[\varepsilon]@_{R[\varepsilon]}R[\varepsilon] \end{array} \tag{29.1}$$

of $R$-linear injections. Whereas the left lower corner parametrizes the self-addressed tones in a simple form of module $R$, and the right upper corner parametrizes the contrapuntal intervals in the module $R$, the right lower corner parametrizes the self-addressed contrapuntal intervals, as opposed to the zero-addressed tones in the module $R$, or the prime counterpoint intervals, respectively.

The four-dimensional $R$-module $R[\varepsilon]@_{R[\varepsilon]}R[\varepsilon]$ can also be viewed as a left $R[\varepsilon]$-module by the ordinary composition $a.x = e^0.a.x$ with linear endomorphisms $e^0.a, a \in R[\varepsilon]$. Then we have a direct decomposition in one-dimensional $R[\varepsilon]$-modules:

$$R[\varepsilon]@_{R[\varepsilon]}R[\varepsilon] = R[\varepsilon].e^1 0 \oplus R[\varepsilon].e^{-\varepsilon}1 \tag{29.2}$$

i.e., two two-dimensional $R$-modules. Moreover, this decomposition is also a kernel-image decomposition with respect to the idempotent[5] right multiplication endomorphism $?.e^\varepsilon 0$, of $R[\varepsilon]$-modules, i.e.,

$$Ker(?.e^\varepsilon 0) = R[\varepsilon].e^{-\varepsilon}1, Im(?.e^\varepsilon 0) = R[\varepsilon].e^1 0.$$

Now, the image is exactly the image of $R[\varepsilon]$ under $c[\varepsilon]$, whereas $?.e^\varepsilon 0$ maps the image of $R@_R R$ under $[\varepsilon]$ isomorphically (as $R$-module) onto the image $Im(?.e^\varepsilon 0)$. in other words:

**Theorem 32** *With respect to the embeddings of diagram (29.1), the projection $?.e^\varepsilon 0$ associates bijectively the self-addressed tones with the contrapuntal intervals and leaves the original tones in $R$ invariant. This means that self-addressed tones and contrapuntal intervals are put under a canonical algebraic correspondence in the large space of self-addressed arrows*

## 29.6 Change of Orientation

**Summary.** Orientation within a contrapuntal sequence may change, and enforce a special treatment of contrapuntal steps from sweeping orientation to hanging orientation or vice versa. We show how such a change can be reduced to an orientation preserving situation by means of the regular embedding of the algebra of dual numbers in the linear endomorphism ring.

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[5] See also appendix C.2.3, example 68.

The left-regular embedding[6]

$$(?) : R[\varepsilon] \rightarrowtail \mathbb{M}_{2,2}(R)$$

identifies the two-dimensional dual number algebra with a subspace of the ring of two-by-two matrices over $R$. In this embedding, the dual number algebra is generated by the dual number multiplication

$$(\varepsilon) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with the relation $(\varepsilon)^2 = 0$. The sweeping and hanging orientations are interpreted as a projection

$$\alpha_+ = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \alpha_- = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

which are related by the multiplicative relation

$$\alpha_- = -\alpha_+(1+\varepsilon)^{-2}. \tag{29.3}$$

The matrix algebra $\mathbb{M}_{2,2}(R)$ is generated by two indeterminates $(\varepsilon), \alpha_+$ with relations $(\varepsilon)^2 = 0, \alpha_+^2 = \alpha_+$, and $\alpha_+.(\varepsilon)+(\varepsilon).\alpha_+ = (1+\varepsilon)$, and it is spanned by the linear basis $1, (\varepsilon), \alpha_+, \alpha_+.(\varepsilon)$.

Relation (29.3) can be used to reinterpret hanging counterpoint in terms of sweeping counterpoint as follows: Suppose that we have a sequence $x_1 + \varepsilon.i_1, x_2 + \varepsilon.i_2$ with the first interval in sweeping, but the second one in hanging orientation. This means that the evaluation via orientation projections produces the two discantus instances $\alpha_+(x_1+\varepsilon.i_1), \alpha_-(x_2+\varepsilon.i_2)$. In order to change the interpretation of orientations, use (29.3) and rephrase the second discantus as

$$\alpha_-(x_2+\varepsilon.i_2) = -\alpha_+(1+\varepsilon)^{-2}(x_2+\varepsilon.i_2) = \alpha_+(-(1+\varepsilon)^{-2}(x_2+\varepsilon.i_2))$$

so that we are dealing with sweeping orientation related to the new interval

$$-(1+\varepsilon)^{-2}(x_2+\varepsilon.i_2).$$

This technique will be used to deduce contrapuntal steps while orientation changes. Observe that the mediator factor $(1+\varepsilon)$ in formula (29.3) enhances the musical meaning of the fourth generator symmetry $s_4$ in section 29.4: *The multiplication by generator $s_4$ helps to reinterpret hanging orientation in terms of sweeping orientation.*

[6]See appendix D.1.

# Chapter 30

# Interval Dichotomies as an Expression of Contrast

*enim vero sicut vitium mala virtus*
*a nullo umquam morali philosopho dictum fuit,*
*ita nec musicus umquam litteratus*
*discordiantiam malam concordantiam nuncupavit.*
Johannes Tinctoris [527, p.90]

**Summary.** For contrapuntal composition and theory, consonant and dissonant intervals are a dichotomic concept. We present the mathematical restatement and fundamental properties of the basic concept of an *interval dichotomy*. For the classification of dichotomies, a strong condition on unique *symmetries of polarity* between the two halves of dichotomies is added. It reveals a distinguished role of the *consonance/dissonance* dichotomy of classical counterpoint and of the *major* dichotomy (associated with the major scale). We discuss evidence of the consonance/dissonance dichotomy from theoretical and empirical points of view and open the discourse to an intercultural perspective guided by the classification of dichotomies, as investigated by Jens Hichert [223].

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**Remark 16** In the following counterpoint chapters, we shall tacitly work in the fifth pitch and interval spaces, i.e., we take the automorphism $q = (7)$ in the identifiers. We therefore shall—for example—speak of the fifth when addressing the unit 1. Whenever we deviate from this convention, the reader should be warned.

We shall also stick to the forms built upon $\mathbb{Z}_{12}$ and tacitly carry over the distance structures of the isomorphic Sylow torus representation $\mathbb{Z}_3 \oplus \mathbb{Z}_4$ in order to keep notation simpler.

## 30.1 Dichotomies and Polarity

**Summary.** The technical definition of *interval dichotomies* in $IntMod_{12,q}$ and in the counterpoint arrow form $IntMod_{12,q}[\epsilon]$ is given[1]. We discuss canonical *polarities*, in fact automorphisms of such dichotomies, as well as their topological behavior which is measured by *diameter* and *span* of a dichotomy. This is used to classify the system of dichotomies. Among the 26 classes, six classes admit unique polarities. These classes of *strong* dichotomies show a topologically distinguished position of the *consonance/dissonance* and the *major* dichotomy, the latter being defined by the six tonic-rooted (proper) intervals of major tonality. Among the strong dichotomies, the consonance-dissonance dichotomy itself (not only its class) is distinguished by a number of characteristics.

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Let $A$ be an addressed $\mathbb{Z}$-module. Then an objective, $A$-addressed local composition $X$ in ambient space $IntMod_{12,q}$ is called a ($A$-addressed) *marked interval dichotomy* iff it is equipollent[2] to its complement $C(X)$ in the total local composition $A@IntMod_{12,q}$. Observe that the latter is not necessarily finite. Hence we have the *complement action* $i \mapsto C^i(X)$ of $\mathbb{Z}_2$ on the set $DiM(A, IntMod_{12,q})$ of $A$-addressed marked dichotomies. An *interval dichotomy* is a $\mathbb{Z}_2$-orbit of marked interval dichotomies, the orbit set is $Di(A, IntMod_{12,q})$.

A second left action is defined by the automorphism group $\overrightarrow{GL}(\mathbb{Z}_{12})$ of the ambient space $IntMod_{12,q}$. If $g \in \overrightarrow{GL}(\mathbb{Z}_{12}), X \in DiM(A, IntMod_{12,q})$, then we set $g.X = \{g.x | x \in X\}$, and the set $CiM(A, IntMod_{12,q})$ denotes the orbit space of this action, its elements are called the ($A$-addressed) *marked dichotomy classes*. Clearly the two actions commute, i.e., we have a left action of $\mathbb{Z}_2 \times \overrightarrow{GL}(\mathbb{Z}_{12})$ on $DiM(A, IntMod_{12,q})$, an induced action of $\overrightarrow{GL}(\mathbb{Z}_{12})$ on $Di(A, IntMod_{12,q})$, and one of $\mathbb{Z}_2$ on the space of marked dichotomy classes $CiM(A, IntMod_{12,q})$. The total orbit space

$$Ci(A, IntMod_{12,q}) = \mathbb{Z}_2 \times \overrightarrow{GL}(\mathbb{Z}_{12}) \backslash DiM(A, IntMod_{12,q})$$

is the space of ($A$-addressed) *dichotomy classes*.

The same constructions (mutatis mutandis) can be made on the space of counterpoint arrows $IntMod_{12,q}[\epsilon]$. Let $A$ be an addressed $\mathbb{Z}$-module. Then an objective, $A$-addressed local composition $X$ in ambient space $IntMod_{12,q}[\epsilon]$ is called a ($A$-addressed) *marked counterpoint dichotomy* iff it is equipollent to its complement $C(X)$ in the total local composition $A@IntMod_{12,q}[\epsilon]$. On the set $DcM(A, IntMod_{12,q}[\epsilon])$ of $A$-addressed marked counterpoint dichotomies, we have the *complement action* $i \mapsto C^i(X)$ of $\mathbb{Z}_2$. A *counterpoint dichotomy* is a $\mathbb{Z}_2$-orbit of marked counterpoint dichotomies, the orbit set is $Dc(A, IntMod_{12,q}[\epsilon])$.

A second left action is defined by the automorphism group $\overrightarrow{GL}(\mathbb{Z}_{12}[\epsilon])$ of the ambient space $IntMod_{12,q}[\epsilon]$. If $g \in \overrightarrow{GL}(\mathbb{Z}_{12}[\epsilon]), X \in DcM(A, IntMod_{12,q}[\epsilon])$, then we set $g.X = \{g.x | x \in X\}$, and the $CcM(A, IntMod_{12,q}[\epsilon])$ denotes the orbit space of this action, its elements are called the ($A$-addressed) *marked counterpoint dichotomy classes*. Clearly the two actions commute, i.e., we have a left action of $\mathbb{Z}_2 \times \overrightarrow{GL}(\mathbb{Z}_{12}[\epsilon])$ on $DcM(A, IntMod_{12,q}[\epsilon])$, an induced action

[1] This means the interval form with the identifier defined by the automorphism $q$ of $\mathbb{Z}_{12}$, i.e., $IntMod_{12,q} \xrightarrow[Id]{} \mathbf{Syn}(PiMod_{12,q})$, etc.

[2] ...has same cardinality as...

of $\overrightarrow{GL}(\mathbb{Z}_{12}[\epsilon])$ on $Dc(A, IntMod_{12,q}[\epsilon])$, and one of $\mathbb{Z}_2$ on the space of marked counterpoint dichotomy classes $CcM(A, IntMod_{12,q}[\epsilon])$. The total orbit space

$$Cc(A, IntMod_{12,q}[\epsilon]) = \mathbb{Z}_2 \times \overrightarrow{GL}(\mathbb{Z}_{12}[\epsilon]) \backslash DcM(A, IntMod_{12,q}[\epsilon])$$

is the space of ($A$-addressed) *counterpoint dichotomy classes.*

**Definition 92** *A marked interval dichotomy $X$ is called* autocomplementary *if it is isomorphic to its complement $C(X)$, i.e., iff its dichotomy class coincides with its marked dichotomy class. The marked dichotomy $X$ is called* rigid *if its symmetry group is trivial. It is called* strong *if it is autocomplementary and rigid.*

If a marked dichotomy $X$ is autocomplementary, so is its complement. If a marked dichotomy is rigid, so is its complement. Hence, if $X$ is strong, so is its complement. So *autocomplementarity, rigidity, and strength are invariants of the dichotomy classes.* From the classification of zero-addressed objective local compositions in $PiMod_{12}$ in appendix L.1, we know that there are 34 classes of zero-addressed marked interval dichotomies (26 classes numbers 63 to 88, complements not counted twice, count twice the 8 numbers without $^*$). There are 26 interval classes, 8 autocomplementary classes, and 6 strong classes.

We often denote a marked dichotomy by $(X/C(X))$ and its class by $[C/C(X)]$, whereas a dichotomy is denoted by $(X|C(X))$ and its class by $[C|C(X)]$. Since the group of symmetries acts on the set of dichotomies, one can say that a symmetry stems not only from an isomorphism of the underlying marked dichotomies but is also associated with an isomorphism of the associated dichotomy. More precisely, if we are given a symmetry $f : (X/Y) \to (U/V)$ between two marked dichotomies, we get an isomorphism of the interpretations that are associated with the partitions $X \sqcup Y$ and $U \sqcup V$. Conversely, an isomorphism between such two interpretations is induced by two isomorphisms on two pairs of charts. But each such isomorphism gives automatically rise to an isomorphism of the other chart pair—the only thing which we lose is the order of charts, i.e., we are left with a transformation among the (non-marked) dichotomies. In this sense the unique non-trivial inner symmetry $p$ of a strong dichotomy $(X|Y)$, i.e., $p(X/Y) = (p(X)/p(Y)) = (Y/X)$ is also called its *polarity.*

**Example 50** Here are the six strong (marked and unmarked) dichotomies, the numbers referring to the classification table of local compositions in appendix L.1. The polarity of dichotomy number $n$ and identifier number $q$ is denoted by $p_{q,n}$, or $p_n$ for $q = 7$, if the context is clear. If an index is omitted, we tacitly suppose the fifth representation with $q = 7$.

1. The dichotomy Nr. 64

   $$\Delta_{7,64} = (I_7/J_7) = (\{1, 2, 3, 4, 5, 11\}|\{0, 6, 7, 8, 9, 10\})$$

   with polarity $p_{7,64} = 7.p_{1,64}.7 = e^{11}.11$ corresponding to the dichotomy

   $$\Delta_{1,64} = (I_1/J_1) = (\{2, 4, 5, 7, 9, 11\}|\{0, 1, 3, 6, 8, 10\}),$$

   in the semitone representation ($q = 1$), with polarity $p_{1,82} = e^5.11$. The dichotomy $\Delta_{7,64}$ arises when considering all proper (non-vanishing) intervals in a major scale when counted from the tonic.

2. The dichotomy Nr. 68

$$\Delta_{7,68} = (\{0, 2, 7, 8, 9, 11\}|\{1, 3, 4, 5, 6, 10\}),$$

with polarity $p_{7,68} = 7.p_{1,68}.7 = e^6.5$ corresponding to the dichotomy

$$\Delta_{1,68} = (\{0, 1, 2, 3, 5, 8\}|\{4, 6, 7, 9, 10, 11\})$$

in the semitone representation with $q = 1$, with polarity $p_{1,68} = e^6.5$.

3. The dichotomy Nr. 71

$$\Delta_{7,71} = (\{0, 2, 7, 8, 9, 11\}|\{1, 3, 4, 5, 6, 10\}),$$

with polarity $p_{7,71} = 7.p_{1,71}.7 = e^5.11$ corresponding to the dichotomy

$$\Delta_{1,71} = (\{0, 1, 2, 3, 6, 7\}|\{4, 5, 8, 9, 10, 11\})$$

in the semitone representation with $q = 1$, with polarity $p_{1,71} = e^{11}.11$.

4. The dichotomy Nr. 75

$$\Delta_{7,75} = (\{0, 2, 4, 7, 8, 11\}|\{1, 3, 5, 6, 9, 10\}),$$

with polarity $p_{7,75} = 7.p_{1,75}.7 = e^5.11$ corresponding to the dichotomy

$$\Delta_{1,75} = (\{0, 1, 2, 4, 5, 8\}|\{3, 6, 7, 9, 10, 11\})$$

in the semitone representation with $q = 1$, with polarity $p_{1,75} = e^{11}.11$.

5. The dichotomy Nr. 78

$$\Delta_{7,78} = (\{0, 2, 4, 6, 7, 10\}|\{1, 3, 5, 8, 9, 11\}),$$

with polarity $p_{7,78} = 7.p_{1,78}.7 = e^3.11$ corresponding to the dichotomy

$$\Delta_{1,78} = (\{0, 1, 2, 4, 6, 10\}|\{3, 5, 7, 8, 9, 11\})$$

in the semitone representation with $q = 1$, with polarity $p_{1,78} = e^9.11$.

6. This is the classical dichotomy Nr. 82

$$\Delta_{82} = \Delta_{7,82} = (K_7/D_7) = (\{0, 1, 3, 4, 8, 9\}|\{2, 5, 6, 7, 10, 11\}).$$

of contrapuntal consonances (left) and dissonances (right) in the fifth system. Its polarity is 'the' *autocomplementarity function* $p_{7,82} = 7.e^2.5.7 = e^2.5$ deduced from the known[3] autocomplementarity function $p_{1,82} = e^2.5$ of the consonance-dissonance dichotomy

$$\Delta_{1,82} = (K_1/D_1) = (\{0, 3, 4, 7, 8, 9\}|\{1, 2, 5, 6, 10, 11\})$$

in the semitone representation, as discussed in section 24.1.1.

[3] See [336], for example.

These strong dichotomies can also be represented as partitions of the interval torus in the Sylow representation $IntThirds_{3,4,q} \underset{(q)}{\longrightarrow} \mathbf{Syn}(IntThirds_{3,4})$ with the $q$-identifier (usually the fifth). Recall that we have the third distance $d(x, y)$ on this torus, and that it is the same as the distance with identifier $q = 1$.

**Definition 93** *Let $(X/Y)$ be a strong marked dichotomy in $IntThirds_{3,4,q}$. Then its* diameter *is defined by*

$$\delta(X/Y) = \frac{1}{2} \sum_{u,v \in X} d(u, v).$$

*By use of the polarity of a strong dichotomy one sees that $\delta(X/Y) = \delta(Y/X)$ so that we may define this number as the* diameter $\delta(Y|X)$ of the strong dichotomy $(X|Y)$. *Since evidently all dichotomies of a dichotomy class have the same diameters (recall that symmetries of the torus are isometries for the distance), we may define the diameter $\delta[X|Y]$ of a dichotomy class $[X|Y]$ by the diameter of any of its representatives.*

The diameter measures the average distance between points of one half of the dichotomy.

**Definition 94** *Let $(X/Y)$ be a strong marked dichotomy in $IntThirds_{3,4,q}$ with polarity $p$. Then its* span *is defined by*

$$\sigma(X/Y) = \sum_{u \in X} d(u, p(u)).$$

*Since any polarity is an involution in our context, one sees that $\sigma(X/Y) = \sigma(Y/X)$ so that we may define this number as the* span $\sigma(X|Y)$ of the strong dichotomy $(X|Y)$. *Since evidently all dichotomies of a dichotomy class have the same span (recall that symmetries of the torus are isometries for the distance), we may define the span $\sigma[X|Y]$ of a dichotomy class $[X|Y]$ by the span of any of its representatives.*

Diameter and span of our six strong classes are visualized in figure 30.1. Intuitively, the minimality of $\delta(K/D)$ means that the subsets $K$ and $D$ are separated in an optimal way on the torus (figure 30.1). The maximality of $\sigma(I/J)$ means that $I$ and $J$ are optimally mixed on the torus. If we stay on a point on $I$, and we want to go to another point of $I$ on a shortest path, then we often have to traverse a point of $J$, a phenomenon which never happens for $(K/D)$: Between any two consonant intervals there is always a shortest path which does not leave the consonant half. By the given polarities, all these statements about $K$ and $I$ are also valid for $D$ and $J$. The possible connections within $K$ and within $I$ (and the complements, respectively) are shown in figure 30.2.

In contrast to the graph of $I$, the graph of $K$ has no inner symmetry, i.e., every consonant interval is uniquely determined by its position on the graph. This means that the consonance-dissonance dichotomy has a privileged position among all strong dichotomies. But this does not yet exhibit the precise representative $(K/D)$ within the class $[K|D]$. The selection of the marked dichotomy $(K/D)$ is in fact realized by an algebraic condition: to require that the first half $X$ of an element $(X/Y)$ of the class $[K|D]$ be a multiplicative monoid. This condition was discovered by Noll [400]. This in fact exhibits the marked half of consonances against dissonance, these two parts are not equivalent from this point of view. Until now it is however not clear what are the structural consequences of the predicates which uniquely exhibit the consonances.

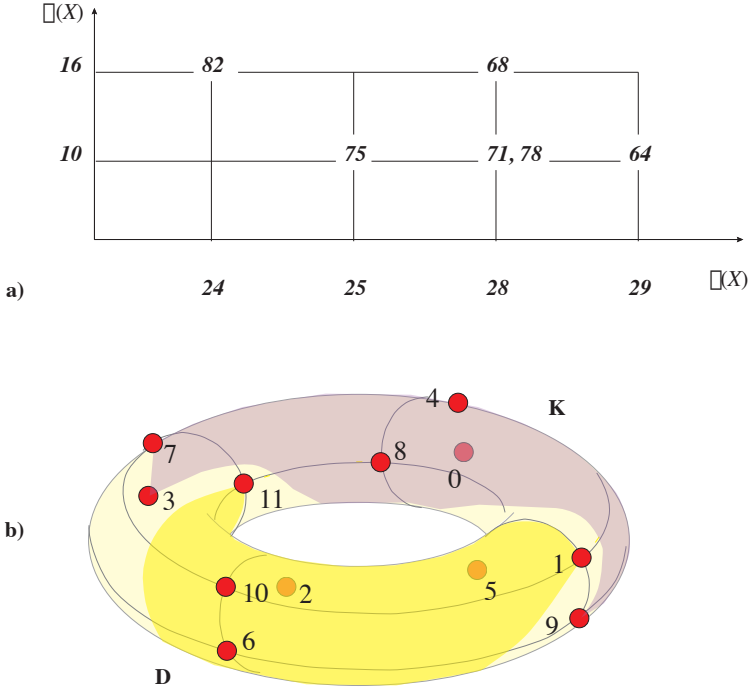


Figure 30.1: (a) Span and diameter of the six strong dichotomies. The polar position of the consonance-dissonance dichotomy (Nr. 82) against the major dichotomy (Nr. 64) is visible. (b) The geometric meaning of the minimal diameter and the maximal span in $(K/D)$ is evidenced by an optimal separation of the two halves of the dichotomy.

## 30.2 The Consonance and Dissonance Dichotomy

**Summary.** The consonance-dissonance dichotomy is in a canonical bijection with the Riemann consonances. We also give empirical evidence of this dichotomy from brain research. The section concludes with a cognitive interpretation of the specific role of music for the individual psyche.

– Σ –

Recall from section 24.1.1 that the consonance-dissonance dichotomy also appears in the context of just intonation and relates to the 12-tempered case by means of the enharmonic projection. We now want to investigate further remarkable properties of the $(K/D)$ dichotomy. To begin with, we define a (marked) dichotomy $(X[\varepsilon]/Y[\varepsilon])$ of contrapuntal intervals for every (marked) dichotomy $(X/Y)$ on the space form $S$ by the rule $(X[\varepsilon]/Y[\varepsilon]) = (coord(S) + \varepsilon.X/coord(S) + \varepsilon.Y$, where $coord(S)$ is the coordinator module of $S$.

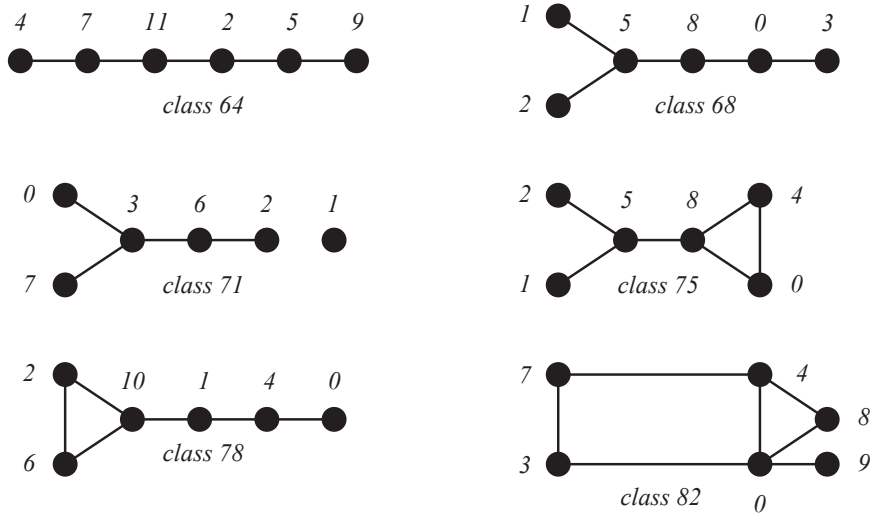


Figure 30.2: The graphs of all strong dichotomies, regarding the third connections. The graphs among the $I$ (major, class 64) and the $K$ (consonance, class 82) intervals show characteristic differences. While in $I$, all intervals are positioned on a line, the consonance graph shows two minimal paths $(0 \to 3 \to 7; 0 \to 4 \to 7)$ between $0, 7$, for example. Observe that when starting from the prime, all imperfect consonances (thirds, sixths) can be reached directly, whereas the perfect fifth is the only non-neighbor of the prime.

### 30.2.1 Fux and Riemann Consonances Are Isomorphic

**Summary.** We present the Riemann dichotomy and its 1-1 correspondence with the Fux dichotomy based on the diagram discussed in 29.5.

– Σ –

Recall the commutative diagram (29.1) of $\mathbb{Z}_{12}$-linear injections for the special ring $R = \mathbb{Z}_{12}$.

$$\begin{array}{ccc} \mathbb{Z}_{12} & \xrightarrow{\ i\ } & \mathbb{Z}_{12}[\varepsilon] \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c[\varepsilon]} \\ \mathbb{Z}_{12}@_{\mathbb{Z}_{12}}\mathbb{Z}_{12} & \xrightarrow{\ [\varepsilon]\ } & \mathbb{Z}_{12}[\varepsilon]@_{\mathbb{Z}_{12}[\varepsilon]}\mathbb{Z}_{12}[\varepsilon] \end{array} \tag{30.1}$$

We may vew this configuration as a realization of determined forms as follows:

$$\begin{array}{ccc} 0@PitchMod_{12,q} & \xrightarrow{\ i\ } & 0@IntMod_{12,q}[\epsilon] \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c[\varepsilon]} \\ \mathbb{Z}_{12}@PitchMod_{12,q} & \xrightarrow{\ [\varepsilon]\ } & \mathbb{Z}_{12}[\varepsilon]@IntMod_{12,q}[\epsilon] \end{array} \tag{30.2}$$

Observe that the space in the right bottom corresponds to the larger space $\mathbb{Z}_{12}[\varepsilon]@\mathbb{Z}_{12}[\varepsilon]$. Also, the lower horizontal arrow is not a natural one in terms of ambient morphisms $f/\alpha$ for local compositions.

To begin with, let us look at the *Riemann dichotomy* which was first defined in [400]. For a justification of this naming that relates to Riemann's concept of relative consonances and dissonances, see [400]. The Riemann dichotomy is the monoid $Trans(D,T)$ generated by the "transporter" set $Tr(D,T)$ consisting by definition of all not necessarily invertible symmetries $f : D = \{1,2,5\} \to T = \{0,1,4\}$ from the dominant triad $D$ to the tonic triad $T$ (in the fifth system). The monoid $Trans(D,T)$ has 72 elements. So it naturally gives rise to a dichotomy on $\mathbb{Z}_{12}@PitchMod_{12,q}$ and also defines a point in the intension topology $InTop(PitchMod_{12,q})$.

**Exercise 67** Show that $card(Trans(D,T)) = 72$.

On the other hand, consider the consonant contrapuntal intervals $K[\varepsilon] = \mathbb{Z}_{12} + \varepsilon.K$ in $0@IntMod_{12,q}[\epsilon]$. The intension $Int(K[\varepsilon])$ is canonically identified to a local composition in $\mathbb{Z}_{12}[\varepsilon]@IntMod_{12,q}[\epsilon]$ since the address $\mathbb{Z}_{12}[\varepsilon]$ is faithful (take the identity). The intersection $W$ of $Int(K[\varepsilon])$ with the subspace $\mathbb{Z}_{12}@PitchMod_{12,q}$ consists of those endomorphisms $e^x y$ of $PitchMod_{12,q}$ that induce endomorphisms of $K[\varepsilon]$. Using the isomorphism

$$\nu = ?.e^{\varepsilon}0|_{\mathbb{Z}_{12}@_{\mathbb{Z}_{12}}\mathbb{Z}_{12}} : \mathbb{Z}_{12}@_{\mathbb{Z}_{12}}\mathbb{Z}_{12} \tilde{\to} \mathbb{Z}_{12}[\varepsilon]$$

from section 29.5, $W$ identifies to the monoid $R \in InTop(PitchMod_{12,q})$ of left stabilizers of $\nu^{-1}(K[\varepsilon])$. In [400] it is shown that $Trans(D,T) = W$. More precisely, with the above identification, we have this theorem:

**Proposition 50** *Let $e^x.y$ be an endomorphism of $PitchMod_{12,q}$. Then it is in the left stabilizer*[4] *of $\nu^{-1}(K[\varepsilon])$ iff it is a product $e^x.y = \prod_i e^{x_i}.y_i$ of endomorphisms*[5] *which transport the dominant triad $D = \{1,2,5\}$ into the tonic triad $T = \{0,1,4\}$. In particular, every element of $\nu^{-1}(K[\varepsilon])$ can be written as such a product of transporter endomorphisms.*

The last statement follows from the fact that the consonance interval numbers $K$ are a multiplicative monoid, and therefore, each element $e^x.k, k \in K$ is a stabilizer of $\nu^{-1}(K[\varepsilon])$. This implies the following:

**Corollary 19** *Let $f = 0 + \varepsilon.1$ the fifth interval at the C tonic 0. For any consonant interval $c = x+\varepsilon.k$, there is a sequence $t_i = e^{x_i}.y_i, i = 1,\dots m$ of transporter endomorphisms $t_i : D \to T$ such that $c = t_m.t_{m-1}.\dots.t_1(f)$.*

Evidently, there is a deep relation between Riemann theory (as it appears in Noll's perspective) and the Fux dichotomy of consonance-dissonance. Presently, we do not know more about the harmonic/contrapuntal implications of the above results. The straightforward hope is that the transition from polyphonic counterpoint to homophonic harmonic relates to the above mathematical facts. But neither the systematic nor the historical consequences of these facts

[4] I.e., for all endomorphisms $e^t.k$ with consonant component $k$, $e^x.y.e^t.k = e^s.k'$ has also a consonant component $k'$.

[5] Including the "empty" product, i.e., the identity.

are at reach. It is however true that musicology has never understood the theoretical relations between counterpoint and harmony. There must be a fundamental relation because the development of homophony out of the polyphonic tradition cannot be a rupture without any inner coherence. Even if such a rupture were a historic fact, it would be a primordial question of systematic musicology to explain the structural, system-immanent rationales for such a rupture. The present results give first hints for the explanation of this lacuna.

### 30.2.2 Induced Polarities

**Summary.** We describe the autocomplementary functions induced on the contrapuntal dichotomies $(X[\varepsilon]/Y[\varepsilon])$ which are deduced from strong dichotomies $(X/Y)$.

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Suppose that we are given a strong dichotomy $(X/Y)$ which bears the polarity $e^u.v$. Then the contrapuntal dichotomy is no longer rigid, but still autocomplementary. The precise situation is described as follows.

**Proposition 51** *Let $\Delta = (X/Y)$ be a strong dichotomy with polarity $p_\Delta = e^u.v$. Choose a cantus firmus point $x$. Then there is exactly one symmetry $p_\Delta^x$ on the counterpoint interval space $IntMod_{12,q}[\varepsilon]$ which is a polarity of $(X[\varepsilon]/Y[\varepsilon])$ and fixes the "tangent space" $I_x = x + \varepsilon.\mathbb{Z}_{12}$ at $x$. Call this the* polarity at $x$. *We have*

$$p_\Delta^x = e^{x(1-v)+\varepsilon.u}.v, \text{ and } p_\Delta^{x+y} = e^x.p_\Delta^y.e^{-x},$$

*and under this polarity, a tangent space $I_y$ is mapped onto the tangent space $I_{x+v(y-x)}$.*

**Exercise 68** Give a proof of proposition 51.

### 30.2.3 Empirical Evidence for the Polarity Function

**Summary.** A review of neurophysiological verifications of the presence of the Fux polarity in human depth EEG is exposed.

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Although this book is not a report on physiological or psychological correlates to music structures, it is important to give an overview of a pronounced evidence of electrophysiological correlates of the consonance-dissonance dichotomy. This is by no means a justification of even a proof of the adequacy of the mathematical investigations, but it must be considered as a fundamental relativization of traditional consonance-dissonance theories. These ideas never produce dichotomies but yield degrees of consonance or dissonance, a quality which is completely irrelevant to the musical counterpoint dichotomy.

The aim of a project at the Neurology Department of the Zürich University Hospital which the epileptologist Heinz-Gregor Wieser, the author, and their collaborators conducted during the years 1984–1988 was to test mathematical principles of classical counterpoint by means of depth EEG responses to musical stimuli. In particular, it was planned to test the

relative results of the EEG to different musical inputs (consonances vs. dissonances) and not the relation of responses to musical stimuli versus non-musical stimuli. The latter problem has been investigated with much success by Hellmuth Petsche and his collaborators [415]. It is important to stress that from our results, we do not draw any kind of conclusions concerning a possible genetic nature of musical understanding or a possible universal validity of classical European interval categories. Our investigations show that in some defined regions of the brain of some European humans, certain significant reactions take place—nothing more and nothing less. There is no reason to generalize whatsoever, but there are enough reasons to try to repeat these investigations in other research sites with a comparable infrastructure. This is all the more desirable since the qualitative results of the investigations by Wieser and the author (namely the prominent role of limbic structures for the judgment of musical pleasantness) have been confirmed by others; see [58], for example.

For a more complete report of our results, we refer to [336, 337, 353, 570, 571, 572]. In this short review, we shall restrict ourselves to the two subtests concerning (a) isolated successive intervals and (b) the polarity between simultaneous consonances and dissonances. Our results confirm our hypothesis on (1) a significant differentiation of EEG responses to consonant vs. dissonant intervals in limbic and auditory brain areas and (2) a pronounced sensitivity of these areas in EEG responses to the fundamental polarity between consonances and dissonances. In particular, the quantitative measurement of these responses by use of the "spectral participation vector" has confirmed our belief that this vector may carry some of the semantic charge of EEG signals.

#### 30.2.3.1 The EEG Test

The test concerned different contexts of consonances and dissonances as well as the test of the polarity $e^2.5$. We used EEG from the scalp (Hess system), stereotactic depth EEG following [569], and multipolar foramen ovale recordings [573]. The tests were applied to the rare cases of patients suffering from medically intractable complex partial epilepsy seizure of suspected mediobasal temporal lobe origin and underwent presurgical evaluation with a view towards surgical epilepsy therapy. None of the total 13 patients considered the voluntary 30 minute music test through monophonic earphones as being disagreeable.

There are several reasons why, despite the particular state of epileptics, the tests remain comparable to tests with normal humans which cannot be conducted for evident reasons. First, our tests were performed during interictal periods. Second, localization of the focus gives a good estimation of its possible influence. Third, epileptiform potentials are easily distinguished from others by the expert.

For each patient, we recorded 700 time windows for fast Fourier transform (FFT) spectral analysis with 256 samples per second, each window for different EEG channels, different power windows ($\delta = 0 - 4Hz, \theta = 4 - 8Hz, \alpha = 8 - 14Hz, \beta = 14 - 40Hz$) and repetitions, totally 11'000 raw spectral data per patient. Unfortunately, the project could not be completed for extrascientific reasons and hence, only two patients have been thoroughly evaluated. These patients were C. J.-L., a 35-year-old academic, and V.S., a 31-year-old artisan. Both are male Europeans ad prefer standard classical, light, and folklore music. Figure 30.3 shows the positions of bipolar depth EEG recordings which we are going to discuss. Notice that C. j.-L.'s recordings RCA, RH, LCA are homologous to V.S.'s recordings 4, 10, 14. Recording RCA lies within the

right hippocampus, recording RH is positioned within Heschl's gyrus (auditory cortex), and recording LCA lies within the left hippocampus.

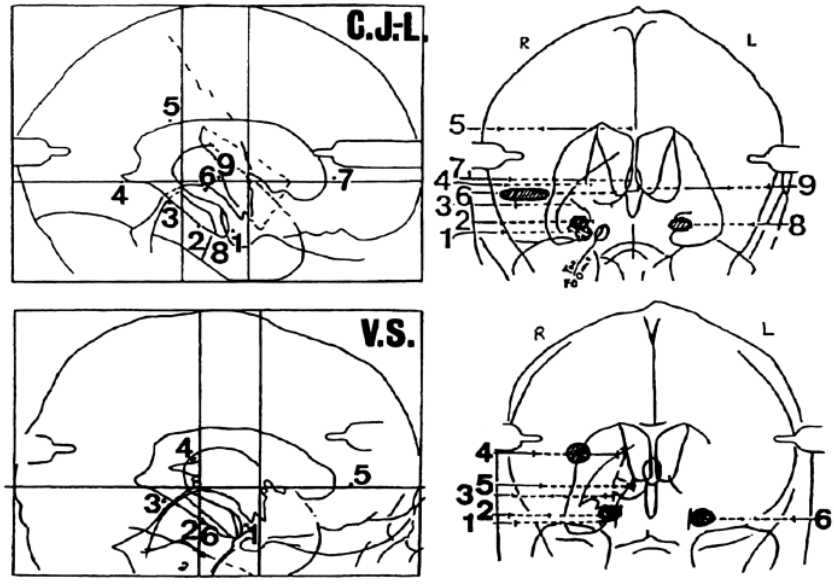


Figure 30.3: Implantation scheme for the two patients C. J.-L. and V.S. following X-rays showing the topographic position of depth electrodes. The three homologous regions are indicated by dashed ovals. For C. J.-L., RCA is electrode 2/1-3, RH is 6/5-6, and LCA is 8/1-3. For V.s., 4 is electrode 2/1-2, 10 is 4/5-6, and 14 is 6/1-2. (Numbers after the slashes indicate precise positions on the electrodes, where 1 = deepest position and 10 = position near surface.

Nevertheless, we have been able to observe a great deal of visual evidence for EEG response to music stimuli in neo- and archicortical regions of all patients; for details, see [336, 570].

#### 30.2.3.2 Analysis by Spectral Participation Vectors

We used four sounds for these tests: piano, sine wave, cello (without vibrato), and "test", a clear, organ-like sound, all synthesized from a Yamaha TX7 synthesizer and CX5M voicing program in order to avoid possible emotional artifacts associated with natural sounds from sociocultural premises. The music program was written on a precursor of the commercial composition software *presto*® [338].

The spectral analysis was executed on a CDC Cyber computer. We made use of the *spectral participation vector*

$$S(E) = (P(E), P(E)/P_\delta(E), P(E)/P_\theta(E), P(E)/P_\alpha(E), P(E)/P_\beta(E))$$

of an event $E$ and its associated *participation value*

$$v(E) = P(E)/P_\theta(E) + P(E)/P_\alpha(E) + P(E)/P_\beta(E)$$

which is a length measure. Here, $P(E)$ is the total spectral power [423], $P_\delta(E)$ is the $\delta$-power of event $E$, $P_\theta(E)$ is the $\theta$-power, $P_\alpha(E)$ is the $\alpha$-power,and $P_\beta(E)$ is the $\beta$-power, see also [336].

From the results obtained thus far we conclude that this representation is well suited to give an adequate picture of possible semantic charge of EEG signals. It is a measure motivated by,among others, the well-known vigilance-related $\alpha$-participation $S_\alpha = P(E)/P_\alpha(E)$ observed by Berger. We also use the delta participation $S_\delta = P(E)/P_\delta(E)$, the $\theta$-participation $S_\theta = P(E)/P_\theta(E)$, and the $\beta$-participation $S_\beta = P(E)/P_\beta(E)$, the latter has been recognized as being strongly related to higher cognitive brain activity by Giannitrapani (see [182]) and Petsche et al. (who also focus on the $\gamma$-band $30 - 50Hz$) [416].

#### 30.2.3.3 Isolated Successive Intervals

We first focus our attention on a subtest concerning musically isolated successive intervals, i.e., the two tones of an interval are played one after the other without interruption. All the intervals were played in three orders: (1) all consonances, ordered according to their size; all dissonances, also ordered according to their size. (2) All consonances, ordered according to complementarity size (if possible); then all dissonances, ordered according to complementarity (if possible). (3) A mixed succession of all intervals according to a particular dodecaphonic all-interval series.

Having fixed a frequency band, $\theta$, say, and an interval with first tone event $E_1$ and second event $E_2$, we consider the quotient

$$Q_\theta(E_1, E_2) = S_\theta(E_1)/S_\theta(E_2)$$

of the theta participations of the first and second tones. If $Q_\theta(E_1, E_2) > 1$ or $Q_\theta(E_1, E_2) < 1$, respectively, then theta participation lowers or increases, respectively, from the first to the second tone. In order to compare these ratios for consonances and dissonances, we take the quotient

$$Q_\theta(K/D) = S_\theta(K)/S_\theta(D)$$

of the mean value $S_\theta(K)$ of all values $Q_\theta(E_1, E_2)$ for consonances $(E_1, E_2)$ and the analogous mean value $S_\theta(D)$ for dissonances. This construction is repeated for all recording positions and all frequency bands, including $\theta, \alpha, \beta$; band $\delta$ is omitted since it may be affected by noise. This test was performed four times with patient C. J.-L. and six times with V.S. A one-sided Wilcoxon test shows significantly higher quotients for consonances compared to dissonances, i.e., $Q_?(K/D)$ is significantly larger than 1 for many recording positions and frequency bands, see figure 30.4, and observe the similarity of distribution of these quotients for our patients with respect to homologous recordings and frequency bands. This means that for consonances, participation lowers more when the second tone appears than for dissonances.

#### 30.2.3.4 Polarity

To test the response to the $(K/D)$-polarity, we confronted each consonant interval $X$ with all the dissonant intervals $Y$, and we looked for particular responses in cases where $Y$ was the interval which should correspond to $X$ according to the polarity formula $Y = 2X + 5$. Here, we looked at simultaneous intervals. For each given consonance $X$, we played a sequence of six confrontations, i.e., immediate successions $(X, Y_1), (X, Y_2), \ldots (X, Y_6)$ of $X$ with each of the six dissonances $Y_1, Y_2, \ldots Y_6$, see also figure 30.5. The duration of each interval was 0.68 seconds.

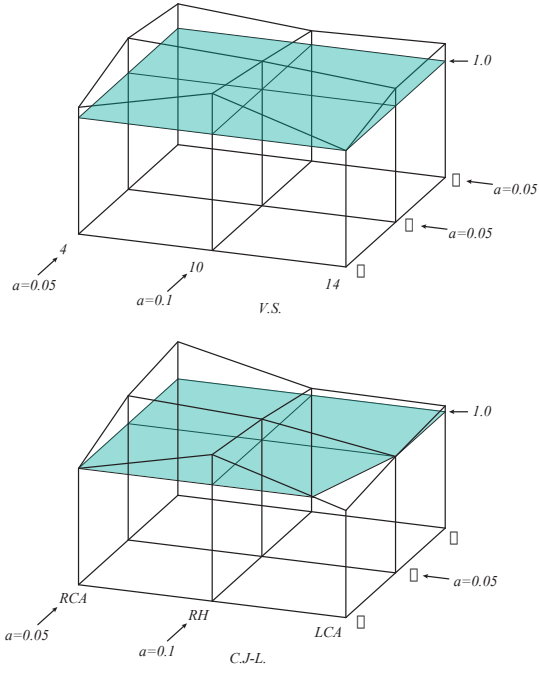


Figure 30.4: Graphical representation of the quotients $Q_\theta(K/D)$, $Q_\alpha(K/D)$, and $Q_\beta(K/D)$ for all locations and patients C. J.-L. and V.S. These numbers show that for consonances, participation lowers more when the second tone appears, compared to dissonances. The 1-level is indicated in the graphics, observe the places where this level is exceeded.

We looked for the least participation values among the dissonant intervals $Y_1, Y_2, \ldots Y_6$, when confronted with a fixed consonance $X$. We then compared the effective hits to the a priori chance to hit the correct dissonance. This method was applied to every recording position and to the three above frequency bands. Figure 30.6 shows the numbers compared to 100%, the measure for a priori chance. Due to the small number of samples for this subtest, we did not apply any statistical test here. However, as figure 30.6 shows, the results are remarkable and similar for both patients, and we conclude that this pilot investigation strongly supports the presence of the $(K/D)$-polarity as a foundation of contrapuntal processes.

### 30.2.4 Music and the Hippocampal Gate Function

**Summary.** The neurophysiological results, in particular their localizations in the emotional brain and the auditory cortex, are interpreted from the cognitive perspective. The gate function of the hippocampal formation suggests a key function of music in opening subconscious—preferredly emotional—memory contents.

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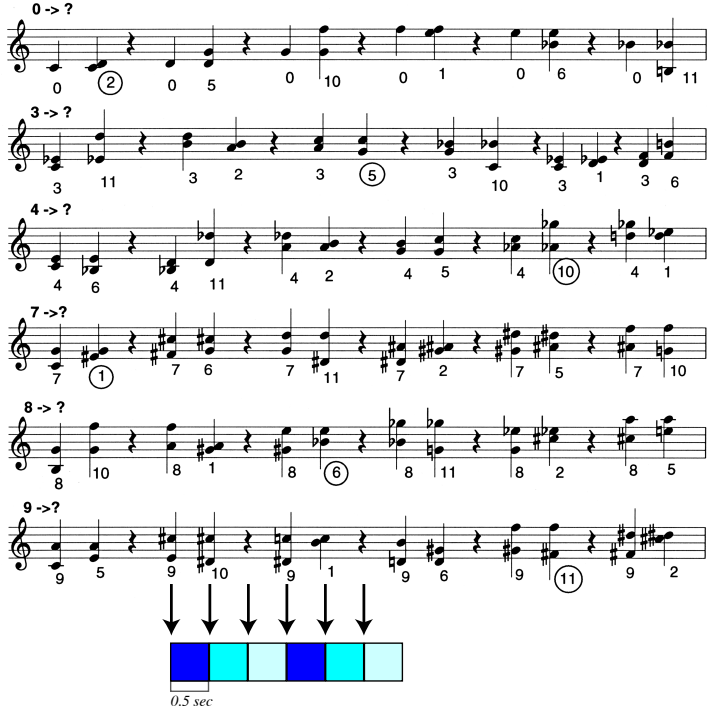


Figure 30.5: The score of the confrontation test of each consonance with all dissonances; duration of a note in the test is 0.68 seconds.

Music and emotions are intimately related, this is common knowledge. The above results suggest a mechanism that could explain this relation on the neurophysiological and cognitive level. We have seen that the emotional brain in its hippocampal structures has a pronounced response to elementary structures of harmony and counterpoints: the intervals in their consonance-dissonance dichotomy, and this is so independently of any sound color physics. Now, the classical thesis of Papez and MacLean [315] states that the limbic system, a prominent part of the archicortex, is responsible for emotional human behavior, this is why it is also called the emotional brain. So the hippocampal sensitivity to consonances vs. dissonances could relate to the emotional function of music, i.e., of musical intervals in our case. The question is, how musical signs which are by no means emotions by themselves (although Schönberg and other prominent music experts constantly evoke the notes' emotional and erotic life) can evoke and signify emotions in humans, and why this is done in such a way that the same music may evoke a great variety of such reactions and significations. Evidently these outputs are the result of a determined sample of music *plus* an individual human ingredient.

The point is that the hippocampal formation has been recognized as a key structure for memory [501]. The neuroscientist Jonathan Winson has proposed a more specific theory of the hippocampal memory function [579], in that he argues that the hippocampus performs a gate

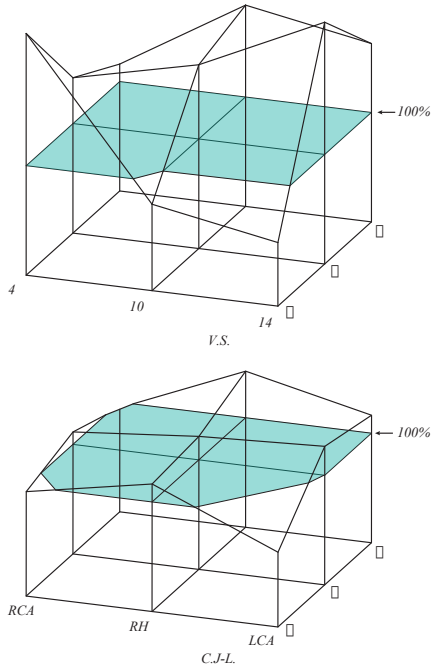


Figure 30.6: Graphical representation of hitting frequencies for the polarity subtest for patients C. J.-L. and V.S., three homologous locations as well as $\theta, \alpha, \beta$ frequency bands. The frequent and pronounced values above 100% show that for both patients there was a strongly affirmative EEG response to the test of the correct values for the polarity. In addition, the topographic/spectral distribution of values for $\alpha$ and $\beta$ bands is comparable for these patients.

function to the subconscious (he even evokes Freud's "Unbewusstes"), i.e., to memory contents of emotional character. This means that the hippocampus is a structure that plays the role of a gateway to hidden memory contents. It is well known that humans do not have a free or controlled access to their memory contents, in particular not on the level of long-term and emotional memory, concerning early childhood, for example.

This suggests that special mechanisms must be activated in order to open the hippocampal gate to unveil locked memory contents. It is straightforward from our neurophysiological findings and the gate function of the hippocampus that its musical stimulation could yield such a "key" to open the gate to hidden memory contents. If this were the case, two specifica of the relation of music and emotion would be explained at once: (1) The emotional contents are not generated by music, they are merely retrieved and evoked from a memory database, whence the individual emotional response to one and the same music would receive a logical explanation. (2) The musical stimulation of the hippocampus is very probably not independent of the human individual who undergoes this process, in other words: If the music is a key, each individual is likely to have his/her individual key to the "subconscious". This would explain why there are so many different musical tastes—beyond musical education and culture. This would also

explain why it is often a specific tune or musical mood that is the personal preference: If this tune played a role in the encoding of a specific emotional memory content, the same key-tune could play a role in the decoding process. Summarizing, we have this thesis:

**Thesis 5** *Consonant and dissonant intervals and associated harmonic or contrapuntal structures evoke a hippocampus state/process which activates a gateway to mainly subconscious memory contents. In other words, Winson's gate hypothesis of the hippocampal formation must also be stated in the sense of the* existence of a musicogenic key to the gate.

*This thesis does not mean that music produces emotions, it only retrieves and reactivates them from a memory database. So it acts on the brain like a drug and produces psychic effects. In this metaphor, the 'chemical formula' of the music drug corresponds to the involved musical structure.*

# Chapter 31

# Modeling Counterpoint by Local Symmetries

*Der Rangunterschied zwischen den*
*perfekten und den imperfekten Konsonanzen*
*ermöglichte die Formulierung*
*genereller Konsonanzfolgeregeln für eine*
indeterminata positio.
Klaus-Jürgen Sachs [468, p. 114]

**Summary.** This chapter presents the counterpoint model in form of a counterpoint theorem which guarantees the existence and exhibits an arsenal of admitted contrapuntal steps that come in extremely close to the rules of classical counterpoint. The theorem is based on the concept of a contrapuntal symmetry and follows the paradigm of local symmetries as a rationale for forces in physics. Because of its generic concept framework, the theorem, which in this general form[1] was proved by Jens Hichert [223], is also valid for non-European scales. We discuss these extensions.

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## 31.1 Deformations of the Strong Dichotomies by Contrapuntal Symmetries on $IntMod_{12,q}[\varepsilon]$

**Summary.** In the core theory of counterpoint [468], the concept of contrapuntal "tension" between successive, perfect and imperfect consonant intervals plays a crucial role. This idea is made precise in the framework of *contrapuntal* symmetries which deform the strong dichotomy. The *separation property* of contrapuntal symmetries is proven.

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[1] The original theorem is presented in [336, 340] and deals with the classical consonances and dissonances, when applied to European diatonic scales.

The following counterpoint theorem (section 31.3) is concerned with the elementary and core situation of classical counterpoint: "note-against-note". It is a theory which describes a system of rules for "allowed" sequences $(x_s + \varepsilon.i_s, \alpha_s)_s$ of oriented intervals, a system which essentially boils down to a set of rules for allowed successor pairings $(x_s + \varepsilon.i_s, \alpha_s), (x_{s+1} + \varepsilon.i_{s+1}, \alpha_{s+1})$ within such contrapuntal sequences. We shall also restrict our investigation to sequences of constant sweeping orientation $\alpha_s = \alpha_+ =$ const., changes in the orientation are handled as described above in 29.6.

The first elementary rule of counterpoint "note-against-note" says that we are not allowed to take other intervals than the consonances $\xi \in K[\varepsilon]$. This seems evident, but it imposes a strong obstruction against another, more hidden directive: the idea of creating a tension between each interval and its successor. More precisely, the meaning of "contra" is not only that of a vertical opposition between cantus firmus and discantus. As Sachs has remarked in [468], the preposition "contra" equally means a horizontal opposition between successive intervals in the given sequence. This requirement is not very explicit, but it is reflected in the distinction between perfect consonances (prime, fifth, octave) and the others, the imperfect sixths and thirds, and the idea of changing between perfect and imperfect consonances in order to create tension. This conceptual distinction seems to evoke a dissonant ingredient in the consonant character, although it does not really abolish the consonance, it is a kind of coloring effect.

So, the idea of contrapuntal tension is in some sense a contradictory requirement against the primordial rule of forbidden dissonances: We should like to behave as if there were dissonances within consonances and to create a tensed movement from consonances to dissonances and vice versa. In order to solve this requirement in our mathematical remake of the contrapuntal rules, we introduce this technique: Given a symmetry $g$ of $IntMod_{12,q}[\varepsilon]$, we may apply $g$ to the consonance-dissonance dichotomy $(K[\varepsilon]/D[\varepsilon])$ and "deform" it to the dichotomy $g(K[\varepsilon]/D[\varepsilon])$. In general, the deformed dichotomy will have its parts in such a position that some real consonances $\xi$ are also *$g$-deformed consonances*, i.e., $\xi \in g(K[\varepsilon])$, and some are *$g$-deformed dissonances*, i.e., $\xi \in g(D[\varepsilon])$. This implies that we may restate the directive of creating contrapuntal tension in the sense that for a given pair of successive consonances $\xi, \eta$, the first one is a $g$-deformed dissonance while the other is a $g$-deformed consonance (or vice versa) for a determined symmetry $g$, in which case we say that *the (unordered) pair $\xi, \eta$ is $g$-polarized*. We shall see below in section 31.2 that the symmetries which we shall exhibit for this role are indeed local symmetries which in physics are responsible for creation of forces, i.e., deformational tension.

Of course it is not evident that there is always a symmetry $g$ that polarizes two consonances $\xi, \eta$ in the above sense. Let us first discuss this topic. It shows that the strong dichotomies are exactly what is needed to guarantee this polarization property.

**Proposition 52** *Let $(X[\varepsilon]/Y[\varepsilon])$ be a strong dichotomy and let $\xi, \eta$ be two different intervals. Then there is a symmetry $g$ such that the pair $\xi, \eta$ is $g$-polarized.*

If $\xi, \eta$ lie in different halves of $(X[\varepsilon]/Y[\varepsilon])$, then $g = Id$ does the job. So we may suppose that both, $\xi$ and $\eta$ lie one half, say in $X$, the other case is settled by a transformation of the pair of intervals to a pair within $X$ via the polarity of $(X[\varepsilon]/Y[\varepsilon])$. Let $\xi = i + \varepsilon.j, \eta = k + \varepsilon.t$ with $j, t \in X$. The symmetry is $g = e^{l+\varepsilon.m}(n + \varepsilon.o), n^2 = 1$. Then we have these applications:

$$g.\xi = l + ni + \varepsilon(m + jn + oi), g.\eta = l + nk + \varepsilon(m + tn + ok).$$

We want the coefficients of $\varepsilon$ to stay in $X$ and $Y$, respectively. If we try with $o = 0$, this means that $e^m.n(j)$ and $e^m.n(t)$ stay in different halves. Suppose that both are always in the same half. Try further $n = 1$. Then we have $j + m, t + m \in X$ or $j + m, t + m \in Y$ for all $m \in \mathbb{Z}_{12}$. Take any $a \in X$ and set $m = a - j$. Then $j + m = a$, and $t + m = (t - j) + j + m = (t - j) + a$. So adding the difference $t - j$ to any $a \in X$ is again an element of $X$. But then the symmetry $e^{t-j}$ is a non-trivial automorphism of $X$, contradicting the rigidity of $X$, and we are done. QED.

We shall give another proof of this fact below. Let us now formulate the properties of symmetries which we want to use for the deformations for a counterpoint rule set:

**Definition 95** *Let $\Delta[\varepsilon] = (X[\varepsilon]/Y[\varepsilon])$ be a strong dichotomy. Let $\xi = x + \varepsilon.i \in X[\varepsilon]$. A symmetry $g$ is* contrapuntal for $\xi$ *iff*

(i) $\xi \notin g(X[\varepsilon])$,

(ii) $p^x_\Delta$ *is a polarity of* $g.\Delta[\varepsilon]$,

(iii) *The cardinality of $g(X[\varepsilon]) \cap X[\varepsilon])$ is maximal among those $g$ which have properties* (i) *and* (ii).

The reason for these requirements is this. We have seen that every pair of intervals can be polarized by a specific symmetry. But we are interested in a rule set which guarantees more than the mere possibility of separation. We want to have those symmetries which admit a maximal number of polarized couples starting from a fixed interval. The second condition is introduced in order to relate the polarizing symmetry $g$ to the given polarity $p^x_\Delta$ of the dichotomy at the cantus firmus $x$. It can be shown [333] that for $x = 0$ this property is also equivalent to the commutativity condition $g.p^0_\Delta = p^0_\Delta.g$, so it is a generalized commutativity condition.

**Definition 96** *If a strong dichotomy $\Delta[\varepsilon] = (X[\varepsilon]/Y[\varepsilon])$ and an interval $\xi \in X[\varepsilon]$ are given, an interval $\eta$ is called an* admitted successor of $\xi$ *if it is contained in an intersection $g(X[\varepsilon]) \cap X[\varepsilon]$ for a contrapuntal symmetry $g$ for $\xi$.*

## 31.2 Contrapuntal Symmetries Are Local

**Summary.** A closer look at contrapuntal symmetries shows their local character. In complete analogy with modern physics, local symmetries produce the looked-for forces of contrapuntal tension. Thus, the melodic variation of the cantus firmus in counterpoint is perfectly interpreted as a deformation *caused* by the "forces" from local symmetries.

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Before we deal with the counterpoint theorem, we should explain the *local* character of contrapuntal symmetries. We shall consider the example of classical consonances and dissonances, i.e., the dichotomy $\Delta[\varepsilon] = (K[\varepsilon]/D[\varepsilon])$. We compare two interpretations of the local composition of all zero-addressed intervals $I = 0@IntMod_{12,q}[\varepsilon]$. The first $I^{C_1}$ is induced by $\Delta$ an has the atlas

$$I_x, K_x = x + \varepsilon.K, D_x = x + \varepsilon.D, x \in \mathbb{Z}_{12}.$$

The second $I^{C_g}$ is the analogous construction built on the deformed dichotomy $g.\Delta[\varepsilon]$, i.e., its atlas is

$$I_x, gK_x = I_x \cap g.K[\varepsilon], gD_x = I_x \cap g.D[\varepsilon], x \in \mathbb{Z}_{12}.$$

Although the autocomplementary function $p^x_\Delta$ for the cantus firmus $x$ is also a polarity of the $g$-deformed dichotomy $g.\Delta[\varepsilon]$ according to definition 95, its action on the two interpretations is qualitatively different (see also figure 31.1).

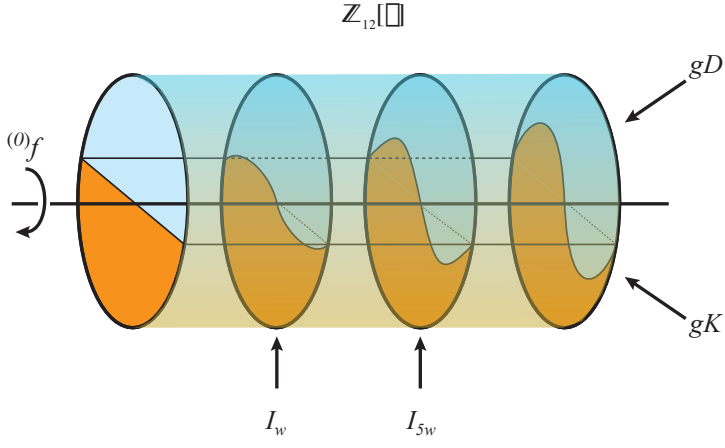


Figure 31.1: The autocomplementary symmetry on the deformed consonance-dissonance dichotomy is a local symmetry, whereas it is a global one on the original dichotomy. This fact resembles physical forces being induced by local symmetries.

Consider the example $g = e^{\varepsilon.8}(5 + \varepsilon.4)$ for the consonance $\xi = \varepsilon.9$, i.e., at cantus firmus $x = 0$. We have

$$g.K[\varepsilon] = (1 - \varepsilon.4)\mathbb{Z}_{12} + \varepsilon.e^8.5K.$$

For a cantus-firmus point $w$, this means

$$gK_w = w + \varepsilon.(5K + 8 - 4w).$$

But under $p^0_\Delta$, $w$ is transported to $5w$, and we have

$$gK_{5w} = 5w + \varepsilon.(5K + 8 + 4w).$$

So we recognize the following: On the first interpretation, $p^0_\Delta$ acts on $K_w$ via *translation* on the cantus firmus: $K_{5w} = e^{4w}(K_w)$, followed by the autocomplementarity symmetry on $I_{5w}$. But on the second interpretation, $p^0_\Delta$ does not operate on $gK_w$ by a translation plus autocomplementarity symmetry on $I_{5w}$, in fact, $gK_{5w}$ is different from $e^{4w}(gK_w)$! This can be understood in the sense that $p^0_\Delta$ acts on the second interpretation via $g$-deformation in the spirit of physics, i.e., as a *local symmetry*, instead of a global symmetry as it is the case for the first interpretation. The latter can be viewed as a 'spiral turn': 'rotation' (=autocomplementarity symmetry on

each chart $I_w$) plus translation ($K_{5w} = e^{4w}(K_w)$) 'along the rotation axis'. The former however does not shift the part $gK_w$ of $I_w$ to $e^{4w}(gK_w)$, but deforms it to

$$gK_{5w} = 5w + \varepsilon.(5K + 8 + 4w) = e^{\varepsilon.8w}(e^{4w}(gK_w))$$

by the factor $\varepsilon.8w$. This phenomenon is analogous to physics in the sense that forces in physics are induced by local symmetries [162]. In this understanding, local symmetries on the interval space seem to be responsible for the tension which controls the progression from interval to interval in our model.

## 31.3 The Counterpoint Theorem

**Summary.** This section presents the counterpoint theorem and its corollaries.

– Σ –

In this section, we prove the general counterpoint theorem. "General" means that we deal with all strong dichotomies and construct the lists of admitted interval successors for these cases, including the classical case as a special item. The general theorem was proved in Jens Hichert's thesis [223] and sheds a new light on the general problem of what counterpoint is about. We shall discuss the special case of the consonance-dissonance dichotomy in section 31.4.1.

**Remark 17** In order to cope with Hichert's calculations and tables we shall use the interval space relative to the identity identifier, and not to the fifth identifier. In particular, the consonance quantities are $\{0, 3, 4, 7, 8, 9\}$ in this section.

### 31.3.1 Some Preliminary Calculations

**Summary.** We prove technical lemmata for the exhibition of Hichert's algorithm to be introduced in section 31.3.3.

– Σ –

We fix a marked dichotomy $\Delta = (X/Y)$. Let us first come back to the rigidity property of $X[\varepsilon]$:

**Lemma 44** *The symmetry group of $X[\varepsilon]$ is $Sym(X[\varepsilon]) = e^{\mathbb{Z}_{12}}$.*

Proof. Clearly $e^{\mathbb{Z}_{12}} \subset Sym(X[\varepsilon])$. Therefore,

$$e^{z+\varepsilon.t(u+\varepsilon.v}X[\varepsilon] = X[\varepsilon] \text{ iff } e^{\varepsilon.t(u+\varepsilon.v)}X[\varepsilon] = X[\varepsilon].$$

This implies $t + vz + uk \in X$ for all $z \in \mathbb{Z}_{12}, k \in X$. Whence $t = 0, u = 1$ since $X$ is rigid. Therefore, for $z = 1$, we imply $v + k \in X$, all $k \in X$, i.e., $v = 0$. QED.

**Proposition 53** *Let $H = e^{\varepsilon.\mathbb{Z}_{12}}.GL(\mathbb{Z}_{12}[\varepsilon])$. Then $X[\varepsilon]$ is $H$-rigid, i.e., the orbit application*

$$H \to ObLoc_0^{\mathbb{Z}_{12}[\varepsilon]} : g \mapsto g.X[\varepsilon]$$

*is injective, i.e., a cadence of the group $H$.*

**Exercise 69** Show that each consonant part $X_x = x + \varepsilon.X$ of the tangent space $I_x$ is rigid.

**Lemma 45** *With the notation of corollary 53, if $g = e^t(u + \varepsilon.v) \in H$, and if $z \in \mathbb{Z}_{12}$, we set $g^{(z)} = g.e^{\varepsilon.vz}$ Then we have*

(i) $(g^{z_1)})^{z_2} = g^{z_1+z_2}$,

(ii) $e^z.g.X[\varepsilon] = g^{(z)}.X[\varepsilon]$, *and* $e^z.g.Y[\varepsilon] = g^{(z)}.Y[\varepsilon]$.

Proof. The first formula is clear. The second follows from $e^z g = e^{z+\varepsilon.t}(u + \varepsilon.v) = e^{\varepsilon.t}(u + \varepsilon.v).e^{z(u-\varepsilon.v)} = g^{(z)}.e^{zu}$. QED.

**Corollary 20** *For $g \in \overrightarrow{GL}(\mathbb{Z}_{12}[\varepsilon])$, there is a symmetry $h \in H$ such that $g.X[\varepsilon] = h.X[\varepsilon]$.*

In fact, there is a $u \in H$ such that $g = e^z.u$, so by lemma 45, $g.X[\varepsilon] = u^{(z)}.X[\varepsilon]$, and we have the solution $h = u^{(z)}$.

**Lemma 46** *Let $\xi = x+\varepsilon.k, g \in \overrightarrow{GL}(\mathbb{Z}_{12}[\varepsilon])$, and $z \in \mathbb{Z}_{12}$. Then, if $\xi \notin g.X[\varepsilon]$ and $p_\Delta^x : g.X[\varepsilon] \xrightarrow{\sim} g.Y[\varepsilon]$, we also have $e^z\xi \notin e^z.g.X[\varepsilon]$, $p_\Delta^{z+x} : e^z.g.X[\varepsilon] \xrightarrow{\sim} e^z.g.Y[\varepsilon]$, and $e^z.g.X[\varepsilon] \cap X[\varepsilon] = e^z.(g.X[\varepsilon] \cap X[\varepsilon])$, in particular $card(g.X[\varepsilon] \cap X[\varepsilon]) = card(e^z.g.X[\varepsilon] \cap X[\varepsilon])$.*

Proof. It is clear that $e^z\xi \notin e^z.g.X[\varepsilon]$, while by lemma 51,

$$p_\Delta^{z+x}.e^z.g.X[\varepsilon] = e^z.p_\Delta^x.e^{-z}.e^z.g.X[\varepsilon] = e^z.p_\Delta^x.g.X[\varepsilon] = e^z.g.Y[\varepsilon],$$

and finally, $e^z.g.X[\varepsilon] \cap X[\varepsilon] = e^z.g.X[\varepsilon] \cap e^z.X[\varepsilon] = e^z.(g.X[\varepsilon] \cap X[\varepsilon])$. QED.

**Proposition 54** *The contrapuntal symmetries can be calculated if one knows the contrapuntal symmetries $g \in H$ at cantus firmus $x = 0$. More precisely, if $\xi = x + \varepsilon.k \in X[\varepsilon]$ and if $g$ is any symmetry, such that properties* (i) *through* (iii) *of $g$ in definition 95 are true, then they are also true with unchanged set $g.X[\varepsilon] = h.X[\varepsilon]$ for a symmetry $h \in H$. Furthermore, to check this property for $h$, we may verify the properties* (i) *through* (iii) *for the interval $\varepsilon.k$, the symmetry $h^{(-x)} \in H$, and the polarity $p_\Delta^{(0)}$. Finally, the intersection $h.X[\varepsilon] \cap X[\varepsilon]$ coincides with the translate $e^x(h^{(-x)}.X[\varepsilon] \cap X[\varepsilon])$, which means that we may just look for relative cantus firmus steps when building the rules of admitted steps.*

Proof. The replacement of $g$ by $h$ follows from corollary 20. By lemma 45 we have $e^{-x}.h.X[\varepsilon] = h^{(-x)}.X[\varepsilon]$, and by lemma 46, with $z = -x$, we can verify the contrapuntality of $h$ on $e^{-x}\xi = \varepsilon.k$, and on $p_\Delta^{(-x+x)} = p_\Delta^{(0)}$. The last statement follows from the translation formula for intersections in lemma 45. QED.

### 31.3.2 Two Lemmata on Cardinalities of Intersections

**Lemma 47** *Let $(X/Y)$ be a strong dichotomy in $0@IntMod_{12,1}$, and $0 \leq i \leq 6$ and integer. Setting $G_i = \{g \in \overrightarrow{GL}(\mathbb{Z}_{12}) |\ card(g.X \cap X) = i\}$, we have $card(G_i) = card(G_{6-i})$.*

Proof. Let $p$ be the polarity of $(X/Y)$, the right multiplication with $p$ induces a permutation of order 2 $?p : \overrightarrow{GL}(\mathbb{Z}_{12}) \xrightarrow{\sim} \overrightarrow{GL}(\mathbb{Z}_{12})$. For $g \in G_i$, consider the dichotomy $g(X/Y)$ and its intersection with $X$. This gives $X = g.X \cap X \cup g.p.X \cap X$, whence $card(g.p.X \cap X) = 6 - card(g.X \cap X)$. Therefore, $g.p \in G_{6-i}$, and by the order two permutation $?p$ and the finiteness of all the involved sets, we have $p.G_i = G_{6-i}$. QED.

The next lemma basically guarantees the existence of admitted contrapuntal successor intervals, as we shall see in the next section.

**Lemma 48** *Let $K$ be a zero-addressed objective local composition in a finite cyclic group $M \xrightarrow{\sim} \mathbb{Z}_n$ and let $U \in GL(M)$. Then*

$$\sum_{m \in M} card(e^m.U(K) \cap K) = k^2.$$

Proof. Let $U(K) = \{u_1, \ldots u_k\}$ the image set with its $k = card(K)$ elements. Let $r$ be a generator of $M$. Then

$$\begin{aligned}
& \sum_{m \in M} card(e^m.U(K) \cap K) \\
= & \sum_{t=0}^{n-1} card(e^{tr}.U(K) \cap K) \\
= & \sum_{t=0}^{n-1} \sum_{s=1}^{k} card(\{e^{tr}u_s\} \cap K) \\
= & \sum_{t=0}^{n-1} \sum_{s=1}^{k} \chi_K(e^{tr}u_s) \\
= & \sum_{s=1}^{k} (\sum_{t=0}^{n-1} \chi_K(e^{tr}u_s))
\end{aligned}$$

with the characteristic function $\chi_K$ for an element being in $K$ or not. Since $r$ is a generator of $M$, the expression $\sum_{t=0}^{n-1} \chi_K(e^{tr}u_s)$ adds up to $k$, and we have the result. QED.

### 31.3.3 An Algorithm for Exhibiting the Contrapuntal Symmetries

**Summary.** This section discusses Hichert's algorithm for the calculation of all contrapuntal symmetries and admitted contrapuntal steps by use of a specific software.

– Σ –

This section restates the three conditions (i) through (iii) of the contrapuntal symmetries in order to provide an algorithm for the calculation of all admitted contrapuntal interval successors. Given the strong dichotomy $\Delta = (X/Y)$, and according to proposition 54, we may start from an interval $\xi = \varepsilon.k, k \in X$ and restrict to symmetries $g = e^{\varepsilon.t}(u + \varepsilon.uv) \in H$ ($uv$ instead of $v$ without restriction). Let us first reformulate conditions (i) and (ii). We have

$$\begin{aligned} g.X[\varepsilon] &= \bigcup_{x\in\mathbb{Z}_{12}} g.(x + \varepsilon.X) \\ &= \bigcup_{x\in\mathbb{Z}_{12}} ux + \varepsilon.(uvx + t) + \varepsilon.uX \\ &= \bigcup_{y\in\mathbb{Z}_{12}} y + \varepsilon.(vy + t) + \varepsilon.uX \\ &= \bigcup_{y\in\mathbb{Z}_{12}} y + \varepsilon.(e^{vy+t}u.X). \end{aligned}$$

Setting $h(y) = e^{vy+t}u$, we have

$$g.X[\varepsilon] = \bigcup_{y\in\mathbb{Z}_{12}} y + \varepsilon.h(y).X.$$

Therefore, we have

$$g.X[\varepsilon] \cap X[\varepsilon] = \bigcup_{y\in\mathbb{Z}_{12}} y + \varepsilon.(h(y).X \cap X).$$

This means that condition (i) is equivalent to $k \notin h(0).X$ which means $k \in h(0).p.X$, and this is equivalent to

$$\exists s \in X \text{ such that } k = h(0).p(s) = u.p(s) + t.$$

This is equivalent to the statement that there is an $s \in X$ such that

$$g.X[\varepsilon] = \bigcup_{y\in\mathbb{Z}_{12}} y + \varepsilon.(e^{vy+k-u.p(s)}u.X) \tag{31.1}$$

so that we have

$$card(g.X[\varepsilon] \cap X[\varepsilon]) = \sum_{y\in\mathbb{Z}_{12}} card(e^{vy+k-u.p(s)}u.X \cap X). \tag{31.2}$$

Further, condition (ii) means $p^0_\Delta.g = g.p^0_\Delta$, see the remarks after definition 95. If we have $p = e^r.w$, this means that

$$wt + r = ur + t. \tag{31.3}$$

Therefore, conditions (i) and (ii) are equivalent to equations 31.1 and 31.3, whereas the maximal number is calculated upon formula 31.2.

In order to calculate the number 31.2, we have to distinguish three cases concerning the values of $v$:

1. $v$ is invertible. Then we have

$$card(g.X[\varepsilon] \cap X[\varepsilon]) = \sum_{y \in \mathbb{Z}_{12}} card(e^{y}.u.X \cap X). \tag{31.4}$$

2. $v = 0$. This gives

$$card(g.X[\varepsilon] \cap X[\varepsilon]) = 12 card(e^{k-u.p(s)}.u.X \cap X). \tag{31.5}$$

3. $v = \pm\rho, \rho \in \{2, 3, 4, 6\}$. This gives

$$card(g.X[\varepsilon] \cap X[\varepsilon]) = v \sum_{j=1}^{12/v} card(e^{(j-1)v+k-u.p(s)}.u.X \cap X). \tag{31.6}$$

If we recall lemma 15, the first case with formula 31.4 implies:

**Fact 15** *There are always at least 36 successors for a fixed given interval $\xi \in X[\varepsilon]$.*

We are now ready to state the algorithm which was implemented on Turbo-Pascal by Hichert [223]. The algorithm starts from the fixed value $k \in X$ and first calculates all the possible coefficients of $g$. This means that we have to go through the loop which transgresses all $g = e^{\varepsilon.t}(u + \varepsilon.uv)$ via $u \in \{1, 5, 7, 11\}, s \in X, v \in \mathbb{Z}_{12}, t = k - up(s)$, and then for each such value set calculate $card(g.X[\varepsilon] \cap X[\varepsilon])$ according to the three cases 31.4, 31.5, 31.6. For each case, we update the set $C_k$ of intermediate candidates for contrapuntal symmetries, i.e., we add a new $g$ to the existing set if its intersection number is maximal among the already given candidates in $C_k$, and we remove all previous $g$ where intersection cardinalities are smaller than the actual maximum.

**Exercise 70** Write a C or Java program which implements the above algorithm. If you do not speak C or Java, write a Mathematica or Maple program. If you do not speak these languages either, interrupt reading this book, learn one of these languages, and proceed.

The complete lists of all contrapuntal symmetries, together with their intersection configurations, can be found in appendix O.1. The admitted successors will be listed below. This information yields the following

**Theorem 33** *Let $\Delta = (X/Y)$ be a strong dichotomy, and let $\xi \in X[\varepsilon]$. The number of admitted successors of $\xi$ is always at least equal to 36, in particular, there exists always a contrapuntal symmetry $g$ for $\xi$. An admitted successor of $\xi$ also always exists if one prescribes the cantus firmus of the successor interval.—For each of the six strong dichotomy classes, a list of forbidden successor intervals is exposed below, after this theorem.*

We shall say that *an interval $\xi \in X$ is a "cul-de-sac" under determined conditions* if there is no admitted successor under these conditions.

The following table is meant as follows: Each table section is related to a fixed representative $(X/Y)$ of the indicated class. The first column indicates the interval quantity $k$ of

the "consonant" start interval $\xi = x + \varepsilon.k \in X[\varepsilon]$. The obstructions to successor interval $\eta = y + \varepsilon.l \in X[\varepsilon]$ are visible in columns 2–13 where the difference $d = y - x = 0, 1, 2, \ldots 11$ leads the column in the first row. For each couple $k, d$, we see the forbidden interval quantities $l$ of the target interval. For example, in class 64, the steps $x + \varepsilon.4 \mapsto x + 6 + \varepsilon.l$ is forbidden for $l = 2, 4, 7, 9$ and admitted for all other $l$.

| k | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Forbidden Successors for Dichotomy Class Nr. 64 | | | | | | | | | | | | |
| 2 | 2 | - | 2 | - | 2 | - | 2 | - | 2 | - | 2 | - |
| 4 | 2,4,7,9 | - | 2,4,7,9 | - | 2,4,7,9 | - | 2,4,7,9 | - | 2,4,7,9 | - | 2,4,7,9 | - |
| 5 | 5,11 | 5,11 | 5,11 | 5,11 | 5,11 | 5,11 | 5,11 | 5,11 | 5,11 | 5,11 | 5,11 | 5,11 |
| 7 | 2,4,7,9 | - | 2,4,7,9 | - | 2,4,7,9 | - | 2,4,7,9 | - | 2,4,7,9 | - | 2,4,7,9 | - |
| 9 | 9 | - | 9 | - | 9 | - | 9 | - | 9 | - | 9 | - |
| 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 |
| Forbidden Successors for Dichotomy Class Nr. 68 | | | | | | | | | | | | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | - | 1 | - | 1 | - | 1 | - | 1 | - | 1 | - |
| 2 | 2,8 | 2,8 | 2,8 | 2,8 | 2,8 | 2,8 | 2,8 | 2,8 | 2,8 | 2,8 | 2,8 | 2,8 |
| 3 | 1,3,5 | - | 1,3,5 | - | 1,3,5 | - | 1,3,5 | - | 1,3,5 | - | 1,3,5 | - |
| 5 | 5 | - | 5 | - | 5 | - | 5 | - | 5 | - | 5 | - |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| Forbidden Successors for Dichotomy Class Nr. 71 | | | | | | | | | | | | |
| 0 | 0,3 | 3 | 0,3 | 3 | 0,3 | 3 | 0,3 | 3 | 0,3 | 3 | 0,3 | 3 |
| 1 | 1,2,7 | - | 1,2,7 | - | 1,2,7 | - | 1,2,7 | - | 1,2,7 | - | 1,2,7 | - |
| 2 | 2,3 | - | 2,3 | - | 2,3 | - | 2,3 | - | 2,3 | - | 2,3 | - |
| 3 | 2,3 | - | 2,3 | - | 2,3 | - | 2,3 | - | 2,3 | - | 2,3 | - |
| 6 | 3,6 | 3 | 3,6 | 3 | 3,6 | 3 | 3,6 | 3 | 3,6 | 3 | 3,6 | 3 |
| 7 | 1,2,7 | - | 1,2,7 | - | 1,2,7 | - | 1,2,7 | - | 1,2,7 | - | 1,2,7 | - |
| Forbidden Successors for Dichotomy Class Nr. 75 | | | | | | | | | | | | |
| 0 | 0 | - | - | - | - | - | 0 | - | - | - | - | - |
| 1 | 0,1,4,5 | - | 0,1,4,5 | - | 0,1,4,5 | - | 0,1,4,5 | - | 0,1,4,5 | - | 0,1,4,5 | - |
| 2 | 2,5 | - | - | 2,5 | - | - | 2,5 | - | - | 2,5 | - | - |
| 4 | 0,1,4,5 | - | 0,1,4,5 | - | 0,1,4,5 | - | 0,1,4,5 | - | 0,1,4,5 | - | 0,1,4,5 | - |
| 5 | 2,5 | - | - | 2,5 | - | - | 2,5 | - | - | 2,5 | - | - |
| 8 | 2,8 | 2 | 2 | 2,8 | 2 | 2 | 2,8 | 2 | 2 | 2,8 | 2 | 2 |
| Forbidden Successors for Dichotomy Class Nr. 78 | | | | | | | | | | | | |
| 0 | 0,2,6 | - | 0,2,6 | - | 0,2,6 | - | 0,2,6 | - | 0,2,6 | - | 0,2,6 | - |
| 1 | 1,2 | - | 1,2 | - | 1,2 | - | 1,2 | - | 1,2 | - | 1,2 | - |
| 2 | 1,2 | - | 1,2 | - | 1,2 | - | 1,2 | - | 1,2 | - | 1,2 | - |
| 4 | 1,4 | - | - | 1,4 | - | - | 1,4 | - | - | 1,4 | - | - |
| 6 | 0,2,6 | - | 0,2,6 | - | 0,2,6 | - | 0,2,6 | - | 0,2,6 | - | 0,2,6 | - |
| 10 | 1,10 | - | 1 | 1,4 | 1 | - | 1,10 | - | 1 | 1,4 | 1 | - |
| Forbidden Successors for Dichotomy Class Nr. 82 | | | | | | | | | | | | |
| 0 | 0 | - | - | - | - | - | 0 | - | - | - | - | - |
| 3 | 3,9 | - | - | 3,9 | - | - | 3,9 | - | - | 3,9 | - | - |
| 4 | 0,4,8 | - | 0,4,8 | - | 0,4,8 | - | 0,4,8 | - | 0,4,8 | - | 0,4,8 | - |
| 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 8 | 8 | - | - | - | - | - | 8 | - | - | - | - | - |
| 9 | 3,9 | - | - | 3,9 | - | - | 3,9 | - | - | 3,9 | - | - |

### 31.3.4 Transfer of the Counterpoint Rules to General Representatives of Strong Dichotomies

**Summary.** The above counterpoint theorem was made explicit for one selected representative of a strong dichotomy. Following [223], we give rules to transfer these results and tables to arbitrary representatives of strong dichotomies.

– Σ –

Let $\Delta = (X/Y)$ be a strong dichotomy with polarity $p_\Delta$. Take an interval $\xi = \varepsilon.k \in X[\varepsilon]$, and let $\mathfrak{R} \subset H$ be the set of contrapuntal symmetries of $\Delta$ and $\xi$. Take any symmetry $g = e^t u \in \overrightarrow{GL}(\mathbb{Z}_{12})$, set $g_\varepsilon = e^{\varepsilon.t} u$, and consider the transformed dichotomy $(L/M) = g.\Delta$. We have $g_\varepsilon.X[\varepsilon] = L[\varepsilon], g_\varepsilon.Y[\varepsilon] = M[\varepsilon]$.

**Proposition 55** [223, Satz 3.2] *With the above notation, the conjugate set $\mathfrak{R}_g = g_\varepsilon.\mathfrak{R}.g_\varepsilon^{-1}$ is exactly the set of contrapuntal symmetries of $g.\Delta$ and $\xi_g = g_\varepsilon\xi$. The number of admitted successors of $\xi$ and of $\xi_g$ coincide.*

Attention: In general, the fact that $\eta$ is a successor of $\xi$ does not imply that $g_\varepsilon\eta$ is a successor of $g_\varepsilon\xi$!

This result can be used to transform the tables of forbidden successors as given in the table after the counterpoint theorem 33 for other representatives of the dichotomy classes. To this end, suppose that in such a table (for dichotomy representative $\delta = (X/Y)$), the couple $\xi = \varepsilon.k, \eta = b + \varepsilon.j$ is forbidden. This means that in our table, on the location of row with interval quantity $k$ and column with interval quantity $b$, the coefficient $j$ appears as a forbidden quantity. According to proposition 55, in the transformed table, the row $g.k$ and the column $g.b$ must show a forbidden quantity $g.j$. So we have a recipe for transforming a table for $\Delta$ under a symmetry $g = e^t u$:

1. Permute the 12 columns of the given table by a multiplication of the column head numbers $0, 1, 2, 3, \ldots 11$ by $u$ (mod 12) and rearrange the new numbers by increasing values.

2. Replace the leading column interval numbers $k$ by $g.k$ and rearrange the corresponding 6 rows by increasing values of the leading numbers.

3. Replace each forbidden item $j$ by the item $g.j$.

## 31.4 The Classical Case: Consonances and Dissonances

**Summary.** This section deals with the classical case of the counterpoint theorem for the consonance-dissonance dichotomy of Palestrina–Fux Theory.

– Σ –

We give a specialized counterpoint theorem for the consonance-dissonance dichotomy, the sweeping orientation, and relating to ecclesiastical modes as defined in section 13.4.2.

**Theorem 34** *Let $\Delta = (K/D)$ be the consonance-dissonance dichotomy, and let $\xi \in K[\varepsilon]$. The number of admitted successors of $\xi$ is always at least equal to 36, in particular, there exists always a contrapuntal symmetry $g$ for $\xi$. An admitted successor of $\xi$ also always exists if one prescribes the cantus firmus of the successor interval.—If one restricts the admitted pitch classes to an ecclesiastical mode (see section 13.4.2), then an admitted successor of $\xi$ always exists, even if the cantus firmus is prescribed. So, under these conditions, there is no cul-de-sac.*

*Parallels of fifth ($x + \varepsilon.7 \mapsto y + \varepsilon.7$) are generally forbidden. For all other parallels, no general obstruction exists.*

*The admitted relative progressions are listed in the table after theorem 33, Class Nr. 82, whereas the progressions for the C-major scale are listed in appendix O.2.*

We shall discuss below the relation between strong dichotomies and scales, in particular the appearance of culs-de-sac.

### 31.4.1 Discussion of the Counterpoint Theorem in the Light of Reduced Strict Style

**Summary.** The concluding section gives an overview of the (strong) commonalities with and (weak) diversities from the classical rules of the Palestrina—Fux system and its reduction to pitch classes.

– Σ –

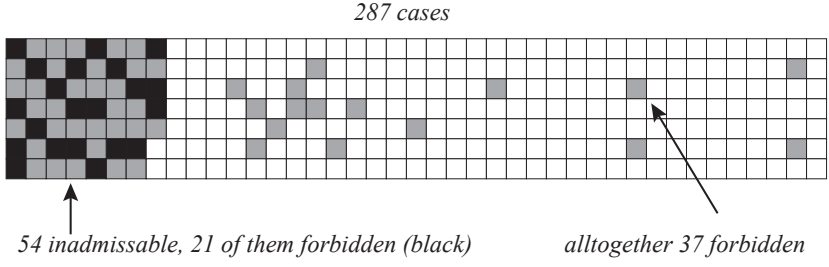


Figure 31.2: Out of the 287 possible progressions with a mode (modulo translations of the cantus firmus), the reduced strict style exhibits 54 inadmissible cases. According to the mathematical model of counterpoint, 37 progressions are not admitted. Out of these, 21 cases are inadmissible in Fux' sense. intuitively, the commonalities of the two approaches can be described by use of a probabilistic argument: If somebody tries to hit at least 21 of the 54 inadmissible cases of the reduced strict style without knowing anything about counterpoint by 37 trials, the chance is less than $2.10^{-8}$.

In this discussion, we refer to the codification of the strict style by Fux [174] (see also [528]) since its explicit and rigorous rule system is particularly useful for a qualitative and quantitative comparison of the mathematical model with the classical counterpoint rules. In order to establish a basis of comparison for the mathematical model, we first have to transfer

the Fux rules to the interval space $IntMod_{12,1}$. This yields a model of Fux' rules 'modulo octave', a rule system which we call the *reduced strict style*.

The upshot of a detailed investigation [342][2] states this:

**Fact 16** *In the reduced strict style, only the rule of forbidden fifth parallels and the tritone rules have an unrestricted validity.*

Within an ecclesiastical mode, there are 287 a priori possible progressions [342]. According to the consonance-dissonance counterpoint theorem 34, 37 of them are forbidden. Among them, 21 coincide with the 54 Fux-inadmissible progressions. The remaining 16 forbidden progressions of the mathematical model deal with progressions which are bad or allowed by Fux. Out of the ten allowed ones, four concern tritone movements of the cantus firmus. Three of the remaining obstructions (in the mathematical model) concern progressions from the major third into the prime, from which one may lead to an "ottava battuta". The remaining three obstructions deal with progressions which leave unaltered the pitch material, see also figure 31.2.

We should stress that the mathematical model does not formalize a switch between perfect and imperfect consonances. The concrete shape of a polarization which induces a determined progression is redefined for each individual progression. Further, the model can be applied to any scale. This unveils an interesting fact concerning the dominant role of the major scale: In the analysis for the three seven-element scale classes which consist exclusively of minor and major second steps (classes $\widehat{38.1}$: major, $\widehat{47.1}$: melodic minor, and $\widehat{62}$: whole-tone scale, extended by one pitch class) the major scale is by far optimal for the degree of freedom in the choice of admitted successors. There is no cul-de-sac. Only for two progressions with prescribed start interval and cantus firmus progression is the successor uniquely determined. The melodic minor scale has less successor freedom, but there are no culs-de-sac. In 16 cases there is only one possible solution. In the extended whole-tone scale, there are 18 culs-de-sac. The freedom of choice is minimal. We shall discuss this item in more generality in section 31.4.2.

### 31.4.2 The Major Dichotomy—A Cultural Antipode?

**Summary.** Among the six strong dichotomy classes, we look for others than the Fux dichotomy and discuss their relative positions. Among the possible alternatives we especially focus on the major dichotomy, a topological antipode to the Fux dichotomy. The possibility to associate the major dichotomy with classical Indian scales is discussed.

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In his thesis [223], Hichert has observed a number of interesting topological properties of the six strong dichotomies. Here is a representative record of these observations:

The major and consonance-dissonance dichotomies are not only polar with respect to diameter (definition 93 in section 30.1) and span (definition 94 in section 30.1), see figure 30.1. They are also polar with regard to their number of contrapuntal symmetries and interdictions, see figure 31.3.

In other words:

[2] The paper [342] was accepted for publication in the journal *Musiktheorie*, but never published for marketing reasons (!), an anecdote about German musicology which shares a particular flavor.

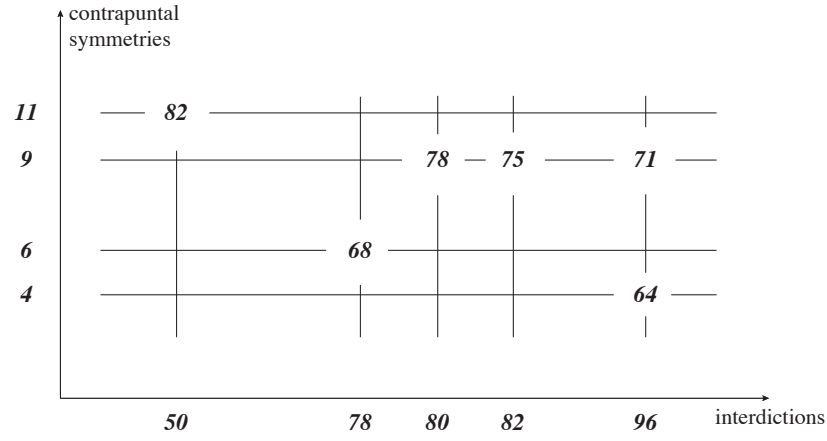


Figure 31.3: From the proof of the counterpoint theorem, one deduces the numbers of contrapuntal symmetries and interdictions. These again position the major and consonance-dissonance dichotomies in a polar relation.

**Fact 17** *The consonance-dissonance dichotomy has a maximum of contrapuntal symmetries and a minimum of interdictions, as opposed to the polar major dichotomy.*

We have already observed in section 31.4.1 that, among the 'diatonic' scales (only minor and major second steps), the major scale has a maximum of freedom of choice for the consonance-dissonance dichotomy. Conversely, the major scale has only culs-de-sac for the major dichotomy.

So we are also interested in the scales with seven tones with respect to the major dichotomy! It is a further fact that among these scales with *no cul-de-sac for the major dichotomy* no European scale appears. There is one such scale, namely $K^* = \{0, 3, 4, 7, 8, 9, 11\}$ of class $\widehat{60}$, which is very interesting. To begin with, it is the consonance set $K$ plus an added 'leading' note 11. So if the major scale is good for consonances and dissonances, the major dichotomy is good for a scale which is intimately related to consonances!

This scale really shows a character which relates to Indian raga music. The basic melakarta framework for ragas is known [103, Bd.8, p.265ff] to be built by the 72 melas. The small number of seven of these mela scales have been used until the present days. One of them, "mayamalavagaula", $Nr.15 = \{c, d_\flat, e, f, g, a_\flat, h\}$, which is class $\widehat{61}$ in our chord classification, can be represented by $\{0, 3, 4, 7, 8, 9, 1\}$. But this is very similar to the above scale $K^*$, we only have to switch the 11 to the 1, as opposed to 11 in 0. So we are led to the question of how far there is a polarity not only in mathematical relations between consonances-dissonance and the major scale as opposed to the major dichotomy and a 'consonance-dissonance' scale $K^*$ which is akin to mela Nr.15 in raga music, but also a global polarity in musical cultures between European and Indian tradition. See also figure 31.4 for this polarization.

*This concluding discussion is an excellent example of the anthropic principle on the level of scale and interval interplay. Seemingly, the historical selection tends to optimize certain abstract*

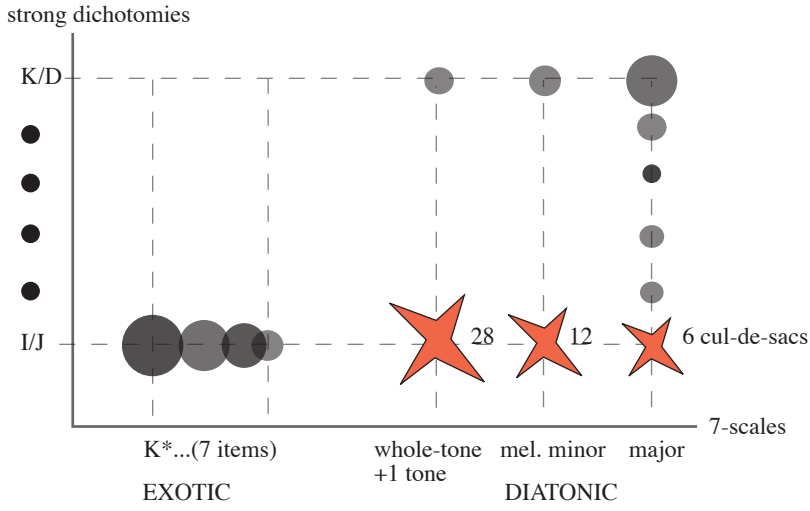


Figure 31.4: Polarity between seven-tone scales and strong dichotomies shows a polarization between European and exotic (in particular: akin to Indian) scales. The size of darkened disks shows the number of progressions while the stars show culs-de-sac.

*a priori properties of topological and transformational character. The possibility to compare such different musical cultures as European and Indian traditions opens a wide field of comparative musicology (relative ethnology) which is based upon systematic results instead of historiographic and ethnographic contingencies.*

# Part VIII

# Structure Theory of Performance

# Chapter 32

# Local and Global Performance Transformations

*C'est l'exécution du poème qui est le poème.*
Paul Valéry

**Summary.** Performance includes a non-trivial transformation from the mental reality of a score to the physical reality of acoustic and, in the limit, gestural realization. We discuss a model of local and global transformation structures and present a preliminary discourse on the need to "shape mental reality" in performance. Local performance transformation structures, together with their syntactic combination to global structures, are formally developed. They involve extrapolation from discrete to continuous or differentiable data; the latter are induced by use of spline techniques. We give a justification of such a procedure from the musical and mathematical points of view, in particular with stress on expressive coherence.

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This part, structure theory of performance, is a turning point in the entire theory of the topos of music. In fact, the preceding parts dealt with general structure theory and then, on a more musicological and music-theoretic focus, with mental perspectives of rhythm, motives, harmony, and counterpoint. In contrast, performance is concerned with the transformation of mental structures into physical ones. This is what traditionally happens in a concert where human artists are performing on physical instruments, including all the richness of human expression on the gestural, emotional, or structural level of physiological, social, and physical parameters.

Evidently, a comprehensive theory of performance is out of reach as long as major constituents have not even been attacked on a scientific level. For example, there is no deeper understanding of the emotional function of music, one knows some extremely elementary facts, for example those concerning the emotional impact of contrapuntal intervals on the emotional brain, as described in chapters 29, 30, and 31. Also on the level of instrumental factors in the quality of a performance, very little is known, for example, concerning the role of instrumental parameters in the communication of musical contents. Worse than that: There is not even a

commonly accepted structure theory of performance, i.e., a theory which deals with the precise and general description of what is a performance in its most elementary shape. For example, the discussion of tempo has not yet been carried to a point of general acceptance of what tempo can be, including a fundamental disagreement on its hierarchical ramifications. One of these discussions could germinate around the distinction between mental (symbolic, logical, call it as you wish) time and physical time. Naively, mental time, as it is encountered on the score notation, looks like something discrete, to be encoded on integers, or some isomorphic submodule $\delta.\mathbb{Z}$ of the real numbers, whereas physical time is parametrized by the full line of real numbers. This is also the point of view of Desain and Honing in [125]: They reduce mental time to

> *discrete time intervals of a metrical structure,*

leaving the smooth part to the

> *continuous time scales of tempo changes and expressive timing.*

This procedure is mathematically incorrect because

- the metrical time is infinitely divisible in itself: No positive lower limit for mental durations has ever been envisaged, metrical time is a topologically dense, not a discrete set in the field of real numbers. Hence, any reasonable (more precisely: uniformly continuous) time function from mental time $E$ to physical time $e$ can uniquely be extended to a time function on the reals (see [261]). There is no conceptual reason to restrict metrical time to a discrete subdomain of the reals.

- Tempo does not deal with something more continuous than metrical time. It is another concept[1]: the inverse differential quotient of a function $E \mapsto e(E)$ between two copies of the real number axis with irreducibly different ontological specifications, namely the musical mental status of the score and the physical status of performed music.

Other misunderstandings floating around in musicological environments [189] maintain that tempo is a locally constant function, i.e., a step function, much like the medieval theories on velocity in the spirit of Oresme (see also our discussion in chapter 4). He would decompose accelerated movements into a succession of uniform movements [485, contribution of Isabelle Stengers on Galileo Galilei].

In what follows, we have not tried to downsize the complexity of performance, it is indeed the most complex subject in musicology. It involves all kinds of considerations concerning the three basic realities of physics, mentality, and psychology. But beyond this ontological diversity, it is of a sophisticated structural nature, involving differential geometry, ODEs, and PDEs, and evidently all the music theory which is presupposed in any reasonable performance theory (though: not in everybody's performance, but blurred gesticulations of musicians "playing their ass off"[2] are not the subject of any performance theory).

We shall however not discuss proper psychological aspects, but restrict ourselves to the mental and physical perspectives. For psychological concerns, refer to [272]. It may appear that we thereby omit an essential and basic point of view, and that such a restriction would

[1] We shall introduce and discuss such concepts in section 33.1.1.

[2] Answer of free jazz saxophonist Werner Lüdi to Cecil Taylor's question: "What's your concept?"

hamper the entire discourse. This is true insofar as we omit an essential perspective. But it is false that this hampers the discourse. The performance discourse has to deal with structural descriptions of performance: What happens if a score is played in acoustical and gestural reality? This is beyond psychology. Also do the rationales of a performance not uniquely rely on the psychological reality (of emotions as stressed in the naive romantic approach), but, in the spirit of Theodor Wiesengrund Adorno or Walter Benjamin, on analytical facts or, in another approach, on gestural paradigms, as investigated by Johan Sundberg, for example.

Performance research is only in its initial phase, in particular with respect to the high level of performing artists. But this is no reason to abbreviate the scientific path with the risk of misunderstanding the beauty and complexity of performance, and to fall into some crevasse of oversimplification. Hopefully, this should help the reader to follow chapters on performance theory with due patience.

## 32.1 Performance as a Reality Switch

**Summary.** There is a sharp dichotomy of realities, communication, and semantical levels between a score and its performances. We expose these facts and their consequences for a theory of performance *versus* mental music theory.

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Performance is more than simply playing an instrument "all improviso"; even in free jazz (if it merits that name), musicians always refer to an inner score. The concept of a score is used here in its generic meaning, i.e., a *generic score* is any written, imagined or conceived scheme for the execution of a musical composition (see [361] for details). In European or Japanese classical music for example, a score is realized as a denotator structure of more or less complex form (see [378], for an explicit score form englobing classical Viennese composition). In improvised music, the scheme is less of a structural nature than of a processual one. The jazz musician, for example, follows action and reaction patterns and rules according to blues schemes and individual dictionaries of motivic, rhythmic, and harmonic elements. We propose this informal but reliable definition:

**Definition 97** *Performance is defined as the physical realization of an interpretation of a generic score.*

We leave the meaning of "interpretation" open in its common understanding, but we do include, and this is a core issue of the subsequent discourse, interpretation in the technical sense of interpretations of local compositions. We also stress the generic attribute "physical" which does include the acoustic realization, but does not exclude further performance parameters, such as the gestural dynamics of a performer. But we do, as already mentioned, exclude psychic parameters and therefore mean strictly physical performance. Notice that this subsumes intermediate technological strata as a special subsystem of physical realization.

## 32.2 Why Do We Need Infinite Performance of the Same Piece?

**Summary.** This section deals with the *a priori* necessity of infinite performance. The argumentation relates to infinite analysis—due to the Yoneda philosophy—and to its communication on the rhetoric level of expressive performance.

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The basic problem here is a very common one: Why do we need infinite performance? We are talking about real performances in real concerts. Do we, and why do we need new performances, again and again? Couldn't it arrive that all possible or—at least—all relevant performances are definitively played at a given moment, and that all successive performances are doomed to the existence of superfluous variants of the arsenal of core performances.

Seemingly, a musical composition, as it is fixed in a score and an associated denotator, is a finite object. So it would be a logical consequence of this finiteness that infinity is not inscripted in a musical composition and its performance. We therefore have two questions: Is there a substantial infinity in the interpretative variety of a given score? And, if this is the case, is there an associated infinity of performances, and why should such varieties have a sense for the listener or for the artist?

The first question is easily answered: Yes, there is an infinity of interpretations of a given finite score denotator. This was already shown in the discussion of iterated interpretations in section 13.4.2.

This is an affirmative answer on the mental level, and one may easily add an infinity of evaluations of any such finite or infinite interpretation, for example on the level of rhythmic, motivic, or harmonic analyses as described previously in the respective chapters. One may also enrich the possibilities of viewing a given interpretation by a variation of the address, a technique which evokes Yoneda's lemma, of course, but which is also very concretely developed in the context of harmonic topologies.

We shall see later in chapter 44.7 on performance operators that the analytical propaedeutics to performance also includes an important class of analyses: the analytical weights, i.e., numerical functions associated with more abstract analyses (such as harmonic topologies). This type of numerical evaluation which was already introduced in the power series $\lambda_{w_0}$ of section 13.4.2 is not just a "boiled-down" version of 'serious analysis', it is just another way of looking at complex configurations. The point is that the categories of 'understanding' a (finite) musical composition are of an infinite character in many respects, and that there is no reason to claim complete understanding by any finitistic argument. So the expression of mental content relies upon an infinite arsenal:

**Fact 18** *Performance conveys an infinite message.*

But infinity in performance also comes in from a second point of view: Performance is inevitably an experiment in understanding or advancing comprehension. The realization of a piece of music in physical spaces creates a view, a flight through a virtual landscape which may reveal new insights to the involved actors, insights which are not anticipated but emerge from a particular visual angle of a determined 'performance flight'. Such an experience is a *mental*

*experiment* as it has been described in chapter 4, and as such it is a creative investigation, not only a reproductive activity for the sake of social coherence and psycho-hygiene.

In this sense, performance need not be agreeable or pleasant—recognizing new aspects within an infinite repertory of analytical structures can be painful, but healthy. Glenn Gould's work is a brilliant piece of history in this investigative field of musical performance. Even his most awkward performances of Beethoven's piano music (such as the strange, cranky performance of the "Hammerklavier" sonata op. 106 or the funny and blasphemic review of the "Appassionata" sonata op. 57) are masterpieces in revealing new aspects of compositions, be it solely to learn how *not* to play them...

## 32.3 Local Structure

**Summary.** Just as with the analytical work, the performance task is also composed of local subtasks which constitute the morphemes of performance transformations. This section describes the emergence of such units and their structure.

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In a first approximation, performance $p$ could be seen as a set map which associates with every element $X$ of a local composition (in a score-related form) a physical event, encoded by an element $x = p(X)$ in a second local composition whose parameters pertain to a form of physical signification.

The coherence of such a performance map is however very rarely a global one. For example, the tempi may change instantaneously, by the indication *istesso tempo*, forcing the artist to restart from the initial tempo after a deformation via a sequence of agogical indications. Or else, left and right hands may follow their separate tempi—except for some meeting points where the onsets should coincide (in Chopin rubato, for example). In orchestral music, the tuning is a function of the instrument, although everything is written in the same orchestra score. Or it may happen that the local performance of an ornament (a complicated trill, say) is an autonomous shaping map, operating independently of the map on the events which neighbor the ornament. We therefore observe a global structure, a patchwork of local chart maps, much like morphisms on global compositions.

### 32.3.1 The Coherence of Local Performance Transformations

**Summary.** Arguments for building local units—based on local compositions—of performance transformations are given. The idea of making performance on local units *coherent* is explicated: The mapping on the unit's domain is defined by a coherent rule, typically described by continuously differentiable transformations on connected neighborhoods of the local unit.

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In the theory of local and global compositions, the definition of morphisms was roughly this: You are given a set map between two sets of music objects and you want to express that this map is not just any set-theoretic map, but shares a type of coherence. This is done by the extensibility of such a set map to the ambient space by an affine space map. This is one

way of expressing coherence between the images of different points of the local composition domain: they are mapped under a formula of affine type. This kind of coherence is also required in classical algebraic geometry when defining morphisms between algebraic sets.

For performance maps, the narrow affine formalism and also the formalism of polynomial maps are both too algebraic in nature. Let us give some arguments to make this clear. If we want to model a fermata tempo shape, we can evidently not restrain to linear expressions since the fermata tempo must decrease in the beginning and increase towards the end of the fermata duration. Moreover, the cognitive rules of logarithmic perception of differences[3] would enforce changes in tempo in the sense that the change of physical onset as a function of mental onset, i.e., $de/dE$ at onset $E$ proportional to the actual value of physical onset $e$, i.e., $de/dE \sim e$ which means that $dE/de \sim 1/e$ and therefore $E \sim \ln(e)$, i.e., $e \sim exp(E)$ so we cannot use polynomials for such a shaping of physical onset. The general shape of physical parameters against mental ones is also strongly documented in glissando and crescendo effects, i.e., changes of pitch $h$ or $l$ as functions of mental time $E$. These two effects are also a strong argument for continuous mental time: Glissando and crescendo must be defined at each moment of a continuous time parameter, otherwise they cannot be realized, not more than the continuous (even differentiable) movement of the conductor's baton.

Of course, such general function requirements could be worked around by gluing of polynomial functions to polynomial spline functions. But this is exactly what we do not want here: We want a unique class of functional expressions for the local coherence. It is this principle: What we can achieve by gluing together local pieces should be left to the global theory instead of hiding the gluing technique on the level of function classes!

### 32.3.2 Differential Morphisms of Local Compositions

**Summary.** This section describes the categories of local compositions with *differential morphisms* on real vector spaces. This formalism captures unreflected prerequisites of musicological approaches on performance instances, such as tempo or tuning.

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Before constructing the categories of differential morphisms, we should specify their objects. As the existing performance theory is only developed for the zero address and for objective compositions, we shall only look at such objects. In general, score denotators live in complex form spaces, including components which are far from being performable, such as bar-lines or pauses. We will not consider these full-fledged structures, but only local compositions which consist of objects that are candidates for events to be performed in physical spaces.

Such local compositions live in space forms which are limits (products are sufficient here), whose factor spaces are submodules of real vector spaces. For example, pitch can be encoded by integers (such as is the case for MIDI key numbers), or by tuples of rationals (as is the case with the Euler spaces built from (just) octaves, fifths, thirds, sevenths etc.), whereas onset is encode by a module $\mathbb{Z}.x$ or $\mathbb{Z}[1/2]$ or $\mathbb{Q}$. Dynamics is usually encoded by integers (such as suggested by the MIDI key velocity numbers $0, 1, 2, \ldots 127$. With these usual values, one may consider local compositions $(K, M)$ with $M$ a submodule of $\mathbb{R}^n$. The same observations are valid

[3] See also appendix A.2.2.

when representing local compositions which are related to physical parameter space forms. We shall therefore, without loss of generality, assume that we are given local compositions $(K, S)$ in simple space forms $S \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}^n)$ of modules[4] $\mathbb{R}^n, n \in \mathbb{N}$. Later, we shall specify the form names when more concrete parameters will be discussed.

Let us first explain the general construction of tangent compositions and their morphisms:

**Definition 98** *Given an objective, zero-addressed commutative local composition $(K, S)$ in a simple ambient space $S \underset{Id}{\longrightarrow} \mathbf{Simple}(M)$ with R-module $M$, then the* tangent composition $TK$ of $K$ *is the composition $(K \times R.K, S^2)$, where $S^t \underset{Id}{\longrightarrow} \mathbf{Limit}(S, \dots S)$ is the t-fold product. The local composition $K$ is called* the basis of $TK$. *For a natural number $t$, the* $t$-fold tangent composition $T^tK$ *is the tangent composition $T(T^{t-1}K) = K \times R.K \times (R.K \times R.K)^{t-1}) \subset S^{2t}$, with special value $T^0K = K$.*

*The* tangent space $T_kK$ of $K$ at point $k \in K$ *is the subset $\{k\} \times R.K$, identified with $R.K$ for its module structure.*

*Given two tangent compositions $TK$ of $K$ and $TL$ of $L$ (both compositions over the same ring $R$). A* tangent morphism $Tf : TK \to TL$ *is a set map $Tf : TK \to TL$ between local compositions $TK, TL$ which factors through the canonical projections $p_K : TK \to K, p_L : TL \to L$ and a (necessarily uniquely determined) map $f : K \to L$, and such that all the fiber maps $Tf_k : T_kK \to T_{f(k)}L$ are linear. For $1 < t$, a t-fold tangent morphism $T^tf : T^tK \to T^tL$ is a tangent morphism $T(T^{t-1}K) \to T(T^{t-1}L)$ whose basis map is a $(t-1)$-fold tangent morphism.*

*The obvious category of t-fold tangent compositions and t-fold tangent morphisms is denoted by $\mathrm{Tan}^t_R$.*

**Lemma 49** *Let $ComLoc^0_R$ be the category of commutative objective local compositions over the commutative ring $R$. Let $f : (K, S) \to (L, U)$ be a morphism in ${}_1ComLoc^0_R$. Then the map*

$$f \mapsto Tf = f \times R.f : TK \to TL$$

*defines an injective natural transformation $ComLoc^0_R \to \mathrm{Tan}^1_R$ onto the subcategory of those tangent morphisms $Tf : TK \to TL$ between tangent compositions such that the fiber maps $Tf_k$ are all the same linear map $T$, and we have $f(k_1) - f(k_2) = T(k_1 - k_2)$ for all couples $k_1, k_2 \in K$, i.e., the map $f$ is defined via $T$ and the value on one single point.*

The proof of this lemma is left to the reader. For the real number field $R = \mathbb{R}$, we are not only interested in the tangent categories, but in those morphisms which extend to differentiable maps on the underlying vector space, or, more generally in maps which extend to any maps of a specific category $Cat$:

**Definition 99** *A* $t$-fold tangent morphism $T^tf : T^tK \to T^tL$ *of t-fold tangent compositions with positive $t$ in $\mathrm{Tan}^t_{\mathbb{R}}$ is said to be t-fold differentiable iff there is a t-fold differentiable map $F$ between the underlying vector spaces such that $T^tF|T^tK = T^rf$. For $0 \leq t$, Such a map is said to be $C^t$ iff it may be extended to a $C^t$ map $F$ between the underlying vector spaces.*

[4] The case $n = \infty$, the countable direct sum of copies of $\mathbb{R}$ for possible Fourier coefficients or similar parameters in sound color spaces will not be considered except for some explicitly described special considerations

*Clearly, for positive $t$, the $t$-fold differentiable and the $C^t$ morphisms, respectively, define subcategories $LocDiff^t$ and $LocC^t$ of $Tan^t_{\mathbb{R}}$, respectively.*

*More generally, if Cat is a category of morphisms on real vector spaces, we denote by $Tan^t_{Cat}$ the category of $t$-fold tangent morphisms which extend to morphisms from Cat. In practice, this mathematical catechism is not very practical, we shall rather use the wording of "an extension of a given $t$-fold tangent morphism to a morphism of the respective category Cat".*

**Example 51** In traditional performance research, tempo is an important feature. It is usually described via Mälzel's metronomic tempo indication (M.M.) of the type "$x$ quarters per minute". This means that we compare mental time $E$ (quarters) to physical time $e$ (minutes) via the quotient $\Delta E/\Delta e$ as a function of $E$. Forgetting about the other parameters for the sake of simplicity, we have a (not necessarily finite) local composition of (real-valued) mental onsets $MyOnsets : 0 \rightsquigarrow OS(\{E_1, \ldots E_i, \ldots\})$ with form $OS \underset{Id}{\longrightarrow} \mathbf{Power}(Onset)$ (see list of mental forms in formula 6.69 in section 6.6). Suppose that we have a performance set map $p$ associating each $E_i$ with a physical onset $p(E_i) = e_i$ of a denotator $MyPhysOnsets : 0 \rightsquigarrow PhysOS(\{e_1, \ldots e_i, \ldots\})$ in the form

$$PhysOS \underset{Id}{\longrightarrow} \mathbf{Power}(PhysOnset)$$

over the physical onset form $PhysOnset \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R})$ (see 6.6). Suppose that the tempo indications are given on $MyOnsets$ by the values $T_i$ at the onsets $E_i$. This data defines a tangent morphism $Tp : MyOnsets \to MyPhysOnsets$ whose linear fibers over $E_i$ are the linear maps $E \mapsto T_i.E$

For the other onsets, tempo is not declared. This is the usual situation in music scores. However, it is not clear what should be the tempo in between the indicated onsets. The point is that one would like to turn the tangent morphism into a differential morphism, at least into a piecewise differential morphism, ie., one that extends to a piecewise differentiable function having the derivatives $T_i$ as required at the arguments $E_i$.

Some musicologists even do not agree with this extensibility requirement, but maintain that tempo is not defined except at the given arguments. We have already countered that position, however, if tempo has to be extended in the sense that we ask for a differentiable morphism giving rise to the present tangent morphism, there are many ways to do so. For example, tempo could be set to a piecewise constant function, giving rise to a step tempo curve which integrates to a piecewise linear map $P$ that extends $p$.

As to the mental contents of a score, this question could be viewed as secondary, but if we agree with Paul Valéry in saying that *c'est l'exécution du poème qui est le poème* (see the catchword of this chapter), then the *total* content of a composition must include the extension of $p$ to $P$.

It should also be stressed that the European understanding of musical time is a type of negative account, time is only interesting if it is over, i.e., the time between two successive notes is non-existent, or, at least, of no existential relevance. Tempo does not exist between two notes since there is no time feeling, performed music has no time except when a note onset intervenes. This is a severe lack of understanding of what musically happens. Such a music understanding is poor and proves a negative, immature relation to time. Positively stated, European time is more a kind of trigger, a Turing machine unit slot transporting the logical processes (typically in harmony).

It is the merit of performance research, in particular the Swedish school of Johan Sundberg and his collaborators, to have pointed at this delicate time question. We shall come back to the extension type problem for tangent morphisms in chapter 36 on expressive performance.

**Exercise 71** Let $Tf : TK \to TL$ be a tangent morphism of $Tan^1_{\mathbb{R}}$ over a basis $K = \{x, y\}$ of cardinality 2 in $\mathbb{R}$. Then $Tf$ is extensible by exactly one 1-fold differential morphism whose coordinate functions $P_i(t)$ are polynomials in $t$ of degree three. Moreover, these polynomials are rational functions in $x, y, f_i(x), f_i(y), t$ which are polynomials for fixed $x, y$. If $Tf_x = Tf_y = 0$, we have $min(f_i(x), f_i(y)) \leq P_i(t) \leq max(f_i(x), f_i(y))$ for $x \leq t \leq y$.

**Exercise 72** Let $p : K \to L$ be a set map of local compositions with $card(K) = m + 1$, then there is a polynomial morphism extension $P$ of $p$ whose coordinate functions have degree $\leq m$. (Hint: Find a straight line through the origin and not parallel to any of the finitely many lines connecting pairs of points of $K$. Take the projection $q$ of $K$ into the orthogonal space to this line. This is a bijection onto the image $q(K)$. Within this orthogonal space, repeat the same procedure until you have projected $K$ into a one-dimensional subspace, call $K^*$ the image of $K$ in this subspace. Then, looking at the map $p^* : K^* \to L$ induced by $p$ on this one-dimensional projection of $K$, the claim is a classical result of polynomial interpolation.)

**Exercise 73** If the points of $K$ in exercise 72 are in general position, then $p$ is a morphism of local compositions and therefore automatically $t$-fold differentiable for every $t$.

#### 32.3.2.1 A Recursive Interpolation Algorithm

The next result is related to what we shall call analytical weights, a key structure in the theory of performance operators. But it pertains to the theory of differential morphisms, so we present it right here. Since the essential statement is a recursive algorithm which is of essential use in programming performance theory, we do not present the result as a lemma but as a construction.

The situation is this: We are given a finite local composition in a simple ambient space $S \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}^n)$, consisting of points $K = \{x_1, \ldots x_s\}$, and contained in the open $n$-cube $C^n = ]u_1, v_1[ \times \ldots ]u_n, v_n[$ of $\mathbb{R}^n$. For $m = 1, \ldots n$, we denote by $C^m = ]u_1, v_1[ \times \ldots ]u_m, v_m[$ the projection onto the first $m$ coordinates, and by $C_m = ]u_m, v_m[$ the projection onto the $m$-th coordinate.

On $K$, we are given a tangent morphism $Tf : TK \to TL$, where the codomain tangent composition $TL$ sits over a local composition $L$ in an ambient space $W \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R})$ of "weights", and where the linear components $Tf_k$ vanish for every $k \in K$. We want to find a $C^1$ extension of $Tf$ which adds no extra extrema to the extrema required per definition on the $K$ points, and which evaluates to the constant value 1 on the complement of the frame cube $C^n$. The meaning of this requirement is that we want to extend the "discrete weight" $f$ to a function $F$ on the entire space $S$ such that the extension is $C^1$, normalizes to the constant value 1 when the arguments tend to infinity, and has values very close to the given values in a small neighborhood of $K$ without producing new extremal values not defined on $K$.

For the following construction, we concentrate on the module $\mathbb{R}^n$ and forget about the underlying space forms. The construction is a recursive one, and we start with $n = 1$. In this case, we suppose that the $s$ points of $K$ are ordered by size, i.e., we have these $n + 2$ points

on the real axis: $u_1 < x_1 < \ldots x_s < v_1$. Extending $f$ to the frame arguments $u_1, v_1$, these arguments are mapped to the values $1 = f(u_1), f(x_1), \ldots f(x_s), 1 = f(v_1)$. From exercise 71 we have a (unique) cubic polynomial extension of the tangent map on every pair of successive points $\{u_1, x_1\}, \{x_1, x_2\}, \ldots \{x_{s-1}, x_s\}, \{x_s, v_1\}$ with zero fibers $Tf_{u_1} = Tf_{x_1} = \ldots Tf_{x_s} = Tf_{v_1} = 0$. We now consider the spline function defined by gluing all the polynomial extensions at the common points of the pairs, i.e., at the elements of $K$. Outside the frame, we extend this spline function by the constant function of value 1. This function is $C^1$ and adds no extra extremal values to the values on $K$ inside the frame.

Suppose now that we have succeeded by induction to construct the following extension: We are given the decomposition $x_i = (z_i, w_i), z_i \in \mathbb{R}^{n-1}, w_i \in \mathbb{R}$. Let $p_n(K) = \{w_{i_1} < w_{i_2} < \ldots w_{i_r}\}$ be the $n$-th projection of $K$ with the different values in increasing order. For every hyperplane $H_x = p_n^{-1}(x), x \in \{u_n, w_{i_1}, w_{i_2}, \ldots w_{i_r}, v_n\}$, we are given either the recursively defined $C^1$-functions $P_x : H_x \to \mathbb{R}$, or the constant function 1 for the frame points $x = u_n, v_n$. For any point $v = (z, w) \in \mathbb{R}^{n-1} \times \mathbb{R}$, the value $F(v)$ is defined as follows. If $w \notin [u_n, v_n[$, we set $F(v) = 1$. Else, there is exactly one interval $[a, b]$ from the successor pairs $(u_n, w_{i_1}), (w_{i_1}, w_{i_2}), \ldots (w_{i_{r-1}}, w_{i_r}), (w_{i_r}, v_n)$ such that $a \leq w < b$. We then evaluate to the value

$$F(v) = P_{a,b,P_a(z),P_b(z)}(w)$$

of the rational function P defined in exercise 71. Since $a, b$ are fixed here, the function is a polynomial in the arguments $P_a(z), P_b(z)$, and $w$. So the function is a $C^1$-function in $v = (z, w)$. On each hyperplane $H_x$, the derivatives of neighboring functions coincide, and the entire function is $C^1$. Finally, since the values are constant 1 outside the frames on the hyperplanes, the constant 1 value is guaranteed outside the $n$-cube $C^n$ and we are done.

This construction has the disadvantage of depending on the order of the coordinates used in the recursion. For every permutation $\pi$ of the $n$ coordinates, we have such a function, call it $F_\pi$. Each such function extends one and the same original tangent function and is 1 outside the cube $C^n$. So we can symmetrize the construction by the weighted sum $F_{sym} = \frac{1}{n!} \sum_\pi F_\pi$. However, for programming tasks, such a symmetrization is very time-consuming.

## 32.4 Global Structure

**Summary.** The global structure of a performance transformation is a patchwork of local performance transformations. The combination and gluing data of local units expresses the syntax of performance. In turn, this performance syntax defines an interpretation of what has been recognized on the analytical level. This "interpretation of the interpretation" obeys its own rules and constitutes a relatively autonomous *rhetorical* shaping of the given "text" and its analytical comprehension.

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Evidently, the local performance structures described above are far from sufficient to grasp realistic performance situations. Basically, there are four reasons for this insufficiency:

- *Instrumental variety*

  The events of a score are not always in one and the same parameter space, especially if we deal with compositions for different instruments. We typically work in a colimit space

$$ScoreOrchestra \underset{Id}{\longrightarrow} \mathbf{Colimit}(ScoreInstr_1, \dots ScoreInstr_k)$$

  of instrumental spaces

$$ScoreInstr_i \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}^{n_i})$$

  with individual coordinator modules. For example, the piano notes have a four-dimensional space associated with pitch, onset, loudness, and duration, whereas the violin space adds crescendo and glissando to the piano space. Ditto for the physical codomain spaces

$$PhysInstr_i \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}^{m_i})$$

  where the mental instrumental events are mapped. But here, the codomains may be very different in dimension according to the real physical instrument which is addressed on the mental score. The typical colimit space over the physical instruments is

$$PhysOrchestra \underset{Id}{\longrightarrow} \mathbf{Colimit}(PhysInstr_1, \dots PhysInstr_k).$$

  This means that performance starts from a local composition $K$ in the ambient space $ScoreOrchestra$ and maps into the local composition of physical events $L$ in the ambient space $PhysOrchestra$. The local compositions $K, L$ are disjoint unions of the subcompositions $K_i, L_i, i = 1, \dots k$ corresponding to the instrumental cofactors in both, $K, L$, and the performance splits into a coproduct of $k$ individual local instrumental performances $p_i : K_i \to L_i$. Even if the mental score level composition $K$ is not a disjoint union of instrumental subcomposition, i.e., a proper colimit, it can be lifted to the coproduct of its cofactors to obtain a disjoint family of performances, so our hypothesis is not restrictive. On the other side, the proper colimits on the physical level are superfluous since it is always (technically) possible to realize the disjoint union of instrumental voices if necessary. It follows that the above splitting is necessary and sufficient for the *orchestral* globalization of local performances.

- *Gluing of local extension strategies*

  The local approach is insufficient when sudden, non-continuous changes of the performance map happen. For example, if starting with tempo M.M. quarter = 120 per minute, and then performing a chain of accelerandi and rallentandi, it may be asked by the composer that we reset the tempo after this tempo variations, by the command "istesso tempo", meaning to return to the original M.M. quarter = 120 per minute. It would be artificial to construct a continuous tempo curve of transition to the reset value. In this case, it is natural to split the performance map into two contiguous domains with their own tempi. Or else, it may happen that the extension category *Cat* is rather strict, allowing only polynomial maps, say. Such a restriction may intervene for reasons of cognitive nature, or because there are hypotheses about the dynamics of performance, such as mechanical

models (see the work of McAgnus Todd [532], for example). Then it is better to glue the performance map from parts which are conformal with *Cat* and coincide on the intersection of charts of the atlas patchwork. This latter strategy is nothing else than the well-known spline approach to the extension of discrete map data, see [499, 500] for a typical reference.

- *Special roles of selected parameters of a given local performance map (hierarchy)*

  This type of globalization effects is perhaps the most interesting. We stated that in performance theory, we are only interested in the performance map on the mental event which will effectively be played, and not in abstract objects such as bar-lines or pauses. But abstract objects may emerge from effectively played event in the following sense: Suppose that a local composition $K$ in some space $S$ has to be performed, for example a space parametrizing onset $E$, duration $D$, and pitch $H$ and suppose that a space $S'$ of selected parameters of $S$, for example onset and duration. We then have the projection $\pi_{S'} : S \to S'$ and the associated projection $K \to \pi_{S'}(K)$. In a great number of performances, it happens that the performance of $K$ factorizes through $\pi_{S'}$, which means this: We look at the performance map $p : K \to L$, and if $L$ lives in the corresponding physical space $PhS$ with the corresponding projection $\pi_{PhS'} : PhS \to PhS'$ and the induced projection $L \to \pi_{PhS'}(L)$. Factorization means that we have a performance map $p_{S'} : \pi_{S'}(K) \to \pi_{PhS'}(L)$ which commutes with the projections, i.e., $\pi_{PhS'} \circ p = p_{S'} \circ \pi_{S'}$.

  Here, the time events in $S'$ are not really played, but their performance determines the time performance of the really played events. Hence, it may be reasonable to add the projected mental event set $\pi_{S'}(K)$ to the real set $K$ (plus some projection and commutativity conditions as shown above) in order to describe the overall situation. This idea is the basis of the so-called performance hierarchies which control the special roles of parameters.

- *Stemmatic deployment of performance*

  Performance is never realized on the spot or by means of a unit process which grasps the analytical data and presents their "rhetorical" shaping at once. Humans have to rehearse again and again, continuously refining their results until they reach the final (or provisionally final) performance.

  This stemmatic deployment process from the sight reading (primavista) performance to the artistically well-devised presentation could be just a human learning process which a machine can achieve at once if it is able to learn the performative substance in abstracto. But this is erroneous, since the logic of performance, the anatomy of the shaping process is a stemmatic one, a multilayered unfolding of deformations of the mental score symbols.

  Rather than a learning process rehearsal is a meditation on the refinement of understanding. So performance is also global in the sense of a multilayered time-dynamic process. Each stemmatic layer is a logical step in the understanding of the rhetorical expressivity.

### 32.4.1 Modeling Performance Syntax

**Summary.** We review the syntactical mechanism of global performance transformation: orchestration, contiguity, hierarchy, and stemmatic layers.

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As was shown above, performance syntax is a bundle of four streams of completely different nature. Contiguity is perhaps the most obvious stream. It reminds us of the syntactic juxtaposition of units within the stream of ordinary language.

The orchestration stream is much more difficult to understand since it conveys a layering of the performative mapping which creates the interplay of voices. The structure of this interplay must be viewed in a multidimensional space of geometric parameters, such as onset, duration, pitch, loudness, of sound color parameters for envelopes, Fourier, FM, or wavelet coefficients and the like, and of gestural coefficients for curves of body movements.

The hierarchical stream is of high cognitive relevance since it describes the leading and slave parameters of performance. Such a hierarchy is likely to produce orientation in the perception of a performance. But it is also of a purely structural relevance since hierarchies are the turning point and key to the stemmatic layering logic. In fact, the successive refinement process of performance is often managed via deformations of hierarchies, i.e., adjustments of relations and functional dependencies among the members within a given hierarchy.

Whereas the three preceding streams are "in praesentia" as semioticians would say, the fourth: stemmatic layering, is of completely different nature. It is not, and this is decisive, of paradigmatic nature, since it describes a development of logical enchainment, i.e., of a juxtaposition of logical stages. But this type of syntagm is not unfolded in time, it is a kind of encapsulated history whose presence cannot be unveiled without a huge amount of ambiguity and uncertainty (see chapter 47 on inverse performance theory), its presence is virtual.

The complexity of global performance is not only due to the four-fold stream of global structures, it is enforced by the completely divergent extension morphisms underlying any two different local charts within a particular stream, and even more dramatically when distributed over different streams.

### 32.4.2 The Formal Setup

**Summary.** This section introduces the formal performance transformation setup via the category $GlTan^t_R$ of global tangent compositions with *t-fold tangent morphisms* and, for $R = \mathbb{R}$, with the associated subcategories $GlDiff^t$ and $GlC^t$ of $GlTan^t_{\mathbb{R}}$.

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From the four types of globalization phenomena, instrumental variety, gluing of local parts, hierarchy of parameters, and stemmatic deployment, the first three will be covered by the following categorical description, while the last one—being of more processual nature—will be treated in chapter 38. Here is the definition which is coined on the definition 36 of a global objective composition.

**Definition 100** *For a positive integer $t$, a* global $t$-fold tangent composition *over a commutative ring $R$ is defined by the following data:*

(i) *A set $G$ and a finite, non-empty covering $I$ of $G$,*

(ii) *a family $(T^tK_u, S_u)_{u\in U}$ of local $t$-fold tangent compositions $T^tK_u$ over $R$, with bases $K_u$,*

(iii) *a surjection $I_? : U \to I : u \mapsto I_u$,*

(iv) *a bijection* $\phi_u : T^tK_u \xrightarrow{\sim} I_u$ *for each* $u \in U$,

(v) *for each couple* $u, v \in U$, *the induced subcomposition*

$$T^tK_{u,v} = \phi_u^{-1}(I_u \cap I_v)$$

*equals the tangent subcomposition of* $T^tK_u$ *induced on the basis*

$$K_{u,v} = p_{T^tK_u}(T^tK_{u,v}),$$

(vi) *for each couple* $u, v \in U$, *the induced bijection*

$$\phi_{u,v} = \phi_u^{-1} \circ \phi_v : T^tK_{u,v} \to T^tK_{u,v}$$

*is an isomorphism of local tangent compositions.*

*The data* (ii) - (vi) *are called an* atlas $\Phi$ for the covering $I$ of $G$. *Two atlases* $\Phi, \Psi$ *for the covering* $I$ *of* $G$ *are called equivalent iff their disjoint union is also an atlas for the covering* $I$ *of* $G$. *A* global tangent composition over $R$ *is a covering* $I$ *of* $G$, *together with an equivalence class of atlases, a fact which we abbreviate by the symbol* $G^I$ *or even by* $G$ *if the atlases or the covering, respectively, are clear from the context.*

*If two global tangent compositions* $G^I$, $H^J$ *over* $R$ *are given, a* $t$-fold tangent morphism from $G^I$ to $H^J$ *is a couple* $(f, \iota)$ *where*

1. $f : G \to H$ *is a set map,*
2. $\iota : I \to J$ *is a set map such that* $f(i) \subset \iota(i)$ *for all covering sets* $i \in I$,
3. *for any atlases* $(T^tK_u, S_u)_U$ *for* $G$ *and* $(L_v, M_v)_V$ *for* $H$, *if we take the chart isomorphisms* $\phi_u : K_u \xrightarrow{\sim} I_u$ *and* $\psi_v : L_v \xrightarrow{\sim} J_v$ *for some pair* $u, v$ *of indexes which correspond under the map* $\iota$ *(i.e.,* $\iota(I_u) = J_v$*), then the induced maps* $f_u : T^tK_u \to T^tL_v$ *define a morphism of local tangent compositions.*

*The global* $t$*-fold tangent compositions and their morphisms define the category* $GlTan^t_R$.

For the real number, $R = \mathbb{R}$, we may consider those global $t$-fold tangent compositions and morphisms which are $t$-fold differentiable or $C^t$, respectively, on the local data, and thereby define the subcategories $GlDiff^t$ and $GlC^t$ of $GlTan^t_{\mathbb{R}}$.

**Example 52** Let $G^I$ be zero-addressed objective global composition which is interpretable by a simple ambient space $S$ of module $M$ over the commutative ring $R$. WLOG we have $G \subset M$, and the atlas is $I = \{G_1, \dots G_n\}$. The transition morphisms $\phi_{i,j}$ between intersections $G_{i,j} = G_i \cap G_j$ are the identities. We then have the inductive system $G_{..} = (G_i, G_{i,j}, \rho_{i,j}, \phi_{i,j})$ with the inclusions $\rho_{i,j} : G_{i,j} \subset G_i$, and $G = colim(G_{..})$. Using the functor from lemma 49, we have an induced inductive system $TG_{..} = (TG_i, TG_{i,j}, T\rho_{i,j}, T\phi_{i,j})$, together with a morphism of inductive systems $p_{..} : TG_{..} \to G_{..}$. This gives the colimit map $p : colim(TG_{..}) \to G$. The domain is $TG^I = colim(TG_{..})$ is covered by the images of the charts $TG_i$. Using theorem 12, we see that the canonical maps $\phi_i : TG_i \to TG^I$ are injective. In fact, the transition morphisms are the identities, therefore the relation $\sim$ of that theorem is an equivalence relation, and the

theorem can be applied. Call $TG^I$ the tangent interpretation associated with $G^I$. This is an interpretation with ambient space $S \times S$, and the projection $p : TG^I \to G^I$ is a morphism of global compositions.

Since the transition morphisms are the identities here, one may also refine the tangent construction to one of differentiable nature.

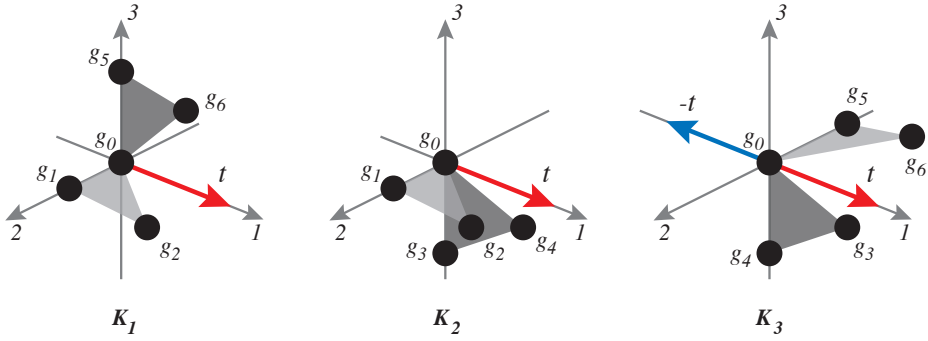


Figure 32.1: The seven-element global composition of which the three charts are shown here is a Möbius bottle. It may not be extended to a global tangent composition by the obvious colimit of its tangent charts and their intersections since, for example, a tangent vector $t$ is equivalent to its negative $-t$ under the colimit construction.

**Exercise 74** For a non-interpretable composition the colimit does not yield a reasonable global tangent composition. The following example illustrates this fact, see also figure 32.1. We consider a global composition $G^I$ whose support

$$G = \{g_0, g_1, g_2, g_3, g_4, g_5, g_6\}$$

consists of seven different points, and whose covering consists of three 5-element charts $G_i$ having the common point $g_0$:

$$G_1 = \{g_0, g_1, g_2, g_5, g_6\}, G_2 = \{g_0, g_1, g_2, g_3, g_4\}, G_3 = \{g_0, g_3, g_4, g_5, g_6\}.$$

They are in bijection with these local compositions $K_i$ in $\mathbb{R}^3$:

$$\phi_1 : G_1 \to K_1 :$$
$$g_0 \mapsto (0,0,0), g_1 \mapsto (0,1,0), g_2 \mapsto (1,1,0), g_5 \mapsto (0,0,1), g_6 \mapsto (1,0,1)$$

$$\phi_2 : G_2 \to K_2 :$$
$$g_0 \mapsto (0,0,0), g_1 \mapsto (0,1,0), g_2 \mapsto (1,1,0), g_3 \mapsto (0,0,-1), g_4 \mapsto (1,0,-1)$$

$$\phi_3 : G_3 \to K_3 :$$
$$g_0 \mapsto (0,0,0), g_3 \mapsto (1,0,-1), g_4 \mapsto (0,0,-1), g_5 \mapsto (0,-1,0), g_6 \mapsto (1,-1,0)$$

The transition isomorphisms are these:

$$\phi_{12} = Id$$

$$\phi_{23}:$$
$$g_0 \mapsto g_0, (0,0,-1) \mapsto (1,0,-1), (1,0,-1) \mapsto (0,0,-1)$$

$$\phi_{13}:$$
$$g_0 \mapsto g_0, (0,0,1) \mapsto (0,-1,0), (1,0,1) \mapsto (1,-1,0)$$

In the colimit of $TK_i, TK_{ij}$, the tangent $t = (g_0, (1,0,0)) \in TK_1$ identifies to $T\phi_{12}(t) = t$, and then to $T\phi_{23}(t) = -t \in TK_{32}$, whereas $T\phi_{13}(t) = t \in TK_{32}$, which means $t \sim -t$, i.e., the canonical maps $TK_i \to colim(TK_i, TK_{ij})$ are not injective—gluing tangent spaces is not feasible. Clearly, this phenomenon is due to the non-interpretability of $G^I$, in fact, this is a kind of Möbius bottle on which any global affine function must identify the values of the pairs $(g_1, g_2), (g_3, g_4), (g_5, g_6)$.

**Exercise 75** However, if any global composition is such that its nerve is one-dimensional, the colimit construction yields a global tangent composition with the expected chart injections. Give a proof of this fact.

Let us discuss the reason why the three globalization perspectives are included in the above definitions. Clearly, the disjoint union of the interpretation morphisms on different instrumental parts is represented by a disjoint union of morphisms on local (or global, if they exist) tangent compositions on each of the instrumental parameter spaces.

Let us give an example of a global gluing operation between local tangent compositions. To keep the discussion simple, let us consider the performance map on the onset axis. Suppose that $G = \{E_0, E_1, E_2\}$ is a sequence of three increasing onsets $E_0 < E_1 < E_2$ which are mapped onto the set of the three physical onsets $g = \{e_0, e_1, e_2\}$, $e_i = P(E_i)$, under the performance map $P : G \to g$. Suppose that the musical conditions are such that one starts with an initial velocity $de/dE|_{E_0} = V_0, de/dE|_{E_2} = V_2$, whereas the velocity is $de/dE|_{E_1} = V_1$ when we arrive at onset $E_1$. But the successive path should be started with velocity $de/dE|_{E_1} = V_1^*$, and end on velocity $de/dE|_{E_2} = V_2$.

This apparent incompatibility at onset $E_1$ can be resolved by the coverings $I = \{G_1 = \{E_0, E_1\}, G_2 = \{E_1, E_2\}\}$, $i = \{g_1 = \{e_0, e_1\}, g_2 = \{e_1, e_2\}\}$ of $G, g$, respectively. Consider the tangent compositions $TG^I, Tg^i$ of the two interpretable compositions $G^I, g^i$, respectively, see figure 32.2.

Consider an extension of the global morphism $TP : TG^I \to Tg^i$ associated with the map $P$ and the above velocity conditions as linear maps on the four tangent spaces

$$TG_{1,E_0}, TG_{1,E_1}, TG_{2,E_1}, TG_{2,E_2}$$

to a given category of maps on the real axis, $C^1$, say. Then we have a "gluing" of two velocities, one from below, one from above, at onset $E_1$.

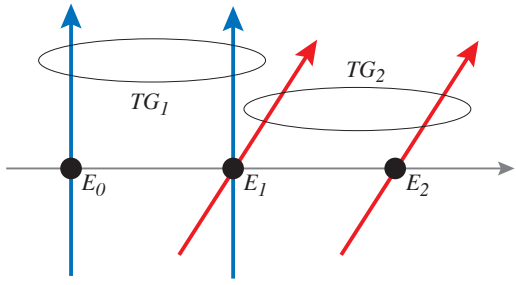


Figure 32.2: The global tangent composition built from two charts $TG_1, TG_2$ of successive tempo regions.

### 32.4.3 Performance qua Interpretation of Interpretation

**Summary.** We finally compare the analytical interpretation in the framework of categories *Glob* of global compositions with associated global performance transformations.

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Recalling the analytical interpretation of a given score denotator, this one a priori regards a variety of involved object-types, such as bar-lines, notes, macros, rests—whatever is needed. Such an interpretation is guided by the analytical approach, such as the topological interpretation using local meters, or motives, or chords, as exposed in the respective chapters.

In contrast, the performance-guided interpretation of the given score is firstly restricted to special event types. Usually, for example, bar-lines are not subjected to performance transformations. Secondly, the interpretation within a performable event-type is not primarily guided by analytical considerations, rather it is related to considerations of coherence in the performative process. For example, the agogical variations in a tempo curve, say, can be shaped by complex weight functions as defined via the maximal meter topology or the nerve topology on the composition's nerve. Such a tempo curve will not refer to the complex covering by maximal meters, but define a unique tangent morphism on a single tangent chart for performance.

So the interpretations of analytical and performative nature can and usually will be very different. Nonetheless, the latter interpretations are related in a complex way to the analytical background interpretations. We shall devote the following chapters to the explication of these relations which belong, to our belief, to the most fascinating challenge in mathematical music theory.

# Chapter 33

# Performance Fields

*La musique mathématiquement discontinue*
*peut donner les sensations les plus continues.*
Paul Valéry [538, I]

**Summary.** Performance fields are the core of an in-depth theory of performance structure. They are a distinguished type of vector fields which give an infinitely precise, i.e., infinitesimal, account of the 'shaping forces' of a given local performance transformation. Although performance fields are not recognized as such in musicology and traditional performance research, they arise in a completely natural way in the traditional context of *tempo*, *intonation*, and *dynamics*. We give a careful account of this basic fact. A closer look at *articulation* and further sound parameters (apart from onset, pitch and loudness which are used for tempo, intonation, and dynamics) reveals that performance fields should be viewed within a fairly general approach. We define the formal setup. In order to provide a deeper understanding of the semiotic signification process of performance fields, we review the performance philosophies of Theodor W. Adorno, Benjamin, and Diana Raffman.

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## 33.1 Classics: Tempo, Intonation, and Dynamics

**Summary.** This section recapitulates and analyzes the concepts of tempo, intonation, and dynamics. We take the opportunity to make the point of blurred concepts in musicology, in this case in the sense of a fascinating quest for expressive precision, paired with denial of formal explicitness.

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### 33.1.1 Tempo

**Summary.** Tempo is one of the best 'known' features of performance. However, its concept is blurred and far from standardized among musicians and performance scientists. We analyze the

state of the art and its deficiencies. It is shown that tempo is a local concept which is charged with a large amount of semantics. We proceed in filtering out the semantics from the structural data. After this, a precise definition of tempo as a one-dimensional performance field in the onset axis is given.

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Tempo is the Italian word for "velocity". In musical notation and performance, tempo refers to the pace at which the mental events, typically written on a score or imagined by the performer, are projected into physical reality. This is done either to describe the relative velocity or the velocity change with respect to a given absolute tempo situation; typically indicated by notation such as verbal annotation "accelerando", "rallentando", or by corresponding pictorial signs. Or else, the tempo is described with an absolute meaning. The prototype of this second situation is Mälzel's Metronome (M.M.), quantified by a quotient such as "quarter = 120", i.e., a quarter note is given the physical duration of 0.5 second, resulting in 120 quarters per minute. In contrast to this precise indication, a relative tempo sign within a Mälzel context does not define precisely how much the tempo should change, and in which way this should happen., e.g., by a linear curve or in a quadratic way, etc. The relative tempo signs are massively ambiguous. We shall have to spend quite some time in order to set up a reliable handling of such blurred signs in algorithmic contexts.

Absolute tempo is also indicated by more poetic indications, such as "andantino", "prestissimo", "maestoso". Clearly, such wordings are loaded with a good portion of connotation which exceeds the mere tempo and targets emotional refinement, expressed via other parameters, such as articulation (legati, staccati), dynamics (loudness variations) or intonational deformations (on violins or for the human voice).

Whatever the specific expression, *tempo relates to the transformation of mental onsets to physical onsets*. Note that this does not imply that such a transformation is independent of the other event parameters. We shall learn that the performance of onsets may involve all other parameters. A simple example of such a relation is given by the so-called "Chopin rubato" which lets the left hand perform a "mother" tempo while the right hand performs local deviations of the left-hand tempo in order to generate the effect of temporal tension ("daughter" tempi). In this case, the common onset of left and right hand notes may lead to different images in physical time, as a function of the interpretation of the total score by the left-hand and right-hand charts.

Formally, local tempo is related to a tangent morphism $Tf : TK \to Tk$ of local tangent compositions $TK, Tk$ derived from compositions $K \subset Onset, k \subset PhysOnset$ in onset spaces. Tempo is then defined iff the linear fiber maps $Tf_X, X \in K$ are invertible and there, we define $Tempo_X = (Tf_X)^{-1}$, the inverse slope at $X$. Moreover, tempo is always supposed to be positive, while negative tempi are without evident musical meaning.

In such a situation, tangent morphisms are extended to differentiable morphisms with positive, continuous derivatives at every point. The *tempo curves* associated with such differentiable morphisms are the continuous, positive functions whose values are the inverse derivatives at all points in an open interval containing the points of $K$. Usually, the tangent morphisms are not of immediate interest in performance theory, this means that one is given differentiable morphisms, ie., the morphisms $F$ inducing tempo curves as their inverse derivatives $Tempo_X = (TF(X)^{-1})$. However, it may then also be required that these functions be differentiable extensions of tangent morphisms, but this is not the mandatory situation.

Nonetheless, the music(ologic)al point of view which negates differentiable extensions is not correct: tempo is also present in between the points of $K$, and its structure is an essential information of the performance maps. Formally speaking, we are dealing with a local composition $K \subset Onset$, a local composition $k \subset PhysOnset$, two closed intervals $OFrame = [A, B] \subset Onset, POframe = [a, b] \subset PhysOnset$ such that $K \subset OFrame, k \subset POframe$, and a $C^1$ diffeomorphism $F : OFrame \tilde{\to} POframe$ with positive derivative $TF(X)$ at each point $X \in OFrame$, and such that $F(K) = k$ (and possibly $F$ extending a given tangent morphism $Tf : TK \tilde{\to} Tk$ if that is required). The *tempo field of $F$ on $OFrame$* is the continuous field $Tempo$ with values $Tempo_X = (TF(X)^{-1}), X \in OFrame$.

By construction, if the performance map $x_0 = F(X_0)$ is defined on any point $X_0 \in OFrame$, the performance $F(X)$ on $X \in K$ is defined by the integral[1]

$$F(X) = x_0 + \int_{X_0}^{X} \frac{1}{Tempo} \tag{33.1}$$

of the inverse tempo function. Musically, the "initial value" $x_0 = F(X_0)$ means that the "conductor" defines a starting time $x_0$ of performance from which the remaining performance onsets can be deduced by means of the given tempo curve.

**Exercise 76** With the above notation, calculate the onset function $F$ for these tempo types: (1) $Tempo_X = q_0 + q_1 X$, (2) $Tempo_X = q_0 + q_1 X + q_2 X^2$, (3) $Tempo_X = e^X$, (4) $Tempo_X = 2 + \sin(X)$. Discuss the possibilities of coping with these tempo types with tangent conditions.

In general, the tempo must also cope with more than one given performance value, $x_0 = F(X_0), x_1 = F(X_1), \ldots x_t = F(X_t)$, and then, we have to ask for the conditions

$$x_j - x_i = \int_{X_i}^{X_j} \frac{1}{Tempo} \tag{33.2}$$

for $i, j = 0, 1, \ldots t$ to guarantee compatibility of the tempo integral with the given values.

### 33.1.2 Intonation

**Summary.** Intonation deals with pitch just as tempo deals with onset time. The definition of intonation as a one-dimensional performance field in the pitch axis is given.

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Intonation, including the specialization of tuning, deals with the relation between mental pitch and physical pitch. It is analogous to tempo insofar as it suggests a map from the pitch space $Pitch$ to the pitch space $PhysPitch$. But the situation is not so easy, if we observe the common pitch spaces in music theory. Recall that symbolic pitch is usually represented in the Euler module space $EulerModule$, and that we have a space morphism $E2M : EulerModule \to MathPitch$ defined in equation (6.28). This is an injective $\mathbb{Q}$-linear map on the supporting

[1] We follow the Douady notation and do not write the old-fashioned infinitesimal "$dY$" in the integrand if this one is clear.

modules $\mathbb{Q}^3$, and $\mathbb{R}$, respectively. If we want to apply calculus to this situation, we have to extend the scalars to real numbers and then consider the induced map $E2M \otimes \mathbb{R} : Euler\mathbb{R}Module \to Pitch$ with underlying vector spaces $\mathbb{R}^3$, and $\mathbb{R}$, respectively. This latter morphism is no longer injective (in fact: surjective with two-dimensional kernel $pv^\perp$, see formula (6.27)). If we consider the performance map composed from the given performance of pitch, $p : Pitch \to PhysPitch$, and the above $E2M \otimes \mathbb{R}$, we are confronted with a performance map $p \circ E2M \otimes \mathbb{R}$ from Euler space to physical pitch which is far from being a diffeomorphism. This cannot be repaired since there is no physical space corresponding to the Euler construction, at least not in the simple physical pitch dimension. The solution could consist in a construction of a direct performance map from $Euler\mathbb{R}Module$ to a physical "pitch" space of higher dimension. This one would evidently not restrict to mere physical pitch (logarithm of frequency, see appendix A.2.3), but include other parameters, such as sound color or the like. This is however an open problem in performance research: How should we perform Euler space pitch?

We will stick to the state of the art and build pitch performance upon the maps of type $F : Pitch \to PhysPitch$. Such maps should be the $C^1$-extensions of tangent maps $Tf : TS \to Ts$ over bijective basis maps $f : S \to s$. Here, $S \subset Pitch, s \subset PhysPitch$ are a number of pitches and corresponding physical pitches. This situation is formally equivalent to the tempo situation. We may indeed take over mutatis mutandis the statements and formulas established for tempo as follows. We are given a mental pitch frame $PFrame = [U, V]$, a physical pitch frame $PPframe = [u, v]$, inclusions $S \subset PFrame, s \subset PPframe$, and a $C^1$-diffeomorphism $G : PFrame \xrightarrow{\sim} PPframe$, the *intonation curve*, with positive derivative $TG(X)$ for every $X \in PFrame$ (and possibly extending a given tangent morphism $Tg : TS \xrightarrow{\sim} Ts$ if that is required). The *intonation field of* $G$ *on* $PFrame$ is the continuous field $Intonation$ with values $Intonation_X = TG(X)^{-1}, X \in PFrame$.

By construction, if the performance map $x_0 = G(X_0)$ is defined on any point $X_0 \in PFrame$, the performance $G(X)$ on $X \in S$ is defined by the integral

$$G(X) = x_0 + \int_{X_0}^{X} \frac{1}{Intonation} \tag{33.3}$$

of the inverse intonation function. Musically, the "initial value" $x_0 = G(X_0)$ means that the piece is played from a starting pitch, i.e., the "chamber pitch" $x_0$ of performance from which the remaining performance pitches can be deduced by means of the given intonation curve.

In general, the intonation must also cope with more than one given performance value, $x_0 = G(X_0), x_1 = G(X_1), \ldots x_t = G(X_t)$, and then, we have to ask for the conditions

$$x_j - x_i = \int_{X_i}^{X_j} \frac{1}{Intonation} \tag{33.4}$$

for $i, j = 0, 1, \ldots t$ to guarantee compatibility of the intonation integral with the given values. This is, in particular, the case when we are given a specific intonation between semitones in a fixed tuning mode, such as well-tempered or just tempered tunings.

Again, musicological approaches only deal with intonation values on the given tangent composition $TS$, while the intermediate values of the intonation curve are neglected (or even negated—in the worst case). However, if glissando effects are present, discrete intonation fails to give information about the intermediate values, much like the intermediate tempo is necessary to perform glissandi in their development along the time axis!

### 33.1.3 Dynamics

**Summary.** Dynamics deals with loudness just as tempo deals with onset time, and intonation deals with pitch. The definition of dynamics as a one-dimensional performance field in the loudness axis is given.

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Dynamics is the physical shaping of loudness symbols such as *ff, mf, ppp, mp, sf* and words such as *sforzato, meno forte, diminuendo, crescendo*. In this classical setup of score notation, the scope is much less quantified than with tempo or intonation. There is no such norm as Mälzel's metronome or the chamber pitch in dynamics. Also, the shaping of dynamics is dramatically different, i.e., refined, with respect to the written prescriptions. It may happen that a section is written in *mezzo forte*, but within this section, the performance of *mezzo forte* is quite variable, within a certain tolerance bandwidth of what *mezzo forte* can be felt. However, just for these reasons, it is much more accepted by musicologists that dynamics is a continuous phenomenon when compared to tempo and intonation.

Sticking again to the state of the art, we build loudness performance upon the maps of type $F : Loudness \to PhysLoudness$. Such maps should be the $C^1$-extensions of tangent maps $Tf : TL \to Tl$ over bijective basis maps $f : L \to l$. Here, $L \subset Loudness, l \subset PhysLoudness$ are a number of loudness values and corresponding physical loudness (logarithms of pressure amplitude, see appendix A.2.2). This situation is also formally equivalent to the tempo situation. We may indeed take over mutatis mutandis the statements and formulas established for tempo as follows. We are given a mental loudness frame $LFrame = [U, V]$, a physical loudness frame $PLframe = [u, v]$, inclusions $L \subset LFrame, l \subset PLframe$, and a $C^1$-diffeomorphism $H : LFrame \xrightarrow{\sim} PLframe$, the *dynamics curve*, with positive derivative $TH(X)$ for every $X \in LFrame$ (and possibly extending to a given tangent morphism $Th : TL \xrightarrow{\sim} Tl$ if that is required). The *dynamics field of $H$ on $LFrame$* is the continuous field $Dynamics$ with values $Dynamics_X = TH(X)^{-1}, X \in LFrame$.

By construction, if the performance map $x_0 = H(X_0)$ is defined on any point $X_0 \in LFrame$, the performance $H(X)$ on $X \in L$ is defined by the integral

$$H(X) = x_0 + \int_{X_0}^{X} \frac{1}{Dynamics} \tag{33.5}$$

of the inverse intonation function. Musically, the "initial value" $x_0 = H(X_0)$ means that the piece is played from a starting dynamics, i.e., the "mezzo forte" $x_0$ of performance from which the remaining performance dynamics can be deduced by means of the given dynamics curve.

In general, the dynamics must also cope with more than one given performance value, $x_0 = H(X_0), x_1 = H(X_1), \ldots x_t = H(X_t)$, and then, we have to ask for the conditions

$$x_j - x_i = \int_{X_i}^{X_j} \frac{1}{Dynamics} \tag{33.6}$$

for $i, j = 0, 1, \ldots t$ to guarantee compatibility of the dynamics integral with the given values. This is, in particular, the case when a specific dynamics range between successive symbols, such as *ppp, mpp, pp, mp, p, mf, f, mff, fff*, is required for acoustical reasons. In the precise technical

sense this is a rare situation which occurs more in studio situations than in live performances. But also in common (what is that in the age of complex man-machine interfaces?) human performance, it may occur that one musician or instrument dictates the dynamical ranges by individual approaches, and that other musicians in the orchestra have to cope with these constraints. For example, the enhanced dynamics of a strong beating drummer can trigger dramatically the dynamical range of a whole jazz band.

## 33.2 Genesis of the General Formalism

**Summary.** Tempo, intonation and dynamic fields are one-dimensional special cases of performance fields. We discuss the a priori two-dimensional performance field of tempo and articulation. From this special case, a general type of performance fields is deduced and motivated by a set of representative examples. We conclude the section with a rigorous definition of the concept of a performance field.

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If we consider a local composition $K \subset Onset \oplus Pitch$ (see section 6.4.1 for this space), the performance of such a composition is a bijection $h : L \xrightarrow{\sim} l$ with codomain $k \subset PhysOnset \oplus PhysPitch$. Suppose that this bijection is induced by bijections on the onset and pitch axes as described above. With the above notation, we consider the projections $p_{Onset} : Onset \oplus Pitch \to Onset, p_{Pitch} : Onset \oplus Pitch \to Pitch$. Consider the two projections $L \to K = p_{Onset}(L)$, and $L \to S = p_{Pitch}(L)$. Suppose that these projections are performed conforming to the above rules, i.e., there are frames $OFrame, PFrame$ with $K \subset OFrame$, and $S \subset PFrame$. Suppose also that we have diffeomorphisms $F, G$ as above. Let our performance $h$ be defined by the projection morphisms, i.e., for $(X, Y) \in L$, $h(X, Y) = (F(X), G(Y))$. This means that we are given a product $C^1$-diffeomorphism $H = F \times G : OFrame \times PFrame \xrightarrow{\sim} POframe \times PPframe$ which restricts to $h$ on $L$. Evidently, the two one-dimensional vector fields $Tempo, Intonation$ induce a two-dimensional vector field $Tempo \times Intonation$ on the product frame $OFrame \times PFrame$, see also figure 33.1. Whereas the factor fields are derived from the tangent morphisms $TF, TG$, the product vector field is evidently not derived from the tangent morphism $TH$. In fact, the latter is a two-dimensional linear transformation

$$TH_{X,Y} = TF_X \times TG_Y$$

and not a vector. We recognize however that the product field verifies this equation:

$$Tempo_X \times Intonation_Y = (TH_{X,Y})^{-1}.\Delta \tag{33.7}$$

where $\Delta = (1, 1)$ is the diagonal unit vector in the tangent space at the $H$-image $(x, y)$ of $(X, Y)$. This means that *the tempo-intonation field is derived from the diffeomorphism $H$ which extends the map $h$ on the underlying composition $L$ via the inverse image of the diagonal field $\Delta$ on the image product frame $POframe \times PPframe$.*

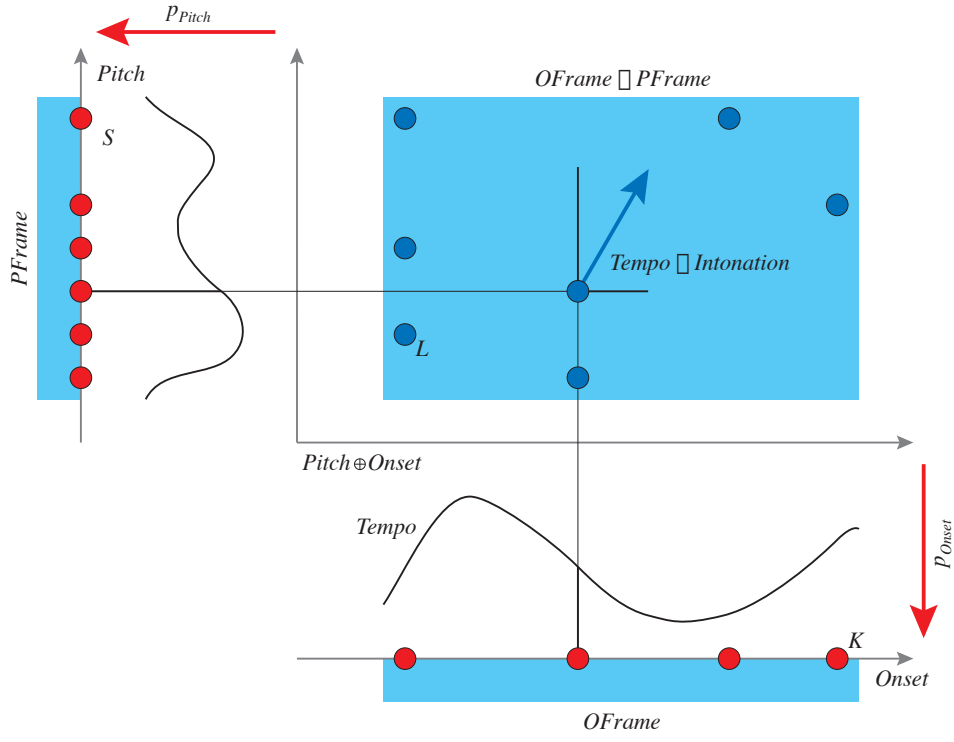


Figure 33.1: The two one-dimensional vector fields $Tempo, Intonation$ induce a two-dimensional vector field $Tempo \times Intonation$ on the product frame $OFrame \times PFrame$.

### 33.2.1 The Question of Articulation

**Summary.** Special attention is given to the delicate relation between tempo and articulation. It reveals that one-dimensional performance fields are a priori insufficient to describe local performance transformations. Articulation requires a priori two-dimensional fields.

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It may appear as if the two-dimensional tempo-intonation field were an artificial generalization of an essentially one-dimensional situation. This is however not the case as we now show for the two-dimensional performance on the plane of onset and duration.

This situation is as follows. Suppose that we have a local composition $K \subset Onset \oplus Duration$ which should be performed with respect to a tempo performance of the onset projection. More precisely, call such a performance $F(E, D), (E, D) \in K$, and recall the sweeping alterator already used in counterpoint (and introduced in section 7.5):

$$\alpha_+ : Onset \oplus Duration \to Onset : (E, D) \mapsto E + D$$

with the omission of the dual number formalism here because we do not need this algebraic enrichment of structure in performance theory. Suppose that the onset projection $K_O = p_{Onset}(K)$ of $K$ is performed according a performance map $f : K_O \to k_O$ onto an image set $k_O$. Where should we map the duration components? The canonical recipe is to consider the "offsets" of our events, i.e., for $(E, D) \in K$, we take its alteration $Off = \alpha_+(E, D) = E + D$ and look for its image $f(Off)$. We may then calculate the duration $d$ in the image $(e, d) = F(E, D)$ by the formula $d = f(Off) - f(E)$.

This is however not well defined if we do not know what is the image of onsets outside $K_O$! This is another strong argument for the existence of tempo outside a composition's onsets. So we have to embed the onset performance $f$ in a $C^1$-differentiable extension, to be defined on an onset frame $Frame$ as above. Call this extension again $f$. Observe that $Frame$ must now also contain the local composition of alterates $\alpha_+(K)$, i.e.,

$$K_O \cup \alpha_+(K) \subset Frame. \tag{33.8}$$

With these conventions, we may calculate the tangent morphism $TF$ associated with the map

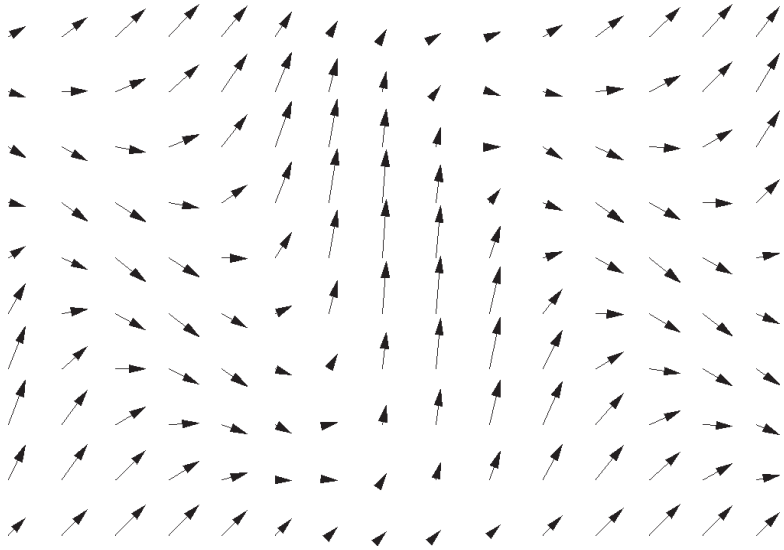

Figure 33.2: This parallel articulation field $\partial Tempo$ is derived from the tempo curve $Tempo_E = 1 + 0.4\sin(E)$. The horizontal axis is onset, the vertical axis is duration.

$F(E, D) = (e(E, D), d(E, D)) = (f(E), f(E + D) - f(E))$. The tangent map at $(E, D)$ is described by the Jacobian

$$JF_{(E,D)} = \begin{pmatrix} \partial_E e & \partial_D e \\ \partial_E d & \partial_D d \end{pmatrix} = \begin{pmatrix} 1/Tempo_E & 0 \\ 1/Tempo_{E+D} - 1/Tempo_E & 1/Tempo_{E+D} \end{pmatrix}$$

which yields the inverse value

$$(JF_{(E,D)})^{-1} = \begin{pmatrix} Tempo_E & 0 \\ Tempo_{E+D} - Tempo_E & Tempo_{E+D} \end{pmatrix}$$

whence the inverse image vector field of the constant diagonal field $\Delta$ on the physical plane:

$$(JF_{(E,D)})^{-1}.\Delta = \begin{pmatrix} Tempo_E \\ 2Tempo_{E+D} - Tempo_E \end{pmatrix} = \partial Tempo_{E,D} \tag{33.9}$$

which we call the *parallel field $\partial Tempo$ of articulation* since the duration component is calculated in parallel to the onset by use of offset values. Figure 33.2 shows such a parallel field. We see that the direction of such an articulation field is from $-\pi/4$ to $\pi/2$.

From this example, we learn that the performance field of the parallel articulation map on the onset-duration plane is not a product of one-dimensional fields, even under the completely innocuous assumption of durations being induced by offset data.

**Exercise 77** Calculate the performance field of articulation if the performance map has a dilatation by $0 < \lambda$ in the duration component, i.e., $F(E, D) = (f(E), \lambda(f(E + D) - f(E)))$.

The example of a parallel field in articulation can be taken over to the pitch domain: Instead of duration we have to think of glissando. This is the proportion of pitch shift with respect to the pitch coordinate if the pitch at the end of an event must have a different pitch with respect to the onset pitch. This generates the parallel glissando performance field $\partial Intonation$ with exactly the same formalism as for articulation. The same method can be applied to generate a parallel crescendo field $\partial Dynamics$ if $Dynamics$ is the field associated with loudness performance. The details are left to the reader. Whenever we deal with basis and pianola spaces, we use this notation: If $B = B_1 \oplus \dots B_k$ is a product space of basis parameters (such as onset, pitch, loudness), and if $P = P_1 \oplus \dots P_k$ is the product space of corresponding pianola parameter spaces, we denote by $\alpha_+$ the alteration map $B \oplus P \to B$ defined by $(b, p) \mapsto b + p$ on each basis-pianola component, whereas $p_B$ is the first projection. So parallel performance maps and the corresponding parallel performance fields are defined by use of the alteration $\alpha_+$. If the basis field at the basis point $X$ is $Z_X$, the parallel field at the basis-pianola point $Q$ is

$$\partial Z_Q = (Z_{p_B(Q)}, 2.Z_{\alpha_+(Q)} - Z_{p_B(Q)}), \tag{33.10}$$

a linear operator in $Z$. We shall use this operator not only for basis-pianola couples, but in general situations of a direct product of isomorphic simple spaces $B, P$ with the associated alteration map.

### 33.2.2 The Formalism of Performance Fields

**Summary.** This section puts the previous considerations and special cases into a generic formalism, the concept of a performance field in general musical parameter spaces.

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The general situation is this. We suppose that a local composition $K \subset S$ is contained in a closed rectangle $R = [a_1, b_1] \times \dots [a_n, b_n]$ of the underlying real n-space. We also suppose that a bijective performance map $\wp : K \to \wp(K)$ onto a local composition $\wp(K) \subset PS$ in a simple physical space $PS$ of same dimension $n$ is given, and that this map can be extended to a $C^1$-diffeomorphism $\wp : R \to \wp(R)$, i.e., a $C^1$-diffeomorphism which is defined on a neighborhood of $R$. The *performance field* of this diffeomorphism is a continuous vector field[2] $\mathbf{Ts}$ on $R$ which is defined by the inverse Jacobian

$$\mathbf{Ts}^{\wp}_X = (J\wp_X)^{-1}.\Delta \tag{33.11}$$

applied to the diagonal unit vector $\Delta = (1, \dots 1)$, i.e., $\mathbf{Ts}^{\wp} = T\wp^{-1}.\Delta.\wp$.

If $x_0$ is a point in $PS$, the integral curve[3] $\int_{x_0} \Delta$ of $\Delta$ through $x_0$ evaluates to $\int_{x_0} \Delta(t) = x_0 + t.\Delta$. For the existence and uniqueness of maximal integral curves of $\mathbf{Ts}^{\wp}$, we suppose that the performance field $\mathbf{Ts}^{\wp}$ is locally Lipschitz[4]. Then, evidently, integral curves of $\mathbf{Ts}^{\wp}$ are transformed into integral curves of the diagonal field under $\wp$. Therefore, if $X \in R$, if $X_0 = \int_X \mathbf{Ts}^{\wp}(t)$, and if the image $x_o = \wp(X_0)$ is known, then we have

$$\wp(X) = x_0 - t.\Delta. \tag{33.12}$$

This means that the performance map $\wp(X)$ can be calculated via the integral curve $\int_X \mathbf{Ts}^{\wp}$ of the performance field if there is at least one point on such a curve for which the performance map $x_0 = \wp(X_0)$ is known. This generalizes what we have already learned for tempo curves and initial performances on selected onsets.

Therefore, performance can be calculated from a Lipschitz-continuous performance field on all points $X$ whose integral curves hit a set $Initial \subset R$ of points whose performance is known in advance. We may now forget about the performance map $\wp$ and start from the performance field and a 'good' initial set, *defining* the performance map via the integral curves and equation (33.12). This performance map will usually only be calculated on the local composition $K \subset R$, or at least for a selected set of points of that frame, and not for any point in $R$, but this is exactly what we want, see figure 33.3.

## 33.3 What Performance Fields Signify

**Summary.** The very complexity of performance fields parallels a strong impact on semantic layers of performance. We want to lay bare this crucial relation. In a first approach, we discuss the contributions of Adorno, Benjamin, and Raffman to the very ineffability of performance nuances. We then deduce the adequacy of performance fields to deal with this "ineffability" and to control it on the level of the sophisticated language of calculus. Finally, we deal with the tension between structural and performative parameters in music. We expose and discuss Helga de la Motte's thesis that, historically, there is an increasing number of performance parameters being transformed into structural score parameters.

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[2] $\mathbf{Ts}$ stands for German "Tempo-Stimmung" and is symbolized by the Hebrew letter "tsadeh".

[3] For integral curves of vector fields, see appendix I.2.3.

[4] See appendix I.2.1.

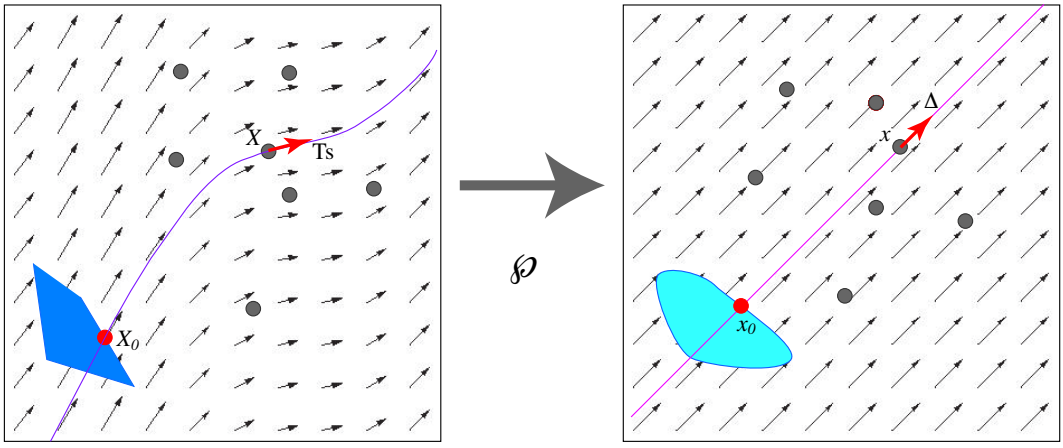


Figure 33.3: Performance $x = \wp(X)$ can be defined upon the performance field Ts and on an initial set (left polygon) where the performance is known in advance.

### 33.3.1 Th.W. Adorno, W. Benjamin, and D. Raffman

**Summary.** Adorno and Benjamin [110] have associated performative adequacy with an activity of "infinitesimal precision". We make plausible that their language suggests the language of vector fields—though not explicitly stated by these authors. Diana Raffman's argument for ineffability of musical nuances in performance [432] is discussed. We relate this admittance of ineffability to the search for a powerful language as an extension of the powerless common language.

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The previous discussion of musicological approaches to performance might have given the impression that in general, musicologists share the tendency to oversimplify the complexity of performance, be it in discrete tempo concepts, be it in correspondingly simplistic understanding of intonation. In fact, the usual understanding of articulation is not better than that. It is defective to the point of not realizing that performance of duration is related to the onset performance plus some deformation of the duration according to articulation rules. We have not encountered any such structural description to date—worse: discussions about these phenomena were dominated by a complete ignorance of this kind of effects.

However, on a non-quantitative level, very intelligent observations have been advanced by the most sophisticated theorists of performance: Theodor W. Adorno, and Walter Benjamin in [7]. Here is their basic text which introduces the "micrological procedure":

> *(Dieses mikrologische Verfahren) darf nicht als ein dem künstlerisch produktiven Entgegengesetztes verstanden werden. Walter Benjamin hat 'das Vermögen der Phantasie' als 'die Gabe, im unendlich Kleinen zu interpolieren' definiert. Das beleuchtet blitzhaft die wahre Interpretation. Der Forderung, Phantasie, als Medium des Lebens der Werke, und Genauigkeit als das ihrer Dauer, zu vereinen,*

> *der Grundfrage, welcher der verantwortliche Interpret sich gegenüber sieht, wird genügt nur durch den gebannten und bannenden Blick auf den Notentext der Werke. In seinem dicht gewobenen Zusammenhang sind die minimalen Hohlräume zu entdecken, in denen sinnverleihende Interpretation ihre Zuflucht findet. (...) Das Medium künstlerischer Phantasie ist nicht ein Weniger an Genauigkeit sondern das noch Genauere.*

As the wording is chosen, micrologic is a logic in the smallest dimensions of a composition and its performance. The text suggests that this procedure could be misunderstood as opposed to artistic fantasy. Adorno evokes Benjamin's observation that fantasy is involved in the infinitely small ("im unendlich Kleinen"), more precisely in the interpolation towards the infinitely small. For Adorno, this is a revealing insight into the ultimate, true performance. Adorno asks for the discovery and inspection of the innermost interspaces ("sind die minimalen Hohlräume zu entdecken"). Infinite interpolation is the tool to do so. And this is not contrary to artistic fantasy, it is, so to speak, the strongest microscopic instrument we have, and should use. Artistic fantasy is not the pseudo-romantic blurredness, but a maximum of precision, of intensity and interplay of minimal movements and forces.

This absolutely central insight of Adorno and Benjamin is not only astonishing in the musicological environment (though not as a category of Adorno's and Benjamin's discourse), it is also a very problematic approach insofar as the humanities—where their text belongs—do not have any means of making such allusions precise. The text is a kind of schizophrenic claim of non-mathematical experts in the words of mathematical concept frameworks: Interpolation, infinitely small, etc.

To the mathematically trained, the allusion to calculus is straightforward. No doubt, the language of the infinitely small is calculus. Is it this kind of language which Adorno and Benjamin were aiming at? What is intriguing is that they are talking of infinite interpolation. Between what? The score is a radically discrete symbolism. The infinite interpolation is not a priori inscripted into the score structure. And, what are those cavities ("Hohlräume") in terms of music parameters and processes? In between the discrete score events, there must be some infinitely divisible space which encompasses the cavities, Adorno and Benjamin are zooming in and penetrating. A solution of this conceptual approach could in fact be the continuous and differentiable interpolation suggested in the previous considerations of extensions of local compositions and maps. We claim that our theory is the mathematically adequate concept framework to the Adorno-Benjamin approach.

More precisely, performance fields which include infinitesimal information about the performance process "between the score units" seem to conceptualize this world of infinitely refined reading of what is happening in the cavities of time, pitch, loudness, articulation, glissandi, and crescendi.

Without anticipating the expressive power of performance fields, it appears that performance fields in their very rich structure could englobe a deep semantic richness towards human expression of all the intentions which human performance commits. We have to imagine that given a score with its discrete event set, the layer of a performance field which is superposed to that score adds an infinitely fine interpolation in the sense of Adorno–Benjamin. It is like an optical lens system which deforms the "mechanical, rigid" score data into a rhetorical expression of the interpreter's understanding.

Performance fields implement a powerful language of performative rhetorics which transcend the discrete vocabulary of common scores. This opens a substantial discussion about ineffability in music. Recall that musical performance is still a strong argument for the ineffability of musical reality: In [432], Diana Raffman has argued that ineffability is a characteristic feature of musical expression, and that this is related to the quale objects as defined by Clarence Irving Lewis in [302]. Quales are those qualities of immediate human experience which cannot be conceptualized and are of private, individual, irreproducible and antilexical nature, such as colors, sounds, hunger, anger, sadness, or happiness. Raffman argues that musical experience is strongly related to quales and therefore shares strongly ineffable characteristics.

However, ineffability means escaping the power of language. And this is the critical point: which language is escaped, transcended, what is the boundary to effability? Clearly, the infinitely small, the infinite interpolation are such ineffabilities to the common language. But they are not to the mathematical language of calculus. Ineffability is a relative concept and not a static verdict. This means that ineffability is a challenge for language extension: Are we able to find a richer language which captures phenomena which were—hitherto—ineffable?

**Principle 24** *We argue that performance fields are precisely such an extension of the music description language which turns ineffable instances of musical expressivity into regions which may be controlled by such a language enrichment.*

After all, vector fields are a very romantic subject: The experience of wind and weather, of stormy rains, of water streams and lava breakouts is a valid metaphor of the forces and processes of our souls. It is not miraculous that performance fields are the exact counterpart of musical expressivity in its most refined appearance as preconized by Adorno and Benjamin in their visionary text.

### 33.3.2 Towards Composition of Performance

**Summary.** Helga de la Motte has hypothesized [122] that, historically, there is an increasing number of performance parameters being transformed into structural score parameters. We discuss this argument and make a picture of its consequences for the composition of performance fields; a language reflecting such a refined performance data must be radically different from the known "digital" language as is common in western score notation.

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It is undeniable that from the early days of neumes to the present, or at least, to the classical European notation, an increasing number of music parameters can be observed in the score notation. For example, the bar-lines were only introduced around 1420, whereas the dynamic signs or the instrumental specification were not present in Bach's *Art of Fugue*, and the metronomic indications became standard only after Mälzel's invention (though sometimes in a problematic way, such as in Beethoven's "Hammerklavier" sonata).

This fact must be seen in the context of the very concept of a score. The concept of a score is that of a mediator between musical ideas and their physical execution. As such, the score of a music piece cannot be narrowed to the level of a notated sign system. The score begins in the mental layers of the composer and musician and is only supported by, not identified with, the

schematic and digitized material form. The distribution between exterior and interior[5] score depends on the specific music culture. In jazz, the major part of the score is present on its interior level, the rudimentary lead-sheet notation sketches only the most elementary ingredients such as harmonic progression and melodic core structures.

This has deep implications on the concept of a composition. If one agrees that a composition is all that is fixed on the "neutral level" of the score, the stratification into inner and exterior score makes the composition concept a bit difficult: Where does the composition begin, where does it end? In fact, the interior part of the score may include mental sketches as well as schemes of performance or construction, such as realized in jazz improvisation or processual schemes in the sense of Cage. If we agree that the composition is everything that can be traced either on the exterior or the interior score, then composition may very well include performative instances if the interior score shapes the mapping from mental to physical reality by a well-defined concept.

In this sense, Helga de la Motte's observation of an increasing shift of performative instances to the score level can be made more precise: It is a shift from inner score instances to those of the exterior score. Now, such a shift is a function of two factors: First, as in jazz, and even more in free jazz, it may not be the objective of a musical approach to aggregate exterior score signs, and then, no such shift is needed. Nonetheless, transcriptions of jazz improvisations may be desired (such as the famous transcriptions of John Coltrane soli by Andrew White), and also to a very refined degree. Second, the composer may want to give a precise description of instances of hitherto interior scores, such as happened with the performance signs of *accelerando*, *crescendo*, etc.

In both cases, the shift can only be achieved by a refined language since the interior score may comprise ineffable spots, regions that cannot be conceptualized on the verbal level: Non-verbal concepts are a widespread phenomenon among musicians! Performance fields may be an extension of the exterior score language which helps shifting non-verbal concepts to the verbal level. But we do not insist on the verbal character of a vector field, at least not in the common sense of verbalization. Mathematical concepts are beyond common language. And they are also beyond quales, they are an effective extension of language which can help thinking things which in the past were completely out of reach to the human intellectual power. Evidently, the score concept which includes such sophisticated objects as vector fields will look completely different from the traditional discrete system. Perhaps it will also per default and as a mandatory condition be related to computer-aided representation and editing. In this sense, the medium computer may help to profile a message which musicians of all cultures have always dealt with and tried to communicate so desperately.

[5] To my knowledge, the term "interior score" was introduced by the Jazz theorist and musician Jacques Siron in his remarkable book "La partition intérieure" [487] on jazz theory.

# Chapter 34

# Initial Sets and Initial Performances

*Jeder Anfang ist ein Ende.*
Hermann Hesse (1877–1962)

**Summary.** Performance has to start somewhere. The theory of this initialization deals with initial sets and initial performance. Naively speaking, initial sets are the first notes of a performance. Semiotically, initial theory describes a turning point from lexicality to reality in music which is supported by shifters; we explain this rationale. Since music deals with many parameters, initialization has to be specified in all dimensions. We first comment on the classical initial sets in onset, pitch, and duration. On an initial set, the performance cannot be calculated from previous performance, an initial performance has to be defined. We discuss ways to do so. On a more technical level, we introduce the hit point theory, a mathematical account for the control of performance field flows as their curves approach initial sets. Strategies of guessing best approaches to initial sets are presented.

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We now situate our investigations in the framework developed in section 33.2.2. This means that we are given

1. a frame $R = [a_1, b_1] \times \ldots [a_n, b_n]$ in the $n$-space $\mathbb{R}^n$,
2. a local composition[1] $K \subset R$,
3. a performance field $\mathbf{Ts}$, i.e., a locally Lipschitz vector field defined[2] on $R$,

[1] We are sloppy about the underlying forms here and just consider the coordinates and modules since the forms are not of primary interest here. It is however subtended that everything happens within well-defined forms that may be evoked if necessary, e.g., if the mathematical methods need a justification via parameter forms beyond numerical coordinates.

[2] By definition, this means that the field is defined in a neighborhood of $R$. But the portions of integral curves of this field contained within the frame $R$ are only a function of the field values within the frame.

4. and an initial set $I \subset R$.

According to the previous theory, we shall suppose that the points $X_0$ of the initial set $I$ are given an *initial performance* $\wp_I(X_0)$, and that we may define the performance $\wp(X)$ for $X \in K$ by means of the integral curve $\int_X \mathbf{Ts}$ if this one hits a point $X_0 \in I$ via formula (33.12).

In this chapter we shall first discuss the meaning of the initial set approach and then techniques to find initial points for polyhedral initial sets.

## 34.1 Taking off with a Shifter

**Summary.** When music performance takes off, a magic moment takes place. This well-known effect relates to shifter signs in the semiotic system of music. We make a point of this magic, in fact a moment with deep consequences for performance as a whole which is fully appreciated by the auditory on an emotional level. This "juncture of fiction and reality" is singular but does in fact happen in several places of a composition's performance. The magic is distributed among the entire performance, we introduce this subject as an interface between semiotics and psychology.

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Even before discussing the different parameter-specific initial sets, it is important to understand the deep meaning of the fact of initial performance. We have known performance as a transitional process (formally described by the performance map $\wp$) from mental to physical reality. On the level of documentation this is a transition from the score to the acoustical realization, to be archived on sound media such as a CD. Following the valid doctrine—as preconized by Valéry[3] and Adorno[4], for example—the performance is an integral part of the work of art, and this means that, in the sense of communication theory of art as described by Jean Molino (see our introduction in section 2.2), performance is part of the semiosis of the work, its meaning is not complete except when it is performed. Put it the other way round: The mental score (interior as well as exterior) conveys a part but not the whole significate, and only via performance can we complete the work's semiosis. Performance englobes a kind of usage of the mental score sign by a performer, much like the usage of a sign in the pragmatic dimension is part of its semiosis. More specifically, those signs whose significate are not only instantiated but substantially depend on the user, are the well-known shifters. Performance of a mental score is such a shift from lexicality to full-fledged meaning, since the pure score is *essentially* less than the work of art. In other words, performance is a shifter characteristic of the score semantics. Production of full-fledged meaning is only possible by means of performance, and this adds a semantic aspect to the sign which is a non-trivial function of the performer(s).

So the anchorage of a score in the physical performance is not only a transformation but also a completion of the score's meaning. It is a completion of shifter type, i.e., adding a new, user-dependent value of the score sign, a value which turns the abstract score structure into a concrete, existential entity.

Within this dramatic transformation process which the conductor Sergiu Selibidache has so violently defended against the musical reproduction industry [476], there is a particularly

[3] "C'est l'execution du poème qui est le poème." [538]

[4] "Die Idee der Interpretation gehört zur Musik selber und ist ihr nicht akzidentiell." [6]

dramatic moment of initiation of the existential kernel. This is what happens in the beginning of a performance.

This beginning is when the conductor (or the soloist in the case of a solo performance) appears on stage, steps to the conductor's desk, takes the baton and freezes every movement in order to get off with the first onset gesture. The moment, when the conductor lowers the baton and unfreezes the time process, is the real magic of performance: Some moments ago, the work was still in its lexical potential state, everything could have happened. But now, we are getting off into reality. The first note puts an end to the potentiality, the shifter level has come into life. We do not know whether it is this switch of existentialities which so incredibly fascinates the auditory (and the orchestra), but it is an objectively dramatic event. It might be compared to the reading of the article on the first person singular pronoun "I" in an encyclopedia, and pronouncing the word "I" as a living person with all the shifted meaning pointed at when you say and mean "I".

The question of what really initiates when the performance gets off is not simple. It was possibly suggested in the previous discussion that it is only a time initialization, but this is as wrong as it would be wrong to claim that music reduces to onset time. In the initial moment of a musical performance, many different settings are instantiated, in fact as many as we have parameters to describe the sound events (and the gestural parameters in an extended performance theory) in our piece of music. In what follows, we want to investigate these initializations in more detail.

## 34.2 Anchoring Onset

**Summary.** The most elementary and important initialization is that of onset. We give the overview of its structure and function, including the shifter nature of onset initialization. The problem of multiple onset initialization is discussed.

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This section uniquely deals with the *Onset* space form and performance fields on this space, i.e., tempo curves. Musically speaking, this approach is a naive one, in fact more involved performance of time does not happen independently of other parameters and therefore cannot be described by tempo curves. For example, if we are involved in a performance where onset is a function of pitch, a situation which may happen in an arpeggio. This is viewed as a temporary onset distortion as a function of pitch. A typical such performance map $\wp$ is as follows:

$$\wp(E, H) = (E - 4e^{-(H-5)^2}, H).$$

The corresponding performance field $\mathbf{Ts}^{\wp}$ is shown in figure 34.1. Observe that onset components of the field may also be negative, according to the retard of onset in the middle arpeggio region of pitch around $H = 5$ in this generic example.

The data for our pure onset performance is as follows: We are given a frame interval $R = [a, b] \subset \mathit{Onset}$, a tempo "field", i.e., a continuous positive tempo curve $\mathbf{Ts}(E) = T(E)$ defined on $R$, and a finite initial set $I = \{E_0 < E_1 < \ldots E_f\} \subset R$ within the frame. On this set, we are given performance data $\wp(E_i) = e_i, i = 0, \ldots f$ with $e_0 < e_1 < \ldots e_f$. The meaning of this data is that the composition $K$'s onset set $K_E = p_E(K)$ is also a subset of

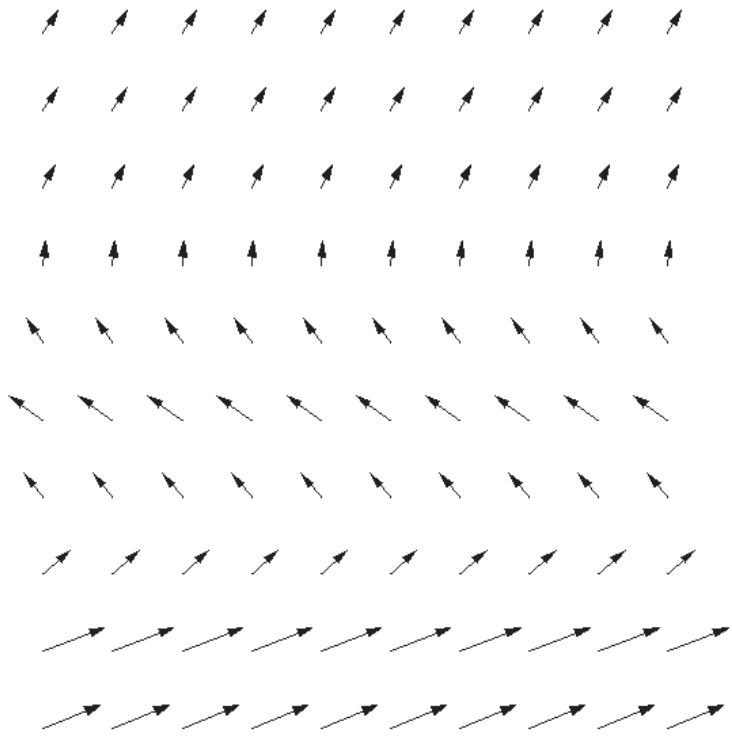

Figure 34.1: The performance field on the plane of onset (horizontal axis) and pitch (vertical axis), corresponding to a prototypical arpeggio. This field cannot be built upon a tempo curve.

the frame, $K_E \subset R$, and that its performance is calculated by the performance formula (33.1). More precisely, in the common situation, the initial set is the singleton $I = \{E_0\}$ consisting of the composition's first referential onset. This could be the first element in $K_E$ or else the first bar-line's onset. In this case, no further discussion is required, the integral exists, and we can calculate all required onset performances.

However, realistic situations are more involved. It may happen that one is given several initial points, for example if the left hand plays in constant tempo, whereas the right hand is allowed to vary locally against the left hand in a "Chopin rubato", i.e., in such a way that the onset performances coincide on each bar-line, but it may differ locally. In this case, the initial points for the left hand would have a fixed performance whereas the right hand tempo curve would have to fit the left hand onset values. This will in fact happen in our discussion of tempo hierarchies, see section 38.2.

This means that we have additional conditions, the integrals of inverse tempo must coincide with the given initial onset performances, i.e., for $0 \leq i < j \leq f$, we must have

$$e_j - e_i = \int_{E_i}^{E_j} \frac{1}{T}. \tag{34.1}$$

Under these conditions, the calculation of any onset performance $\wp(E)$ is clearly independent of the reference initial point $E_i \in I$. In this case we say that *the tempo curve is adapted to the initial performance* $\wp_I$ *on the initial set* $I$.

**Proposition 56** *With the above notation and hypotheses, there is always a unique continuous tempo curve* $T(E)$ *which is linear on each interval* $[E_i, E_{i+1}], i = 0, \dots f-1$, *constant outside* $[E_0, E_f]$, *and prescribes an arbitrary positive tempo* $T(E_i)$ *for one index* $0 \leq i \leq f$.

Proof. We prove the case of a fixed start tempo, the general case works in complete analogy. By induction on $f$, we may restrict the proof to $f = 1$. It is sufficient to show that we may find a final positive tempo value $x = T(E_1)$ such that the linear tempo $T_x(E) = (x - T(E_0))(E - E_0) + T(E_0)$ fulfills $d = e_1 - e_0 = q(x) = \int_{E_0}^{E_1} \frac{1}{T_x}$ for any positive value $d$. But $q(x) = (E_1 - E_0)\frac{\log(x) - \log(T(E_0))}{x - T(E_0)}$, which is a continuous function of $x$, converging to $\frac{E_1 - E_0}{T(E_0)}$ as $x \to T(E_0)$, and ranging from $\infty$ to 0 for positive $x$. Therefore there is exactly one positive $x_1$ such that $d = q(x_1)$, and $T(E_1) = x_1$ solves our problem, QED.

**Corollary 21** *Suppose that for the increasing sequence* $E_0 < E_1 < \dots E_f$ *of symbolic onsets, we are given positive tempi* $T_i = T(E_i), i = 0, \dots f$, *and that the tempo curve* $T(E)$ *on the interval* $[E_0, E_f]$ *is the polygon through these values. Let* $\Delta = \int_{E_0}^{E_f} \frac{1}{T}$, *and select a positive real scalar* $\sigma$. *Then there is a positive scalar* $\tau$ *such that the polygonal tempo curve* $T_\tau$ *for the vertex values* $T_\tau(E_0) = T_0, T_\tau(E_f) = T_f, T_\tau(E_i) = \tau T_i, 0 < i < f$, *has* $\int_{E_0}^{E_f} \frac{1}{T_\tau} = \sigma.\Delta$.

Proof. The integral $\int_{E_1}^{E_{f-1}} \frac{1}{T_\tau} = \frac{1}{\tau} \int_{E_1}^{E_{f-1}} \frac{1}{T}$ assumes any positive value if $\tau$ varies. Further, proposition 56 guarantees that the initial and terminal integrals $\int_{E_1}^{E_0} \frac{1}{T_\tau}, \int_{E_f}^{E_{f-1}} \frac{1}{T_\tau}$ can also be adapted to any positive value with varying $\tau$, so we are done.

This means that given a tangent morphism $Tg$ on the initial values $E_0, E_f$, and a polygonal tempo curve $T$ which extends $Tg$, we can "deform this curve" to a new polygonal curve $T_\tau$ without altering the tangent data $Tg_{E_0}, Tg_{E_f}$, but such that the total duration is stretched by any positive value $\sigma$. In practical situations—such as the composition software *presto*®'s Agologic module (see also chapter 49)—this has the following application: We are given a polygonal tempo curve as in the corollary, with duration $\Delta = \int_{E_0}^{E_f} \frac{1}{T}$. By graphically interactive editing, the curve may be altered in that the vertical position (the tempo coordinate) of one inner vertex of the polygon is augmented or diminished. This changes the duration from $\Delta$ to $\rho.\Delta$, but we do want to conserve duration. This can be achieved by a deformation of the graphically altered polygon as in the lemma, setting $\sigma = 1/\rho$, and we recover the original duration. So the new shape of the polygon can be conserved as far as possible.

## 34.3 The Concert Pitch

**Summary.** In the pitch dimension, initialization deals with concert pitch, i.e., initial intonation or tuning. We compare this initialization with the onset's take-off and discuss the specific difference in deicticity: Concert pitch is dominated by a lexical dimension.

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In this first approach, we shall view pitch performance as a parallel situation to onset performance as discussed in the previous section. This means that pitch performance is viewed independently of other parameters, in particular, independent of time. As with tempo curves, this is a musically naive, but basic approach. In fact, pitch performance may be a function of onset, according to the tonal system of the given composition, for example in modulatory parts.

So we are given a frame $R = [a, b] \subset Pitch$, an intonation field, i.e., a continuous intonation curve[5] $\mathbf{Ts}(H) = I(H)$ on $R$, and a finite initial set $I = \{H_0 < H_1 < \dots H_f\} \subset R$. On this set, we are given performance data $\wp(H_i) = h_i, i = 0, \dots f$ with $h_0 < h_1 < \dots h_f$. In the standard case, this initial set is not as variable as for tempo. In fact, there are several constraints of tuning which are above all given by the orchestral instrumentation. For the piano, to begin with a common reference for orchestral tuning, we are given all semitone pitches $I = \{H_0 < H_1 < \dots H_f\}$ of the chromatic 88 keys[6], together with their rigid performance pitches which are $\wp(H_i) = h_i = h_0 + i.\log(2)/12$ in the common 12-tempered tuning. The underlying intonation curve must cope with all 88 values. We have the following result which replicates proposition 56 for intonation:

**Proposition 57** *With the above notation and hypotheses, there is always a unique continuous intonation curve $I(H)$ which is linear on each interval $[H_i, H_{i+1}], i = 0, \dots f - 1$, constant outside $[H_0, H_f]$, and prescribes an arbitrary positive intonation slope $I(H_i)$ for one index $0 \leq i \leq f$.*

The less rigid case is the tuning of the (continental European) chamber pitch $a' \sim 440\ Hz$, together with the octave periodicity condition on the integral, i.e.,

$$\int_H^{H+12} \frac{1}{I} = log(2)$$

for all pitches $H$ and the semitone encoding of pitch. Such a tuning is independent of local variations of intonation due to intonation specificities for just fifths and the like which violinists and singers may prefer. It is also open to glissandi which include all pitch values in a determined real interval.

There is, however, a qualitative difference between intonation initialization and onset initialization. The latter is a "magic shifter" which by its very construction instantiates the fictitious music score time in real time. The former correspondingly instantiates fictitious score pitch in physical pitch, but its value is not a question of individual construction, it is lexicalized on the standards of music culture and tradition. It is also lexicalized by the absolute pitch perception of a number of musicians, for example Herbert von Karajan, which makes it virtually impossible to alter this conventional initialization on a shifter basis of individual, spontaneous usage. The usage of intonation and its initialization is therefore much more restricted than the usage of onset initialization. One may distinguish the local initialization within a fixed tonal context, between two successive modulations in just tuning of octaves and selected fifths (tonic, dominant for example), say, and the more individual shaping of the individual intonation of less prominent intervals within one such tonal context.

[5] We are sorry for the homonymous symbol $I$ for initial sets and intonation fields, a confusion is however very unlikely.

[6] For the added keys on Bösendorfer's Imperial model, extend the key numbers by $-1, -2, \dots$.

It is clear that such a one-dimensional intonation field component is as artificial as the one-dimensional tempo field. It is not artificial in the sense of a useless academic exercise, but in the sense of first approximation to artistic performance—something like a zero-state of performance—which needs refinement. This is an important observation insofar as it suggests an investigation of the problem of refining performance, of seeking for paradigms of unfolding performance shaping. The theory of performance stemmata in chapter 38 will deal with this approach.

## 34.4 Dynamical Anchors

**Summary.** In the loudness dimension, initialization deals with reference dynamics. This is a dramatic and completely shifting phenomenon which stays in contrast to the onset's take-off and the lexical pitch initialization.

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Instead of repeating the propositions presented in the previous sections, which are, of course still valid mutatis mutandis, we should rather focus on the specific character of dynamical initialization. Above all, the performance of loudness in its physical expression is a complete shifter: There is no lexical normalization since every concert determines its dynamical initialization as a function of the concert hall, the orchestra, its disposition, and the public. Whereas the intonation curve may remain more or less the same over different performances of a determined orchestra, the initial anchoring will vary considerably, though not as fundamentally as with onset, because onset will be an existentially different one in each performance.

There is another shifting character of dynamics initialization: the reproduction of a performance or the live broadcast from an electronic media, such as a radio, TV or internet concert broadcast or the simple playing of a CD on the private sound equipment. In these cases, the individual user's preferences may initialize a very specific dynamic anchorage. And, more dramatically: During the ongoing performance, the initialization may be reset according to hearing dispositions and temporary irritations from disturbing environmental noise, including the abrupt lowering of dynamics while a phone-call or a verbal intervention of another person happens.

Similar to onset initialization, dynamics initialization is quite strong and shifting, in contrast to pitch initialization. But it is also more existential, together with onset initialization: It reflects the human condition of when and how and why music is enjoyed. In this shifter process, evidently, the lexical musical content seems not to suffer, it is more the anchorage in human life which is profiled. Whereas absolute pitch seems to play a certain role for the understanding of the musical message—even to the vast majority of non-absolute pitch listeners—absolute onset and dynamics play the contrary role: They give the listener his/her coordinates of existence where they want to meet this particular music.

## 34.5 Initializing Articulation

**Summary.** The initial theory becomes less trivial when applied to articulation. Initial articulation reveals the complex recursive structure within initial data, i.e., sets and performance.

We explicate different approaches to initial articulation, as based upon initial onsets.

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Whereas initialization on one-dimensional performance fields is a question of selecting finite point sets, initialization on higher-dimensional frames is dramatically more complex. Let us have a look at the most elementary two-dimensional situation: the articulation field, see figure 34.2.

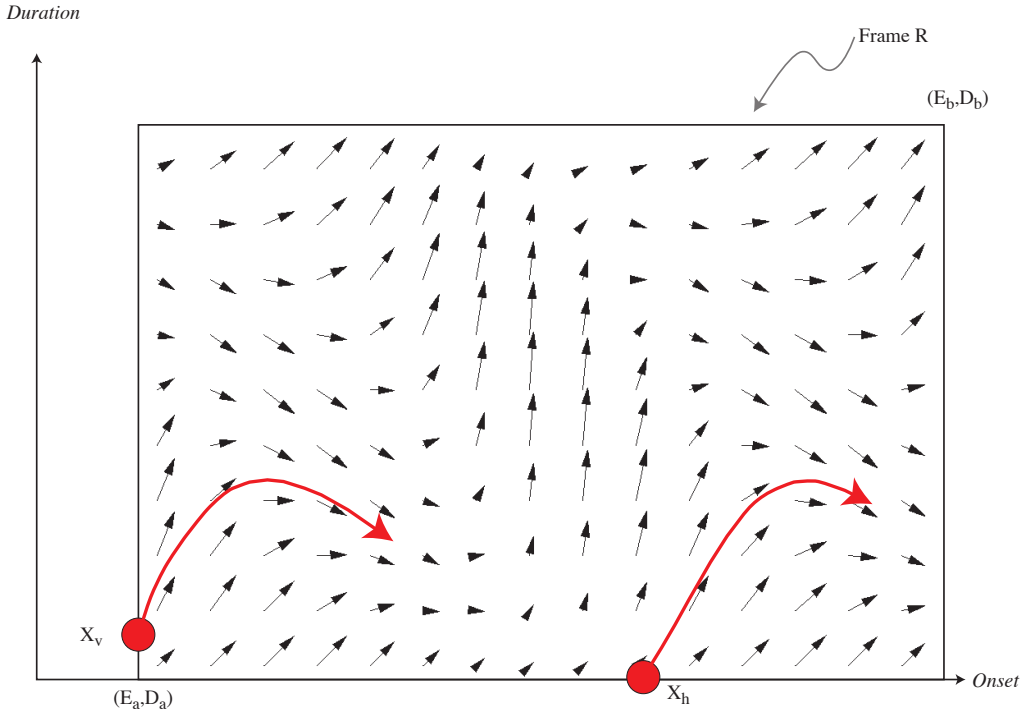


Figure 34.2: The integral curves of an articulation field may typically hit boundary points of the given frame on different positions: the horizontal lower boundary line or the vertical left one. This entails dramatically more complex initialization data than for the one-dimensional case of tempo and intonation.

In this example, we are given a frame $R$ defined by its low left vertex $(E_a, D_a)$ and the high right vertex $(E_b, D_b)$. (In this figure, we even suppose that $D_a = 0$, but this is not the general case.) Let us first look at the right integral curve (terminating at the point on its arrow-head to be performed) which hits the frame boundary on $X_h = (E_h, D_a)$. Initialization can be decomposed in two partial problems: Initialization of onset can be reduced to the one-dimensional case since we have a tempo curve here. This means that we may consider the projection of the articulation field onto the tempo curve and accordingly the projection of

the integral curve onto its onset component. Suppose further for simplicity that the onset initialization is defined on the frame boundary value $E_a$. Then we are done with the onset performance, and we may proceed with duration. Let us look at initial duration in $X_h$. Here, we have the onset performance and should define performance of duration $D_a$. If this value vanishes, it seems natural to perform it to physical duration zero, too. But if the lower bound $D_a$ is positive, it is not clear what should be done. The canonical idea would be to identify $D_a$ as a difference $D_a = (E_h + D_a) - E_h = \alpha_+(X_h) - p_E(X_h)$. We could then apply the well-known construction of duration performance as a difference of onset performances, i.e.,

$$\wp(X_h) = (\wp_E(p_E(X_h)), \wp_E(\alpha_+(X_h)) - \wp_E(p_E(X_h))) \tag{34.2}$$

which is built upon the onset performance $\wp_E$ via the tempo curve and the initial performance on $E_a$. This is not mandatory though: Initial duration performance need not be without articulation, be it legato, be it staccato. Whatever: We have to be sure that not only the onset component of $X_h$ is on the frame, the alteration component $\alpha_+(X_h)$ must also be within the frame's onset bounds! This implies that score points to be performed on this basis must stay to the left of the descending diagonal through the lower right corner of the frame. Other points must be given a different initial performance data.

If, on the other hand, the frame is hit in a point $X_v = (E_a, D_v)$ on the left boundary line, its (initial) performance can be settled by the same formula as above

$$\wp(X_v) = (\wp_E(p_E(X_v)), \wp_E(\alpha_+(X_v)) - \wp_E(p_E(X_v))), \tag{34.3}$$

but now, the initial duration can be any long duration, not just the lowest admitted duration of our frame. In this setup, the initial performance of the left boundary line is a function of the onset performance of the whole onset interval $[E_a, E_a + (D_b - D_a)]$, and not only of the initial onset performance at $E_a$. This is quite dramatic as a contrast to the formula (34.2) which is local on the onset of $X_h$ and the slightly shifted onset $\alpha_+(X_h)$. Here, we have to know a lot about the future onset performance for the initial values at the beginning of the frame.

An even more dramatic effect happens if the frame has an upper boundary line which enforces integral curves to hit the upper boundary when reaching the rectangle's boundary $\partial R$. In our example, this could happen if the frame ended at the half height (with the same field). There, the initial performance of duration can fail to be controllable by onset performance alone, and we have to design new initial performance strategies.

Difficult initial performance problems can also happen if the field doesn't have a positive onset component, as in figure 34.1. Here, initial performance must be defined on the right boundary hyperplane of the frame, i.e., for the last, not for the first events. And if the performance configuration is such that initial values happen to be positioned anywhere within the frame, initial performance has to be defined by use of general strategies which work for any initial set configuration. We shall review this topic in chapter 38 after the discussion of the basic problem: How can we effectively know where an integral curve hits the initial set?

## 34.6 Hit Point Theory

**Summary.** Hit point theory deals with the control of access to initial sets along the integral curves of performance fields. These curves describe the flow associated with performance fields.

In general, they can be quite "wild" so that a generic strategy of seeking hit points of such curves with initial sets is required.

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The general setup for initial sets is a (locally Lipschitz) performance field $\mathbf{Ts}$ defined on (a neighborhood of) a (closed) frame $R$ in a parameter space which—up to isomorphism of functors—is defined by $\mathbb{R}^n$. In many situations, the initial set for the performance $\wp$ is not any set, but will be a polyhedron defined as a union $I = \bigcup_i S_i$ of a finite family $(S_i)_i$ of simplexes $S_i \subset R$. Each simplex $S_i$ is given by a sequence $s^i. = (s^i_0, s^i_1, \dots s^i_{n(i)})$ of (pairwise different) points in $R$. The points need not be in general position, i.e., $dim(\mathbb{R}.s^i.) < n(i)$ is admitted[7]. If we view this data as being the vertexes of the simplex $S_i = \{\Sigma_j \lambda_j s^i_j | 0 \leq \lambda_j \leq 1, \Sigma_j \lambda_j = 1\}$, this means that we admit degenerate simplexes as constituents of initial sets. Also is it not required that the simplexes build a simplicial complex, their intersections can be arbitrary. We also do not require that these simplexes have $n+1$ vertexes: it is allowed to have a number of isolated points and straight lines in three-space, for example. Clearly, this type of initial sets allows virtually every shape—up to approximations by triangulations.

The previous examples are: a series of zero simplexes in $\mathbb{R}$ for tempo and intonation curves, or the sequence $(\partial_E R, \partial_D R)$ of left and lower sides of $R$ in onset-duration space. More generally, given a coordinate $Y$ (or its index $j$ if $Y$ is indexed with respect to $\mathbb{R}^n$) we shall denote by $\partial_Y R$ (or $\partial_j R$) the simplex defined by the vertexes of $R$ which have the lowest $Y$-coordinate (or $j$-th coordinate). Correspondingly, highest $Y$-coordinates define the simplexes denoted by $\partial^Y R$ (or $\partial^j R$). Observe that these simplexes are degenerate for $n > 2$.

The hit point problem is this:

**Problem 1** *Given an integral curve, we have to decide whether and for which curve parameter value the curve will hit the initial set $I$. More precisely, since an integral curve $\int_X \mathbf{Ts}$ is defined starting from an event $X \in R$, we ask for the smallest parameter $|t|$ such that $\int_X \mathbf{Ts}(t) \in I$, i.e., $\int_X \mathbf{Ts}(t) \in S_i$, one of the initial set's simplexes.*

### 34.6.1 Distances

**Summary.** This section is dedicated to the calculus of distances between points on integral curves and polyhedral initial sets.

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Given an polyhedral initial set $I$ which is defined by a family $(S_i)_i$ of simplexes, and any point $X \in R$, the distance $d(X, I)$ is the minimum of the distances $d(X, S_i)$, therefore distances of points $X$ and simplexes $S$ must be calculated. Suppose that $S$ is given by the sequence $(s_0, s_1, \dots s_m)$ of points in $\mathbb{R}^n$.

If $S$ is degenerate, it is the union of its non-degenerate sub-simplexes. This follows from the fact that it is a projection of a non-degenerate simplex, and that any point of this projection stems from a point of a side of the non-degenerate pre-image. So we are left with the calculation of the distance $d(X, S)$ for a non-degenerate $S$. If $dim(S) = n$, clearly, either $X$ is in $S$ or the

[7] The vector space $\mathbb{R}.s^i.$ is the module associated with the local composition defined by the vertexes $s^i.$ of $S_i$.

distance is achieved on a point of $\partial S$. So in the latter case, we are in a recursive situation (the zero-dimensional simplex being trivial).

To decide upon the position of $X$ relative to $S$, we consider the unique representation

$$X - s_0 = (X - s_0)^{\perp} + \sum_{j=1}^{m} \lambda_j.(s_j - s_0) \tag{34.4}$$

of the difference $X - s_0$ as a linear combination of the basis vectors $s_j - s_0$ and the vector $(X - s_0)^{\perp}$ of the (Euclidean) orthogonal space $\mathbb{R}.s.^{\perp}$. Considering the non-singular symmetric quadratic form of scalar products

$$Q = ((s_j - s_0) \cdot (s_i - s_0))_{i,j=1,\dots m}$$

and the vectors $U = ((X - s_0) \cdot (s_i - s_0))^{\tau}_{i=1,\dots m}, \Lambda = (\lambda_j)^{\tau}_{j=1,\dots m}$, we have

$$\Lambda = Q^{-1}.U.$$

With this, the barycentric coordinates are defined by adding the coefficient $\lambda_0 = 1 - \Sigma_{i=1,\dots m}\lambda_i$; we have

$$X = (X - s_0)^{\perp} + \sum_{i=0}^{m} \lambda_i.s_i,$$

and the component $y = \sum_{i=0}^{m} \lambda_i.s_i$ in the affine space spanned by $S$ is in $S$ iff $0 \leq \lambda_i$ for all $i = 0, \dots m$. We now have $d(X, S) = \|(X - s_0)^{\perp}\| + d(y, S)$. If the second distance is not zero ($y \notin S$), we can proceed with the recursive calculation of $d(y, \partial S)$, and we are done. *Observe that this algorithm also gives us the coordinates of a point of $S$ which has minimal distance to $X$.*

In computer programming practice, however, it is not reasonable to check for vanishing of distances $d(X, S)$ because of rounding and number representation errors. We therefore should prefer to calculate whether $d(X, S) < \epsilon$ for a selected positive neighborhood variable $\epsilon$. The corresponding routines are obvious.

Denote the resulting point in $S$, which is found by this algorithm, and which has the shortest distance of points within $S$ to $X$ by $Min(X, S)$, whereas its barycentric coordinates are denoted by $Min(X, S)_i, i = 0, 1, \dots m$, i.e.,

$$Min(X, S) = \sum_{i=0}^{m} Min(X, S)_i.s_i. \tag{34.5}$$

As a consequence, one can use this algorithm to calculate the distance of a straight line $L \subset \mathbb{R}^n$ to a simplex $S$ as follows: Take the projection $p(S)$ of $S$ on an affine hyperplane $H_L$ orthogonal to $L$. Then $p(S)$ is defined by the projected points $p(s_i)$, and our algorithm applies to the singleton $Y$ of the projection $p(L) = \{Y\}$ and to $p(S)$, giving $d(L, S) = d(Min(Y, p(S)), p(S))$. Moreover the coordinates $Min(Y, p(S))_i$ give us a point

$$Min(L, S) = \sum_{i=0}^{m} Min(Y, p(S))_i.s_i, \tag{34.6}$$

which evidently lives in $S$ and has minimal distance to $L$. In our applications for integral curves, the line $L$ is parametrized by a point $X$ and a directional vector $D$, i.e., $L(t) = X + t.D$. Then the parameter $\lambda$ on $L$ such that we have $d(L(\tau), Min(L, S)) = d(L, S)$ is denoted by $\tau(X, D, S)$.

### 34.6.2 Flow Interpolation

**Summary.** This section deals with selection algorithms for searching points on an integral curve of the performance field which are successively approximated to initial sets.

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After splitting the approximation to an initial set $I$ to the approximation of one of its simplexes, the problem is this: We are given a positive $\epsilon$, a point $X$, and a simplex $S$ in $\mathbb{R}^n$, and we have to decide whether, and in which parameter value, the integral curve $\int_X \mathbf{Ts}$ hits the $\epsilon$-neighborhood[8] $U_\epsilon S$ of $S$. Theoretically, one could just calculate the integral curve and search for a parameter that does the job—if there is any. But in computer programs, such a procedure is illusory. Numerical integration of ordinary differential equations is a time-consuming, expensive task. One cannot afford to calculate all curve points for any score event of a normal composition, which usually contains $10^4 - 10^5$ events. We shall see further in section 39.4.4 that fields of any complexity may occur by chains of successive reshapings of given fields via arbitrary performance operators—same for the complexity of initial sets. Also the performance fields and the initial sets are of a completely arbitrary relative position.

So the situation is this: We are given a point $X$ and a simplex $S$. We know in what direction—namely $\mathbf{Ts}_X$—the integral curve starts (with positive or negative curve parameter values). Nothing more. So we have to guess where the curve could approach $S$ as well as possible. We have to evaluate the guess, and then start with another guess, etc. Eventually, we find a curve point in $U_\epsilon S$ and we are happy, or else we will have to give up the search and decide that the curve did not hit $U_\epsilon S$. This could be a wrong decision, but (calculation) time runs out and we have to resign. Intuitively, the search is best described by the following scenario: We have a paper dragon being suspended at a fixed position in the air. We also have a nervous fly, flying around in its random zig-zag (however differentiable) manner where you never know which will be the next turn. This scenario is traced on a video recorder, and the video really shows whether and when the fly hits the dragon. But we are not in the state of viewing the entire film. Rather are we given a determined sequence $VF(0) = (VideoFrame(t_{i_0}), VideoFrame(t_{i_0+1}))$ of two successive frames. This gives us the fly's position and its velocity vector at time $t_{i_0}$ (supposing that the video is binocular...). We now have to guess, which video frame sequence could be the next best that shows the fly as near as possible to the dragon. We then forward the video to this next frame sequence $VF(1)$ and judge the new situation, and so on, until we find a hit point time or else we run out of time.

The first action is to guess a good curve time from the starting point $X$ and the field vector $\mathbf{Ts}_X$. Denote by $x(t)$ the integral curve $\int_X \mathbf{Ts}(t)$. As nothing is known about the curve's future directions, we draw the straight line $L_X(t) = X + t.\mathbf{Ts}_X$ and look for the parameter value $t_1 = \tau(X, \mathbf{Ts}_X, S)$ defined above, which gives the nearest point to $S$ on $L_X$, see figure 34.3. We now have to compare $d(x_1, S)$, and $d(X, S)$. If the former is smaller than the latter, we may proceed, if not, we are in a bad position. Of course, this linear first approximation is not necessarily the best one to get off, since the field may be rather circular than linear in this region. We could therefore try other first approximations, such as a circle in the plane[9] spanned by the barycenter of $S$, $X$, and $\mathbf{Ts}_X$, for example the circle through the barycenter of $S$ and $X$,

[8] This is $U_\epsilon S = \{x |\ d(x, S) < \epsilon\}$.

[9] Since the linear approximation failed, this must be a plane!

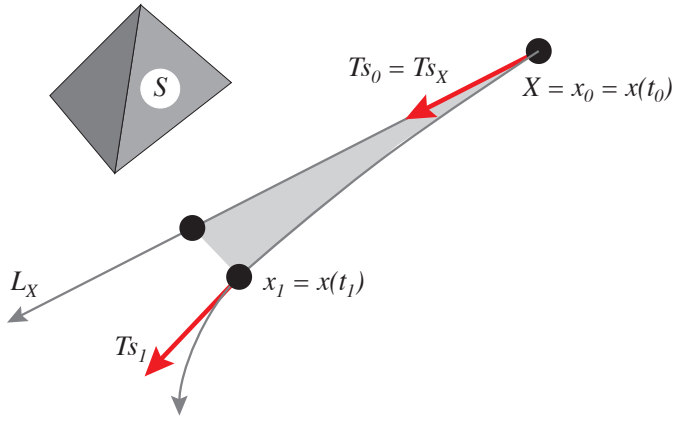


Figure 34.3: The first approximation to a hit point on a simplex $S$ when starting from the point $X$ of the given composition is found by a linear approximation $L_X$.

and tangent to $\mathbf{Ts}_X$. We could follow this alternative, but we refrain from this because it does not demonstrate qualitatively new problems.

For the following interpolations, recall exercise 71 in section 32.3.2 on cubic splines. This tells us that, given any two points $x(s), x(t)$ on our integral curve, there is exactly one cubic interpolation function $P_{s,t} : [s,t] \to \mathbb{R}^n$ with $P(s) = x(s), P(t) = x(t)$, and tangents $TP_s = \mathbf{Ts}_{x(s)}, TP_t = \mathbf{Ts}_{x(t)}$. We shall use these approximation curves to guess optimal points since we cannot calculate all curve points for an interval $[s,t]$.

We now proceed as follows. We are given the curve parameter $t_1$ whose point $x_1 = x(t_1)$ is nearer to $S$ than $x_0$. We now repeat our linear approximation procedure with the line $L_{x_1}$ and get a new curve parameter $t^*$. We also calculate the cubic interpolation point (this is a cheap calculation effort) for $P_{t_0,t_1}$ at parameter $(t_0 + t_1)/2$. If $x(t^*)$ is nearer to $S$ than $x_1$ and $P_{t_0,t_1}((t_0 + t_1)/2)$, we set $t_2 = t^*$, else, if $P_{t_0,t_1}((t_0 + t_1)/2)$ is nearer than $x_1$, we calculate $x((t_0 + t_1)/2)$, and, if this latter is still nearer than $x_1$, we set $t_2 = (t_0 + t_1)/2$. Else, we are stuck and have to quit.

The general situation is this: We have calculated a sequence $X = x_0, x_1, \ldots x_k$ of successively nearer points with curve parameters $t_i$, i.e., $x(t_i) = x_i$. Essentially we now have to check what happens to the left and to the right[10] of the best point $x_k$. Suppose that $t_r, t_s$ are the right and left neighbor times to $t_k$. We now calculate the interpolation points $P_{t_k,t_r}((t_k + t_r)/2)$ and $P_{t_l,t_k}((t_l + t_k)/2)$. If one of them, with parameter $t^*$, is better than $x_k$, we calculate nearer one's curve point $x(t^*)$. If this is still nearer than $x_k$, we have found $t_{k+1} = t^*$. If not, we are stuck and give up.

This procedure will be repeated until a maximal admitted number $\sigma$ of steps, and as long as the points $x_k, k \leq \sigma$ are not in $U_\epsilon S$. The procedure stops if we run out of steps or if we eventually hit the given neighborhood of the simplex. This algorithm has been implemented in

[10]It could happen that $x_k$ is the right or left extremal one. Then, we have to make linear interpolation on that side, but this situation has been explained for the construction of the third point $x_2$ above.

the PerformanceRUBETTE® of the RUBATO® software, see section 41.4.

It seems that the general situation of performing points $X$ meets serious problems for points whose integral curve does not hit any initial point (or, at least, a point in a small neighborhood of the initial set $I$). This is however not so tragic since in reality, performance is not a one-step process: Stemma theory (chapter 38) will show us that any performance is unfolded from a previous, less artistic performance, starting on the quasi-mechanical reproduction of the score. Therefore, if at a given stage, a point $X$ cannot be performed, its performance which is already defined on a less artistic level can always be called in order to save the performability of the given score.

A final technical remark about the above algorithm: We implicitly supposed that the space where point $X$ and simplex $S$ live is the space of a performance field $\mathbf{Ts}$. However, we used only an integral curve through $X$, and not the performance field itself. *This means that we only needed a curve $x(t)$ through a point $X$, and no information about the curve's genealogy. So the algorithm is also valid for any such "isolated" curve.* This will be used later in our discussion of hierarchies of local performance scores in chapter 35.

In the next chapter, we shall use this notation: If $X$ is in the given frame $R$, we denote by $\int_X^- \mathbf{Ts}$ ($\int_X^+ \mathbf{Ts}$) the restriction of $\int_X \mathbf{Ts}$ to the maximal interval contained in the interval $]-\infty, 0]$ ($[0, +\infty[$) such that all values of the curve on this interval are contained in $R$. Given a performance field $\mathbf{Ts}$, $\epsilon \in \{+, -\}$, and an initial set $I$, we set:

$$I \int^\epsilon \mathbf{Ts} = \{X \in R |\ I \cap \int_X^\epsilon \mathbf{Ts} \neq \emptyset\} \tag{34.7}$$

Here is a sorite concerning this symbol:

**Sorite 11** *In the following statements, direct products of performance fields $\mathbf{Ts}_1, \mathbf{Ts}_2$, initial sets $I_1, I_2$, and frames $R_1, R_2$ refer to the product (limit) space $SS$ built from the given factors $SS_1, SS_2$.*

(i) *The operator $I \mapsto I \int^\epsilon \mathbf{Ts}$ conserves inclusions, commutes with unions and is idempotent.*

(ii) *Setting $I_1 \int^\epsilon I_2 = I_1 \times (I_2 \int^\epsilon \mathbf{Ts}_2) \cup (I_1 \int^\epsilon \mathbf{Ts}_1) \times I_2$, we have*

$$(I_1 \int^\epsilon I_2) \int^\epsilon \mathbf{Ts}_1 \times \mathbf{Ts}_2 = I_1 \int^\epsilon \mathbf{Ts}_1 \times I_2 \int^\epsilon \mathbf{Ts}_2.$$

Proof. Statement (i) is straightforward by the uniqueness of integral curves. We show statement (ii) for $\epsilon = -$, the other case is analogous. For the inclusion "$\subset$", suppose WLOG that we have $(X, Y) \in (I_1 \times (I_2 \int^- \mathbf{Ts}_2)) \int^- \mathbf{Ts}_1 \times \mathbf{Ts}_2$. This means that we can reach a point of $I_1 \times (I_2 \int^- \mathbf{Ts}_2)$ from $(X, Y)$ on the integral curve $\int_{(X,Y)}^- \mathbf{Ts}_1 \times \mathbf{Ts}_2$. Since integral curves of a product of vector fields project to integral curves of their factors, and by the idempotency of $\int^- \mathbf{Ts}_2$, this implies $(X, Y) \in I_1 \int^- \mathbf{Ts}_1 \times I_2 \int^- \mathbf{Ts}_2$. As to the inclusion "$\supset$", suppose that $(X, Y)$ is such that $\int_X \mathbf{Ts}_1(s) \in I_1, V = \int_Y \mathbf{Ts}_2(t) \in I_2$, and WLOG $s \leq t \leq 0$. Setting $W = \int_X \mathbf{Ts}_1(t)$ we conclude $(W, V) \in I_1 \int^- \mathbf{Ts}_1 \times I_2$. By the following lemma, we can reach $(X, Y)$ from $(W, V)$ on an integral curve of $\mathbf{Ts}_1 \times \mathbf{Ts}_2$ from $t$ to 0, and we conclude $(X, Y) \in (I_1 \int^- I_2) \int^- \mathbf{Ts}_1 \times \mathbf{Ts}_2$, QED.

**Lemma 50** *If we are given two locally Lipschitz vector fields* $\mathbf{Ts}_1, \mathbf{Ts}_2$ *on respective domains* $D_1, D_2$ *and two (maximal) integral curves* $x_1 : J_1(0) \to D_1, x_2 : J_2(0) \to D_2$*, the diagonal curve* $x : J_1(0) \cap J_2(0) \to D_1 \times D_2 : t \mapsto (x_1(t), x_2(t))$ *is the (maximal) integral curve of* $\mathbf{Ts}_1 \times \mathbf{Ts}_2$ *on* $D_1 \times D_2$ *through the couple* $x(0) = x_1(0), x_2(0))$ *of initial points.*

Proof. Clearly, the diagonal is an integral curve since differentiation goes factorwise. On the other hand, if we had a proper extension of $x$ in the product space, its projections $p_1 \circ x, p_2 \circ x$ would yield two integral curves, one of which would have an extended domain, which contradicts the choice of $x_1, x_2$, QED.

# Chapter 35

# Hierarchies and Performance Scores

*On trouve toujours l'homogène à un certain degré de division.*
Paul Valéry [538, I, p.209]

**Summary.** As a synthesis of the structural parts described in chapter 32 through chapter 34 we establish the overall structure of performance. The objects of performance structure are understood as being an *additional* score type, called *performance score*, layered over the given "symbolic" score like a system of optical lenses which 'deform' the rigid configuration of note symbols. The performance score is a *global* object built from an atlas of local performance scores. A local performance score is built from a hierarchy of performance cells, the very core of performance structures. We first describe the category of performance cells. Local performance scores are defined by a hierarchical construction principle: They are particular diagrams in the category of performance cells. The conceptual and musical background of such *local hierarchies* is evidenced through a series of examples, including the piano and violin hierarchies. We end up with the definition and exemplification of the concept of a (global) performance score.

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## 35.1 Performance Cells

**Summary.** Performance cells are the very local data of performance. They comprise the cell's *frame* (a domain of definition), the *symbolic kernel* (a set of prima vista objects), the *performance field*, the *initial set*, and the *initial performance*.

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The innermost local structure of performance is the performance cell. We have known all of its ingredients and shall now set up the formal definition of such a cell.

**Definition 101** *Given $\epsilon \in \{+,-\}$, a performance cell is a five-tuple*

$$C = (K, R, \mathbf{Ts}, I, \wp_I)$$

*consisting of the following objects:*

1. *a local composition $K \subset SS$ whose space is simple of underlying module $\mathbb{R}^n$, $K$ is called the* symbolic kernel *of the cell;*
2. *a closed frame $R = [a_1, b_1] \times \dots [a_n, b_n]$ contained in $\mathbb{R}^n$ and containing $K$;*
3. *a locally Lipschitz-continuous performance field $\mathbf{Ts}$ which is defined[1] on $R$;*
4. *an initial set $I$ such that $K \subset I \int^\epsilon \mathbf{Ts}$ (see equation (34.7));*
5. *an initial performance $\wp_I : I \to PS$, with codomain a physical space $PS$ with module $\mathbb{R}^n$, and such that for any point $X \in K \cap I \int \mathbf{Ts}$ and any two points $a = \int_X^\epsilon \mathbf{Ts}(\alpha), b = \int_X^\epsilon \mathbf{Ts}(\beta)$ in $I$, we have $\wp_I(b) - \wp_I(a) = (\alpha - \beta).\Delta$, where $\Delta = (1, \dots 1)$ is the diagonal vector in $PS$'s $\mathbb{R}^n$.*

Often, $I$ is given as a union of a finite family $(S_i)_i$ of possibly degenerate simplexes $S_i$, but this is not the general case. A performance cell is visualized by a tetrahedron as shown in figure 35.1. We therefore also call $(R, \mathbf{Ts}, I, \wp_I)$ a *performance body for $K$*.

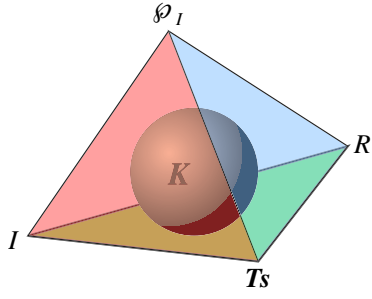


Figure 35.1: Visualization of a performance cell by a tetrahedron, the inner ball symbolizes the cell's symbolic kernel, whereas the 4-tuple of the tetrahedron's vertexes is called the *performance body for the symbolic kernel*.

Given a performance cell $C$, we automatically have a well-defined performance map $\wp_C : K \to PS$ which we shall always refer to when talking about the performance $\wp$ associated with $C$.

Without stressing the contrary, all our cells will be $-$-cells, the theory for $\epsilon = +$ is the same. We shall only occasionally consider mixed signatures.

[1] Recall that this means that $\mathbf{Ts}$ is defined on an open neighborhood of $R$, but we only identify the field on $R$.

## 35.2 The Category of Performance Cells

**Summary.** Performance cells constitute the objects of the category *PerCell* of performance cells, the morphisms being the technical expression of relations between performance data on different parameter spaces.

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In nuce, we have already seen phenomena of related performance cells in the discussion of parallel fields as they arise in the $Onset \oplus Duration$ space. There, we had the projection $p_{Onset} : Onset \oplus Duration \to Onset$ which was compatible with the parallel field $\partial Tempo$ and the tempo curve $Tempo$.

We shall now set up the formal statement behind those incipits. What we want is a concept of a morphism $C_1 \to C_2$ which defines the *category PerCell of performance cells* such that the associated performances are compatible with this morphism.

**Definition 102** *(Recall that we have fixed the signature $\epsilon = -$ in the following discussion!) Let $C_1 = (K_1, R_1, \mathbf{Ts}_1, I_1, \wp_{I_1})$ and $C_2 = (K_2, R_2, \mathbf{Ts}_2, I_2, \wp_{I_2})$ be two performance cells living in spaces $SS_1$ and $SS_2$. Suppose further that we have a projection $p : SS_1 \to SS_2$ of the parameter space $SS_1$ onto the parameter space $SS_2$ such that the underlying projection $p : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ is the projection onto a subset of coordinates. Then $p$ is a* morphism of performance cells $p : C_1 \to C_2$ *iff the following conditions are verified:*

1. $p(K_1) \subset K_2$ *(in other words: $p : K_1 \to K_2$ is a morphism of local compositions);*
2. $p(R_1) \subset R_2$*;*
3. $Tp \circ \mathbf{Ts}_1 = \mathbf{Ts}_2 \circ p$*, i.e., a morphism of vector fields $p : \mathbf{Ts}_1 \to \mathbf{Ts}_2$;*
4. $p(I_1) \subset I_2 \int \mathbf{Ts}_2$*;*
5. $p \circ \wp_{I_1} = \wp_2 \circ p|I_1$ *(here, $p$ denotes the corresponding projection on the physical spaces $p : PS_1 \to PS_2$);*

**Lemma 51** *With the above notation, if $p : C_1 \to C_2$ is a morphism, then we have*

$$p \circ \wp_1 = \wp_2 \circ p. \tag{35.1}$$

*with the homonymous $p$ symbol for the mental and physical projections.*

Proof. Let $X \in K_1$, and suppose that $\int_X \mathbf{Ts}_1$ hits $I_1$ at the point $Y = \int_X \mathbf{Ts}_1(t)$. (By axiom 5 of the definition 101 of a performance cell, it does not matter, which hit point we are selecting.) Then, $\wp_1(X) = \wp_{I_1}(Y) - t.\Delta_1$, and by linearity of $p$, $p \circ \wp_1(X) = p \circ \wp_{I_1}(Y) - t.\Delta_2$. But $p \circ \wp_{I_1}(Y) = \wp_2(p(Y)) = \wp_{I_2}(Z) - s.\Delta_2$, if $Z$ is the hit point for $p(Y)$ in $I_2$, $s$ is the curve parameter for this hit point. Hence, $p \circ \wp_1(X) = \wp_{I_2}(Z) - (s+t).\Delta_2$. But integral curves of $\mathbf{Ts}_1$ are projected into integral curves of $\mathbf{Ts}_2$ by axiom 3 of definition 102 and by the uniqueness of integral curves (see the fundamental theorem of ODE 78, appendix I.2.2). Therefore, the curve parameter $s + t$ is also the parameter of the integral curve $\int_{p(X)} \mathbf{Ts}_2$ where it hits $Z$, and this means that $\wp_{I_2}(Z) - (s + t).\Delta_2 = \wp_2(p(X))$, QED.

This means that performances of performance cells behave in the sense of natural transformations with respect to the category *PerCell*.

**Exercise 78** Work out the example of a parallel articulation field projection and its associated performance body data, as discussed in section 34.5, to obtain a morphism of performance cells.

The next proposition gives us means to construct product performance bodies and product cells from two given cells.

**Proposition 58** *Suppose that we are given two performance cells*

$$C_1 = (K_1, R_1, \mathbf{Ts}_1, I_1, \wp_{I_1}), C_2 = (K_2, R_2, \mathbf{Ts}_2, I_2, \wp_{I_2})$$

*living in spaces $SS_1$ and $SS_2$. Let $SS$ be the product of $SS_1$ and $SS_2$, with projections $p_i : SS \to SS_i, i = 1, 2$. Then the following data defines a performance cell $C$ on $SS$:*

1. *the frame is $R = R_1 \times R_2$;*
2. *the field is $\mathbf{Ts} = \mathbf{Ts}_1 \times \mathbf{Ts}_2$;*
3. *the symbolic kernel is any sub-composition $K \subset K_1 \times K_2$;*
4. *the initial set is $I = I_1 \int^\epsilon I_2$;*
5. *on $I$, the initial performance $\wp_I$ is defined as follows: if $(x, y) \in I_1 \times I_2 \int^\epsilon \mathbf{Ts}_2$, then we set $\wp_I(x, y) = (\wp_{I_1}(x), \wp_2(y))$, and symmetrically for the other case $(x, y) \in I_1 \int^\epsilon \mathbf{Ts}_1 \times I_2$.*

*With his data,we have two morphisms $p_i : C \to C_i, i = 1, 2$ of performance cells.*

Proof. As to the construction of $C$, we only have to check the relation $K \subset I \int^\epsilon \mathbf{Ts}_1 \times \mathbf{Ts}_2$. From sorite 11, we know that $I \int^\epsilon \mathbf{Ts}_1 \times \mathbf{Ts}_2 = I_1 \int^\epsilon \mathbf{Ts}_1 \times I_2 \int^\epsilon \mathbf{Ts}_2$. But the latter evidently contains $K_1 \times K_2$, and a fortiori $K$, whence the claim. The morphisms $p_i$ are now straightforward by construction, QED.

The performance cell $C$, together with the two morphisms $p_1, p_2$ will be called the *product of the cells* $C_1, C_2$ and denoted by $C_1 \times C_2|K$, or simply $C_1 \times C_2$ if $K = K_1 \times K_2$. Although the initial set $I_1 \int^\epsilon \mathbf{Ts}_1 \times I_2 \int^\epsilon \mathbf{Ts}_2$ of the product performance cell is not a union of simplexes in general, the product formula of sorite 11 guarantees that the calculation method for hit points on simplexes discussed in section 34.6.2 may be applied to each component of an event $X = (X_1, X_2)$ if the factor cells have initial sets which are built from simplexes.

## 35.3 Hierarchies

**Summary.** Hierarchies are space diagrams arising in local performances. They trace the functional organization among the arguments of performance fields.

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In order to obtain more concrete results, we shall first consider a particular system of symbolic and physical spaces. We suppose that we are given a series $B_i, i = 1, 2, \ldots$ of pairwise different simple spaces, called *basis spaces*, as well as an equipollent series $P_i, i = 1, 3, \ldots$ of spaces, called *pianola spaces*, with $P_i \underset{Id}{\longrightarrow} \mathbf{Syn}(B_i)$, while $B_i \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}^{n_i})$. Often, we shall consider finite products $B_{i_1} \times B_{i_2} \times \ldots B_{i_k} \times P_{j_1} \times \ldots B_{j_l}$ of such spaces, and always ordered with basis spaces first, and pianola spaces second, and each space type ordered by increasing index (no repetitions!). This means that we in fact parametrize such products with finite subsets of the name set $BP = B \cup P, B = \{B_i | i = 1, 2, \ldots\}, P = \{P_i | i = 1, 2, \ldots\}$.

We shall then identify these product spaces with the simple space $B_{i_1} \oplus B_{i_2} \oplus \ldots B_{i_k} \oplus P_{j_1} \oplus \ldots B_{j_l} \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}^N)$ associated with the direct sum $\mathbb{R}^N$ of all involved copies of real vector spaces. If the defining sequence $B_{i_1}, \ldots B_{i_k}, P_{j_1}, \ldots P_{j_l}$ is denoted by $U$, we shall also rename the space $B_{i_1} \oplus B_{i_2} \oplus \ldots B_{i_k} \oplus P_{j_1} \oplus \ldots B_{j_l}$ by $\oplus U$. By definition, the space associated with the empty sequence $\emptyset$ is the simple space $\oplus\emptyset \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}^0)$ of the zero module.

This generalizes the nomenclature introduced in the description of standard spaces (see equations (6.69) ff.), such as the piano space $Onset \oplus Pitch \oplus Loudness \oplus Duration \xrightarrow{\sim} Onset \times Pitch \times Loudness \times Duration$ referring to (symbolic) onset, pitch, loudness, and duration, and represented by $\mathbb{R}^4$. For any subsequence of symbols $V = B_{u_1}, B_{u_2}, \ldots B_{u_r}, P_{v_1}, \ldots B_{v_s}$ of $U = B_{i_1}, B_{i_2}, \ldots B_{i_k}, P_{j_1}, \ldots B_{j_l}$, we have a canonical projection

$$p_{U,V} : \oplus U \to \oplus V$$

of such standard spaces, also denoted by $p_V$ or $p$, if no ambiguities are possible.

For any two such sequences $V, W$, we denote by $V \cup W$ ($V \cap W$) the sequence defined by the union (intersection) of the basis and pianola symbols in $BP$. Similarly for other set-theoretic operations, such as complement in $BP$, differences $V - W$, etc. This means that we consider the Boolean algebra $Sub(BP)$. When considering the spaces $\oplus U$ associated with such sequences $U$, we often speak loosely about the sequences and mean the spaces, for example, we speak of the "union of spaces" $\oplus U, \oplus V$ and mean the space $\oplus(U \cup V) = (\oplus U) \oplus (\oplus V)$, etc., but no confusion should occur...

**Definition 103** *Given a finite space collection BP, a* space hierarchy *in BP is a non-empty sublattice $H \subset Sub(BP)$ (closed under finite unions and finite intersections), with maximal element (the* top space*) $Top(H)$; $H$ is viewed as a category with $U \to V$ iff $V \subset U$. The minimal non-empty elements of $H$ are called* fundamental spaces*, their set is denoted by $Fund(H)$. For any non-empty space $U \in Sub(BP)$ which is contained in $Top(H)$, we denote by $Cl_H(U)$ the unique smallest space in $H$ which contains $U$, and call it the* hierarchy closure of $U$.

*A hierarchy space $U \in H$ is called* indecomposable *if it is not the union of two disjoint non-empty subspaces of $H$.*

The motivation for the hierarchy concept is that we want to consider systems of performance cells which are related by morphisms, such as the parallel field morphisms or the product cells and their projections. Moreover, we have to group performance cells which are dominated by a root cell since all score events will be performed in one big space whose projections are however compatible with the root space.

**Lemma 52** *Given a space hierarchy $H$, let $Fund(H) = \{F_1, \dots F_k\}$ be the set of fundamental spaces of $H$. Then*

(i) *Each $F_i$ is contained in a unique maximal indecomposable space $T(F_i)$ of $H$.*

(ii) *Any two different maximal indecomposable spaces in $H$ are disjoint.*

(iii) *The top space $Top(H)$ is the disjoint union of its maximal indecomposable subspaces.*

Proof. The union of two indecomposable superspaces of each $F_i$ in $H$ is indecomposable, therefore there is a unique maximal indecomposable superspace $T(F_i)$ for each $F_i$. Evidently, the maximal indecomposable spaces are mutually disjoint, i.e., for $T(F_i) \neq T(F_j)$, $T(F_i) \cap T(F_j) = \oplus\emptyset$. The existence of a decomposition of $Top(H)$ as a product of maximal indecomposable subspaces follows by induction on the cardinality of $Top(H)$, QED.

The set of fundamental spaces contained in one maximal indecomposable space of $H$ are called *the blocks* of the fundament. Hence the blocks are in bijection with the maximal indecomposable subspaces.

**Definition 104** *If an indecomposable space contains a unique (proper) maximal subspace—including the zero space—it is called* irreducible.

Hence, every indecomposable space is either irreducible or it is the union of its maximal subspaces. Fundamental spaces are irreducible, the zero space is not.

With this we may now proceed to define the performance hierarchies. To this end, we consider the zero performance cell $C_\emptyset$ which is defined by a frame over the zero space $\oplus\emptyset$ and has everything trivial: unique zero field, trivial frame, one-point initial set, zero initial performance. Here is the relation of space hierarchies to performance theory (see also figure 35.2):

**Definition 105** *Given a space hierarchy $H$, a* cellular hierarchy *is a diagram (in fact, a functor on the category $H$) $\mathfrak{h} : H \to PerCell$ with values in the category of cells $PerCell$ such that for each $U \in H$, $\mathfrak{h}(U)$ is a cell in the space $\oplus U$, with morphisms being denoted by $\mathfrak{h}(p_{U,V})$, $p_{U,V}$, $p_V$, or $p$ if no ambiguities are likely. The domain $H$ of $\mathfrak{h}$ is called* the type *of the cellular hierarchy.*

The meaning of a cellular hierarchy is that we are given a performance on the kernel of the top cell $Top(\mathfrak{h}) = \mathfrak{h}(Top(H))$, which is compatible with the performances on all other cells $\mathfrak{h}(U)$ of $\mathfrak{h}$. So the parameters of top kernel events which are also grouped on lower hierarchy spaces can be performed independently of the other parameters. Correspondingly, the performance field components grouped on a lower hierarchy space are independent of the other space parameters, a situation already encountered for the articulation hierarchy $Onset \oplus Duration \to Duration$, for example. Moreover, if we have two cells $\mathfrak{h}(U)$, $\mathfrak{h}(V)$, their performance fields $\mathbf{Ts}_U, \mathbf{Ts}_V$ evidently define performance fields $\mathbf{Ts}_{U\cup V}, \mathbf{Ts}_{U\cap V}$ on the arguments of the union and intersection of their spaces, and this is met by the lattice structure of $H$.

The signification of a space being irreducible now also becomes more evident: If a hierarchy space is the union of two proper subspaces, any coordinate of the performance of an event may be calculated via projection of the event into an appropriate subspace. However, for an irreducible

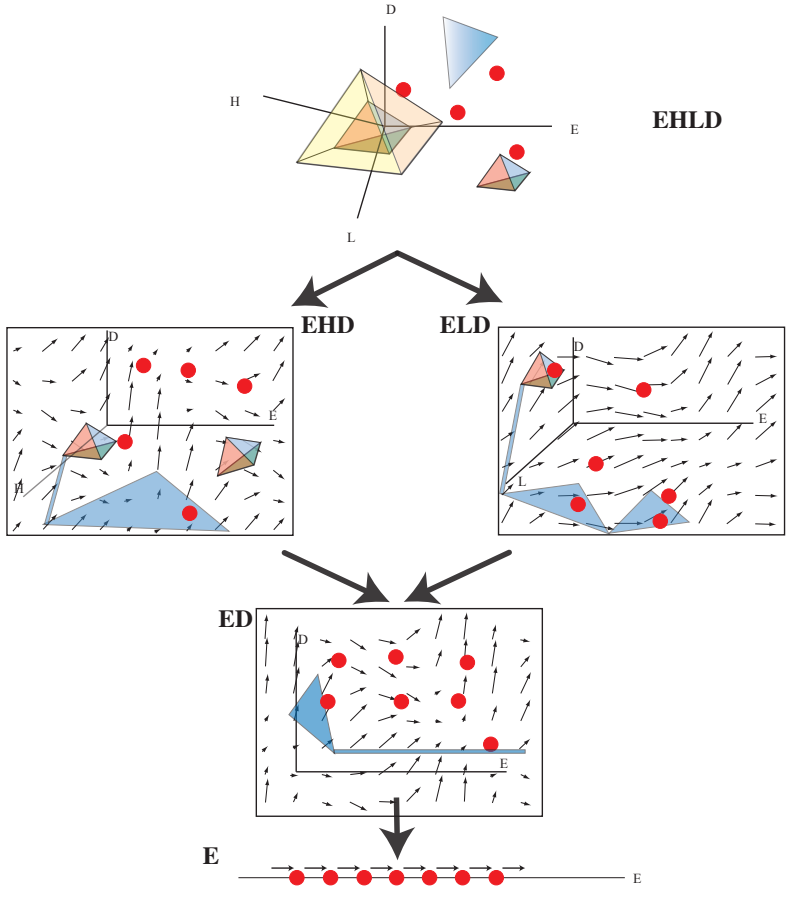


Figure 35.2: A cellular hierarchy is shown, together with the initial sets (blue simplexes), the performance fields, the projections, and the symbolic kernels (red points).

space, we have to make an extra effort to know how its kernel events are performed since no subspace will give us full information. The parallel field construction from section 33.2.1 is a standard example for this: The duration component of the articulation field is not reducible to the cell of a subspace, whereas the onset component is. So the hierarchy does not tell us how to compute the duration component. The parallel construction is one possibility among an infinity of others to deduce the duration component from the fundamental tempo field by means of a special "formula" which in more generality has been explicated in formula (33.10).

If we are given a space hierarchy $H$ with $Top(H) = \oplus BP = \oplus B \oplus P$, *we shall henceforth assume that for each pianola component* $P_i$ *of* $P$*, the corresponding basis component* $B_i$ *is contained in the basis* $B$*, whereas a basis component* $B_i$ *may happen to live alone without its pianola counterpart in* $P$*. Such hierarchies are called* standard*, if we have to distinguish them from other, non-standard hierarchies.* For a standard space hierarchy, we therefore always have

the alteration projections $\alpha_{\pm} : \oplus BP \to \oplus B$. It is the usual alteration on the couples $\oplus B_i \oplus P_i$, and the identity on the unparalleled basis components.

### 35.3.1 Operations on Hierarchies

**Summary.** Hierarchies can be altered and recombined according to a set of standard operations. Such operations intervene in practical calculations of performance transformations in software algorithms.

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Given a cellular hierarchy $\mathfrak{h}$, if $U$ is any subspace of the underlying space hierarchy $H$, we may restrict $\mathfrak{h}$ to the sublattice $H|U$of $H$ whose top space is $U$. We may then restrict the cellular hierarchy $\mathfrak{h}$ to this sublattice. The restriction is denoted by $\mathfrak{h}|U$.

If we are given two cellular hierarchies $\mathfrak{h}, \mathfrak{k}$ with space hierarchies $H, K$, and whose spaces pertain to the given space collection $BP$, and such that their top spaces are disjoint, then we have the product cellular hierarchy $\mathfrak{h} \times \mathfrak{k}$ whose domain is the space lattice with spaces $U \cup V, U \in H, V \in K$. Clearly, in this product space hierarchy, a pair $U \cup V$ extends $U' \cup V'$ iff each component does so in the respective space hierarchy $H$, $K$, respectively. So for each such relation and the associated morphism $p : U \cup V \to U' \cup V'$, we have the corresponding morphism of performance cells

$$p : \mathfrak{h}(U) \times \mathfrak{k}(V) \to \mathfrak{h}(U') \times \mathfrak{k}(V').$$

The only non-trivial statement within this fact concerns property 4 of definition 102 and follows from the idempotency of the operator $?\int^{\epsilon}$ **Ts** (sorite 11, (ii)).

Given a space $U$ in the name space $B$ of basis spaces, we call a *parallel space to* $U$ and denote by $\partial U$ the space $U \cup P|U$ consisting of $U$ and the space $P|U$ of all the corresponding pianola space components. If a cellular hierarchy $\mathfrak{h}$ is such that for each projection $\partial U \to U$ of spaces in its space hierarchy $H$, the performance field over $\partial U$ is the parallel field $\partial$**Ts** to the field **Ts** of $U$ (i.e., of the cell $\mathfrak{h}(U)$) in the sense defined in formula (33.10), then we say that $\mathfrak{h}$ *is a parallel hierarchy.* Parallel hierarchies are the default hierarchies where performance is initiated.

### 35.3.2 Classification Issues

**Summary.** For small sets of parameters, frame structures of cellular hierarchies are completely classified. We describe the classification for hierarchies involving tempo, intonation, and dynamics.

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We do not claim classification of the full-fledged cellular hierarchies, this is a much too difficult task, and it is not of primary interest; it is easier and perhaps more relevant for practical reasons to classify "frame structures" for concrete cellular hierarchies. Such a frame structure is the hierarchical organization of the hierarchy's performance fields. We shall present a complete classification of hierarchies sitting over the set $B = \{E, H, L\}$ of the three usual basis spaces $E =$

$Onset, H = Pitch, L = Loudness$. For each hierarchy space $U = E, H, L, EH, EL, HL, EHL$, we denote the corresponding field as built from the symbols $T = Tempo, I = Intonation, D = Dynamics$. Hence, $T$ is the field over $E$, $I$ over $H$, $D$ over $L$, whereas $TD$ denotes the field over $EL$, etc., and $TID$ the field over $EHL$. We write $TI \times D$ for a product field corresponding to the space $EHL$ which is decomposable into the subspaces $EH$ and $L$ in the given hierarchy $\mathfrak{h}$. Figure 35.3 shows the classification Hasse diagram, with a straight line from every hierarchy to its next specializations. Here, specialization means that field components become independent from certain parameters with increasing split space hierarchies.

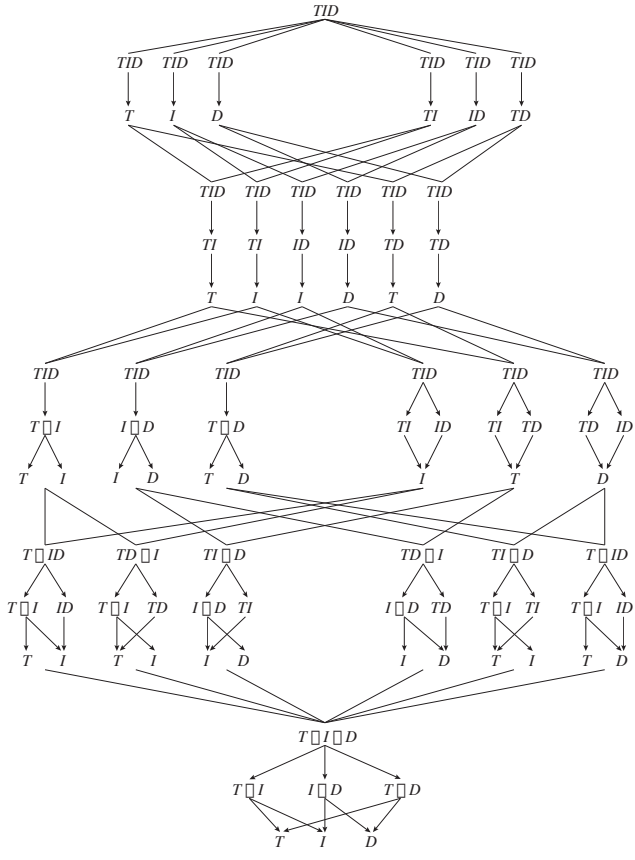


Figure 35.3: The complete classification Hasse diagram of basis hierarchy frame structures for onset ($E$), pitch ($H$), and loudness ($L$) in terms of corresponding fields, including specialization (straight lines) to more split fields according to functional independency of parameters.

For a performance field $\mathbf{Ts} = \mathbf{Ts}_U, U \in H$ of a cellular hierarchy $\mathfrak{h}$ with space hierarchy $H$, we may ask for its functional dependence within $\mathfrak{h}$. Let $\underline{\mathbf{Ts}} = \mathbf{Ts}_{\underline{U}}$ be the field of the maximal subspace $\underline{U}$ of $U$ in $H$. This is called the *territory of* $\mathbf{Ts}$, and describes the portion

of $\mathbf{Ts}$ which is completely determined by lower hierarchy data. By the *extraterritorial part of* $\mathbf{Ts}$, denoted by $\mathbf{Ts}_{ex}$, we mean the projection of $\mathbf{Ts}$ to the complementary space $U - \underline{U}$, so that we have $\mathbf{Ts} = (\underline{\mathbf{Ts}}, \mathbf{Ts}_{ex})$. In general, the extraterritorial part of $\mathbf{Ts}$ is a function of its territory and of a system $Para$ of parameters which are external to the hierarchy; we write $\mathbf{Ts}_{ex} = \mathbf{Ts}_{ex}(\mathbf{Ts}, Para)$ to indicate this dependency. So the performance fields of the fundamental spaces, the *fundamental fields*, play a primordial role in the construction of the hierarchical architecture, which is enriched by external parameters in the case of irreducible spaces (for others, the parameters are non-existent).

**Example 53** Suppose we are given a parallel hierarchy with just one parallel projection $\partial\mathbf{Ts} \to \mathbf{Ts}$ over the space projection $\partial U \to U$, according to the formula $\partial\mathbf{Ts} = (\mathbf{Ts} \circ p_U, 2.\mathbf{Ts} \circ \alpha_+ - \mathbf{Ts} \circ p_U)$ (33.10). Here, we have the empty parameter set $Para = \emptyset$, and $\underline{\partial\mathbf{Ts}} = \mathbf{Ts}, \partial\mathbf{Ts}_{ex} = 2.\mathbf{Ts} \circ \alpha_+ - \mathbf{Ts}$. If we generalize this configuration by the extraterritorial part, then

$$\partial\mathbf{Ts}_{\lambda,ex} = \frac{1+\lambda}{\lambda}.\mathbf{Ts} \circ \alpha_+ - \mathbf{Ts}.$$

The parameter set is $Para = \{\lambda\}$, and we recover the old situation by a specialization $\lambda \to 1$. We shall see in the discussion of performance operators that this parametrization is a natural one in the context of operators for articulation (legato, staccato).

These concepts apply to the standard situations while performing special musical effects which we list here:

1. Pianola deformations[2] of parallel pianola fields of tempo:

   $$\mathbf{Ts} = (T, Z_{ex}(T, Para))$$

   with specialization to $\partial T$. This is used for articulation as discussed above, for local performance cells of ornaments and ties.

2. Deformation of dynamics over tempo:

   $$TD = (T, TD_{ex}(T, Para))$$

   with specialization to $T \times D$. This is used for ondeggiando (bow vibrato) effects and prima vista dynamics accentuation following bar-lines and time signatures.

3. Deformations of tempo over dynamics:

   $$TI = (I, TI_{ex}(S, Para))$$

   with specialization to $T \times I$. This is used for arpeggios if we agree to quit simultaneity of arpeggio events and to change their physical onsets.

[2]Deformation and specialization are reciprocal processes. The first is the embedding of a particular structure in a topologically dominant set of variants (usually in some Zariski topology of algebraic geometry), the second is the restriction of an irreducible set of variants to a special, closed subset, or even a single point, usually by specialization of parameters, see also Appendix F.

The last two deformation types are shown in the lower part of figure 35.3. More systematically, this classification shows these types of specialization (we refer to figure 35.3 in this list):

1. Type: shown on the lowest line, no extraterritorial part, only the one-dimensional fundamental fields $T, I, D$.

2. Type: shown on the second lowest line, on top no extraterritorial part, in the middle of the hierarchy one two-dimensional field (e.g.,

$$ID = (D, ID_{ex}(D, ID_{ex}(D, Para))$$

in the left extreme hierarchy) with a one-dimensional territory (in the example: $D$), and two fundamental spaces (in the example: $T, D$).

3. Type: shown on line three from below. One dimension is extraterritorial and we have two fundamental one-dimensional fields (e.g., $D$ in the left extreme hierarchy, with[3]

$$TID = (T, I, TID_{ex}(T \times I, Para))$$

and the fundamental fields $T, I$.)

4. Type: line three from above. Only one fundamental field, above which a one-dimensional extraterritorial field component is built, and the same for the top space, it is a one-dimensional extension of its two-dimensional territory (e.g., the left extreme hierarchy with $TID = (TI, TID_{ex}(TI, Para)), TI = (T, TI_{ex}(T, Para)))$.

5. Type: second line from top, right half. The fundament is two-dimensional and the top field has a one-dimensional extraterritorial part (e.g., the right extreme hierarchy with

$$TID = (TD, TID_{ex}(TD, Para)).$$

6. Type: second line from top, left half. The fundament is a one-dimensional field, and the top field has a two-dimensional extraterritorial part (e.g., the left extreme hierarchy with

$$TID = (T, TID_{ex}((T, Para)).$$

7. Type: top line. Here we are left with the single total field $TID$ without any subspaces; it is at the same time its own irreducible territory and fundamental field.

Nonetheless, complete classification of cellular hierarchies should be a major issue of future research, although a difficult one. To this end, one notices that a morphism $\mathfrak{h} \to \mathfrak{k}$ between cellular hierarchies should be defined as a natural transformation of these functors which, by definition, starts from one and the same space hierarchy.

However, the "horizontal" natural morphisms of performance cells should be generalized beyond simple projections of parameter spaces since it is natural to say that a performance

[3]We just write down the whole sequence of territorial fields in this formula.

cell is isomorphic to another cell, if the second is generated by any reasonable[4] diffeomorphism between their frames. So cellular hierarchies have only projections as morphisms, while the general concept of a morphism between performance cells is a more general one. We cannot digress on this issue since nothing non-trivial is known to the date. Let us nonetheless denote the category of cellular hierarchies by $\mathfrak{H}$ although the morphisms are not made precise here, whereas the objects are.

### 35.3.3 Example: The Piano and Violin Hierarchies

**Summary.** We describe the default hierarchy associated with piano and with violin scores.

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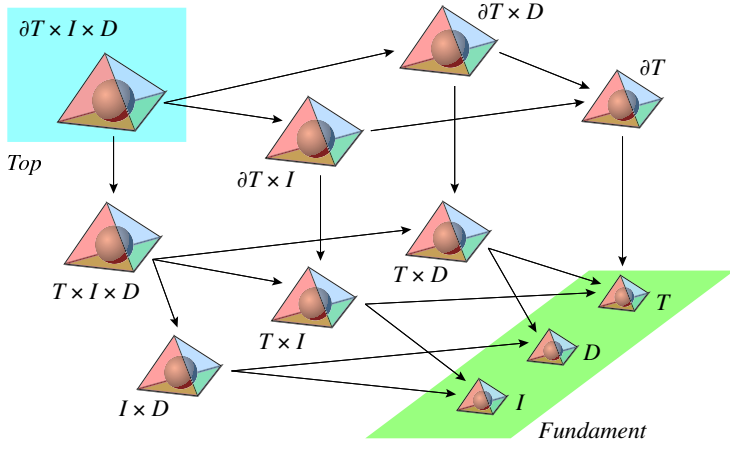


Figure 35.4: The default piano cellular hierarchy.

Figure 35.4 shows the default hierarchy for piano music. This is the hierarchy which one has to start with, when shaping performance. We see that pianola parameters are not given, except for duration, since on the piano, glissando or crescendo parameters for single notes are not feasible. The basis fields are also completely independent (no coupling), and we have no parameters for extraterritorial parts, since the parallel field $\partial T$ is determined by the underlying territory $T$. Figure 35.5 shows the default hierarchy for violin music. It is first characterized by a double parameter extension: on one hand, we now have the crescendo extension, on the other, we have the glissando extension. Thirdly, the fundament is reduced to onset $E$ and pitch $H$. This system also includes two primavista parameters $\lambda, \Lambda$. The point here is that there are primavista predicates for violin, such as "ondeggiando", an $E$-periodic change of loudness, which ask for refined parametrization already in primavista hierarchies. Same for the articulation field $\partial T_\lambda$. Details of these performance operators will be explained in chapter 44.7.

[4]For example an affine deformation of the frames.

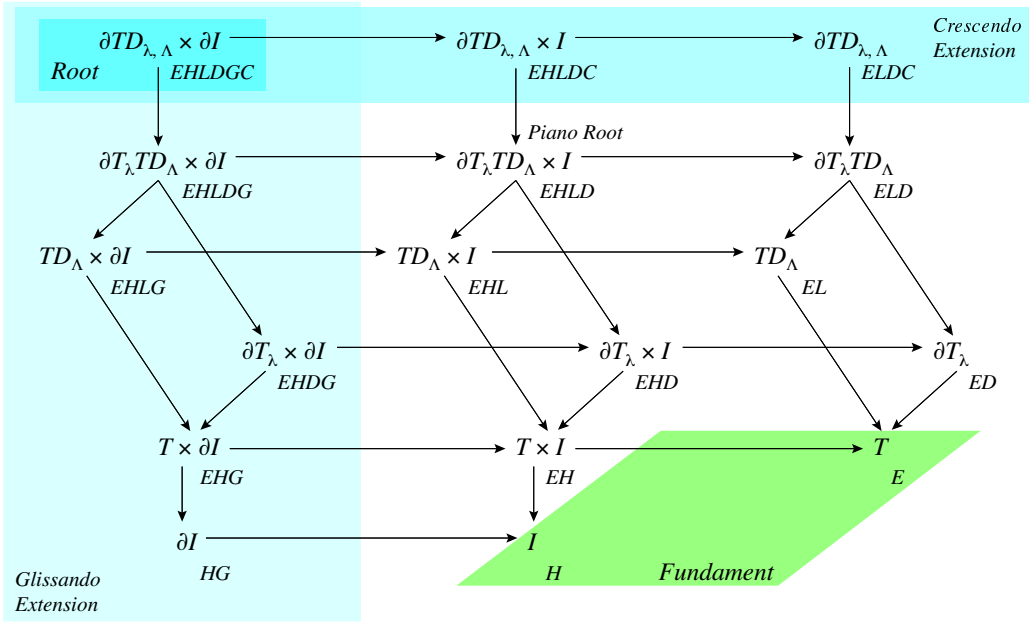


Figure 35.5: The default violin cellular hierarchy.

## 35.4 Local Performance Scores

**Summary.** Cellular hierarchies are the core ingredient of the local performance units, but we are still not in state of controlling all the locally relevant parameters. Moreover, we have to get prepared for the future unfolding processes (stemma theory). Local performance scores are the complete local structures for performance, we give the technicalities in the language of the space form *LocPerScore* of local performance scores.

– Σ –

So far, a cellular hierarchy lacks several specifications which are mandatory in order to be able to perform on an instrumental basis, and in order to hook a single hierarchy into a chain of unfolding performance stages. We also do not have any reference to parameters which might contribute to the specific performance hierarchy. In what follows, we shall for several reasons set up such an environment in the language of denotators. First, we ought to englobe the language of performance theory in the general denotator concept framework, homework which we have not done to this point. This is particularly important for any software developments of performance tools since the denotator language is the lingua franca of all our theoretical perspectives when they are implemented on the software level. Second, it turns out that the very definition of a local performance score is circular and therefore cannot be expressed in usual terms of mathematical theory. It can be easily expressed on the level of object-oriented programming, and this is one reason why denotators are so useful: Their formalism fits perfectly in the object-oriented paradigm.

The following definition of the performance score space will be given top-down in the sense that it has certain spaces in its ramification tree which are not yet made fully explicit. See figure 35.6 for its visualization by a double tetrahedron. This will be completed in subsequent chapters, but this is the right place to introduce this tree structure since the role of its components will be clearer if we have presented them in their functionality rather than in a full-size space definition.

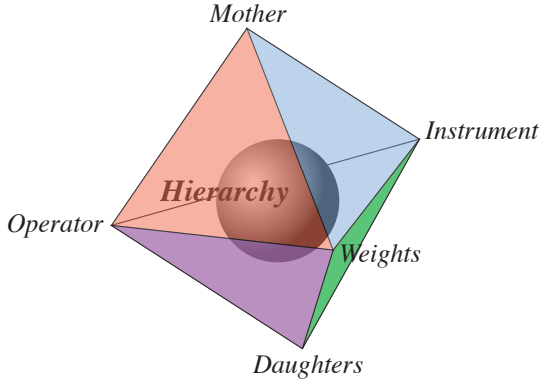


Figure 35.6: The visualization of the six instances of a local performance score (LPS) on a double tetrahedron.

We shall henceforth abbreviate LPS = Local Performance Score. Here is the space for LPS. We thereby still suppose that we are working in a double sequence $B, P$ of basis and pianola spaces, as introduced in section 35.3, including this notation.

$$\begin{aligned} LocPerfScore_{BP} \underset{Id}{\longrightarrow} \mathbf{Limit}(\mathbb{D}) \\ \mathbb{D} = & Mother, \\ & Daughters, \\ & CellHierarchy_{BP}, \\ & Instrument, \\ & Operator, \\ & WeightList_{BP}. \end{aligned} \tag{35.2}$$

Here is the meaning of these factors:

**Mother.** The mother form is the reference to another LPS, from which the given one may inherit a number of properties. It may also happen that there is no mother, i.e., our LPS is already a "primary mother", in which case a denotator of a score form $ScoreForm$, such as the common piano score form defined in [378], will be set instead of the referenced

mother, i.e.,

$$Mother \underset{Id}{\longrightarrow} \mathbf{Limit}(ScoreForm, LocPerfScore) \tag{35.3}$$
$$ScoreForm \underset{Id}{\longrightarrow} \ldots(\ldots) \text{ (any adequate score form).}$$

**Daughters.** The present LPS may be related to a finite set of other LPS which are derived from this LPS. The members of such a set are termed daughters of this LPS, the corresponding form is that of finite local compositions over $LocPerfScore$:

$$Daughters \underset{Fin(F) \rightarrowtail \Omega^F}{\longrightarrow} \mathbf{Power}(LocPerfScore) \tag{35.4}$$
$$\text{with } F = Fun(LocPerfScore).$$

**CellHierarchy$_{BP}$.** This space parametrizes cellular hierarchies. A cellular hierarchy may be parametrized as a set of performance cell denotators $\mathfrak{h}(U), U \in H$, $H$ the space hierarchy of $\mathfrak{h}$, such that the space of any such denotator also identifies the space name $U$. Since space names are supposed to be unique for spaces, we may just retain the space names $U$ to nominate these spaces. In this nomenclature, we shall also denote by $PU$ the physical space associated with $U$ within our total space system $BP$. So cellular hierarchies are modeled by

$$CellHierarchy_{BP} \underset{Fin(F) \rightarrowtail \Omega^F}{\longrightarrow} \mathbf{Power}(Cell_{BP}) \tag{35.5}$$
$$\text{with } F = Fun(Cell_{BP})$$
$$\text{and } Cell_{BP} \underset{Id}{\longrightarrow} \mathbf{Colimit}(\mathbb{D}) \tag{35.6}$$
$$\text{with } \mathbb{D} = (Cell_U)_{U \in BP}.$$

So we are left with the space $Cell_U$ of performance cells of space $U$. By definition, we have this structure:

$$Cell_U \underset{Id}{\longrightarrow} \mathbf{Limit}(InitSet_U, Frame_U, Kernel_U, Field_U, InitPerf_U). \tag{35.7}$$

The initial set space $InitSet_U$ parametrizes either sets of simplexes or just any local composition in $U$, i.e.,

$$InitSet_U \underset{Id}{\longrightarrow} \mathbf{Colimit}(I_U, Simplexes_U) \tag{35.8}$$
$$I_U \underset{2^{Fun(U)} \rightarrowtail \Omega^{Fun(U)}}{\longrightarrow} \mathbf{Power}(U) \tag{35.9}$$
$$Simplexes_U \underset{Fin(F) \rightarrowtail \Omega^F}{\longrightarrow} \mathbf{Power}(Simplex_U) \tag{35.10}$$
$$\text{with } F = Fun(Simplex_U)$$
$$Simplex_U \underset{Fin(G) \rightarrowtail \Omega^G}{\longrightarrow} \mathbf{Power}(U) \tag{35.11}$$
$$\text{with } G = Fun(U).$$

The *frame* is just a pair of points in $U$, i.e.,

$$Frame_U \underset{Id}{\longrightarrow} \mathbf{Limit}(U, U), \tag{35.12}$$

designating the lower and upper extremal points $R_{min}, R_{max}$ of the cell frame.

The *field* is something rather mathematical which we leave in its encapsulated form as an element of the vector space[5] $Der(R)$ of vector fields over the function on the frame $R$ of the cell, i.e., we have this space form:

$$Field_U \underset{Id}{\longrightarrow} \mathbf{Colimit}(\mathbb{D}) \tag{35.13}$$

$$\text{with } \mathbb{D} = (Field_{U,R})_{\text{frame } R \subset U}$$

$$Field_{U,R} \underset{Id}{\longrightarrow} \mathbf{Simple}(Der(R)). \tag{35.14}$$

The *kernel* is a local composition in $U$, i.e., we have this space for kernels:

$$Kernel_U \underset{Fin(Fun(U)) \to \Omega^{Fun(U)}}{\longrightarrow} \mathbf{Power}(U). \tag{35.15}$$

The *initial performance* is given by its graph, i.e., as a (usually infinite) local composition in the product space of $U$ and $PU$:

$$InitPerf_U \underset{2^F \to \Omega^F}{\longrightarrow} \mathbf{Power}(U \oplus PU) \tag{35.16}$$

$$\text{with } F = Fun(U \oplus PU)$$

$$U \oplus PU \underset{Id}{\longrightarrow} \mathbf{Limit}(U, PU). \tag{35.17}$$

**Instrument.** The space for instruments is not specified in this general setup. However, if a concrete instrumental specification is needed, we shall make this component more precise. For instance, if the output could be intended to be piped to an MIDI device, a MusicN family member, a physical synthesis device, or a real physical instrument like a piano, a violin, etc.

**WeightList$_{BP}$.** The weight space $WeightList_{BP}$ has as denotators finite lists of weights on symbolic spaces $U \in BP$. A weight is not really something new, it is only a special kind of textual predicate which we discussed in chapter 18. The relation is this: Given a predicate $E$ and a denotator $x$, we associate a truth value $x/E$ in $\mathbb{T}_I^A$. This truth value can be any abstract truth-oriented data, but the generality of the truth modules also includes fuzzy and similar evaluation. In particular, if we select $I = \mathbb{R}$, the truth value may be any subset of $\mathbb{R}$ (for the zero-address), and, more specifically, any open interval $]-\infty, a[$. If we identify the latter with the upper bound $a$, this just means that the predicate is a real-valued weighting of denotators. This is the interpretation of the weight concept here. But we do include predicates in their truth-oriented meaning. For example, one may define a set of notes $K$ in the space $U$, i.e., a local composition which is the symbolic kernel of

[5] The space of derivations on the functions $R$ is identified with the space of vector fields, see Appendix I.2.4.

some cellular hierarchy. This can be achieved by a characteristic function $\chi_K : U \to \mathbb{T}_{\mathbb{R}}$ which takes the value $\chi_K(k) = \emptyset$ iff $k \notin K$, and $\chi_K(k) = \mathbb{R}$ else. But this also opens the path to weights which also have relevant values between the notes where they originated. Then we have

$$WeightList_{BP} \underset{Id}{\longrightarrow} \mathbf{List}(Weight_{BP}) \tag{35.18}$$

$$Weight_{BP} \underset{Id}{\longrightarrow} \mathbf{Colimit}(Weight(U), U \in BP) \tag{35.19}$$

with the space macro **List** for finite lists over a given space. The space $Weight(U)$ is this:

$$Weight(U) \underset{Id}{\longrightarrow} \mathbf{Colimit}(Weight_n(U), n = 1, 2, 3, \ldots) \tag{35.20}$$

with the indexed spaces

$$Weight_n(U) \underset{Id}{\longrightarrow} \mathbf{Power}(WP_n(U, \mathbb{T}_{\mathbb{R}})), \tag{35.21}$$

$$WP_n(U, \mathbb{T}_{\mathbb{R}}) \underset{Id}{\longrightarrow} \mathbf{Limit}(U_n, \mathbb{T}_{\mathbb{R}}) \tag{35.22}$$

$$U_{n+1} \underset{Id}{\longrightarrow} \mathbf{Limit}(U^n, U_n), \text{ with } U_1 = U \tag{35.23}$$

$$U^n \underset{Id}{\longrightarrow} \mathbf{Power}(U_n), \tag{35.24}$$

where values of weights are given by truth denotators on $\mathbb{T}_{\mathbb{R}}$, and the weight arguments live in the ambient space $U$ or in one of its powers $U_n$. So the weight predicate is evaluated on single objects in the parameter space $U$, or in one of its mixed powers: local compositions in $U$, local compositions of local compositions, mixed with points in $U$, etc. Usually, weights reflect structures stemming from rhythmical, melodic, harmonic and similar music analysis, see section 44.7 for details.

**Operator.** Operators are new in this setup. The point is that the cellular hierarchy which as such completely describes the performance—together with the instrumental data. But we have not dealt with the problem of generating such cellular hierarchies from system data. The operator instance has precisely this functionality: To define the cellular hierarchy. We shall deal with this very complex component in section 44.7. Mathematically, a performance operator $\Omega$ is a map

$$\Omega : \mathfrak{H} \times \mathfrak{W} \to \mathfrak{H}, \tag{35.25}$$

where $\mathfrak{W}$ is the space of weight lists. The first argument, a cellular hierarchy, will be taken from the mother's data and inherited to the actual LPS by the operator $\Omega$. The second argument, a selected list of weights, is usually conceived as a contribution of given musical analyses to the shaping of the present cellular hierarchy. This subject will also be dealt with in section 44.7. But it is also conceived as a source of symbolic kernels when we need to access them via their characteristic function, see section 38.3.2 for this approach.

## 35.5 Global Performance Scores

**Summary.** Global performance scores are atlases of local performance scores, defined for reasons of performance syntax, such as, for example, instrumentation. We describe this global approach.

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We have already stressed in section 32.4 that performance is a four-fold global phenomenon: There is instrumental variety, gluing of local charts, hierarchies of parameter sets, and stemmatic inheritance.

The concept of a cellular hierarchy meets the hierarchical aspect, whereas the broader LPS concept meets also the inheritance aspect by the instances of mother and daughters, and the operator—together with its weights. So we are left with the local character regarding a) instrumentation, b) local charts and strategies.

With respect to these two local aspects, a global performance score should be a finite local composition of LPS, i.e., a finite set of LPS which cover different instrumental specifications as well as local charts of a global composition which is to be performed. So we may state formally the space of global performance scores:

$$GlobPerfScore_{BP} \underset{Fin(F)\to\Omega^F}{\longrightarrow} \mathbf{Power}(LocPerfScore_{BP}) \qquad (35.26)$$

with $F = Fun(LocPerfScore_{BP})$.

Henceforth, we abbreviate "global performance score" by "GPS". The moral of this construction is that it serves to perform all the kernel events within each top space in the respective cellular hierarchies of the LPS.

### 35.5.1 Instrumental Fibers

**Summary.** If we have the same instrument appearing for several LPS, limit constructions are necessary.

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GPS denotators are just sets of LPS without any specific instrumental relations. If we however want to group several LPS around one and the same instrument, we have to look for limit tools. In fact, if we have $k$ LPS with a common instrument, this is controlled by the projection $p^{BP}_{Instrument} : LocPerfScore_{BP} \to Instrument$ onto the instrumental form. We then have to take the $k$-fold fiber product $LPS^{BP,k}_{Instrument} = \prod^k_{p^{BP}_{Instrument}} LocPerfScore_{BP}$. The coproduct

$$LPS^{BP}_{Instrument} = \coprod_{k=1,2,\dots} LPS^{BP,k}_{Instrument}$$

of all these $k$-fold fiber products gives us the possibility to build global performance scores with instrumental grouping specifications on the space type $BP$. If we accumulate all space types which are of interest, in a sequence $BP. = BP_1, \dots BP_n$, say, we get the more general coproduct

$$LPS^{BP.}_{Instrument} = \coprod_{k=1,2,\dots,\ l=1,\dots n} LPS^{BP_i,k}_{Instrument}$$

and finally the global performance score space

$$GlobPerfScore_{BP.} \underset{Fin(F)\to\Omega^F}{\longrightarrow} \mathbf{Power}(LPS^{BP.}_{Instrument}) \tag{35.27}$$

with $F = Fun(LPS^{BP.}_{Instrument})$.

# Part IX

# Expressive Semantics

# Chapter 36

# Taxonomy of Expressive Performance

*This last album is not titled as a memorial album or as an album in tribute because it was titled by Coltrane himself the Friday before his death on Monday, July 17, 1967. He and Bob Thiele were considering words that might apply to the sense of this album, and finally Coltrane said, "Expression. That's what it is."*
Nat Hentoff [219]

**Summary.** Performance structure describes a semiotic fact: expression of meaning by shaping of score data. These expressive semantics are classified according to the three layers of reality captured by the topographic cube: Psychic, physical, and mental (see section 2.4). The first one means that performance expresses emotions, the second one deals with expression of gestural contents, and the third one—the musicologically most interesting one—aims at giving one's understanding of the musical text a rhetoric expression. In a realistic performance, all three expressive semantics will participate, however, a theory of expressive semantics must first of all deal with the "pure" types which are, each in its own way, difficult subjects of ongoing research. We will not deal with the psychological, cognitive or neurophysiological *esthesic* aspect of performance since this is part of music psychology and would exceed our subject.

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To the common music lover, it is by no means clear whether and in which way performance should express contents, and what kind of contents could be addressed. The most widespread belief is that music expresses emotions, or even that "music is" emotions. The latter approach is sometimes contended by music psychologists (see 36.1). This is due to the common usage of music as a carrier for external contents. Music often just ornaments events, ceremonies, feasts, and as such it is not intended to ask music for whatever content.

Perhaps the most interesting such phenomenon is film music. It is a common saying among film music experts [118] that the best film music is the one of which the spectator does not even take notice while watching the movie. But this is a very superficial judgment, since it is in contradiction to the fact that muting the music channel in a movie virtually destroys

the movie. Without music, most of its message vanishes. This is not only the case where in a criminal story of one of those cheap TV productions music has to announce that something dangerous is imminent. Omitting music evaporates the very atmosphere, the parfume of the best film! So music is quasi-absent but essential to the movie.

The point of this apparent contradiction is that the common music expert saying does not really observe what we experience while watching a movie: why are we really sitting in a movie show? First of all, we replace everyday reality by an artificial, virtual reality which is projected into the dark environment of the classical *camera obscura*. Such a reality switch needs a strong booster to work, and this booster is music. Music is well known as the exemplary environment for a counter-world, a force that lets us forget about the common things and transports us to lost, hidden, and subconscious layers of existence, see also our comments on the Depth-EEG experiments in counterpoint theory in section 30.2.4.

In reality, *the core of a movie is its music*, not its visual process and textual story, the latter are only the pretext of what is being communicated. Pretext in its literal sense: ante textum, not irrelevant, but not the kernel either. One could even require from good film that *as a work of art it should transfigure the visual process and textual story into music.*

In short, this pleading for music states that music expression in performance is expression of something, not pure, self-sufficient artistry. If this is acceptable, performance as a rhetoric category should take care of how to express the contents. Therefore, on the one hand, the question must be posed about which contents can be conveyed in performance, and, on the other, about the equilibrium between contents and the medium of performance.

The following discussion deals with the basic strategies of creating performance of something. It does not claim exhaustivity, but sketches prototypical approaches to contents of expression in order to position our research approach in the field of the young science of performance. For a more extended discussion of performance research history and also psychological streams, see Reinhard Kopiez' excellent survey in [272].

## 36.1 Feelings: Emotional Semantics

**Summary.** Several authors, such as Susan Langer, Manfred Clynes, Alf Gabrielsson, Jörg Langner, and Reinhard Kopiez, have also focused their research ona relation between emotions and performance. Whereas Gabrielsson [177] contends with Langer [288] that there is an isomorphism between musical structure and emotion, Langner and Kopiez [270] develop a theory of oscillating systems that is supposed to find a physiological counterpart on the neurological level.

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In [288, p.27], Susan Langner states that "music is a tonal analogue to emotive life", a statement which is interpreted by Alf Gabrielsson in [177, p.35] as the basic idea of "an isomorphism between the structure of music and the structure of feelings". This is also the doctrine which Gabrielsson adapts: "In summary, we may consider emotion, motion, and music as being isomorphic." This latter statement also includes motion as one of the isomorphic structures. This is related to Manfred Clynes's stress of emotion and its expression as an integrated system where motion, i.e., the gestural dimension, plays a crucial role [91]: Emotion, he calls it a "sentic

state", may be expressed by "gestures, tone of voice, facial expression, a dance step, musical phrase, etc."

While this conjecture may please psychologists, it is completely useless to scientific investigation. In fact, such an isomorphism is a piece of poetic literature as long as the components, emotions, motion (gestures), and music, are not described in a way to make this claim verifiable. Presently, there is no hope for a realistic and exhaustive description of emotions. Same for gestures (see next section 36.2), and as to music, the mathematical categories of local and global musical objects are so incredibly complicated that the claim sounds far-out, see also figure 36.1 for an attribution of emotions to articulatory ambitus. For example, the number of isomorphism classes of 72-element motives in pitch and onset (modulo octave and onset period) is $\approx 2.23.10^{36}$ as we have seen in section 11.4.1.3. How could the claimed isomorphism fit in this virtually infinite arsenal? Such a *terrible simplification* however does not contradict the

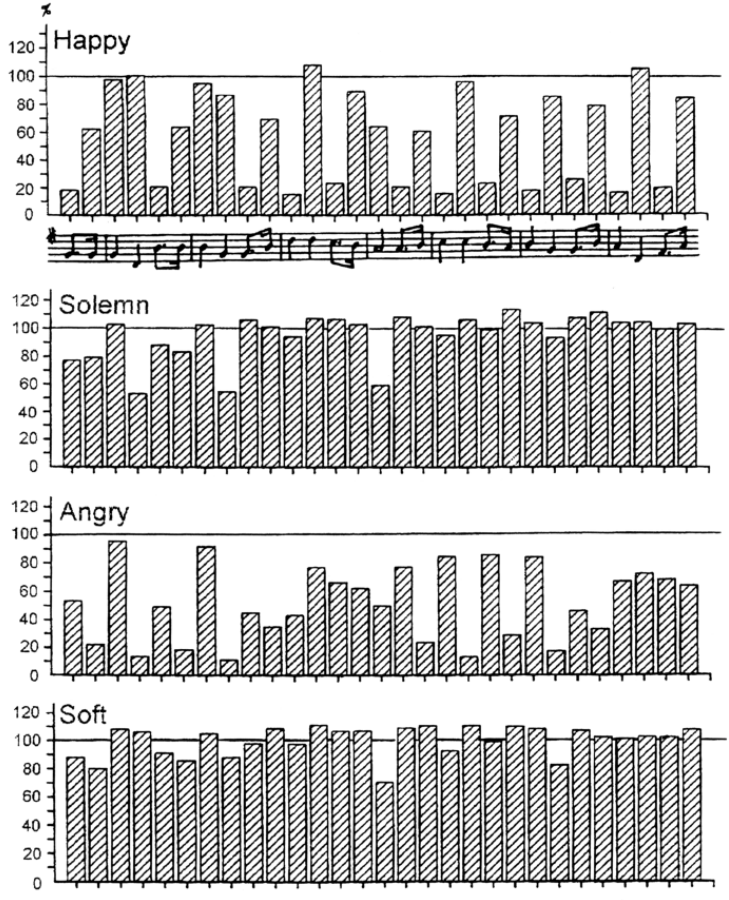


Figure 36.1: Attribution of emotions to the articulation ambitus [177, p.42] for the song "Oh, my darling Clementine".

generic insight that expression in performance may be motivated and grounded in emotional categories. But we should be careful on this point: Our present discussion is not about emotional effects of music in the listener's psyche, we are talking about emotional rationales for expressive performance, i.e., the question of how performance could be induced by emotions. This aspect is typically addressed when the score annotations require performance actions in the mood of "amoroso", "languissant", "beklemmt". Here, the performing artist has to play in and through such a mood. To be honest, to play in such a mood is not a concrete way of telling a performer what to do. The mechanism very probably works by a feedback: The performer plays and hears his/her performance so that the output may be adapted to the received impression in the artist's own ears compared to what the artist conceives as being an impression of the given type.

Evidently, such a rationale is extremely difficult to handle on a scientific level. And it is also difficult to understand a relation to the given score which transgresses the explicit annotations mentioned above. Since general feelings as a motor for performance are no good point of departure, one should (truly in the spirit of Langner, Clynes, and Gabrielsson) at least try to relate the score's contents (not the annotations, the structural facts) to emotions, which, in turn, could then be used to shape performance via the above feedback mechanism. But this is a very complex task. How should a determined emotion (and which?) be incited by a given structural fact, a cadence, a melody, a harmonic configuration, a rhythmic process? So the emotional rationale for performance splits into the

- association problem between emotions and performative shaping;
- and the association between non-emotive score contents and emotions.

This story has not yet been written.

A less totalitarian and more quantified approach than Langner, Clynes, and Gabrielsson to emotional rationales for performance has been proposed by Jörg Langner and Reinhard Kopiez [270, 271, 272] with what they call TOS (=Theory of Oscillating Systems). The TOS postulates a system of 120 oscillators with a determined frequency each, reaching from 8 $Hz$ to 0.008 $Hz$, and distributed in logarithmic steps. In a kind of Fourier analysis (the authors have not to date published the precise formulas), the dynamical curve of a recorded piece of music is decomposed and shows the contributions of these oscillators to the given curve. The TOS is not only meant as a formal description but the authors argue that the musical progression really triggers a series of oscillators in the cognitive stratum of the human brain.

It is contended that the TOS spectrum is stored in human memory and that comparison of performances is enabled by comparison of such spectral data. The authors conclude [271, p.33] that "production and perception of expressiveness in music are essentially one entity." No reference is however made to the score as such, i.e., TOS just measures the performance output and represents the spectral development in time by graphical means (called the oscillogram, see figure 36.2). So TOS does not measure the performance map as such, only the image of the map. It is evidently subtended in the TOS that the spectral decomposition bears a semantics of time processes which—independently of the hidden score—transports the neural activities of the musical brain. So expressivity is correlated to a neuronal oscillator system (expressed by firing rates of neuron populations). This is a type of rationale which refers to a score-independent instance which encodes expressivity. This is a dramatic tournament since

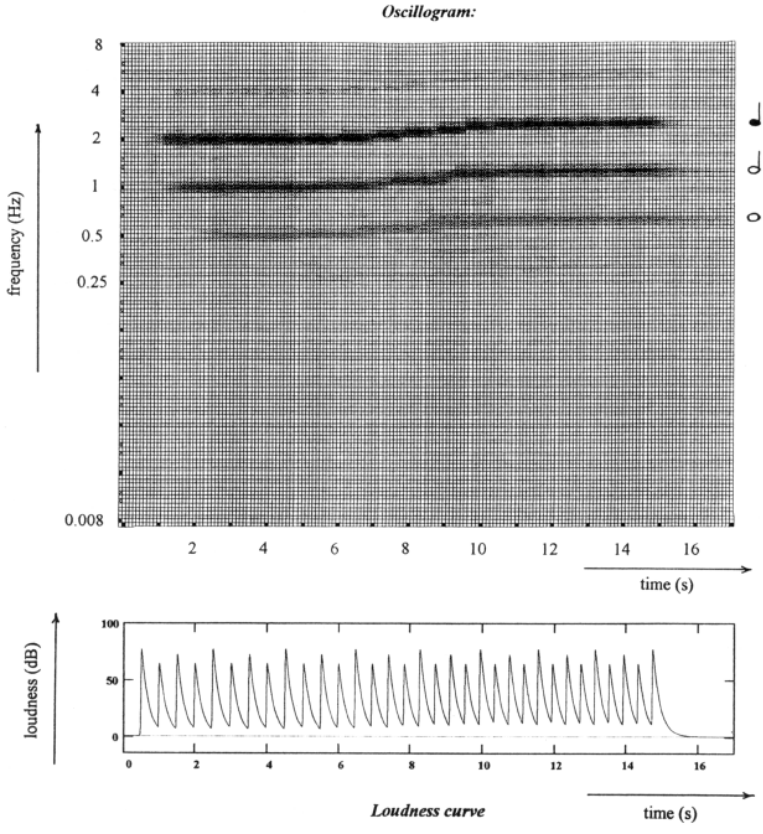


Figure 36.2: Oscillogram (above) and loudness curve (below) belonging to quarter notes played in a 4/4-time signature by a drum computer with an additional accelerando at the end. This accelerando leads to a parallel upward movement of the dark bands in the oscillogram, which means that the activation changes to higher oscillator frequencies.

semantics, i.e., expressivity of something, is uncoupled from the text. This rationale is a kind of "pure expressivity", not a result of reflection or analysis, nor a result of gestural structures. It must therefore be an emotional rationale, although the authors are not precise on the cognitive category and the topographic origin of such neuronal oscillators (it could be that the limbic system is meant), and an experimental verification of the existence of such oscillators in human brain outstanding.

## 36.2 Motion: Gestural Semantics

**Summary.** Besides—but connected to—emotions, motion, as mediated by gestural paradigms, is a widespread rationale of expressive semantics. Neil McAgnus Todd [532], then Johan Sund-

berg, Violet Verillo [518], and Ulf Kronman [280], and also David Epstein, Jacob Feldman, and Whitman Richards [146] have proposed physically motivated descriptions of retards at group endings. Richard Parncutt, Jan Sloboda, and Eric Clarke [412] have added the concrete aspect of anatomical constraints in piano performance. We describe and analyze these approaches. We shortly account for the work by Shuji Hashimoto and Hideyuki Sawada [210] concerning the Japanese kansei processing research in the musical domain of gestures.

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The emotional argument of performance is often paralleled by the argument that gestures, "motion", are also present as a parallel phenomenon to emotion, we have already seen this in Gabrielsson's isomorphism statement in section 36.1. It is also stressed by Clynes' sentic concept which measures the human input by a gestural device of joystick-character, and by Kopiez–Langner [271, p.32]. Citing Francois Delalande [115], they state that "body movements and gesture are in close relation to musical timing. Our neuro-psychological oscillation model implies that body movements can be triggered by musical events." And of course vice versa: Body movements trigger expressive performance. Also, in the framework of Japanese "kansei information processing"[1], Shuji Hashimoto and Hideyuki Sawada stress that "gesticulation is often employed in musical performance to express the performer's emotion." Using the data-glove, they have implemented applications which transform gestures into "MIDI control units to improve the performance in real-time" [210]. This regards expression of crescendi, vibrati, or pianissimo, for example.

Richard Parncutt has investigated the intuitively evident fact that expressive performance is strongly conditioned and induced by physical constraints from fingering [411, 412]:"good fingering is a crucial ingredient in the preparation of performances that are both technically reliable and appropriately expressive." In [410], he has also maintained the thesis that "the most important sounds conditioning the perception of rhythm may be the sounds associated with the heartbeat and walking movements of a mother, as heard by her unborn child." So good performance should been adapted to this motion trigger.

In accordance with these general observations, several performance scientists have proposed models of performance which derive performance fields from mechanical principles of accelerated motion. In particular, this has been set forth regarding tempo curves. Analyzing the first experimental studies of final retard phenomena [518] by Sundberg and Verillo, Kronmann and Sundberg [280] propose a model of final retard which is completely derived from the mechanical analogy of constant deceleration of tempo, as if it were induced by the action of a constant force upon a physical mass. However, tempo $T$ is seen here as a kind of velocity that is a function of physical time, not symbolic time, as it is standard. So what is constant is the force as a function of physical time. We then have $T(e) = dE/de = c.e, c = const.$. The resulting formula is $T(E) = T_0\sqrt{1 - E/E_0}$, where the tempo $T(E)$ at onset $E$ is related to the starting tempo $T_0$, the total onset interval $E_0$ until total stop (which is not included in the really onging music, but has to be set as an ideal endpoint of motion). If such a formula is derived, by the same reasoning, one could also derive formulas with different, not necessarily constant force function; this has also been observed by these authors. However, there is

[1]This is the term coined to stress a special application of information technology to implementation of emotional contents against the classical Artificial Intelligence and other, more logically and mathematically oriented approaches.

no indication of which force should act and why so. So in principle, any tempo curve can be constructed by an adequate acceleration and therefore force function. More generally, it is not clear, why the supposed mass should be constant. As with special relativity, the mass could vary as a function of tempo. It seems that the mechanical motion model is only the construction of an intermediate layer to the real question: What are the basic forces which shape tempo? If the straightforward mechanics (constant mass, constant deceleration) are maintained, however, this explanation completely standardizes the final retard phenomenon and uncouples it from the underlying score and composition. This type of approach has also been proposed by Jacob Feldman, David Epstein, and Whitman Richards [146]. Their paper models tempo $T(E)$ as a velocity function of symbolic time $E$, and its derivative is meant to be determined by a quadratic force function $F(E) \sim E^2$. The Newtonian equation $F(E) = m.T(E)$ yields a cubic polynomial function $T(E) = a.E^3 + \ldots$ for the tempo. Of course, this is a completely different mechanical situation, here the force really acts on the symbolic level instead of the physical action described by Kronman and Sundberg. While that one means $T(E) \sim E^{1/2}$, this one yields $T(E) \sim E^3$. Unfortunately, the latter approach is not congruent with the examples shown in [146]: They refer to the reciprocal value $1/T(E)$ instead of $T(E)$! So we should have $T(E) \sim e^{1/3}$, but that requires another mechanical situation. In terms of Kronman and Sundberg, this requires a force which, as a function of physical time $e$, is proportional to $e^{1/2}$.

A less simplistic approach which also includes structural analysis (see also below in section 36.3) and not only mechanical generalities is presented in Neil McAgnus Todd's paper [532]. He rightly observes that the final retard is only a very special agogical situation, and therefore models his tempo curves according to a superposition of accelerando/ritardando units which are defined by a triangular sink potential $V$. Accordingly, tempo is defined as a velocity $v$, and the total energy of the system $E = \frac{1}{2}mv^2 + V$, supposed to be constant (why?) gives the velocity formula $v = \sqrt{2(E-V)/2}$. Todd further supposes that there is an intensity variable $I$ for loudness, with a relation $I = K.v^2$ that is common to many physical systems. This yields the relation $I = 2K(E-V)/m$ and sums up to an aggregated formula $I = \sum_l 2K(E-V_l)/m_l$ if the grouping of the piece is taken into account. Thus, the idea is that there is a physical energy and intensity parameter system that controls the "surface" of the tempo (= velocity) via classical energy formulas. So the background structure is an energetic one, i.e., the tempo curve and loudness are an expression of mechanical dynamics. The author comments on his method as follows [532, p.3549]:

> *The model of musical dynamics presented in this paper was based on two basic principles. First, that musical expression has its origins in simple motor actions and that the performance and perception of tempo/musical dynamics is based on an internal sense of motion. Second, that this internal movement is organized in a hierarchical manner corresponding to how the grouping of phrase structure is organized in the performer's memory.*

The author also suggests a physiological correlate of this model (loc. cit.):

> *...it may be the case that expressive sounds can induce a percept of self-motion in the listener and that the internal sense of motion referred to above may have its origin in the central vestibular system. Thus, according to this theory, the reason why expression based on the equation of elementary mechanics sounds natural is that the vestibular system evolved to deal with precisely these kinds of motions.*

Todd refers to insights of neurophysiologists according to which the vestibular system is also sensitive to vibrational phenomena. So the musical expressivity is understood as an effect of transformed neurophysiological motion.

The drawback of this approach is that finer musical structures are not involved in the structuring of the energy which shapes tempo/intensity. And even if that could be done, there is an essential kernel of this shaping method which should be based upon paradigms of motion. These paradigms do however not appear clearly in the above approach. More precisely: The complex motion dynamics of the vestibular system cannot easily be mapped onto the structures of performative expressivity. What is the operator which transforms whatever structures of motion into expression parameters? If music were isomorphic to motion, no such isomorphism could be recognized from Todd's clever approach.

Beyond general motion paradigms there is the more visible level of gestural structures which can be implemented in operators for musical expressivity. Unfortunately, the existent classification of gestures is anything but detailed. Classification of gestures is only settled on a prototypical basis. But this is precisely not what music needs so urgently. For example, the extremely refined gestures of Glenn Gould's performance[2], the movements of his hand and arms, his head and thorax, this is beyond any scope of present classification.

Figure 36.3: Glenn Gould while performing.

It is the same with the incredibly refined movements of Herbert von Karajan's hands, and especially their single fingers: It is known that these gestural directives to the members of the

[2]The reader should be careful on this point: In [115], Francois Delalande has recognized the expressive role of Gould's gestures (analyzing a film record of Gould's performance of Bach's *Kunst der Fuge*, see figure 36.3) with respect to musical structure. This is an extension of the performance parameters from sound to gestural parameters, but this is not what we are discussing here. We are discussing the role of gestures as a cause for musical performance, not a media thereof. Evidently, a pianist's (and a fortiori a violinist's) performance is strongly and essentially driven by gestural shapes which are not only mediators of structural facts.

orchestra were observed and followed with extreme attention. To measure such details is not what we can control at present.

To be clear: We are not contending that gestural and motion information is irrelevant to performance. On the contrary, this is an essential contribution, but it is too difficult for scientific research as long as classification of gestures is so far from being settled. And that alone does not solve the problem, since operators for shaping performance parameters must be defined from the information provided by gestural input data. All this seems to be a bit easier than the far-out emotional rationales, but still is subject of advanced research.

## 36.3 Understanding: Rational Semantics

**Summary.** As opposed to "low level" emotional and gestural expressivity, rational semantics deal with expression of rational interpretations of the score structure. This means that the text is analyzed from different points of view, such as harmony, rhythmics, motivic content. These analyses are used as an input to shape the performance structure. This aspect is dealt with in research by the group of Anders Askenfelt, Anders Friberg, Lars Frydén, and Johan Sundberg [23, 163, 164, 166], then by Neil McAgnus Todd [530], Gerhard Widmer [567, 568], then by Jan Beran, Guerino Mazzola, Joachim Stange-Elbe, and Oliver Zahorka [346, 347, 348, 349, 350, 357, 360].

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Already Hugo Riemann [452] had stressed that the scope of rehearsal should be to support the communication of the motives' comprehension. And it is a fact of music psychology (see for example [87] or the excellent overview [272]) that performance as an expression of the score's structure is better understood than performance which disregards structure. Also, in Theodor W. Adorno's theory of performance [6, 7], the analytical point of view, i.e., the purpose of performance to transmit analytical insights, is prominent.

In fact, the most explicit starting point of any performance is the given score. This is a text that abounds with structure that must be shaped in a physical performance space. The reference to this structure as a rationale for performance is a straightforward logic which is completely standard in literature: Interpretation of a text is one of the most recognized and widely practised methods in text performance, especially in the actors' interpretation of dramas. Besides emotion and motion it is therefore logical to refer to the rational text analysis in order to shape performance.

Probably the first explicit and quantitatively stated contemporary approaches in this vein is the "analysis-by-synthesis" method of Sundberg and his collaborators which was first presented in [517]. Analysis-by-synthesis means that a bunch of performance rules for the shaping of different parameters is defined in a software environment, and then applied to the production of a synthetic performance (on an MIDI-instrument, say). The result is then analyzed by an expert (in Sundberg's group this was the professional violinist Lars Frydén) who proposed alterations of the given rules and/or new ones to the programmer. In this experimental cycle, the rules are always of a general character which is based on structural data, not on direct—emotionally or gesturally driven—interventions on the performance data of the individual composition in question. This approach has been implemented on the basis of eleven rules

in the computer application *Rulle*, later, renamed to *Director Musices* [166], with an extended repertory of rules.

The rules of this approach are grouped into three categories: the

- **differentiation category**, including these rules: Duration contrast, High sharp, Double duration, High loud, Accents, Melodic charge, Melodic intonation;
- **the grouping rules**, including these rules: Punctuation, Phrase arch, Leap articulation, Phrase final note, Leap tone duration, Harmonic charge, Faster uphill, Chromatic charge, Amplitude smoothing, Final retard, Inégales, Repetition articulation;
- **the ensemble rules**: Melodic synchronization, Bar synchronization, Mixed intonation, Harmonic intonation.

We shall discuss these rules in section 37.1.1 of chapter 37 on performance grammars. Let us just give two examples here: The rule "Faster uphill" requires in a melodic context, that if a note is followed by a higher pitch note, its physical duration is shortened by $2.k\ ms$, where $k$ is a system variable in order to give its rules a variability in strength, default is $k = 1$. A second example is "Melodic charge", which depends on a non-negative numerical weight attribution for every pitch as a function of its position in the selected tonality. The weights are roughly proportional to the distance of the pitch from the central pitch of the tonic (its weight is zero). Proportionally to the weight, amplitude (in Decibel), duration (percents), vibrato amplitude (percents), and unevenness smoothing are effected, also multiplied with the omnipresent strength factor $k$.

These rules are all very elementary in their mathematical as well as in their music-theoretic approach, but they are completely concrete, and this makes this early attempt so precious. However, the present presentation and formal statement of these rules lacks a clear-cut distinction between symbolic and physical reality. For example, the above mentioned rule "Faster uphill" does not ask for the absolute duration of these notes, the change of duration is independent of the physical data, and produces either too large or too small duration changes in extremal tempi. And this is the delicate point here: A distinction between symbolic and physical reality must rely on absolute tempi. Tempi are however not mentioned in the entire machinery, although in his PhD thesis, Anders Friberg makes a comment on tempi. These rules act not in the sense that they define a map from the symbolic reality into the physical one, but they act on an already given physical image of the symbolic reality. This seems to be a kind of "prima vista" performance, but no information on this is given. *There is no performance of a score which lives outside some tempo specification*[3]*, there is only performance of performance.* Moreover, the lack of agogical shaping operators is also manifest from the absence of onset in the tone parameters. The shaping of onset as such (independently of articulatory shaping) is not defined.

The example of a motion-triggered performance model by Todd which was discussed in section 36.2 above is a realization of Todd's generic approach to rational semantics in performance [530] which we shall now describe. The background structure of that motion paradigm in fact relies on structural data (of grouping), as we have already mentioned above. Todd's generic performance model is designed upon a bidirectional transformation pairing from a score representation $\Psi$ to a performance $P$ and backwards by means of:

[3] Or something of this type if simple tempo curves are not possible in complex hierarchies.

1. a *performance procedure* $\Pi$ acting on $\Psi$ and an *encoding function* $\gamma$:
   $P = \Pi(\Psi, \gamma)$,

2. a *listening procedure* $\Lambda$ acting on $P$ and a *decoding function* $\delta$:
   $\Psi = \Lambda(P, \delta)$.

In this generality, "the theory... is sufficiently general to cover any variable of expression. At the same time, it is agnostic as to what is being communicated, be it structure, emotion, or extra-musical reference" [530, p.407]. The generic character of Todd's approach hides an asymmetry of the transformation pairing which is due to its semiotic background; see also [361] for a modern survey on music semiotics. In fact, performance is a poietic process issued by the performer from the composer's score. In other words, a performance is *caused* by its creators and must be *understood* by the listener, not vice versa. Hence, the performance transformation has to be specified as a semiotic mechanism. This is the difficult part of the business.

Without entering into details here (see chapter 46 for a detailed discussion) it can be said that the critical subject of performance theory—a problem which Todd thematizes in the spirit of cognitive science—is a reconstruction problem: Given a performance $P$, how many representations $\Psi$ and encoding functions $\gamma$ can you find such that $P = \Pi(\Psi, \gamma)$? In mathematical terms, we are looking for the fiber $\Pi^{-1}(P)$ over $P$. This is the so-called inverse image of $P$, and therefore, this branch of performance theory is called *inverse performance theory*. The listening procedure in [530] is just a formal setup for a section $\Lambda$ to $\Pi$, i.e., the selection of an element in the fiber over $P$ as a function of the decoding data $\delta$.

Clearly, the fiber cannot be described in effective mathematical terms if one does not assume a well-defined transformation model. And even for very special models, the so-called locally linear performance grammars (see [352] and section 46.2), fibers turn out to be high-dimensional algebraic varieties. Further, the encoding function must be meaningful enough to reflect the score's structure and its relations to the above categories of expressive semantics. Otherwise, performance cannot claim to interpret the selected score. In other words, the big problem of performance theory is to propose models of *adequate generality* that cope with *expressive semantics*.

In Todd's singular example to his theory, he restricts to hierarchical grouping data for the shaping of duration. Commenting on the inverse problem of listening procedure, he states that "the durations used in the calculations are from only one metrical level. Much information about tempo is given at metrical levels below the tactus and in the durations of actual notes. The representation needs to be extended downwards to include note timing, which would mean that a rubato handler would have to work in cooperation with a metrical parser, one feeding the other. Clearly, a lot of work is needed in this area." Concluding, he notes that "the known algorithms make no reference to any tonal function. Therefore, a rubato handler could be a vital component of any theory of grouping in the perception of atonal music. A complete theory must of course include dynamics, articulation and timbre."

Methodologically, this approach is tightly bound to cognitive science in that any algorithm is first of all tested upon its immediate fitting into human perception mechanisms, within real-time constraints, say. We believe this is a too narrow approach for two reasons: First of all, the investigation of general structural facts must be carried out before any relevance to human perception is taken into account. There is the general problem of getting an overview of possible models and their classification. Second, the cognitive knowledge is all but settled, more

precisely: We do not know, by what processes cognition of performative expression is handled in the human brain. It could happen that a rather abstract invariant of the geometric structure of a mathematically complex fiber $\Pi^{-1}(P)$ can easily be detected by the cognitive machinery, but that this invariant would not have been detected if we were only permitting fibers which allow an immediate access by the cognitive capacities. For example, the mathematical structure of a Möbius strip shaped fiber may be too complex to be grasped by the cognitive machinery, whereas its lack of orientation may be an easy task to be tackled by a small test routine built on a neuronal basis.

Gerhard Widmer's work (e.g., [567, 568]) is based on the machine learning method. In this context, structural features of given scores, such as chords, or small motives, are correlated to the performance data of a given human performance. This data is then used to shape a second score. Here, the performance rule is found by an investigation of a given performance. However, there is no background theory to this, it is purely imitative with respect to human competence. We cannot see any deeper value of such an approach since nothing is really learned beyond parrot-style imitation, although such an imitation may sound quite attractive.

A completely orthogonal approach, which is more akin to Todd's intelligent setup, was undertaken by the Zurich school[4] of the author and his collaborators in [50, 51, 52, 346, 347, 348, 349, 350, 357, 360]. In our setup, the structure theory of performance in its full-fledged concept framework as developed in chapter 35 is used as an output level for performance, whereas the LPS operators are designed to implement different output data from musical analyses, such as metrics/rhythmics, motives/melodies, harmony, counterpoint, or grouping. This approach was first sketched in [341], then presented at the SMAC in [345], at the ICMC in [347], in a paper [346] concerning tempo hierarchies, and above all in the SNSF reports "Geometry and Logic of Musical Performance I,II,III (1993–1995) [348]. This approach is completely general in the sense of Todd's scheme [530] described above. However, we have stressed a specific approach to the communication of analytical facts to performance operators which eases many methodological questions concerning the interplay and concerted action of an entire collection of analytical results. This is the method of weights, i.e., it is required that the analysis of the music-theoretic procedure $A$ be delivered in the form of weight functions $w_A : K \to \mathbb{R}$, where $K$ is a local composition in some parameter space associated with the score, and more directly with the underlying cellular hierarchy of the LPS which implements a performance operator. For example, a metrical weight function $w_M : K_E \to \mathbb{R}$ typically associates a non-negative weight $w_M(E)$ for each onset $E$ in a given local composition $K_E$ of (symbolic) onsets. At first sight, this seems to be utterly restrictive, but the explanation is this: Ultimately, performance has to be defined via numerical indications of how to specify the instrumental parameters. So even if the analysis is of a more symbolic character, it has to be filtered or transformed into numerical values sooner or later. Of course, this could be done in the innards of a specific operator, and there is no obstruction to defining such operators. However, if we want to combine different analyses to be used as nurture for a determined operator, the problem of uniformity of such an input combination arises. But if the input is a priori a weight, the analytical arguments of an operator can be designed in a much more weight-independent format. For example, one may then feed a linear combination of a number of given weights into an operator, without being concerned about where these weights stem from.

[4]The term was coined by Thomas Noll.

Apart from these analytical input strategies, the approach of the Zurich school also stresses the generative nature of performance. Its differentia specifica to other approaches is that the unfolding of performance has been thoroughly formalized in the concept of a performance stemma, the genealogical tree of performance rehearsal and development history (see chapter 38 for details). So Todd's hierarchical examples of refined grouping are not only extended to a cascade of LPS, their hierarchical nodes are also turned into autonomous agents of performance shaping processes which *trace and group the entire performance unfolding as a compound historical and logical process in the large.*

This refined input policy, extending the elementary grouping rationales as exposed by Todd, yields a more sophisticated option to test correlations between measured performances and analytical insights. In this vein, different investigations have been executed with a good success by Beran and Mazzola [50, 51, 52] on the level of statistical methods. The point of this approach is that performance is so complex that it requires a faithful representation of the analytical structure of a score. And that this analytical structure must be an essential input qua significant function of the given individual score. We argue that principles of performance shaping which are completely unspecific towards the concrete score structure cannot provide us with relevant performance directions.

## 36.4 Cross-semantical Relations

**Summary.** Since music has a communicative dimension between poiesis and esthesis, each of the three above semantical directions on the poietic level may influence the esthesic level in one or several of the other semantical directions. For example, a rational expressivity may produce a gestural understanding or vice versa. We give an account of these "cross-semantical" phenomena.

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As was already observed above, it may (and will in many practical situations) very well happen that one type of semantic rationale may act only indirectly upon the shaping of performance. For example, an emotional rationale may first "shape" a gestural object which in turn will shape musical performance. There are no restrictions to that. It is only important that there be one semantical modality at least which is able to convey its contents to the effective shaping of musical parameters. The rest may be arbitrarily complicated: For instance, a rational instance (score analysis, say) may produce an emotional object as its consequence, and the emotional object may produce in turn a gestural object which may evoke a second emotional object, completely different from the first! And then it may happen that this second emotional instance acts directly upon the shaping of performance. It is not clear at present, how much and on which basis the interrelation of different types of performance rationale could be implemented as computer applications, since the cross-modal assignment procedures (e.g., emotion from ratio, ratio from motion) may be hard to realize. Undoubtedly, this question is very interesting, be it for Japanese kansei research (as discussed in Hashimoto's approach in section 36.2), be it for the general problem of harmonizing divergent semantic directions in music, above all: harmonizing the emotional direction with the rational semantics of the text.

# Chapter 37

# Performance Grammars

*Why care for grammar as long as we are good?*
Artemus Ward (Charles Farrar Browne) (1834–1867)

**Summary.** The idea of basing performance on rules in analogy with linguistic grammar goes back to Mathis Lussy [311]. In modern performance research, this terminology was recovered by Johan Sundberg and his school [520]. We discuss the principles for a grammar of performance and give an overview of representative approaches to this theory.

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To our knowledge, the term *performance grammar* was coined by Johann Sundberg on the occasion of a performance theory conference in Aarhus [520]. The reason for such a conceptualization is that the specialists became aware that performance can be shaped in various ways, but not from an amorphous design rationale, on the contrary: It became evident that there are entire organisms for shaping performance from the given score, its different semantic approaches, and the way these approaches are transformed into concrete performance instructions.

The idea that performance should be executed along certain regular patterns that remind us of a language structure comes from the fact that performed music is viewed as a rhetoric vehicle of contents, and that these contents are, by the very nature of musical semantics, hidden, difficult and ambiguous. In other words, the way they are expressed is an essential condition for their communication. For a number of expressive methodologies, their architecture in fact resembles a language although, at present, only very elementary grammatical patterns are known. In [519], Sundberg proposed the creation of a dictionary of expressive rules where the patterns of the performance language can be looked up. We distinguish rule based approaches from rule learning procedures (and we shall not deal with chaotic ad hoc performance being taught in the vast majority of music conservatories!). Representative research is reviewed and classified according to the semantical perspective, as discussed in chapter 36.

But the very need for a performance "language" has also a deeper explanation. If we listen to performance, it is not just the concrete piece being performed and the concrete way of performing without further context which are perceived and judged. In fact, one cannot understand the expression of a coherent and extended text without having an access to the

method of shaping contents. Understanding performance really means that we gain access to a system which is applied in this concrete expressive work, and this system is a language with its grammatical rules which shape the message. In other words, *understanding the performance of a musical composition amounts to understanding its system, the language and, in particular, the grammar which guides performance.*

## 37.1 Rule-based Grammars

**Summary.** The generic scheme of rule-based grammars and three representative directions in rule-based grammars are presented: the KTH school, Niel Todd's approach, and the Zurich School.

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The known rule-based performance languages share a generic grammatical structure which can be described as follows. Basically, there are three components which shape the performance transformation $\wp : K \to \wp(K)$ of a given local composition $K$, i.e., the kernel of the top cell in a hierarchy, see figure 37.1. The first component, which we call the rationale *Rat*, yields the

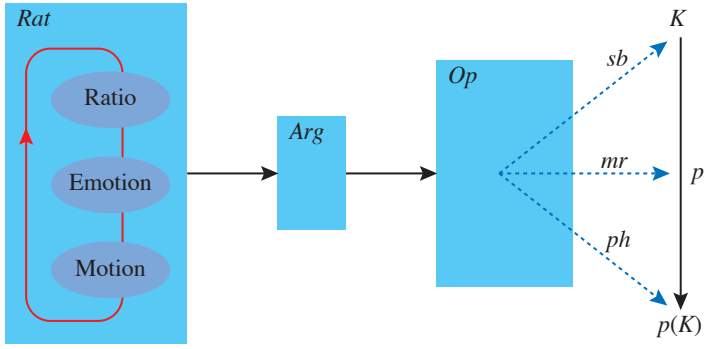


Figure 37.1: The three components of rule-based performance grammar: the rationale *Rat*, the argument *Arg*, and the operator *Op*. They contribute to the construction of the performance transformation $\wp$ of the kernel $K$. The action of an operator on the performance map can be either symbolic (sb), morphic (mr), or physical (ph).

raw material which is intended to act on $\wp$. As we have seen in chapter 36, *Rat* is a complex organism which may include emotional, motional/gestural, and rational (analytical) agents which may interact and result in a final statement to be delivered to the subsequent shaping actors. The output of these operations is the second component, call it *Arg*, the argument of performance. For example, in the analytical approach, this may be a weight function. In the motional situation, it could be an object which parametrizes a physical movement, and in the

emotional setup, this may be a verbal description of a feeling, for example. The third component is an operator $Op$ which "understands" the $Arg$ and which, when fed with this argument, yields a determined function $Op(Arg)$ which defines the performance transformation $\wp = \wp(Op(Arg))$. If we view this situation in the rich context of LPS theory, the argument may of course include the mother LPS and thereby determine $\wp$ not only from the LPS' proper rationale, but also from the LPS' inherited data from previous shaping activities as they are traced on its mother LPS.

The operator's action may typically be targeted to one of the three ingredients of $\wp$: its domain $K$, its codomain $\wp(K)$, or the map (the functional expression) as such. The first type is called a *symbolic* operator because it alters the domain. This is a very strong action since the original notes, i.e., the score's genuine structure, are changed. Symbolic operators create a new composition such that performance is defined on new input data. Whether this type is really a case of performance seems somewhat critical.

But suppose we are given the portion of a score which is written in *fortissimo*, and suppose that this dynamical attribute is a part of the kernel's specification. Then, if an additional dynamical annotation, such as *diminuendo*, is inserted in the score text, this may be seen as a performative prescription: Change the dynamics of the specified group of notes by a successive lowering of *fortissimo* dynamics in some determined range. In this case, a symbolic operator would do the job since it is an action which is required before any artistic shaping begins. We shall call this a *primavista* operator in the operator theory of chapter 44.7.

The second operator type acts directly upon the given physical output $\wp(K)$, this is why it is termed *physical* operator. It may deform the physical data without altering the kernel $K$ or the function $\wp$. We have to explain this seemingly contradictory argument, because it effectively changes $\wp$. We have in mind that a physical operator is a successive map $ph : \wp(K) \to ph(\wp(K))$, i.e, the original map as such remains what it is, but it is composed with the physical operator's action and yields $ph \circ \wp$.

The third operator type alters the functional description of $\wp$ into $Op(\wp)$ and therefore is termed *morphic*. This is typically the case if $\wp$ is defined via a performance field as it is implemented in the cellular hierarchy of an LPS.

Summarizing, we may restate this principle for a rule-based performance grammar:

**Principle 25** *The described scheme englobes the generic framework wherein the performance language is structured. The contents which are shaped by this grammatical structure are cast in the argument Arg, whereas the grammatical structure is centered around the operator instance Op. The operator is the rhetoric element, it tells how the shaping works, whereas the argument Arg is the codified message to be conveyed after an encapsulated process of semantical elaboration in Rat.*

### 37.1.1 The KTH School

**Summary.** The KTH school's system is dominated by local rules which are based on low-level structural analysis of the text. Semantics of this analysis are rational and—to a lesser degree—gestural. They are found by the characteristic empirical "analysis-by-synthesis" method.

The grammatical structure of the KTH school has been described very clearly in [165], see figure 37.2 for a reproduction of that grammatical scheme. According to this scheme, the surface

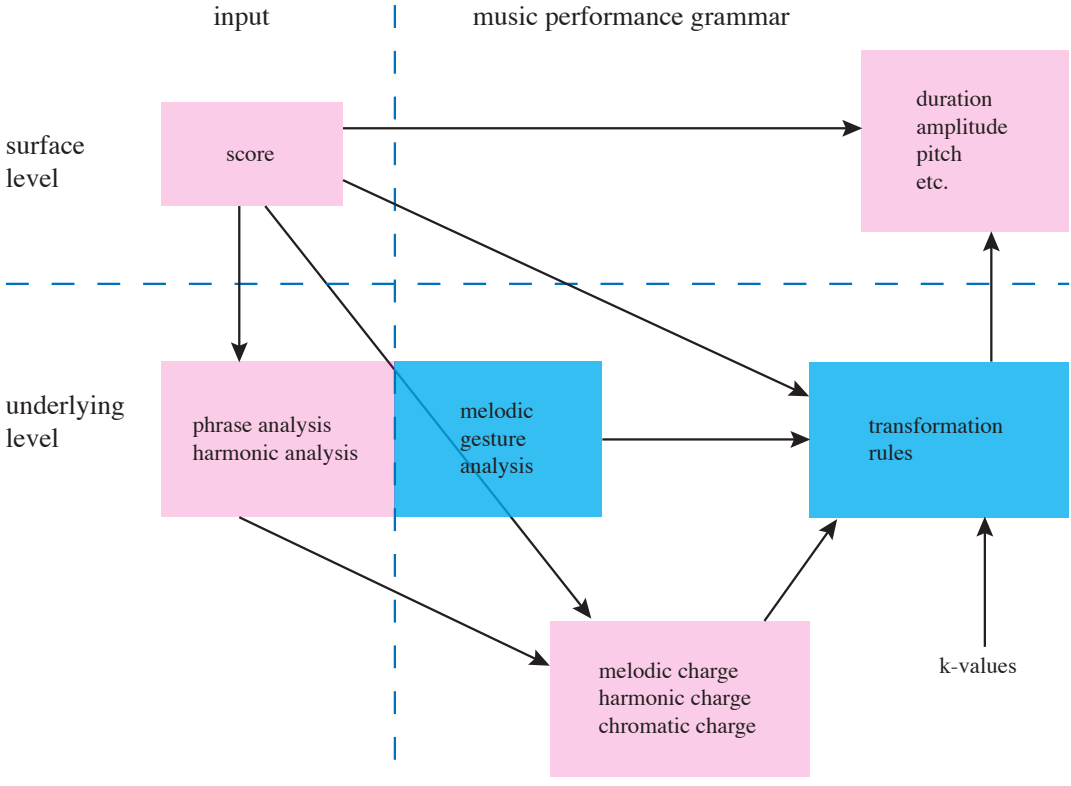


Figure 37.2: The KTH scheme following [165] shows a clear congruence with the general scheme which was described above in 37.1.

level shows the transition from the symbolic score to the physical performance (horizontal arrow). This transition (in fact: the performance map $\wp$) is shaped by an analytical rationale (left lower group in the underlying level) which comprises phrase, harmony, and melody. The output of this rationale is given in the charges for melody, harmony, and chromatics. These charges are then fed into the operator unit, the transformation rules (right lower group).

The performance transformation (horizontal arrow) is visibly factorized via the transformation rules, i.e., these rules define the entire performance map. It is however not clear from this scheme, which kind of action: symbolic, morphic, or physical is taken in the concrete cases. As was already mentioned in the first discussion of the KTH system, the original transformation of symbolic data into physical data is not made explicit in these rules. It is supposed that by

some underlying procedure, there is already a physical image of the score symbols before the explicit rules are activated. So the explicit rules seem to be physical operators. They act on a "primavista performance" which is implicitly assumed.

The rationales are purely analytical in their majority, but melodic rules also refer to gestural rationales. For example, "the leap tone duration modifies duration of tones in singular leaps" ([163, Rule GMI 1B]) in the sense of an expression of a gestural constraint of hand and arm movements when playing a keyboard. In fact, the duration of a high tone after a low one (or vice versa, same rule) is shortened by $\Delta DR = 4.2\sqrt{\Delta N}.k\ msec$, with the absolute pitch difference $\Delta N$ of the leap, ad $k$ the system constant.

The structural penetration of these rules is however quite poor. For example, the harmonic analysis does not take into account longer syntagmatic units of harmony. Only values of isolated chords within a predefined tonality are evaluated. Cadences or modulations are not considered. Moreover, rhythmic structures are completely neglected as a rationale for dynamic accents, for example. Ditto for contrapuntal structures.

Further, these rules are not inductive in the sense that they are not built to shape already given performances. They just act on the physical output and do not take into consideration the special character of the already given data. And an interaction between different arguments in combined rules is not developed. A hierarchical perspective is also not envisaged in the KTH approach. But this grammar is nonetheless a very clear scientific method that can be verified/falsified upon the audible quality of its output.

### 37.1.2 Neil P. McAgnus Todd

**Summary.** Todd's approach is backed by a systematic formalism of performance as a function of structure and specific grammatical arguments. It relates simple structural data, such as grouping boundaries, to expression by means of physically oriented transformation rules.

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As we have seen in section 36.3, Todd's approach to performance is a symmetric one using a performance procedure $\Pi$ which acts on the input $\Psi$ and an encoding function $\gamma$, whereas its inverse is a listening procedure $\Lambda$ which acts on a performance output $P$ and a decoding function $\delta$. The latter is in fact meant to be a section which determines the parameters of the encoding function that lead to the given encoding values.

This distinguishes Todd's approach from the KTH approach since the requirement of a listening procedure which accompanies the performance procedure is a strong restriction to the entire theory. However, this restriction is not really needed, i.e., one may also investigate the performance procedure without knowing whether there is an inverse solution. This is also the way Todd has viewed his scheme in [532].

As an example of an analytical rationale, Todd has described a rubato encoding formula

$$\rho_B(t, \phi_.) = \phi_1 + \frac{\phi_2}{(1-\phi_6)^2}\{\frac{t}{\phi_3} - \frac{(\phi_4 - 1)}{\phi_5} - \phi_6\}^2$$

at time $t$, and with parameters

1. $\phi_1$ = tempo,

2. $\phi_2$ = (rubato) amplitude,
3. $\phi_3$ = length of phrase,
4. $\phi_4$ = upper limit of boundary strength,
5. $\phi_1$ = offset of parabola minimum.

in [529]. This formula is determined by a hierarchy of time span reduction segmentations in the spirit of Jackendoff's and Lerdahl's Generative Theory of Tonal Music [243], and is applied in a recursive way, between the surface at hidden levels.

This formula defines an operator which is qualitatively different from KTH operators: It includes the basic tempo variable $\phi_1$ which could be interpreted as a previously defined performance information, i.e., this operator may be seen as a refinement of an already given "mother tempo $\phi_1$". Although this variable is not meant as a time-dependent quantity, the formula works independently of such a restriction and therefore is a real refinement operator of the tempo field. This means that in our above scheme, the Todd operator acts in a morphic way on the already given tempo field.

Inserted in our scheme, the argument of Todd's rubato operator is $\phi.$, and this one includes the mother tempo $\phi_1$, which is in fact a reference to the mother LPS.

This approach to rubato is nevertheless poorly rooted in the composition's musical structure since it does not include the possibility to have harmonical, melodic, rhythmical or contrapuntal arguments in the rubato formula. The existence of a tempo curve is also not mandatory, as we have already seen in the discussion of arpeggio effects in section 34.2.

### 37.1.3 The Zurich School

**Summary.** We give an overview of the approach of the Zurich school. The details will be discussed in the subsequent chapters.

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Essentially this approach is centered around the key concept of *weights*. These are numerical functions that encode analyses and serve as an input to the core of grammatical instances: stemmata and operators. Accordingly, a performance is generated by the stemma, a genealogical tree of nodes representing local performance scores of successively refined performance quality. The generation of such node "daughters" from antecedent "mothers" involves performance operators in the role of "fathers". The latter are charged with weights and realize grammatical rules of different flavors. The nature of these rules is not further specified, and may include any of the systems proposed by other approaches as long as they are based upon weights (in particular the KTH and Todd proposals).

The qualitative difference to the KTH and Todd systems is that a clear primavista performance is defined as a starting point of successively refined performances, and therefore, the initial transition from symbolic score data to physical data is anchored. Further, the weight system is conceived in such a way that a combinations of weights may define new weights to be fed into an operator, thereby allowing a simultaneous combination of different analyses to act on performance. Further, the LPS approach is so rich that any performance situation can

be dealt with: tempo hierarchies, abolition of tempo in arpeggio and rubato effects, combined deformation of parameters, and also the treatment of gestural output (beyond physical sound parameters). Finally, the stemmatic genealogy with sexual propagation from mother LPS and father operator guarantees an in-depth simulation of the process of rehearsal, where the spiritual unfolding of understanding a score may be modeled. All this has been implemented in the RUBATO® workstation, see chapter 40.

## 37.2 Remarks on Learning Grammars

**Summary.** This section gives a very short remark about grammatical patterns generated by machine-based learning from empirical performance data.

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For this section, we refer to [520]. We do in fact not believe that machine-delegated statistical methods such as neural networks or proper machine learning algorithms for rule learning are of proper scientific value, since when machines learn, we do not. Of course, this is an ideological point of view, but we cannot follow methods which delegate decisions to structured ignorance: understanding cannot be delegated to engineered devices.

For example, Gerhard Widmer's approach starts with a relatively detailed structural analysis of the score, including motives, groupings etc. It then correlates these structures to empirical performance data, such as dynamics or articulation, in order to apply machine learning algorithms for extrapolation to other scores.

# Chapter 38

# Stemma Theory

*O matre pulchra filia pulchrior.*
Horace (65–8 B.C.)

**Summary.** The stemma theory is introduced from its musicological and practical motivation. Semiotically speaking, performance is a result of a diachronic process. This is traced on the structure of a genealogical tree, the *stemma* of a performance. The stemma formalizes the diachronic process of rehearsal and practising. We describe the structure of stemmata as "family trees of performance", together with the corresponding *genetic* and *environmental* principles.

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When we compare the performance grammars developed by the KTH school and Todd to what happens when a musician learns to perform a new composition or when a conductor rehearses a composition with his/her orchestra, there is a tremendous difference of procedures. In the KTH system, the analysis-by-synthesis makes this particularly evident: The rules are not given a priori, but have to be derived via human criticisms and successive revision. This diachronic process is however not part of the grammar, it is a meta-theoretic construct. There is no trace of this successive improvement of rules in the system. History is annihilated by the uncontrollable criticism of a human expert (the violinist Lars Frydén in the KTH methodology). No trace of how the improved rules are produced from the old ones and their result is retained.

The KTH methodology is loaded by a meta-theoretical historical dimension in the analysis-by-synthesis loop. This dimension is not present in Todd's approach, although his hierarchical technique suggests a construction of the surface level (single beats in his prototypical example) from hidden levels. These hierarchies are more of a generative nature from global to more local structures, and not historically guided, though.

If one studies the way a musician rehearses a performance, it seems that a decisive component of this process is the successive improvement of performance which is built upon an added value with respect to the respective previous stages of perfection.

The formal theory of such a diachronic process is the following stemma theory. This theory is already prefixed in the definition of the LPS spaces, since the variables "mother" and "daughters" create the connection to inheritance structures. This theory cannot grasp the

totality of diachronic strategies, though. While a stemma comprises genealogical LPS trees, the selection strategies for the construction of determined LPS are not formalized in the stemma concept, only the result is.

Also, the arsenal of performance operators is not under control in this setup, it is supposed that a number of operators is available, but their systematic construction is not dealt with in this framework. The classification of performance operators is far from settled, see chapter 44.7 for this subject.

We are also aware that the present stemma structure is far from general with respect to feedback options, although the circularity of the LPS definition includes quite a lot of feedback techniques. But there is a serious limit to generalization: If one is going to implement denotators for stemmata by way of LPS denotators, their finite character must be assured in order to create performance outputs in a finite calculation time.

## 38.1 Motivation from Practising and Rehearsing

**Summary.** This section analyzes the development of an artistic performance through practising and rehearsal. We exhibit structural ramification and hierarchy, together with shaping mechanisms.

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When a performance is realized, this is never the result of a one-step process. Performance results from a development of successively improved intermediate performances. To understand performance, we need to understand its genealogy. A number of experiments have been undertaken in order to analyze the process of performance preparation via rehearsal and preparation. For example, Kacper Mikaleszewski has investigated this process via video recordings of the preparation of the *XII Prélude* from the *Second Book of Préludes* by Claude Debussy [375] by a professional pianist. Mikaleszewski introduces his investigations by the remark that it is not mandatory that pianists may clearly separate stages of their preparatory work of new compositions. In an investigation of A.A. Wicinski [566], ten famous pianists[1] were interviewed on their strategies. Seven of them comprise the first group of pianists to distinguish separate stages in their work. The second group of three pianists were not able to separate any such stages. This does not demonstrate that the pianists of the second group do not really follow unconscious strategies. The incapability of verbal description of such strategies is in fact a problem known among musicians: Very often, they are not able to verbalize their activities.

The first observation about this experiment is that "the divisions of the musical material introduced by the subject (the pianist) agree with the basic formal units of the composition, here related tightly to its texture. This characteristic agrees with earlier notions about the role of the structure of music in performance, and the tendency to practise longer compositions divided into shorter units (. . . ). At the same time, more complex textures led to the selection of shorter fragments for separate practice, and to making more divisions of the musical material. (. . . ) we may say that what he (the pianist) was been doing is to prepare effective sub-routines of a more complex programme which in turn would make him able to perform the musical composition at a satisfactory level of proficiency." [566]

[1] Among them were: Sviatoslav Richter, Emil Gilels, and Harry Neuhaus.

Although the general picture is somewhat ambiguous, it can be deduced from these findings that the subdivision of the given score structure, together with a development of local strategies of performance, are crucial. This can be viewed as a strategy for the acquisition of a global "performance plan". The analysis is that "the majority of comments concerning the text of the composition, fingering, hesitations, error corrections, and memorization seem to be in agreement with the general objective of the first stage of work mentioned by Wicinski: to work out a general idea of the composition and to become able to perform it with sketchy interpretation." *In other words, the strategy is not to start working out details and local performance, but to go top-down from the overall picture (as sketchy as it might be) to more and more detailed aspects in the fragments of the subdivision of the given score. So one exhibits a hierarchy of performance development which starts with the global sketch and successively ramifies to sub-routines of local aspects.*

### 38.1.1 Does Reproducibility of Performances Help Understanding?

**Summary.** Psychologically, the structure and function of performance generation is far from inscripted within a conscious memory. We discuss the value of an explication and memorization of such a process: Why is reproducibility of performance processes of scientific interest? Memorization relates to the question of identity of a performance (process). This leads to the question whether human precision is different from "machine" or "mathematical" precision.

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As we have already mentioned above, a number of excellent musicians cannot (or do not want to) control their performance generation, they just rehearse by some instinctive activity and do not care about strategies and conscious plans. So why try to make such processes explicit, since a good number of artists just do not care. The question really is whether there can be a culture of performance without reflection of the conditions of good performance, good in the sense of Adorno: expression of contents that are discovered via analyses of the underlying text. Now, if this goal is accepted, we need to know about the unfolding of such a performance along the nerve of the inner logic of an artist's elaboration—supposing that such an inner logic subsists. We insist that the absence of such a logic would result in a random walk to performance, an agnosticism driven by blind admiration of an artist's instinct, genius, call it as you like. But then, understanding performance would reduce to plain admiration of a miraculous phenomenon which does not meet our concept of a performance culture.

The scientific treatment of performance culture seems to face still another objection, i.e., the problem of objectivization of performance structures: The uniqueness and magic of artistic performance is sometimes viewed as being in contradiction to objective description, of conceptualization in the framework of scientific experimentation where reproducibility of objectively given conditions is mandatory. This skepticism culminates in the claim that objective, "mathematical" precision misses the precision of a human artist, that the latter precision is of another nature.

This is to say that you may draw a faithful trace of a performance and miss the essence, because a human may reproduce the performance in another way, however maintaining the core of "human precision" as opposed to machine precision. This is however not an argument against "mathematical" precision since its claim is an invariance argument: Although mathematical

changes in a performance may occur, the "human precision" is maintained. This means that we would have to search for invariants within the scientifically precise variety of performances. So this is not in contradiction to "mathematical" precision, it simply states that the latter may be too precise. But as long as we do not know what the "human precision" is about, we better stick to the "mathematical" precision which englobes a faithful trace of performance.

Another argument against scientific performance analysis is that its very concept of performance is illdefined. But this means that something has been forgotten in this conceptualization, i.e., one ought to include other components of a performance to grasp its identity. This could be: the audience, the space (concert hall, studio, etc.) of performance, the individual conditions under which a performance is perceived (listening in a bad mood, listening repeatedly to a recording, etc.), and the general historical and cultural background of such an experience. It is evident that the effect of a performance depends on these factors. And that therefore, a complete image of the phenomenon must include these factors. But this does not change the problem of giving a precise description and pursuing an in-depth analysis of the performance in the sense developed so far. *This is a fundamental objection we make against the methodology of the humanities: that they refuse to investigate parts of a phenomenon because they are related to other parts. We do not contend that understanding the whole can be reduced to the understanding of its parts, we contend that understanding the whole cannot refrain from understanding its parts, and that this latter task is the first step of any scientific procedure.*

## 38.2 Tempo Curves Are Inadequate

**Summary.** We discuss the conceptual and technical inadequacy of "flat" tempo structure for performance construction and derive the ramified hierarchical tempo trees as realized on the *presto*® software.

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It is a commonplace in performance research that a single tempo curve cannot control non-trivial tempo configurations. We therefore implemented a module for hierarchical tempo configurations, called *AgoLogic*, within the composition software *presto*® which was developed for the Atari® computer from 1988 to 1994 [335, 338, 340]. Tempo hierarchies[2] are a preliminary version of stemmata in the onset domain.

In order to make clear the scope of tempo hierarchies, we first give three examples of tempo hierarchies which are of practical significance in the historical context. After that, we shall give a formal definition of a tempo hierarchy in terms of corresponding denotator spaces.

**Example 54** The first example is an exercise in tempo curves taken from Carl Czerny's "Pianoforte Schule" [98], see also figure 38.1 . Czerny's exercise proposes to play this short composition with different tempo curves: first without any tempo change, second with an accelerando in the middle, third with a strong rallentando at the end. We have simulated these proposals

[2]This terminology is antiquated now, since we reserve hierarchies for cell hierarchies, and what we call a tempo hierarchy here is in fact a tempo stemma in the present terminology. We nonetheless conserve this antiquated terminology in this special discussion of tempo in the *presto*® software.

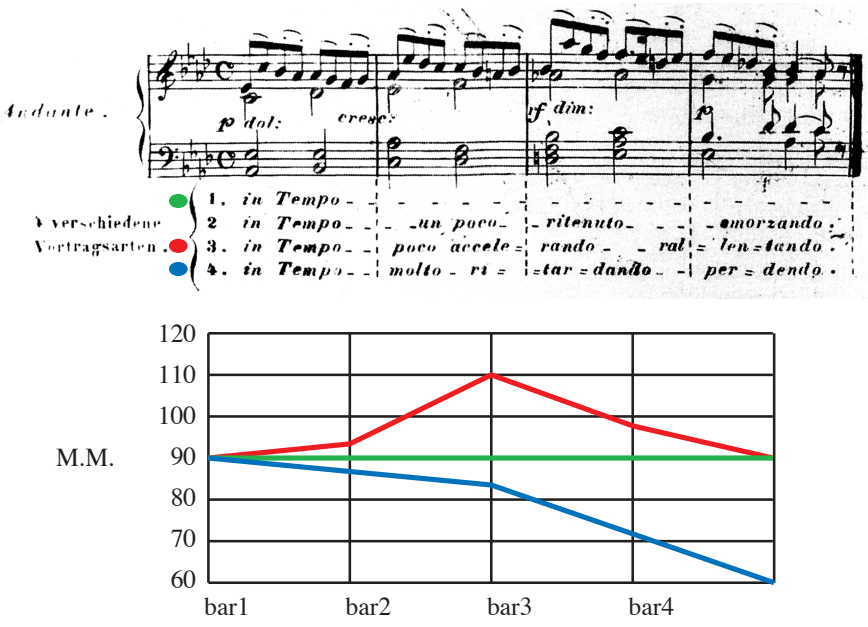


Figure 38.1: Czerny's exercise for tempo curve testing.

on *presto*®, and it turned out that the result is quite deceiving: No really relevant tempo experience results. In order to construct a less poor tempo structure, we have split the tempo levels in order to achieve the so-called *Chopin rubato*[3]. The three tempo setups as well as the Chopin rubato version can be heard in the first four samples on the audio-file `Czerny` on the book's CD-ROM, see page xxx. This is a very classical technique of tempo shaping. In *presto*®, this works following a hierarchical construction. In our example, we want the left hand to play a constant master tempo $T$. The right hand is the slave in tempo; we ask that the right hand tempo may vary anyhow under the condition that both hands coincide on each bar-line. To do so in *presto*®'s AgoLogic module, we may split the onset domain $I = [a_0, a_n[$ of the right hand into a sequence of one-bar portions $I_1 = [a_0, a_1[, I_2 = [a_1, a_2[, \dots I_n = [a_{n-1}, a_n[$. Then we have a hierarchy $I \to I_1, I \to I_2, \dots I \to I_n$. On each bar portion $I_i$ of the right hand, the user may reshape tempo via graphically interactive editing. The user can define any polygonal tempo curve $T_i$ within the limits $a_{i-1}, a_i$ of this interval, provided the integral is conserved, i.e., $\int_{I_i} \frac{1}{T} = \int_{I_i} \frac{1}{T_i}$, see figure 38.2. The routine taking care of the boundary condition of invariance of the above integrals was presented in corollary 21 of section 34.2. Of course, the mother tempo of the left hand need not be constant, any polygonal tempo curve can be produced for the left hand, and the daughters intervals $I_i$ can be given corresponding polygonal variations with the above boundary condition $\int_{I_i} \frac{1}{T} = \int_{I_i} \frac{1}{T_i}$.

[3]Sometimes also called "bound rubato" since one hand is playing a trigger tempo, whereas the other is bound to cope with the master every bar-line onset, say. In contrast to this concept, "free rubato" means that both hands play rubato, but exactly the same, and this frees them from following a master tempo.

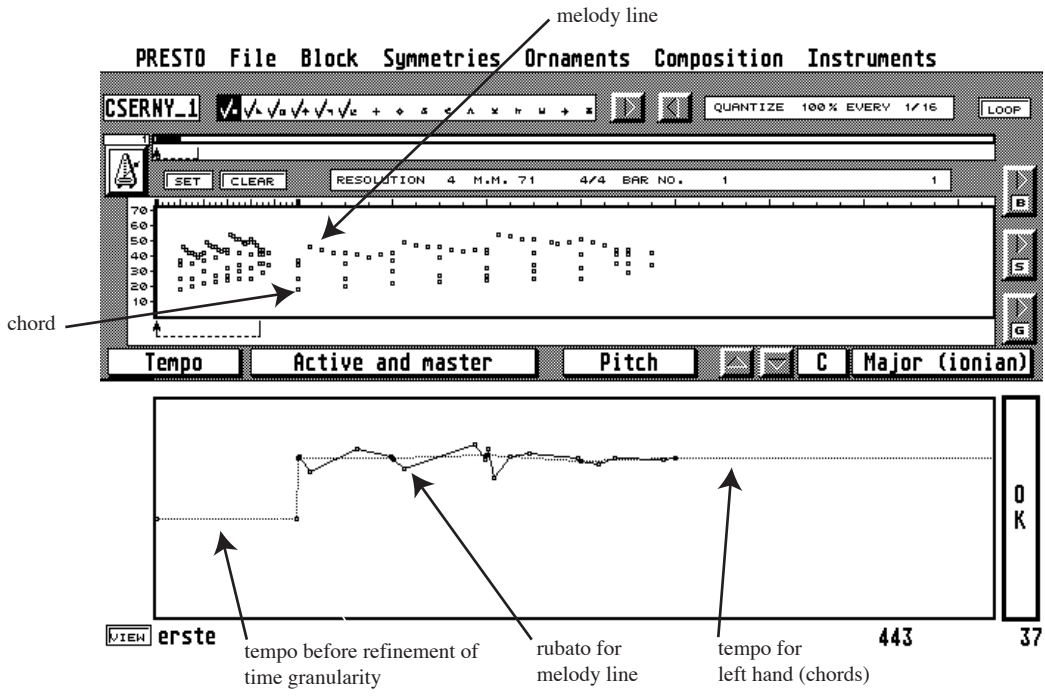


Figure 38.2: Czerny's example with a depth 1 hierarchy of tempo curves where each right hand bar is a daughter tempo polygon of the right hand tempo which is constant here.

**Example 55** The second example is taken from the Chopin *Impromptu* op.29, refer to figure 38.3, bars No.78-80. This situation shows a series of trills and arpeggi. The tempo of bars SS is confronted with the tempi of the trills and the arpeggi. This is not the situation of a bound[4] Chopin rubato, since the trill ornament is only one portion of a number of notes to be played by the right hand, it is more of a hierarchy in a scenic arrangement where the ornamental notes add a supplementary level of structure. The trill notes are not even explicitly denoted, the artist has to fill up the trill sign in the spirit of Chopin style tradition.

The second auxiliary structure in this example is the arpeggio. This is also an incomplete notation insofar as you have an anchor note and several "satellite" notes, i.e., the chord's notes attached to the anchor note. These must be played in a temporal succession. The temporal succession and the anchor note are not always clear: It could be that the arpeggio succession is read top-down in pitch, or vice versa. Also could the onset of the last or that of the first note be the anchorage onset. These things being selected, the speed and shape of temporal succession

[4] A *bound* Chopin rubato is one where one hand plays a trigger tempo whereas the other hand's tempo may vary locally, as a tempo slave, but coping with the right hand trigger tempo on a number of master events. A *free* Chopin rubato is one where the rubato is played synchronically for both hands.

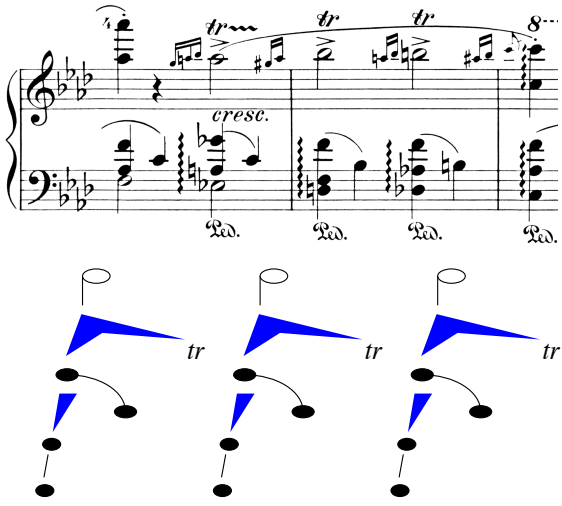


Figure 38.3: Chopin's *Impromptu* op.29, bars No. 78-80 shows a series of trills and arpeggi.

in the arpeggio are not well defined. As with the trill notes' temporal distribution, the arpeggio development is a "satellite" phenomenon in the note hierarchy.

We have chosen this hierarchy in *presto*®'s Agologic module (see again figure 38.3): The mother tempo is present on the top level represented by the half notes in the graphic below the score. The top level has two daughter tempi: the trill daughter (to the right below the half notes) for the trill tempo, as well as the arpeggio daughter (to the right below the half notes). The latter controls the tempo of the descending interval notes. It has a daughter tempo for the expression of the arpeggio tempi.

This hierarchy does not determine the concrete tempo curves on the mother level, on the trill, and on the arpeggios; to this end, the graphically interactive input by the user is needed. This means in particular that the same tempo hierarchy can express very different ways of performing a piece: from the beginner to a virtuoso. On the book's CD-ROM (see page xxx), four samples of performances of this tempo hierarchy are traced. They are heard on the last four samples (after the Czerny samples) of the audio-file `Czerny`.

**Example 56** The third example illustrates the tempo hierarchies as they are needed in the performance of a large orchestra with special instrumental groups. Our example is a large orchestra which is controlled by a conductor. Within this orchestra, we suppose given a string group (violins, say) which has to obey the conductor's indications, but within two time windows (the curva1, and the curva2 windows) may follow an individual tempo curve (as indicated by the concert master, say). Within each such slice, on a small time window (curvetta3, curvetta4), there is a soloist part (played by a first violin, say) which may realize a cadence-like small expression, but has to cope with the string group (curva1 for curvetta4, curva2 for curvetta3, respectively). The tempo window of *presto*® shows the individual curves with their local deviations which, according to the implemented algorithm, yield the same total physical durations

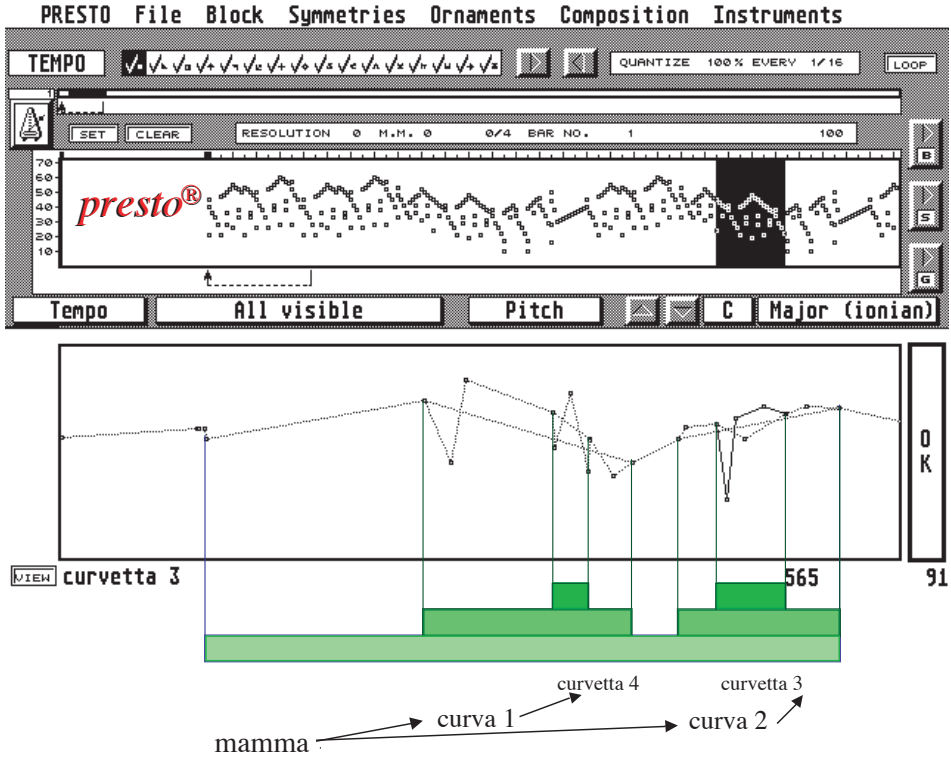


Figure 38.4: The tempo hierarchy of an orchestra is shown. The mother tempo curve (mamma) is ramified into two daughters (curva1, curva2) which in turn have one daughter each (curvetta4, curvetta3, respectively). This interdependence of local and global tempi is used to differentiate roles in orchestral time control.

as their respective mothers.

## 38.3 The Stemma Concept

**Summary.** The formalism of performance stemmata is introduced. A stemma is a rooted directed graph which carries on its nodes local performance scores. The generative principles as well as the entire structure are modeled after a matrilineal scheme: There are only mothers and daughters on the stemma, starting with the primary mother. But the proliferation of these families is sexual: Fathers, formally represented by performance operators, do contribute to their daughters together with their mother. The detailed "sexual behavior" is described and turns out to be quite similar to the veritable biological/sociological behavior in life.

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This section terminates the chapter with a formal construction of stemmata. This is a remarkable subject for three reasons:

First, the stemma concept is probably one of the first to grasp historical processes on the formal level in the mathematical sense, where rather than "formal" we should say: "precise", as contrasted to the notoriously poor conceptualization in the humanities. This has tremendous consequences for the experimental modeling of historical processes: One can now perform historical developments which never have taken place, more concretely: rebuild real and simulate fictitious performance history of the rehearsals of a pianist, compare these directions and draw conclusions on the quality of the real history against the way of fiction: Are we, and why are we living in the best of possible worlds.

Second, this formalization is not only formal in the mathematical sense, but also on the level of implementation in computer software. In fact, we have seen in the course of the intricate definition of an LPS in section 35.4 that this concept can only be defined by means of a circular form $LocPerfScore_{BP}$, since a standard mathematical definition would not allow circularity, circularity being only an accepted technique in implicit equations, not in conceptualization—although, as we have seen in chapter 9, set theory (probably unconsciously) conceptualizes sets in a circular form. But this nature perfectly fits the nomenclature of object-oriented programming languages: An instance variable can very well be an object of the same class as the object whose instance variable it declares. This observation once more evidences a turning point between mathematics and object-oriented programming, where the denotator and form concepts have been derived.

Third, the very nature of this formalization of historicity appears to fit with biological inheritance principle of sexual propagation. This is not only a happy coincidence, rather is it a mandatory direction if one wants to model learning: By sheer life experience, inheritance and evolution are the best proven models of successful learning.

Although it is possible from the preset concepts, we do not diverge on global performance score constructions for the stemma theory and leave this segment to future research.

*All the LPS denotators will be situated at the zero address as long as we do not stress the contrary.*

### 38.3.1 The General Setup of Matrilineal Sexual Propagation

**Summary.** This section describes the overall mechanism of stemma construction. In particular, we discuss the reason for the matrilineal approach.

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Given a sequence of basis and pianola spaces $B, P$, recall the definition of the LPS space (35.2) with the six factors *Mother* $\mu$, *Daughters* $\Delta$, $CellHierarchy_{BP}$ $\mathfrak{h}$, *Instrument* $\iota$, *Operator* $\Omega$, *Weight* $w$.

**Definition 106** *If* $\Lambda : 0 \rightsquigarrow LocPerfScore_{BP}(\mu, \Delta, \mathfrak{h}, \iota, \Omega, w)$ *is a local performance score, let* $\Lambda\downarrow$ *(resp.* $\Lambda\uparrow$*) the directed graph of all LPS that can be reached by finite descent from* $\Lambda$ *to the daughters, to their daughters, etc. (resp. the set of all LPS that can be reached by finite ascent from* $\Lambda$ *to the mother, to its mother, etc.), together with the mother-daughter arrows. Denote*

*by $\Lambda\updownarrow$ the union of the graphs $\Lambda\downarrow$ and $\Lambda\uparrow$. Call $\Lambda^\infty$ the directed graph of all LPS that can be reached from $\Lambda$ by finite ascent and descent, together with the mother-daughter arrows.*

**Definition 107** *A local performance score $\Lambda : 0 \rightsquigarrow LocPerfScore_{BP}(\mu, \Delta, \mathfrak{h}, \iota, \Omega, w)$ is called* a stemma[5] *if its graph $\Lambda\updownarrow$ is finite and defines an undirected tree (no undirected cycles), and if it is not the daughter of its mother, if that mother exists*[6]*. The leaves of $\Lambda\updownarrow$ (which are also the leaves of $\Lambda\downarrow$ in this case) are called* the leaves of $\Lambda$*, whereas $\Lambda$ is called* the primary mother of the stemma*. If $\Lambda$ has no mother, it is called* a prime stemma*. The set of leaves of $\Lambda$ is denoted by $\underline{\Lambda}$.*

Intuitively speaking, a stemma is a primary mother, together with its daughters, its granddaughters, etc., until we reach the leaves which are the output LPS that will eventually yield the data to be performed. It is important to have the mother of a stemma being disconnected from its daughter, i.e., not pointing at the latter in its daughter list. The primary mother of a stemma is meant to define a new tree of unfolding LPS which cannot be accessed by any other stemma. Although it can access another stemma, it is invisible to this latter. We have seen a stemmatic structure in the previous discussion of "tempo hierarchies" which are, in the present terminology, a kind of tempo field stemma, although usually there is more in a stemma than just onset-related fields and onset kernels.

**Exercise 79** Restate the "tempo hierarchies" in terms of stemmata with

$$B = \{Onset\}, P = \emptyset.$$

The matrilineal terminology is not really the whole truth here. In fact, each LPS contains its operator, the patrilineal component which is responsible for the generation of this LPS. But the result, more precisely: the hierarchy $\mathfrak{h}$, returns the output information for performance. And it is this result which will also be used to produce further daughters, and not the operator. This justifies the matrilineal terminology, whereas the operator is only a hidden generative instance.

Let us look at a historical example of a stemma: the stemma for the composition *Kuriose Geschichte*, the second *Kinderszene* in Robert Schumann's synonymous collection op.15 [482]. This stemma was constructed on the performance platform RUBATO® in 1996 at the *Staatliche Hochschule für Musik*, Karlsruhe by the author, Oliver Zahorka, and Joachim Stange-Elbe. It took us three days to realize the whole setup and performance on a Bösendorfer MIDI grand. The performance of the piece is documented on a CD, see [360], and in a broadcast of the Austrian TV [161]. Although the stemma is quite primitive, the shaping results were satisfactory and taught us a lot about the empirical aspects of computer-assisted performance research. The stemma is visualized in figure 38.5. Although we see that each single refinement layer is controlled by one and the same operator (horizontal arrow), the layer did not form a grouping in the technical sense to be discussed in section 38.3.3: Each daughter had to be performed as an isolated instance, since no grouping methods were implemented at that stage. The construction of this stemma first follows the splitting of right (RH) and left hands (LH), then, after the shaping of primavista dynamics and agogics, global agogics is constructed on these two LH and RH symbolic kernels. The splitting for operators $\Omega_5, \Omega_6, \Omega_7$ regards a small

[5] "Stemma" is synonymous to "genealogical tree".

[6] This is a slightly irritating subtlety of our conceptualization.

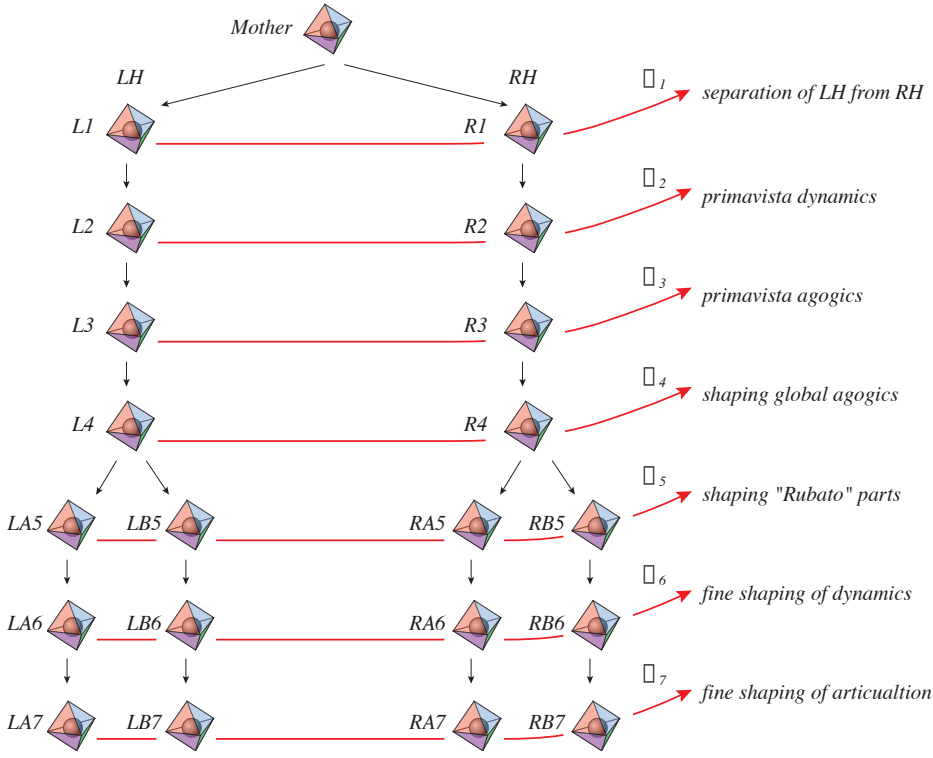


Figure 38.5: The stemma of the first performance of Schumann's second *Kinderszene: Kuriose Geschichte* that was constructed and performed 1996 on the Bösendorfer grand piano at the *Staatliche Hochschule für Musik* in Karlsruhe.

number of bars which have to undergo a more differentiated rubato. The final shaping regards fine "tuning" of dynamics and articulation in all leaves.

### 38.3.2 The Primary Mother—Taking Off

**Summary.** The primary mother represents the performance score which is deduced from the score data as they are inscripted on the predicate level. We make the deduction process explicit, together with the set of prima vista parameters.

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The primary mother $\Lambda$ of the stemma is the starting point of a stemmatic evolution process, it is used to derive all the LPS in $\Lambda\downarrow$, and eventually leave set $\underline{\Lambda}$. There are two situations in such a primary mother: Either it has a mother LPS $\Lambda_0$ or else it is a prime mother with the score form mother denotator $S : 0 \rightsquigarrow ScoreForm(\sigma)$.

If it has a mother LPS, this means the following: The stemma which it defines is not an autonomous structure, but it is derived from another stemma $\Lambda_0$. This other mother stemma can however not access $\Lambda$, this one is invisible to $\Lambda_0$. The idea of this asymmetry is that we want to separate stemmatic processes in their information flow. A stemma is a closed unit of unfolding performance stages. It can be used to induce a new stemma, but it is not related by a daughter pointer to the new one. This prevents mixtures of stemmatic unfolding: Once a stemma is defined, it can help another stemma to build refined performances, but it does not refer to this one, while vice versa, a new stemma may refer to the old one via the mother pointer of its primary mother. This technique creates a series of successively improved stemma $\Lambda_0, \Lambda, \Lambda', \Lambda'', \ldots$

If $\Lambda$ has just the score denotator $S$, there is no reference to other previous stemmata, and *we are in the properly termed primavista situation*: There is a score, which is formally expressed in terms of the denotator $S$, and this score is the only reference to create the first performance. This is a complex operator which we shall now discuss.

The construction of all data of this prime mother is the scope of a special operator which we call the *PrimavistaOperator*[7]. The PrimavistaOperator is a macro-operator since it has to deal with all possible score signs incorporated in $S$, but this is more a question of software implementation than a mathematical problem.

The problem is this: recall that in the KTH theory, there was no such a thing as a primavista setup. This prime mother construction must be dealt with somewhere in the performance process. In the unfolding of a performance, it is the first action to be taken: to establish a first version of a performance which is uniquely based upon score data. It could be argued that the score data can be supposed to be introduced in advance via the symbolic kernel. This could even be admitted, but then, all the other data, such as rallentandi, fermatas, slurs, etc., where should they be piped in order to contribute to a primavista performance?

For example, a fermata sign must be taken into account while it defines a sensible tempo sink, and this must be done before any refined shaping activity of performance is set forth. Of course, it is not sufficient to just know that this fermata sign is situated at a determined place, we also have to provide the data for the exact shape of this specific tempo sink. So, first of all, the input must list all the possible relevant signs, i.e., the primavista predicates in the sense of section 18.3.3.

The input data are all based on the first information about the events of the score $S$ in the different parameter spaces of $BP$ that will participate in different special predicates. These may be ordinary notes, bar-lines, pauses, and the like. In order to produce a set of events that cope with the hierarchy induced by condition 1 in definition 102, i.e., that in a cellular hierarchy, kernels project into kernels. This means that the symbolic kernel event set $Evt_S(U)$ associated with space $U \in BP$ must project into $Evt_S(V)$ if $V \subset U$. We shall here speak of "events $x$ in space $U$" in the sense of zero-addressed denotators $x : 0 \rightsquigarrow U(\xi)$. This induces the Boolean predicate $Evt_S(U), U \in BP$, via

**Definition 108** *For $U \in BP$ and an event $x$ in $U$, we set $x/Evt_S(U) = \top$ iff $x : 0 \rightsquigarrow U(\xi)$ is such that there is an event $y : 0 \rightsquigarrow W(\eta)$ in the score denotator $S$, living in a (not neces-*

[7] In accordance with code naming conventions in object-oriented programming, the nomenclature is this: every operator is named by a special name *Specialname*, directly followed by the postfix "Operator", yielding "*Specialname*Operator".

*sarily strict) superspace $W$ of $U$ in $BP$, such that $p_U(y) = x$ (i.e., $p_U(\eta) = \xi$), else we set $x/Evt_S(U) = \perp$.*

This presupposes that the events of the score denotator $S$ have been identified a priori. These are denotators in spaces of $BP$ that can be found upon inspection of the $S$. This is a knowledge which ultimately exceeds the formalized knowledge base we are dealing with in our mathematical framework, it needs an instance which can create a score denotator $S$ from the given score. This could be an optical character recognition (OCR) software for scores, or any machine that collects events from MIDI files, for example, or just a human expert in score reading. *In any case, we may suppose that the score is transformed into a score denotator $S$ in a score form, and that the events are deduced from $S$. The latter is a standard task in logical and geometric motivation predicates.*

To define the prime mother LPS, we have to define all its constituents: Instrument, weights, operator, and hierarchy. As to mother and daughters, the first is $S$, and the latter will be added when the stemma is made explicit in a later stage of the historical process—presently, it is empty. The most important data is the hierarchy. We have to construct it by use of the PrimavistaOperator and the given weights. The hierarchy defines also the performance map on its top space, and we suppose that the instrumental specifications are sufficient to transform the physical parameter vectors of the image of the performance map $\wp$ into sound objects. So we may concentrate on the hierarchy construction here.

In the hierarchy $\mathfrak{h}$ to be constructed, we have to define the diagram of cells. This first of all means defining the projections of kernels. We instantiate all the event sets as the first bunch of predicates $Evt_S(U)$ in the weight list of our prime mother. For any space $U \in BP$, we select the kernel $K_U = supp(Evt_S(U))$, by definition of the predicates $Evt_S(U)$, the kernels map into each other. Also, the frames $R_U$ may be pre-defined as the smallest cubes containing all the predicate supports $K_U$. Since we are starting with an instrumentally well-defined local situation (only one instrument), we may also suppose that there is a top space $Top_S$ in $BP$ where the kernel $K_{Top_S}$ is not empty. To define the space hierarchy, start taking all spaces with non-empty kernel, these are just all subspaces of $Top_S$. This hierarchy is much too large, in general. We first have to restrict to the standard hierarchy requirement described in section 35.3, requiring that for any pianola space within a hierarchy space $U$, the corresponding basis space must also be in $U$. For example, in piano music, duration cannot be a reasonable hierarchy space. One further has to restrict this hierarchy to a subhierarchy which is reasonable for the given instrument. There is no general algorithm for such a procedure, one has to observe two things, however:

- the hierarchy must be standard with respect to basis and pianola spaces;
- if a $K_U$ contains events that are not proper projections of other events, this space must be retained in $\mathfrak{h}$;
- there are default space hierarchies for specific instruments, such as the piano and violin hierarchies described in section 35.3.3.

With respect to these constraints, one will choose as space hierarchy the smallest standard subhierarchy $\mathfrak{h}$ of the total hierarchy of $Top_S$ such that it contains the default space hierarchy and the spaces containing events that are not proper projections. For piano music this means

taking notes in $EHLD$, pauses, slurs, accelerandi and similar events in $ED$, bar-lines, time and key signatures in $E$, but observe that these event types are all contained in one of the spaces of the default piano hierarchy. The selection of such boundary conditions for hierarchies is a typical system parameter in the PrimavistaOperator, a parameter which, together with the weights, creates concrete hierarchies.

A further set of system parameters: the selection of initial sets, performance fields, and initial performance, is a bit more delicate. The performance field to start with depends on the physical values attributed to the symbolic ones before any refined performance takes place. Typically, this can be solved by an affine isomorphism if we convene that on the physical level, we have the common space of pitch, corresponding to the logarithm of frequency, loudness, corresponding to the logarithm of intensity and time corresponding to onset units, etc., see appendix A for such common spaces. Then the performance field is a constant one (the derivative of an affine isomorphism being constant, i.e., its linear part). Since in common situations, the linear part of $\wp$ is also diagonal (just some positive gauging constants), *the prime mother performance field is constant with positive coordinates.* This setup evidently guarantees that the projections commute with the constant performance fields $\mathbf{Ts}_U$ on the spaces $U$ (condition 3 in definition 102 of performance cell morphisms), which we have to define in their respective coordinates in each space $U$ of the defined space hierarchy. This is what we now assume.

Consequently, the entire frame volume can be reached on these constant prime mother performance fields from the left "bottom walls" of the frames $R = [a_1, b_1] \times \dots [a_n, b_n]$. There are $n$ bottom walls for $R$, i.e., the $(2^{n-1} - 1)$-simplexes

$$W^R_{n,i} = \{(x_1, \dots x_{i-1}, a_i, x_{i+1}, \dots x_n), \text{with } x_k = a_k \text{ or } x_k = b_k, k \neq i\} \tag{38.1}$$

for $i = 1, \dots n$; they are degenerate for $n > 2$. The initial set $I$ on an $n$-dimensional space $U$ with frame $R$ is the family $(W^R_{n,i})_i$ of bottom-wall simplexes. Since we have $I \int^{\epsilon} \mathbf{Ts}_U = R$ (with $\varepsilon = +$), condition 4 of the definition of a morphism of performance cells is also fulfilled. We are left with the initial performance condition 5 in the definition of a morphism of performance cells. This initial performance can be defined by the same data which we have used to define the performance fields. Therefore, condition 5 is automatically fulfilled. So we are left with the construction of the weight system and the PrimavistaOperator that acts on the defined hierarchy in order to integrate the information from the primavista score predicates.

For events with $x/Evt_S(U) = \top$, or for local compositions of such events, or for local compositions of such local compositions, etc., as required by the weight definition of an LPS, we now look for special predicates according to the musical notation conventions[8]. We have already listed the common prima vista predicates in section 18.3.3. Here, we give a selection to show how these predicates can be restated in terms of weights. If needed, such weights are added to the weight list of the LPS.

**Slurs.** There are two types of slurs: normal legato slurs and articulation slurs. Both are boolean predicates which are evaluated on sets of events in a space $U$. Call the legato predicate $LegatoSlur_U$ such that $x/LegatoSlur_U = \top$ iff $x$ is a local composition in $U$ which is embraced by a legato slur. As we are specifically interested in $x$ being also a local

[8] By that we mean the conventions of European tradition. For other traditions, special predicates and parameter spaces have to be introduced, but the procedure is the same, though not always an easy one, as the Japanese Noh nomenclature may illustrate (see the Noh example in 18.3.3.2).

composition stemming from $S$, we set $LegatoSlur_{U,S} = LegatoSlur_U \& Evt_S(U)$. We shall henceforth abbreviate the logical combination $?\&Evt_S(U)$, by an added index $?_S$; for example, the articulation slur is denoted by $ArtiSlur_{S,U}$.

**Articulation.** The following predicates are self-explanatory by their names and all relate to evaluation on single events of $U$ in $S$:

$$Staccato_{U,S}, Staccatissimo_{U,S}, Marcato_{U,S}, Tenuto_{U,S}, Accent_{U,S}$$

**Fermata.** A fermata lives in the $U = E$ space of onset; no duration of the fermata is explicitly defined. The predicate weight is denoted by $Fermata_{E,S}$, and an onset $x$ is a fermata iff $x/Fermata_S = \top$.

**Value Change.** A value change $p/q = r/s$ is a change in the time signature on a special barline, from $p/q = 2/4$ to $r/s = 3/8$, say. So we first need a time signature predicate for time signature $p/q$. This is an onset-located predicate, call it $TimeSig(p/q)_S$, and then a predicate for two-element onset sets $x = \{a, b\}$ of time signature predicates, i.e.,

$$\begin{aligned} &x/ValCh(p/q = r/s)_S = \top \text{ iff} \\ &card(x) = 2 \ \& \\ &\exists a \in x, a/TimeSig(p/q)_S = \top \ \& \\ &\exists b \in x, b/TimeSig(r/s)_S = \top, \end{aligned}$$

a predicate that is motivated by a mathematical predicate and the already given time signature predicate. Observe that the structure of the mixed powerset spaces $U$ enables us to use ordered pairs of any points of $U$ as they are defined in classical set theory, see appendix C, definition 114. So it is also possible to redefine the above predicate via ordered pairs $a, b$ of objects with $a/TimeSig(p/q)_S = \top, b/TimeSig(r/s)_S = \top$. The details are left as an exercise.

So the weights in our weight list of the prime mother LPS is the list of a) the symbolic kernels, and b) the appended list of all primavista predicates, as discussed above, with their restatement as weights with truth values in $\mathbb{R}$.

The *PrimavistaOperator* now has to add operations on the given hierarchy $\mathfrak{h}$ which stem from a paratextual meaning of the weight predicates, i.e., an interpretation beyond the abstract textual trace which the score denotator has induced. For example, a fermata predicate has to induce a tempo sink. This and all other actions of the *PrimavistaOperator* will be discussed in chapter 44.7 about operator theory.

### 38.3.3 Mono- and Polygamy—Local and Global Actions

**Summary.** The typology of actions that operators may take in order to unfold the stemma includes monogamic coupling with one mother and production of one or several daughters, or else polygamic coupling with simultaneously several mothers. We give the formal description and its justification in terms of practising and rehearsal.

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A performance operator $\Omega$ of an LPS $\Lambda : 0 \rightsquigarrow LocPerfScore_{BP}(\mu, \Delta, \mathfrak{h}, \iota, \Omega, w)$ has the function to define the cellular hierarchy and the initial performance $\iota$ of $\Lambda$. This data is calculated by use of the weight input $w$ and—except for the primary mother—by the already calculated mother LPS $\mu$. This is also represented by the equation

$$\Omega \propto \mu = \Lambda.$$

In general, a mother may have several daughters with the same father operator, i.e.,

$$\Omega \propto \mu = \Lambda_i,$$

which seems to be an inconsistent notation since the index is absent in the left part of the equation. In fact, we have to be more precise here. The operator for daughter $\Lambda_i$ must have a system parameter $i$ which specifies that daughter from one and the same mother $\mu$. So the correct notation would be

$$\Omega(i) \propto \mu = \Lambda_i,$$

where the system parameter is specified. This is also what in software design is realized: For each parameter $i$, the corresponding daughter is instantiated as a function of that operator with the $i$-value in its instance variables.

This ordinary christian family life is however not the most economic and reasonable in many performance situations. For example, it may happen that one has already developed the stemma to a strongly ramified tree, and that one wants to apply the same refinement procedure to all or a large number of leaves $\Lambda_k, k = 1, \ldots K$. One could of course apply the same operator $\Omega$ to each $\Lambda_k$ independently of the other leaves. But then, changing some parameters of this operator would require us to go through each leaf and alter the data step by step. In order to avoid this "copy and paste" process, it is better to let the leaves know that it is one and the same operator and not just a set of clones $\Omega^*$ of $\Omega$ which have to produce daughters $\Omega^* \propto \Lambda_k$.

The problem here lies in the definition of a denotator's identity. In fact, if we assign to all leaves $\Lambda_k$ the same denotator $\Omega$ as their operator, then any change in this operator is simultaneously carried out on all these leaves. But this is not so trivial: If we had a situation as it is known from object-oriented programming, i.e., if the operator were an instance variable of a LPS class, then its change would be an automatic change in every instance $\Lambda_k$. In the context of denotators, however, we do not have object-oriented structures, and the change has to be declared explicitly according to some kind of concept surgery, as it was discussed in section 6.9. The surgical intervention to be carried out here would be this: Search for all LPS $\Lambda_k$ in the given stemma, or else in some more specific hierarchical position, such that their operator is named $\Omega$. Then define new LPS $\Lambda_k^*$ by replacement of the coordinate (named) $\Omega$ by a new coordinate (named) $\Omega^*$. This only works if names identify denotators, otherwise, we have to search for other keys to retrieve the wanted denotators. One could denote this intervention by the symbol $\Lambda_k^* = \Lambda_k/\Omega \pitchfork \Omega^*$. A global notation of a grouping of LPS within a stemma is to write $G/\Omega$ or simply $G$ for a set $G$ of LPS which is operated by one and the same performance operator $\Omega$, we call such sets *stemmatic groupings*, they are evident denotators in the powerset space of the present LPS space. Accordingly, we write $G/\Omega \pitchfork \Omega^*$ for the replacement of $\Omega$ by $\Omega^*$ in the grouping $G$. Clearly, any two groupings within a given stemma are disjoint. Observe also that the mothers of a grouping do not automatically define a grouping for *this* reason! Moreover, we ask for the following transitivity (collective responsibility) axiom:

**Axiom 4** *If $G$ is a stemmatic grouping, and if a daughter $\delta$ of a mother $\mu \in G$ is a member of a group $H$, then every daughter of a mother in $G$ is also a member of the group $H$.*

For example, if $G$ is grouped by a tempo operator which imposes a new tempo curve on each member of $G$, and if we create a group $H$ of daughters with a refinement of their tempo curves, then, if a daughter $\delta \in H$ has its mother $\mu \in G$, it is reasonable to have the same refinement of the tempo curve for all daughters of all members of $G$, because the shared tempo curve from $G$ should inherit its refinement.

**Axiom 5** *It is always possible to resolve a grouping in the sense that any further changes to the stemma only affect the grouping's former members instead of the entire grouping. However, the descendants of a grouping's members are not affected in their grouping memberships.*

This is a typically historical process in the stemmatic construction: The final stemma is a sequence of intermediate stemmata, i.e., we really have the time parameter entering the world of denotators, or, rather (and more precisely) the world of predicates.

### 38.3.4 Family Life—Cross-Correlations

**Summary.** Apart from genetic interaction as described by mono- and polygamic propagation, the shape of a single daughter can also be determined by cross-correlations with its sisters or other relatives. We again give the formal description and its justification in terms of practising and rehearsal.

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Suppose that a mother $\mu$ in a stemma $\Lambda$ has a number $\delta_1, \ldots \delta_m$ of daughters. What is the common situation for such a family? We have seen in the example of the composition *Kuriose Geschichte* (figure 38.5) that such a set of daughters can occur if we have to split the whole composition or a part of it into mutually disjoint subcompositions, such as left hand and right hand, which must be treated in individual ways. Or if some bars require a different performance shaping than the other bars, as is the case for the splitting of left and right hand, respectively, in that stemma.

This procedure might also be applied in a more systematic way with respect to natural grouping structures of a composition: For example, in Schumann's *Kinderszene 7: Träumerei*, we have four periods $A_1$, $A_2$ (= the repetition of $A_1$), $B$ $A_3$ (= recapitulation of $A_1$), eight bars each. This could be the basis of an operator which splits the entire composition into these four daughter. In each period, we have the grouping into the eight bars, and that can provide us with granddaughters $A_{1,1}, A_{1,2}, \ldots A_{1,8}$,$A_{1,1}$, $A_{2,2}, \ldots A_{2,8}$, $B_2, \ldots B_8$, $A_{3,2}, \ldots A_{3,8}$. This situation is however not what is really relevant in a realistic performance. The point is that we need operators that deal with relations between different sisters and not only between daughters and mothers. Musically speaking, this means that an artist must take into account what will be played in future bars and what has been played in previous bars of the present period, when shaping a particular bar performance. One must also take into account what is the performance of future and of previous periods *qua* periods. So we have to face operators which take into account more distant relatives in this large family of stemmatic nodes.

In fact, from the artistic point of view, a performance must testify a sophisticated coherence in the large and also within more restricted neighborhoods of musical events. This is a common place in musicological performance theory and in the feuilletonistic music criticism. But such a general insight is very difficult to make precise for several reasons.

First, it is not clear which performance structures should be involved when being set into a coherent whole. Must we think of coherence among local tempo curve segments, or, more generally: coherence among local parts of performance fields? Or should we separate coherence questions along the strata of cellular hierarchies? And if we are searching for such a coherence, how should the analytical background, more precisely: the weights which aliment operators, be screened for coherence? In other words: What is a coherent analysis? Evidently, without such a concept, coherent performance fields can be defined, but they risk failing in their rational task: to reflect the analytical background. Can they be coherent just with respect to themselves without expressing coherence of analyses?

Second, the mathematical variety (in the non-technical sense of the word) of coherence structures is virtually infinite: One could imagine linear, polynomial, differentiable, analytical, and—hélas—statistical, any kind of non-linear relations that would be involved to define coherence. The influence of other family members on a determined LPS could also come from its sisters, from cousins, from more distant relatives, anything is imaginable. We shall make explicit models of stemmatic coherence in chapters 44 and 46. These models are essentially linear models on every set of sisters, but in their combination among the whole stemmatic inheritance, they accumulate non-linear phenomena which lead to non-trivial algebro-geometric phenomena, and phenomena pertaining to second-order differential operators.

These perspectives should make clear that we scarcely understand the genuine concepts of performative coherence in their musical phenomenology and accordingly cannot construct mathematical models on the basis of such a blurred phenomenology. Adorno and Benjamin have given us the catchwords for a deeper investigation of performance, catchwords which we could very well transform into adequate mathematical concepts. But these theorists did not elaborate their conceptual germs to a degree of differentiation that could help describing and understanding the concrete artistic shaping of performance. We can hardly understand and even less forgive the tremendous lack of musicological conceptualization and knowledge about performance in view of the overly fluffy and too often ridiculously blasé music criticism in the feuilletons of our newspapers.

# Chapter 39

# Operator Theory

*If I chance to talk a little wild, forgive me;*
*I had it from my father.*
William Shakespeare, Henry VIII.

**Summary.** Operators are the substance of shaping performance. They refer to mothers, but they are the only instances capable of altering, refining, or ruining what has been achieved. They are also the pipes where exterior information, be it from score predicates, from analytical data, or from general system parameters, can be channeled and transformed into performance structures. We describe and motivate the concept of a weight (function). This is the turning point between "exterior" and "interior" strata to performance. We discuss different exigences for performance operators to cope with 'primavista' and 'analytical' data. A series of common primavista and analytical weights is discussed. We then expose a taxonomy of operators, followed by some special examples of the existing realizations, regarding tempo and articulation, as well as theoretically founded generalizations which are based upon Lie derivatives. The chapter concludes with a discussion of more "social" types of operators which correspond to "family life" correlations introduced in section 38.3.4. The final subject is a prospective to 'continuous stemmata', i.e., generalized stemmata based upon infinitely small coupled space portions.

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Whereas the previous theory culminated in the matrilinear stemma theory, we now have to face the masculine contribution to sexual propagation of performance shaping. This part is driven by the performance operators $\Omega$ which are a coordinate of each LPS $\delta$ and generate this LPS from its mother $\mu$: $\delta = \Omega \propto \mu$.

The information which is used to feed an operator is a list of weights in the sense of the space form $WeightList_{BP}$ of section 35.4. Recall that a weight is a real-valued predicate on an iterated powerset of local compositions derived from the kernel in the hierarchy spaces. We have to justify this approach and will do that in section 39.1.

But from our discussion of expressive performance in chapter 36 it is clear that operators have to use some rationale to shape their LPS' performance maps. In our following examples, we shall restrict such rationales to rational operators, i.e., such operators which use exclusively score-related primavista and analytical information.

Despite this restriction, we are far from understanding the general nature of rational operators. Clearly, the primavista operators are not the problem: they are straightforward (though not always easily formalized) translations of musical score competence into mathematically valid operations. The problem is rather on the side of general formalisms which could englobe the processing modes of analytical weights for producing reasonable (expressive) performance deformations of a given mother LPS data. A coarse qualitative classification of operators is nevertheless possible according to where the operator intervenes in the entire transformation process from the mental (symbolic) kernel, through the performance field (and the associated performance transformation), to the physical output data.

It is also possible to show that a number of special formulas for rational operators, such as the tempo operator, or some types of articulation operators, are special cases of the so-called Lie-type operators (section 39.7). "Lie-type" means that the Lie derivative of a weight function along a performance field is responsible for the field deformation of the daughter with respect to its mother. These Lie operators then specialize to basis operators and pianola operators according to their action being on basis or pianola parameters, respectively.

The chapter is concludes with a subject that extends the discrete character of a stemma to continuous parameters for the description of families of daughters. It is in the same vein as the introduction of continuous performance maps to replace discrete maps on discrete sets of notes: Although the material is discrete, the mental construction behind the shaping of this material is more than that: Human cognitive activities can operate on a continuous (or differentiable, etc.) paradigm when generating discrete effects. In fact, we could state this as a very principle of musical activity:

**Principle 26** *Music notation and also the very expression of musical material is essentially a trace of infinitesimal forces on a discrete reduction*[1]*.*

This is, what Valéry was alluding to when he stated[2]: "La musique mathématique-ment discontinue peut donner les sensations les plus continues." In our understanding, this is also valid in the sense of a poietic principle, the words "donner les sensations" being replaced by "provenir des forces".

## 39.1 Why Weights?

**Summary.** Within the transformation process from abstract and conceptual analysis and representation of score data, weights are an *inevitable* instance charged with the production of numerical performance data.

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Usually, in musicology, analysis does not end up with numerical data. A harmonic analysis, for example, yields a sequence of harmonic functions which take their values in abstract symbols, such as $D_F, Am7$. Or a motivic analysis ends up on a verbal description of certain important motives. However, when an artist has to perform a score, the abstract description level is of

[1] This reduction is in fact—among others—a reduction of originally gestural neumatic signs!

[2] See the catchword of chapter 33.

no direct use: Only the numerically quantized information can immediately help in defining shaping processes of instrumental parameters. *The final condensation of any analytical facts in performance has a numerical appearance.*

Our presentation of weights is in some sense an ideal compromise between abstract predicates, such as harmonic symbols, and the naive numerical evaluation of analytical properties. We understand that weights are predicates, i.e., truth-valued functions on local compositions and their powerset constructions, but the truth module is at the same time a numerical one, or at least a truth value can be associated with a numerical value if required.

### 39.1.1 Discrete and Continuous Weights

**Summary.** A priori, weights are functions on discrete sets of points within determined parameter spaces. In order to insert weights into general tasks of performance, in particular those referring to infinitesimal or continuous character, one has to consider extrapolation methods.

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We recall that we have defined a cubic interpolation formula of class $C^1$ for a real-valued (discrete) weight $w$ with zero derivative on local compositions $K$ in an $n$-dimensional simple form space $S$ over the reals, according to defintition 99 of section 32.3.2. We also supposed that $K$ is contained in the $n$-cube $C^n$ and that the interpolation is constantly equal to 1 outside $C^n$. In our present situation, we want to make a slight generalization to general weights with truth values in $\mathbb{T}_{\mathbb{R}}$. We suppose that the truth values $w(x), x \in K$ are all non-empty sets, i.e., do not have the value $\perp$, and that $K$ is still in the cube $C^n$. Usually, this is the case since we are given weights which are open intervals $]-\infty, a[$ with non-negative upper bound $a$. Suppose that $w_x \in w(x)$ is a selection of values from all truth sets $w(x), x \in K$. Supposing that an interpolation function F has been chosen (e.g., a cubic interpolation with respect to a specific permutation of coordinates, or the weighted sum of all such functions, or still another adequate candidate), we have an interpolation function $F_{w.}$ for each such selection $w.$, which is constant with value 1 outside $C^n$ and with zero derivatives in all points of $K$. Then we obtain a predicate on every element $s \in S$ by the definition

$$F(s) = \{F_{w.}(s) | w. \in \prod_{x \in K} w(x)\}. \tag{39.1}$$

In case the truth values are intervals $w(x) = ]-\infty, a(x)[$, the interpolation yields the value intervals associated with the interpolation values stemming from the upper bounds $a(x)$.

The fact that such a generalized interpolation is not defined for higher powerset arguments is not really a problem. Some operators in fact take these weights and boil their information down to more common weights as discussed above. We may then apply interpolation formula (39.1) to these boiled-down weights.

Behind such an interpolation procedure there is the non-trivial question about the justification of the usage of continuous weights in performance operators. There are two reasons we can find for this: The first is a very practical one: calculation precision. Suppose that we are given a local composition $K$ and that we want to calculate the weight $w(x)$ at a point $x \in K$. In the context of computer programs, it is often not clear whether we can really catch $x$ in its

numerical identification, for example for numerical calculation rounding effects. Therefore, it would be good to have the weight being a non-discrete function which varies little around a given argument. The second argument is that the continuous character of performance fields suggests continuous methods to shape such fields. For example, the commonly accepted continuous character of tempo would ask for a continuous weight function in order to determine a tempo change for all values where the tempo curve is defined. The inherent idea is that anayt-ical facts that are calculated on discrete events of a given score should have their presence in a neighborhood of such events, in the sense of a field action of such predicates.

### 39.1.2 Weight Recombination

**Summary.** We describe and motivate a set of procedures for building new weights from given ones.

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A further advantage of the numerical truth value set $W \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R})$ for weight predicates is that one may recombine such weights on standard operations from linear algebra. This type of building new predicates from given ones is a special item in the methodology of predicate calculus as exposed in section 18.3.4, i.e., it is a special logical motivation since it operates on the codomain of predicates.

Given a scalar $\lambda \in \mathbb{R}$ and a weight $w$ on a local composition $K$, or its differentiable extension $F_w$ by a cubic interpolation formula as described above, we can define

$$\lambda.w(x) = \{\lambda.\nu |\ \nu \in w(x)\}, \tag{39.2}$$
$$\lambda.F_w(x) = \{\lambda.\nu |\ \nu \in F_w(x)\}, \tag{39.3}$$

and it is clear that $\lambda.F_w = F_{\lambda.w}$, and $(\kappa.\lambda).F_w = \kappa.(\lambda.F_w)$, $(\kappa.\lambda).w = \kappa.(\lambda.w)$, respectively. If $\lambda$ is positive, and if $w$ is a weight whose values are intervals $]-\infty, a[$, then $\lambda.w$ is still of this type, viz, its intervals are the shifted intervals $]-\infty, \lambda.a[$. If we are given two weights $v, w$, then we can define

$$(v+w)(x) = \{\mu+\nu |\ \mu \in v(x), \nu \in w(x)\}, \tag{39.4}$$
$$(F_v+F_w)(x) = \{\mu+\nu |\ \mu \in F_v(x), \nu \in F_w(x)\}; \tag{39.5}$$
$$(v.w)(x) = \{\mu.\nu |\ \mu \in v(x), \nu \in w(x)\}, \tag{39.6}$$
$$(F_v.F_w)(x) = \{\mu.\nu |\ \mu \in F_v(x), \nu \in F_w(x)\}, \tag{39.7}$$

and we also have $F_v + F_w = F_{v+w}, F_v.F_w = F_{v.w}$. Sum and product are associative, but $(\kappa+\lambda).F_w \neq \kappa.F_w + \lambda.F_w$, in general. If both, $v$ and $w$ (and therefore also the differentiable extensions) have intervals $]-\infty, a[$, then so is their sum.

The philosophy of these combinations is that the action of weights may be better if their influence is mixed and weighted by scalars such as with cooking, the delicate dosage can be controlled.

A more general type of recombination is a non-linear deformation of a weight according to a deformation function $\delta : \mathbb{R} \to \mathbb{R}$. With the previous notation, we define a scalar multiplication

by

$$\delta.w(x) = \{\delta(\nu)|\ \nu \in w(x)\} \tag{39.8}$$
$$\delta.F_w(x) = \{\delta(\nu)|\ \nu \in F_w(x)\}, \tag{39.9}$$

and it is clear that for two such functions $\delta, \epsilon$ we have $(\delta \circ \epsilon).F_w = \epsilon.(\delta.F_w)$.

In practical cases, we often have the weight $w$ being defined by intervals $w(x) = ]-\infty, a(x)[$ for $x \in K$, and therefore also $F_w(x) = ]-\infty, \alpha(x)[$ for any $x \in S$. Since $K$ is supposed to be finite, and since by construction $\alpha(x) = 1$ outside the defining cube $C^n$, the image set $\alpha(x), x \in S$ is a finite interval $[\alpha_{min}, \alpha_{max}]$. We then consider a continuous deformation function $\delta(\alpha_{min}, \alpha_{max}, \tau)$ such that $\delta(\alpha_{min}, \alpha_{max}, \tau)(t) = t$ for $t \notin [\alpha_{min}, \alpha_{max}]$. For $t \in [\alpha_{min}, \alpha_{max}]$, the deformation parameter $\tau$ describes a one-parameter family of continuous, monotonically increasing deformations of the interval $[\alpha_{min}, \alpha_{max}]$ with $\delta(\alpha_{min}, \alpha_{max}, 0)$ being the identity. Typically, the deformations for $\tau$ and $-\tau$ are related to each other by a reflection of their graph at the main diagonal in $\mathbb{R}^2$. For example, we may take affine images of hyperbolas $y = -1/x$ on intervals $[-u, -1/u]$, where $u = e^\tau$, yielding the typical formula $\delta(0, 1, \tau)(t) = \frac{t}{e^{2\tau} - t(e^{2\tau} - 1)}$ for non-negative $\tau$, and the symmetric (diagonal reflection) formula for negative $\tau$.

The philosophy of non-linear deformations is that often, the action of an analytical weight on performance is qualitatively correct, but its quantitative influence should be distorted in order to yield a good perception. This effect can then be achieved via non-linear deformation functions.

## 39.2 Primavista Weights

**Summary.** Many of the traditional score data are not codified in numerical values. Numerical quantification is, however, a conditio sine qua non for any performing artist. We give an overview of common transformation procedures for non-numerical score parameters, including their extrapolation to continuous weights.

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### 39.2.1 Dynamics

**Summary.** We discuss the quantification and syntax of absolute and relative dynamics.

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Here, we need some preliminary remarks concerning the parametric interpretation of dynamical score signs since these are very coarse and of different types. We distinguish three types:

- *Absolute* dynamical signs such as *ppp, mf, fff, sempre pp*, etc. They give information at a determined onset.
- *Relative punctual* signs such as *frz, sf*, etc. They indicate a momentous change of dynamics *as a function of* the momentous dynamical level.

- *Relative local* signs such as the *crescendo* signs, giving verbal or pictorial indications on their domain of validity or having the shape of short wedges.

To begin with, absolute dynamic signs are ambiguous verbal descriptions of intended loudness. The first step towards a performance-adequate representation of such symbols is to assign them numerical values in a symbolic loudness scale. The good thing to do here is to assign them values which are in an affine relation to physical (Cents) units[3]. The precise value relations are however not codified and must be left to the free decision of the formalizing instance (in instrumental practise: the performer, the conductor, in computer-systems: the user). So we start with the usually sufficient setup of an increasing sequence

$$\begin{aligned} &\gamma(\mathit{ppppp}) < \gamma(\mathit{mpppp}) < \gamma(\mathit{pppp}) < \dots \\ &< \gamma(\mathit{mp}) < \gamma(\mathit{p}) < \gamma(\mathit{mf}) < \gamma(\mathit{f}) < \dots \\ &< \gamma(\mathit{fff}) < \gamma(\mathit{mfff}) < \gamma(\mathit{ffff}) \end{aligned} \tag{39.10}$$

of 19 real numbers in the *Loudness* space. The naive setting would be an equidistant series, such as suggested by MIDI and other technological codes. If we could limit loudness quantification to such absolute signs, this would be the end of the story: We would just define a parser for verbal dynamics signs into the numerical format of the space *Loudness*.

However, the meaning of the absolute signs is more than the above gauging convention. In fact, when we are given a sequence of absolute dynamic signs, we are also facing relative dynamic signs which have extensions, i.e., onsets and "offsets" within the range of the given absolute signs. For example, we may see a crescendo wedge, followed by a second such wedge, and then by a decrescendo wedge. What is the loudness curve for this situation?

The basic data is a sequence of absolute loudness symbols which are specified by their onsets and the symbols, i.e., a sequence of denotators in the limit space

$$\begin{aligned} &\mathit{AbsDynamicEvents} \underset{Id}{\longrightarrow} \mathbf{Limit}(\mathit{AbsDyn}, \mathit{Onset}, \mathit{Duration}) \\ &\mathit{AbsDyn} \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z} < \mathit{UNICODE} >). \end{aligned} \tag{39.11}$$

The duration of such an event is the difference from its onset and the onset of the next event, except for the last event, whose duration defines the end of the given composition. So we have the sequence $\mathit{AbsDynSequ} =$

$$\begin{aligned} &\mathit{Evt}_1 : 0 \rightsquigarrow \mathit{AbsDynamicEvents}(\mathit{Dyn}_1, E_1, D_1), \\ &\mathit{Evt}_2 : 0 \rightsquigarrow \mathit{AbsDynamicEvents}(\mathit{Dyn}_2, E_2, D_2), \\ &\dots \\ &\mathit{Evt}_m : 0 \rightsquigarrow \mathit{AbsDynamicEvents}(\mathit{Dyn}_m, E_m, D_m), \end{aligned} \tag{39.12}$$

with $D_1 = E_2 - E_1, \dots D_{m-1} = E_m - E_{m-1}$. This data induces the weight $w_{\mathit{AbsDynSequ}}$ on onset events $E$ with these values $w_{\mathit{AbsDynSequ}}(E) = ]-\infty, a(E)[$:

$$\begin{cases} a(E) = -\infty \text{ if } E < E_1 \text{ or } E_m + D_m \leq E, \\ a(E) = \gamma(\mathit{Dyn}_i) \text{ if } E \in [E_i, E_i + D_i[, i = 1, 2, \dots m-1. \end{cases} \tag{39.13}$$

[3]See appendix A.2.3 for this gauging question.

Over this absolute dynamics data, we now tilt the relative dynamic events: On each interval defined by event $Evt_i$, we are given a sequence of events of this form:

$$\begin{aligned} &RelDynamicEvents \underset{Id}{\longrightarrow} \mathbf{Limit}(RelDyn, Onset, Duration) \\ &RelDyn \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z} < UNICODE >). \end{aligned} \tag{39.14}$$

The symbols in *RelDyn* are such as *crescendo,molto crescendo, decrescendo, molto decrescendo*, for example. Their onset and duration are visible from the position of the wedges or the dashed lines in the score notation. So we are given sequences $RelEvt_{i,.} =$

$$RelEvt_{i,j} : 0 \rightsquigarrow RelDynamicEvents(RelDyn_{i,j}, rE_{i,j}, rD_{i,j}), \tag{39.15}$$

$j = 1, \ldots m_i, 0 \leq m_i$, with $m_i = 0$ for empty sequences. This time, the conditions on the respective onsets and durations are

$$\begin{aligned} &rE_{i,j} + rD_{i,j} \leq rE_{i,j+1}, 1 \leq j < m_i, \\ &E_i \leq rE_{i,1}, rE_{i,m_i} + rD_{i,m_i} \leq E_i + D_i. \end{aligned} \tag{39.16}$$

This is the framework for redefining the finer dynamics according to the relative signs sequences. The problem here is that we do not know how much a crescendo may increase absolute dynamics in order to remain within interval $E_i, E_i + D_i$ of the given absolute value $\gamma(Dyn_i)$ of the absolute event $Evt_i$. To this end, one needs other information which is implicit in the interpretation of a score: this is the *tolerance* of dynamical variation around $\gamma(Dyn_i)$ such that we still accept the label $Dyn_i$. So we have to provide each symbol $Dyn$ of absolute dynamics with a tolerance number $0 < \tau_{Dyn}$, and this means that all values in the half-open interval $[\gamma(Dyn) - \tau_{Dyn}, \gamma(Dyn) + \tau_{Dyn}[$ will be accepted as variations of the label $Dyn_i$. This does not exclude that these intervals may overlap, so the labels can be ambiguous and contradictory: a high *pp* value can be higher than a low *mp* value.

Upon this tolerance system, we now have to define the meaning of relative dynamics in the sequences $RelEvt_{i,.}$. To this end, we have to fix a quotient $0 < \kappa_{RelDyn}$ of dynamical increase or decrease of each relative dynamical symbol. For crescendi, we suppose $1 < \kappa$, and for decrescendi, we want $\kappa < 1$. For each relative dynamical event $RelEvt_{i,j}$, we calculate the dynamical values (i.e., the upper bounds b of the predicate's intervals $]-\infty, b[$) at $rE_{i,j}$ and $rE_{i,j} + rD_{i,j}$, and then, we interpolate linearly between these cornerstones, whereas the value remains constant inbetween two relative dynamics events.

The first relative event $RelEvt_{i,1}$ has the starting value at $rE_{i,1}$ equal to the absolute value $\gamma(Dyn_i)$ of $Evt_i$. We now suppose inductively, that the starting value $v_j$ of the relative event $RelEvt_{i,j}$ has been defined within the *open* interval $]\gamma(Dyn_i) - \tau_{Dyn_i}, \gamma(Dyn_i) + \tau_{Dyn_i}[$. This defines the quotient $\frac{v_j - (\gamma(Dyn_i) - \tau_{Dyn_i})}{(\gamma(Dyn_i) + \tau_{Dyn_i}) - v_j}$ which we want to increase by the factor $\kappa_{RelDyn_{i,j}}$ defined by the relative dynamical event at index $i, j$, i.e.,

$$\kappa_{RelDyn_{i,j}} \frac{v_j - (\gamma(Dyn_i) - \tau_{Dyn_i})}{(\gamma(Dyn_i) + \tau_{Dyn_i}) - v_j} = \frac{v_{j+1} - (\gamma(Dyn_i) - \tau_{Dyn_i})}{(\gamma(Dyn_i) + \tau_{Dyn_i}) - v_{j+1}} \tag{39.17}$$

defines the new dynamical value $v_{j+1}$ at the end of the $(i, j)$th relative sign. The new value is still in the open interval $]\gamma(Dyn_i) - \tau_{Dyn_i}, \gamma(Dyn_i) + \tau_{Dyn_i}[$, and we may go on inductively until

all relative dynamical signs are parsed. Observe that a succession of an increase factor $\kappa$ by its inverse neutralizes the dynamical value change. More generally, the succession of two factors $\kappa_1$ and then $\kappa_2$ results in a value change by factor $\kappa_1.\kappa_2$. So we can manage any succession of relative dynamical signs of this "crescendo/decrescendo" type without falling out of the prescribed tolerance interval. Denote the weight on all of the onset axis which is defined in this way by $w_{Evt.,RelEvt.\cdot}$.

So we are left with punctual relative dynamical signs *pct* at a note $x$, such as $pct = accent$ or $pct = marcato$. We may define the predicate

$$pct(x) = \begin{cases} \{1\} \text{ if } x \text{ is not marked by } pct, \\ \{\lambda(pct)\} \text{ else,} \end{cases} \tag{39.18}$$

where $\lambda(pct)$ is the increase factor for *pct*. Then, the punctual dynamical sign influences the already given absolute and relative dynamical weight *dyn* by the product *pct.dyn* of weights.

Of course this implementation need not live for ever, but it is one reasonable solution of a non-trivial problem of making blurred score signs precise. Observe that this kind of weight is not a performance map, it is just a weight on the symbolic (mental) events that must be used for operators to be defined later. Observe also that the above weights are not continuous functions of onset if several absolute dynamical signs are present.

### 39.2.2 Agogics

**Summary.** We discuss the quantification and syntax of absolute and relative agogical indications. A special attention is payed to general curve types for retards, fermatas, and general pauses.

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These are the common agogical indications:

- *Absolute* tempo, such as Mälzel metronome, or anterior verbal indications of type *andante, adagio*, etc. Formally, they correspond to absolute dynamic signs.
- *Relative punctual* tempo signs such as fermatas and general pauses. Remarkably, there is no relative punctual acceleration sign corresponding to the fermata.
- *Relative local* tempo signs are the following:
  1. Coarse indications concerning agogics, e.g., *ritardando, rallentando, accelerando, stringendo* etc.
  2. notation of correspondence between two adjacent tempi, such as "$2/4 = 3/8$",
  3. rest signs such as *a tempo*.

The essential difference to dynamical signs is that symbolic onset and duration are precisely codified, so agogical signs relate to proper performance transformations, and not to making blurred signs precise.

We have already discussed some of the agogical predicates (fermata, value change) in the presentation of weights for the prime mother LPS in stemma theory, section 38.3.2.

Let us first look at the Mälzel metronome sign $x\ quarter/Min.$. This is a predicate which takes its $\top$ values exactly on onsets $E$ and tempi $x$ where a Mälzel indication is $x\ quarter/Min.$, and $\bot$ else. The reader may easily make the underlying forms precise. Verbal absolute tempo indications need a parsing instance to render them in numerical (Mälzel) terms.

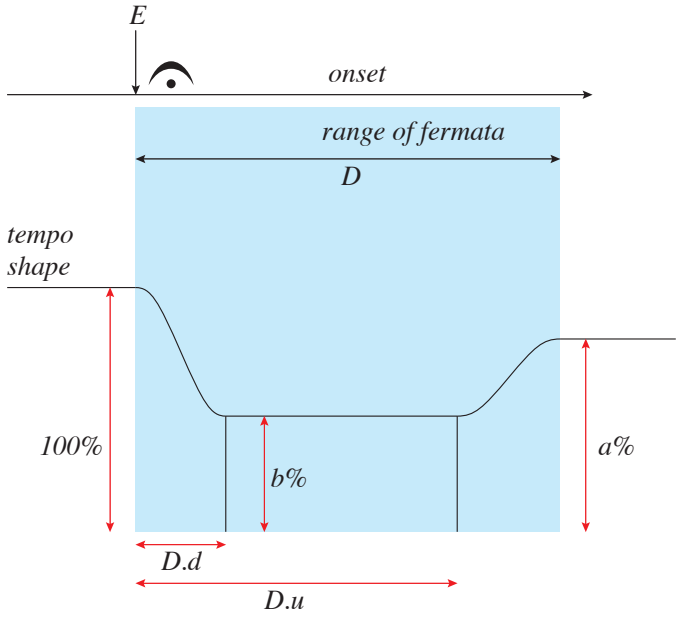


Figure 39.1: The interpolation curve of a fermata weight. See the text for the explanation of the symbols.

A fermata has only its onset $E$ made precise, the rest is blurred. We need several additional parameters to generate a viable weight. We first need a duration $D$, then a shape parametrization to describe the fermata's tempo sink. We may for example take this form:

$$Fermata \underset{Id}{\longrightarrow} \mathbf{Limit}(Onset, Duration, Bottom, Down, Up, After) \qquad (39.19)$$
$$\text{with } Bottom, Down, Up, After \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{R}).$$

A fermata $Ferm : 0 \rightsquigarrow Fermata(E, D, b, d, u, a)$ is parametrized by (1) the percentage number $B$ of maximal tempo lowering with respect to the given tempo, (2) the percentage $d$ of the duration $D$ from the beginning to reach the lowest tempo, (3) the percentage $u$ from the beginning to restart getting back to the following tempo, (4) the percentage $a$ of the original tempo which is resumed after getting back. This data is used to define a cubic spline interpolation as follows: We have four onsets: $E, E + b.D, E + u.D, E + D$ and corresponding relative tempo values

$1, b, b, a$. This defines a discrete weight $w(Ferm)$ and a corresponding interpolation function $F_{w(Ferm)}$, see figure 39.1. Of course, many different curves are possible, but with this shape, the main quality of a fermata can be imitated.

Weights for accelerando- or ritardando-typed predicates follow the building scheme of a fermata, except that the duration of these signs is defined. We leave this as an exercise to the reader.

**Exercise 80** Give an explicit description of accelerandi and ritardandi in terms of denotators and associated weights.

The general pause (G.P.) predicate is also undetermined like the fermata. It does however not imply a smooth recapitulation of the original tempo: After the general pause, the previous tempo is reset. We will have to deal with this in corresponding operators. The same phenomenon of a tempo reset is the case for the *a tempo* predicate.

### 39.2.3 Tuning and Intonation

**Summary.** We discuss the weights for pitch values.

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Tuning and intonation is a delicate subject for the PrimavistaOperator since it it not clear from the beginning how much the settings depend on the instrument and how they depend on individual instruments. For piano, the situation is easy since we have a fixed tuning and very often, it is even well-tempered. For violins, this is much more complicated: Should the primavista (!) tuning be a just tuning for each tonality which is encountered in the score, or should it be just one "default" just tuning? It is wise to let a special operator, not necessarily the PrimavistaOperator, do the work of delicate tuning and to just operate the minimum on the first process level. The tuning information is twofold: We have the chamber pitch which is the initial set and initial performance, this is ok. And we have the tuning data for all pitch events of a sufficiently large chromatic scale (88 keys for common pianos). This is completely analogous to a step tempo function given by a number of absolute tempo settings. One may use the form

$$\begin{aligned} &Tune \xrightarrow[Id]{} \mathbf{Limit}(Pitch, StepTune) \\ &\text{with } StepTune \xrightarrow[Id]{} \mathbf{Simple}(\mathbb{R}) \end{aligned} \tag{39.20}$$

being the tuning quantity that measures the "pitch velocity" between to neighboring pitches. Examples for default tuning for some classical cases are found in appendix K. The common well-tempered case has the denotators

$$x : 0 \rightsquigarrow Tune(x, 1/100\ Semitone/Ct)$$

for each pitch $x$ of the well-tempered chromatic scale.

### 39.2.4 Articulation

**Summary.** We present the parameters for articulation types, from molto staccato to molto legato.

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On one hand, articulation is a predicate type which regards single notes, for example molto staccato, staccato, legato, or molto legato. All these predicate types can be encoded by weights on events $x$ of a space $U$ within the $BP$ hierarchy which contain duration and take their values $]-\infty, a(x)[$ according to whether articulation stretches durations or compresses them. For staccato and other compressing signs, we take $0 < a(x) < 1$, whereas for legato and other stretching signs, we take $1 < a(x)$.

On the other hand, we have articulation in the sense of grouping of a set $G$ of usually consecutive notes which are grouped by an articulation slur via form $ArtiSlur_{U,S}$ as introduced in section 38.3.2. Such a predicate may be made more precise by a weight which takes into account the group $G$ as well as the single notes $x$ with their position within $G$. On a space $U$, this is covered by the form $U_2$ with the notation from formula (35.18). The weight $w_{GrpArti}(G, x) = ]-\infty, a(G, x)[$ is parametrized by the number $a(G, x)$ which tells how much the note's $x$ duration is altered relatively to its nominal duration.

### 39.2.5 Ornaments

**Summary.** Historically, ornaments form a complex set of constructions. We introduce a unified language for ornaments which is based on macro-events (see section 6.7).

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Recall from section 6.7 that we had defined macro-events by the form $Makro_{Basic}$ which is based on an event form $Basic$. Here, we take $Basic = U$, one of the event spaces in the hierarchy of our stemma. An ornament (one of the many forms of a trill, for example [66]) of events in $U$ can be described by a special denotator of the form $Makro_U$.

An ornament is first of all anchored at an event $a$ of $U$, this is the note which is ornamented by a specific sign, with a reference onset $a$, see figure 39.2. So we start our ornament denotator by a singleton:

$$Ornament : 0 \rightsquigarrow Makro_U(\{D\})$$

with

$$D : 0 \rightsquigarrow Knot_U(a, SmallAnchors).$$

Here, the next ramification denotator $SmallAnchors$ encodes the three anchor events of the three structural units $O_1, O_2, O_3$, each possibly with repetitions within:

$$SmallAnchors : 0 \rightsquigarrow Makro_U(O_1, O_2, O_3).$$

Each of these knots $O_1, O_2, O_3$ has a reference event $a_1, a_2, a_3$:

$$O_i : 0 \rightsquigarrow Knot_U(a_i, P_i), i = 1, 2, 3,$$

with macros $P_1, P_2, P_3$. Each of these macros is a set of knots:

$$P_i : 0 \rightsquigarrow Makro_U(q_{i,j}, j = 1, \ldots m_i)$$

with multiplicities $m_i$. Each such knot has the shape

$$q_{i,j} : 0 \rightsquigarrow Knot_U(d_{i,j}, o_i),$$

where the macros $o_i$ are independent of the second index $j$: They signify a repetition of the same macro according to the shift quantities $d_{i,j}$. The terminal macros $o_i$ have this meaning: The first, $o_1$, is the starting "micro-motif" of the ornament, it is played before the reference onset of $a$ and ends with the beginning of the second micro-motif $o_2$ which usually starts from $a$. This inner motif is repeated several times, usually something like eight times, and very fast. The ornament terminates on the third micro-motif $o_3$ which is a tail to the ornament and may be played much slower than the middle sequence. If we apply successive flattening operations $Flatten^n, n =$

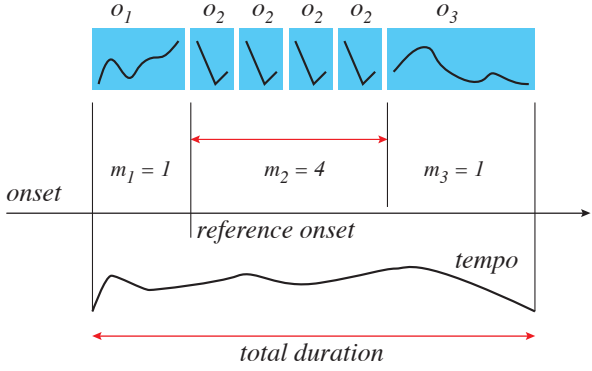


Figure 39.2: The structure of an ornament, showing a start shape (sequence) $o_1$, a sequence of middle shapes $o_2$, and a final shape (sequence) $o_3$.

$0, 1, 2, 3$ to the *Ornament* denotator, we obtain a hierarchy of successively appearing $U$-events which gives rise to different refinement levels of tempo curves which an *OrnamentOperator* in the PrimavistaOperator has to manage.

This perspective suggests that the hierarchy spaces $U$ in $BP$ should really be extended to the spaces $Knot_U$ in order to couple stemmatic refinement with hierarchies of sounds, but this theory is not developed so far.

It transpires however that macro-events fit in the general weight scheme which is defined by successive powerset constructions from the spaces $U$ in $BP$. So we can assign to each of the involved macro-events or knots a weight in $\mathbb{T}_{\mathbb{R}}$ in order to weight the performance role of the ornament's components.

## 39.3 Analytical Weights

**Summary.** We give a generic view on analytical weights. Their concrete shape depends on the analysis which is available. Several types are realized on analytical RUBETTE®modules, see chapter 41.

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Analytical weights are crucial to rational performance and intervene from analyses of the score $S$ relating to metrical, rhythmical, motivic, thematic, harmonic, contrapuntal, grouping and other structural perspectives. We shall sketch the construction for metrical, motivic, and harmonic weights. In all these descriptions, we shall abbreviate the nomenclature of predicates and simply write down the numerical values $a$ of such weights, the predicative statement being implicit, either in form of intervals $]-\infty, a[$ or in form of singleton predicates $\{a\}$, the information is the same, and the usage of the operators does not depend on which predicative encoding is chosen. Also we restrict to the discrete weights, the continuous associated weights being automatic from our previous discussions (subsection 39.1.1)

1. *Metrical weights* $w^S_{metro}$: We have a sober weight on the onset space $U = Onset$ for which we have prepared a formula in example 43. In that formula, the weight $nW(\sigma)$ is calculated on a simplex $\sigma$ of maximal local meters. For an onset $E$, we then take the simplex $\sigma_S(E)$ consisting of all maximal meters within the onset kernel $S_{Onset}$ of $S$ which contain $E$ and set $w^S_{metro}(E) = nW(\sigma_S(E))$ (with the evident singular value $nW(\emptyset) = \perp$ for onsets outside $S_{Onset}$).

2. *Motivic weights* $w^S_{motif}$: Following the preliminary discussion in section 22.9, we take an interpretation $S^\mu_{Onset\oplus Pitch}$ of the onset-pitch space of $BP$ by motives. More precisely, $\mu$ may be chosen as the set of all motives within a range of cardinality and extent of onsets between the first and last event. We define sober motif weight which is induced by the weight on motives from the atlas $\mu$. So we are left with the definition of the weight on motives $M \in \mu$. To this end, we consider a more precise motif theory framework as defined in chapter 22. We choose a shape type $\Gamma_t$ and a distance $d$, on which an equivariant, isometric group action of a paradigmatic group $P$ is defined, and such that we have the inheritance property fulfilled (for example, the elastic and diastematic types with Euclidean distance and counterpoint groups). We then have the epsilon topology $\mathfrak{T} = \mathfrak{T}_{t,P,d}$ and choose a neighborhood radius $0 < \epsilon$. For a motif $M \in \mu$, we consider the $\epsilon$-disk neighborhood $D^\mu_\epsilon(M)$ of $M$ in the relative topology $\mathfrak{T}_\mu$. We define the *$\epsilon$-presence of $M$* by the weighted sum

$$p_\epsilon(M) = \sum_{N \in D^\mu_\epsilon(M)} 2^{card(M)-card(N)} \tag{39.21}$$

which visibly counts the elements of the $\epsilon$-neighborhood of $M$ with a weight according to the cardinality difference to $M$'s cardinality. This is roughly speaking[4] the presence implemented in the MeloRUBETTE®.

[4] In our implementation of this function, we have also taken into account the multiplicity of submotives $X$ of the $N$ in the neighborhood with $card(X) = card(M)$, which are in the distance less than $\epsilon$ to $M$.

We then also define a *$\epsilon$-content of $M$* function which is the following weighted sum:

$$c_\epsilon(M) = \sum_{N,M \in D^\mu_\epsilon(N)} 2^{card(N)-card(M)}, \tag{39.22}$$

and we therefrom get the weight of $M$, i.e., the product

$$w^S_{motif}(M) = p_\epsilon(M).c_\epsilon(M) \tag{39.23}$$

which—again roughly speaking the presence function which is also implemented in the MeloRUBETTE®—measures the presence of $M$ combined with the content of $M$. Intuitively, the weight of $M$ is a measure of where $M$ appears in other motives—up to $\epsilon$ distance—and how much motives are contained in $M$—up to $\epsilon$ distance. The coefficients are a weight for the cardinality distance form $card(M)$.

3. *Harmonic weights* $w^S_{harmo}$: This weight is also a predicate whose values are non-false only on the onset-pitch space $S_{Onset \oplus Pitch}$ of $BP$. To define it, we have to start from an interpretation $S^\eta_{Onset \oplus Pitch}$ by chord events. The construction of a chord atlas is however not uniquely determined since it is not clear what is a chord. We may just take those local compositions of all events having one and the same onset. But we may also consider all the onsets where some chord notes end and some others still last, and then take the still lasting notes as constituents for new chords at those ending onsets.

   Given such an interpretation $\eta$, we take its abstraction in form of a temporally defined sequence $a. = a_0, a_1, \ldots a_k$ of length $k$ of chords, i.e., local compositions in the $Pitch$ space. For any such sequence $a.$, we have defined weight functions in the context of the Riemann algebra, as discussed in section 27.2. There, formula (27.13), with its interpretation as a truth denotator-valued weight, as explained in the example of section 27.2.2, gives us a criterion for finding a best path $p_{opt} = (v_0, f_0, a_0) \to (v_1, f_1, a_1) \to \ldots (v_k, f_k, a_k)$ in the Riemann quiver. We now use this best path to calculate the *tension* $t_{p_{opt}}(a_i)$ of each chord $a_i$ with respect to this path. The immediate tension function would be $t_{p_{opt}}(a_i) = \Omega(p_{opt}|a_i)$, where we denote by $p_{opt}|a_i$ the optimal path from the beginning to (inclusively) chord $a_i$. However, the global tension is a problematic quantity as discussed in section 27.2.1. We therefore renormalize the function $t_{p_{opt}}(a_i)$ by the unique shearing of its graph in $\mathbb{R}^2$ such that the first point $(0, t_{p_{opt}}(a_0))$ goes to a predefined initial tension $(0, t_0)$ whereas the last point $(0, t_{p_{opt}}(a_k))$ goes to $(0, t_k)$. This has the advantage that the global tension can be cast to initial and final values which the local methods cannot predict and control. Call $t^{t_o,t_k}_{p_{opt}}$ this new tension function.

   In order to calculate the weight of single notes $x$ within a chord $a_i$, we look at the Riemann matrix predicates $TF_{f,t}(a_i) = [0, \phi(a_i)[$ and $TF_{f,t}(a_i \backslash x) = [0, \phi(a_i \backslash x)[$ where $a_i \backslash x$ is the chord $a_i$ after omission of $x$. We know that $\phi(a_i)$ is not zero. We may then map the interval $[0, \phi(a_i)]$ onto the interval $[IP, 1], 0 < IP < 1$ by an affine map $Q$ for a normalization purpose. Then, we get the relative importance $rel(x, a_i)$ of $x$ within $a_i$ by the formula

$$rel(x, a_i) = \frac{1}{Q(\phi(a_i \backslash x))}. \tag{39.24}$$

This gives us the final expression for the weight of the note $x$: We take the tension of the underlying chord $a_i$ and multiply it by the relative importance of $x$ within this chord, i.e.,

$$w^S_{harmo}(x) = rel(x, a_i).t^{t_o,t_k}_{p_{opt}}(a_i). \tag{39.25}$$

This is the harmonic weight which is implemented in the HarmoRUBETTE®.

An important technique to produce new weights from analytical weights is the *boiling down* method: Often, a weight is given on a space where the operator at hand does not work. For example, the TempoOperator (to be discussed later in this chapter) needs a weight on the *Onset* space. If a weight lives in a space with more dimensions, we should be able to boil it down to a weight on *Onset*. Here is the procedure: We have two spaces $U, V$ in the system $BP$, and a projection $p_V : U \to V$. A weight $w$ is given on the kernel $S_U$, and we would like to get a boiled-down weight $BD_V(w)$. We make these two definitions:

$$BD_V(w)(x) = \sum_{y \in p_V^{-1}(x)} w(y), \tag{39.26}$$

$$BD^{mean}_V(w)(x) = \frac{1}{card(p_V^{-1}(x))} BD_V(w)(x), \tag{39.27}$$

where $x \in S_V$, and with the value $-\infty$ for empty fibers.

In general, we have this situation: we would like to have a weight $w$ on a specific space $V$ of the hierarchy, but the given weight lives either in a larger space $U$ which projects onto $V$, or it lives in a smaller space $U$ onto which $V$ projects. The former case is solved by the boiled-down construction $BD_V(w), BD^{mean}_V(w)$, whereas the latter is straightforward by the formula $w_U(x) = w(p_U(x)), x \in V$. In the future, *whenever we use a weight which is possibly defined in the "wrong" space, these constructions are referred to, and we simply write w instead of the above correct symbols if no ambiguity is likely.*

## 39.4 Taxonomy of Operators

**Summary.** Though a general description of operator principles is risky (since there is no general theory of how a stemma can be altered), we want to give a preliminary classification of the ways an operator may choose to intervene in the existing configuration. This taxonomy is guided by the generic description of how performance works: as a transformation from mental to physical reality, and by means of its description via performance fields given in section 33. This means that an operator can intervene on the level of mental or physical reality, on the frame of the local performance cell, see chapter 35.1, and on the performance field together with its initial set.

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The basic data of a performance is encoded in the performance map $\wp : K \to \wp(K)$ on the given performance cell. An operator has to change any of the involved structures: $K$, $\wp$, or $\wp(K)$. This is understood in the sense that K is a set of arguments upon which the prescription $\wp$ acts, and a set $\wp(K)$ which is the output of that map.

The first operator type acts on $K$. It does this in two ways: First, named splitting operator, it just divides the kernel $K$ into sub-kernels, nothing else. Second, it alters the kernel's events and submits these new events to the given prescription $\wp$. This one is termed symbolic operator. It is as if the artist would play in the same "mood" with a changed score.

The third type takes the input $K$, the map $\wp$, and the output $\wp(K)$ as given and now changes the output without regard to the previous process. This is the physical operator type.

The fourth type takes the input kernel $K$ for granted and just changes the map's "formula". For example it changes the tempo curve or the intonation, etc. This is the so-called field operator type. This is the most complex operator type and has not yet been understood in its different outfits.

Let us now have a closer look at these types.

### 39.4.1 Splitting Operators

**Summary.** A splitting operator is rather simple. It restricts to partitioning the performance score's kernel into sub-kernels without any further change on the performance score. Splitting may also be operated on a group of selected instruments in a global performance score.

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The idea of a splitting operator is this: You are given a score with different groups of notes that you want to perform in an individual way. For example: right hand and left hand, or onset-driven grouping: split the score at a given parameter, such as onset or loudness or duration etc. This may happen at a grouping line such as the beginning and the end of an eight-bar period. Very often, this also happens if a special group needs a special performance procedure because of its inherent structure, such as a trill or another ornament. Whatever you will do to this group is irrelevant, you decide later. So the splitting operator is a propaedeutical operator intending to prepare more in-depth operator actions.

A prototypical realization of the splitting operator may be implemented by use of an essentially mathematical predicate $Split_{U,\nu}$, where $U$ is one parameter (such as pitch, onset, etc.) in the given hierarchy $\mathfrak{h}$, i.e., of the top space $BP$, and where $\nu$ is a real number parametrizing a coordinate value in the $U$ parameter. Then if $X$ is an event in one space of the hierarchy, the predicate $X/Split_{U,\nu} = \top$ iff either $X$ does not share this parameter $U$, or if it shares it and the $U$-value $X_U$ of $X$ verifies $\nu \leq X_U$. So $X/Split_{U,\nu} = \bot$ iff $X_U$ exists and $X_U < \nu$. We then get the logical combination predicate $Split_{U,\nu,\mu} = Split_{U,\nu}\&\neg Split_{U,\mu}$ which selects those events with either no $X_U$ coordinate or else with $\nu \leq X_U < \mu$. In the RUBATO® software, the splitting operator has been implemented such that it may perform any logical conjunction of splitting operators of the types $Split_{U,\nu}$ and their negations $\neg Split_{U,\nu}$. For example, this enables us to select all events within a half-open onset interval $[E_1 = \nu, E_2 = \mu[$ and a half-open loudness interval $[L_1 = \xi, L_2 = \chi[$, and having durations $D$ with $D_1 = \delta \leq D$. So the splitting operator $Split$ (we omit further specifications in this notation) is defined by a Boolean predicate which extracts a given set $K'$ of events from a given kernel $K$. This means that $Split$ produces two daughters, one with the $K'$ kernel, the other (by logical negation of the former) with the remainder kernel $K - K'$. In the nomenclature of section 38.3.3, we have a coupling

$$\Omega(i) \propto \mu = \Lambda_i, i = 1, 2$$

which is also written more intuitively as

$$Split \propto \mu \tag{39.28}$$

$$Remainder_{Split} \propto \mu. \tag{39.29}$$

This operator may also be applied to a GPS $(\mu_i)_i$ just by applying the operator simultaneously to all the member LPS $\mu_i$ of this GPS. We should add that all the other data of the mother(s) in $\mu$ (in $(\mu_i)_i$) are left as they are, and everything still works fine. The new LPS just have some points of the original kernel being removed.

In the formal setup of the weight system of an LPS, the predicate *Split* would be viewed as being a weight of both daughters and the operator would then act on the mother via the Boolean selection and its negation which is defined by this weight.

### 39.4.2 Symbolic Operators

**Summary.** A symbolic operator affects the score data *before* they are performed whatsoever. This means that the operator really changes the *composition* which is to be interpreted. Such an operator type is seemingly contradictory to the very objective of performance. Nonetheless, primavista weights suggest an intervention of operators before any real performance, i.e., performance is already initiated on the very level of the score's *interpretation*.

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Symbolic performance operators are a delicate species since they intervene at a very early stage of the performance process: on the kernel level of the LPS hierarchy $\mathfrak{h}$. One could see this fact as a natural completion of the performance philosophy in that composition is the first stage of performance, so why not alter the kernels, i.e., make a new composition out of an old one.

We adopted this integrative point of view while developing RUBATO®. From the software engineering perspective, this is indeed tempting since it would in the limit yield an integrated software for performance *and* composition. In this generic setup the program routine must, however, recalculate virtually every ingredient of the given LPS. The change of the basic score elements in a recomposition entails that the frames, the initial sets, the initial performances, the fields, and the weight list must be updated, or, rather: rewritten from scratch. Potentially, nothing will be the same again. So the operator then would just be a reset of the complete LPS data for a new composition. And this is not what performance was meant to do. After all, having rehearsed on a given piece and then being told that you get a new score, but you may go on with the old tempo curves, is not the kind of thing you will enjoy, since it will not make sense to perform on this schizophrenic data. For example, if a tempo curve has been developed, and then all onsets and durations are reset to half of the former values, the tempo curve becomes useless unless you also redefine the curve by a so-called time stretching operation (multiply the time arguments by 0.5).

In order to avoid such risks, one should really recalculate all weights whenever a symbolic operator has changed the kernel data. But even then this would not necessarily be the right solution since some primavista weights of analytical nature really need to be calculated on an unrefined score, i.e., on symbolic data which do not yet share some sophisticated symbolic

explications, for example in dynamics. This problem could be attacked by an implementation of weight calculation routines which average out the over-refined symbolic data, but this is not very elegant. It remains a fact that weights should also be given the parameter of the stemma LPS where the weight is instantiated. By this method, one would then be sure that a weight is related to a determined kernel in the stemmatic inheritance tree and not exclusively to the primary kernel data.

With these caveats in mind, we may nevertheless define some useful (though also risky!) symbolic operator. We name it *SymbolicBrueForceOperator* to remind you of this dangerous enterprise. It takes as arguments a "directional factor" $d \in W$, $W$ a space of the hierarchy $H$, with components $d_R$ for parameters $R$ of $W$, and a weight $w$. Then, if $X \in U$ is an event of $Kernel_U$, we set

$$SymbolicBruteForceOperator_{d,w}(X) = w_d(X) \tag{39.30}$$

where $w_d(X)$ is the new event in $U$, whose $R$-coordinate $w_d(X)_R$ is the product $w(X).d_R.X_R$, if $R$ is a coordinate of $W$, and $X_R$ else. So we only change $X$-coordinates for the directional factor, and there, we scale the coordinates of $X$ by the weight and by a fixed directional coordinate.

The risk here is that for $d$ or $w(X)$ values far from 1, the image $w_d(X)$ is likely to fall out of the given frame. But one may then easily redefine the frame, the field, etc., since usually this symbolic operator is applied in a stemma stage where we are just given the default data. As we have seen in section 38.3.2, this data can be adapted to special kernels without difficulty: The frame is extended to include new events, the performance field is the constant field, and the initial set and performance are not a function of special kernel data. Under general stemmatic conditions, this operator is risky, however, but it is also an elegant solution of some nasty parameter problems which arise from the fuzzy score notation. For example, if we are given a weight for primavista dynamics $w_{Evt.,RelEvt..}$ (see section 39.2.1), then the directional factor $d = (1) \in Loudness$ gives the complete dynamics values if the given loudness was set to a preliminary value 1, say. Same for other primavista weights relating to symbolic parameter values.

For dynamics, it seems to be clear that the symbolic operator is the right one to be applied. For other primavista weights this not so easy to decide. For example, the articulation weights for slurs, legati, or staccati could be seen as symbolic prescriptions to alter duration within the score event framework. We contend that this is not analogous to the dynamics situation because absolute dynamical signs (such as *ff, mf*) are not quantified and even less are relative dynamical signs (such as *crescendo*). We therefore have a clear mission to transform such verbal signs into quantitative data. In contrast, duration is a perfectly quantified parameter, so it needn't be generated on that level. We therefore decided to deal with these weights in the context of field operators which alter the mother's articulation field.

This minor dilemma demonstrates (once more) that one may have the same performance output with very different infrastructure in the transformation process from the score to the physical events. This subject has not been dealt with from a more theoretical point of view: It pertains to the inverse performance theory to be dealt with in part XII, i.e., the theory of the variety of performance scores leading to a given performance output. But there is no result on the variety of local or global performance scores with *variable* score data inducing a given performance output; everything to date supposes that we are given a fixed score. Especially in ethnomusicological contexts, where the very concept of a score is uncertain, this is not

acceptable.

For completeness of this taxonomy, we should add that we also call symbolic operator an operator which changes the initial set of a performance cell. This may be necessary if we want to enrich certain boundary conditions on the performance of selected groupings of notes.

### 39.4.3 Physical Operators

**Summary.** Physical operators affect given results of performance which are inherited from the *mother's* data. They act in a very simplistic way since they do not relate to the deeper process of performance; they only take for granted what the mother had already produced as a performance output.

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At the other end of the operator scale, we have the physical operators. While symbolic operators should be applied in the very beginning of a stemmatic unfolding, physical operators should be applied after all other operators have been deployed and no risk to have them applied once more is likely. An exception to this general rule is the case where after a physical operator another operator is applied, but not in any of the coordinates to which the physical operator has made changes. But the one who applies such exceptional stemmatic operations must know very well what is being done—as with pointers in C-style programming.

Formally, what we call the $PhysicalBruteForceOperator$ is quite analogous to the operator $SymbolicBruteForceOperator$ discussed in equation (39.30). We are again given a directional factor $d \in W$, $W$ a space in the hierarchy $H$, with components $d_R$ for parameters $R$ of $W$, and a weight $w$. Then, if $X \in U$ is an event of $Kernel_U$, we set

$$PhysicalBruteForceOperator_{d,w}(X) = w_d^{\wp}(X) \tag{39.31}$$

where $w_d^{\wp}(X)$ is an event in the physical space $u$ corresponding to $U$, whose $r$ coordinate ($r$ corresponding to $R$ according to our general nomenclature) is the product $w(X).d_R.\wp(X)_r$, if $R$ is a coordinate of W, and $\wp(X)_r$ else.

So this operator really changes the performance at the point $X$, but the symbolic operator only affects the symbolic point and does not intervene on the $\wp$-level or the output. There is another very pleasant difference: Whereas the symbolic operator is risky as to the frame, field, and initial conditions, the physical operator is completely indifferent against these conditions. It only needs the mother's performance and the rest will always work! This operator is also very fast on the programming level, it produces very little changes and therefore is quite comfortable to handle. This preconizes another application of $PhysicalBruteForceOperator$ in the sense of a test operator. In some situations, we would just like to know how an alteration of the output looks. Instead of applying a complicated operator (acting on the performance field and consuming much calculation power, say), we first apply this physical operator and obtain a first impression of how the given weight might influence the mother's performance. If we like the effect, we may go on, first resetting the stemma to the mother's level, and then apply a more sophisticated operator with the weight in question.

For completeness of taxonomy, we should add that operators which alter the initial performance map are also called physical operators. They tell initial events where to be mapped in the

physical space. This is however not completely satisfactory since it usually depends upon the way the mother told its initial events where to be mapped. And this could have been done on a more field-theoretic level. The actual change could also involve field-theoretic considerations.

### 39.4.4 Field Operators

**Summary.** Field operators are the very essence of shaping performance. They include a variety of more or less general operators according to the direction where the field has to be altered. The theoretical complexity is paralleled by a computational complexity (numerical integration of ordinary differential equations) which is very important for software implementation of such operators.

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Whereas symbolic and physical operators are an easy chapter in performance classification, field operators constitute an entire world of various approaches to the definition and deformation of performance fields. There are several reasons for this. The first is the mathematical complexity of fields, the definition and deformation of a field requires differential-geometric tools of local and global nature. The second is that performance fields have to be inserted in the hierarchies of their LPS. Therefore, the change of a field is always coupled to the fellow fields in the hierarchy. The third reason is that the stemmatic inheritance enforces operators which are capable of acting upon given mother fields of quite general nature: the stemmatic framework has to face unexpected field inheritance. Therefore, the definition of field operators is not completed if a particular type of deformation is implemented, one must also allow for mother fields which do not fit in the naive setup of such a deformation type. The fourth reason is that there is no such a thing like a performance field theory in traditional performance theory, and not even in modern approaches, such as the KTH school or the Todd approach. Performance fields are a new paradigm to performance theory, although, as we have shown, they are a completely natural and mandatory setup to control performance on an artistic level and from the more esthetic point of view as put forward by Adorno and Benjamin.

We shall start our investigation of performance field operators by the classical tempo operator. We then generalize its setup to cope with more general mother fields. Then we generalize this approach to a more systematic framework, the so-called scalar operator.

Another approach will be discussed under the title of basis-pianola operators, which arose from a systematic generalization of a physical operator to quite general situations. It turns out that this type is an interesting general approach involving Lie derivations of weights along the given mother fields. We then specialize this setup to operators for basis parameter field and pianola parameter field deformations.

Although the field operators look very attractive, they have a big practical drawback: If one has to compute integrals of ordinary differential equations in order to solve the integral curve problem for performance fields, one has to control the type of vector field. In fact, not every wild field can be integrated by one and the same numerical ODE routine. For example, we have implemented Runge–Kutta–Fehlberg routines in RUBATO®'s PerformanceRUBETTE®. We often experienced that the application of a field operator, such as the tempo operator or the scalar operator (see sections 39.5 and 39.6, respectively, for these operators), causes a breakdown

of these numerical routines. So the routines should be adapted to the field, but this is beyond this discussion and should be left to numerical ODE specialists.

## 39.5 Tempo Operator

**Summary.** The tempo operator is the classical field operator which alters the mother's tempo as a function of given weights. We expose the different variants of such an intervention.

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This operator acts on the tempo field $\mathbf{Ts}_E$ of a performance cell, so we suppose that $E$ is a member of the space hierarchy $H$ of $\mathfrak{h}$. We are given a weight $w$ and set

$$TempoOperator_w(\mathbf{Ts})_E(X_E) = w(X_E).\mathbf{Ts}_E(X_E) \tag{39.32}$$

for a $E$-event $X_E$. This operator is only defined if the weight is a strictly positive continuous function on the given onset frame, which we shall from now on tacitly assume. If the tempo operator is clear, we shall write the shorthand

$$TempoOperator_w(\mathbf{Ts}) = \mathbf{Ts}_w.$$

This operator is as easy as problematic if we do not take into consideration other parameters which are canonically tied to onset. Duration is such a parameter. Let us look at the combined articulation field that is influenced by the weight in the tempo operator. Let us first start with the parallel articulation field (see section 33.2.1), i.e., $\mathbf{Ts}_D = 2.\mathbf{Ts}_E \circ \alpha_+ - \mathbf{Ts}_E$. So if we deform the tempo field $\mathbf{Ts}_E$ to $\mathbf{Ts}_{w,E}$, the $D$-component of the parallel field is deformed to a $D$-component of the parallel field with the weight contribution, i.e., $\mathbf{Ts}_{w,D} = 2.\mathbf{Ts}_{w,E} \circ \alpha_+ - \mathbf{Ts}_{w,E} = w \circ \alpha_+.2.\mathbf{Ts}_E \circ \alpha_+ - w.\mathbf{Ts}_E$. However, this formula is not satisfactory since it only works if we are given a parallel field situation. Else, we would destroy the given mother $D$-component and retain only the mother's tempo information.

In order to take the mother's duration field component into consideration, we need a more invariant formula. To this end, rewrite

$$2.\mathbf{Ts}_E \circ \alpha_+ = \mathbf{Ts}_D + \mathbf{Ts}_E \tag{39.33}$$

and then get the formula

$$\begin{aligned} \mathbf{Ts}_{w,D} &= w \circ \alpha_+.(\mathbf{Ts}_D + \mathbf{Ts}_E) - w.\mathbf{Ts}_E \\ &= w \circ \alpha_+.\mathbf{Ts}_D + (w \circ \alpha_+ - w).\mathbf{Ts}_E \end{aligned} \tag{39.34}$$

or for the total articulation field $\mathbf{Ts}_{w,ED} = (\mathbf{Ts}_{w,E}, \mathbf{Ts}_{w,D})$:

$$\mathbf{Ts}_{w,ED} = \begin{pmatrix} w & 0 \\ w \circ \alpha_+ - w & w \circ \alpha_+ \end{pmatrix} \mathbf{Ts}_{ED}. \tag{39.35}$$

The new articulation field therefore results from the old one by a linear action

$$Q_w = \begin{pmatrix} w & 0 \\ w \circ \alpha_+ - w & w \circ \alpha_+ \end{pmatrix} \tag{39.36}$$

on the tangent bundle of the $ED$-frame[5]: $\mathbf{Ts}_{w,ED} = Q_w.\mathbf{Ts}_{ED}$. The shape of the operator $Q_w$ is precisely that which one obtains for the parallel field at tempo $w$. This means that instead of a real tempo, we take the 'weight tempo' $w$ and the associated 'performance' $\wp_w$. Its inverse 'is' $Q_w$, and the new articulation field $\mathbf{Ts}_{w,ED}$ is the inverse image of the old articulation field $\mathbf{Ts}_{ED}$ under the 'weight performance' $\wp_w$.

Usually, the change from the given tempo curve $\mathbf{Ts}_E$ to $\mathbf{Ts}_{w,E} = w.\mathbf{Ts}_E$ changes also the duration $\Delta = \int_{E_0}^{E_1} \frac{1}{\mathbf{Ts}_E}$ to the duration $\Delta_w = \int_{E_0}^{E_1} \frac{1}{\mathbf{Ts}_{w,E}}$. If the integration limits $E_0, E_1$ are initial onsets, this is bad news, and we have to adapt the tempo deformation to the condition that these integrals should coincide. Moreover, one usually requires also that the tempi at the initial points should not change. To this end, one introduces a continuous support function *supp* which vanishes outside $[E_0, E_1]$, is identically 1 in a slightly smaller interval $[E_0 + \beta, E_1 - \beta]$ for a relatively small positive real number $\beta$, and is the cubic spline with zero slopes on the boundary intervals $[E_0, E_0 + \beta], [E_1 - \beta, E_1]$. Then

$$s(\mathbf{Ts}_E, w, t) = \mathbf{Ts}_E(t)(1 + supp(t)(w(t) - 1))$$

coincides with $w$ within $[E_0+\beta, E_1-\beta]$ and is 1 outside $[E_0, E_1]$. We then try the new duration

$$J(\mathbf{Ts}_E, w) = \int_{E_0}^{E_1} \frac{1}{s(\mathbf{Ts}_E, w, t)}$$

and compare the durations:

$$r(\mathbf{Ts}_E, w) = \frac{J(\mathbf{Ts}_E, w)}{J(\mathbf{Ts}_E, 1)}.$$

We then try the obvious correction $w_1 = r(\mathbf{Ts}_E, w).w$ and get a weight which gives us nearly the right duration since within the interval $[E_0 + \beta, E_1 - \beta]$, the error is corrected. We look for the new error $r(\mathbf{Ts}_E, w_1)$, set the new weight $w_2 = r(\mathbf{Ts}_E, w_1).w_1$, and repeat this procedure until the error becomes small enough. This procedure has been implemented in RUBATO®'s PerformanceRUBETTE®, but we do not know whether it converges[6].

## 39.6 Scalar Operator

**Summary.** The scalar operator is a first generalization of the tempo operator to articulation. The conceptual background is that tone parameters can be split into *basis* and *pianola* parameters. Onset, pitch, loudness are basis parameters. The corresponding pianola parameters are duration, glissando, crescendo. Pianola components of performance fields are coupled to the corresponding basis components of the performance fields. This coupling can be distorted by making operator weights act on specific parameter sets. The scalar operator does this job.

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[5] Recall that performance fields are defined on open neighborhoods of frames.

[6] Perhaps this could be settled by use of Banach's fixpoint theorem, we do however not know whether the map is a contraction.

Suppose that we are given a space $B$ of basis parameters, and that $P$ is the corresponding space of pianola parameters. We suppose given a weight $w$ on the $B$ space. The scalar operator $ScalarOperator_w$ deforms a performance field $\mathbf{Ts}_{BP} = (\mathbf{Ts}_B, \mathbf{Ts}_P)$ to a new field $ScalarOperator_w(\mathbf{Ts}_{BP}) = (\mathbf{Ts}^*_B, \mathbf{Ts}^*_P)$ as follows. We have two parameters: parallel/not parallel, and a two-bit parameter $B$ = yes/no, $P$ = yes/no. The *notparallel* case is a deformation of the components of the given field according to the second parameter:

1. $B = P =$ no: We leave the field unchanged.
2. $B = P =$ yes: We set $ScalarOperator_w(\mathbf{Ts}_{BP}) = w.\mathbf{Ts}_{BP}$.
3. $B =$ yes, $P =$ no: We set $ScalarOperator_w(\mathbf{Ts}_{BP}) = (w.\mathbf{Ts}_B, \mathbf{Ts}_P)$.
4. $B =$ no, $P =$ yes: We set $ScalarOperator_w(\mathbf{Ts}_{BP}) = (\mathbf{Ts}_B, w.\mathbf{Ts}_P)$.

The *parallel* case runs as follows. It means that we refer to the parallel field structure in one way or another. The four cases run as follows:

1. $B = P =$ no: No weight influence, but the parallel structure is installed, i.e.,

$$ScalarOperator_w(\mathbf{Ts}_{BP}) = \partial\mathbf{Ts}_B \tag{39.37}$$

2. $B = P =$ yes: This is the generalized tempo operator, i.e.,

$$ScalarOperator_w(\mathbf{Ts}_{BP}) = Q_w.\mathbf{Ts}_{BP} \tag{39.38}$$

   with the matrix being understood as a block matrix of scalar endomorphisms on the basis and pianola spaces, respectively. The original tempo operator is the special case $B = E, P = D$.

3. $B =$ yes, $P =$ no: We set $ScalarOperator_w(\mathbf{Ts}_{BP}) = \partial(w.\mathbf{Ts}_B)$.
4. $B =$ no, $P =$ yes: We set $ScalarOperator_w(\mathbf{Ts}_{BP}) = w.\partial\mathbf{Ts}_B$.

We should remark that the *ScalarOperator* and a fortiori the *TempoOperator* do not change any standard hierarchy. But the initial data could be affected. First of all, the field changes in the articulation plane of onset and duration could throw some kernel event out of the reach of the given initial set, excluding them from being performed. Second, the initial performance could be turned into an incompatible state, a problem we have already dealt with at the end of the *TempoOperator* discussion. For the general situation of the *ScalarOperator*, one must develop specific routines to control initial data, not a trivial task.

## 39.7 The Theory of Basis-Pianola Operators

**Summary.** The tempo and scalar operators can be generalized to operators which act on the basis and pianola parameter grouping and their coupling on the level of performance fields. The formalism is deduced from the standard situation and stated in its general shape. The point of this approach is the introduction of Lie derivative as a general device for the production

of deformations of performance fields by use of scalar potential funcions. The Lie formalism is applied when stressing on basis rather than on pianola parameters. Although it is a general formalism for operators, we do not know whether its application can generate all important deformation operators needed for a musically satisfactory performance construction.

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The standard situation which motivates the basis-pianola operators is this: We are given a performance map $\wp(X,Y) = (x(X), y(X,Y))$ on variables $X, Y$ of two disjoint spaces $U, V$, with a hierarchy projection $U \oplus V \to U$. For example, this could be an articulation ($U = E, V = D$) or a direct product ($U = EL, V = H$), etc. We now want to deform this performance on the second factor, and we are using a positive deformation weight function $\lambda$ on $U$. The deformed performance map is defined by $\wp_\lambda(X,Y) = (x(X), \lambda(X).y(X,Y))$. So this is a kind of physical operator we are familiar with. Let us calculate the Jacobian of this transformation (supposing that everything is smooth). The Jacobian for $\lambda = 1$, the start situation is

$$J_{\wp 1} = J_\wp = \begin{pmatrix} \partial_X x & 0 \\ \partial_X y & \partial_Y y \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \tag{39.39}$$

where $\partial_X x$ is the submatrix with the partial derivatives of all $x$ coordinates with respect to the $X$ arguments, etc. The inverse Jacobian is

$$J_\wp{}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{pmatrix}. \tag{39.40}$$

And the general inverse Jacobian is

$$J_{\wp_\lambda}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} - \lambda^{-1}C^{-1} \cdot d\lambda \otimes y \cdot A^{-1} & \lambda^{-1}C^{-1} \end{pmatrix} \tag{39.41}$$

with the dual gradient $d\lambda = grad\lambda^*$ and the identification[7] $Lin(U,V) \tilde{\to} U^* \otimes V$. The performance field is calculated by use of the diagonal unit vector $\Delta = (\Delta X, \Delta Y)$ and yields

$$\mathbf{Ts}_\lambda = J_{\wp_\lambda}^{-1}\Delta = \begin{pmatrix} A^{-1}\Delta X \\ -C^{-1}BA^{-1}\Delta X - \lambda^{-1}C^{-1} \cdot d\lambda \otimes y \cdot A^{-1}\Delta X + \lambda^{-1}C^{-1}\Delta Y \end{pmatrix}.$$

If we set $\mathbf{Ts}(X,Y) = (Z(X), W(X,Y))$, and $\mathbf{Ts}_\lambda(X,Y) = (Z(X), W_\lambda(X,Y))$, we get

$$\begin{aligned} W_\lambda(X,Y) &= W(X,Y) - \lambda^{-1}C^{-1} \cdot d\lambda \otimes y \cdot Z(X) + (\frac{1}{\lambda} - 1)C^{-1}\Delta Y \\ &= W(X,Y) - L_Z(\Lambda)C^{-1}y + (\frac{1}{e^\Lambda} - 1)C^{-1}\Delta Y \end{aligned} \tag{39.42}$$

with the Lie derivative $L_Z(\Lambda)$ of the function $\Lambda = \ln(\lambda)$, knowing that $\lambda$ is a strictly positive function. Everything is taken at the argument $X, Y$, or $(X,Y)$, respectively.

[7] See appendix E.3.2.

This gives us formula

$$\mathbf{Ts}_\Lambda = \mathbf{Ts} - L_Z(\Lambda) i_V C^{-1} y - (1 - e^{-\Lambda}) i_V C^{-1} \Delta Y \tag{39.43}$$

where $i_V$ is the canonical injection $V \to U \oplus V$. So we obtain a deformation of the original field $\mathbf{Ts}$ by two additive terms $L_Z(\Lambda) i_V C^{-1} y$ and $(1 - e^{-\Lambda}) i_V C^{-1} \Delta Y$, which we want to interpret in the following sections.

Both terms involve the weight function. The first term $L_Z(\Lambda) i_V C^{-1} y$ involves the basis field $Z$, while the second term only involves contributions from the $Y$ component of the field. Therefore, we call the first term the *basis deformation*, whereas the second is called the *pianola deformation*.

### 39.7.1 Basis Specialization

**Summary.** The general basis-pianola theory is specialized to basis parameters and explicated through concrete formulas.

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If we have a weight $\Lambda \in U_\epsilon(0)$ in a very small neighborhood of 0 (or else $\lambda$ in a small neighborhood of 1), we may neglect the pianola term in formula (39.43). If further the performance map in the second variable is affine, i.e., $y = A.Y + B$, and does not depend on the first variable $X$ as is the case with pianola coordinates $Y$, then $C^{-1} = const. = A^{-1}$, i.e., $C^{-1} y = Y + const.$ So we have a deformation of type $L_Z(\Lambda) i_V C^{-1} y = L_Z(\Lambda) i_V (Y + const.)$.

For example, consider the two-dimensional situation $U = E, V = L$ of two basis spaces. The performance map for a weight $\Lambda \in U_\epsilon(0)$ on $E$ is $\wp_\Lambda(X, Y) = (x(X), e^\Lambda.y(Y))$, where we start from a primavista performance map $\wp = \wp_0$. We may therefore suppose that $y = A.Y$. Then, one has the primavista product field $\mathbf{Ts} = T \times D$ of tempo $T$ and dynamics $D$. Then the deformed field is

$$(T \times D)_\Lambda = T \times D - L_T(\Lambda) i_L Y = T \times D - T.\frac{d\Lambda}{dX} i_L Y.$$

In other words, we deform the given tempo by the gradient of $\Lambda$ and project this component to the loudness field component via a field $i_L Y$. So musically speaking, the loudness field deformation is essentially controlled by the change of $\Lambda$ and the mother tempo.—Let us now propose a generalization of the basis deformation: We are given

- two not necessarily disjoint spaces $U, V$ of the space hierarchy,
- a weight $\Lambda$ on the first (the basis) space $U$,
- an affine *directional endomorphism* $Dir(Y) \in V@V$ of the $V$ space.

With these data, we define a field deformation of the performance field $\mathbf{Ts}$ on a superspace of $U \cup V$ in the hierarchy by the formula

$$\mathbf{Ts}_{\Lambda, Dir} = \mathbf{Ts} - L_{\mathbf{Ts}_U} \Lambda . i_V Dir \tag{39.44}$$

where the argument of $Dir$ is the $V$-component, and that of the Lie-derivative is the $U$-component of the total argument, whereas $i_V$ is the embedding of $V$ in the space of $\mathbf{Ts}$. The following is immediate from the linearity of the Lie derivative in both arguments.

**Lemma 53** *With the above notation:*

(i) *The basis operator is linear in the performance fields, i.e.,*

$$(\mu\mathbf{Ts}_1 + \nu\mathbf{Ts}_2)_{\Lambda,Dir} = \mu\mathbf{Ts}_{1\Lambda,Dir} + \nu\mathbf{Ts}_{2\Lambda,Dir}.$$

(ii) *If $U$ and $V$ are disjoint, and if $\Lambda_1, \Lambda_2$ are two weights, then*

$$(\mathbf{Ts}_{\Lambda_1,Dir})_{\Lambda_2,Dir} = \mathbf{Ts}_{\Lambda_1+\Lambda_2,Dir}.$$

**Example 57** An example of a non-disjoint union of spaces $U, V$ is the elementary deformation of tempo by a positive $C^1$ weight $\gamma$. On onset arguments $X$ in a frame of positive onsets, we set $\Lambda = \int \frac{1-\gamma}{X}$. This implies $\gamma T = T - L_T(\Lambda).1_E$, and we have another example of a basis operator.

**Example 58** For this example, we suppose given a two-dimensional $C^2$ performance map $\wp_{EH} : EH \to eh$ on the plane of onset and pitch with a hierarchy $EH \to E$ over the onset performance $\wp_E : E \to e$ with tempo curve $T$. So the performance field $\mathbf{Ts}_{EH}$ is $C^1$. Setting $I_\wp = (\partial_H h)^{-1}, T = (\partial_E e)^{-1}$, the field reads

$$\mathbf{Ts}_{EH} = (T, I_\wp(1 - \partial_E h.T)). \tag{39.45}$$

We want to view such a field as a deformation of the primavista product field $T \times I$ which is composed of the same tempo factor $T$ and an intonation factor $I$, which is also $C^1$ and strictly positive, as usual for primavista fields. We are looking for a weight $\Lambda$ on the onset-pitch frame $R$ and a pitch shift $e^{-Y_0} \in H@H$ such that the deformation $L_{T\times I}(\Lambda)i_H e^{-Y_0}$ of the primavista field yields $\mathbf{Ts}_{EH}$. This defines a linear partial differential equation (PDE) in the onset variable $X$ and the pitch variable $Y$:

$$T\partial_E\Lambda + I\partial_H\Lambda = (Y - Y_0)^{-1}(I_\wp(\partial_E h.T - 1) + I) = Q. \tag{39.46}$$

This equation has $C^1$ coefficients $T, I, Q$, if we let $Y_0$ be smaller than the lower boundary value of the pitch frame interval. Using the method of characteristic curves (see appendix I.6) in pseudo-linear PDEs, we can see that there is a solution of equation (39.46). In fact, the characteristic curve projection ODE reduces to the pair $d_t X = T(X), d_t Y = I(Y)$, whereas the third curve component $Z(t)$ is defined by the ODE $d_t Z = Q(X, Y)$. This means that the characteristic curve projection onto the $XY$-space is an integral curve of the primavista field $T \times I$. Clearly, a transversal curve $\Gamma$ to the characteristic curves exists since the Jacobian criterium $T\Gamma \nparallel T \times I$ can be met for the non-vanishing field $T \times I$. We therefore have this result:

**Proposition 59** *Let $\wp_{EH} : EH \to eh$ be a two-dimensional $C^2$ performance map on the plane of onset and pitch with a hierarchy $EH \to E$ over the onset performance $\wp_E : E \to e$ with tempo curve $T$. For the primavista product field $T \times I$ which is composed of the same tempo factor $T$ and an intonation factor $I$, which is also $C^1$ and strictly positive, there is a $C^1$ weight $\Lambda$ on the $EH$-frame of this performance such that the performance field $\mathbf{Ts}_{EH}$ is a deformation of the primavista field $T \times I$ by a basis operator:*

$$\mathbf{Ts}_{EH} = T \times I - L_{T\times I}(\Lambda)i_H e^{-Y_0}. \tag{39.47}$$

The basis operator is designed for distributing weight information from the "basis" subspace $U$ over any other space $V$, independently whether this one is also in the hierarchy or not. This creates a considerable freedom of shaping performance. The way this shaping is related to the given weight is the Lie derivative with respect to the performance field $\mathbf{Ts}_U$ on $U$. This one also measures the angle between the weight's gradient and $\mathbf{Ts}_U$. If their mutual position is perpendicular, the derivative vanishes and the operator has no effect.

#### 39.7.1.1 Deforming Hierarchies

**Summary.** We discuss the change in an existing hierarchy after the application of a basis operator.

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The most dramatic effect of the basis operator is the deformation of the given hierarchy. For example, in the above example of proposition 59, the projection $EH \to H$ is destroyed since the weight $\Lambda$ is not only a function of $H$ or of $E$, but of both, in general. So only the projection $EH \to E$ survives this deformation. Let us describe more systematically which hierarchy spaces disappear a priori by a basis operator deformation.

Suppose that we have a hierarchy space $U$, a weight $\Lambda$ on $U$ any other space $V$ within the top hierarchy space, a directional endomorphism $Dir = e^B.A$, and any hierarchy space $W$ which we want to test for survival after the subtraction of the basis deformation $L_{\mathbf{Ts}_U}(\Lambda)i_V Dir$. Consider the projection $p : V \to V \cap W$ onto the intersection space, including the empty intersection which then defines the projection onto the zero space. Let $[p, A] = p.A - A.p$ denote the commutator endomorphism of $p$ and $A$ on V.

**Proposition 60** *With the above notation, the mother hierarchy space $W$ remains a priori alive (i.e., member of the daughter hierarchy deduced from the mother hierarchy) after the basis deformation $L_{\mathbf{Ts}_U}(\Lambda)i_V Dir$ iff either $p = 0$, i.e., $V \cap W = \emptyset$, or $p[p, A] = 0$ and $U \subset W$. In particular, for $[p, A] = 0$ and $U \subset W$, $W$ remains alive.*

Proof. Suppose that $W$ remains alive. If $p \neq 0$, we have at least one coordinate that is common to $V$ and $W$. On that coordinate, the functional dependence of $U$ is inherited via $L_{\mathbf{Ts}_U}(\Lambda)$. Therefore we must have $U \subset W$. Since no functional dependence from $W - V$ arguments can be the case on the coordinates in $V \cap W$, and since the constant part of $Dir$ is irrelevant here, we must have $p.A.(1 - p) = 0$. On the other hand, if $p = 0$ everything is clear, and under the conditions $p[p, A] = 0$ and $U \subset W$, the deformation arguments all stem from $W$, and we have saved the life of $W$, QED.

**Example 59** Let the top coordinate space be all six usual basis and pianola parameters $EHLDGC$. Let a weight $\Lambda$ act on $U = EH$, take $V = ED$, whereas the directional endomorphism is

$$e^B.\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with a rotation $A$ as its linear part. Suppose that the hierarchy is the parallel hierarchy, i.e., the hierarchy generated by the basis hierarchy $TID, TI, TD, T$ and the parallel fields

$\partial TID, \partial TI, \partial TD, \partial T$ with the corresponding projections. Since the rotation $A$ has no proper invariant subspaces, we must have either $ED \cap W = \emptyset$ or else (because of $p[p, A] = 0$) $ED \subset W$. But every field in our hierarchy contains $E$, hence $U \cup V = EHD \subset W$. This is only the case for the sub-hierarchy $EHD, EHDG, EHLD, EHLDG, EHLDC, EHLDGC$.

#### 39.7.1.2 Lie Derivatives

**Summary.** Basis-pianola operator theory leads to Lie derivatives as a device for operator definition. We discuss the Lie formalism in its realization as a component of a performance grammar.

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The appearance of the Lie derivative in this context is quite surprising. Its usage is well known from classical mechanics, for example, where the Lie derivative of a function with respect to a Hamiltonian vector field is related to the Hamiltonian function $H$ via the Poisson bracket [2]. Presently, we do not know of any analogous structures of Hamiltonian or Lagrangian type in performance field theory. But it is good to have this perspective in mind for a future 'dynamics of performance'. We should however observe that the Lie derivative $L_{\mathbf{Ts}}$ associated with a vector field $\mathbf{Ts}$ induces an isomorphism $L_? : \mathfrak{X}_R \xrightarrow{\sim} Der(\mathcal{F}(R))$ between the real vector space $\mathfrak{X}_R$ of smooth vector fields over the frame $R$ and the vector space $Der(\mathcal{F}(R))$ of derivations on the real algebra $\mathcal{F}(R)$ of smooth functions (see appendix I.2.4). So $L_{\mathbf{Ts}}$ can be identified with $\mathbf{Ts}$, and the basis deformation means taking into account the 'weight' $L_{\mathbf{Ts}}(\Lambda)$ deduced from the weight $\Lambda$. In this sense, weights become natural mathematical objects associated with performance fields: They are just the natural objects, these fields act upon qua derivations, they are not only justified by the quantification argument given when we introduced weights in section 39.1. In other words:

**Thesis 6** *A performance field is not only a construction principle for the performance map, but equivalently an 'interpretation of weights'—this is effectively the mathematical transfiguration of the rational approach to performance.*

This thesis suggests that one should study the natural properties of the Lie algebra $Der(\mathcal{F}(R))$ with respect to performance theory, in particular the question of what is the musical interpretation of the Lie bracket $[\mathbf{Ts}_1, \mathbf{Ts}_2]$ of two performance fields. We have to pass this subject to future research.

**Exercise 81** Consider the performance $\wp : ED \to ed : (X, Y) \mapsto (x(X), f(d)), d = x(X + Y) - x(X)$, which is a functional change in duration, built upon the parallel performance, with a $C^1$ invertible deformation function $f$ of physical duration $d$. This is an example of a physical operator. Show that its field is

$$\mathbf{Ts} = \begin{pmatrix} T \\ \frac{(f'+1)}{f'} T \circ \alpha_+ - T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{(1-f')}{f'} \frac{T \circ \alpha_+}{T} & 1 \end{pmatrix} \partial T, \tag{39.48}$$

i.e., a linear automorphism of the tangent bundle of which we have already seen an example for the tempo operator. Observe however, that the automorphism is also a function of the tempo

curve $T$! Show by use of the characteristics method for quasilinear PDEs that this deformation of the parallel field $\partial T$ can also be obtained by a basis operator with $U = ED, V = D$.

### 39.7.2 Pianola Specialization

**Summary.** The general basis-pianola theory is restricted to pianola parameters and explicated through concrete formulas.

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Compared to the basis deformation, the second contribution $(1 - e^{-\Lambda})i_V C^{-1}\Delta Y$ in the general deformation formula (39.43) plays a different role. Whereas the basis contribution is sensitive to the gradient of $\Lambda$, i.e., its local changes, the second contribution is sensitive to the absolute values of the weight, so this contribution is relevant if the weight changes little, but has values different from zero. We give a more precise interpretation of this contribution in the case of a basis-pianola-space situation, i.e., $U$ is a space of basis parameters, and $V$ is the corresponding space $P_U$ of pianola parameters. We then have the alterator $\alpha_+ : U \oplus P_U \to U$, and we have a canonical isomorphism $\tau : U \xrightarrow{\sim} P_U$. If the original field is defined on $U \oplus P_U$ and is a parallel field $\partial \mathbf{Ts}_U$, then we have $C^{-1}\Delta Y = \tau \circ \mathbf{Ts}_U \circ \alpha_+$. Therefore we obtain the *pianola operator* for this special space configuration:

$$\mathbf{Ts}_{\Lambda,U} = \mathbf{Ts} - (1 - e^{-\Lambda})i_{P_U} \circ \tau \circ \mathbf{Ts}_U \circ \alpha_+. \tag{39.49}$$

This formula only involves a weight on *any subspace* of the top space and a hierarchy space $U$ consisting of basis parameters, together with the corresponding pianola space $P_U$ which must also be a subspace (but not necessarily hierarchic!) of the top space.

## 39.8 Locally Linear Grammars

**Summary.** According to section 38.3.4, interaction between "inherited" performance score structures of sisters—or farther relatives—can be envisaged. We describe this formalism which is a basic approach in inverse performance theory (see section 46.2).

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Until now, we have only considered operators which are directly related to the mother data, and not to farther relatives, such as sisters, or daughters of sister, etc. In the following discussion, we shall present an essentially linear model for such a more global interconnection of a stemma's LPS.

We start with a stemma, i.e., a local performance score $\Lambda$ whose graph $\Lambda\updownarrow$ is an undirected tree. We want to forget about the mother of $\Lambda$ and concentrate on the tree $\Lambda \downarrow = T_0$. For the following construction, we need the *stemma quiver*[8] $T = T_\Lambda$ associated with $\Lambda$:

**Definition 109** *The* stemma quiver *is a finite directed graph $T = (V, A)$ with vertex set $V$ and arrow set $A$, including multiple arrows and loops. It is constructed as follows. We start*

[8] See definition 123 in appendix C.2.2 for the quiver concept.

*with a directed tree $T_0 = (V, A_0)$ with root $r$, i.e., each vertex can be reached by a unique path starting from the root. If $x \to y$ is an arrow of $T_0$, we say that $x$ is the* mother *of $y$ and $y$ is a* daughter *of $x$ (in combinatorics they are known as 'father' and 'son', respectively, here we try to be politically correct). If we have a path $x \to y \to z$, then $z$ is a* granddaughter *of $x$, while $x$ is the* grandmother *of $z$, and so on. For a vertex $x$ of $T_0$, the set of vertexes which are daughters of $x$ is denoted by $D_x(T_0)$. The vertexes $x$ which are not mothers, i.e., $D_x(T_0) = \emptyset$, are called* final *(the 'leaves' in graph theory). Similarly for each vertex $x \in V(T)$ we define $M_x(T_0) \subseteq V(T_0)$ as the set of vertexes lying on the unique path from $x$ to the root ($x$ included).*

*In order to define the stemma quiver $T$ the directed tree $T_0$ is enriched by the following set of arrows (no vertexes added): First, each vertex $x$ is given a loop $x \circlearrowleft$, and for any couple of sisters $x_1, x_2$, i.e., of daughters of a common mother $y$, we add an arrow $x_1 \to x_2$. The resulting quiver $T = (V, A)$ is called a stemma quiver of the tree $T_0$, which is uniquely determined by the stemma quiver and is called the* stemma tree. *We therefore may define $D_x(T) = D_x(T_0)$ and $M_x(T) = M_x(T_0)$.*

**Definition 110** *With these graph-theoretical data, a* locally $\mathbb{R}$-linear grammar *is a family of $\mathbb{R}$-linear representations*[9] *of a stemma quiver $T$ which is defined by the following data:*

1. *For each vertex $x \in V(T)$ let $A_x, B_x$ be two real vector spaces, $B_x$ of finite dimension $s_x$.*
2. *Each mother–daughter arrow $x \to y$ is represented by a surjective linear map $r_{x,y} : A_x \to A_y$.*
3. *For all $x \in V(T)$ let $\varphi_x : B_x \to End(A_x)$ be an affine map, i.e., a linear map $\varphi^0_x$, followed by a displacement by $\varphi^t_x \in End(A_x)$. So each loop $x \circlearrowleft$ is represented by a family of endomorphisms $(\varphi_x(b) : A_x \to A_x)_{b \in B_x}$ parametrized by the* parameter space $B_x$.
4. *For all pairs of sisters $x, y \in D_m(T)$, the sister arrow $x \to y$ is represented by an isomorphism $i_{x,y} : A_x \to A_y$ with $i_{y,x} = i^{-1}_{x,y}$ and $i_{x,x} = Id_{A_x}$.*

For concrete performance field configurations associated with $\Lambda$ this axiomatic setup is realized as follows. For the LPS of the stemma tree, we suppose a number $n$ of real parameters for the top space $S$ of the LPS hierarchies. We concentrate on this top space $S$ of all the LPS, and do not discuss the other cell data of these space hierarchies. For each vertex $x$ of $T_0$, we have the frame $R_x = [l_{x_1}, u_{x_1}] \times \dots [l_{x_n}, u_{x_n}]$ of the top space of its hierarchy. For each mother $m$ and daughter $x$, we suppose that $R_x \subset R_m$, and that for each couple of sisters $x_1, x_2$, $R_{x_1} \cap R_{x_2} = \emptyset$. This corresponds to a restriction of a larger portion of a musical score to a disjoint grouping of smaller portions. Here is the realization of our above system (properties 1–4) of quiver representations:

Consider the vector space $\mathcal{F}_x$ of $C^\infty$ functions on $R_x$. We then set $A_x = Der(\mathcal{F}_x)$, the space of derivations, i.e., the $C^\infty$ vector fields on $R_x$. The surjective maps $r_{x,y}$ are defined as the restrictions of vector fields on the mother's rectangle $R_x$ to the daughter's rectangle $R_y$. To define the representations for a sister arrow $x_1 \to x_2$, consider the unique affine morphism $a_{x_2,x_1} : R_{x_2} \to R_{x_1}$ on the sisters' rectangles such that the respective vertexes are mapped onto

[9] These are representations of the quiver algebra over $\mathbb{R}$, see appendix E.2.1 for the concept of a linear representation.

each other. Then the sister arrow representations are isomorphisms $i_{x_1,x_2} : A_{x_1} \to A_{x_2}$ defined by the transport of a vector field $Z$ on $R_{x_1}$ to $Z \cdot a_{x_2,x_1}$ on $R_{x_2}$.

To define the operation of a parameter family, first take a vector field $Z \in Der(\mathcal{F}_x)$, and a weight function $\Lambda \in \mathcal{F}_x$. For $Dir \in S@S$, consider the corresponding vector field $\Gamma_{Dir}(t) = (t, Dir(t))$ on $R_x$. Then we have a new vector field $Z - L_Z\Lambda.\Gamma_{Dir}$, corresponding to a basis deformation. This is an $\mathbb{R}$-linear operator on $Z$, and the 'deformation' part $L_Z\Lambda.\Gamma_{Dir}$ is $\mathbb{R}$-bilinear in the weight function and the affine endomorphism.

We now take a finite dimensional subspace $W_x$ of $\mathcal{F}_x$ which in performance theory represents the weight functions issued from analyses of metrical, motivic, and harmonic structures of the given score. We now set $B_x = W_x \otimes_{\mathbb{R}} S@S$, and we obtain an $\mathbb{R}$-linear map $\varphi_x^0 : B_x \to End(A_x)$ defined by $\varphi_x^0(\Lambda \otimes Dir)(Z) = -L_Z\Lambda.\Gamma_{Dir}$. Setting $\varphi_x^t = Id_{A_x}$, we have defined the required affine map $\varphi_x = \varphi_x^t + \varphi_x^0 : B_x \to End(A_x)$ with

$$\varphi_x(\Lambda \otimes Dir)(Z) = Z - L_Z\Lambda.\Gamma_{Dir}.$$

Let $m \in V(T)$ be a mother. Then for each daughter $x \in D_m(T)$, we define a triaffine (affine in each argument) map

$$f_x : A_m \times \mathbb{C}^{\#D_m(T)} \times \prod_{y \in D_m(T)} B_y \to A_x, \tag{39.50}$$

by

$$\begin{gathered}(a_m, (c_{y,x}^m)_{y \in D_m(T)}, (b_y)_{y \in D_m(T)}) \mapsto \\ \textstyle\sum_{y \in D_m(T)} c_{y,x}^m i_{y,x}(\varphi_y(b_y)(r_{m,y}(a_m))) = \\ \textstyle\sum_{y \in D_m(T)} c_{y,x}^m i_{y,x}(\varphi_y^0(b_y)(r_{m,y}(a_m))) + \sum_{y \in D_m(T)} c_{y,x}^m i_{y,x}(\varphi_y^t(r_{m,y}(a_m))).\end{gathered}$$

Referring to the above example, this formula describes the following: In order to determine the performance field on the frame $F_x$, we use the field of its mother $m$, we first restrict that field to any of its daughters $y$ and get the fields $r_{m,y}(a_m)$. These sister fields to daughter $x$ are then deformed under the endomorphisms $\varphi_y(b_y) = \varphi_y^t + \varphi_y^0(b_y)$ induced by the system parameters $b_y$. These deformed fields are then transported to $x$ and weighted by the factors $c_{y,x}^m$. These sister fields influence the final value of the field at daughter $x$. Musically, this means that the performance at $x$ is influenced by surrounding sister fields, which are typically the fields of past and future times (past or future periods, bars, etc.).

We shall pursue this model in the course of the inverse performance theory of chapter 46. A final word on a perspective of generalized stemmata which seem to be suggested by the locally linear grammars as discussed above: One could envisage *continuous stemmata*. They are based on a generalization of the stemma's ramification structure to one-parameter families and narrowing of the daughters' extents of 'infinitesimal' quantities. This construction would take care of the fact that psychologically, interaction between neighboring performance moments is continuously 'updated'. A theory of continuous stemmata is still pending.

# Part X

# RUBATO®

# Chapter 40

# Architecture

*The most rigorous test of the efficiency of theories in*
*modern cognitive science is the production of a*
*working computer program whose*
*external behaviour mimics*
*that to be explained.*
John Sloboda [491]

**Summary.** RUBATO® is a metamachine designed for representation, analysis, and performance of music. It was developed on the NEXTSTEP environment during two SNSF grants from 1992 to 1996 by the author and Oliver Zahorka [348, 347, 350, 357, 588, 590]. From 1998 to 2001, the software was ported to Mac OS X by Jörg Garbers in a grant of the Volkswagen Foundation. RUBATO®'s architecture is that of a frame application which admits loading of an arbitrary number of modules at run-time. Such a module is called RUBETTE®. There are very different types of Rubettes. On the one hand, they may be designed for primavista, compositional, analytical, performance stemma or logical and geometric predication tasks. On the other, they are designed for subsidiary tasks, such as filtering from and to databases, information representation and navigation tasks, or else for more specific subtasks for larger "macro" Rubettes. A RUBETTE® of the subtask type is coined OPERATOR and implements, for example, what we have called performance operators in section 44.7. The RUBATO® concept also includes distributed operability among different peers. This software is conceived as a musicological research platform and not a hard-coded device, we describe this approach. Concluding this chapter, we discuss the relation between frame and modules.

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In the original concept of RUBATO®[345], we had defined RUBATO® as being a software for analysis and performance, divided into two submodules: one for "structuring" a score, and the other for "shaping" this score. This meant that structuring would yield analytical structures, whereas the other would yield a shaped performance transformation, alimented by analytical data from the structuring process.

In the course of the software development, we learned that no data model for music objects known to the developers at that time would be sufficient for all requirements of a comprising

music analysis and performance. This led to the concept of denotators and forms, as realized under the title of the "PrediBase" database management system (DBMS) of the first RUBATO® implementation in 1994, as described in [589]. At that time, it became clear that under such a universal data model, RUBATO® would split into an application framework comprising the "PrediBase" DBMS and a series of dynamically loadable software modules as implemented in the Objective C language of those NEXTSTEP OS driven NeXT computers. According to the diminutive convention for modules, such a module was coined "RUBETTE®".

In 1996, at the end of a grant of the Swiss National Science Foundation where RUBATO® was realized, three analytical Rubettes and one for performance have been developed[1], which will be described in chapter 41. The PerformanceRUBETTE® is connected to five OperatorRubettes (at those times still named "OPERATORS"). The PrimavistaRUBETTE® takes care of paratextual score predicates.

In 2001, this software has been ported (and improved in many data management aspects) to Mac OS X by Jörg Garbers and is now available as an open source project on the internet [357], or on the CD appended to this book, see page xxx. The RUBETTE® screenshots in chapter 41 are all taken from this version of RUBATO®. Figure 40.1 shows the info panel of the Mac OS X version. However, the version included in the book's CD-ROM is the latest version before the book went into production, whereas the screenshots are somewhat older. We hope that the reader will excuse us for this slight asynchronicity.

Figure 40.1: The info panel of the Mac OS X version of RUBATO®.

In the following sections of this chapter we shall however not describe the Mac OS X implementation, we will rather expose the more advanced and flexible architectural principles of the ongoing Java-based implementation of the distributed RUBATO® version.

## 40.1 The Overall Modularity

**Summary.** RUBATO® is a modular engine for metamachine rationales and because research is

[1] For NEXTSTEP, RUBATO® as well as these Rubettes are available on the internet, see [357]. The source code is GPL and is contained in the book's CD-ROM, see page xxx.

itself increasingly modular. Built upon the denotator language, the RUBATO® concept is fully modular, all parts that can be split into modules *have* been split in this way.

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The modularity of RUBATO® has two aspects: First, it shows a composition of the software from an arbitrary, a priori undetermined, number of functional units—the RUBETTE® modules. Second, the available Rubettes are a dynamic factor: According to the research progress, new Rubettes of any flavor may be added to the existing arsenal.

Modularity is an old principle, in fact, the traditional disciplinarity of science preconizes modules of scientific activities which have or pretend to have a relative autonomy in knowledge production. What is new with respect to the traditional disciplinarity is that this modularity is a dynamical one, at any time new modules of knowledge processing may be added or old ones removed. Discipline becomes a task-driven decision instead of being a rigid preset splitting.

Such a modularization can only work on the common ground of unrestricted cross-communication among any subgroup of knowledge modules. Without a common language ground, which in our case is the denotator and form data model, dynamic disciplinarity would inevitably collapse since a new module would require language modifications, adaptations, and extensions.

To be clear, we view dynamic disciplinarity as the idealized version of inter- and transdisciplinarity. The unity of knowledge cannot be achieved without a temporary and task-driven compartmentation of research fields, grouping and regrouping is inevitable; there is no direct path to the unity of knowledge. This credo can however not be realized without a common language basis. Otherwise, language engineering frictions would paralyze any major effort of dynamical grouping of disciplines. Evidently, the present RUBATO® environment is limited to musical and musicological scopes. Here dynamic disciplinarity is not that utopic. After all, the denotators and forms are universal language approaches issued from music(ologic)al requirements. But it has been shown in [464] that denotators are not only a priori applicable to non-musical concept modeling, but also in concrete cases such as geographic information systems. This suggests that dynamic disciplinarity could be realized on RUBATO® for modules of completely general scopes as long as the denotator data model is joined. This is one of the most intriguing vectors of future developments concerning the RUBATO® environment.

The principle of dynamic disciplinarity has its social form: a so-called collaboratory. According to Bill Wulf, this is "a 'center without walls' in which the nations researchers can perform their research without regard to geographical location, interacting with colleagues, accessing instrumentation, sharing data and computational resources, and accessing information in digital libraries"[278]. To collaborate in this way requires adequate software platforms, and RUBATO® is precisely this type of software in the field of musicology.

## 40.2 Frame and Modules

**Summary.** Modularity has been realized on the basis of a frame application which offers interfaces to an arbitrary number of modules. This is one of the technical core features in the realization of a metamachine. We describe its splitting interfaces and their functional positions.

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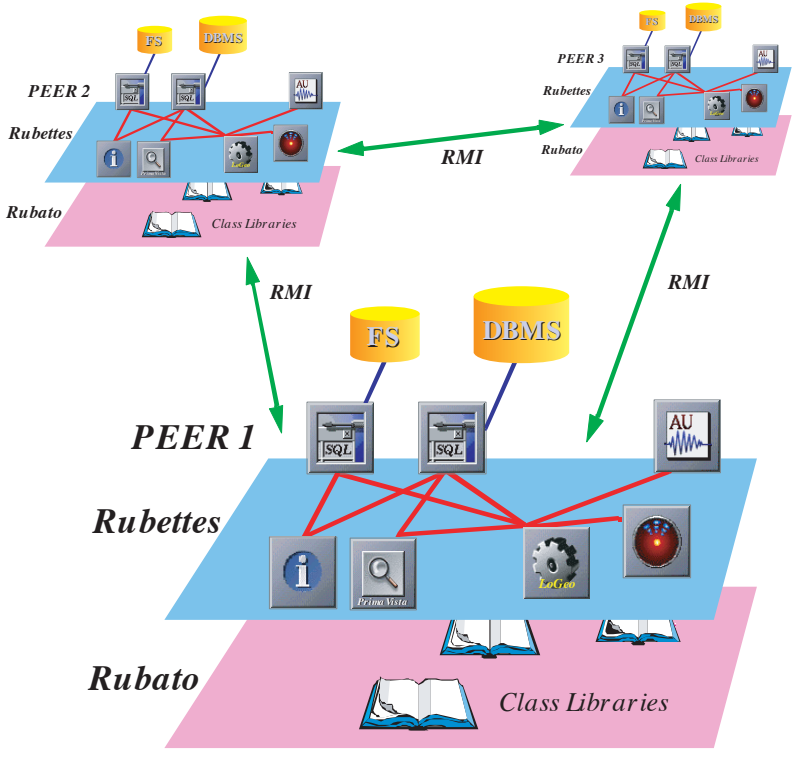


Figure 40.2: The RUBATO® layers of Rubettes with denotator communication, and the RUBATO®Framework layer with class libraries that are related to different Rubettes. Each peer has one such configuration. Different peers are interconnected via remote method invocation (RMI).

The RUBATO® platform consists of a number of installations of the software on different peers which may communicate via Java's remote method invocation protocol (RMI, see SUN's Java documentation on the internet). For each peer, RUBATO® contains two layers: the RUBETTE® layer and the RUBATO®framework layer. The first contains a number of Rubettes which are autonomous Java applications that communicate with each other and with Rubettes of another peer exchanging denotators via RMI. These are instances of the denotator class. The class library on the RUBATO® framework layer contains corresponding basic Java classes for denotators, forms, diagrams of presheaves, and modules. It also contains other classes which provide Rubettes with the necessary routines. The concept of these libraries is that they should contain all classes and methods that are of *general interest*, while classes and methods with specific interest for a RUBETTE®'s functionality should be installed in that RUBETTE®.

There are a number of mandatory Rubettes: The InfoRUBETTE® (with an "i" in figure 40.2) is the initialization RUBETTE®. It is automatically started when RUBATO® starts and informs the user about available peers and Rubettes on the distributed environment.

The visualization of all Rubettes' content and manipulation structures is managed by the PVBrowserRUBETTE®, whose functionality is to visualize any denotator in 3D space via Java3D classes, in figure 40.2, this RUBETTE® is represented by a lens symbol. The concept of this RUBETTE® has been described in chapter 20. The advantage of this centralization is that no other RUBETTE® designer has to take care of graphical and other multimedia representations, every denotator is piped to the PVBrowserRUBETTE® in case a multimedia representation is required. And every such representation is uniform according to this Rubette's visualization routines, which makes orientation much easier than individual design for every RUBETTE®. Nonetheless, as explained in chapter 20, the flexibility of the *Satellite* form for multimedia objects allows an unlimited multiplicity of shape and behavior. Two further Rubettes are devoted to the storage of denotators. The first, to the upper left of the RUBETTE® layer, we have the DenotexRUBETTE®. It takes care of the storage and editing of denotators which are given in the Denotex ASCII format. The second storage RUBETTE® is a filter to a SQL DBMS such that SQL databases can be transformed into denotators for RUBATO®. A last central RUBETTE® is the LoGeoRUBETTE®, as shown by a toothed wheel symbol to the right front of the RUBETTE® layer. It manages the logical and geometric operations on denotators (see section 18.3.4) and can be used by any RUBETTE® for its specific needs. The methods of this RUBETTE® are encoded in the class library of the RUBATO® layer.

# Chapter 41

# The RUBETTE® Family

*Vögel, Vieh und alles, was auf Erden kriecht,*
*die laß heraus mit dir, daß sie sich tummeln auf der Erde*
*und fruchtbar seien und sich mehren auf Erden.*
The Holy Bible, Genesis 8

**Summary.** We give an overview of the analytical MetroRUBETTE®, MeloRUBETTE®, HarmoRUBETTE®, the PerformanceRUBETTE®, and the PrimavistaRUBETTE®, which have been realized on the NEXTSTEP and then on the Mac OS X environment.

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Originally, the Rubettes as such were not the central concern of the RUBATO® project, this was rather to establish their collaboration and the realization of the whole transmission process from analytical data to the performance shaping operators of the PerformanceRUBETTE®. Each of these Rubettes was more an experimental prototype without the claim of a high-end tool in the specific domain.

The interest in such experiments lies in the fact that when one starts the design of a RUBETTE®, it turns out that musicology and music theory do not offer any reasonable support, be it in conceptual, be it in operational aspects. The path from the given score to a specific analysis reveals an incredible complexity of what in musicology and music theory looks like an easy enterprise. For example, in the design of the HarmoRUBETTE®, the mere definition of what is a chord cannot be traced from traditional literature. Should we only look for local compositions of pitches that stem from notes with a common onset, or should one also consider durational aspects? The standard answer—or rather: excuse—states that it depends on the particular context, and the context of the context, but this is no way out if one has to implement clear concepts and methods rather than rhetorics. So the design of a RUBETTE® is always a very good test of the validity of a model and of its adequacy with traditional fuzzy understanding. But it is also a test for the tradition: After all, who decides what is a good model for harmony? Here, the alternative between general speculative nonsense theories and concrete, but possibly non-sufficiently general implementation and operationalization becomes dramatic. At least, one can hope that this confrontation will force everybody to rethink ill-defined approaches.

## 41.1 MetroRUBETTE®

*Die Zählzeiten (Schlagzeiten, rhythmische Grundzeiten)*
*gewinnen unter allen Umständen erst*
*reale Existenz durch ihre Inhalte.*
Hugo Riemann [453]

**Summary.** The MetroRUBETTE® is an elementary analysis module which shows that seemingly simple approaches yield complex but informative results. We also make evident that operationalization of abstract concepts reveals unexpected insights into generically not foreseeable structures.

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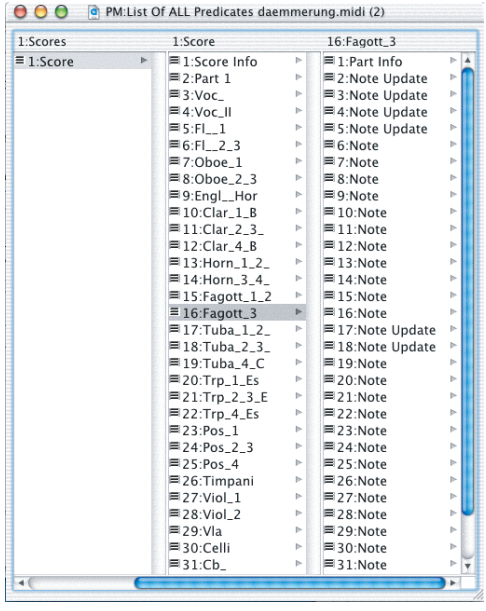


Figure 41.1: The 31-part score denotator deduced from Richard Wagner's composition "Götterdämmerung".

The MetroRUBETTE® is built on the sober weight calculation which we developed in chapter 21 in the frame of the maximal meter nerve topologies. The formula in example 43 of section 21.2 is realized, except that there is no upper length limitation (i.e., it is put to $\infty$). The mixed weight formula $W(x) = \sum_{X_i \in Sp_I(x)} W_i(x)$ is also realized in this RUBETTE®. The input is a score denotator (although still in the early shape of the PrediBase data model, which is the very special denotator form of a list of lists, which start from the simple form of strings). Figure 41.1 shows the 31-part score deduced from Richard Wagner's composition "Götterdämmerung".

This data is imported to the RUBETTE® and then evaluated according to the said formulas. For example, the weight of the union of all the onsets of the 31 parts is shown in figure

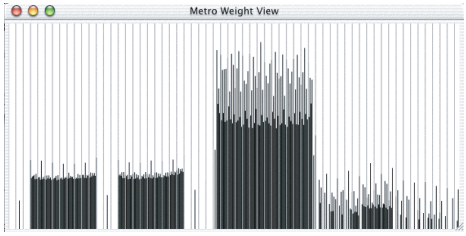


Figure 41.2: he weight graphics of the above composition with profile = 2, minimal length = 2. Horizontal axis: onset, vertical axis: (relative) weight. The unit of the onset grid in the graphical representation of the weight is 1/8.

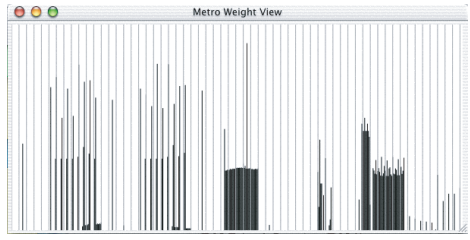


Figure 41.3: The mixed weight graphics of four parts clarinet (part 11), bassoon (part 16), horn (part 14), and violin (part 27) with scaling factors 3, 2.5, 1, and 0.1 for the respective parts. The unit of the onset grid is again 1/8, as in figure 41.2.

41.2. Its parameters are: profile = 2, minimal length = 2, and the unit of the onset grid in the graphical representation of the weight is 1/8.

The mixed weight is also an option of this RUBETTE®. Figure 41.3 shows the mixture of the parts: clarinet (part 11), bassoon (part 16), horn (part 14), and violin (part 27). The parameters are same as above, the weights are scaled by the factors 3, 2.5, 1, and 0.1 for the respective parts. For a more in-depth application of this RUBETTE®in musicology, we refer to [155]. This paper is an excellent example of an unexpected musicological application of a RUBETTE® which was not intended to be interesting on its own. The metrical analysis turns out to be quite sophisticated, although the concept of a local meter is a very elementary one. The surprising effect of this setup is that the combination of the simplistic concept reveals unprecented insights into the time structure of classical works. This is a good hint to the musicologists, teaching them that interesting insight can result from a complex aggregation of simple ingredients. In this case, the simple elements are provided by the maximal local meters, whereas their complex aggregation is conceived by the nerve of the covering they define. A second very interesting application of this RUBETTE® has been presented in [349]. It was recognized that the longest possible minimal lengths of local meters for the left hand part of Schumann's "Träumerei" yields a 3+5 quarters periodicity over two bars, whereas the same analysis of the right hand yields the expected periodicity of 4 quarters with stress on the barlines. The sonification of this fact can be heard on the audio example in the book's CD-ROM, see page xxx.

In this RUBATO® version, however, the output is a weight which is by no means a denotator. This output is a final data and must be used in its special format. This will be the case for operators of the PerformanceRUBETTE®. For the Java-based distributed RUBATO®(figure 40.2, such a restrictive usage would be forbidden.

## 41.2 MeloRUBETTE®

*In general, the author does not believe in the possibility or even desirability of enforcing strict musical definitions.*
Rudolph Reti on the concept of a motif [444]

**Summary.** The MeloRUBETTE® is an excellent example of the tension between abstract concepts and operational implementation. We expose the routines for motivic analysis, the interface concept, and we discuss the performance problem, including proposals for performance improvement and their theoretical limits.

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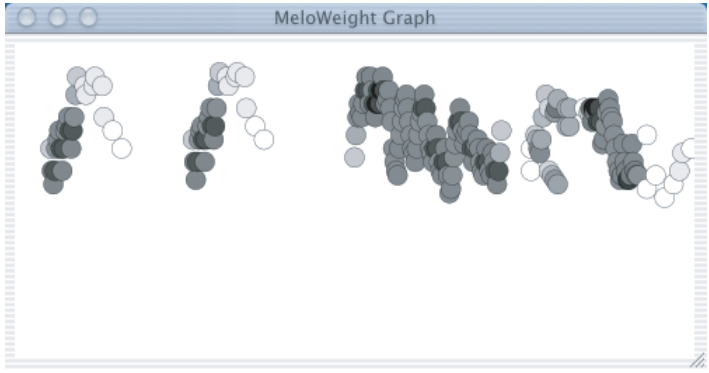


Figure 41.4: The weight graphics for the celli part 30 in the above score denotator deduced from Richard Wagner's "Götterdämmerung".

The MeloRUBETTE® refers to the theory of motivic topologies in chapter 22 and, in particular, to section 22.9 about motivic weights. The score is loaded as for the MetroRUBETTE®, its projection to the onset-pitch space is then analyzed and yields a numeric weight for each onset-pitch event. One such weight is visualized in figure 41.4. It corresponds to part 30 (celli) of the above composition "Götterdämmerung". The weight values are encoded in gray-levels of the discs which represent the events in onset and pitch.

The calculation relies on these parameters which relate to melodic topology:

- Symmetry Group. This is the paradigmatic group of the shape type. In each group, we include the translation group in pitch and onset. The choice is then between the translation group (encoded by "trivial"), the one generated by the translations plus the retrograde, or the one generated by the translations plus inversion, and the full counterpoint group, i.e., generated by the inversion and retrograde over the translations.

- Gestalt Paradigm. This is one of three possible shape types: diastematic, elastic, and rigid. The first candidate is in fact the type which we called "diastematic index shape

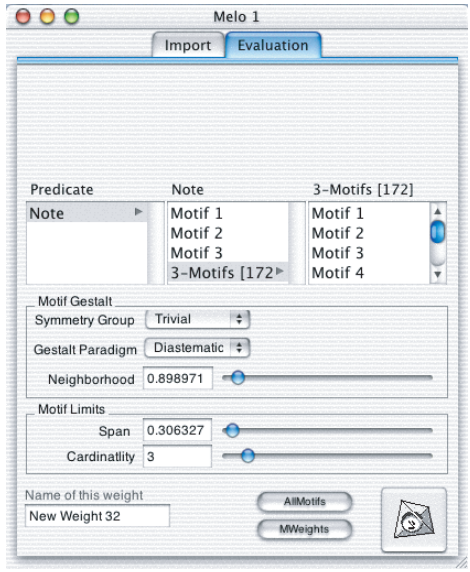


Figure 41.5: The main window of the MeloRUBETTE®.

type" in chapter 22. Observe that we have no topology for the diastematic type, but may nevertheless define neighborhoods!

- Neighborhoods. This is the neighborhood radius $\epsilon$ which was used in the expression $D^{\mu}_{\epsilon}(M)$ in section 22.9.
- Span. This is the maximal admitted onset difference between motive events.
- Cardinality. This is the maximal admitted cardinality of motives. Together with the Span, this condition, defines the selection $\mu$ of motives which is addressed in the approach from section 22.9.

The presence and content functions are defined as follows:

1. Presence. For a motif $M$, the presence value $pr_{\mu,\epsilon}(M)$ is the sum of all these numbers: For each $N \in D^{\mu}_{\epsilon}(M)$, we count the number $m$ of times where $M$ has a submotif $M'$ of $N$ that at a distance less than $\epsilon$ from $M$. We also look for the difference of cardinalities $d = card(N) - card(M)$. This gives a contribution $p(N) = m.2^{-d}$, and we add all these numbers.
2. Content. Similarly, for each motif $N \in \mu$ such that $M \in D^{\mu}_{\epsilon}(N)$, we take $p(N) = m.2^{-d}$ with $d = card(M) - card(N)$ and $m$ the number of times, where $N$ has a submotif $N'$ of $M$ that at a distance less than $\epsilon$ from $N$. We add all these numbers $p(N)$ and obtain the content $ct_{\mu,\epsilon}(M)$.

3. Weight. Given a motif $M \in \mu$, this is the product

$$nW_\epsilon(M) = pr_{\mu,\epsilon}(M).ct_{\mu,\epsilon}(M),$$

i.e., taking the function $\omega(x, y) = x.y$ from section 22.9.

We have already given musicological comments on this construction in chapter 22. It is however remarkable to see the overwhelming amount of calculations which arise in this routine. For example, we have calculated the number $C$ of comparisons of motives (for distance measurements) Schumann's "Träumerei" ("Kinderszene" number 7) which comprises 463 notes. If we take Span = 1/2 bar and Cardinality = 4, we obtain $25'745$ motives and $C = 1'023'490'904 \sim 1.023.10^9$. This is beyond any explicit human calculation power. It demonstrates that the task of finding a dominant motif is a very hard one, and that this one, if it is recognized by a human listener, can at most be present in a very hidden layer of consciousness. This becomes even more dramatic for larger pieces, such as an entire sonata, say! Here, the combinatorial extent of motivic units exceeds any calculation power of humans and machines, as is easily verified. This means that a huge composition bears a motivic complexity that will escape to (human or machine-made) classification forever. Nonetheless, the usage of statistical methods, of simplified approaches to motivic topology, or of topological invariants that are more easily perceived could help find a rough orientation in the virtually infinite motivic variety of music.

This implementation also makes evident the tension between fuzzy concepts in musicology and implementation of a precise model. Although the concept of a motif is rather elementary, it entails a very sophisticated motivic analysis which could eventually converge to the intensions hidden[1] behind those fuzzy motive theories.

[1] We are not sure whether they are really hidden and not only faked...

## 41.3 HarmoRUBETTE®

*Eine Theorie aber, die gerade dort versagt, wo auch das Phänomen, das sie erklären soll, ins Vage und Unbestimmte gerät, darf als adäquat gelten.*
Carl Dahlhaus on Hugo Riemann's harmony [100]

**Summary.** The HarmoRUBETTE® makes clear that a vague theoretical approach does not reflect a vague phenomenon but an extremely complex one. The implementation of this RUBETTE® reveals several deep deficiencies of traditional "messy" analysis in harmony. We account for this on the level of preferences that have to be defined in order to get off ground with the analysis. The chapter concludes with a discussion of combinatorial problems due to the global complexity of harmony, and to the local character of tonal paradigms.

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The HarmoRUBETTE® is probably the most interesting RUBETTE®, since it is situated on a turning point of several critical issues in harmony. To begin with, the context problem in harmony is a multilayered and ramified one which is (we said it repeatedly) not clarified by music theory. This is manifest in the preliminary question of what is a chord in a given score. Should one only look at groups of notes as a common onset, should one also consider onset groups which are not manifest, but can be deduced from plausible rules, or is the selection of the relevant set of chords within a score also a function of the harmonic statements which could result thereof? We have implemented two variants. The first one takes as the sequence $(a_i)_i$ all maximal zero-addressed local compositions $a_i = \{a_{i,j} | j = 1, \dots t_i\}$ of pitches of note events with identical onsets. The second one is less naive. Within the given score, we take all local compositions $a_i = \{a_{i,j} | j = 1, \dots t_i\}$ of pitches with this property: There is at least one onset which is the offset time of an event, and the chord $a_i$ is the non-empty set of pitches of all note events which either start or still last at this offset time. This option is chosen by the button "use Duration" on the RUBETTE®'s main window, see figure 41.6. This second variant encodes all changes of chord configurations, not only the onset-commonalities. Using either of these methods, the generated chord sequence is the basis for the following analysis which at the end will yield a harmonic weight for each note event, and for which we refer to the harmonic tension theory presented in section 27.2.2.

According to that approach, each chord $a_i$ of the chord sequence $(a_i)_i$ must be given a Riemann matrix $(TF_{f,t}(a_i) = \phi_{f,t}(a_i)\hat{})_{f,t}$, from which we deduce the weights $\omega(f, t, a_i) = \ln(\phi_{f,t}(a_i))$ and also call this data the Riemann matrix of $a_i$. According to that discussion, we may also downsize weights below a threshold $\phi_{min}$ to $-\infty$. This is what the user sets when defining the "Global Threshold" in the main window. The local threshold is just the same for relative weights within a given Riemann matrix. The percentages in the main window mean that we downsize values below a defined percentage relative to the global or local (only within a fixed Riemann matrix) value range. Following the rules for the value $-\infty$, we may neglect any path through a chord which has this value. So this singular value means that a chord is "inharmonic" insofar as it cannot contribute to a positive harmonic evaluation. This is a mathematical rephrasing of the classical, but fuzzy concept of *inharmonic chords*. Here it just means that the harmonic weight of a chord is too small to be considered as a contributon to the global harmonic path, and that the minimal size of allowed weights is set without further

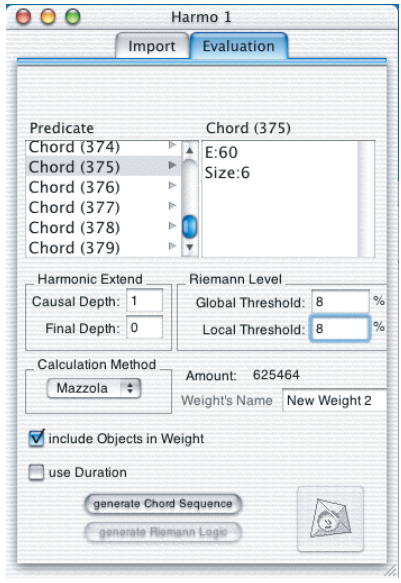


Figure 41.6: The main window of the HarmoRUBETTE®.

theoretical justification. It is a regulatory limit for the sensitivity of the path maximization with respect to the involved weights. We should stress that this matrix $(\omega(f,t,a_i))_{f,t}$ defines Riemann weights without any contextual considerations. This is information that comes along from the isolated calculation on the chord $a_i$. There are different calculation methods for this matrix, one by the author, and using chains of thirds, as discussed in section 25.3.3, and one by Noll, using self-addressed chords—the user may choose his preferred method by a pull-down button on the RUBETTE®'s main window. The Riemann matrix $(\omega(f,t,a_i))_{f,t}$ is visualized on the "ChordInspector" window for each chord (see figure 41.7). In this window, the non-logarithmic values $\phi(a_i)$ are shown. The window also shows the chord's pitch classes as well as all its minimal third chains.

Once the Riemann matrices $(\omega(f,t,a_i))_{f,t}$ are calculated, the optimal path in the Riemann quiver is calculated. This one follows the method discussed in 27.2.2. To this end, we need preferences for the matrices $T_{\mathrm{VAL}_{type}}, T_{\mathrm{VAL}_{mode}}, T_{\mathrm{TON}}$. The matrices $T_{\mathrm{VAL}_{mode}}, T_{\mathrm{TON}}$ are defined in the windows shown in figure 41.8. To the left, we have $T_{\mathrm{TON}}$, however as a distance value according to the amount of fourth between pairs of tonalities, to the right, we have $T_{\mathrm{VAL}_{mode}}$. The matrix $T_{\mathrm{VAL}_{type}}$ is shown in the left lower corner of the window in figure 41.9. In the middle lower part of that window, we have check buttons for every Riemann locus (i.e., the position in the Riemann matrix), meaning that if the check is disabled, no path is possible through such a locus. The large upper matrix encodes the third weights needed for chord weight calculations according to the formulas (25.8) and (25.9).

With these settings, the best path is calculated. This is however a tedious task, to say the least. In fact, if we are given 200 chords (a very small example), we may choose from a number of

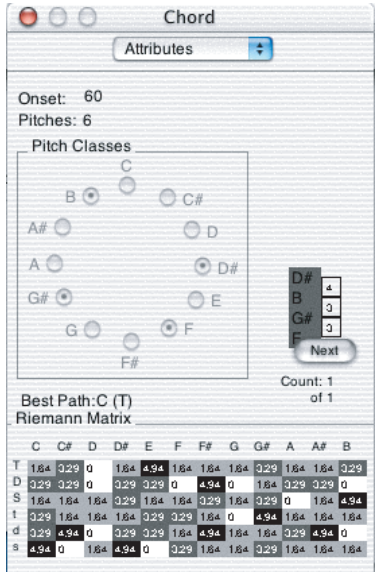


Figure 41.7: The ChordInspector of the HarmoRUBETTE® shows each chord of the chosen chord sequence with its pitch classes, the third chains, and the Riemann matrix according to the chosen calculation method. The grey level of the values is proportional to their relative size.

$12^{200} \sim 6.8588169039290515.10^{215}$ paths. This number exceeds any calculation power of present computers. This exuberant number is due to different factors. First of all, no larger paths are taken into account, i.e., we have not implemented cadences as preferred paths, nor have we implemented modulatory constraints. More precisely, we do not give preference to maximal subpaths within a fixed tonality. We only take into account tonality changes a posteriori, i.e., via the weights of paths of length 1, when they show a tonality change. So we have to calculate the entire path and then hope that the negative points for tonality changes rule such paths out.

It is also not clear whether human harmonic logic really can take into account such global path comparisons. In other words, it is more likely that humans only consider local optimization of paths. This is what we have in fact implemented in the following sense. In each index $i$ of a chord $a_i$, we consider only a local part of the entire chord sequence. Such a part is defined by two non-negative entire variables $CD = \textit{Causal Depth}$, $FD = \textit{Final Depth}$. This means that we look at the subsequence of chords from index $i - CD$ to $i + FD$ (inclusive) and therein select an optimal path $p_{i,CD,FD}$. Within this path, chord $a_i$ is positioned at a determined Riemann locus $(f(i), t(i))$. The path which we finally select is the path $p$ through all triples $(f(i), t(i), a_i)$. The causal part is a tribute to the influence of preceding chords down to index $i - CD$ on the harmonic position of chord $a_i$. The final part influences the harmonic position of $a_i$ relating to the future chords up to index $i + FD$. The result is visualized in the *Riemann Graph* which is shown in figure 41.10.

This is however not the end of the job. We do not have the weights of the single notes. To

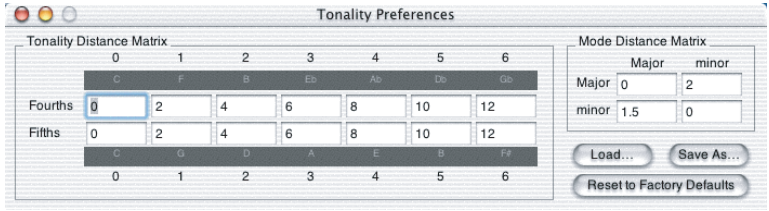


Figure 41.8: This preference window (for the author's third chain method) shows the tonality distance matrix $T_{\mathrm{VAL}_{mode}}$ (left) and the mode matrix $T_{\mathrm{TON}}$ (right).

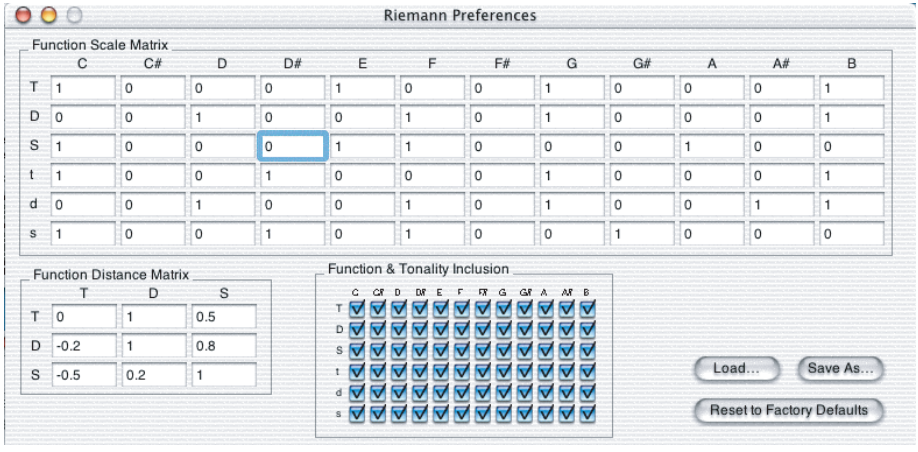


Figure 41.9: The upper matrix encodes the weights of thirds (relative to a fixed reference tonic, the lower left shows the matrix $T_{\mathrm{VAL}_{mode}}$. The lower middle matrix encodes the a priori allowed Riemann matrix locus position.

this end, we first need the globally calculated weight of each chord $a_i$. In a provisional form, the weight of chord $a_i$ is defined by the weight $\omega(p_i)$ of the chosen path $p$ from the first chord to $a_i$. This value is finally modified by a global slope $\sigma$ preference such that the final and the first weight can be set to build a defined slope. Thereby, we meet the requirement of a control over the global tension which cannot be deduced from the locally (only 0 and 1 lengths of sub-paths) defined weights. Let us denote by $\omega_i$ the weight $\omega_\sigma(p_i)$ corrected by the slope preference.

Using the weight of a chord $a_i$, we finally may define the weight[2] $\omega(x)$ of a determined event $x$ in $a_i$. To this end, the weight $\omega(f(i), t(i), a_i)$ is compared to the weight $\omega(f(i), t(i), a_i - \{x\})$ of the chord $a_i$ which contains $x$. With a positive preference quantity $0 < d \leq 1$, we consider the factor

$$\lambda(x) = \frac{1}{d + (1-d)e^{\omega(f(i),t(i),a_i-\{x\})-\omega(f(i),t(i),a_i)}} \tag{41.1}$$

[2]In this notation, we omit all the preferences.

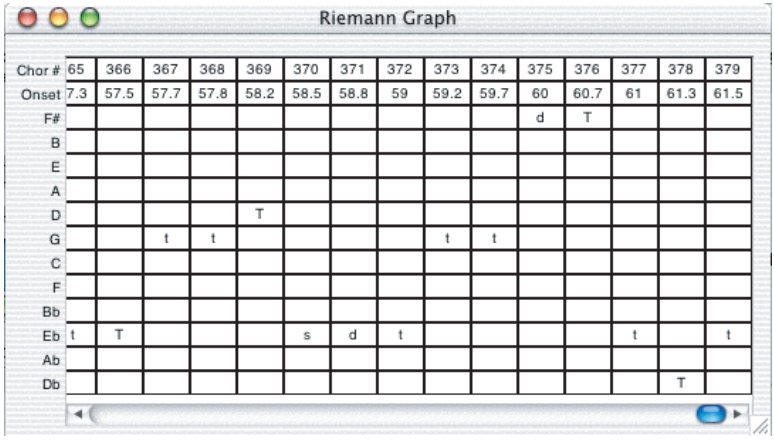


Figure 41.10: The Riemann graph is the sequence of chords, together with their functional values as they result from the optimal path.

which measures the weight differences. It evaluates to 1 for the difference zero and yields $1/d$ for the weight $\omega(f(i), t(i), a_i - \{x\}) = -\infty$. This means that the influence of $x$ in the building of the chord's weight is accounted for. If the weight decreases after omission of $x$, its influence is important and the factor increases $\lambda(x)$. So we finally get the weight

$$\omega(x) = \lambda(x).\omega_i.$$

The graphical representation of this weight is the same as for the MeloRUBETTE®, and we may omit this window of the HarmoRUBETTE®.

Whereas the Riemann graph is conformal to the usual function-theoretic analysis (although it need not provide the common data in general), the weights of chords and events are far beyond the usual harmonic analysis and therefore cannot be compared without caution to established knowledge in harmony. It is however a common approach to harmony in its performance aspects to weight chords or notes in a more or less metaphoric way. Our present approach in the HarmoRUBETTE® is a concretization of these metaphors and also a point to be discussed with traditional performance theorists.

## 41.4 PerformanceRUBETTE®

*All words,*
*And no performance!*
Philip Massinger (1583–1640)

**Summary.** The PerformanceRUBETTE® is a 'macro' RUBETTE®: It manages the stem-ma generation, the weight input and recombination, the operator instantiation, and the production of output of performance data on the level of music technology.

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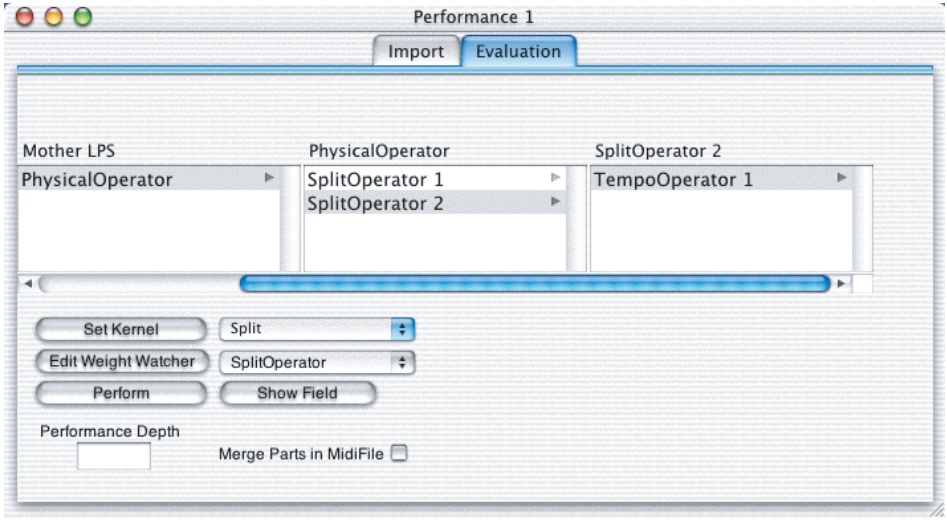


Figure 41.11: The main window of the PerformanceRUBETTE®shows the stemmatic inheritance, descending from left to right. In the figure, the "Mother LPS" has a daughter named "PhysicalOperator", and this one (all mothers are indicated on top of the daughters' column) has two daughter "SplitOperator 1", "SplitOperator 2", generated by the SplitOperator, etc.

Originally, the PerformanceRUBETTE® was the very focus of RUBATO®. Its purpose was the implementation of a type of performance logic with arguments from an analytical output. Although the analytical Rubettes have earned a growing importance, one of the cornerstones of analysis is its success in the construction of a valid performance. In fact, *playing a good performance is a way of demonstrating one's understanding of music.* Therefore the performance theory implementation is important beyond its autonomous interest.

The PerformanceRUBETTE® implements the stemma theory of chapter 38. The starting point is a selection of a score denotator. This will play the primary mother's role, i.e., we are constructing a primary mother LPS in the sense of definition 35.3. The score form is provided in the same format that we have known as input for the other Rubettes. For this RUBETTE®, the kernel is always given as a (zero-addressed) local composition in the space form $EHLDGC$.

This local composition is the kernel in the top space of a cellular hierarchy pertaining to the primary mother's LPS. The primary mother's LPS is instantiated according to "hard coded" default parameters in the Objective C source code[3]. However, for each specific performance operator, the LPS data are adopted and define specific daughters. After setting the kernel (see figure 41.11) of the primary mother, the main window shows the stemmatic ramification with individual names for each LPS and arranged in a browser, stemmatic inheritance running from the left to the right. In the figure, the "Mother LPS" has a daughter named "PhysicalOperator", and this one (all mothers are indicated on top of the daughters' column) has two daughters, "SplitOperator 1", "SplitOperator 2", generated by the SplitOperator, etc.

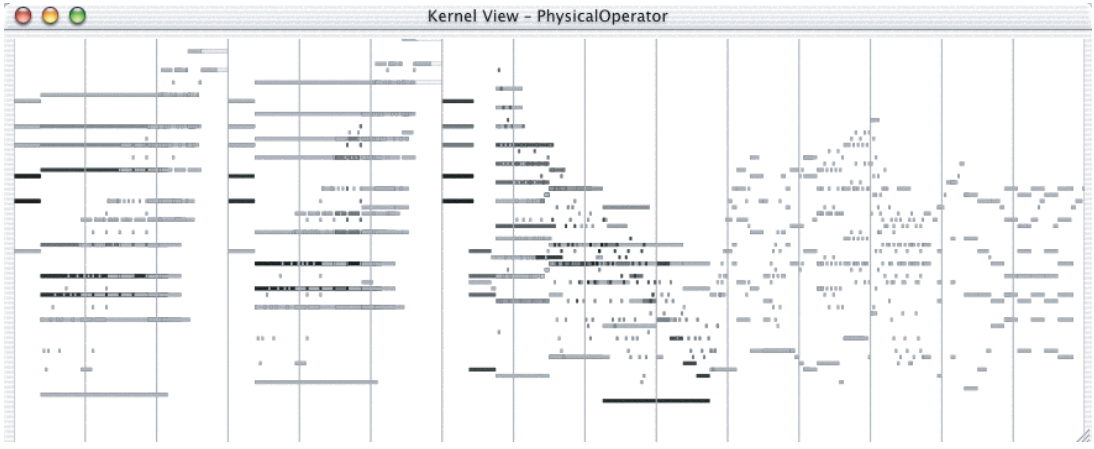


Figure 41.12: The Kernel View window shows the top kernel of the hierarchy of a selected LPS (here the LPS named "PhysicalOperator" in the above stemma browser) in common pianola (piano roll) rectangles, loudness being codified by grey levels. The vertical bars are set to four bar intervals in the given score.

[3]Objective C is a programming language for the NEXTSTEP-, OPENSTEP-, and Mac OS X-based RUBATO® projects.

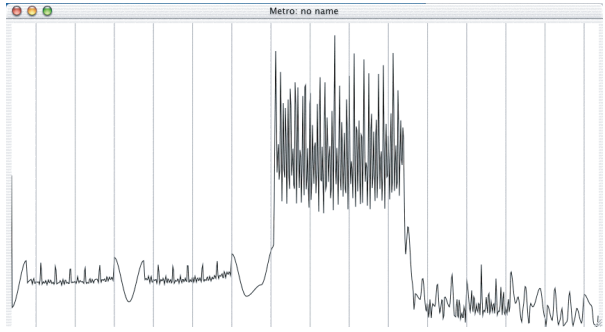


Figure 41.13: In the PerformanceRUBETTE®, weights are used in their splined interpolation shape. Here, we see a metrical weight issued from the MetroRUBETTE®'s analysis.

Figure 41.14: The Weight Watcher window shows the loaded weights (top), the upper and lower limit of their range, the non-linear deformation, and the Boolean flag of inverting/non-inverting the weight (button in the lower left corner).

For each ramification, one or two daughters are generated according to a chosen operator. For example, in figure 41.11, the SplitOperator generates two daughters, whereas the TempoOperator always produces one single daughter. The operators need only be loaded at run-time as they are needed. So this RUBETTE® is non-terminal in the sense that it allows further ramifications via an arbitrary number of dynamically loadable performance operators. For every highlighted LPS in the stemma browser of the main window, we can visualize the top kernel of its hierarchy on the Kernel View window, as shown in figure 41.12, by means of the usual pianola graphics. Here, the grey level indicates the loudness. In order to apply an operator to a given LPS, one next needs a list of weights, this is conformal with the operator theory exposed in chapter 44.7.

The management of available weights as well as their concrete application are the business of the Weight Watcher system. Figure 41.13 shows a metrical weight in its splined interpolation shape. The weights to be used for a given operator can be loaded into the Weight Watcher, see top of figure 41.14. The loaded weights are then added or multiplied (according to the Boolean flag button "Combine as Product" to the right, below the weight list), and the resulting weight combination is applied to the given operator. For each weight, one can set the upper and lower limit of range (High Norm, Low Norm), the non-linear deformation quantity (Deformation), the inversion/non-inversion flag (Inverted Weight button to the lower left corner), the influence in a combination of several weights (Influence), the slope of decrease to weight value 1 as the arguments tend to infinity (Tolerance). The moral of this Weight Watcher system is a gastronomic one: Weights may be mixed and dosed at will in order to experience their influence on a given operator. This is not merely a lack of theory, it is above all an experimental environment for effective performance research. In fact, since virtually nothing is known about the influence of weights on performance, we have provided the user with a great number of possibilities in order to realize an optimal testbed for future theory.

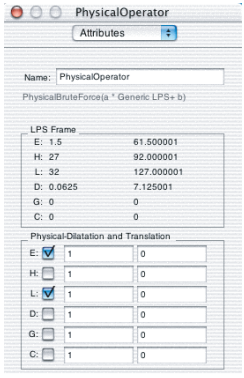


Figure 41.15: The PhysicalOperator Inspector allows us to select a number of physical output parameters where the weight changes the values.

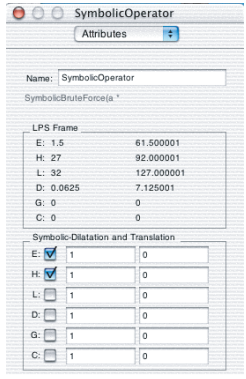


Figure 41.16: The SymbolicOperator Inspector allows us to select a number of symbolic input parameters where the weight changes the values.

Now, given a weight watcher combination of weights, an operator is fed by this combined weight and acts on the mother LPS to yield a new daughter LPS (or two in the case of the SplitOperator, where however no weight is needed). The detailed operation of a specific operator has already been described in chapter 44.7, we need not repeat these details here.

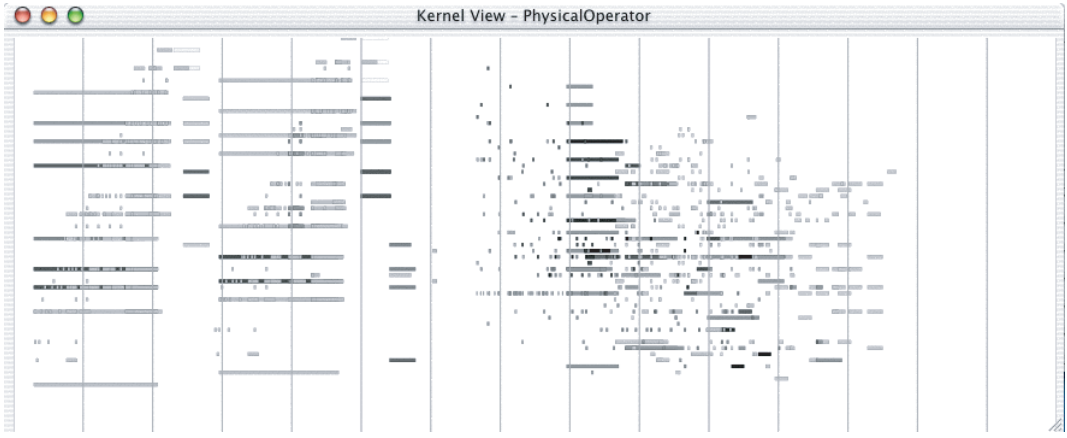


Figure 41.17: The effect of an operator, here a PysicalOperator, is shown in the Kernel View of the performed kernel. This figure shows the performed symbolic kernel as shown above in figure 41.12.

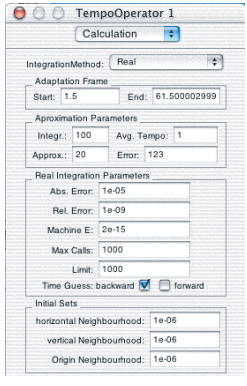


Figure 41.18: The TempoOperator inspector allows us to select different integration methods. The "Real" method uses Runge–Kutta–Fehlberg routines, whereas the "Approximate" method uses simple numerical integration.

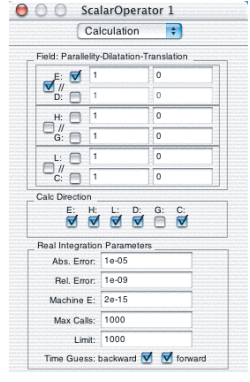


Figure 41.19: The ScalarOperator inspector allows us to select different options as defined in the scalar operator theory, but Runge–Kutta–Fehlberg ODE integration is mandatory in this situation.

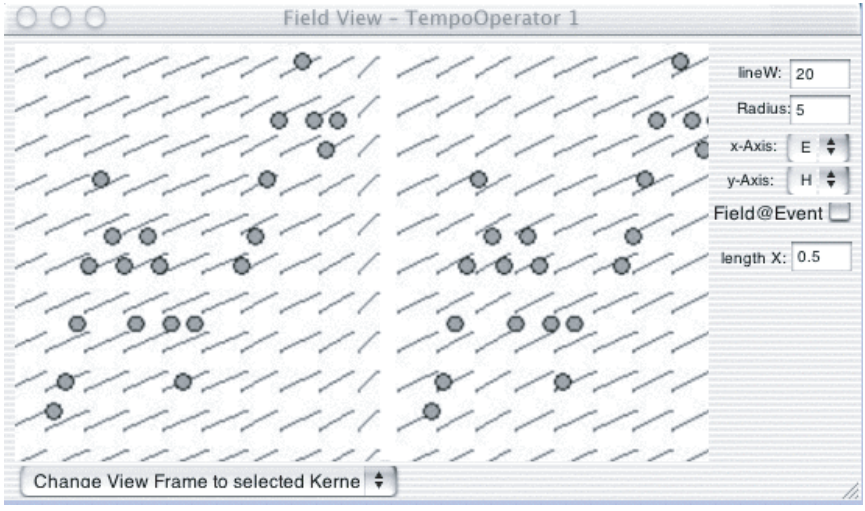


Figure 41.20: The performance field of a selected LPS can be visualized. The user may select two parameters whereon the six-dimensional field is projected.

Figure 41.16 shows the inspector of the SymbolicOperator. The weight acts on selected parameters which are defined by Boolean buttons. The same procedure is performed for the PhysicalOperator, whose inspector is shown in figure 41.15. The action of an operator on the symbolic score is shown in the performed kernel in the same pianola representation as for the symbolic kernel. The action of a physical operator is shown in figure 41.17. A funny application of this operator to Schumann's "Träumerei" can be heard on the book's CD-ROM under the title `Alptraeumerei`, see page xxx. This piece is the dead-pan version of the score with the melodic weight being applied to pitch via the physical operator, and everything being played with Schumann's original tempo indication. The inspector windows of the tempo-sensitive operators, the TempoOperator and the ScalarOperator, are shown in figures 41.18 and 41.19, respectively. The TempoOperator implements basically two methods: "Approximate", and "Real". The former is a direct integration method, whereas the latter uses Runge–Kutta–Fehlberg numerical ODE routines, including different parameters for numerical precision. The ScalarOperator uses exclusively Runge–Kutta–Fehlberg numerical ODE routines since it is an operator that acts on two or more parameters, where the naive approximation method cannot work. For the visualization of performance fields, the window shown in figure 41.20 is available.

Finally, the parameters for the SplitOperator are determined on the window shown in figure 41.21. Here, one may define those lower and upper parameter limits of the total six-dimensional frame, where the subframe of the split daughter is cast.

Figure 41.21: On this window, the user may define those lower and upper parameter limits of the total six-dimensional frame, where the subframe of the split daughter is cast for the SplitOperator.

## 41.5 PrimavistaRUBETTE®

*It is hard if I cannot start some game on these lone heaths.*
William Hazlitt (1778–1830)

**Summary.** Several musical predicates from score notation are paratextually loaded. The PrimavistaRUBETTE® takes care of the paratextual signification for the most important predicates regarding dynamics, agogics, and articulation.

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The PrimavistaRUBETTE® serves a different task insofar as it is neither analytic nor performance oriented. It deals with paratextual information as it is provided by verbal indications for dynamics, tempo, and articulation. It basically does this: it transforms verbal information into weights which may then be used to shape the symbolic data and the tempo before performance in the proper sense is shaped.

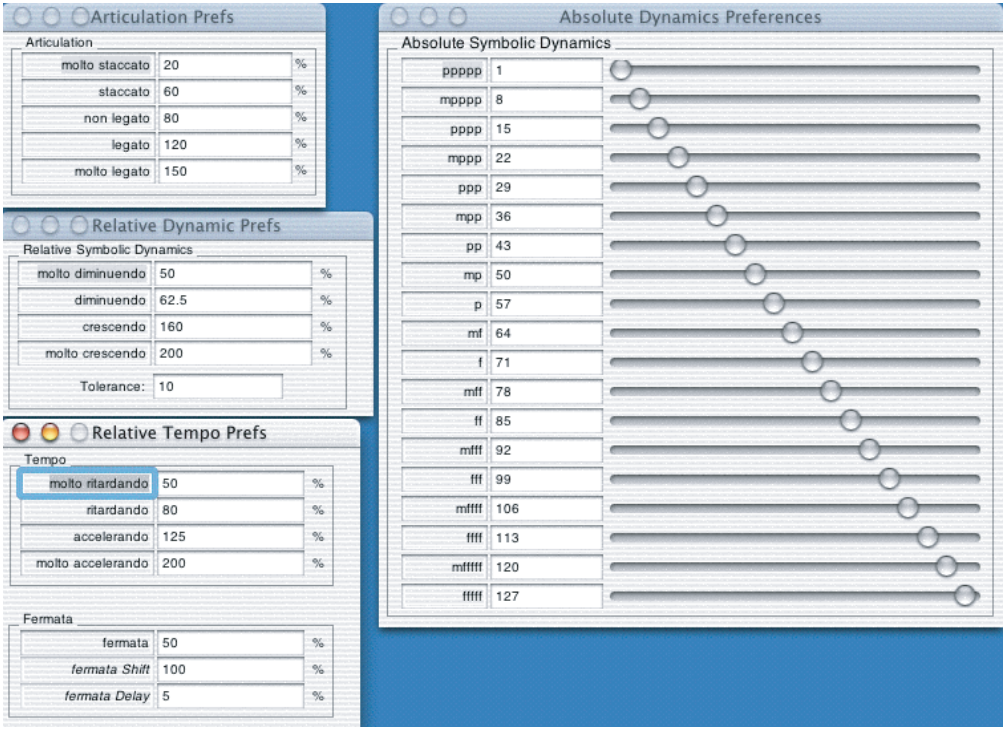


Figure 41.22: A number for preference windows for dynamics, articulation, and tempo, allow us to define numerical values of paratextual predicates.

The input of this RUBETTE® is a local composition whose elements are events with verbal specification such as absolute dynamics (figure 41.22 right preference window), relative dynamics

(figure 41.22 left middle preference window), articulation (figure 41.22 left upper preference window), and relative tempo (figure 41.22 left lower preference window). The functionality of this RUBETTE® is to transform these data into weights, this is performed on the window for primavista operations as shown in figure 41.23, and according to the numerical data that are defined the above preference windows. These methods have been discussed in detail in section 39.2.

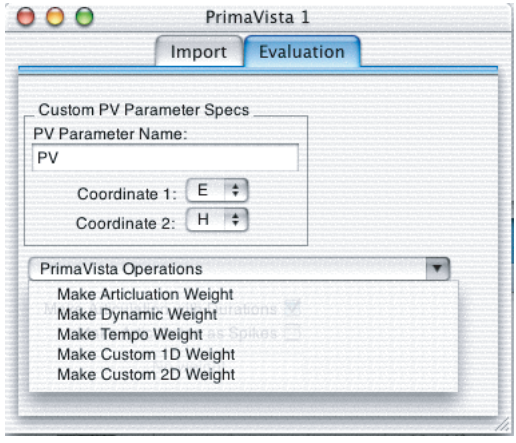


Figure 41.23: The main window of the PrimavistaRUBETTE® manages the transformation of verbal (paratextual) predicates into weights.

# Chapter 42

# Performance Experiments

*Learning by Doing.*

**Summary.** This chapter traces the analyses and syntheses processes which led to the historically first full-fledged RUBATO®-driven performance in July 1996 on the MIDI-Boesendorfer at the Staatliche Hochschule für Musik in Karlsruhe, as well as to the qualitatively high performance of contrapunctus III in Bach's *Kunst der Fuge*. We report the technical prerequisites, the analytical background generated by RUBATO®, and the step-by-step realization of the stemma and the overall parametrization.

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## 42.1 A Preliminary Experiment: Robert Schumann's "Kuriose Geschichte"

*Nicht eine Geschichte wird hier erzählt,*
*sondern der Eindruck, den eine solche*
*bei einem Zuhörer weckt,*
*wird charakterisiert.*
Thomas Koenig [267]

The first realistic experiment with RUBATO® took place in July 1996 on the MIDI-Boesendorfer grand piano at the Staatliche Hochschule für Musik in Karlsruhe. The experiment lasted three days and was led by the author, together with his assistant Oliver Zahorka and the musicologist Joachim Stange-Elbe. The experiment was executed on a NeXTStation. The results were digitally recorded on DAT and everything was protocolled. The technical support for the MIDI-Boesendorfer was offered by Sabine Schäfer. In what follows, we want to give a very sketchy account of that experiment, however recalling the essentials of initial experiments in performance theory and the consequences thereof. The more elaborate discussion of such experiments is left to the following section 42.2.

The preparatory work for this experiment was above all the analytical part, i.e., the metrical, melodic, and harmonic analyses on the respective Rubettes. We also experimented on the stemmatic and operator strategy, including the WeightWatcher mixtures as discussed in section 41.4. The selection of the MIDI-Boesendorfer was also a consequence of preliminary experiments on the MIDI-Yamaha grand insofar as this instrument turned out to respond in a much too coarse way to the input. In contrast, the Boesendorfer has a refined calibrating interface allowing a realistic MIDI input (with roughly a thousand dynamical values instead of 128 from MIDI), in particular for soft dynamics, where the Yamaha grand is completely inappropriate.

The first day was used to adapt the different technical conditions, such as the calibration of the Boesendorfer, the weight ranges, and the dynamical limits. The second day was devoted to the evaluation of our preparatory material and strategy. The third day was devoted to the production of all stemmatic levels, as well as their recording up to the final output level.

The most significant experiences were twofold: On the one hand, the judgment of the three experts concerning which weight mixture to use at which stemmatic knot, and the way to use it, with all the variables from range to inversion/non-inversion and deformation, was very precious! We learned that the effects of such strategies were almost constantly judged in three different ways—except for trivial failure or success situations. This means that performative coherence is a very personal affair, even when using very explicit techniques, analyses, and shaping tools.

On the other hand, we had to face a very disagreeable side effect of such an extensive performance shaping: Let me call it *the zoom of supersensitivity*. The effect is this: After having listened to performances of a selected part of the piece, the ear begins to recognize a steadily refined differentiation in the different parameters, such as agogics, dynamics, and articulation. For example, if in the beginning phase, one only recognized a huge change of a weight influence to agogics, with progressing trials, one would believe that the slightest further change in weight influence could change the performance of tempo in an unsupportable way. Maybe one was eventually incapable of really identifying the details of a performance, i.e., it is either simply too complex to be grasped by humans because the attention to different parameters is always selective, or it is not possible to memorize the samples and to compare them, or—and this is the worst case—our perception really changes so much form case to case that we cannot rely on the individual samples. The latter would mean that performance is greatly co-determined by the performance of the perceptive system.

The result was nonetheless acceptable and strongly differs from the so-called dead-pan version, see [360, CD attachment], or this book's CD-ROM (cf. page xxx), where the performance and the dead-pan version can be inspected.

## 42.2 Full Experiment: J.S. Bach's "Kunst der Fuge"

*Die Kunst der Fuge erschien wohl etliche Monate nach Bachs Tode und kostete vier Taler. Sie fand keinen Absatz. (...) Enttäuscht verkaufte der Sohn (Emanuel) die Platten, auf denen das letzte Werk seines Vaters geätzt war, um den Metallwert. Das war das Schicksal der Kunst der Fuge.*

Albert Schweizer [484]

This experiment is fully accounted in [504]. Here, we give a concise presentation, the version of RUBATO® used in this experiment is the one compiled for OPENSTEP/Intel. The "contrapunctus III" in Bach's *Kunst der Fuge* has these characteristics: It is a four-voice composition, comprises 72 bars, has time signature 4/4, and tonality *d*-minor. The main theme of *Kunst der Fuge* is only used in its inversion and appears the first time in a rhythmically dotted and syncopated variant; the fugue starts with the theme in its comes shape and contains three complete developments (bars 1-19, 23-47, and 51-67).

## 42.3 Analysis

**Summary.** We give an account of the metrical and melodic analyses, whereas the harmonic analysis has not been done.

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### 42.3.1 Metric Analysis

For the metric analysis of the "contrapunctus III", the calculations were made for each single voice, including the sum of the voice weights, and for the union of all voices. Please, refer to the discussion of the MetroRUBETTE® in section 41.1 for the following discussion.

The settings of the weight parameters are these: *Metrical Profile* is 2; *Quantization* is 1/16; *Distributor Value* is 1. Since the metrical profile of all voices should be viewed under the same valuation, the *Distributor Value* was set to a common neutral value; the value 2 for the *Metrical Profile* resulted from several trials of analyses and yields a well-ordered distribution of the weight profile.

The value for *Minimal Length of Local Meters* was successively decremented starting from the length of the largest local meter, descending until value 2 where the smallest cells are caught in their signification for the metrical overall image.

When considering single local meters, it is above all the onset time and the step size of the single time grids[1], which matters. Based upon the summation formula, this can however not be fully understood since from a certain superposition of periodical onset sequences, the observation of a single local meter becomes very difficult if not impossible (see also the consideration of the union of all onset times).

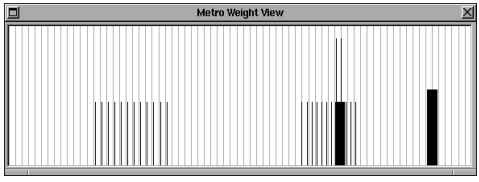
#### 42.3.1.1 Single Voices

Like all other contrapuncti, "contrapunctus III" shows no regularities in the compositional structure, in the repetition of whole parts or single bars. At least for the four bars of the theme, the onsets of themes in the developments of fugues do however create structural incisions, which only become relevant for the motivic analysis—the metric analysis remains unaffected. As to the analysis of single voices, it is interesting to observe that the longest found local meters nearly exclusively relate to relatively short parts, including only a few bars. These parts usually consist of uninterruptedly pulsating sequences of quavers or semiquavers, which in their regular sequence of notes differ significantly from the otherwise pronounced principle of tying stressed

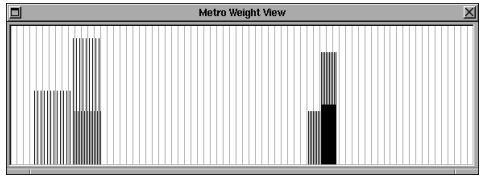
[1] In this discussion, "grid" is synonymous with "local meter".

bar onsets to unstressed precursors. This principle, having its origin in the shaping of the main theme with its sequences of quavers that lead to the interluding sections, can be recognized as a rhythmical pattern in the interludes and the countersubjects of the theme. By the tying of this sequence of quavers with the preceding half note, the theme breaks the hitherto confirmed bar accent scheme, a peculiarity which will be further confirmed in the sequel of this contrapunctus. It even has effects on the theme onsets which—unusual in a 4/4-bar—are partly shifted by a quarter note (see also the theme variant which is extended by dots and quaver transitions in the bars 23, 29, and 35).

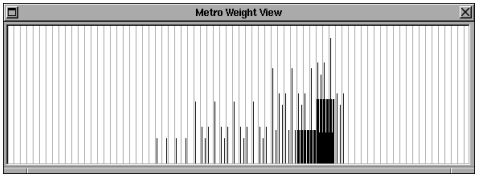
When considering the single voices, one realizes that most of the longest grids (i.e., those with highest weights, according to our choice of system parameters) catch exclusively onsets on unstressed bar times in the middle voices. If the initial value lies on a stressed bar time, a dotted duration value as a step length for local meters also weights more onsets on unstressed times. Only the metrical weight of the bass voice acts in a contrary way, in fact an increasing confirmation of the bar meter is observed, see figure 42.1.



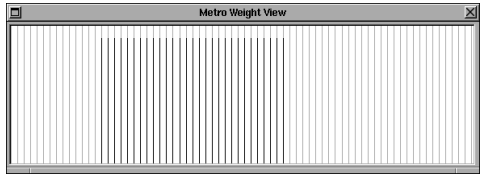

*Contrapunctus III (Soprano): metrical weight (Min. Length of Local Meter: 11)*




*Contrapunctus III (Tenor): metrical weight (Min. Length of Local Meter: 17)*




*Contrapunctus III (Alto): metrical weight (Min. Length of Local Meter: 19)*




*Contrapunctus III (Bass): metrical weight (Min. Length of Local Meter: 28)*

Figure 42.1: *Contrapunctus III*: metrical weights for the four voices.

#### 42.3.1.2 Weight Sums of All Voices

When considering the weights of single voices, and in particular the sum of their weights, observe that the longest weights are the most important contributions. This means that the most important weights stem from the bass, followed by the alto, the tenor, and then the soprano voice. The latter has a weakened contribution since in the middle of the contrapunctus, there is a pause of thirteen bars, which breaks down the coherence of the soprano to the metrically weakest voice, see figure 42.2 for the lengths of local meters for the four voices. Further, we observe that the bass voice prevails against the other voices in the confirmation of the bar

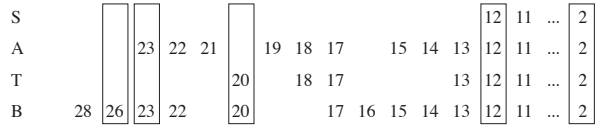



Figure 42.2: The lengths of maximal local meters for the four voices.

meter since it creates high metrical weights by the longest metrical grids. Nearly all peaks of the metrical weight sum—except some cases in the middle part—coincide with stressed bar times.

Generally speaking, the ambitus of the metrical weight profile increases towards the middle of the contrapunctus—an observation which is valid for all pieces of the "Kunst der Fuge" which we have analyzed, see also figure 42.3. The reasons for weak initial metrical profile lies in the formal construction of a fugue, where the voices do not appear simultaneously, but one after another, such that the full voicing only appears after a number of bars. Another reason can be seen in the structure of the theme of the fugue, which does not develop its full motional impulse before the last five notes. These are used to shape—until the next thematic onset—counter-subjects and interludes. The equally very frequent decrease of the metrical profile towards the

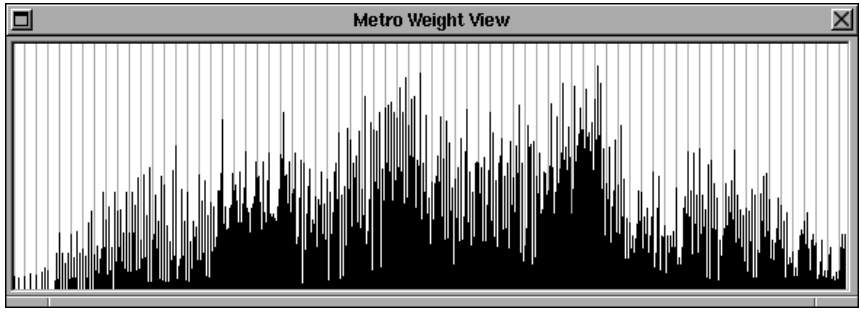



Figure 42.3: The ambitus of the metrical weight profile increases towards the middle of the contrapunctus; minimal local meter length is 2.

end of a piece is explained by different formal facts. In the case of the "contrapunctus III", this is due to long pauses of the single voices—in particular bass, soprano, and tenor—as well as the pedal point in the bass voice (bars 68-71).

#### 42.3.1.3 Union of All Voices

Relating to the union of all onsets, the "contrapunctus III" shows at first sight dotted step lengths in metrical grids. However, by the two longest grids (*Minimal length of Local Meters* being 502 and 271), starting at the final notes of the exposition of the theme in the tenor and at

the end of the exposition of the theme in the alto, an uninterrupted pulse of quavers and quarter notes is established, and the only remaining grid (*Minimal length of Local Meters* is 128) with step length a half note also drops at the non-stressed bar times of $4^{th}$ and $8^{th}$ quaver. All other grids have step lengths of different dotted durations, however dominated by the doubly dotted half note, the said dotted quarter note, and the half note with tied quaver, as well as once the dotted half note. What is remarkable in this contrapunctus is the three 'double grids' of

| Length of metrical grid | Start of metrical grid (bar, note) | | Step length | End of metrical grid (bar, note) | |
|---|---|---|---|---|---|
| 502 | 9 | 4th Quarter | 1/8 | 72 | 3rd Quarter |
| 271 | 4 | 4th Quarter | 1/4 | 72 | 3rd Quarter |
| 172 | 8 | 1st Quarter | 3/8 | 72 | 3rd Quarter |
| 171 | 8 | 2nd Quarter | 3/8 | 72 | 4th Quarter |
| 128 | 8 | 4th Quarter | 1/2 | 72 | 4th Quarter |
| 108 | 4 | 8th Quarter | 5/8 | 72 | 4th Quarter |
| 104 | 7 | 3rd Quarter | 5/8 | 72 | 3rd Quarter |
| 102 | 8 | 2nd Quarter | 3/8 | 72 | 1st Quarter |
| | 8 | 3rd Quarter | 3/8 | 72 | 2nd Quarter |
| 91 | 4 | 1st Quarter | 3/4 | 72 | 2nd Quarter |
| 78 | 3 | 4th Quarter | 7/8 | 72 | 1st Quarter |
| 75 | 6 | 3rd Quarter | 7/8 | 72 | 2nd Quarter |
| 74 | 7 | 3rd Quarter | 7/8 | 72 | 2nd Quarter |
| | 7 | 4th Quarter | 7/8 | 72 | 3rd Quarter |
| 73 | 8 | 1st Quarter | 7/8 | 71 | 8th Quarter |
| | 8 | 3rd Quarter | 7/8 | 72 | 4th Quarter |
| 71 | 1 | 1st Quarter | 4/4 | 72 | 1st Quarter |

Figure 42.4: Maximal local meters for the union of all four voices.

*Minimal length of Local Meters* equal to 102, 74, and 73, which within the same bar are shifted to each other by a quarter note, and two quarter notes in the last case.

Therefore, in the *contrapunctus III*, the tendency of 'metrical instability', which was already encountered in the single voices, persists, since by summing up the weight contributions by the grids with dotted step width, a metrical profile is established which breaks the bar meter as well as the pulse of uninterrupted quavers and quarters.

### 42.3.2 Motif Analysis

For the calculation of motivic weights each single voice of the "contrapunctus III" was analyzed separately. We refrained from a motivic analysis of the union of all voices since by the contrapuntal structure of the single and autonomous voices within the polyphonic setting, a motivic setup across the voices seems rather unlikely and therefore was omitted.

The settings for the motivic analysis were chosen as follows: *Symmetry Group*: counterpoint; *Gestalt Paradigm*: elastic; *Neighborhood*: 0.2. By the choice of the counterpoint symmetry group, the theme forms *recta* and *inversa*, as well as their (possibly appearing) retrogrades, were considered as being of equal weight. The neighborhood value has been chosen as based upon analytical experiments during the development period of RUBATO®. The elastic gestalt paradigm was preferred against the rigid and diastematic ones in order to obtain a more elastic point of view.

As to the values for *Motif Limits*, compromises with the calculation power had to be made. By the choice of *Span* equal to 0.625 and *Cardinality* from 2 to 7, motives within a span of a half note plus a quaver were captured; this corresponds exactly to the duration of the theme where the transition of the virtual theme to the interludes must be recognized.

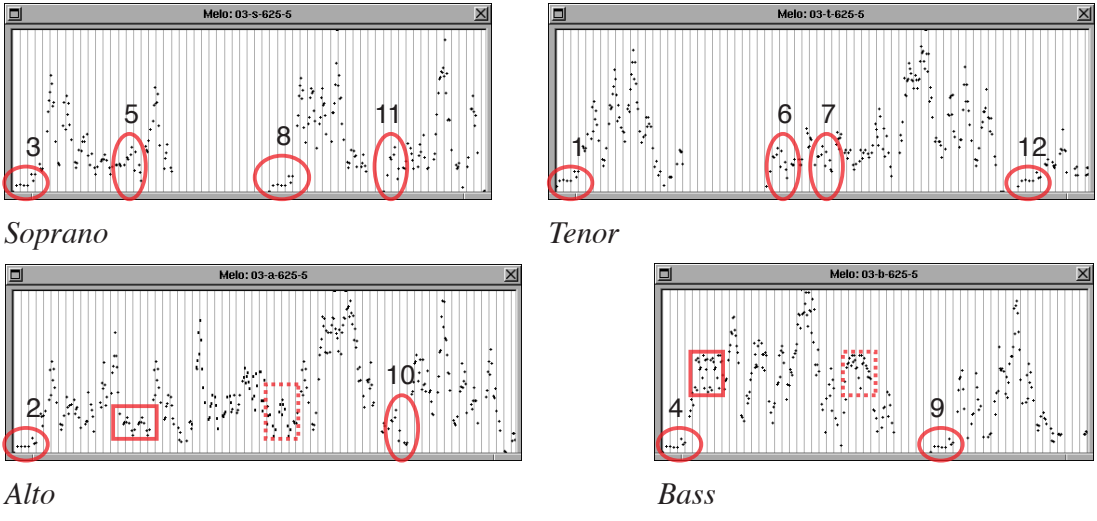



Figure 42.5: Motivic weights for "contrapunctus III".

With the results of the metrical analysis, some regularities in the microstructures can be read at first sight; herein we find in particular the onsets of the theme within a particular development. The representations of the single motivic weights in each voice with a mean value of 5 for the *Cardinality* may be elucidated in detail, see figure 42.5. Within these graphics, the theme onsets are numbered according to their temporal order; equal weights can be recognized on the onsets 1,2,3,4,8,9,12, and 5,6,7, as well as on 10 and 11.

While further considering these weights, the overly long pauses in the soprano, tenor,

and bass voices attract attention. Further, in the length proportion of the single weight representations, the succession of onsets of the single voices (tenor-alto-soprano-bass) is reflected. Moreover, a significantly lower motivic profile at the beginning and after the longer pauses of the respective weights can be observed—due to preceding pauses, this is the case of exposed thematic onsets.

For the weight values, a neat exposition of the inverted gestalt of the original theme is observed, bearing nearly identical weights at the beginning of every motivic weight, here even the differences of comes and dux forms are visible, since the weights of the tenor (first appearance) and soprano (third appearance) differ slightly by the different initial interval of the theme (descending fourth in the comes, and descending fifth in the dux form) from the weights of the alto (second appearance) and bass (fourth appearance), see figure 42.6.

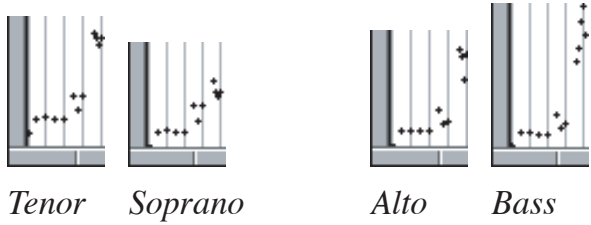



Figure 42.6: Motivic weights for "contrapunctus III", bars 1-4.

Other clearly visible onsets of the theme in inverted shape are recognized after the long pauses in the soprano (eighth appearance), bass (ninth appearance), and tenor (twelfth appearance). Characteristically, the inverted shape always appears after pauses.

At first sight these observations may seem to be tautological. However, if these weights are viewed with respect to their sense and purpose, their force to shape performance, then the transition from a quantitative to a qualitative information content becomes evident: Thus the different onsets of themes can be shaped by these weights in one and the same way; if these weights are used—in inverted form—for the dynamic shaping, then the thematic onsets can be stressed with plasticity.

When trying to evaluate motivic data more in detail, a problem appears that is inherently founded in RUBATO®'s intended purpose: since the platform was originally conceived for performance research, it is sufficient to present the analytical material for its performance shaping and to ascertain its reasonable construction. But these analyses also yield information about the musical structure, which in the case of metrical analysis—by a certain extra effort apart from RUBATO®—can be traced without difficulties. However, for the MeloRUBETTE®, any practical support beyond the brute calculation of weights—which is also sufficient for performance—is absent; the "introspection" of the details of weight calculation is hidden to the user. Thus for the calculation a motif's weight it cannot be known which other motives are responsible for its *presence* and *content*. Although all motives are arranged in the motif browser in their temporal order, but another structured introspection of motives, for example with regard to their weights, is not possible for this version of RUBATO®. This makes the introspection and evaluation of

the 'heaviest' motives a difficult task. All analytical information is present, but it is hidden; a particular difficulty may be seen in the exuberant effort in complexity which a detailed motivic analysis, which extracts knowledge from the score in an immanent way, is loaded.

### 42.3.3 Omission of Harmonic Analysis

A harmonic analysis was omitted in this situation since—to our mind[2]—the approach of Hugo Riemann which is implemented in the HarmoRUBETTE® is not really suited for Bach's harmonies. By use of the Riemann theory which was developed from the Viennnes classics, the specific harmonic structures of a contrapuntal maze, where harmony does not result from progression of fundamental chords but from the linearly composed voices, can only be captured in an incomplete way.

## 42.4 Stemma Constructions

**Summary.** This section discusses the performance construction via the stemmatic paradigm, and using the analytical results investigated in section 42.3.

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Before the stemmatic construction and thereby in particular the problem of performing "contrapunctus III" are discussed, some general remarks regarding the various performance strategies are necessary. In the course of the single performance parcours, two different approaches resulted which would turn the given analytical weights into expressivity: the *target-driven* and the *experimental* strategy.

The *target-driven strategy* has its roots in the knowledge about existing performances, it is stamped by a preliminary experience of how the piece should sound and has been performed. With this procedure, the weights are used in a way which targets a predefined performance. One—just to name a pithy example—was oriented towards Glenn Gould's Bach interpretation; the corresponding weights were selected according to these targets to obtain particular effects. In this procedure, however, the intrinsic structural meaning of analytical weights was ignored! Stamped by the knowledge and the expectation of the existing performances, this strategy did not allow one to judge and categorize those performance constructions which did not suffice for the music-esthetic exigencies.

The other approach, the *experimental strategy*, moves the analytical weight to the center in order to investigate how this weight could 'sound', and which analytical insight it could convey in the listening. With this procedure, which views the main performing agent entirely within the weight, one has to free oneself completely from horizons of expectation for any particular performance target. The working process on such performances, the acquaintance of experience with the most different weights, and the playing with their effects taught us in the course of many experiments that this strategy would give rise to much more interesting performance aspects. Here we also have the freedom to admit extremal positions which disclose more about the inherent musical structure and as 'daring ingredients' may evoke lively musical expression.

[2] Decision by Joachim Stange-Elbe.

Moreover, the experimental approach to single performance aspects, which starts from curiosity about the sonic realization of analytical weights, conveys a deeper insight into to score's musical structure. This path has its take-off in a "sonic analysis", or else in "the sonic analytical structure" and aims at a "musically reasonable performance". It is centered around the researcher's curiosity for a sounding and interpretational realization of analytical weights and for "the never heard", and it is paralleled by a liberation from expectational presets. Moreover, this strategy tries to apply as few weights as possible in order to couple the clearest possible analytical statements with the resulting performance.

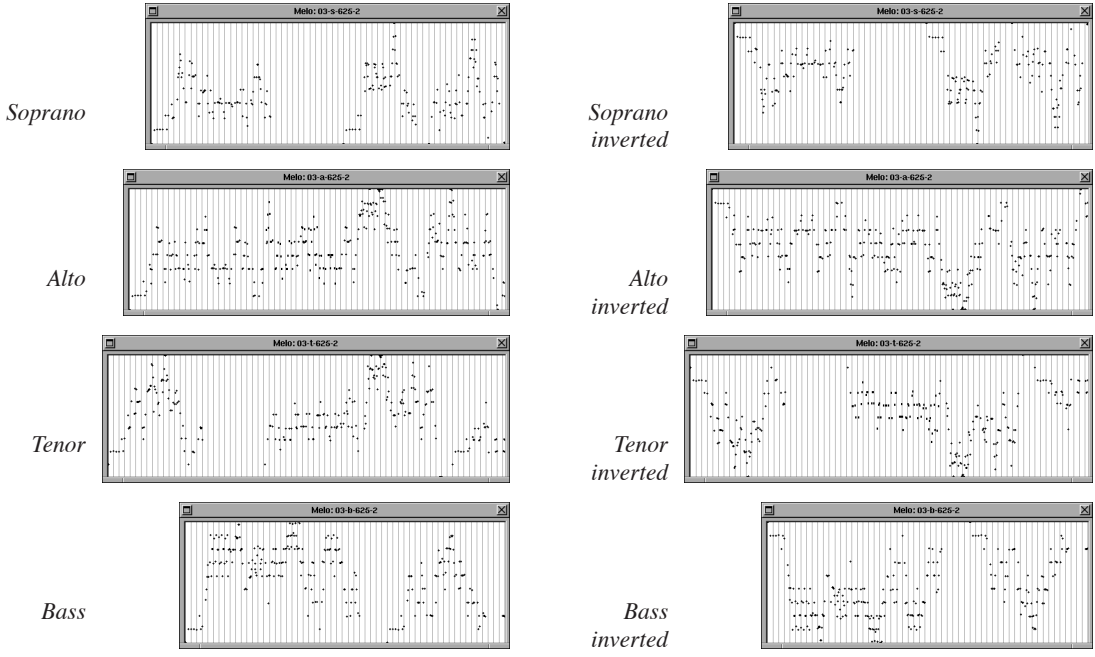


Figure 42.7: Metrical weights for "contrapunctus III".

### 42.4.1 Performance Setup

**Summary.** We discuss the detailed performance construction.

– Σ –

The performance of "contrapunctus III" took place in three parcours, out of which we only report the last two in more detail. The first one was entirely devoted to a target-driven strategy whereas the subsequent ones switched to an experimental strategy which yielded much more successful and conclusive results. Nonetheless, all these approaches contributed results that influenced the final result in a significant way.

Generally speaking, the procedure in all these parcours first focused on isolated single aspects of performance (articulation, dynamics, agogics) and then were put together for the final parcours. For the complete description of all these steps, see [504].

#### 42.4.1.1 Results From First Performance Parcours

As a result of the first parcours which was executed under the paradigm of a target-driven strategy, the usage of motivic weights for a determined shaping of dynamics has been recognized. Using the inverted form of weights in the WeightWatcher for each voice, the thematic onsets for each development of the fugues could be modeled in an excellent way. Simultaneously however, the global usage of these weights lead to a completely disequilibrated passage so that this type of application was eliminated in the subsequent performance construction.

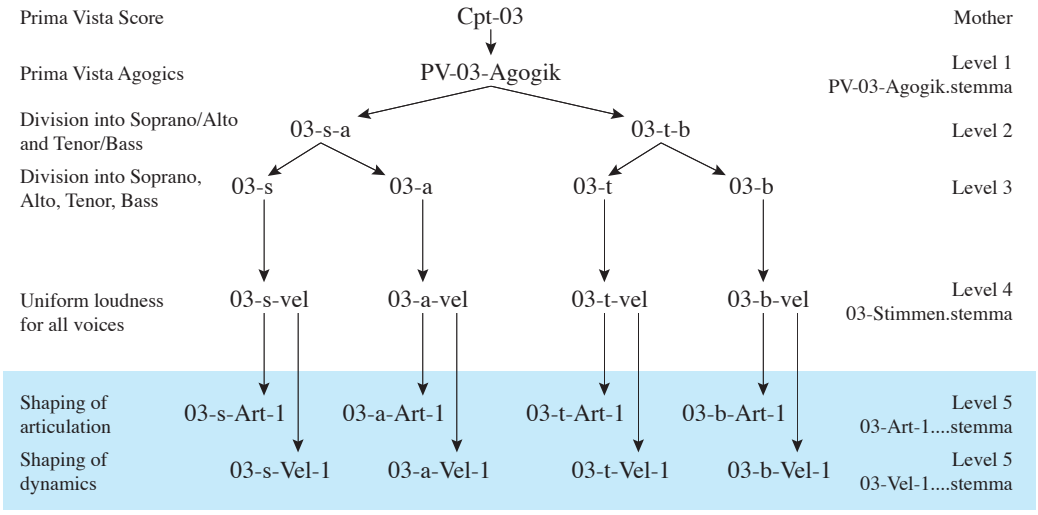


Figure 42.8: Stemma construction for the second parcours.

#### 42.4.1.2 Construction of Second Performance Parcours

In the second parcours, whose rather simple stemma (see figure 42.8) we are going to discuss hereafter, we changed to the experimental strategy, with the aim to investigate to which degree motivic weights can contribute to the elaboration of a determined performance aspect. Those motivic weights of a voice were used which cover a minimal number of motives (Cardinality: 2) within a dotted half note (Span: 0.625); all these weights were applied in inverted form.

As with the subsequent third performance parcours, the core shaping work by means of weights (grey shaded levels in figure 42.8) must be preceded by further steps. To begin with, this concerns the shaping of the score's primavista components, such as dynamic, articulatory,

and agogical prescriptions. In the case of *contrapunctus III*, which contains no performance prescription by the author, this task consisted only in the primavista shaping of a short ritardando before the junction to the final tonical chord (Level 1).

The next steps within the stemma (levels 2 to 4) relate to the horizontal separation into the four voices (by the SplitOperator), which were identified by specific (artificial!) different loudness values (Levels 2 and 3), and which in the fourth step are reset to a uniform loudness. Upon this basis, an attempt to shape articulation and dynamics was undertaken.

In the course of six subsequent performance experiments it turned out that the global application of a single weight was capable of producing three slightly different variants of articulation shaping, however one had to pay attention to change the parameter for the weights' influence (in the WeightWatcher) in a minimal and systematic way. Figure 42.9 shows the final choice. Built upon these insights we tried to use the same motivic weights, the same systematic

| | High Norm | Low Norm | Influence | Deformation | Invert |
|---|---|---|---|---|---|
| Soprano | 1.6 | 0.2 | 1 | -0.75 | YES |
| Alto | 1.5 | 0.3 | 1 | -0.75 | YES |
| Tenor | 1.5 | 0.4 | 1 | -0.5 | YES |
| Bass | 1.4 | 0.6 | 1 | -1 | YES |

Figure 42.9: Choice of WeightWatcher parameters for motivic weights.

and a similar handling of the change of intensities in order to produce comparable results for the shaping of dynamics. Without going into detail, it remains to be stated as a result that these attempts all failed. Typically, the dynamical relations between the voices turned out to be disequilibrated, and although the dynamics was perfectly modeled within single voices, the dynamical profile of thematic parts was sensibly worse. To obtain conclusive dynamics, one had to find another shaping procedure.

#### 42.4.1.3 Construction of Third Performance Parcours

Because of these different dynamical profiles, the principle of former performance experiments—the exclusive usage of a weight and its global extension—had to be given up. In a first step it was recommended to split the single voices at appropriate locations, and in a second step, a regress to the metrical weights already used in the first parcours and their renewed application under other viewpoints (a mixed usage together with motivic weights) seemed reasonable. The shaping of articulation from the second parcours would be conserved.

In a preliminary step, a division of the single voices had to be executed. To this end, one had to find structurally legitimate points from the musical context, such as articulation by harmonic incisions or thematic groupings for developments and interludes.

The first division of all four voices took place in bar 39, legitimated by a harmonic close to the major parallel of the minor dominant (C-major); at the same time this is viewed as a possible ending of the second (however incomplete) development and a beginning of a four-bar interlude.

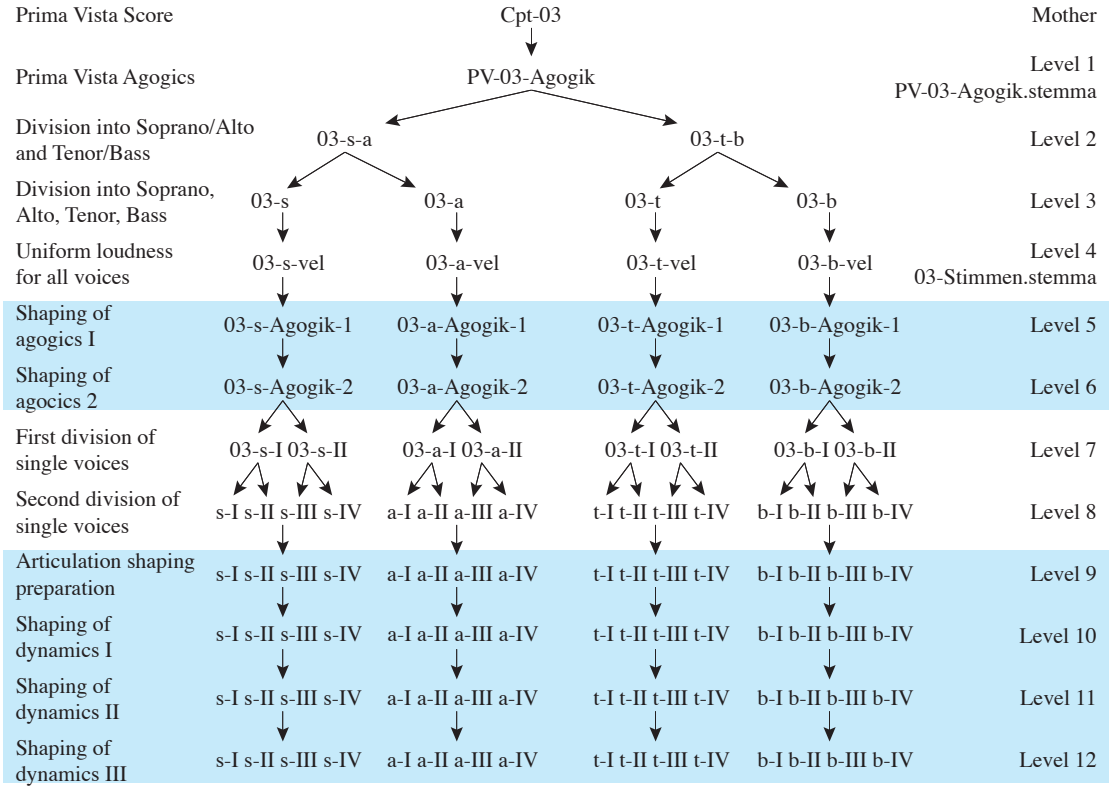


Figure 42.10: The stemma of the third parcours.

In order to equilibrate the dynamical unbalances relating to the interludes from bars 19 and 46, a further division of the two halves of the fugue were necessary. A division of the first half was recommended in bar 19, having a close of the first development (exposition of fugue) and its half close on the dominant (A major).

Because of the too strong dynamic sink of the three-voice interlude from bar 46/47, the division of the second half had to take place not later than at this point. This division was legitimized by the half close on the minor dominant (A-minor) beginning in bar 46 on the one hand, and the simultaneous ending of the second (then complete) development according to the three-part construction of the fugue.

For the subsequent performance shaping, consider figure 42.10. Besides the already known preparatory steps—horizontal division into single voices (Level 3) and equalizing of loudness (Level 4)—two performance steps for the later shaping of global agogics were inserted (Levels 5 and 6). This trick is applied, because agogics needs long calculation time on the global level of single voices and should be calculated after the stemmatically subsequent shaping articulation

and dynamics. The vertical division of the single voices is applied in the previously described steps (Level 7 and 8). For the subsequent shaping of articulation and dynamics, each voice had to receive its separate and individual performance shaping for the four sections. This enabled us to apply different parameter values for the intensity effects, one per used weight.

For the shaping of articulation, the three already elaborated performance steps were inherited.

As is seen in the stemma (figure 42.10), the shaping of dynamics was realized in three consecutive steps. Here, besides the known motivic weights, two additional metrical weights were applied. For the first step (Level 10), we applied the metrical weight from the union of all voices with *Minimal Length of Local Meters* equal to 2, in inverted form, and without deformation, see figure 42.11.

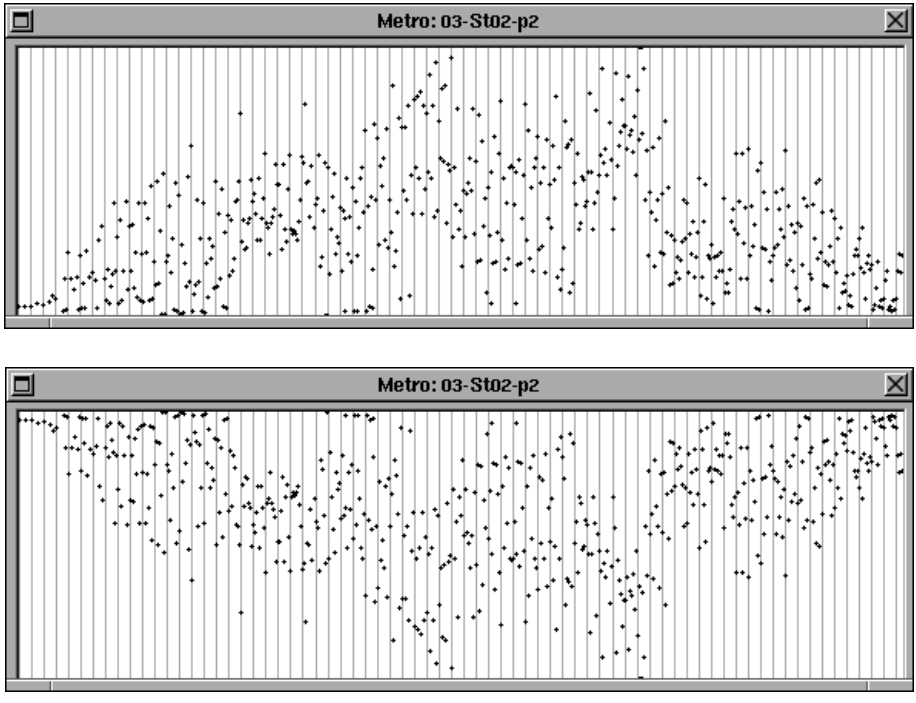


Figure 42.11: Metrical weights in "contrapunctus III", union of all voices, *Minimal Length of Local Meters* equal to 2, in original form (top), and in inverted form (bottom).

Upon this stemma, the second step (Level 11) applied the metrical weights with value 5 for *Minimal Length of Local Meters* for each individual voice in inverted form and also without deformation (the weight graphics were comparable to those from the second performance parcours described above).

For the concluding shaping of dynamics, the already known motivic weights were applied

in order to give the thematic onsets a plastic relief. From the interplay of the various intensity values, we got the constellation documented in figure 42.12. The result of this performance

| | **Soprano** | | | | **Alto** | | | | **Tenor** | | | | **Bass** | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Part 1 | Part 2 | Part 3 | Part 4 | Part 1 | Part 2 | Part 3 | Part 4 | Part 1 | Part 2 | Part 3 | Part 4 | Part 1 | Part 2 | Part 3 | Part 4 |
| **Level 10** | Metro: St02-p2 | | | | Metro: St02-p2 | | | | Metro: St02-p2 | | | | Metro: St02-p2 | | | |
| Weight | | | | | | | | | | | | | | | | |
| High Norm | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 |
| Low Norm | 0.6 | 0.6 | 0.6 | 0.8 | 0.6 | 0.6 | 0.6 | 0.8 | 0.6 | 0.6 | 0.6 | 0.8 | 0.6 | 0.6 | 0.6 | 0.8 |
| Influence | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Deformation | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Invert | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y |
| **Level 11** | Metro: S05-p2 | | | | Metro: A05-p2 | | | | Metro: T05-p2 | | | | Metro: B05-p2 | | | |
| Weight | | | | | | | | | | | | | | | | |
| High Norm | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| Low Norm | 0.7 | 0.7 | 0.7 | 0.8 | 0.7 | 0.7 | 0.7 | 0.8 | 0.7 | 0.7 | 0.7 | 0.8 | 0.7 | 0.7 | 0.7 | 0.8 |
| Influence | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Deformation | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Invert | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y |
| **Level 12** | Melo: s-625-2 | | | | Melo: a-625-2 | | | | Melo: t-625-2 | | | | Melo: b-625-2 | | | |
| Weight | | | | | | | | | | | | | | | | |
| High Norm | 1.0 | 1.2 | 1.4 | 1.2 | 1.0 | 1.2 | 1.05 | 1.2 | 1.0 | 1.4 | 1.2 | 1.2 | 1.3 | 1.7 | 1.3 | 1.3 |
| Low Norm | 0.5 | 0.4 | 0.4 | 0.5 | 0.7 | 0.7 | 0.7 | 0.7 | 0.75 | 0.7 | 0.7 | 0.75 | 0.7 | 0.8 | 0.8 | 0.7 |
| Influence | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Deformation | 0.5 | 0.5 | 0.5 | 0.5 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | -0.25 | -0.25 | -0.25 | -0.25 |
| Invert | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y |

Figure 42.12: Intensity values for the concluding shaping of dynamics.

communicates a relatively balanced dynamics, spread over the whole contrapunctus, the thematic onsets gain a profile, which can also be confirmed in the slight crescendo that leads to the beginning of the third development after the three-voiced interlude (from bar 46/47).

Bringing together the dynamic and the already elaborated articulatory aspects, the result can be stated as a complementary shaping of both performance aspects, which on top of that reveals a musical sense in the elaboration of thematic onsets and the three-voiced passages of the interludes.

For the shaping of agogics, the said levels 5 and 6 of our stemma were reserved. We did two different subsequent performance parcours with two different metrical weights: the sum of all voice weights (*Minimal Length of Local Meters*: 2) and the weight of the voice union (*Minimal Length of Local Meters*: 91 (!)), see figure 42.13. The intensity values are shown in figure 42.14.

As a result we may state that according to the weight structure and a minimal intensity effect a slight increase of the basic tempo towards the middle of the piece happens, but it is

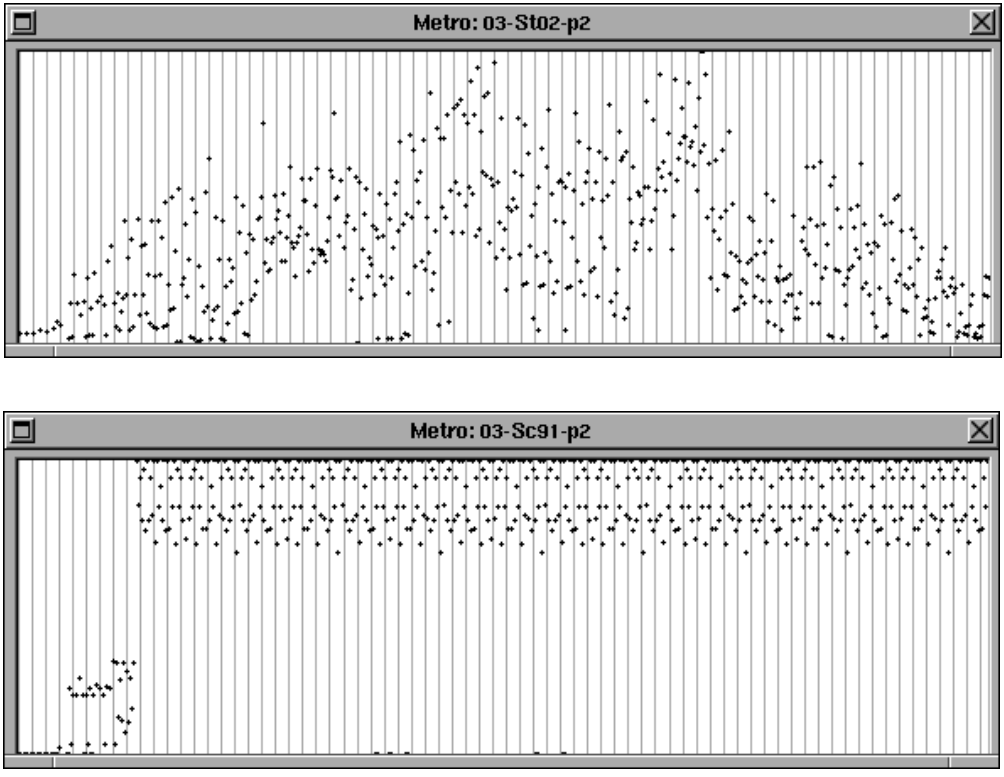


Figure 42.13: "Contrapunctus III"; top: metrical weights, sum of all voice weights (*Minimal Length of Local Meters*: 2); bottom: weight of voice union (*Minimal Length of Local Meters*: 91).

balanced by a slightly slower tempo for the initial and final bars. For the resulting performance as traced on the audio track on the CD-ROM attached to this book, see page xxx. The corresponding stemmata are also found on this location of the CD-ROM.

#### 42.4.1.4 Local Discussion

Starting from the experimental strategy,the performance of "contrapunctus III" was first centered around the shaping with a single weight in order to sound the potential of a single weight. The shaping of articulation showed a reasonable local profile for every voice, i.e., a single analytical structure was capable of giving one performance aspect a reasonable expression.

The global application of weights and the usage of a single weight showed its limits, as we have learned from the dynamical shaping of "contrapunctus III". For example, the global application of weights failed in the different grades between the contributions of the four voices.

| | High Norm | Low Norm | Influence | Deformation | Invert |
|---|---|---|---|---|---|
| Total Contrapunctus | 1.025 | 0.975 | 1 | 0 | NO |
| Total Contrapunctus | 1.05 | 0.95 | 1 | 0 | NO |

Figure 42.14: Intensity values for the metrical weights.

Especially with the motivic weights of the tenor and bass voices, different weight profiles become visible which cannot be eliminated even by suitable deformations. These differing profiles of weights do result from the compositional structure. As this one splits into a number of parts—developments and interludes, groupings by harmonic closes and semi-closes—the division of the voices according to such compositional criteria is legitimized. Within these parts, the selected weights can be applied with different intensities and thusly equalize the disparate shapings.

Dynamics received a special significance in the shaping process: the elaboration of thematic onsets by an inverted motivic weight—this was justified in the structure of the theme. More precisely, the long durations of the initial notes resulted in a weak weight profile, which could be used to stress these incipits by the weight's inversion. This principle which remains valid for almost all contrapunti, can however only be proved for the "Kunst der Fuge".

### 42.4.2 Instrumental Setup

**Summary.** The conditions and influence of the instrumental setup are discussed.

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A discussion of adequate instrumentation must be in the forefront of virtual performance work, since the behavior of the respective sound generator is an essential basis for shaping of the single musical parameters. Since the performance results from RUBATO® are encoded in a MIDI file, we have the possibility to access a MIDI-driven acoustical piano or else to use a corresponding digital device. Here, we should observe some relevant differences which—besides the basic difference in sound—the repetition, the dynamic response, the resonance behavior, and the spatial environment which we cannot discuss here. These differences are not only present between the acoustical and the digital instruments, they also act within each category.

For pianos and grand pianos, the MIDI-driven access is offered by Boesendorfer and Yamaha models. At the time of our experiments, the comparison between these two brands could a priori not be made in a serious way. Even if the shaping of dynamics admits a limited bandwidth of variation within the 128 MIDI velocity values, the *piano* and *pianissimo* ranges (velocities between roughly 30 and 1) for the Boesendorfer were much finer to tune than for the Yamaha piano and grand piano, where these dynamical values make the keys move silently. This defect as well as the extremal differences in the repetition mechanics show the futility of a comparison between such Boesendorfer and Yamaha instruments and imply that the Bösendorfer is the only reference for a reasonable performance. In this sense, the Boesendorfer was already chosen for the first experiment with Schumann's "Kuriose Geschichte" as briefly reported in section 42.1.

In principle, an acoustical piano should be chosen as a reference for computer-assisted performance, but because of restricted availability of such instruments[3], a digital piano had to be selected in our case.

Our experiences with digital pianos and their sounds showed significantly different behavior for identical dynamics and articulation when applying identical weights. And it must be remarked that at the present state of performance work, it is difficult to obtain exact information about the dynamics and articulation shaping in the sense of a possibly reliable performance grammar since the judgment of the dynamical and articulatory aspects *as a function of the available instruments*, see [505] for a detailed presentation of sound experiments. Any conclusion regarding the sources and rationales for the performance shaping of a score are vitally influenced by these instrumental conditions.

In other words, within a strictly scientific framework, the MIDI-encoded performance data enforce an instrumental selection. If research can be realized by means of one and the same MIDI piano, comparative statements can be made exactly for this instrument.

In a somewhat broader sense, this is also true for digital maps of the acoustical piano, i.e., one has to restrict the research to one and the same digital piano, such as Kurzweil's *Micro Piano* as it was used in [504]. Any conclusion from the digital to the analog piano or vice versa is impossible, even among different digital or analog pianos no comparison is possible, see also [505].

One solution out of this dilemma could be the application of the "physical modeling" or "virtual acoustics" technology (see appendix A.1.2.4), where a direct access of the physical technology of an instrument by the virtualization (i.e., the software modeling) of the physical system of an instrument is enabled. This methodology offers flexible instruments which can be deformed seamlessly and can respond without delay. Although only first experiments have been initiated, this perspective opens an encompassing approach of computer-assisted performance research. The sound and resonance environment with its consequences for the shaping of performance aspects would not only yield new insights in the functioning of musical instruments, but also insights in the practice of instrumental playing. In this context, an interdisciplinary research team consisting of instrumentalists/interpreters, musicologists, computer scientists, mathematicians, and physicists would be required.

### 42.4.3 Global Discussion

**Summary.** We summarize the insight drawn from this second experiment.

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In the course of the performance experiments, two different approaches and performance strategies crystallized. We tried to give the score's text an immanent shaping by means of two approaches:

- what is the sound of the analytical structure?
- can the sounding analytical structure yield a musically reasonable performance?

[3] In Germany, the Bösendorfer MIDI grand exists only in two locations: at the conservatories in Lübeck and Karlsruhe. In both cases, access to this instrument is virtually impossible since it is located in rooms that are used for ordinary school activities.

and two contrary performance strategies:

- the target-driven strategy,
- the experimental strategy.

In contrast to objective analytical approaches, for performance, subjective ingredients cannot be completely eliminated. They are present in their feedback with the performance result, while weights and intensity parameters in the WeightWatcher are determined, but they play a fairly reduced role.

From the first performance experiments, which have not been discussed in detail here, until the complete performance as described above, we have known milestones which demonstrated several problematic issues: It was not easy to eliminate the impression of an existing performance—in our case by Glenn Gould, say—and to stick strictly to what is written in the score; the performed version of the piece automatically resonates as a comparison while doing the performance work. This was the situation where we started these experiments with the ambitious task of approaching an artistical and esthetical performance as far as possible.

Therefore, the *target-driven* strategy was to a certain degree determined by the comparison with traditional human performances. Under these conditions, weights were applied and results were judged. This turned the tradition into an obstruction, it positioned the expected performance in the foreground and the shaping weight in the background.

Only the consequent questioning of the analytical structure and the systematic liberation from traditional performance expectations led to a performance strategy which positioned the analytical weights in the center of the investigation. This *experimental* strategy was coined by an as unbiased as possible sounding realization of analytical structures, centered around the question of how a weight, when applied to a particular performance aspect, would sound. Within this procedure it was possible to insert 'unheard' results, to admit purposed over-subscriptions in the sense of the 'still more clear', whereas the question whether an interpreter would play in this way turned out to be completely irrelevant.

From this point of departure, how a determined analytical structure would sound, the experimental approach to shaping a musically reasonable performance was sought. This qualitative determination of what is a "musically reasonable" performance is inevitably a subjective one which as such decided upon the subsequent steps towards the final performance. Similarly to the interpreter who puts up for discussion his provisionally final version while performing in concert—where in the last analysis it is more his personality than the musical performance which is judged—in the case of computer-assisted performance, the subject who works with the performance workstation RUBATO® presents his results as a provisionally final contribution to the ongoing discussion.

When judging all these performances, one has to take into account that only metrical and motivic weights were applied and the effects of harmonic passages were not included in the shaping of performance (except of the motivations for the not machine-made subdivisions from global to more local applications of weights in the third parcours). Furthermore a certain economy in the choice of weights and their application was applied. In this sense, we first had to check out which weights would entail what type of shaping consequences, and how the change of intensity parameters would influence the musical expressivity. It was only after this preliminary work that a systematic application of the weights and a partially purposed work with their intensity parameters became possible.

The portability of the presently described performance technique must be deduced from the compositional structure (a fugue in general and the thematic structure of the "Kunst der Fuge" in particular) as well as from the instrumental context. In nuce it can be said that such systematic statements are still premature. Many more analyses and performances would be necessary, but these can only be realized as soon as RUBATO® has become a common tool of musicology. Then the question of whether general recipes which are valid beyond the limits of single compositions can be stated, or whether performance is rather bound to each individual composition, could be restated.

Whatever is true for the transformation of the analytical structure in a scientific work targeting an artistically valid esthetic performance, one should not forget about the elimination of (and nonetheless omnipresent) emotional and gestural aspects. The realization of a sonification of analytical structures during the interaction with the computer always bears a degree of emotionality, a phenomenon that should be taken into account as a kind of "uncertainty relation".

The judgment of the performance results took place in the same line as the judgment of a human performance, and the work with RUBATO® was also proposed as a provisionally final contribution to the work's discussion.

While describing the performance results, the stress of a scientific analytical performance was central. The feedback to the analysis has a particular significance in that possibly, the conclusive character of a performance could yield an analytical criterium. This implies an absolutely serious attitude towards analysis, and no disclosure from emergent new aspects and innovative analytical ways of hearing.

Therefore we refrain from a discussion of subjects such as "prejudices against results which are produced by a machine", or "performance and the soul of music versus soulless performance machines". Instead, we favor representations of procedures and performance strategies, the exemplary demonstration of connections between analyzed structures, performed results, and the attempt at a generalization of these insights in the form of a performance grammar in its dependency on the instrumental conditions.

# Part XI

# Statistics of Analysis and Performance

# Chapter 43

# Analysis of Analysis

*O sancta simplicitas!*
Jan Hus (1370–1415)

**Summary.** Not unexpectedly, weight analysis turns out to be complex information that cannot always be handled intuitively. This suggests techniques that help analyzing weight analysis. The problem may be tackled by use of statistical methods. We expose the subject and Jan Beran's approach based upon hierarchical decompositions of weights, together with an application to comparison of analyses of Bach, Schumann, and Webern.

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Although the MetroRUBETTE® seems to implement a very simple analysis, the metrical weights turn out to encode quite complex information. For performance applications such as RUBATO®, this may be acceptable, but for an analytical understanding per se, the weights are too complex to be used directly, except for direct visual inspection of evident surface properties. The prejudice that musical analysis should be a simple, intuitive affair, is therefore banned to the fairy tales of auto-incompetent humanities.

In this chapter, we give an account of a statistical approach to understanding weights. More specifically, Jan Beran's method of hierarchical smoothing (see also [50]) is presented.

## 43.1 Hierarchical Decomposition

**Summary.** This section describes and motivates the hierarchical decomposition of weights by use of decreasing sequences of time bandwidths, the so-called *hierarchical smoothing*.

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### 43.1.1 General Motivation

Can additional structural insight into weights be gained by suitable analysis of the analytic weight curves? The idea of the following method is to find a "natural" decomposition of the

weight functions in order to find hidden regularities. In time series terminology, the general problem can be stated as follows: Let $\{x_s(t_i),\ t_i \in R,\ s = 1, ..., k, i = 1, ..., n\}$ be a collection of $k$ time series, measured at the time points $t_i$. The aim is to find a decomposition

$$x_s(t_i) = \sum_{j=1}^{M} x_{j,s}(t_i)$$

such that the components $\{x_{j,s}, s = 1, ..., k\}$ reveal a maximal amount of 'regular structure'. One of the difficulties is to define what is meant by 'regular structures' and to define corresponding meaningful measures of the amount of 'regular structure'. Here, a pragmatic approach is taken, in that the amount of 'regular structure' is judged visually. Clearly, more formal definitions could be used.

Before introducing the idea of hierarchical decomposition, a few general remarks should be made:

**Remark 18** Traditionally, one of the main structures of interest for time series is periodicity. In particular, spectral decomposition based on sines and cosines may be used for this purpose (see e.g., [423], [69]). In our context, this is not applicable, because many compositions are likely to have much more interesting structures than just periodicities. In fact, some scores may not contain any nontrivial periodicities at all. More generally, the problem is that using the same basis of functions, irrespective of the structure of the score, results in focusing on a very limited number of predetermined features that may in fact not be present.

**Remark 19** As a consequence, a nonparametric approach based on kernel smoothing will be proposed here. In a traditional setting, the bandwidth $b$ is chosen by minimizing a criterion such as the mean squared error as $n$ tends to infinity. In particular, $b$ tends to zero with increasing sample size. This concept is not directly applicable in our context. The main reasons can be summarized as follows:

1. Based on the definitions given above, the metric, melodic and harmonic aspects of a score are characterized respectively by one weight function only. In contrast, a composer is likely to have a hierarchical view. For instance, a piece has on one hand a global harmonic shape that makes the piece coherent as a whole, and on the other hand more local structures. Some composers in fact consciously write a score using a hierarchical approach, first defining a global shape and then refining more and more local structures. Similarly, while rehearsing, a performer is likely to focus first on global features of the score and then successively refine more and more local features. This fact was also used in RUBATO to design the process from a primavista performance to the refined artistic result [357]. Here, a genealogical tree, the *stemma* of the performance process, is responsible for successive refinement and localization of the performance. In order to obtain a better picture of the structure of a score it is therefore necessary to "extract" the hierarchy that is hidden in the weight functions. For smoothing, this means that there is not just one optimal bandwidth that is of interest. Instead, there is a hierarchy of relevant bandwidths $b_1 > b_2 > ... > b_M$. Moreover, the structure of the score, rather than an omnibus statistical criterion (such as the mean squared error), is likely to yield the key information about which sets of bandwidths could be interesting.

2. The weight functions obtained from the analysis above are generally rather complex. In particular, the weights often jump abruptly up and down between very small and very large values (see figure 43.4). This can certainly not be carried over linearly to musical performance. For instance the tempo of a "musically acceptable" performance is unlikely to change up and down *drastically and repeatedly* within a few seconds. It is therefore reasonable to assume that a performance is not a linear function of the weights but rather a weighted sum of non-linearly deformed smoothed versions of these functions. Again, there may be a hierarchy of several bandwidths that need to be considered.

These general considerations motivate the idea of *hierarchical smoothing* and *hierarchical decomposition* described below.

### 43.1.2 Hierarchical Smoothing

Let $\{x_s(t_i), t_i \in R, i = 1, ..., n, s = 1, ..., k\}$ be a $k$-dimensional time series observed at time points $t_1, ..., t_n$ and $K_b$ a smoothing kernel with bandwidth $b$ and support $[-b, b]$. Applying the smoothing operator

$$K_b x_s(t) = \sum_{i=1}^{n} K_b(t, t_i) x_s(t_i)$$

$(t \in R)$ for a hierarchy of bandwidths $b_1 > ...b_M$, we obtain a hierarchy of $k-$dimensional curves

$$\{x_{j,s}(t) = K_{b_j} x_s(t), s = 1, ..., k\}, \; j = 1, ..., M.$$

Here, the Naradaya–Watson kernel

$$K_b(t, t_i) = \frac{K(\frac{t-t_i}{b})}{\sum_{j=1}^{n} K(\frac{t-t_j}{b})}$$

with a triangular function $K(s) = 1\{|s| \leq 1\} \cdot (1 - |s|)$ was used. For $b = 0$, we have $K_b x_s(t) = x_s(t)$. Figures 43.1 and 43.2 display hierarchies of smoothed curves for Schumann's "Träumerei" and for Bach's "canon cancricans" (see also figure 8.7), resulting from the metric, melodic and harmonic weights. The figures illustrate that different bandwidths make different features more visible. In particular, for the metric weights, smoothing highlights places where high values occur more frequently. Also, some remarkable similarities between the metric, melodic and harmonic weights become apparent after smoothing.

**Remark 20** The statistical technique of using smoothing kernels deserves a comment from the point of view of stemma and operator theory (section 39.8), a comment which strongly relates to the inverse performance theory to be exposed later in part XII. To begin with, taking into account neighboring values of the analyses by kernel smoothing has a musical meaning: The interpreter is rightly supposed to be conscious of what happened and will happen within the time bandwidth $b$. In inverse performance theory, the idea of kernel smoothing is introduced in the locally linear performance grammars, see section 39.8. There, the mutual influence of different local parts $C_i$ and $C_j$ of the composition $C$ for the performance shaping process is formalized by use of *interaction matrices* $(c_{i,j})$. The coefficient $c_{i,j}$ quantifies the influence of

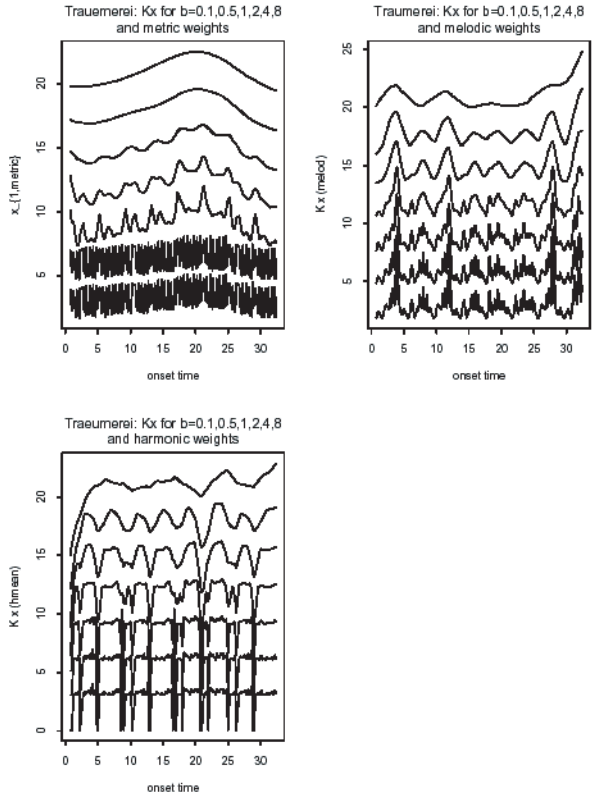


Figure 43.1: Smoothed version of metric, melodic, and harmonic weights for Schumann's "Träumerei".

$C_i$ on $C_j$. The general theory of locally linear performance grammars deals with the description of the de facto algebraic variety of interaction matrices inducing a fixed performance. From this point of view the triangular kernel smoothing means a selection of the a priori shape of interaction matrices, showing a peak around the diagonal.

### 43.1.3 Hierarchical Decomposition

The approach of hierarchical smoothing suggests a decomposition of the weight function into components of varying smoothness. Thus, let $\{x_s(t_i),\ t_i \in R,\ s = 1, ..., k, i = 1, ..., n\}$ be a collection of $k$ time series. As discussed above, the aim is to find a decomposition $x_s(t_i) = \sum_{j=1}^{M} x_{j,s}(t_i)$ such that the components $\{x_{j,s}, s = 1, ..., k\}$ reveal a maximal amount of "regular structure". Structure can be, for instance: symmetry, repeated shapes/periodicities, relationship between different components etc. Note that with respect to cross-correlations, a number of methods are known in the literature for testing dependence between stationary time series

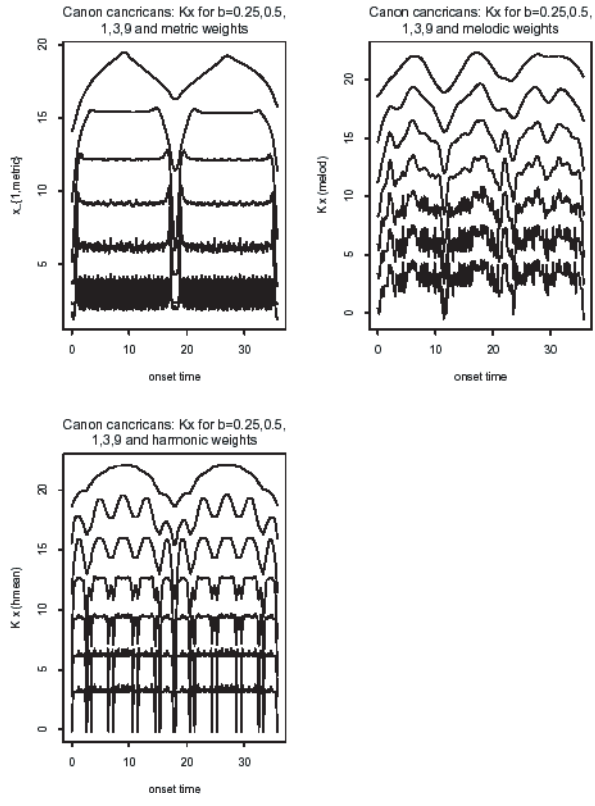


Figure 43.2: Smoothed version of metric, melodic, and harmonic weights for Bach's "canon cancricans".

(see e.g., [181], [211], [230], also see [423] and references therein). A direct adaptation of these methods is not possible for the following reasons: 1) The series considered here are not stationary in a nontrivial way and can, in particular, not be reduced to white noise by applying a linear filter. 2) The time points are not equidistant. 3) The aim is not only to obtain high cross-correlations but also to highlight regular features of the individual series. 4) Not only cross-correlations between "residual" but between all components are interesting. 5) The musical context suggests that the decomposition should be hierarchical in the sense that, with increasing index $j$, $x_{j,s}$ should contain increasingly local features.

We thus define the following decomposition:

1. Define a hierarchy of bandwidths $b_1 > b_2 > ...b_M = 0$, based on structural information from the score.

2. Define the smoothed function $x_{1,s} = K_{b_1} x_s$ and for $1 < j \leq M$, $x_{j,s} = K_{b_j}(x_s - \sum_{l=1}^{j-1} x_{l,s})$.

It should be noted that this decomposition is only one of many possible decompositions

of $x_s$. The problem of choosing a meaningful decomposition of a time series is not new. In particular, in the context of regression analysis (see chapter 44), it is a special case of the general problem of defining meaningful explanatory variables in regression models. Here, subject-specific considerations provide important guidelines. From a pragmatic point of view, a chosen decomposition can be considered reasonable if the subsequent regression analysis leads to meaningful interpretable results. In our context, the above decomposition appears meaningful, since it decomposes $x_s$ in a simple additive way into components of decreasing smoothness. This translates, in a straightforward way, the generally accepted fact that a musical composition as well as a performance may be considered as a superposition of a hierarchy of local and global "shaping features", obtained by different degrees of "zooming in or out". For a given sequence of bandwidths $b_1 > b_2 > \ldots b_M = 0$, the first component $x_{1,s}$ represents the most global view of the score (or more specifically of the metric, harmonic or melodic structure, respectively), $x_{2,s}$ represents the next step of refinement by considering, in a more detailed fashion with a smaller bandwidth $b_2 < b_1$, the remaining information (obtained by subtracting the "global information" $x_{1,2}$, and so on.

## 43.2 Comparing Analyses of Bach, Schumann, and Webern

**Summary.** The statistical method developed in the preceding section is applied to a comparative study of RUBATO-analyses of works by Johann Sebastian Bach, Robert Schumann, and Anton Webern.

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Each of figures 43.3.a through 43.3.d displays the melodic (dotted, middle), metric (full, lower) and harmonic (dashed, upper) weights for Schumann's "Träumerei" op. 15/7 (Kinderszene No.7), Webern's "Variationen für Klavier" op. 27/II, the "canon cancricans" from Bach's "Musikalisches Opfer" BWV 1079, and Schumann's "Kuriose Geschichte" op. 15/2 (Kinderszene No.2). For onset times with more than one value of the melodic and harmonic weight respectively, the average of the values was taken.

It is also interesting to look at scatterplots of the three types of weights against each other. The example for Bach's "canon cancricans" is displayed in figure 43.4. For each of the compositions, some simple regular features of the weights are visible:

- *Träumerei*: From the score it is clear that this composition may be divided into four parts $P_j$, $j = 1, 2, 3, 4$, corresponding to the onset intervals $I_1 = [0, 8]$ and $I_j = ((j-1) \cdot 8, j \cdot 8]$, $j = 2, 3, 4$, respectively. Also is it obvious that these four parts are similar to each other, and that $P_3$ differs most from the other parts. In fact, $P_2$ is, by definition, an exact replicate of $P_1$ (except for the slightly different up-beat). In figures 43.5.a through 43.5.c, the weights for the four parts are plotted on top of each other, i.e., onset time is taken modulo 8. The weights are indeed almost identical to each other. Interestingly, the fact that $P_3$ differs most from the other parts shows only for the melodic weights. Also, the scatter plots do not indicate any strong relationship between the three weight functions. The sample correlations are all in the range $[-0.01, 0.09]$.

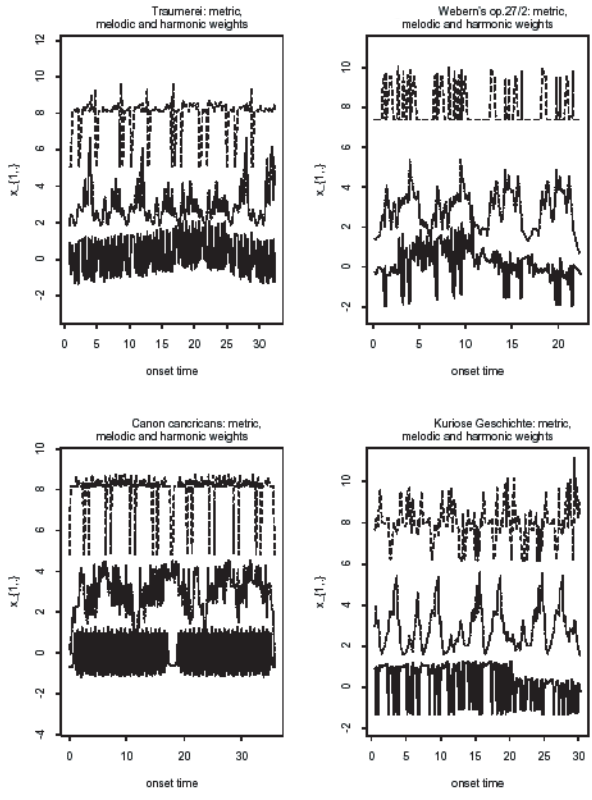


Figure 43.3: Metric, melodic, and harmonic weights for Schumann's "Träumerei", Webern's "Variationen für Klavier", Bach's "canon cancricans", and Schumann's "Kuriose Geschichte".

- *Variation op. 27/II*: With respect to the melodic and harmonic weights and from the score, it is clear that the composition can again be divided into four parts $P_j, j = 1, 2, 3, 4$, corresponding to a division of the onset time into four intervals of equal length. Clearly, the first two parts are almost identical with respect to the melodic and the harmonic weights. The same is true for the last two parts. For the metric weights, however, $P_2$ is not a simple replicate of $P_1$. The same is true for the last two parts. Also, for $P_1$ and $P_2$, the maximal values of the metric weights are much higher than for $P_3$ and $P_4$. Again, no apparent relationship seems to exist between the three weights (not shown). However, the largest correlation (in absolute value) is much higher than in the previous example, namely $-0.31$ between metric and harmonic weights.

- *Canon cancricans*: As expected for a retrograde canon, there is an almost exact time symmetry with respect to the middle of the onset axis. The symmetry is not exact, because the retrograde is not just a reflection of onsets but rather a transvection in the

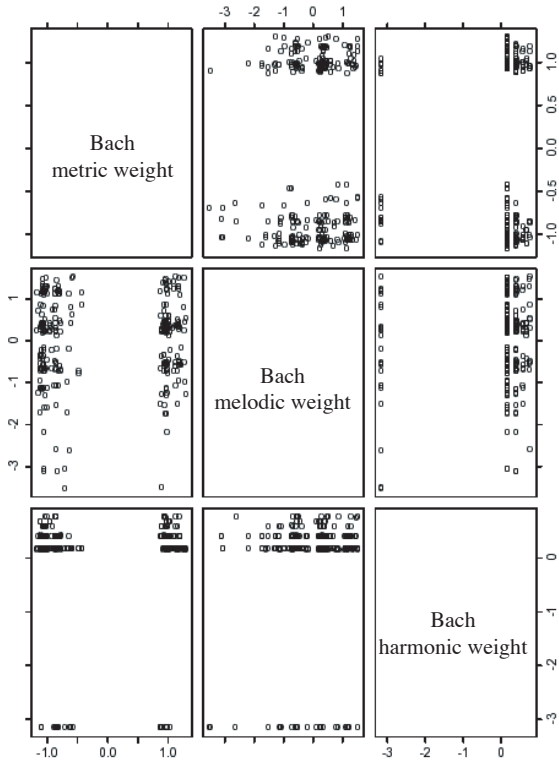


Figure 43.4: Scatterplots of analytical weights for Bach's "canon cancricans".

onset duration space parallel to the onset axis (see section 8.1.1, example 9). Also striking is the clustered nature of the weights and the apparently very regular high frequency oscillation of the metric curve. A high metric weight is almost always succeeded by a low weight and vice versa. Because of the clustered nature of the weights, scatter plots are not very useful in this case. The correlations between the weights are again very small, ranging between 0.03 and 0.04.

- *Kuriose Geschichte*: Here, the score is again divided into four parts corresponding to the onset intervals [0,6], (6,12], (12,21], (21,30], with $P_1$ equal to $P_2$ and $P_3$ equal to $P_4$. Again, it is difficult to tell in how far the three different curves may be related to each other. Note however that the metric weights are much lower for onset times above 21. Thus, for the metric weights, the correspondence between $P_3$ and $P_4$ is much weaker. The reason is the breakdown of local meters at bar 21. Similarly to Webern, the strongest correlation between the weights is quite remarkable, namely $-0.33$ between melodic and harmonic weights.

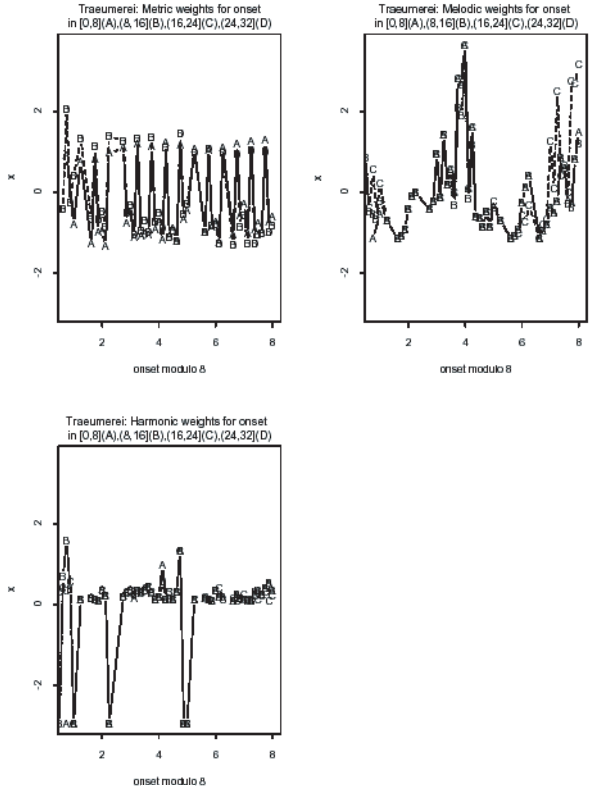


Figure 43.5: Analytical weights for Schumann's "Träumerei" against onset time modulo 8.

It is also interesting to compare the four compositions with each other. The weights of Bach's "canon cancricans" exhibit an extreme high frequency oscillation that is not observed for the other scores. Ignoring that onset times are not exactly equidistant, this can be seen for instance very clearly by comparing the sample autocorrelations of the metric weights (figure 43.6).

Another property of interest is the marginal distribution of the weight functions. Eliminating global 'trends' by taking first differences $x(t_j) - x(t_{j-1})$ the histograms are given in figure 43.7 for the metric weights. For the compositions by Schumann and Bach, the first difference of the metric weights can essentially be classified into three clusters (low, medium, high). For Webern's score, the distribution is completely different and in fact rather close to a normal distribution. In contrast, the distributions of the differenced melodic weights are qualitatively similar for all four scores. For the harmonic weights, all distributions appear to be essentially symmetric. However, while for Schumann's "Träumerei" and the score by Webern there appear to be three clusters, the histograms for Bach and the "Kuriose Geschichte" are essentially unimodal.

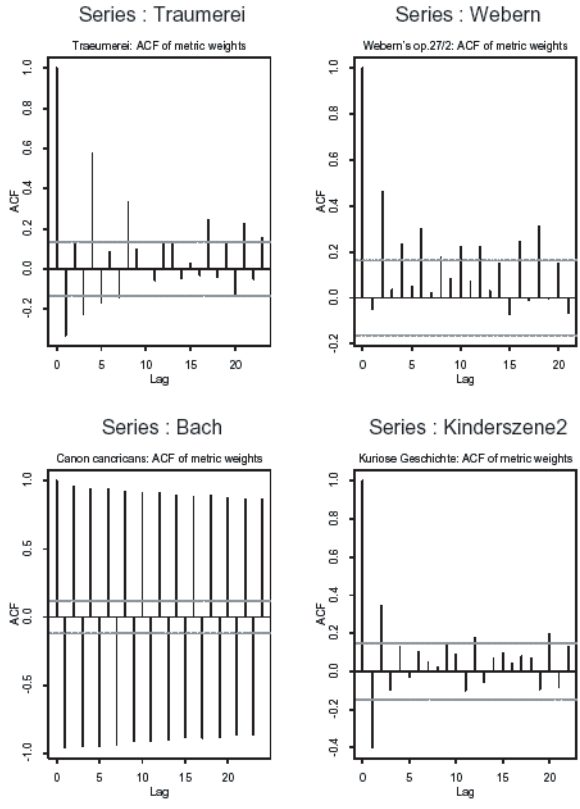


Figure 43.6: Autocorrelograms of metric weights for Schumann's "Träumerei", Webern's "Variationen für Klavier" op.27/2, Bach's "canon cancricans", and Schumann's "Kuriose Geschichte".

In summary, a first look at the weight functions reveals certain elementary features of the score. In the following it will be demonstrated that a more thorough analysis leads to further new insights about the structure of the scores. In particular, note that the three weight functions were defined in a completely different way. It may therefore be expected that there is no strong relationship between the curves. The scatter plots of the weights seem to support this conjecture. But the following analysis will show that certain components of the weight functions are indeed closely related.

Specifically, application to the four examples was carried out using $M = 4$. This choice was based on musicological considerations (time signature and bar grouping) as explained in the following. In this sense, the analysis here is exploratory, since no statistical selection criterion was used for choosing $M$. The following notation will be used here: $x_1 = x_{metric}$ =metric weight, $x_2 = x_{melod}$ =melodic weight, $x_3 = x_{hmean}$ =harmonic (mean) weight, $x_{j,metric} = x_{j,1}$, $x_{j,melod} = x_{j,2}$, $x_{j,hmean} = x_{j,3}$. The choice of the bandwidths was based on the time signature and bar grouping information. Example Schumann/Träumerei is written in 4/4 signature, the

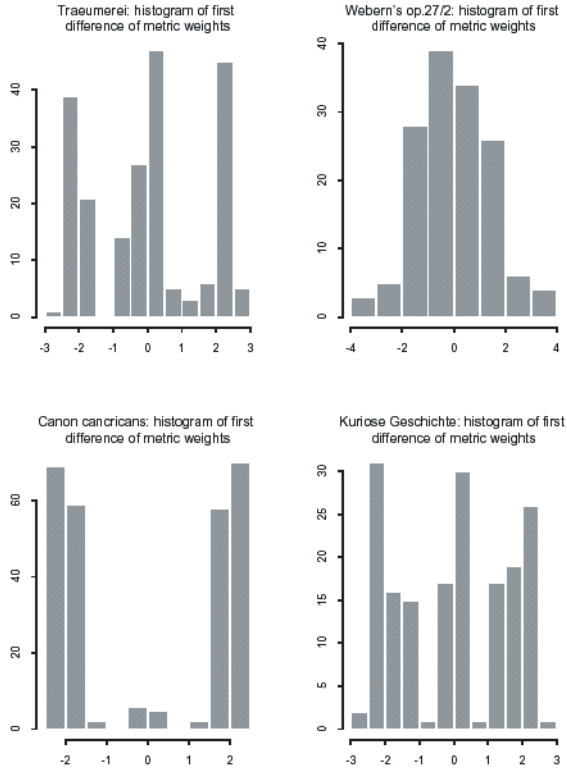


Figure 43.7: Histogram of first difference of metric weights for Schumann's "Träumerei", Webern's "Variationen für Klavier", Bach's "canon cancricans" and Schumann's "Kuriose Geschichte".

grouping is $8+8+8+8$. The chosen bandwidths are therefore 4 (4 bars), 2 (2 bars) and 1 (1 bar). Example Webern is written in 2/4 signature, its formal grouping is $1+11+11+11+11$; however, Webern insists on a grouping in 2-bar portions [562], suggesting the bandwidths of 5.5 (11 bars), 1 (2 bars) and 0.5 (1 bar). Example Bach is written in 4/4 signature, the grouping is $9+9+9+9$. The chosen bandwidths are 9 (9 bars), 3 (3 bars) and 1 (1 bar). For example Schumann/Kuriose Geschichte, the time signature is 3/4, the grouping is $8+8+12+12$. The chosen bandwidths are 3 (4 bars), 1.5 (2 bars) and 0.75 (1 bar).

Figures 43.9 ("Träumerei") and 43.8 ("canon cancricans") show remarkable regularities that have not been observed for the original weights (same for Webern and Schumann/Kuriose Geschichte, which we omit here). In particular, for all four compositions, much stronger similarities between the metric, melodic, and harmonic components can be observed than for the original weights, especially for $j = 2, 3$. Moreover, for the first two scores, the same kind of relationship can be observed for $j = 2, 3$, namely: positive correlation between $x_{j,melod}$ and

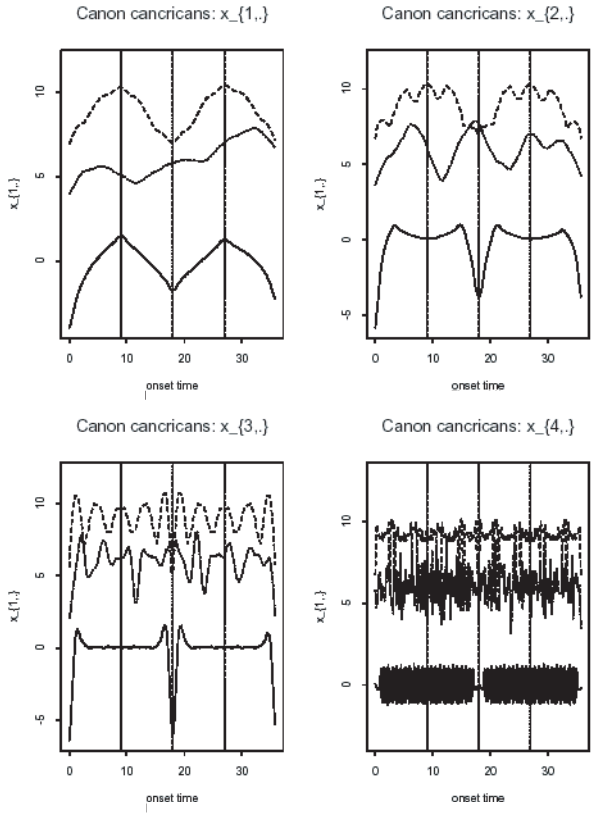


Figure 43.8: Hierarchical components of metric (solid lines), melodic (dotted lines), and harmonic (dashed lines) weights for Schumann's "Träumerei", as defined in section 43.1.2: (a) $b = 4$; (b) $b = 2$; (c) $b = 1$; (d) remaining (residual) series.

$x_{j,hmean}$, negative correlation between $x_{j,melod}$ and $x_{j,metric}$, and negative correlation between $x_{j,hmean}$ and $x_{j,metric}$. Particularly surprising is the fact that Webern's score shows the same type of association as Schumann's "Träumerei". This leads to new insights into different approaches to composition. The weight functions are in fact very complex data and deserve a refined "analysis of analysis". Hierarchical smoothing is a possible approach to this problem.

Webern's piece is written in a completely dodecaphonic way, and thus breaks with harmonic and homophonic tradition. *This deserves a special methodological comment.* The fact that we have nevertheless applied harmonic analysis could be viewed as being in contradiction to Webern's rupture with harmony. Now, we do not claim that this analysis corresponds to Webern's poietic position when composing his "Variationen". Nonetheless, an objective analysis according to the Riemann approach is reasonable for two reasons: (1) Riemann intended to *attribute tonality to any possible chord.* The fact that he did not succeed in his goal is no reason for refraining from completion of his sketch. This is what the HarmoRUBETTE® is about: It

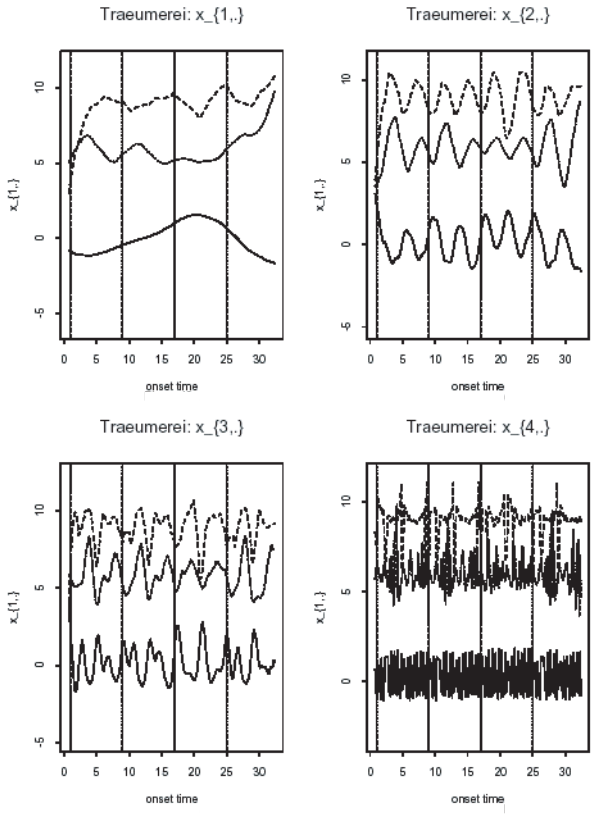


Figure 43.9: Hierarchical components of metric (solid lines), melodic (dotted lines),and harmonic (dashed lines) weights for Bach's "canon cancricans", as defined in section 43.1.2: (a) $b = 9$; (b) $b = 3$; (c) $b = 1$; (d) remaining (residual) series.

is a proposal to discuss possible completions of Riemann's theory. (2) Therefore it is also very interesting to discuss its application to apparently atonal compositions. Such an experiment is likely to yield a testbed for the universality of Riemann's approach. These considerations suggest that *the following fact is not completely surprising*, although it has not been established explicitly elsewhere in the literature: The correspondence between metric, melodic, and harmonic structure in Webern's "Variationen" is very similar to Schumann's "Träumerei". It should be emphasized that this conclusion and in particular its quantitative demonstration is new in the musicological literature.

Schumann's "Kuriose Geschichte" also shows a strong correspondence between the three curves for $j = 1, 2$ and 3. But this time, the relations are different:

For onset times below 12, we have the following:

1. For $j = 1$,
   $cor(x_{1,metric}, x_{1,melod}) = 0.83$,
   $cor(x_{1,metric}, x_{1,hmean}) = -0.71$,
   $cor(x_{1,hmean}, x_{1,melod}) = -0.63$.

2. For $j = 2$, we have the following (rounded) correlation values:
   $cor(x_{2,metric}, x_{2,melod}) = 0.00$,
   $cor(x_{2,metric}, x_{2,hmean}) = -0.31$,
   $cor(x_{2,hmean}, x_{2,melod}) = -0.82$.

3. For $j = 3$, we have
   $cor(x_{3,metric}, x_{3,melod}) = -0.67$,
   $cor(x_{3,metric}, x_{3,hmean}) = -0.20$,
   $cor(x_{3,hmean}, x_{3,melod}) = -0.61$.

Observe in particular that, in contrast to the other scores, melodic and harmonic components are negatively correlated.

After onset time 12, the correlations are:

1. For $j = 1$:
   $cor(x_{1,metric}, x_{1,melod}) = 0.10$,
   $cor(x_{1,metric}, x_{1,hmean}) = -0.38$,
   $cor(x_{1,hmean}, x_{1,melod}) = -0.29$.

2. For $j = 2$:
   $cor(x_{2,metric}, x_{2,melod}) = -0.47$,
   $cor(x_{2,metric}, x_{2,hmean}) = -0.14$,
   $cor(x_{2,hmean}, x_{2,melod}) = -0.11$.

3. For $j = 3$:
   $cor(x_{3,metric}, x_{3,melod}) = -0.75$,
   $cor(x_{3,metric}, x_{3,hmean}) = 0.58$,
   $cor(x_{3,hmean}, x_{3,melod}) = -0.69$.

Finally, for Bach's composition, the only noticeable correlations occur between metric and harmonic weights, namely:

1. For $j = 1 : cor(x_{1,metric}, x_{1,hmean}) = 0.94$,

2. For $j = 2 : cor(x_{2,metric}, x_{2,hmean}) = 0.63$,

3. For $j = 3 : cor(x_{3,metric}, x_{3,hmean}) = 0.61$.

With respect to the shapes of $x_{j,.}$, for $j = 2$ and 3, the two scores by Schumann and the one by Webern are clearly more similar to each other as compared to Bach's shapes. From the point of view of music history, this is quite plausible, since Webern's organic composition principle is more related to Schumann's rankly growing romanticism than to Bach's self-disciplined architectural setup (see also the following remarks).

Finally, note that the scatterplots in figure 43.4 show that Bach's harmonic weights are highly clustered and the smoothed curves in figures 43.8.a through 43.8.d are more 'edgy' than for the other compositions. In this sense, *Bach's composition exhibits a high degree of organization*. This confirms the general belief that the principle of architectural rather than processual construction plays a dominating role in Bach's music.

Overall, we may conclude that hierarchical decomposition reveals interesting properties, in particular strong similarities between the metric, melodic and harmonic weights, that were not visible in the original series. The results are musically plausible in that the analysis of Bach's score turns out to be the most regular one and the analyses of Webern and Schumann appear to be closer to each other than to Bach's. The results are surprising in that (the analysis of) Webern turns out to be closer to (the analysis of) Schumann than expected. Also, the strong relationship between the three analytic curves could not be expected a priori, since the three weights were calculated using completely different aspects of the score and the scatter plots of the original curves did not show almost any association.

*Based on the results, one may conjecture that appropriate matching of metric, melodic and harmonic structure plays an important role in music, independently of musical style.* The tools introduced here provide the possibility of investigating which types of relationships may exist in which musical and historical contexts. An important task for future research will be to investigate such aspects for a larger variety of compositions.

# Chapter 44

# Differential Operators and Regression

*Rejection by common sense, for whatever reason, proves nothing.*
*Other fields of science are built on propositions that*
*seem absurd but in fact are true.*
Donald O Hebb [213]

**Summary.** We give statistical evidence from 28 performances of Schumann's "Träumerei", as measured by Bruno Repp [438] that the rhythmic, motivic, and harmonic analyses provided by RUBATO® are shaping structures for the agogical streams. The statistical model is based on regression analysis and realizes shaping of agogics by a second degree linear differential operator as a function of analytical weights which are averaged over a natural grouping hierarchy (as described in chapter 43) of the score.

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At present, the best investigated aspect of performance theory—including appropriate software—is timing microstructure, i.e., agogics on the level of tempo curves and their hierarchies, see [272, 346] for further reading. This chapter deals with this topic: agogics as an expression of harmonic, melodic and rhythmic structures. So observe that we do not consider emotional or gestural rationales for agogics. This does not mean that these factors are negated. We merely restrict our investigation to the question whether and how strongly agogics could be explained by exclusive causal reference to structural analysis.

Even in this neat reduction is the question neither trivial nor even well defined, since musicology does not offer precise tools for rhythmical or melodic analysis, and even harmonic analysis is far from effective. Therefore, the question is only a scientific one if one specifies the analyses and their output data format. For the general setup for such an explicit and operationalized analysis framework, namely the RUBATO® analysis and performance workstation, we refer to part X. The main concern is an *empirical study* regarding the basic question *whether agogics (the tempo curve of the "Träumerei", also called "timing microstructure") may be expressed in mathematical terms by use of structural data obtained from a specific set of musicological*

*analyses offered by* RUBATO® *'s modules.* The experimental data are taken from Repp's timing measurements of 28 famous performances of Schumann's "Träumerei" [438]. Summarizing our results, we can state that

**Result 2** *There is strong statistical evidence for the timing microstructure in 28 famous performances of Schumann's "Träumerei" as being an expression solely of harmonic, melodic and rhythmic structures furnished by the* RUBATO® *analysis. In other words: These structural rationales are sufficient for explaining agogics, and emotional or gestural rationales can be disregarded for the present experimental material.*

In the framework of RUBATO®, the analytical output of any type of analysis is always a (smooth) weight, i.e., differentiable function of one or several note event parameters, such as onset $E$, pitch $H$, loudness $L$, duration $D$, etc. With this specification, the above result can be restated in more mathematical but less intuitive terms:

**Result 3** *The tempo curve (timing microstructure) can be generated by an* agogical operator $\Omega$ *which is essentially a linear differential operator of second order as a function of harmonic, melodic and rhythmic smooth weights.*

Statistically speaking, this means that in our empirical context, the fiber (in the sense of Todd's approach) of the chosen performance transformation is not empty. Observe that the operator $\Omega$ which plays the role of the transformation $\Pi$ in Todd's theory [530] (see also section 36.3) does not rely on general encoding tempo curve functions. Intuitively, this means that our approach generates agogics from smooth weights as *score-specific* functions, and not general, score-independent, curve types, as proposed in [518, 532]. This does not contradict usage of general encoding tempo functions, it simply suggests that agogics is a superposition of more "primavista"-like tempo functions and of a strong and differentiated timing microstructure stemming from analytical data.—Thirdly, the statistical results suggest

**Result 4** *Essential commonalities and diversities among tempo curves may be characterized by a relatively small number of analytical weight curves. There is in general no unique way of attributing features of the tempo to exactly one cause (harmonic, metric or melodic analysis). Results depend on which of the three analyses is given priority. However, there appear to be a certain number of* canonical curves *that are essentially independent of the priority.*

Overall, a large variety of musically meaningful results is obtained. This is in particular due to the fact that a score-specific basis of curves is used on which the tempo curves are projected. We thus may conclude

**Result 5** *The analytical curves obtained from (1) score-specific harmonic, melodic and rhythmic smooth weights and (2) a score-specific hierarchical decomposition of these weights, yield a natural* score-specific linear basis *in the space of tempo curves, for performances of the considered score.*

We should stress that all our results are intimately related to the concrete analyses which RUBATO® produces—together with the underlying theories. There is no unique analysis, and

therefore, specification and numerical representation of musical analysis is not secondary and will in any case (!) influence the results. We should also remind the reader trained in natural sciences that musical analysis is not a neutral tool but pertains to the unavoidable "artifacts" of analysis in the humanities. *There are no objective laws in human creations which subsist beyond interpretative interaction.* This is a caveat to those who believe that performance can be more than a relation between what we understand (rationally, emotionally, gesturally) and how we express this understanding.

### 44.0.1 Analytical Data

**Summary.** This section describes the analytical weights used in this analysis.

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We shall omit the notification of the used parameter lists $Param$ which are related to the specific RUBETTE®, and of the predicate $P_{Träumerei}$ of Schumann's score and abbreviate

$$\begin{aligned} x_{metric} &= x^{P_{Träumerei}}_{MetroRubette,Param_{metric}} \\ x_{melodic} &= x^{P_{Träumerei}}_{MeloRubette,Param_{melodic}} \\ x_{hmax} &= x^{P_{Träumerei}}_{HarmoRubette,Param_{harmonic/max}} \\ x_{hmean} &= x^{P_{Träumerei}}_{HarmoRubette,Param_{harmonic/mean}} \end{aligned}$$

to denote the four following weights used in our context. We are going to give their description when evaluated on onsets $E$ of note events occurring in $P_{Träumerei}$:

- $x^{P_{Träumerei}}_{MetroRubette,Param_{metric}}$

  This is a metrical weight which measures the rhythmic relevance or "weight" of every onset of a note event in the composition in the lines of Riemann [453], Jackendoff–Lerdahl [243], and Mazzola [340]. A detailed description of the MetroRUBETTE® was given in section 41.1.

- $x^{P_{Träumerei}}_{MeloRubette,Param_{melodic}}$

  This is a "boiled-down" melodic weight[1] which measures the sum of the *melodic weights* $w(Evt)$ of all note events $Evt$ at a given onset $E$. The calculation is extremely complex and time-consuming and goes back to theories of Reti [444], see chapter 22. See section 41.2 for a detailed description of the MeloRUBETTE®.

- $x^{P_{Träumerei}}_{HarmoRubette,Param_{harmonic/max}}$

  This harmonic weight measures the harmonic relevance of a chord $ch$ which occurs at an onset $E$ in $P_{Träumerei}$. It is calculated by the same method as the fourth weight. The only difference is that this weight captures the harmonic relevance of the most important note in $ch$ whereas the fourth weight represents the average harmonic relevance among all notes of $ch$. See section 41.3 for a detailed description of the HarmoRUBETTE®.

[1] See formula (39.26) in section 39.3.

- $x^{P_{Träumerei}}_{HarmoRubette,Param_{harmonic/mean}}$

  As already mentioned, this weight is a variant of the third one, the only difference being an averaging instead of maximizing procedure. We refer to the previous discussion for the basics.

# 44.1 The Beran Operator

**Summary.** In this section, we define the conceptual setup for the following statistical analysis.

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The general idea is that agogics is to be shaped by use of smoothed versions of the boiled-down weights, their first and second derivatives, and corresponding kernel-smoothed versions with respect to hierarchical (triangular) kernel functions.

## 44.1.1 The Concept

As we know, the general RUBATO® concept of shaping performance is built on smooth (actually $C^2$ in this context) weights $x$ where $d_E x$, $d_E^2 x$ denote the first and second derivatives with respect to symbolic time $E$. The kernel smoothing process relates to *kernel* functions $\hat{b}(s) = 1/b \cdot \chi\{|s| \leq b\} \cdot (1 - |s|/b)$ with triangular, zero-symmetric support of extent $\pm b$, and characteristic function $\chi\{P\}$ for a predicate $P$. The linear smoothing operator $b \diamond f$ on a function $f$ is defined by the convolution

$$b \diamond f(E) = \int \hat{b}(t - E) \cdot f(t). \tag{44.1}$$

It averages $f$ around $E$ with weighted center $E$ and *bandwidth* $b$. If this function is a weight, this means that the weight's analysis within the entire bandwidth neighborhood of a given onset is included instead of spiking the analysis to the singular onset. In the following process, this kernel smoothing process has been applied to a hierarchy of bandwidths, starting with $b = 4$ (= eight bars), then $b = 2$, then $b = 1$. The averaging process is taken to define successive remainder functions as follows:

$$f_1 = 4 \diamond f,\ f_2 = 2 \diamond (f - f_1),\ f_3 = 1 \diamond (f - f_1 - f_2),\ f_4 = f - f_1 - f_2 - f_3 \tag{44.2}$$

This means that the decomposition

$$x = x_1 + x_2 + x_3 + x_4 \tag{44.3}$$

for a smooth weight $x$ defines a "spectrum" of that weight with respect to successively refined neighborhoods of its ambit.

**Remark 21** *Musically speaking, as already observed before, this kernel smoothing process is completely natural.* In fact, the kernel function alters the original time function $f(E)$ by a weighted integration of $f$-values in the kernel neighborhood of a given time $E$. This means that we now include the information about $f$ from the neighboring times to make an analytical

judgment. This latter is a well known and common consideration in musical performance: The interpreter looks up a full neighborhood of a time point to derive what has to be played in that point. Moreover, the repeated application of the kernel smoothing process with increasingly narrowed neighborhoods is understood as a succession of a refinement in local analysis: First, the interpreter makes a coarse analysis over eight bars ($b = 4$), then he/she looks for the remainder $f - f_1$ and goes on with refined actions, if necessary.

This procedure is applied to the metric, melodic and harmonic weights and to their first and second derivatives. This gives the following list of a total of 48 *spectral analytical* functions:

$$
\begin{array}{llll}
x_{metric,1} & x_{metric,2} & x_{metric,3} & x_{metric,4} \\
d_E x_{metric,1} & d_E x_{metric,2} & d_E x_{metric,3} & d_E x_{metric,4} \\
d_E^2 x_{metric,1} & d_E^2 x_{metric,2} & d_E^2 x_{metric,3} & d_E^2 x_{metric,4}
\end{array}
$$

$$
\begin{array}{llll}
x_{melodic,1} & x_{melodic,2} & x_{melodic,3} & x_{melodic,4} \\
d_E x_{melodic,1} & d_E x_{melodic,2} & d_E x_{melodic,3} & d_E x_{melodic,4} \\
d_E^2 x_{melodic,1} & d_E^2 x_{melodic,2} & d_E^2 x_{melodic,3} & d_E^2 x_{melodic,4}
\end{array}
$$

$$
\begin{array}{llll}
x_{hmax,1} & x_{hmax,2} & x_{hmax,3} & x_{hmax,4} \\
d_E x_{hmax,1} & d_E x_{hmax,2} & d_E x_{hmax,3} & d_E x_{hmax,4} \\
d_E^2 x_{hmax,1} & d_E^2 x_{hmax,2} & d_E^2 x_{hmax,3} & d_E^2 x_{hmax,4}
\end{array}
$$

$$
\begin{array}{llll}
x_{hmean,1} & x_{hmean,2} & x_{hmean,3} & x_{hmean,4} \\
d_E x_{hmean,1} & d_E x_{hmean,2} & d_E x_{hmean,3} & d_E x_{hmean,4} \\
d_E^2 x_{hmean,1} & d_E^2 x_{hmean,2} & d_E^2 x_{hmean,3} & d_E^2 x_{hmean,4}
\end{array}
$$

For which *musical reasons* are these derivatives added to the analytical input data? The first derivatives measure the local change rate of analytical weights. Musically speaking, this is an expression of transitions from important to less important analytical weights (or vice versa), i.e., a transition from analytically meaningful points to less meaningful ones (or vice versa). This is crucial information to the interpreter: It means that he/she should change expressive shaping to communicate the ongoing structural drama. In the same vein, information about second derivatives is musically relevant since it lets the interpreter know that the ongoing structural drama is being inflected. Evidently, one could add higher derivatives but we argue that an interpreter is already highly skilled if he/she can take care of all these functions, also because different analytical aspects from metrics to harmonics must be observed simultaneously.

Besides these analytical input functions, we add three types of 'sight-reading' functions. They regard the following three instances: ritardandi, suspensions[2], and fermatas. It is clear that any text-sensitive performance should be aware of such information.

[2]Suspensions are notes which are tied by a slur while the harmony changes; we may attach to such events the time interval where the suspension does not start until the harmony change is terminated.

1. **Ritardandi**

   The score shows four onset intervals $R_1$, $R_2$, $R_4$, $R_4$ for ritardandi, starting at onset times $E_o(R_j)$ $(j = 1, 2, 3, 4)$ respectively. We define the four linear functions

$$x_{rit_j}(E) = \chi(R_j) \cdot (E - E_o(R_j)), j = 1, 2, 3, 4. \tag{44.4}$$

2. **Suspensions**

   The score shows four onset intervals $S_1$, $S_2$, $S_4$, $S_4$ for suspensions, starting at onset times $E_o(S_j)$ $(j = 1, 2, 3, 4)$ respectively. We define the four linear functions

$$x_{sus_j}(E) = \chi(S_j) \cdot (E - E_o(S_j)), j = 1, 2, 3, 4. \tag{44.5}$$

3. **Fermatas**

   The score shows two onset intervals $F_1$, $F_2$ for fermatas. We define the two support functions

$$x_{ferm_j}(E) = \chi(F_j), j = 1, 2. \tag{44.6}$$

Summarizing, we have a total of 58=48+4+4+2 onset functions of analytical and primavista types. Call $X$ the *analytical vector* of these 58 functions listed in a fixed order. The present approach is to define the tempo function at onset $E$ as being a linear function of these 58 variables.

For a 'shaping' vector $\omega \in \mathbb{R}^{58}$, the shaping operator $\Omega^X_\omega$ for the tempo curve is defined by the canonical scalar product of $X$ with the shaping vector $\omega$,

$$\Omega^X_\omega = (X, \omega). \tag{44.7}$$

This means that for every onset $E$, we have $\Omega^X_\omega(E) = (X(E), \omega)$. Recapitulating the meaning of the analytical vector $X$, we are dealing with a second order differential operator which we call "Beran operator" since it was introduced by Jan Beran in [52].

On this basis, the central question of the following is whether tempo curves $T$ of the "Träumerei" as they appear in the context measured by Repp in [438] may be approximated via $\Omega^X_\omega$ by appropriate choice of the shaping vector $\omega$. The main result of this approach states that there is strong statistical evidence for the equation

$$\ln(T) = \Omega^X_\omega + C \tag{44.8}$$

for the given analytical vector $X$, a suitable shaping vector $\omega$, and a constant $C$.

This means that the 58 coefficients of the shaping vector $\omega$ are random variables and that we prove a significant statistical correlation—in the mathematical form described by the Beran operator—between a certain subset of the analytical vector $X$ and tempo as it is measured for the 28 performances by Repp.

**Thesis 7** *One may therefore try to use the above formula (44.8) to define tempo as a function of analytical score data in the sense of a general performance grammar as described in chapter 37.*

## 44.1.2 The Formalism

### 44.1.2.1 Tempo Information

In the following, a more detailed description of the tempo data used for the analysis is given.

- **Onset times:** The onset times are on a grid of $1/8^{th}$ beats. Thus, for instance, grace notes are excluded. From this set of onset times, we consider only onset times where at least one note is actually played. This results in a set $T$ of $n = 212$ not equidistant onset times $t_i$ $(i = 1, ..., n)$ which are multiples of $1/8$.
- **Log-transformation:** Instead of the original tempo $y$ we consider its natural logarithm $\ln y$. Intuitively this can be justified by the expectation that a performer may control the tempo in a relative rather than an absolute way. Also, the statistical results were more satisfactory on the logarithmic scale. In the following we refer to the logarithmic tempo as 'the tempo curve'.
- **Standardization of individual curves:** The data consist of tempo measurements (or tempo curves) for $m = 28$ performances. In the current analysis, the interest lies in investigating the shape of the tempo curves rather than the absolute tempo values. Therefore, each of 28 tempo curves is standardized. More specifically, let $y^*(t_i, j)$ be the (natural) logarithm of the tempo of the $j^{th}$ performance at onset time $t_i$ $(i = 1, ..., n; j = 1, ..., m)$. Then the standardized tempo data are defined by

$$y(t_i, j) = \frac{y^*(t_i, j) - \bar{y}^*(j)}{s^*(j)}$$

where $\bar{y}^*(j) = n^{-1} \sum_{i=1}^{n} y^*(t_i, j)$ and

$$s^*(j) = [(n-1)^{-1} \sum_{i=1}^{n} (y^*(t_i, j) - \bar{y}^*(j))^2]^{\frac{1}{2}}.$$

### 44.1.2.2 The Explanatory Variables

The following notation is used: Let $A$ be a $p \times q_1$ matrix and $B$ a $p \times q_2$ matrix, then $C = (A, B)$ denotes the $p \times (q_1 + q_2)$ matrix obtained by 'attaching' $B$ on the right-hand side of $A$. The following steps describe the definition of the matrix of explanatory variables in more detail.

According to the concept in section 44.1.1, the score data (metric, harmonic and melodic weights, additional score information) are given in the form of a design matrix that is used subsequently in a regression analysis. The following definitions are used:

1. **Derivatives.** "Derivatives" are defined as finite differences divided by the difference of the onset times. Thus, for instance,

$$d_E x_{metric,j}(t_i) = \frac{x_{metric,j}(t_i) - x_{metric,j}(t_{i-1})}{(t_i - t_{i-1})}$$

and

$$d_E^2 x_{metric,j}(t_i) = \frac{d_E x_{metric,j}(t_i) - d_E x_{metric,j}(t_{i-1})}{(t_i - t_{i-1})}.$$

2. **Hierarchical smoothing.** Each of the weights and their first and second (discrete) derivatives are decomposed into four components of different smoothness as defined by equations (44.2) and (44.3).

3. **Additional variables.** Additional variables modeling ritardandi, suspensions and fermatas were defined by (44.4), (44.5), and (44.6). The aim is to model these musical events in a "minimal" way. For instance, the resulting linear model for a ritardando is only a crude approximation to a "true" ritardando. The reason for using only the simplest parametrization is that the main purpose here is to examine to what extent the metric, melodic, and harmonic weights alone, together with only *absolutely necessary* additional information from the score, contain enough information to "explain" the tempo of a performance.

4. **Initial design matrix.** Using the definitions above, we define for $j = 1, 2, 3, 4$ the $n \times 4$ matrices

$$\begin{aligned} X_j(harmo) &= (x_{hmean,j}, x_{hmax,j}, x_{metric,j}, x_{melod,j}) \\ X_j(metric) &= (x_{metric,j}, x_{hmean,j}, x_{hmax,j}, x_{melod,j}) \\ X_j(melod) &= (x_{melod,j}, x_{hmean,j}, x_{hmax,j}, x_{metric,j}) \end{aligned}$$

Clearly, there are more possibilities of permuting columns. Here, we consider only the representative permutations above. The first column of $X_j(harmo)$ is $x_{hmean,j}$ so that, due to the orthonormalization to be described in the following section, the main emphases is put on the harmonic mean weights. Similarly, the metric and melodic emphasis are chosen with $X_j(metric)$ and $X_j(melod)$. Furthermore, we define

$$\begin{aligned} d_E X_j(harmo) &= (d_E x_{hmean,j}, d_E x_{hmax,j}, d_E x_{metric,j}, d_E x_{melod,j}) \\ d_E X_j(metric) &= (d_E x_{metric,j}, d_E x_{hmean,j}, d_E x_{hmax,j}, d_E x_{melod,j}) \\ d_E X_j(melod) &= (d_E x_{melod,j}, d_E x_{hmean,j}, d_E x_{hmax,j}, d_E x_{metric,j}) \end{aligned}$$

$$\begin{aligned} d_E^2 X_j(harmo) &= (d_E^2 x_{hmean,j}, d_E^2 x_{hmax,j}, d_E^2 x_{metric,j}, d_E^2 x_{melod,j}) \\ d_E^2 X_j(metric) &= (d_E^2 x_{metric,j}, d_E^2 x_{hmean,j}, d_E^2 x_{hmax,j}, d_E^2 x_{melod,j}) \\ d_E^2 X_j(melod) &= (d_E^2 x_{melod,j}, d_E^2 x_{hmean,j}, d_E^2 x_{hmax,j}, d_E^2 x_{metric,j}) \end{aligned}$$

and the $n \times 10$ matrix

$$X_{add} = (X_{rit}, X_{sus}, X_{ferm})$$

where $X_{rit} = (x_{rit_1}, x_{rit_2}, x_{rit_3}, x_{rit_4})$, $X_{sus} = (x_{sus_1}, x_{sus_2}, x_{sus_3}, x_{sus_4})$ and $X_{ferm} = (x_{ferm_1}, x_{ferm_2})$.

Finally define the $n \times p$ matrices (with $p = 58$)

$$X(harmo) = (X_1(harmo), X_2(harmo), X_3(harmo), X_4(harmo), d_E X_1(harmo), d_E X_2(harmo), d_E X_3(harmo), d_E X_4(harmo), d_E^2 X_1(harmo), d_E^2 X_2(harmo), d_E^2 X_3(harmo), d_E^2 X_4(harmo), X_{add}),$$

$$X(metric) = (X_1(metric), X_2(metric), X_3(metric), X_4(metric), d_E X_1(metric), d_E X_2(metric), d_E X_3(metric), d_E X_4(metric), d_E^2 X_1(metric), d_E^2 X_2(metric), d_E^2 X_3(metric), d_E^2 X_4(metric), X_{add}),$$

$$X(melod) = (X_1(melod), X_2(melod), X_3(melod), X_4(melod), d_E X_1(melod), d_E X_2(melod), d_E X_3(melod), d_E X_4(melod), d_E^2 X_1(melod), d_E^2 X_2(melod), d_E^2 X_3(melod), d_E^2 X_4(melod), X_{add}).$$

5. **Orthonormalization.** Each of the design matrices $X(metric), X(harmo)$, and $X(melod)$ turned out to be singular, since the last column can be expressed as a linear combination of the previous ones. Hence, we omit the last column. For simplicity of notation, the new $n \times 57$ matrices will also be denoted by $X(metric), X(harmo), X(melod)$.

   Figure 44.1 shows that the corresponding columns of the three matrices are closely related, at least for $j = 1, 2, 3$. Intuitively this means that *it is not possible to distinguish exactly whether certain characteristics of the tempo curve stem from the metric, the harmonic or the melodic analysis.* Thus, results may depend on the sequence of orthogonalization. This sequence reflects whether, in our view, the harmonic, the metric or the melodic has priority. Moreover, instead of focusing on names such as "metric weight", "first derivative of the metric weight", etc., we will also try to extract typical weight curves (canonical curves) that appear to be important for the tempo, independently of which of the three analytic approaches (metric, harmonic, melodic) have priority.

   More specifically, the three design matrices are defined in the following way: The columns of $X(harmo), X(metric)$, and $X(melod)$ respectively are orthogonalized and standardized successively. We thus obtain three $n \times 57$ matrices which will be denoted by

$$Z(harmo), Z(metric), \text{ and } Z(harmo).$$

   Each of these matrices has orthonormal columns. The reason for computing three different matrices is that orthonormalization depends on the initial sequence of the columns. An artificial preference of the variables that are accidentally in the first (or first few) column(s) is avoided by carrying out three separate regression analyses with the respective matrices $Z(harmo), Z(metric)$, and $Z(melod)$, and by comparing the common features of the three results.

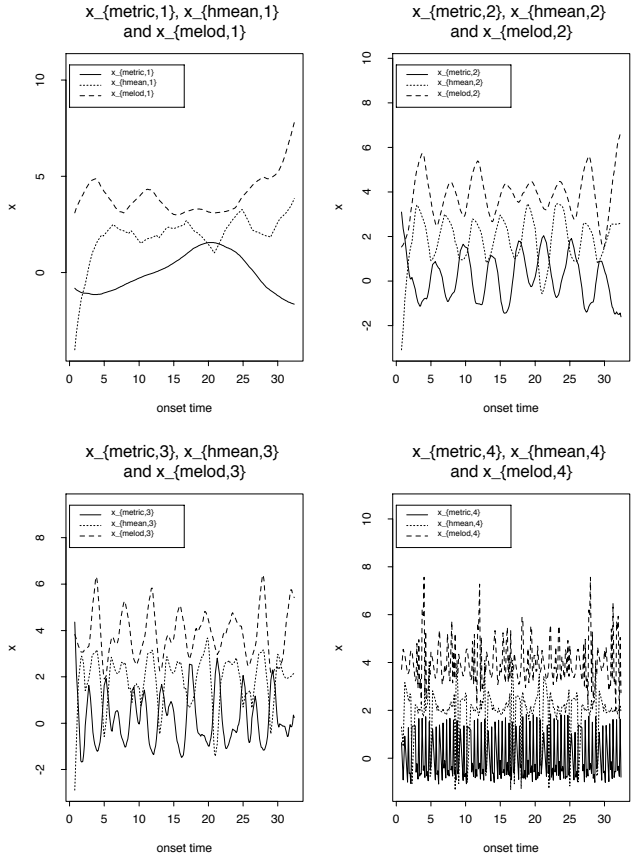


Figure 44.1: The four hierarchical levels of metrical, harmonic, and melodic analyses.

## 44.2 The Method of Regression Analysis

**Summary.** Inspired by visual comparison of logarithmic tempo cures, where strong similarities between the different performances become visible, we applied the following regression model.

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### 44.2.1 The Full Model

Let $Z$ be one of the three matrices $Z(harmo)$, $Z(metric)$, or $Z(melod)$, respectively. The full (i.e., biggest possible) model for the $j^{th}$*individual tempo curve* is

$$y(t_i, j) = Z_i\beta(j) + \epsilon(t_i, j), \ (t_i \in T),$$

where $Z_i$ is the $i^{th}$ row vector of $Z$, $\beta(j) = (\beta_1(j), \ldots \beta_{57}(j))^t$ and $\epsilon(t_i, j)$ $(t_i \in T)$ are (for each fixed $j$) identically distributed zero mean random variables. This means that we assume each performance to be essentially characterized by a 57-dimensional parameter vector $\beta(j)$. Under the present orthonormalized conditions, this vector corresponds to the shaping vector $\omega$ introduced in section 44.1.1. Note that we do not assume the residuals $\epsilon_i$ to be independent, since corrected p-values are used that take into account serial dependence. Also, due to standardization of $y$ and of the columns of $Z$, there is no intercept in the model.

The vector $\beta(j)$ is the parameter vector corresponding to the performance number $j$. Therefore, $\beta(j)$ is assumed to be a random vector, sampled from the space of all "possible" interpretations, with expected value $E[\beta(j)] = \beta$. We then may write

$$\beta(j) = \beta + \eta(j)$$

where $\eta(j)$ is a random vector with $E[\eta(j)] = 0$ and

$$y(t_i, j) = Z_i\beta + Z_i\eta(j) + \epsilon(t_i, j).$$

Intuitively this means that, up to a small unexplained deviation $\epsilon(t_i, j)$, the (logarithmic) tempo of the $j^{th}$ performance at onset time $t_i$ can be expressed as a "mean performance" $Z_i\beta$ plus an individual deviation from the mean that is equal to $Z_i\eta(j)$. For the mean tempo curve

$$\bar{y}(t_i) = m^{-1} \sum_{j=1}^{m} y(t_i, j),$$

we then have

$$\bar{y}(t_i) = Z_i\beta + \tilde{\epsilon}_i(t_i)$$

where $\tilde{\epsilon}(t_i)$ $(t_i \in T)$ are identically distributed zero mean random variables.

### 44.2.2 Step Forward Selection

In the following, the main focus is on the individual curves. Some comments on the mean curve are also given.

In order to decide which components of $\beta$ or $\beta(j)$ respectively are not zero (i.e., which explanatory variables contribute "significantly" to the tempo curve), stepwise forward selection [84] is carried out with F-to-enter level of significance 0.01. For the individual curves, a separate stepwise regression is carried out for each individual. The statistics software S-Plus [467] was used for the calculations.

## 44.3 The Results of Regression Analysis

**Summary.** In the following discussion the main questions are: 1) Is there a relevant association between the analytical weights computed from the score and the observed tempo curves? 2) How complex is the relationship? 3) Are there commonalities and diversities; how can they be characterized?

– Σ –

### 44.3.1 Relations between Tempo and Analysis

**Summary.** The statistical analysis of the relation of tempo data and analytical weights via the Beran operator is carried out.

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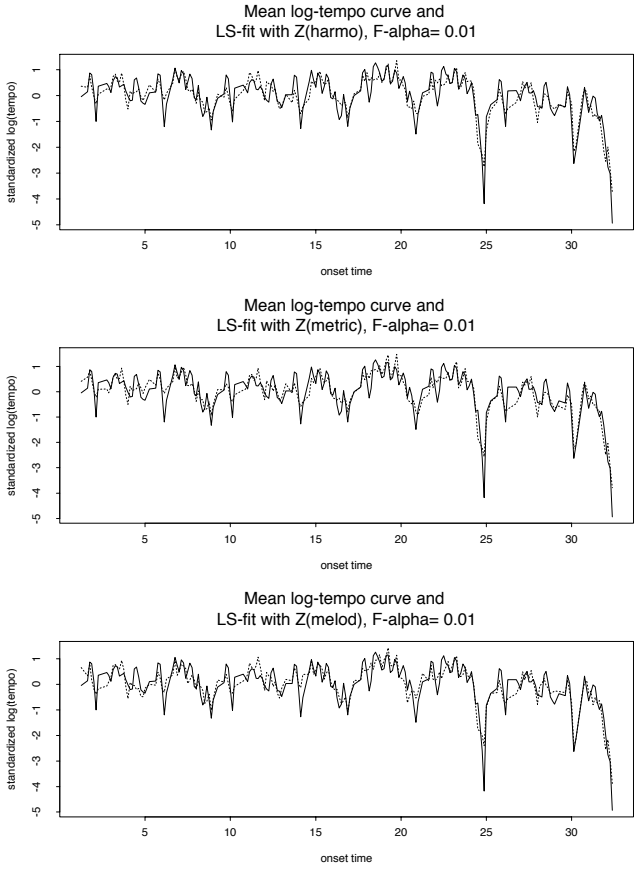


Figure 44.2: Mean tempo curve and LS-fit with orthonormalized $Z$ matrices with harmonic, metric, and melodic emphasis.

To begin with, it is interesting to learn how much can be 'explained' at most by the analytical weights. Recall that in regression, $R^2$ denotes the proportion of the variability of $y$ that is explained by the estimated regression function. Ideally, $R^2$ would be equal to 1.00 which would mean that the (log-)tempo curve can be expressed exactly as a linear function of the analytic information encoded by the design matrix $Z$. Such an exact correspondence between the analytic curves and each tempo curve can hardly be expected. The maximal achievable values

of $R^2$, which are obtained by using the full matrix $Z$ (i.e., without eliminating nonsignificant variables), are quite high however for the mean curve as well as the individual curves. For the mean curve, $R^2$ is equal to 0.84. For the individual curves, we have $0.65 \leq R^2 \leq 0.85$.

It is well known that $R^2$ can be increased by simply including a sufficiently large number of explanatory variables, even if these variables have nothing to do with $y$. It is therefore necessary to investigate which explanatory variables contribute significantly to the response $y$. To do this, we first applied stepwise forward selection with F-to-enter $\alpha = 0.01$. In all cases, all coefficients in the resulting model turned out to be significantly different from zero at the 5% level of significance, even after taking into account the possibility of serial correlations in the residuals.

The values of $R^2$ for the mean curve and the individual curves respectively are still remarkably high: For the mean curve, $R^2$ is equal to 0.79 for $Z(harmo)$, 0.79 for $Z(metric)$ and 0.77 for $Z(melod)$. Figure 44.2 shows that the fit (dotted line) to the mean tempo curve (full line) is very good, even if only significant coefficients are used. For the individual curves, we have $0.46 \leq R^2 \leq 0.79$ for $Z(harmo)$, $0.48 \leq R^2 \leq 0.78$ for $Z(metric)$ and $0.36 \leq R^2 \leq 0.77$ for $Z(melod)$. The low value of 0.36 is obtained for Kubalek. Excluding this performance, we have 0.51 as the lower bound $Z(melod)$. Thus, the quality of the fit is also good in general, but varies individually. This is illustrated further by figure 44.3 to 44.3 for $Z(melod)$. The results on commonalities and diversities given below in section **??** yield further evidence for the existence of a meaningful association between $y$ and $Z$.

### 44.3.2 Complex Relationships

**Summary.** We discuss the complexity of the relationships between the individual results.

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Even when using only significant coefficients, the estimated models are very complex. As an example, consider the performance by Brendel. With $Z(melod)$, the $R^2$ is in this case equal to 0.76. No. 4.4 in figure 44.3-44.6 confirms the good fit.

The following table A summarizes the result:

*Table A: Coefficients of explanatory variables chosen by stepwise forward selection with F-to-enter=0.01 and Z(melod), for the logarithmic tempo curve of Brendel. (The P-values given here do not take into account serial correlations.)*

| $z-$variable | est. coefficient | std. error | t-statistic | P-value |
|---|---|---|---|---|
| $z_{melod,1}$ | -0.3136 | 0.0353 | -8.8760 | 0.0000 |
| $z_{hmean,1}$ | -0.2737 | 0.0353 | -7.7443 | 0.0000 |
| $z_{hmean,2}$ | 0.2781 | 0.0353 | 7.8713 | 0.0000 |
| $z_{hmax,2}$ | 0.2659 | 0.0353 | 7.5248 | 0.0000 |
| $z_{melod,3}$ | -0.1258 | 0.0353 | -3.5597 | 0.0005 |
| $z_{hmean,3}$ | 0.1303 | 0.0353 | 3.6882 | 0.0003 |
| $z_{hmax,3}$ | -0.2800 | 0.0353 | -7.9244 | 0.0000 |
| $z_{melod,4}$ | -0.1562 | 0.0353 | -4.4193 | 0.0000 |
| $d_E z_{melod,1}$ | -0.2663 | 0.0353 | -7.5371 | 0.0000 |
| $d_E z_{melod,2}$ | 0.1567 | 0.0353 | 4.4337 | 0.0000 |
| $d_E z_{hmean,2}$ | 0.1143 | 0.0353 | 3.2356 | 0.0014 |
| $d_E z_{melod,3}$ | 0.2938 | 0.0353 | 8.3143 | 0.0000 |
| $d_E z_{hmean,3}$ | 0.1308 | 0.0353 | 3.7024 | 0.0003 |
| $d_E^2 z_{metric,1}$ | -0.1361 | 0.0353 | -3.8508 | 0.0002 |
| $z_{rit_3}$ | -0.1952 | 0.0353 | -5.5249 | 0.0000 |
| $z_{rit_4}$ | 0.1032 | 0.0353 | 2.9208 | 0.0039 |
| $z_{ferm_1}$ | -0.1425 | 0.0353 | -4.0331 | 0.0001 |

The number of significant coefficients is very large. The model contains all four weight functions, first and second derivatives of various degrees of smoothness and also two ritardando-variables and one suspension. Also note that all degrees of smoothness are used. Formally, even after adjustment for serial correlations, all p-values are below 0.05. (As a cautionary remark, it should be noted however that p-values obtained after model selection can be used as guidelines only.)

### 44.3.3 Commonalities and Diversities

**Summary.** In spite of the high complexity of the selected models for 28 individual tempo curves, there are interesting commonalities and diversities. They are characterized in this section.

– Σ –

Before going into the details of commonalities and diversities among the 28 given performances, we should make a remark on the performance selection and the performers as made available by Repp [438]. Above all, we should emphasize that Repp succeeds in a choice of first quality pianists, among others the celebrated "romantic virtuoso" Vladimir Horowitz, the "analytical mannerist" Alfred Brendel, or the "perfect but utterly cool" omnipresent Vladimir Ashkenazy, to name just three of them[3]. So from the point of view of performance culture,

[3]For a complete list of all performers, see [438]. We shall only name selected performers who are relevant to this analysis.

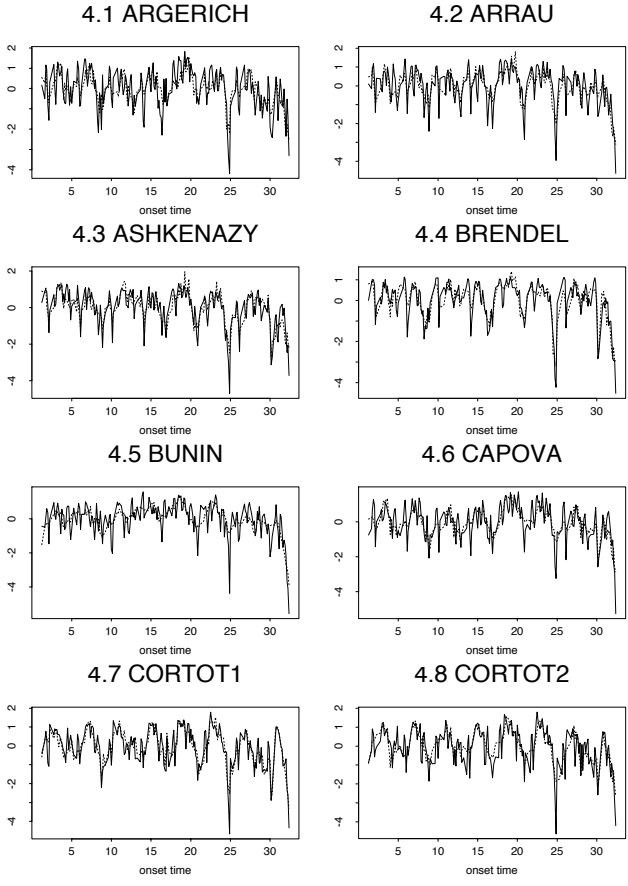


Figure 44.3: Fit for $Z(melod)$.

the election is unprecedented and representative. But in this scientific context, we must refrain from further judgments. This discussion is not about journalistic criticism. However, we should encourage critics to review their understanding of performance by focusing on the question whether and to what degree the analytical structure of a score may be responsible for agogical expressivity. Our present answer to this question—incomplete as it must remain—may seem to position some of the artists in unexpected relative position to each other. But this is not surprising since we do not claim that the overall judgment from common criticism really does represent the strictly analytical perspective of our approach. As mentioned in chapter 36, one should absolutely add emotional and gestural components to reach a complete description, an objective which was out of reach of this discussion.

In spite of the high complexity of the selected models for individual tempo curves, there turn out to be interesting commonalities and diversities that can be characterized by either of the matrices $Z(harmo)$, $Z(metric)$ and $Z(melod)$ respectively. Recall that using $Z(harmo)$

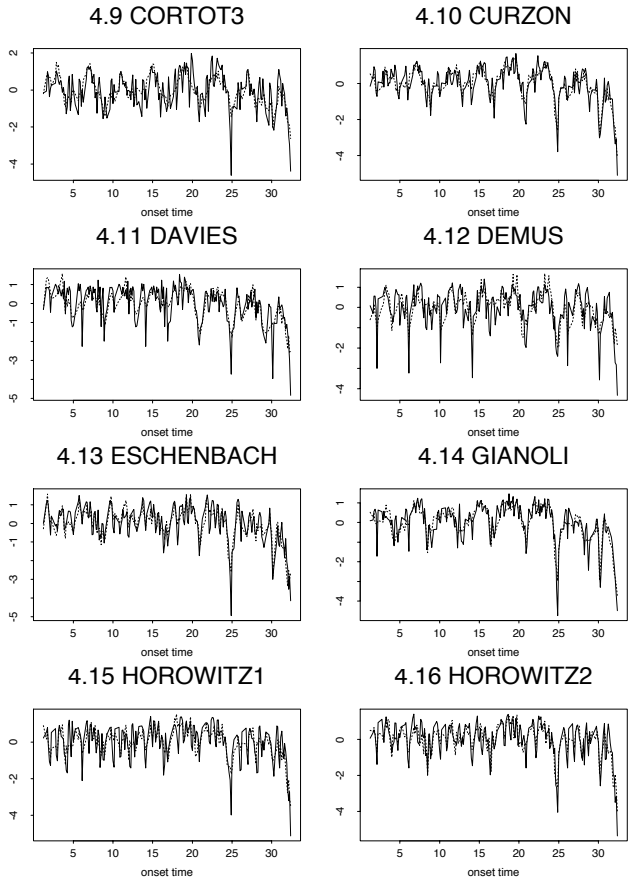


Figure 44.4: (Cont.)

corresponds to an understanding of the score that put a first priority on the harmonic structure. Using $Z(metric)$ corresponds to putting first priority on the metric structure. Using $Z(melod)$ corresponds to putting first priority on the melodic structure. Therefore, depending on which of the three matrices is used, somewhat different results should be expected. The fundamental problem is the ambiguity of a performance. In general, based on one performance, it cannot be decided with certainty whether certain features of the tempo are 'due to' the harmonic, the metric or the melodic content. Nevertheless, the results below show a strong similarity between the three regressive analyses. Thus, there appears to exist at least a core of tempo features that are unambiguously attributable to specific weight functions.

A number of different aspects of commonality and diversity can be considered. Here, three possible aspects are described.

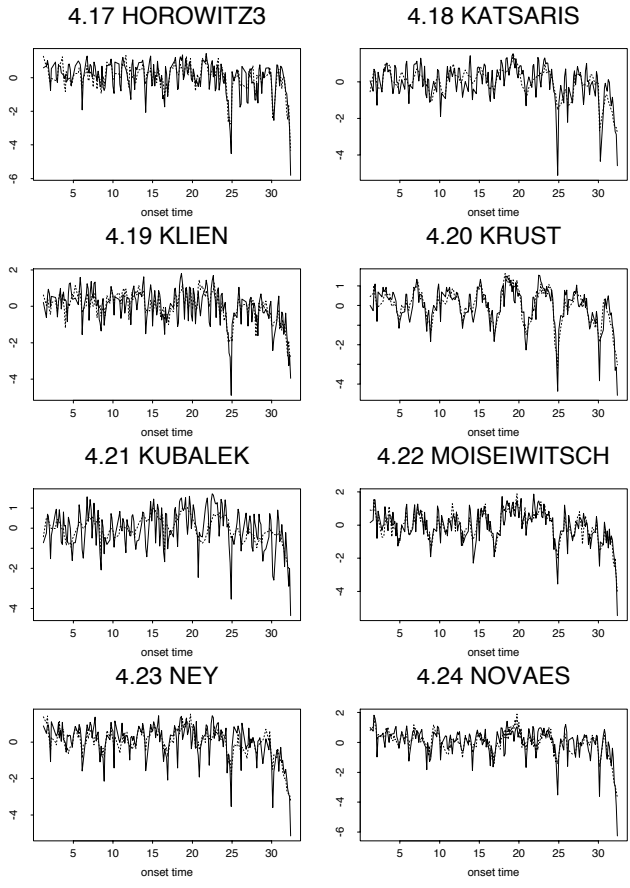


Figure 44.5: (Cont.)

#### 44.3.3.1 Signs of Coefficients

We ask the following question: For which $k = 1, \ldots p$ do we have either $\hat{\beta}_k(j) \geq 0$ for all $1 \leq j \leq m$, or $\hat{\beta}_k(j) \leq 0$ for all $1 \leq j \leq m$? In other words, which coefficients have the same sign for all performances? *The result is quite amazing: For $Z(metric)$, $Z(harmo)$, and $Z(melod)$, all except 3, 2, and 1 coefficients (out of 57) respectively have the same sign for all performances.* Thus, the sign of the coefficients is a very strong commonality. The analytic curves 'act' in the same direction. In particular, the following general tendency can be observed:

- The tempo decreases as the original (not orthogonalized) harmonic weight increases.
- The tempo increases as the original (not orthogonalized) metric weight increases.
- The tempo decreases as the original (not orthogonalized) melodic weight increases.

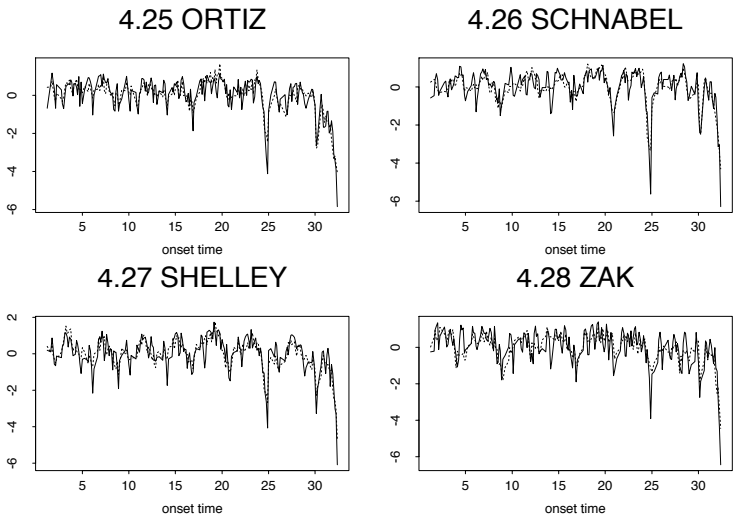


Figure 44.6: (Cont.)

It should be noted however that these conclusions are valid under the assumption that all other variables are kept fixed. As we saw above, for the score considered here, the original weights are strongly correlated. This makes the actual relationship between weights and tempo much more complicated.

#### 44.3.3.2 Frequency of Variable Inclusion

For $k = 1, ..., p$, let $n_k = \sum_{j=1}^{m} 1\{\hat{\beta}_k(j) \neq 0\}$ be the number of performances for which the explanatory variable number $k$ was included in the model.

Figures 44.7-44.9 show, for $Z(harmo)$, $Z(metric)$, and $Z(melod)$ respectively, the curves of variables that were chosen at least 24 times (out of 28). The curves are multiplied by the sign of the coefficient. At least two types of curves are common to practically all performances, independently of the matrix that is used: 1) very smooth 'global' curves, such as $z_{melod,1}$, that shape the overall tendency of the tempo; 2) almost periodic curves, with a period of about four measures, corresponding to the approximate periodicity of the harmonic curve $z_{hmean,2}$.

*Comment on* $Z(harmo)$. Note that $z_{hmean,1}$ is identical with $x_{hmean,1}$, see figure 44.1a. Moreover, $z_{hmean,2}$ is almost the same as $x_{hmean,2}$ in figure 44.1b. Also, $d_E z_{hmean,3}$ exhibits features that are very similar to $x_{hmean,3}$, see figure 44.1c. Thus, analytical weights obtained by local averaging without orthogonalization have a direct impact on the performance. In fact, by the above, the orthonormal curves selected as most relevant by the regression turn out to be closely related to the original curves.

*Comment on* $Z(metric)$. Here, resemblance of curves is similar to $Z(harmo)$. Namely:

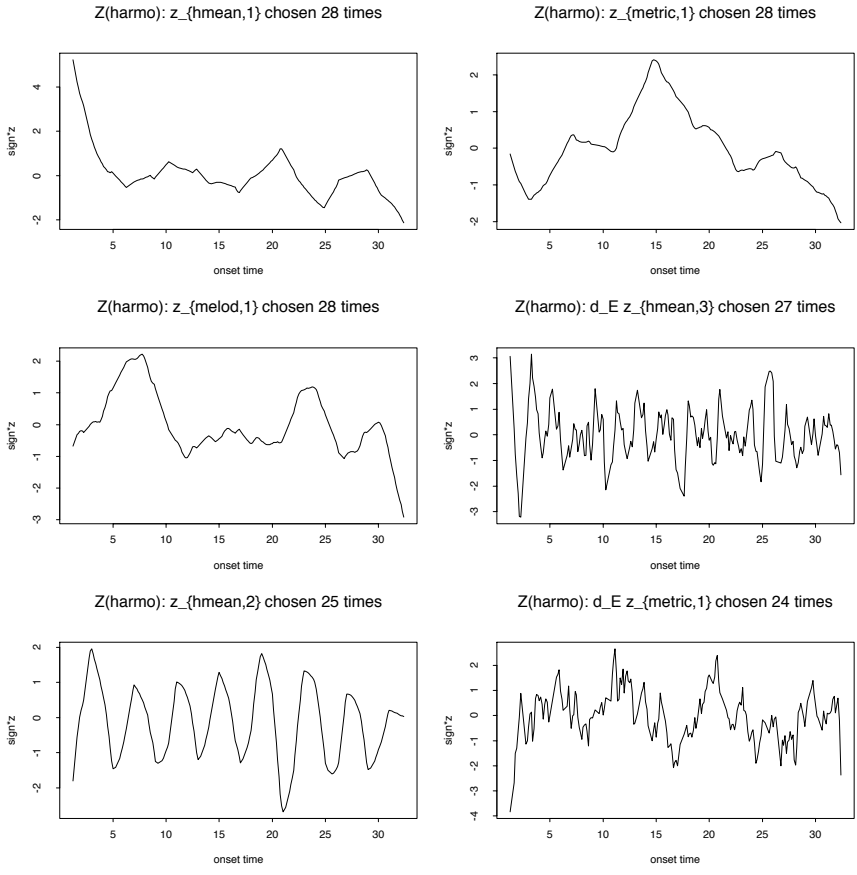


Figure 44.7: Frequently selected curves for $Z(harmo)$.

$z_{hmean,1}$ is almost the same as $x_{hmean,1}$, $z_{metric,1}$ and $x_{metric,1}$ are identical, and $z_{hmean,2}$ is very similar to $x_{hmean,2}$. Also $z_{melod,1}$ is almost the same as $z_{melod,1}$ in $Z(harmo)$, compare figures 44.7 and 44.8. Again, we conclude that the original averaged weight curves influence the performance directly.

*Comment on* $Z(melod)$. Similar comments as for $Z(harmo)$ and $Z(metric)$ apply for $Z(melod)$. Here, $z_{melod,1}$ is the mirror image of $x_{melod,1}$. Further, $z_{hmean,1}$ and $z_{hmean,2}$ are very similar to $x_{hmean,1}$ and $x_{hmean,2}$, respectively.

*Comment on non-linear deformations.* The melodic weights seem to play the most prominent role. Independently of the emphasis, $z_{melod,1}$ is chosen for all 28 performances. For the melodic emphasis, $z_{melod,1}$ is obviously identical with $x_{melod,1}$. For the harmonic and metrical emphasis, the corresponding $z_{melod,1}$-curves turn out to be non-linear deformations of $x_{melod,1}$. The method of non-linear deformations of analytical weights as arguments of refined shaping of performance is also implemented in the PerformanceRUBETTE® of RUBATO®.

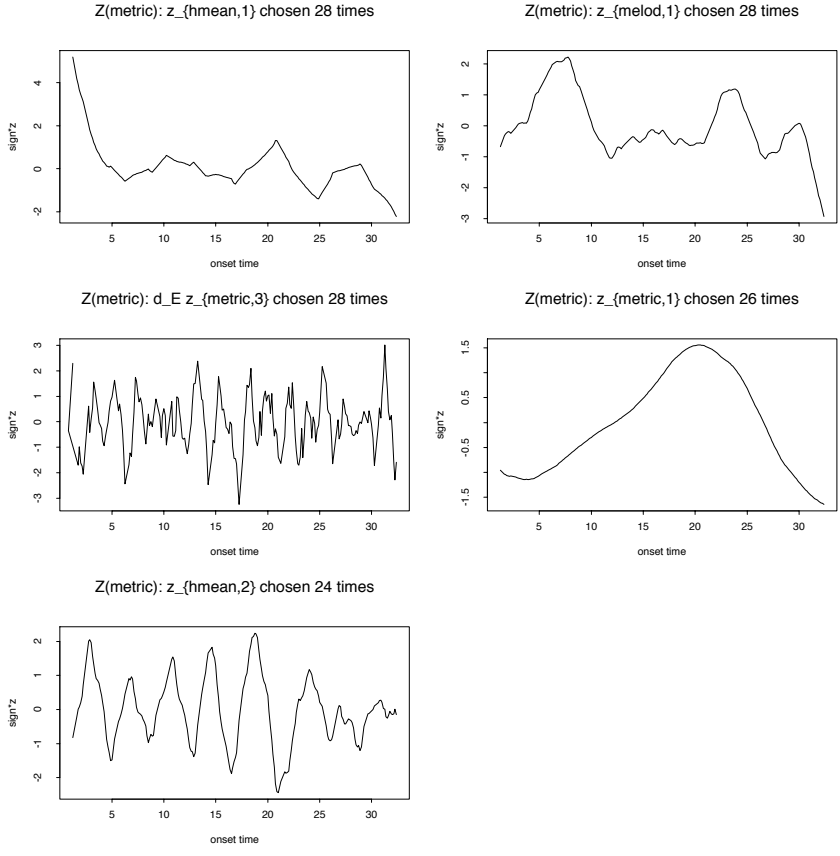


Figure 44.8: Frequently selected curves for $Z(metric)$.

In summary, we conclude that for the score of "Träumerei", *there is a small number of "canonical" analytical weight curves that are relevant for most performances and essentially do not depend on the analytical emphasis.*

#### 44.3.3.3 Largest Coefficients

Since the design matrix $Z$ is orthonormal, the importance of the $k^{th}$ explanatory variable may be assessed by the absolute value of the corresponding $k^{th}$ estimated coefficient (ranked in comparison to the other coefficients). (Also note that, due to orthonormality, all estimated slope components are uncorrelated and their standard deviations are the same.) For fixed $j$, let $r_k(j)$ be the rank of $|\hat{\beta}_k(j)|$ among all coefficients $\hat{\beta}_s(j)$ $(s = 1, ..., p)$. Furthermore, for $1 \leq l \leq p$, let $f_k(l) = \sum_{j=1}^{m} 1\{r_k(j) > p - l\}$. Thus, $f_k(l)$ is the number of performances for which $|\hat{\beta}_k(j)|$ is at least the $l^{th}$ largest.

Consider first $l = 1$. Thus, $f_k(1)$ is the number of performances for which the $k^{th}$ variable

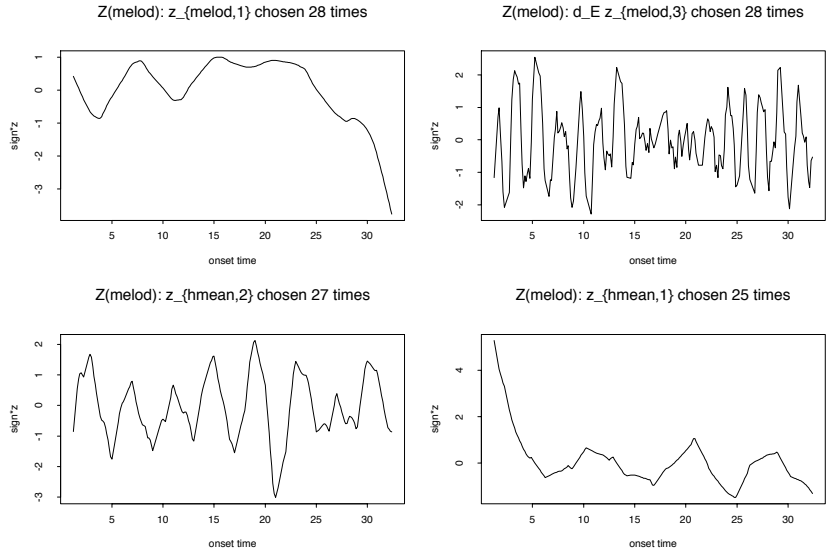


Figure 44.9: Frequently selected curves for $Z(melod)$.

is most important. It turns out that $f_k(1)$ is not zero for a very small set of variables. The results for the three matrices are:

$Z(harmo)$ : figure 44.10 displays the four curves for which $f_k(1)\neq 0$, i.e., variables that are most important for at least one performance. Table B shows the clusters of columns H1 to H4 of performances for which the corresponding variables $z_{hmean,2}$ ($\sim$ H1), $z_{hmean,1}$, $z_{metric,1}$, and $z_{melod,4}$ are the most important ones.

Thus, using $Z(harmo)$ and therefore an approach that gives priority to the harmonic structure, the first cluster of performances has the 4-measures periodicity of the harmonic structure as the dominating feature. In particular, all Cortot performances are included. For the second cluster that includes in particular Horowitz1 and Horowitz2, the more global shaping curve $z_{hmean,1}$ is most prominent. For Bunin and Gianoli, a global curve with a peak around the 15th measure is most important. Finally, for the first performance by Horowitz1, the very detailed local structure of the melodic curve $z_{melod,4}$ seems to dominate.

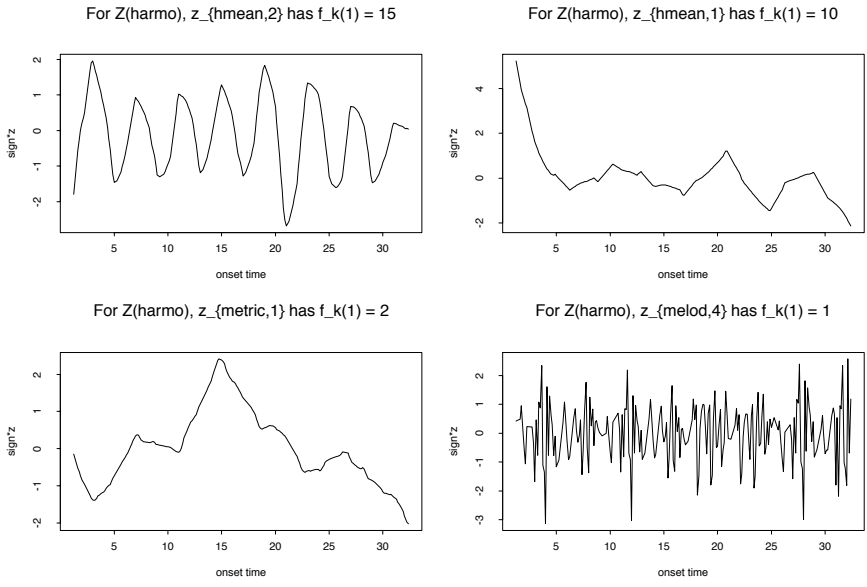


Figure 44.10: Important variables for $Z(harmo)$.

*Table B: Overview of clusters as derived by the above criterion with* $l = 1$.

| Artist | H1 | H2 | H3 | H4 | M1 | M2 | M3 | M4 | M5 | M6 | ML1 | ML2 | ML3 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ARG | ● | | | | | | | ● | | | ● | | |
| ARR | ● | | | | | | | ● | | | ● | | |
| ASH | ● | | | | ● | | | | | | | | ● |
| BRE | | ● | | | ● | | | | | | ● | | |
| BUN | | | ● | | | | ● | | | | ● | | |
| CAP | ● | | | | | | ● | | | | ● | | |
| CO1 | ● | | | | | ● | | | | | | ● | |
| CO2 | ● | | | | | ● | | | | | | ● | |
| CO3 | ● | | | | | ● | | | | | | ● | |
| CUR | ● | | | | | ● | | | | | ● | | |
| DAV | ● | | | | | | | ● | | | ● | | |
| DEM | ● | | | | | ● | | | | | ● | | |
| ESC | | ● | | | ● | | | | | | ● | | |
| GIA | | | ● | | | | ● | | | | ● | | |
| HO1 | | | | ● | | | | | | ● | ● | | |
| HO2 | | ● | | | ● | | | | | | ● | | |
| HO3 | | ● | | | ● | | | | | | ● | | |
| KAT | ● | | | | | ● | | | | | ● | | |
| KLI | | ● | | | ● | | | | | | ● | | |
| KRU | ● | | | | | | | ● | | | | ● | |
| KUB | ● | | | | | | ● | | | | ● | | |
| MOI | | ● | | | ● | | | | | | ● | | |
| NEY | | ● | | | ● | | | | | | ● | | |
| NOV | | ● | | | ● | | | | | | ● | | |
| ORT | | ● | | | ● | | | | | | ● | | |
| SCH | ● | | | | | | | | ● | | ● | | |
| SHE | ● | | | | | ● | | | | | ● | | |
| ZAK | | ● | | | ● | | | | | | ● | | |

$Z(metric)$ : The clusters of columns M1 to M6 in Table B show the results for $Z(metric)$ (see figure 44.11). These six columns correspond to the variables $z_{hmean,1}$, $z_{metric,2}$, $z_{metric,1}$, $z_{hmean,2}$, $z_{hmax,3}$, and $d_E z_{metric,3}$.

Hence, by use of an approach that gives priority to the metric structure, the cluster with Horowitz2 and 3 is exactly the same as for $Z(harmo)$, with the exception of Ashkenazy who

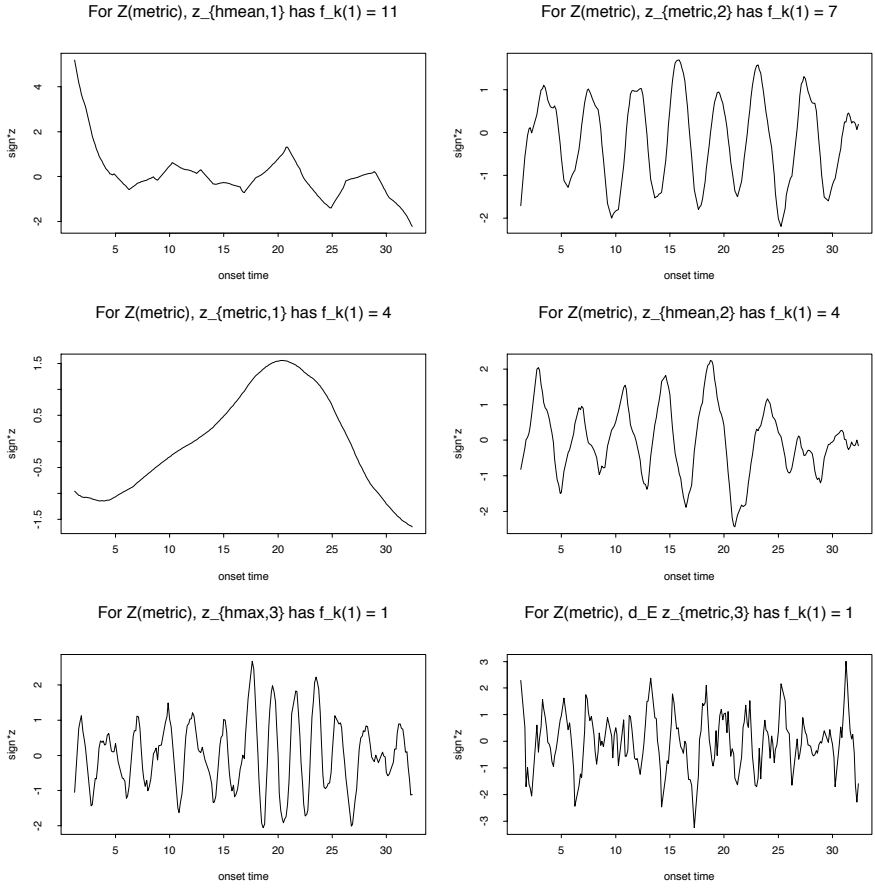


Figure 44.11: Important variables for $Z(metric)$.

was not included before. Also, it corresponds essentially to the same curve. Similarly, the cluster with the three Cortot performances corresponds to a very similar curve. It is however smaller since it is a proper subset of the previous 'Cortot' cluster above. Bunin and Gianoli are again in the same separate cluster, this time together with Capova and Kubalek. The peak of the curve is now around measure 20. The cluster with Argerich contains several of those performances that were previously in the Cortot cluster. The dominating curve is still almost periodic with a period of about four measures. Finally note that Horowitz1 builds again a separate cluster with a locally very refined metric curve corresponding to the derivative of $z_{metric,3}$.
$Z(melod)$ : For $Z(melod)$ (figure 44.12), we obtain the clusters shown in columns ML1 to ML3 in Table B. The corresponding variables are: $z_{melod,1}$, $z_{hmean,2}$, and $z_{melod,2}$.

The melodic approach yields very simple clusters. For almost all performances, including all Horowitz performances, the global shape of $z_{melod,1}$ is the most important feature. For Cortot1 through 3, and Krust, the 4-measures periodicity of $z_{hmean,2}$ is most important. For

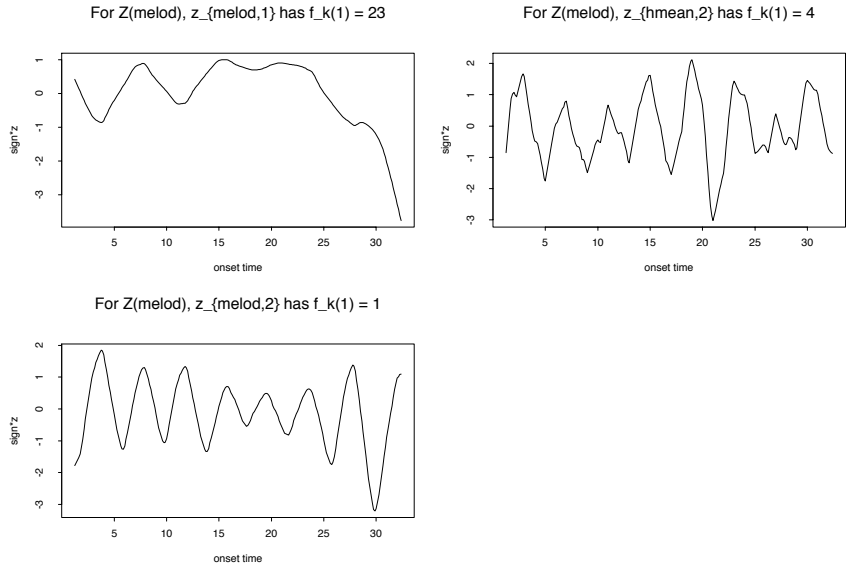


Figure 44.12: Important variables for $Z(melod)$.

Ashkenazy, the 4-measures (almost) periodic curve $z_{melod,2}$ is dominating. Note in particular that the minima and maxima of this curve do not occur at the same place as for $z_{hmean,2}$. In comparison, $z_{melod,2}$ appears to be shifted to the right. Also, $z_{melod,2}$ has an extreme local minimum around the beginning of measure 30.

#### 44.3.3.4 Argerich "Versus" Horowitz

The remarkable first performance Horowitz1 from 1947 evidences a preference of very detailed local information, be it from the melodic or metrical analysis—in typical contrast to the highly coherent Argerich performance. This observation is confirmed by an investigation of the correlation coefficients in the algebro-geometric analysis of the performance genealogy in the sense of RUBATO®'s stemma theory, see chapter 38. When translated into common language these quantitative results are in perfect coincidence with the judgments of experts on Argerich's and Horowitz' specific differences in performance [42]. Let us therefore make these findings more meaningful to the common understanding. How would an interpreter such as Horowitz experience his performance? He would look up a few notes ahead and remember just a few of the past note events when hitting a couple of keys in a given moment. He would then realize a couple of neighboring analytical facts, such as a harmonic step or a melodic contour, in his imagination, and then shape the present note event in its tempo, dynamics, and articulation within this minor context, and thusly express his analytical consciousness. Metaphorically speaking this resembles a near-sighted man who can only see and recognize nearby objects. It is as if he had no significant memory of what was happening several bars ago, or of what will happen in the

larger time span ahead. In contrast, a performer of the Argerich type would be aware of lots of long-range facts in the overall analytical stream of the piece which is being played. She would then remember and plan everything and therefore hit the present notes in full consciousness of what was and what will be. This is what the semantics of the data tells.

*Table C: Overview of clusters as derived by the above criterion with* $l = 3$.

| Artist | M1 | H2 | H1 | DM3 | M2 | DM2 | HM2 | HM3 | M4 | DM1 | DDM1 | SS2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ARG | ● | ● | | ● | | | | | | | | |
| ARR | ● | ● | | ● | | | | | | | | |
| ASH | | | ● | | ● | | ● | | | | | |
| BRE | ● | | | ● | | | | ● | | | | |
| BUN | ● | ● | | | | | | | | | ● | |
| CAP | ● | ● | | ● | | | | | | | | |
| CO1 | ● | ● | | | | ● | | | | | | |
| CO2 | ● | ● | | | | ● | | | | | | |
| CO3 | ● | ● | | | | ● | | | | | | |
| CUR | ● | ● | | | | | | ● | | | | |
| DAV | ● | ● | | | ● | | | | | | | |
| DEM | ● | | | ● | ● | | | | | | | |
| ESC | ● | | ● | ● | | | | | | | | |
| GIA | ● | ● | | | | ● | | | | | | |
| HO1 | ● | | | ● | | | | | ● | | | |
| HO2 | ● | | | | | | ● | | ● | | | |
| HO3 | ● | | ● | | | | | | ● | | | |
| KAT | ● | ● | | ● | | | | | | | | |
| KLI | ● | | ● | | | | ● | | | | | |
| KRU | ● | ● | | | ● | | | | | | | |
| KUB | ● | ● | | | | | | | | | | ● |
| MOI | ● | ● | ● | | | | | | | | | |
| NEY | ● | | ● | ● | | | | | | | | |
| NOV | ● | ● | ● | | | | | | | | | |
| ORT | ● | | ● | | | | | | | ● | | |
| SCH | ● | ● | | | | | | ● | | | | |
| SHE | ● | ● | | | ● | | | | | | | |
| ZAK | ● | ● | ● | | | | | | | | | |

Similar results can be obtained for $l > 1$. Here, $f_k(l)$ is the number of performances for

which the $k^{th}$ variable is among the $k$ most important ones. Consider, for instance, $l = 3$. The following partially overlapping clusters corresponding to variables with $f_k(3) \neq 0$ are obtained (see Table C). The columns correspond to the following variables: M1 $\sim z_{melod,1}$, H2 $\sim z_{hmean,2}$, H1 $\sim z_{hmean,1}$, DM3 $\sim d_E z_{melod,3}$, M2 $\sim z_{melod,2}$, DM2 $\sim d_E z_{melod,2}$, HM2 $\sim z_{hmax,2}$, HM3 $\sim z_{hmax,3}$, M4 $\sim z_{melod,4}$, DM1 $\sim d_E z_{melod,1}$, DDM1 $\sim d_E^2 z_{melod,1}$, and SS2 $\sim z_{sus2}$.

The first cluster consists of all performances except Ashkenazy. Thus, using the 'melodic approach', apart from Ashkenazy, the global melodic curve $z_{melod,1}$ is one of the three most important factors for all tempo curves. Again, the 4-measure periodicity determines clusters with Cortot performances. It is remarkable that, in spite of the large number of overlapping clusters, there is no cluster—except the first one—that contains at the same time Cortot and Horowitz. Moreover, there is one cluster consisting solely of Horowitz1 through 3 corresponding to the complex local melodic structure of $z_{melod,4}$.

Evidently, many more detailed comments about figures 44.10-44.12 could be added. We conclude the analysis by noting that the relative size of the coefficients suggests a natural way of obtaining simplified tempo curves that contain the most important features. For given $j$ and $1 \leq q \leq p$, let the $p \times 1$ vector $\gamma_q(j) = [\gamma_{q,1}(j), \gamma_{q,2}(j), , ..., \gamma_{q,p}(j)]^t$ be defined by $\gamma_{q,k}(j) = \hat{\beta}_k(j) 1\{r_k(j) > p - q\}$. Then $y_q(t_i, j) = Z\gamma_q(j)$ is a simplified tempo curve that corresponds to using the variables (analytic curves) that are among the $q$ most important ones for tempo curve $j$, importance being measured by $r_k(j)$. Thus, the resulting tempo curve is a simplified curve obtained superposition of the $q$ most important features only. Note that, for $q = p$, this yields the complete curve fitted by stepwise regression. Nos. 44.13.1-44.16.28 in figures 44.13-44.16 display $y_q$ $(q = 1, ..., p)$ for $Z(melod)$ for all performances.

### 44.3.4 Overview of Statistical Results

The main statistical conclusions from the analysis above can be summarized as follows:

- There is a clear association between metric, melodic and harmonic weights and the tempo.
- The exact relationship between the analytic weights $Z$ and an individual tempo curve is very complex. However, a large part of the complexity can be covered by our model.
- Commonalities and diversities among tempo curves may be characterized by a relatively small number of curves. There is in principle no unique way of attributing features of the tempo to exactly one cause (harmonic, metric or melodic analysis). Which curves need to be used depends partially on which of the three analyses (harmonic, metric, melodic) has 'priority'. However, there seems to be a small number of canonical curves that are essentially independent of the priorities and which determine a large part of the commonality and diversity among tempo curves. Natural clusters can be defined.
- There is a natural way of reducing an individual tempo curve to a series of simplified tempo curves containing an increasing number of features.

Overall, the proposed method yields a variety of results that are interpretable from the point of view of music and performance theory. In particular, the hierarchic approach of decomposing each of the weight functions into components of different degrees of smoothness seems to

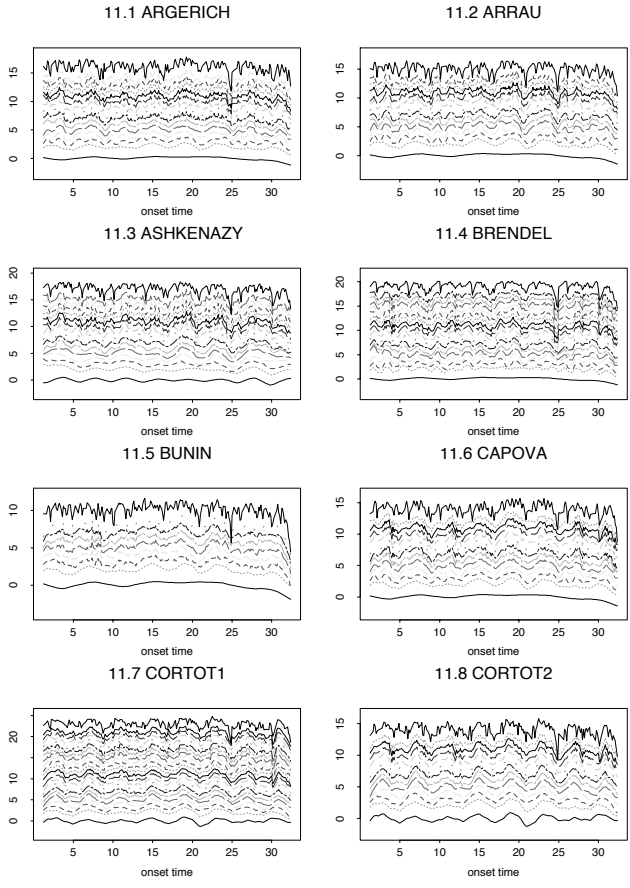


Figure 44.13: (44.13-44.16) Superposition of the $q$ most important features.

be appropriate. The different choices of the bandwidth $h$ correspond to a hierarchic approach to musical performance, starting with the most global features of the score and refining the performance successively in greater detail.

The results here are closely related to Repp's work [438]. Repp applied principal component analysis to the 28 tempo curves. One of his main results is that Cortot and Horowitz appear to represent two extreme types of performances. Thus, in a heuristic way, Repp suggested classifying the performances according to their factor loadings into a Cortot and a Horowitz cluster respectively. Our regression analysis confirms the basic findings. Due to the use of weights obtained from a musical analysis of the score, we obtain further information about the nature of the commonalities, diversities and clusters. For instance, as discussed above, for the Horowitz cluster, the most important feature appears to be the overall descending line of $z_{1,melod}$ and very local variations of the tempo that correspond to the local variations of the analytic weights. On the other hand, for the Cortot cluster, the up and down movement of $z_{3,hmean}$ with a period

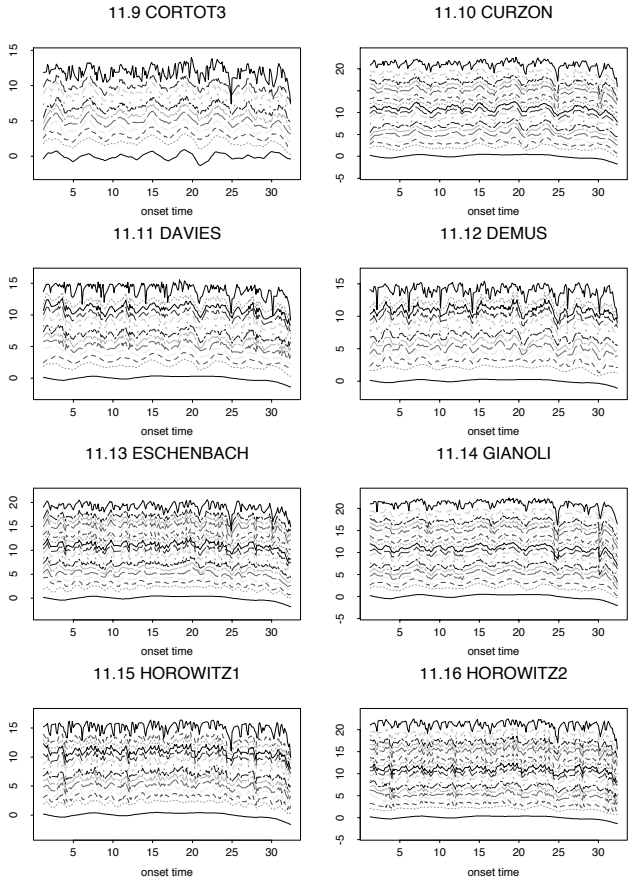


Figure 44.14: (Cont.)

of about four measures is the most important characteristic.

**Result 6** *More generally, this approach reveals a set of canonical curves whose combination yields the most important features of a tempo curve. It should be emphasized that* these curves are score-specific*. Thus, for each score (in our case "Träumerei"), a new set of essential curves is obtained. The original weights as well as the decomposition into parts of different smoothness are based on the specific score that is performed. This is a crucial feature that is in sharp contrast to traditional mathematical 'omnibus-decompositions' such as provided by Fourier or wavelet analysis. The score-specific choice of the $Z-$matrix enables us to relate statistical results directly to the musical/analytic content of the score.*

*The main point here is this: We argue that to understand the character of tempo, it is above all important to refer it to a "basis" of score specific analytical curves and not to curves—such as sinoidal curves in Fourier representation—which have a generic type that tells nothing about*

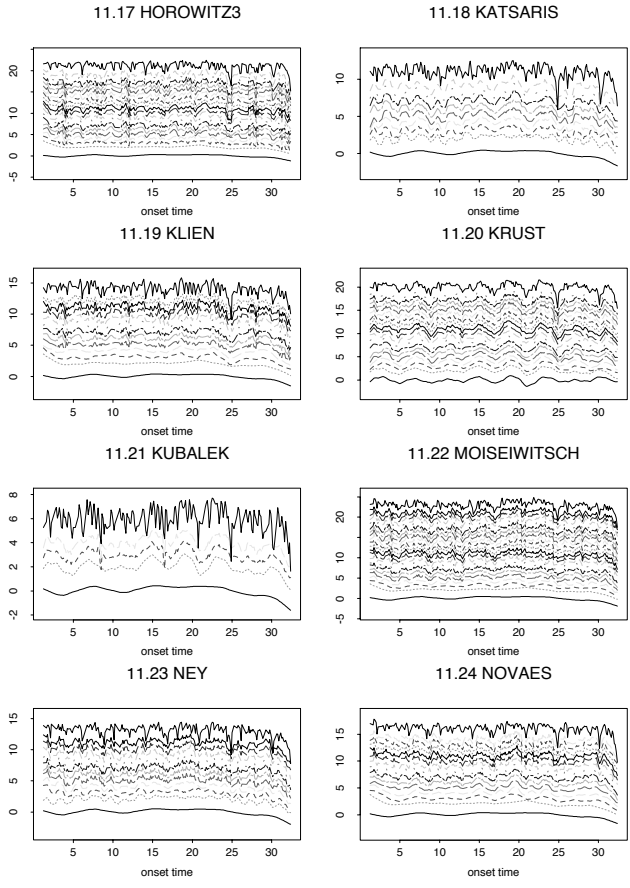


Figure 44.15: (Cont.)

*the particular context of the genealogy for the given tempo curve.*

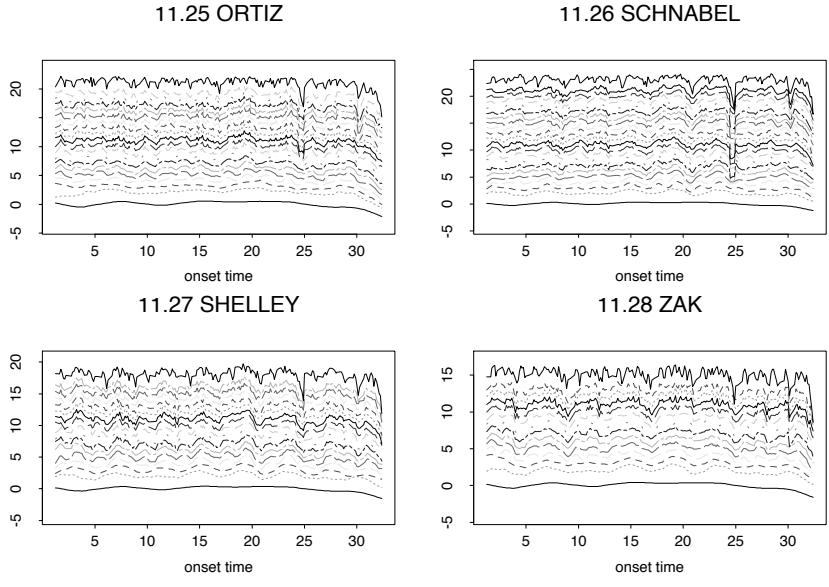


Figure 44.16: (Cont.)

# Part XII

# Inverse Performance Theory

# Chapter 45

# Principles of Music Critique

*Never trust the artist. Trust the tale.*
*The proper function of a critic is to*
*save the tale from the artist who created it.*
David Herbert Lawrence (1885–1930)

**Summary.** Inverse performance theory deals with the critic's problem of how to extract in his critique what is hidden behind a performance output. To initiate this theory, we therefore should question the perspectives which a critic has to envisage. To begin with, we inquire the intriguing task of feuilletonistic critique: why is it a never ending story? Is it substantially necessary—beyond music business? We then position the critic within the sociological context: How do norms intervene in critique? This is exemplified by Glenn Gould's performative redefinition of classics. The chapter terminates with an ethnomusicological view on historicistic performance as it is typically undertaken by Nicolas Harnoncourt.

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## 45.1 Boiling down Infinity—Is Feuilletonism Inevitable?

**Summary.** It is not accidental that music critique has stuck to feuilletonism. Its scope is an infinite one in several dimensions: It has to cope with the infinite interpretative work with respect to a given text and with the infinity of expressive nuances of each given interpretation. We analyze the necessity of boiling down this infinite challenge in view of the poor tools of traditional musicology.

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We have learned in previous chapters, in particular in section 13.4.1 and section 32.2, fact 18, that performance conveys an infinite message. And that this infinity is a double one[1] : that of interpretative perspectives as they are realized in music analysis, and that of performative

[1] Leaving aside the gestural and emotional rationales for the time being.

shaping variants as they may be expressed on the infinitesimal vocabulary of performance fields. It is not clear whether critics are aware of such a variety of background that may produce concrete performances. In particular with respect to (analytical) interpretation, they preferredly stick to the traditional canon of how the structure of a composition should be viewed and interpreted or analyzed, respectively. Of course, it is not clear whether music critics should be cognizant of possibly new interpretations, but once they have gone into their business, a creative dealing with analytical problems should be mandatory. One may understand that this is not necessary ante rem, but after the event, a re-reading of the text should be considered, be it only for comparative handling of the present performance: Could it be that the artist discovered a new interpretation of the given text? In practice, the selection of an analytical interpretation (in the best case, autoincompetent critics excluded...) is just a matter of limitations of time, energy, and interest, besides ignorance of the infinite variety of interpretations. As to the infinity of performance nuances, this is beyond the vocabulary of music critics and it is also beyond the present measurement technology for such data: In a common concert, no performance field reconstruction is feasible. So critics are nolens volens limited to describing performance by use of common language expressions ("elegant diminuendo blended by a mysterious pedaling cloud...") which beyond its imprecision cannot relate expression to interpretation.

So is feuilletonism inevitable? Or rather: Is such a bad feuilletonism inevitable? Is it necessary to play the game of a unique "best" interpretation whose expression has to move along unreflected paths of prejudices? The alternative would be to embed one's judgment in the potential infinity of analytical interpretation and expressive performance. And to keep this embedding omnipresent in the critical discourse. We argue that the most precious role of a music critic would be that of putting the infinity of perspectives on a musical work into evidence in every concert or CD review. These would be the crucial points:

- Infinity of analytical interpretations,
- infinity of expressive performances,
- infinity of correlations between interpretative rationales and expressive performance shaping.

And it is not the question about bad or good quality in these specifications, *in the limit, the only quality is to teach us something about the work in question, and about the relativity of each perspective. Suggesting a boiled-down finitistic or even unicorned view of art is a destructive way of reduction and hinders every deepening or progress in the arts.*

## 45.2 "Political Correctness" in Performance—Reviewing Gould

**Summary.** Given the infinity of critical understandings of artistic performance, norms are easily infiltrated against unlimited variation of expressivity. We make the point concrete with the example of Glenn Gould's eccentric (ab-normal...) performance of classics from Bach to Beethoven.

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A testing ground for a valid music critique is Glenn Gould's performance of classical works from Bach to Webern. His technically unprecedented performances have evoked strong reactions which unveil a number of limitations in common critique styles.

Whereas Gould's Bach performances may be non-conformist, but still acceptable and adequate for Bach's compositions, his performances of Beethoven's sonatas is beyond the supportable deviation from common taste. The famous critic Joachim Kaiser has described in [257] the most famous "mis-performance" of a Beethoven sonata on the example of Gould's presentation of op. 57 "Appassionata":

> *Bei Goulds Wiedergabe des allegro assai dürfte es sich um die verrückteste, eigensinnigste Darstellung handeln, die jemals ein Pianist einem Bethoven-Satz hat angedeihen lassen; und das will etwas heißen. Gould hält es für richtig, demonstrativ langweilig und gelangweit den Kopfsatz so zu bieten, als ob ein Beethoven-Verächter seinen Plattenspieler nur mit halber Geschwindigkeit ablaufen ließe. Tranig langsam, langweilig und gelangweilt, die Triller während des pp im Schneckentempo, während der Fortissimo-Stellen etwas rascher, quält sich die Musik vorbei. Man meint, der Pianist imitiere ein Kind, das mit erfrorenen Fingern die Appassionata vom Blatt spiele. Nur selten vergißt er dabei, daß er ja vergessen machen wollte, der genialische Glenn Gould zu sein.*

This critique is strongly based upon the commonly accepted reading of the Beethoven text as a passionate message which calls for temperament and stormy dynamics in performance, and not for analytical cool vivisection of such a vital piece of literature. In Kaiser's characterization, Gould's production is like a "child with frozen fingers in a sight-reading performance". Here, the different and aberrant performance is incorrect, even forbidden. It is a norm which the politically incorrect Gould has broken and thus made the sonata ridiculous; Kaiser even comments that the sonata "remains silent" when confronted with such a misreading.

The basic hypothesis behind such an outrageous indignation is that Kaiser knows what and when and how the sonata (which is personalized here) would have communicated, and that crazy Gould just destroyed that known and accepted messaging. Kaiser in fact evokes an installed performance grammar which requires a passionate forte seventh degree cascade towards the piano on the dominant in bars 14-16 of allegro assai. Instead, Gould descends like a noble, bored lady and snobbishly sits down on the boring dominant fermata. No passion whatsoever.

The same, even more dramatic viz. ridiculous deformation can be observed in Gould's performance of sonata op.106, "Hammerklavier". This case is even worse since one just thinks that Gould did not understand a single word of the text, he simply was too stupid for the performance task.

What happened? And why was Gould's Bach so much more successful? Evidently, Gould's microscopic performance method works for Bach, and not for Beethoven. Why does this microscopic view fascinate and illuminate Bach's work whereas it virtually kills Beethoven's sonatas? The point is that in Beethoven's work, there is an inbuilt performance grammar which is not engraved in the score but stems from the performance tradition as such, an oral tradition so to speak, an element of rhetoric communication which transcends written code. Instead, Gould reads the same code from the Bach and from the Beethoven scores, and effectively demonstrates,

that there is a huge defect in Beethoven's written code; it is quite trivial, at least locally, the written script is simply boring.

Gould has effectively given a quasi-mathematical demonstration that the same performance strategy cannot be applied to Bach and to Beethoven, that the same analytical insight and the same rhetoric shaping yield completely different results for these composers. To me, this is a sensational lesson to teach a characteristic difference between Bach and Beethoven. This is very clear in the descending seventh passage on bar $14^{th}$ bar, which runs on semiquavers after a triggering triplet of quavers at the end of bar 13. Gould effectively takes the double temporal rate of the semiquavers with respect to the quaver triplet, without any tempo increase, without any dynamic profiling, just letting us see the anatomy of this triadic descent structure. The common reading [534] of this passage is that of an explosion:

> *Die Explosion (a tempo, Auftakt zu T.14) erfolgt im niedersausenden Dominant-Arpeggio und f-Sextakkord (T.15), wird aber sogleich abgedämpft durch einen C-Dur-Sextakkord, p, T.16.*

With Gould, there is no explosion, just the written text, cleanly played, but antagonistic to any such musical drama to which an explosion would testify. The common reading in fact classifies this sonata as a musical drama, and asks interpreters to integrate this semantic into their performance. Gould plays the "Appassionata" minus the commonly implied drama. The question here is whether this dramatic character is implicit in the score structure or whether it is an external determinant which has been added by historical standards—which Gould filters away to lay bare what he believes is a poor structural essence [191]. So the question arises whether the commonly accepted dramatic performance is an expression of Beethoven's work or of an added character. Let us therefore analyze the specific performative shape of the passage in question. To begin, its agogics is profiled against the temporal neighborhood, i.e., not only is the indication "a tempo" valid from the last three quavers of bar 13, but in bar 14, the resumed tempo is again increased. The dramatic performance contains an increase of tempo, and within that level, also an increase of tempo towards the middle of the descent. Further, the dynamics is not only the forte at the end of bar 13, but the target tones of each descending intervallic movement of the descent is played louder, maybe to a *ff* or *sf*. As a whole, this descent (with its added ascending tail in bar 15) is not only a musical structure, but more an explosive gesture whose very beginning goes to the top pitch, falls down and bounces back to the dominant fermata. This is not a written rationale, but it is a semantic unit which can easily be deduced if gestural semantics is to be included in the performance shaping.

So Gould's experiment would demonstrate that Beethoven requires gestural rationales beyond analytical ones. Meaning that Beethoven's compositions have a performative added value of gestural nature which is not (yet?) virulent in Bach's architectural music. Observe however that this gestural character is not on the level of the interpreter's gestures, it is a rationale in the performance grammar, a semiotic layer which is added to the score system. Summarizing, Gould's politically incorrect performance withdraws from the common dramatizing approach and gives us an insight to Beethoven which would not have occurred otherwise.

## 45.3 Transversal Ethnomusicology

**Summary.** Ethnomusicology is essentially confronted with the *synchronic normalization* problem: Given two (simultaneously existing) cultural areas, understanding one of them from the other's perspective means normalizing fundamental categories, e.g., the "score" concept, such that *comparison* of different ethnics becomes feasible. When dealing with performance of works which belong to distant epochs, the ethnomusicological problem of understanding a performance is restated as *diachronic normalization*: what is the common ground for understanding the performance of a work which was written in the spirit of another epoch? We discuss the efforts of Nicolas Harnoncourt in this direction of "transversal" ethnomusicology.

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In its common understanding, ethnomusicology deals with a transformation problem between synchronic music cultures. Such a transformation may cause major problems to the contents and forms of music, for example because of incompatible notation or even incompatible modalities of communication, such as oral traditions, rebuilding instruments for each new musical event, embedding of music in more global forms of art, etc. But synchronic ethnomusicology has the undeniable advantage that a feedback process may help deal with such problems, and eventually solve them in the ideal case[2]. This advantage cannot be claimed for diachronic ethnomusicology.

What is this type of ethnomusicology? It deals with transformation of music cultures which are at a temporal distance instead of a geographic or social distance of contemporaneous cultures. For example, if we play an opera of the late Renaissance composer Claudio Monteverdi in the $20^{th}$ century, this is a diachronic type of ethnomusicology. In fact, the cultures of Monteverdi's time and of the $20^{th}$ century are very different, and the communication between them is restricted to the historical proliferation channel. The historical distance influences strongly the communication of forms and contents. It is by no means automatic that everything is transmitted without loss of information. Above all, the socio-cultural background of a musical composition is easily blurred by historical filters.

But also, the instrumental practice and technology impose dramatic deformations of what was reality in a historically distant context. Nicolas Harnoncourt has with great success restarted a dialog with historically distant traditions in the sense of understanding those conditions and not imposing ours to a diachronically distant culture. His approach is based upon the basic position that instrumental constraints are very important for shaping one's performance and expressivity. Restriction gives one a clear frame, a limited field of activity where a composition must unfold its semantics, and not an unlimited Wagner-tailored orchestra, where the quasi-infinity of instrumental power and colors competes with and actually substitutes the efforts for better expressivity.

All this intelligent effort does not solve the communication problem in the historical dimension: in contrast to synchronic ethnomusicology, diachronic communication is unidirectional:

[2]However, the real case may be far from ideal. For example, if we want to initiate *inverse ethnology*, i.e., the review of our own music culture from the point of view of an 'exotic' culture, major obstacles will occur, from financial ones to the intolerance against another culture which tries to relativize the usual perspective of occidental supremacy.

No real feedback from ancient times is possible, we only have the sources and must try to understand without really having answers to our experiments. Harnoncourt's experiments may be an attempt to transform our performance practice back to Monteverdi's, but it is not a demonstration of anything.

So performance is not only a hic et nunc affair, but also a process which is coupled to spatio-temporal distances of cultures. Within this transformational framework, there are different serious asymmetries of communication. In the synchronic direction, feedback can be dealt with—however in the limits of sociocultural asymmetries for inverse ethnology. In the diachronic direction, the communication is intrinsically unidirectional, and performance remains a challenge for the adequacy of cultural transformation.

# Chapter 46

# Critical Fibers

*To see a World in a Grain of Sand,*
*And a Heaven in a Wild Flower,*
*Hold Infinity in the palm of your hand,*
*And Eternity in an hour.*
William Blake (1757–1827)

**Summary.** Stemma theory offers a model for a critical understanding of performance *as* a complex process between interpretation and the rhetorics of expressive performance grammar. We make the model and its limitations explicit.

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## 46.1 The Stemma Model of Critique

**Summary.** Modeling performance through stemma theory allows us to define inverse images of performances within a well-defined variety of stemmatic situations leading to the given performance. A critical fiber is such a variety. A critique is a choice of a point within this fiber. We discuss criteria to make such a choice, i.e., to select one critique among all possible critiques *within* the stemma model.

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In section 32.4 we discussed the four global aspects of performance, in particular the fourth point: stemmatic deployment of performance. In that discussion, we stressed the fact that stemmata are not just a learning process but much more a logical unfolding of deformation strategies. If we take this point of view for granted[1], music critique should deal with the stemmata which could possibly lead to a given performance, since they would unveil the anatomy and genealogy of a performance, and this must be the central issue of a critique which merits that name.

[1] Other models of performance genealogy are at hand, but we do not know of any such model that is technically as explicit as the stemma model.

Probably the very construction of a stemma with its genealogical factors from cell hierarchies to performance operators and ramification architecture would help a critic to shape his/her style and criteria for judging performances. But beyond that, the general framework of stemma theory is far too generic for any concrete goal. What is the concrete goal?

**Principle 27** *Try to construct a stemma that produces a given performance from the existing score!*

In this form, the problem has trivial solutions if we admit that any performance field is admitted without further articulation and operator constraints. The (non-trivial!) construction of a performance field from experimental performance data (e.g., on a MIDI file record) has been implemented by Stefan Müller, see section 46.3, and can be used to construct a depth-one stemma whose operator just produces this performance field on a monolithic LPS. Such a solution is however not what a critical understanding of performance would preconize since this is nothing more than the brute 'sampling' of a performance transformation, and would not lay bare any of the semantically valid rationales.

The crucial point is that one should impose constraints on the admitted tools for stemma construction. This would lead to the more reasonable

**Principle 28** *Given a defined set of constraints regarding the construction tools for stemmata, try to construct a stemma that produces a given performance from the existing score!*

In this form, the principle becomes a challenge for critics since it thematizes the strategies a critic could imagine to be used by a specific artist. Only after such a strategic preset can the reconstruction of a possible generating stemma be seriously tackled. The lesson to be learned from such an 'exercise' is that in criticism, one should learn to reflect very cautiously the conditions under which critical judgments are made.

The critical business now splits into two subtasks: comparative criticism of one and the same performance, and comparative criticism of a number of different performances of the given score. The first means that we are given a fixed performance and compare different reconstructions of backing stemmata: Which one is acceptable, which one is simpler but still adequate, in what respect could two criticisms be considered as being isomorphic, etc. The second one means that the phenomenological difference of performances is lifted to a genealogical difference of backing stemmata, and this one is the core activity of a music critic. However, it presupposes that usually, the constraints on the stemma reconstruction potential are the same, a condition for comparability which is not automatic.

## 46.2 Fibers for Locally Linear Grammars

**Summary.** Locally linear grammars are a special approach to stemma varieties, see section 39.8. We give a first description of critical fibers in this context. They turn out to be varieties in the sense of algebraic geometry and are called grammatical varieties.

The general inverse problem of reconstructing a stemma for a given performance of a defined score is not solvable for two reasons: we first do not have a general formalism and second, even a reasonable general formalism would imply wild[2] mathematical classification problems.

We therefore want to consider a more tractable situation: the locally linear performance grammars which were introduced in section 39.8. We refer to that section and keep its notation, recalling in particular from definition 110 that a locally $\mathbb{R}$-linear grammar is a family of $\mathbb{R}$-linear representations of the stemma quiver $T$ which are parametrized by $\mathbb{R}$-vector spaces $B_x$ of finite dimension $s_x$, $x \in V(T)$, the vertex set of $T$, the parametrization being given by affine maps $\varphi_x : B_x \to End(A_x)$.

Within this very precise setup, where the stemma and the locally linear grammar are fixed, we may ask for the structure of the fibers lying above the performances which are defined on the stemma's leaves by the given locally linear grammar.

Technically speaking, we proceed as follows: For each final vertex $z \in V(T)$ let $m_{z0} = r, m_{z1}, m_{z2}, \dots m_{zn_z} = z$, be the ordered sequence of elements of $M_z(T)$. For the root $r$ and for each final vertex $z \in V(T)$ let us fix arbitrary $e_r \in A_r$, $e_z \in A_z$, respectively. The main task of this *inverse performance theory* is to study the set of solutions of the system of equalities

$$\begin{aligned} e_z \quad &= \\ & f_{m_{zn_z}}(f_{m_{zn_z-1}}(\dots f_{m_{z1}}(e_r, (c^r_{y_1,m_{z1}}), (b_{y_1})), \dots), (c^{m_{zn_z-1}}_{y_{n_z},m_{zn_z}}), (b_{y_{n_z}})) \end{aligned} \tag{46.1}$$

where for $k = 1, \dots n_z$ the vertexes $y_k$ lie in the set $D_{m_{zk-1}}(T)$.

For the explicit calculation, we select a basis $v_{x1}, \dots v_{xs_x}$ of the vector space $B_x$. This means that for every vertex $x \in V(T)$ we may consider the linear operators $O_{xi} := \varphi^0_x(v_{xi}) \in End(A_x), i = 1, \dots s_x$ as a part of our data. Whence, if we identify $B_x$ with $\mathbb{C}^{s_x}$ through the basis $v_{x1}, \dots v_{xs_x}$, we can define the homomorphism (39.50) as a triaffine map

$$f_x : A_m \times \mathbb{C}^{\#D_m(T)} \times \prod_{y \in D_m(T)} \mathbb{C}^{s_y}_y \to A_x$$

by associating to $(a_m, (c^m_{y,x})_{y \in D_m(T)}, (b_{yj})_{y \in D_m(T), j=1,\dots s_y})$ the sum

$$\sum_{y \in D_m(T)} \sum_{j=1}^{s_y} c^m_{y,x} b_{yj} i_{y,x}(O_{yj}(r_{m,y}(a_m))) + \sum_{y \in D_m(T)} c^m_{y,x} i_{y,x}(\varphi^t_y(r_{m,y}(a_m))).$$

The equations (46.1) become:

$$\begin{aligned} e_z = & \\ & \sum c^{m_{zn_z-1}}_{y_{n_z},m_{zn_z}} b_{y_{n_z} j_{n_z}} \cdots c^r_{m_{y_1,z1}} b_{y_1 j_1} m_{z,y_1,\dots y_{n_z},j_1,\dots j_{n_z}} + \\ & \dots + \\ & \sum c^{m_{zn_z-1}}_{y_{n_z},m_{zn_z}} \cdots c^r_{y_1,m_{z1}} m_{z,y_1,\dots y_{n_z}} \end{aligned} \tag{46.2}$$

[2] In algebra, a classification problem is said to be wild, if its solution would imply the classification of any module category.

where the sum is over all $y_1 \in D_r(T), y_2 \in D_{m_{z1}}(T), \dots y_{n_z} \in D_{m_{zn_z-1}}(T)$ and all $j_k = 1, \dots s_{y_k}$, for $k = 1, \dots n_z$. The leading vector summand of this linear combination equals

$$m_{z,y_1,\cdots y_{n_z},j_1,\dots j_{n_z}} = \tag{46.3}$$
$$i_{y_{n_z},m_{zn_z}}(O_{y_{n_z}j_{n_z}}(r_{m_{n_z-1},y_{n_z}}(\dots i_{y_1,m_{z1}}(O_{y_1j_1}(r_{r,y_1}(e_r)))\dots))).$$

The general vector summand refers to a choice of endomorphisms $X_{\dots} = O_{y_{n_z}j_{n_z}}$ or $X_{\dots} = \varphi^t_{y_{n_z}}$ etc. and is equal to

$$i_{y_{n_z},m_{zn_z}}(X_{\dots}(r_{m_{n_z-1},y_{n_z}}(\dots i_{y_1,m_{z1}}(X_{\dots}(r_{r,y_1}(e_r)))\dots))). \tag{46.4}$$

In order to describe the solution variety we can interpret (46.2) as a system of linear equations with parameters,

$$e_z = \sum_{y \in D_{m_{z,n_z-1}}(T)} c_{y,z}^{m_{n_z-1}} l_{yz}. \tag{46.5}$$

Observe that $e_z, l_{yz}$ are vectors in $A_z$, so the solutions $c_{y,z}^{m_{n_z-1}}$ of equations (46.5) result from simultaneous solutions in the vectors' coordinates if we are given a basis of each $A_z$. The coefficients $l_{yz}$ equal linear combinations of the vector summands (46.3) and (46.4) with coefficients which are monomials in the remaining variables $c_{\dots}$ and the $b_{\dots}$.

If we now assume that all variables $c_{zy}^{m_{n_z-1}}$ are independent, we see that for any two different $z \in D_{m_{n_z-1}}(T)$, the solutions of the equations (46.5) do not interact with each other. This means that the solutions of the system (46.5) for $z \in D_{m_{n_z-1}}(T)$ is either empty (if the value $e_z$ is not in the image of the linear map (46.5)) or equals

$$\prod_{z \in D_{m_{n_z-1}}(T)} L_z \subseteq \prod_{z \in D_{m_{n_z-1}}(T)} \mathbb{C}^{\#D_{m_{n_z-1}}(T)}$$

where $L_z$ are linear subspaces of $\mathbb{C}^{\#D_{m_{n_z-1}}(T)}$. Their codimension can be read from the matrix of the coefficients. Since there is no other condition on the other variables $c_{y,x}^{m_{zk}}$ we obtain altogether the following result:

**Theorem 35** *The solution space of (46.2) is a linear fibration over some appropriate affine space.*

**Corollary 22** *The dimension of the non-empty fibers of this fibration is generically minimal, and it increases along some finite union of proper algebraic subschemes (the loci where the minors of the coefficients' matrices vanish). This means that the non-empty fibers $Fib(e_\cdot)$, i.e., the solution spaces defined by (46.1) over a given output set $e_\cdot = (e_z)_z$ are all generically isomorphic, i.e., isomorphic when restricted to appropriate open subschemes. However, the configuration of the specialization subschemes is not evident and depends on the particular vector summands. Also, the condition for non-empty fibers is not evident in general.*

How can we interpret this result in musicological terms? A quantitative measurement of a performance can be done by recording the values of the parameters that characterize a note, i.e., loudness, duration, and so on. In the last paragraph we have denoted this data by the vectors $e_z$. Equation (46.2) explains that in order to produce given values of the parameters from the weight functions, one has to find suitable values for all $c_{y,x}^{m_{zk}}$. It would be nice if the choice of these coefficients could be small, because this would mean that in order to produce a given performance one does not have a lot of freedom in the choice of system parameters. However from this model this is not the case. In fact, corollary 22 tells us that for a fixed performance one has either no one or else infinitely many possibilities to choose system parameters which produce the given performance. Moreover, we learn that all performances with non-empty fibers $Fib(e_\cdot)$ have open dense subsets $U(e_\cdot) \subset Fib(e_\cdot)$ which are all isomorphic with each other, so these generic subsets $U(e_\cdot)$ are qualitatively equivalent, see also figure 46.1. *In other words, the non-empty fibers only differ on their special loci apart from generic open subschemes* $U(e_\cdot)$.

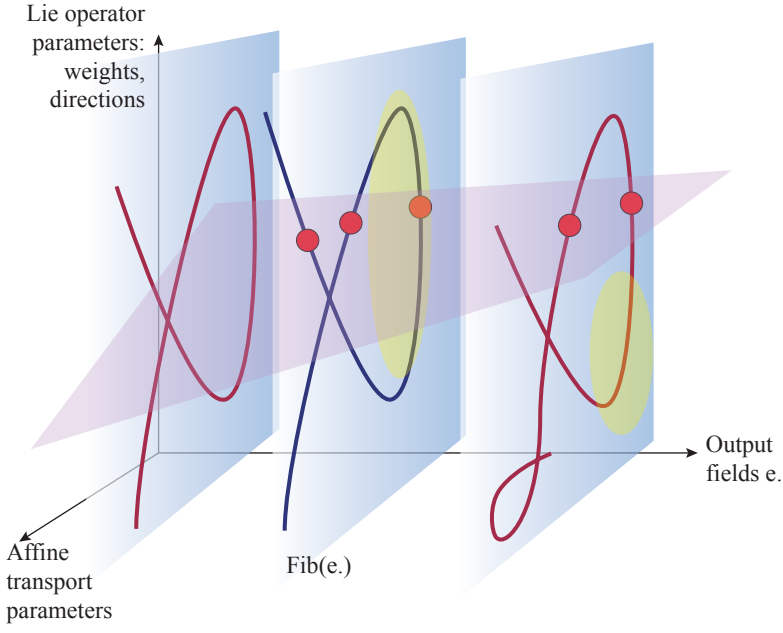


Figure 46.1: All performances with non-empty fibers $Fib(e_\cdot)$ have open dense subsets $U(e_\cdot) \subset Fib(e_\cdot)$ (elliptic regions) which are all isomorphic with each other

In musical terms, a fiber $Fib(e_\cdot)$ could be called a *critical fiber* because its points are the possible background parameters—in the present stemmatic model—which lead to the given performance output data $e_\cdot$, such as local tempi, articulations, dynamics, detunings, etc. So the fiber really includes the possible ways of understanding why a performer is playing his actual performance. In fact, finding out which parameters the interpreter could have used is (or should be) the core activity of a music critic. Having generically isomorphic fibers means that in any

two critical fibers of two given performances, there are "dominant" open sets of "criticisms" (i.e., points in the fiber!) which are isomorphic with each other. This does not mean that the criticisms are the same for the two given performances, but that their structural contexts can be identified. Again, this does not mean that the relevant criticisms may be identified—on the contrary: maybe, these isomorphic contexts just describe the criticisms which are what everybody could say if no supplementary information about the specific performance culture of the interpreter is known.

So we should not discard this model as insignificant, instead we need to look at it in the right way. In fact, it is like pretending to explain the geometry of smooth plane curves, just using lines. The theory of lines can be useful for local questions, but not for global ones. The same thing happens here. The great flexibility of this model enables us to adapt it to the questions that one tries to answer, but they can not be too general. Indeed, one can select subschemes that are appropriate for the study of a particular problem. For example, if one is interested in comparing several performers under the point of view of the local/global way of playing, one could try to restrict the research to the level of the daughters of the root, and to use some more of the structures of the vector space $A_x$ (if any). Another question that could be asked is that of the final/causal way of playing. This requires one to impose mutual dependence conditions upon the variables $c_{y,x}^{m_{zk}}$ for varying $z \in D_{m_{zk}}$ (see 46.4).

## 46.3 Algorithmic Extraction of Performance Fields

**Summary.** The algorithmic extraction of performance fields is a first step for systematic calculations in inverse performance theory. The extracted fields can further be used for visualization. We describe an approach how performance fields can be calculated from given scores and performances and present the tool that implements the theory.

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In this section, we address the question of inverse performance theory which deals with the reconstruction of a performance field for a given performance on a determined score, including an implementation in the RUBATO® framework, named EspressoRUBETTE®. This question generalizes the well-known problem of constructing a tempo curve from a measured performance to form space $S = EHLD...$ of parameters such as onset $E$, pitch $H$, loudness $L$, etc. Such an attempt is first of all an interpolation task where a continuous performance field must be reconstructed from a discrete data set of performed notes. As such it is subjected to (a) the problem of matching symbolic and performance events, (b) ambiguities in the local definition of field vectors, (c) algorithmic constraints for real-time objectives, and (d) visualization options for performance fields.

### 46.3.1 The Infinitesimal View on Expression

The concept of expression is ambiguous as far as the content of the expression and its reality layer are not a priori clear, see chapter 36 for the details. If we aim at analyzing expression, this does not regard the psychological perception of a performance by humans. The psychological aspect is a legitimate one, but it touches a category which relates the performed music to human

categorizations in terms of emotional response. Such a perspective is, for example, dealt with in [231] or in [289]. In contrast, our point of view is expression as a rhetorically shaped transfer of structural score contents by means of the deformation mapping of symbolic data into a physical parameter space. The psychological implications are not the subject of this perspective, it is a purely mathematical description of this mapping, not of the emotional correlates.

The theory of performance fields is derived from the general hypothesis that performance is a smooth (continuously differentiable) isomorphism $\wp = R \to RP$ on a frame neighborhood $R$ of the given local composition $C \subset S$. This is of course a strong hypothesis, but it is, at least locally on the given composition, a reasonable one. In our inverse problem, we are not given $\wp$, but only its restriction $\wp|_C$ to the given local composition. Accordingly, we shall not really construct the performance field $\mathbf{Ts}$ associated with the unknown map $\wp$, but a discrete performance field, defined on the points of $C$, which is determined by the restriction $\wp|_C$.

We shall now construct such discrete fields, their interpolation on a neighborhood of $C$, as well as their visualization by means of color fields. It will also be possible to calculate difference fields in order to compare two performances of the same local composition.

### 46.3.2 Real-time Processing of Expressive Performance

An implementation of the performance field theory should be able to operate in real-time, especially for interactive applications, where immediate feedback, either visible or audible, is desired. As we shall see, the complete calculation of the performance fields can be split up in dedicated, communicating modules for specific tasks. Particularly important for performance is the extraction of tasks that can be processed in advance. Figure 46.2 gives an overview of

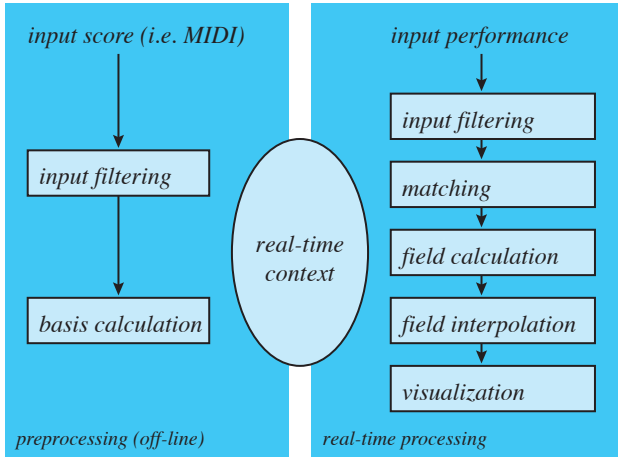


Figure 46.2: Overview of the modules and the flow of control (as shown by the vertical arrows) in our implementation of the EspressoRUBETTE®.

the modules and the flow of control (as shown by the vertical arrows) in our implementation. Modules are notified by events when new data for processing is ready. The modules themselves are *stateless*, they share their information with other modules in the *real-time context*, a data

structure which contains all relevant information for the whole process, thus minimizing the risk of inconsistency. Of course, asynchronous accesses to the context have to be synchronized using locks or a similar synchronization technique.

For increased flexibility and efficiency all modules accept lists of events, therefore making off-line and real-time processing structurally identical. For example, the input filtering modules for the score and for the performance are the same. The former accepts the score as a whole and the latter processes individual events as they are received in real-time. Further, modules can be prioritized and be put to sleep if there is not enough processing power to support all present modules temporarily.

The following items give short descriptions for the modules shown in figure 46.2. Details will be presented in the subsequent sections. It is important to see that the described architecture allows the definition of additional modules as needed. This mostly depends on application requirements. Also some application might not need certain already defined modules, i.e., field interpolation in a computer accompaniment system.

**Input filtering.** This module translates incoming note events to the representation defined in the real-time context. It also processes structural information, such as different voices, tempo changes, etc. The input filtering module must be implemented for any external representation (e.g., MIDI or RUBATO®s denotator format).

**Basis calculation.** The calculation of the bases depends only on the input score, not on the performance and can thus be performed off-line. For each event an appropriate basis has to be calculated. Typically this is a time-consuming process.

**Matching.** The incoming performance events have to be matched to the corresponding score events. As we shall see in the designated section, this is a non-trivial task and has developed to a research field on its own.

**Field calculation.** The individual field vectors for each note are be calculated based on the precalculated basis and the given match.

**Field interpolation.** The field vectors calculated by the former step are typically not aligned on a grid. However, for visualization a 2D or 3D grid-like field, with field vectors defined anywhere in this grid is desired. The interpolation step allows the definition of such a grid and performs the translation from the note field to the interpolated field.

**Visualization.** Finally the calculated field is ready for visualization. Here, many user defined parameters such as scaling, color-specification, ranges, etc., have to be taken into account.

### 46.3.3 Score–Performance Matching

A lot of research effort has been put into score–performance matching techniques. *Score-following*, the real-time matching and tracking of soloists performing a given score was first published by Dannenberg [108] and Vercoe [544]. Puckette [426] presented the methods used on the IRCAM Signal Processing Workstation (ISPW). Heijink et al. [216] have given an evaluation of different approaches to score-performance matching.

Literature has typically differentiated between two types of algorithms: For *real-time algorithms*, mostly used in real-time accompaniment software, good performance had higher priority than matching quality. Off-line algorithms, where calculation time is less important, were mostly used for in-depth analysis applications requiring a high level of matching quality. We however experienced that with todays processing power high quality matching can be performed in real-time, particularly when the algorithms are well suited for extensive preprocessing of the given score. Mathematically, the matching problem is complex and depends upon the difference which one allows between score and performance. For example, if chords remain chords and all notes are played exactly once, the problem is trivial. But normal performance includes more or less strong arpeggiation of chords, omissions of notes or playing additional notes by error or by ambiguous definition of the notes, such as is common for trills and other ornaments. We have implemented an algorithm which will not be described in detail here since it is not our principal subject. The algorithm is a kind of matching along a 'wave front of notes which are defined by the temporal unfolding of the performance and thereby fits in the real-time constraints. We nevertheless should sketch the principle ideas behind our algorithm.

Usually, matching is thought bottom-up in that the performance map of the whole piece is constructed from the performance map $X \mapsto \wp(X)$ on the single elements $X$. We rather tried a top-down strategy, i.e., to rebuild the element images from maps on sets of specific coverings $I$, $J$, respectively of the local composition $C$ and its performance $D$. Typically, one considers the covering of $C$ by hyperplane sections in each parameter (for example onset slices). On $D$, a covering $J$ is defined which is a more fuzzy version of $I$, for example neighborhoods of hyperplane sections (for example $\varepsilon$-neighborhoods in the onset dimension). If $\wp$ exists, different constraints can be imposed on the induced map on the coverings: First, $\wp$ induces a map $n_0(\wp) : I \to J$ such that $\wp(U) \subseteq n_0(\wp)(U)$ for all covering elements $U$ in $I$. This yields a map $n(\wp) : n(I) \to n(J)$ of the simplicial nerves, and thus conditions on the map on the covering sets. Second, the sets of these coverings are linearly ordered[3] by $U \prec V$ iff either $U \subset V$ or both $U - V$ and $V - U$ are non-empty and $min(U - V) < min(V - U)$. In this ordering, we require that $U \preceq V \Rightarrow n_0(\wp)(U) \leq n_0(\wp)(V)$. Third, if one defines a distance $d(U, V)$ between the covering sets (for example the elastic shape distance from motif theory, see section 22.2.1.3), one requires that $d(U, n_0\wp(U)) < \varepsilon$ for a given positive distance limit $\varepsilon$. With these constraints one may define the map, and then recover $\wp$ if every point $X$ in $C$ may be seen as the intersection of all covering sets of $I$ which contain the point. This is evidently the case for the hyperplane sections described above.

Following these observations, an implementation typically makes use of structural properties of a musical score and a corresponding performance. Further, dynamic programming techniques help in coping with the real-time problem: Multiple possible solutions are created, maintained, and discarded as the matching process is running.

### 46.3.4 Performance Field Calculation

Let us consider the score space $S$, the performance space $P$, the performance transformation $\wp : S \to P$ and the constant vector field $\Delta(x) = \Delta = (1, ..., 1)$ for all $x \in P$. Recall from section

[3]This is the usual linear ordering of powerset denotators as defined in section 6.8.

33.2.2, equation (33.11) that the performance field is the inverse image of $\Delta$ and evaluates to

$$\mathbf{Ts}(X) = J(\wp)^{-1}(X)\Delta$$

where $J(\wp)$ is the Jacobian matrix

$$J(\wp)(X) = \left(\frac{\partial x_i}{\partial X_j}\right)\Big|_{X_j=E,H,L,D,\ldots}^{x_i=e,h,l,d,\ldots} = \begin{pmatrix} \frac{\partial e}{\partial E} & \frac{\partial e}{\partial H} & \cdots \\ \frac{\partial h}{\partial E} & \frac{\partial h}{\partial H} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}(X)$$

at $X$. In order to calculate the field vectors in an element $X$ of the given local composition $C$ in $S$, we have to determine $J(\wp)$. Now, assume that we are given a matrix $U_X$ of not necessarily orthogonal basis vectors based in $X$. $J(\wp)$ can be rewritten as:

$$J(\wp) = J(\wp)U_X U_X^{-1} = V_X U_X^{-1}$$

where the basis matrix $V_X$ is the image of the basis matrix $U_X$. Then:

$$\mathbf{Ts}(X) = U_X V_X^{-1}\Delta = det(V_X)^{-1} U_X Adj(V_X)\Delta.$$

The last term is used to identify three cases:

1. $V_X$ is regular, thus $\mathbf{Ts}(X)$ is defined,
2. $det(V_X) = 0, Adj(V_X) \neq 0$: only the direction of the vector is given, not its length, and
3. $det(V_X) = 0, Adj(V_X) = 0$: no information at all is given.

While we are now able to calculate the field vectors, the question of how to find the appropriate bases is still open.

#### 46.3.4.1 Obtaining the Bases

The only information available for basis calculations are difference vectors of the given score notes. Basically any difference vectors could be considered as basis vectors, but due to the following restrictions, the candidates have to be selected carefully: First, only notes in a small neighborhood of $X$ should be considered. This *principle of locality* ensures that the basis consists of notes that are in the local musical context.

The second restriction is of a mathematical nature: we have seen that the transformed basis has to be regular in order to be able to calculate the field vector. Because the performance is allowed to have arbitrary deviations from the score, there is no general solution to this problem. What can be done is to decrease the possibility that the transformation of the basis $U_X$ yields to a non-regular basis matrix: This can be accomplished by making $U_X$ as orthogonal as possible. Thus, the selection of the basis vectors is based on the following two criteria:

1. *Locality:* $|det(U_X)|$ is minimal, and
2. *Orthogonality:* $|det(U_X^{Norm})|$ is maximal (where $U_X^{Norm}$ is the matrix of normalized basis vectors).

Note that the two criteria are to some extent in competition, so they have to be weighted and combined. Consequently, a basis-calculation algorithm has to select bases by searching for

$$min\left(w_{loc}|det(U_X)| + w_{orth}\frac{1}{det(U_X^{Norm})}\right)$$

with $w_{loc}$ and $w_{orth}$ being positive pre- or user-defined weight values.

Unfortunately, there is still one case that has to be dealt with: the case where it is not possible to find a regular basis matrix $U_X$ in a small neighborhood of $X$. This may occur if all notes have the same loudness, if the basis has to be calculated for an isolated chord, where all onsets are equal, or for repeated notes with the same pitch. The only option left here is to construct orthogonal basis vectors that ensure that the basis remains regular.

Finally, the pseudo-code for a basis-calculation algorithm can be given:

```
for(each Note X in Score S) {
  List neighbors = S.getNeighborList(X, maxDist);
  List basisVectors = emptyList;
  for(each Note N in neighbors)
    basisVectors.add(N - X);
  List bases = getCandidates(basisVectors);
  X.basisCost = infinity;
  for(each Basis B in bases) {
    float basisCost = wLoc * abs(B.det()) + wOrth / abs(B.norm().det());
    if(basisCost < X.basisCost) {
      X.basisCost = basisCost;
      X.B = B;
  }
}
```

The function `getCandidates()`, whose pseudo-code was omitted here, generates a list of bases containing the permutations of the basis candidates, and also adding constructed basis vectors if necessary. The above algorithm can be optimized by generating the permutations on the fly, the best expected ones first. In that case, the candidate list can be sorted by increasing distance, and the distance is used to cancel the loop, as soon as it is known that a lower `basisCost` can not be reached anymore.

### 46.3.5 Visualization

One of the most straightforward applications of a calculated performance field is its visualization. Vector field visualization has been successfully used in many science and engineering domains, i.e., in gas and fluid dynamics. Thus, many different techniques and their corresponding implementations are available. Common to all those methods is that they should be *accurate*, *fast*, and display the field in an *intuitive way*. See [76] for an advanced method that is suited for 2D as well as for 3D visualization. This section shows how the calculated field vectors need to be processed in order to make them available to such standard visualization methods.

So far, we have dealt with a score $C$, consisting of a set of notes, the corresponding performance $D$, and the associated set of calculated field vectors $F$. The points in those sets reside in

an $n$-dimensional space, $n$ being the number of symbolic sound parameters. For visualization, $n$ will normally be too large, so as a first step it has to be decided which parameters are to be used for visualization. For instance, we may choose onset $E$ as the horizontal axis and pitch $H$ as the vertical axis in a 2D setup. The remaining sound parameters are omitted. Further, the desired field vector components have to be selected, for example $E$ in horizontal direction and $D$ in vertical direction for a tempo-articulation field.

#### 46.3.5.1 Field Interpolation

Typically, when dealing with vector fields, the field vectors are arranged in a grid of a given resolution. In contrast, our setup implies that the score points reside at arbitrary locations, making it impossible to use standard vector field visualization methods. Thus, a conversion from the calculated field vectors to vectors located on a grid is necessary. This can be accomplished through *interpolation*.

At first sight, a triangulation of the given set could be considered, making it easy to calculate the interpolated grid vectors in the resulting triangles. However, since the different symbolic sound parameters have different meaning, triangulation is not well suited here: interpolation should occur in a musically meaningful way. Therefore, it makes sense to perform interpolation in a defined recursive order. For instance, when interpolating an $ED$ field, first the $D$ axis of the grid is considered and then the $E$ axis. More precisely, one draws hyperplanes $\mathrm{H}_1, \mathrm{H}_2, ..., \mathrm{H}_k$ perpendicular to the n-th axis in the symbolic parameter space S such that every point of the given composition C sits in one such hyperplane. By recursion, we suppose that the interpolation is available for the first $n-1$ coordinates. To get the interpolation value on an arbitrary point $X$, one draws the straight line through $X$ and parallel to the n-th axis. This line cuts two neighboring hyperplanes in points $P$, $Q$. The values in $P$ and $Q$ are then interpolated by a cubic spline with zero slope in $P$ and $Q$. For a detailed description see section 32.3.2.1. What happens at the boundaries of the given set? Since no field vectors are available, boundary vectors have to be defined. When having a look at the theory of the former sections, it becomes clear that outside the boundaries a frame of diagonal vectors has to be placed. The distance between the boundaries and the frame is constant and has to be predefined.

### 46.3.6 The EspressoRUBETTE®: An Interactive Tool for Expression Extraction

The methods for algorithmic extraction of musical expressions have been implemented in a tool called EspressoRUBETTE®. The tool can run as a stand-alone Java application. The Swing and Java2D classes take care of the user interface, and the user can manipulate calculation and visualization parameters through a simple dialog panel.

As an alternative and more flexible approach, the software also implements the RUBETTE® interface and can thus be integrated into the Distributed RUBATO® framework, see chapter 40.

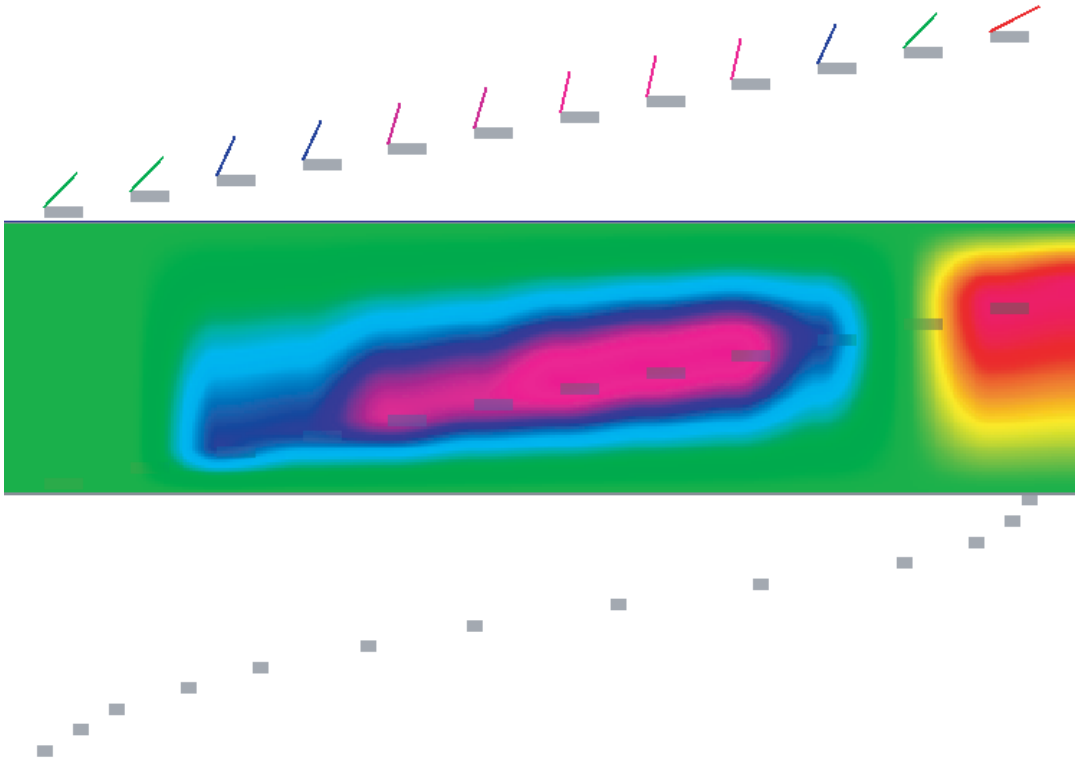

Figure 46.3: A chromatic scale and its performance: Above the field vectors, middle: the color encoded and interpolated field vector, below the performed scale.

#### 46.3.6.1 Example 1: Tempo Field of a Chromatic Scale

Let us now give examples of calculated performance fields. Figure 46.3 shows a chromatic scale and its performance. In this case representation is close to the one of a piano roll: the horizontal axis represents onset, pitch is mapped to the vertical axis. The width of events corresponds to their duration. Note that the EspressoRUBETTE® allows arbitrary redefinition of those mappings. The top section shows the score containing the chromatic scale, twelve note events in increasing pitch order, all with the same duration. A hypothetical performance of the twelve events is shown in the bottom section. The first three events are played at the same speed as the original MIDI score. Then the performance is getting slower, and towards the end it is getting faster again. The last two notes are played faster than the MIDI score.

This situation is depicted by the calculated and visualized vectors in the top section: the first, the second and the eleventh vector are diagonal vectors, stating that the notes are played at the given tempo. The angle of the other vectors depends on the local tempo played at a

given note. The middle section shows the corresponding interpolated field, at a resolution of 400 times 200 cells. Here, the slope of each vector has been mapped to a color, and its length is related to the brightness of a cell.

#### 46.3.6.2 Example 2: Excerpt from Czerny's Piano School

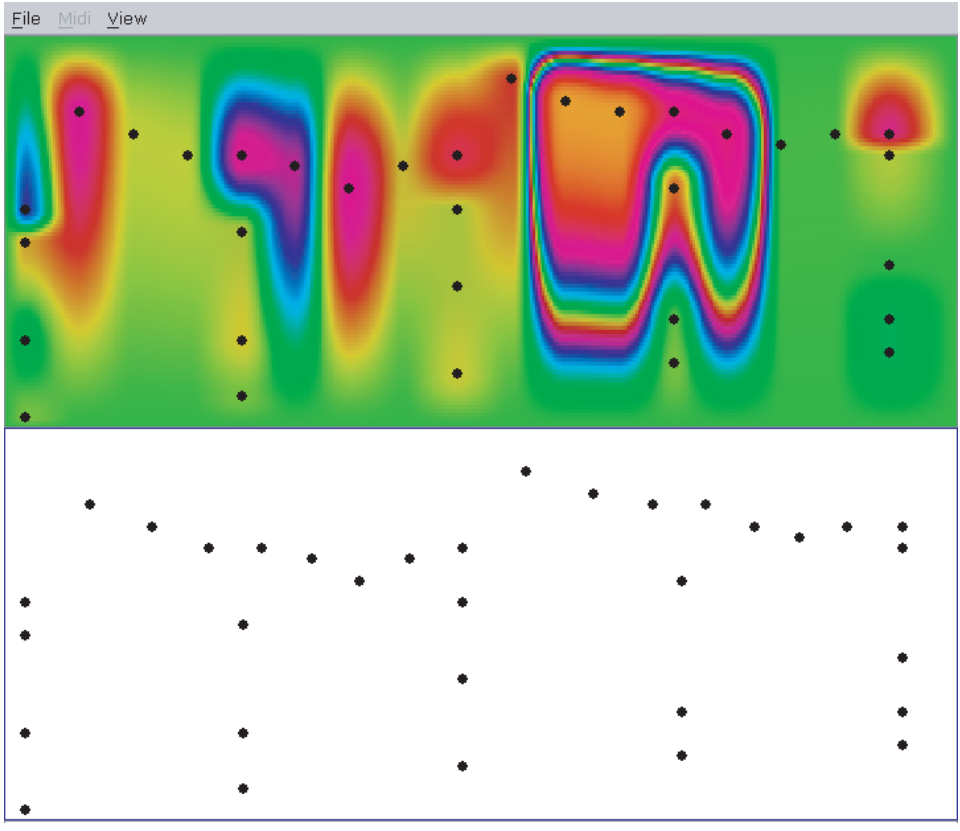


Figure 46.4: A performance field for an exercise from Carl Czerny's piano school.

The second performance is a real-world example, namely a performance field for an exercise from Carl Czerny [98], as recorded on an MIDI file. Figure 46.4 shows the first two bars of the exercise, axes and note representations are as in the previous example. The upper half shows the performance field on the score space $S = EH$, the lower half shows the physical space $S = eh$ of the performed piece (physical parameters being written in small letters). The exercise shows the Chopin rubato, i.e., the right hand plays the melody slightly shifted in time against the firm left hand chords in such a way that synchronization is recovered at the end of bars. The field shows the $E$- and $D$-components of the four-dimensional $EHLD$-performance field, as encoded by colors. One recognizes that the left hand notes are quite near in color to the green color

which encodes the diagonal unit field for mechanical, unshaped performance. We see that there is a right hand rubato effect in the middle of each bar, and significantly more in the second bar. The cyclic coloring effects are due to a multiple covering of the color circle in order to make the fine slope differences of the performance field more visible.

**Remark 22** The results - calculated and interpolated performance fields - contain explicit expressive information and are available for visualization or for other performance analysis tools. The algorithms are not restricted to specific sound parameters, and the method can thus be used for extensive expressive analysis. Currently, basis calculation imposes the biggest limitation. In some cases, the calculated basis of a note does not correspond to its musical context, resulting in field vectors that are - while being mathematically correct - hard to understand. Here, ongoing research will definitely deliver better results.

A promising field of further research is also the insight that performance fields are not restricted to musical data. In medical applications and in computer-aided anthropology, the growth information of human bones and organs can be extracted in a similar manner - in which case we may talk about *Nature's Performance*.

## 46.4 Local Sections

**Summary.** We study canonical sections, i.e., selectors of points in grammatical varieties. The subject deals with problems of how to choose a determined critique out of an entire variety of possible critiques.

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The generic isomorphism of critical fibers as shown in the above corollary 22 suggests that we should look for musically motivated restrictions on the admitted system parameters in order to obtain more specific information. In this section, we propose a model which is based upon the causality and finality of locally linear grammars and gives a manageable approach to the structure of causality and finality in the transition parameters $c^m_{x,y}$ from a daughter $x$ of a mother $m$ to its sister $y$ (notation inherited from section 46.2).

The idea is to parametrize the transition parameters by a small set of shaping parameters which give the system of all $c^m = (c^m_{x,y})_{D_m(T)}$ a causal-final coherence. Suppose that the daughters in $D_m(T)$ are linearly ordered with respect to time. For example, this may happen if these daughters are defined by onset intervals of a sequence of adjacent bars or periods. Then we may just enumerate these daughters $x_1 < x_2 < \dots x_{d(m)}$ in ascending order and temporarily abbreviate $c^m_{x_i,x_j} = c_{i,j}$. In this notation, if $i < j$, this means that $c_{i,j}$ measures the *causal* influence of the prior LPS $x_i$ on the later LPS $x_j$, while $i > j$ yields a measure $c_{i,j}$ for the *final* influence of the later LPS $x_i$ on the prior LPS $x_j$. The diagonal values $c_{i,i}$ measure the "autocorrelation" of LPS $x_i$.

With this in mind, we want to give the value matrix $c^m$ a simplified shape as follows: Suppose that the index set $1, 2, 3, \dots d(m)$ is evenly distributed on the interval $[1, -1]$, i.e., $\lambda(i) = 1 - (i-1)2/(d(m)-1)$. We now define the function $deformLiGr$ on the square $[1, -1] \times [1, -1]$

on the real plane as follows:

$$spl(x) = e^{-(0.4x)^2}$$

$$LiGramm(causal, final, x) = \begin{cases} (1 + 0.1.causal.x^2)spl(x) \text{ if } 0 \leq x, \\ (1 + 0.1.final.x^2)spl(x) \text{ else,} \end{cases}$$

$$LiGr(causal, final, x, y) = LiGramm(causal, final, (x + y)/2)$$

$$deform(start, end, y) = 0.5((start - end)y + (start + end))$$

$$deformLiGr(causalStart, causalEnd, finalStart, finalEnd, x, y) =$$

$$LiGr(deform(causalStart, causalEnd, y), deform(finalStart, finalEnd, y), x, y). \quad (46.6)$$

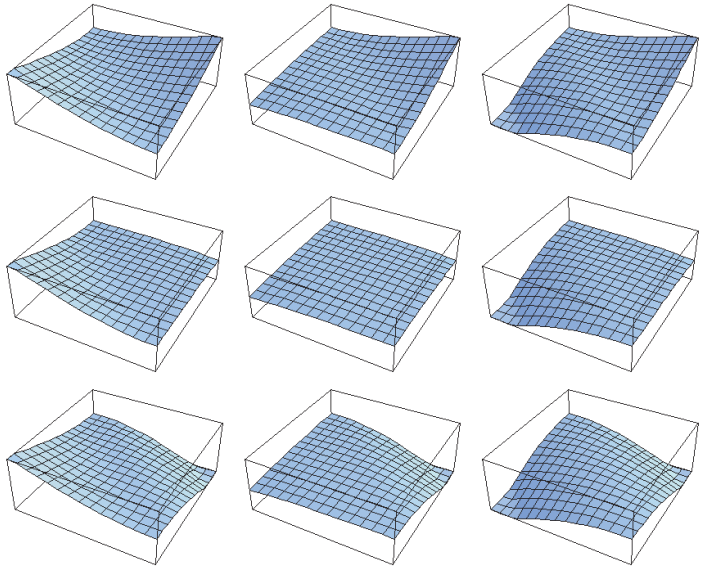

Figure 46.5: Nine shapes of the function $deformLiGr$ for the system parameter 4-tuples $causalStart, causalEnd, finalStart, finalEnd$ (from left top to right bottom): (3,3,3,3), (3,3,1.5,1.5), (3,3,0,0),(1.5,1.5,3,3),(1.5,1.5,1.5,1.5),(1.5,1.5,0,0),(0,0,3,3),(0,0,1.5,1.5), (0,0,0,0). The causal extremum $(1, d(m))$ is to the left, the final extremum $(d(m), 1)$ is to the right of each surface. The horizontal diagonal is the autocorrelation area which has the constant value 1.

For each fixed pair $x, y$, the function $deformLiGr$ is an affine function of the four system parameters $causalStart, causalEnd, finalStart, finalEnd$. The system parameters contain information on the strength of the causal and final correlations at start and end of the given time interval. Figure 46.5 gives an image of the function for nine different system parameter

combination. The function is evaluated on each pair $x = \lambda(i), y = \lambda(j)$ and yields $c^m_{i,j}$ as an affine function of the four system parameters. We see that low parameter causal and final parameter values, respectively, give low correlations in causal and final direction, respectively. For example, the top right "flying carpet" is strong in final, but weak in causal direction.

This means that we are looking at an algebraic variety which is defined by the substitution of the generic transition coefficients $c^m_{i,j}$ by functions

$$c^m_{i,j}(causalStart^m, causalEnd^m, finalStart^m, finalEnd^m)$$

of four system parameters which are a function of the given mother $m$. If $m(T)$ is the total number of mothers of the stemma quiver $T$, this yields a total number of $4.m(T)$ *causal-final variables* which add to the analytical variables from the modules $B_x$.

To be clear, the shape of a "flying carpet" is a transcendental function involving exponential and quadratic components, but the causal-final system variables which define the actual shape of a carpet are only involved in an affine way.

### 46.4.1 Comparing Argerich and Horowitz

**Summary.** This section applies the comparative theory to a particular case of agogics performance in Robert Schumann's "Träumerei" by Martha Argerich and Vladimir Horowitz. It turns out that Argerich's agogical coherence is global compared to a rather local coherence with Horowitz.

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In this section, we shall apply the preceding approach to the inverse problem regarding tempo curves and their shaping by use of motivic weights and associated operators. The piece is Robert Schumann's Kinderszene 7 "Träumerei", the performances are those by Martha Argerich (ARG) and Vladimir Horowitz (HO1 in Repp's list), both measured by Bruno Repp [438] among a total number of 28 performances that were already discussed in chapter 44.

For our example, we take the stemma on the onset space which is defined by the primary mother $R_M$ frame that extends from the first onset through the end of the last event. This mother has four daughters $R_A, R_{A'}, R_B, R_{A''}$. Frame $R_A$ is the time frame of part $A$ from the very beginning to the end of bar 8. The second frame $R_{A'}$ is the time frame of the eight bars of the first period, the frame $R_B$ is that of the third eight bar period, the last frame $R_{A''}$ is that of the reprise eight bar period to the end. Each of these daughters is the mother of eight bars, except the first, which has a first daughter including the upbeat quarter in the first full bar. Let us denote the daughters of $R_A$ by $R_{A,1} \dots R_{A,8}$, then those of $R_{A'}$ by $R_{A',1} \dots R_{A',8}$, those of $R_B$ by $R_{B,1} \dots R_{B,8}$, and those of $R_{A''}$ by $R_{A'',1} \dots R_{A'',8}$.

For the analytical parameters, we take a constant vector space $B_x = B = \langle W_1, W_2, \dots W_t \rangle$ generated by a finite family $(W_i)_i$ of global weights on the onset frame $R_M$. For a given onset frame $R$ of our stemma, we have the linear map $\varphi_x : B \to A_x$ into the vector space of $C^1$ tempo curves on the stemma frame $x$. It maps the weight $W$ to the scaling transformation by the quantity $\varphi_x(W) = |x|^{-1} \int_x W$. This operation leaves constant fields constant, polynomial fields polynomial, etc., it conserves any reasonable special type of tempo curves. Observe that we really restrict global weights to local frames and do not recalculate local weights when averaging.

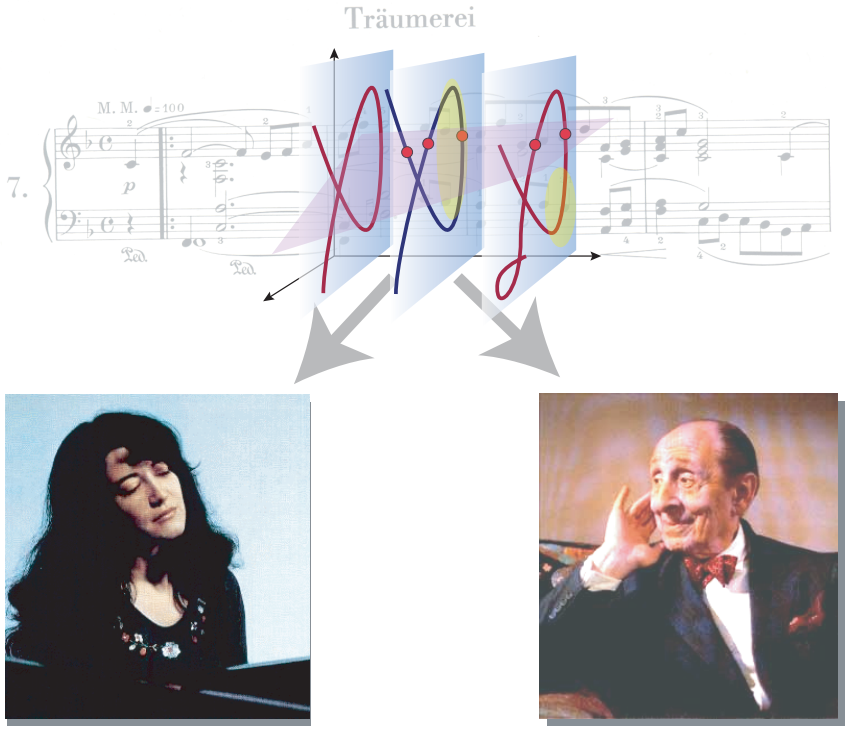


Figure 46.6: The restriction of the generic fiber spaces to 'flying carpet' coherence domains yields characteristic differences between Argerich's and Horowitz' performances.

This being the case, we start by a global constant default tempo $dT$, then take its restrictions to the period daughters, then apply all the scaling transformations to the restricted tempi, then add up all the shifted contributions that are multiplied by the crosscorrelation coefficients from a "flying carpet" with causal-final variables

$$causalStart^M, causalEnd^M, finalStart^M, finalEnd^M$$

on the periods. Then we have the 32 restrictions to the single bars, the application of the scalings according to the averaged weight restrictions to the bars, and then ending by the cross correlations according to "flying carpets" for the four periods and eight bars each, with the total of 16 causal-final variables

$$\begin{aligned}
&causalStart^A, causalEnd^A, finalStart^A, finalEnd^A,\\
&causalStart^{A'}, causalEnd^{A'}, finalStart^{A'}, finalEnd^{A'},\\
&causalStart^B, causalEnd^B, finalStart^B, finalEnd^B,\\
&causalStart^{A''}, causalEnd^{A''}, finalStart^{A''}, finalEnd^{A''}.
\end{aligned}$$

We have applied this system to the one and only motivic boiled-down weight

$$x^{P_{Träumerei}}_{MeloRubette,Param_{melodic}}$$

which was also used in the statistical analysis of chapter 44. This yields a total of 21 variables and a total of 32 cubic polynomials (one for each bar) in these variables, whose solutions yield the variety lying above the average measured tempi on the 32 bars. Therefore, it cannot be expected that we really have zeros of all these equations.

With Mathematica® routines, we have therefore calculated local minima of these polynomial equations and found the following results [348, III] in terms of causal-final variable values:

**Result 7** *Period level:*

- *In the inter-period coherence, Argerich is more final than Horowitz, whereas the causal level is more pronounced by Horowitz.*

**Result 8** *Bar level:*

- Horowitz *plays the first period with pronounced causal and final coherence, whereas the causal coherence decreases to a very low level towards the end of the piece.*
- *The repetition $A'$ of the first period $A$ shows a 'relaxation of coherence' which may be justified by the repetitive situation.*
- *The development section $B$ slightly increases the final character.*
- *The recapitulation seems to be quite 'tired': the causal character is very low, the final character is decreased.*
- *For* Argerich*, the first period has a less coherent ambitus than with Horowitz.*
- *In contrast to Horowitz, the final coherence of Argerich increases as the piece goes on.*
- *The development and the recapitulation are pronouncedly final. The development and the recapitulation shows a consciousness of the end of the piece which is absent with Horowitz.*
- *In other words, Argerich's recapitulation is 'prospective' and not 'retrospective'.*

These calculations are however only locally relevant, and a global solution space with more subtle estimations should also be calculated. Nonetheless, an identical algorithm is applied to two samples and therefore, comparing these inverse performances is admitted.

# Part XIII

# Operationalization of Poiesis

# Chapter 47

# Unfolding Geometry and Logic in Time

*Keeping time, time, time,*
*In a sort of Runic rhyme,*
*To the tintinnabulation that so musically wells*
*From the bells, bells, bells, bells.*
Edgar Allan Poe: The Bells (1849)

**Summary.** Musical poiesis in composition (and performance) is intimately related to a projection of abstract objects into time. We discuss the logical and geometric aspects of this mapping process. The subject is crucial to the entire art of music since music is involved in the creation of autonomous time beyond the physical "tyranny" of real time. This enforces a review of Michael Leyton's theory of time as a philosophical category which is *derived* from spatial symmetry transformations [303]. We also discuss the role of unfolding insight within the syntagmatic discourse of music.

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After the development and analysis of the various aspects of musical structure and process in this book, it should no longer astonish that the creation of music involves an incredibly complex semiotic, communicative, and reality-critical construction. Far beyond a simple representation and performance of elementary sound events, music is a narration of strong logical and geometric categories, or, at least: without such an intense existentiality, music would never reach the status of a valid antiworld which takes us to an autonomous time and space. This is valid all the more than straightforward common signification processes fail, and meaning has to be built without external references.

So the question is legitimate: What is the story that compositional narration is likely to convey to the listener? This question is critical in two regards: first, the narrative discourse without common, external content is not likely to be a remake of the narrative discourse in ordinary language. Second: The absence of an ordinary content (the story) makes it a questionable point whether narration in music is a reasonable category at all, or whether it is rather

a metaphor which one instantiates *à défaut de mieux*, and which one should avoid rather than abuse.

Of course, the musical performance is embedded in the physical time-line, just like the ordinary story which is being told in a piece of literature. But in literature, this time surface transports a hierarchy of time strata which owe their existence to told and telling times of a time-sensitive reality, be it a description of events and movements in a fictitious world, be it the story teller's discursive telling time. In music, this is not possible. Except physical time, there are no timely contents that are referred to and represented, including a beginning, middle, and ending part, as Aristotle has described narration in his *poetics*. This means that if narration does exist in music, it must not only invent an autonomous content layer, but in addition, it has to create the time hierarchy that organizes the narrative stream. In this short chapter, we want to give some remarks on how such a creative process and its contents could be conceived.

## 47.1 Performance of Logic and Geometry

**Summary.** In contrast to linguistic discourse, musical syntax does not, a priori, tell a story which has its own time dimension. We contend that the intrinsic story in music is about *performing logic*. This includes—in particular—that musical time means above all syntagmatic time, and not the material time shared by the parametrized music events. We relate the subject to Algirdas Julien Greimas' theory of narrativity [195].

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Although there is no common storyboard in music, it does nevertheless realize a narrative discourse. We shall see in chapter 52 that the discursive dialog among four humanist persons (as represented by four instruments from the family of violins) is a characteristic feature of the string quartet. This is all the more remarkable that string quartet music is quite the contrary of a program music, it has always been the art form of "absolute music" where an abstract "musical idea" is processed.

So what are these musical personalities talking about? In common musicology, one would argue that they are organizing musical ideas. We suggest that more precisely, they are constructing a poetical work along the principles of Jakobson's poetical function: projecting the paradigmatic axis into the syntagmatic axis, see section 11.6.1 for this function. Such a paradigm could be a pitch or chord class, a tonality, a motif or a melody, a contrapuntal line such as cantus firmus or discant, a rhythm, or an instrumental color, for example. The projection of such paradigms into the syntagmatic axis means (1) selecting representative instances of such paradigms and (2) arranging these instances along the syntagmatic contiguity. So the paradigms in absentia are syntagmatically represented in praesentia. This may be a contiguity in time, such as is the case for a succession of tonalities, or else a contiguity in pitch, such as we have discussed for contrapuntal voices, or a contiguity in a sound color space for a multi-instrumental projection.

The point here is that these syntagmatic relations are not linear, i.e., the contiguity can extend to different dimensions. And it can also take place on relatively abstract levels in the sense that, for example, a tonality need not be the concrete score level, but a preliminary organizational level which is a generic scheme for the score realization. We shall also see in

chapter 51 on the OpenMusic software that the abstract syntagms are completely natural in intelligent implementations of compositional strategies.

The syntagmatic arrangement of paradigmatic representatives creates the first instance of Greimas' fundamental categories of narration, i.e., succession. But we again have to stress that succession is not necessarily one in physical time, it is one in an abstract parameter space, or even in a space form of generic concepts. The concatenation of successive units is given a logical justification by the insertion of transformational process units, such as contrapuntal interval relations, contrapuntal or harmonic rules of progression, modulatory parts between adjacent tonalities, transformations of motives to their variations, for example. This enrichment is associated with Greimas' category of transformation which, together with the states of succession defines his "programme narratif"[1]. In the third stage of the narrative organization, the category of "mediation" is recovered in order to embed the narrative program in a global reasoning, i.e, the declaration of a purpose behind the narrative program. This can be, for example, the overall strategy for a modulatory plan as we have known it in the analysis of the modulatory landscape in Beethoven's op. 106, see section 28.2.

Summarizing, the narrative organization is a performance of the logical strategies which dispatch paradigmatic units in the syntagm of their logical concatenation, the latter being explicated on successive transformations, whereas the whole expresses what one really could call "the musical idea". But the syntagm of this organization is by no means a linear one, and even less one in the physical time of a typical telling instance. And accordingly, the story being told is much more than a temporal succession of events, it is a logical construction of local and global compositions, of morphisms between such objects, of universal constructions in the corresponding categories, short: of logically and geometrically motivated predicate instances. This is why constraint programming has become an interesting field in computer aided composition, but see chapter 51 for details.

At this point we have to a certain degree reached what Eduard Hanslick [206] calls "tönend bewegte Formen". So what is missing? Essentially, there are two major gaps at this stage:

- A narrative time concept and its organization is still missing.
- The unfolding of the syntagmatic logical display into physical performance time is outstanding.

We are now going to discuss these issues.

## 47.2 Constructing Time from Geometry

**Summary.** We sketch predicates of logical and geometric nature which are designed to set up syntagmatic time in music. In this context, Michael Leyton's theory of time is briefly reviewed.

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As the present organism is a purely logical and geometric one, the only hope to generate narrative time is by means of mechanisms which turn logic and geometry into time. On the

[1] "On appelle programme narratif (PN) la suite d'états et de transformations qui s'enchaînent sur la base d'une relation S-O [Sujet-Object] et de sa transformation."

logical level, for example, the implication form $A \implies B$ induces a time category where the condition is antecedent to the implied statement. On the geometric level, there are several symmetry-related "time generators". In fact, a symmetry $S$ can be viewed as a transformation, where objects $x$ are moved around, i.e., $x \mapsto S(x)$, whence a time-stamped relation from $x$ to $S(x)$.

A radical approach to time is proposed by Michael Leyton [303, 304]: In its cognitive *and* physical reality, the time concept is viewed as a derived one. Leyton's theory develops structures to reduce time ontology to a dynamical syntax of spatial symmetries. These symmetry groups are wreath products[2] $F \wr C$ of a control group $C$ and a fiber group $F$. Leyton's approach allows us to understand music as a *natural* reconstruction method of time *via* syntax of symmetries. This syntax aligns symmetries and their broken variants as a ordering relation: first symmetry, then its broken version. Time is generated via symmetry breaking.

This implies that musical time is generated by such symmetry breaking relations. Together with the logical time construction, we obtain a variety of ordering relations which are induced by symmetry breaking processes. More generally speaking, time is generated by logical and geometrical succession relations. For example, if we recall the model for tonal modulation (chapter 27) the modulation quantum is a global composition whose inner symmetries relate the antecedent to the successor tonality, but it is not possible to distinguish these tonalities in their logical role from the structure of this symmetric quantum. Leyton would say [304, symmetry and asymmetry principles p.41] that symmetry is without memory, since it is symmetry breaking that constructs memory. The time ordering of the involved tonalities is only generated by marking the trace of the modulation quantum in the target tonality: which defines in fact the modulation steps. This is also the Greimas transformation part relating the two units in the succession chain. Such a breaking of a symmetry in modulatory degrees creates time and bans the symmetry from the syntagmatic surface.

A good example of memory-less construction by unbroken symmetries is Schönberg's dodecaphonic method. It starts on an original dodecaphonic series and proposes a display of some of its (generically) 48 transformed versions under the known group $D_{12,12,}$ (see the discussion following definition 22 in section 8.1.1). There is no specification of the method regarding the instantiation of the involved symmetries, they are external to the composed syntagm, and there is no symmetry breaking that would create a time-line among the 48 variants in that orbit. So the narrative structure of this method is not specified. Maybe this drawback, which lives in a pronounced contrast to the excellent narrative sonata form, is one of the reasons for the failure of *pure* dodecaphonism, i.e., dodecaphonism that is not enriched by syntagmatic concepts for the narrative construction.

So let us assume that syntagmatic time is generated as an ordering relation between logical and/or geometric units. Such a time is not necessarily linear, there may be several competing time strings. For example, recall from counterpoint theory that "punctus contra punctum" relates to the horizontal interval succession as well as to vertical cantus firmus vs. discantus succession (as the melodic implication). So we have two time strings which are simultaneously present in the "contra" process. In a modulatory process, the time-line of tonalities can be superimposed by a vertical time-line of melodic deduction from the harmonic basement, and a time-line induced by an instrumental hierarchy. Such abstract time-lines are the narrative

[2] See Appendix 73.

bricks of the musical counter-world.

Suppose that we now dispose of such a narrative organization. Then, its unfolding is the following project: In performance, we have to unwind the logical and geometric time-lines in a space of performance time. We have discussed at length the time hierarchies and stemmata of performance in parts VIII and IX. Evidently, these tools must be used to unfold the abstract narrative organization in physical reality. Presently, such a theory is lacking, but the prerequisites are at hand. The main problem with regard to human cognitive capacities is to find an equilibrium between the complexity of the narrative instances of logical and geometric time-lines and the limitation of comprehension of such structures and processes in the physical performance space-time.

This means that we have to name principles that define the basic structures of a global composition with regard to its narrative communication. A global composition has a basic syntagm that is defined by its covering and, more specifically, its nerve. This is already present in Schönberg's harmonic strip between successive chords, which is in fact an elementary but prototypical syntagmatic junction (the transformational part between two adjacent succession units in Greimas' theory). So the composition's nerve is an excellent syntagmatic device in that it includes narrative paths (on the nerve's one-dimensional skeleton) of neighboring local charts in praesentia. For example, if we have the interpretation of a diatonic scale by the seven triadic chords, we obtain the harmonic (Möbius) strip as a global composition. The geometric relation of two neighboring degrees on the strip, $I_C, III_C$, say, can be given a narrative direction in the succession $I_C^6, III_C$ where the common notes $e, g$ of the inversion $I_C^6$ and the third degree may be held such that we only have the time event of a downward movement $c \mapsto b$ to indicate the time-line from $I_C^6$ to $III_C$. We contend that any unfolding of the syntagmatic logical and geometric time-lines should locally *on the physical time axis* produce a small linear storyboard, i.e, minimize the multi-dimensionality of the abstract syntagm, or at least try to stress its most important components. To put it the other way round: Understanding the abstract narrative structure of a composition's nerve amounts to realizing a sequence of paths through the nerve's one-dimensional skeleton such that eventually, all important parcours of the nerve's 'city map' have been exposed. The problem here is analogous to the problem of how the eye's path should look when it observes a painting in order to obtain a good understanding of what is seen, or else to the problem of how we should walk through a house in order to optimally understand it as an architectural organism. It is the problem of curvilinear reduction of high-dimensional objects, a problem which by the human language stream of words has been solved in a very special context and with the success of a very limited textual representation of the world.

## 47.3 Discourse and Insight

**Summary.** This final section locates the musical discourse as a means of gaining insight into complex multidimensional structures by syntagmatic sequentialization (linearization). This contrasts with the linguistic discourse, since the latter has rich semantic rootings which are absent in music, must therefore be balanced by explicit discoursivity.

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The advantage of common language above music is the presence of external semantics, the language stream runs around any kind of complex entities that are *not* part of the language

structure. This makes language an easy business since it does not include its objects, but only points to them. In contrast music cannot point outside (except for trivial special cases of onomatopoiesis), it has to build everything from an inner discourse. This is also its advantage and scope: to be able to invent worlds instead of pointing to them. In this sense it is an excellent exercise to learn what is really needed to create a 'world' in our imagination, i.e., a meaningful spiritual architecture without external roots. The investigation of the narrative discourse should teach us how we gain insight into things when telling their story, meaning: how to understand music while playing it.

# Chapter 48

# Local and Global Strategies in Composition

*Aus den Auswahlkriterien also entsteht die Dialektik der Abfolge oder Verknüpfung von Lokalstrukturen, wobei diese Auswahlkriterien bestimmend sind für die Eingliederung der Lokalstrukturen in die große allgemeine Struktur, die* Form.
Pierre Boulez [60, II]

**Summary.** This chapter sketches the compositional process between paradigmatic selection and syntagmatic combination within the musical sign system. Apart from these semiotic perspectives, the process is characterized by a "dialectic" interrelation of local and global criteria. These features—well known from the general structure theory of global compositions—reappear in the special light of poiesis: The construction of a composition resembles the step-by-step completion of a puzzle of logical units, distributed in syntagmatic time, and selected to optimize association to already placed units. This activity is remunerative and fed back by a successive accumulation of poetical semantics.

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We do not intend to describe the psychological path in the composer's mind, but the fundamental strategic steps, independently of the psychological or cognitive realization. We also do not impose this strategy on any composer, but want to sketch a possible system for composition which can also be implemented on the software level. We shall see that two implementations: *presto*® and OpenMusic realize quite a portion of these ideas.

Essentially, the poiesis of a musical composition is the construction of a global composition, i.e., a patchwork of local charts, together with gluing transformations. Moreover, the charts are selected from a set of paradigms and combined according to a set of syntagmatic rules. But the concrete making of a composition is not a one-step procedure, it results from a successive completion of an ensemble of charts. Ideally, one is

1. given a simplicial complex and would like to realize it as the nerve of a covering by local charts.
2. One also wants the charts to pertain to determined isomorphism classes or else being specializations of representatives of such classes.
3. One finally wants particular gluing transformations on determined parts of the charts.

Point two reflects the analytical approach using the paradigmatic theme of Ruwet and Nattiez, see section 11.7. But it extends it in that the nerve is planned as a syntagmatic design pattern, further specialization and not only transformation of basic paradigms is allowed, and finally, the gluing transformations for the syntagmatic combination are thematized.

## 48.1 Local Paradigmatic Instances

**Summary.** Local paradigmatic strategies can be split into transformational and topological procedures. In each case, we describe the basic options.

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Recall from chapter 10 that paradigmatic relations can be either transformational or topological—or combinations thereof. Whereas the first is related to symmetries, the latter is related to the idea of variation. Usually, transformations will quite brutally change the auditory impression of a local composition and must be used with care, whereas topological variation is more adequate for instantaneous recognition of similarity.

### 48.1.1 Transformations

**Summary.** Starting from Schönberg's dodecaphonic method, we describe varieties of local compositions produced by a determined set of transformation groups, *the paradigmatic themes of poiesis*, acting on a selected set of local "germs".

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This method starts from a sequence $S = (G_1, \dots G_c)$ of local germs, for instance thematic units such as the singleton $S = (G_1)$ containing a dodecaphonic series, or a two-element set $S = (G_1, G_2)$ containing a main motif and a side motif of a sonata. Then, for each local germ $G_i$, we are given a transformation group $P_i$, the paradigmatic group of this germ, which describes the a priori allowed transformations. For instance, for dodecaphonic compositions, we have $P_1 = D_{12,12}$. It is however not necessary to realize the entire orbits $P_i.G_i$ of these group actions, a composition will usually select a small number of such transformed germs. In the construction of the germinal melody of the "Synthesis" composition in section 11.6.3, see also figure 11.16, we have 26 isomorphism classes of three-element motives and take one representative of each class, where the transformation group is the same for all classes, i.e., the full affine group on $\mathbb{Z}_{12}$.

The main point in this procedure is the instantiation of the transformations within the composition. In fact, it is very rarely sufficient to place just two objects $G_i, p.G_i$ in contiguous position in order to communicate the particular symmetry $p$ which is responsible for the

succession. It is part of the failure of dodecaphonism that its exponents never did care about instantiating $p$. A nice example of such an instantiation in Beethoven's op. 106 was given in section 28.2.5.

### 48.1.2 Variations

**Summary.** Instead of transformation groups, a set of local "germs" can be deformed according to a set of similarity paradigms (e.g., gestalt topologies)

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In this approach, there is no group action and the original germ $G_i$ must be associated with one of its specializations by use of deformation procedures. For example, one might superimpose a force field on the given germ and then move its points around according to the local force action. Such a procedure has been implemented in the OrnaMagic module of the *presto*® software, see section 49.3. This variational change of perspective is very precious for understanding a germ's potential, but in general will not give us back the original germ. The quantity of change is a function of the used gestalt topology whose choice depends on the composer's preferences.

## 48.2 Global Poetical Syntax

**Summary.** According to rules of contiguity and semantic "added value", a variety of transformed and/or deformed local germs is distributed along the syntagmatic space. We systematize the possible procedures according to horizontal and vertical poetical functions.

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The construction of a global assembly of local charts which are provided by the local techniques is very delicate, see also Pierre Boulez' reflections [60, II]. For instance, the germinal melody in "Synthesis" was built from the generic motif class representative, then gluing it with a representative of the first different motif class that fits with the first representative on a subset of two points. But the possibility to find such a solution until every class was represented exactly once is not obvious, it is a constraint to the compositional material that could have ended in an incomplete solution. The normal procedure is a puzzle reconstruction: One begins on any interesting germ and adds other germs at any syntagmatic position (not necessarily adjacent), and successively densifies the display, thereby observing constraints on the nerve that one wants to construct. For example, in the sonata "L'essence du bleu" [368], the plan as described in [328] was to construct a global composition on a motivic zig-zag $\{c, c_\sharp, d, d_\sharp, e, d_\sharp, d, c_\sharp, c\}$ by use of nine three-element submotives in such a way that the nerve turns out to be a M
bius strip.

The construction of a nerve is not only complicated as a constraint problem, it is also a problem of narrative character: the intersection configurations must also be mediated on the concrete time axis, and in such a way that the transformations between associated units can be made explicit, e.g., the Möbius strip must be boiled down to the time axis such that its

one-dimensional skeleton (we stick to the ideas from section 47.2) is transgressed on reasonable paths.

Evidently, the main point of this global patchwork is the syntagmatic distribution of paradigmatic representatives in order to achieve an added poetical value in the sense of Jakobson's poetical function. This objective splits into two subtasks: horizontal and vertical poetical functions according to Jakobson and Posner, respectively.

### 48.2.1 Roman Jakobson's Horizontal Function

**Summary.** Distribution of a variety of transformed and/or deformed local compositions can follow Jakobson's correspondence [245] along the syntagmatic axis.

– Σ –

Jakobson's poetical function was introduced in section 11.6.1. Counterpoint, harmonic syntax or dodecaphonic row distribution are examples of this mechanism. The most elementary realization of this function is the action of a translation group on rhythmical units in order to produce a cyclic character of our composition. This semantic enrichment creates a coherence of the compositional corpus which is basic to all other elements. It may, for example, connect motives, harmonies and similar units if they are distributed on the regular positions of the rhythmic basis. But it also creates a time circle that abolishes the straight physical time-line and rebuilds and "good" cyclic eternity (in Hegel's sense) which our finite life was to negate.

### 48.2.2 Roland Posner's Vertical Function

**Summary.** Jakobson's poetical function is orthogonal to Posner's vertical function which relates different signification levels in denotation and connotation. We discuss strategies which make use of this functionality in the production of poetical "added value".

– Σ –

Jakobson's poetical function relates signs of the same semiotic system, for example, two phonological signs such as "bad", "dad" which are positioned in syntagmatically equivalent places, in a rhyme, say. But this structure is enchained with the semantic sign system (by the double articulation of language, see [361]) where the phonological equivalence in the rhyming places produces a paradigmatic equivalence between the bad and dad, i.e., "dad is bad", a meaning which is neither inherent in "bad" nor in "dad". So the level of equivalence is shifted from the phonological to the semantic one. This shifting of equivalence in Jakobson's function illustrates the vertical function as proposed by Roland Posner [421].

In music, such a construction is not obvious since connotative semantic levels are not automatic, but must be constructed or simulated. This can be achieved by a small global composition which is attached to a local 'signal' composition whose equivalence to another local 'signal' composition on the horizontal Jakobson function induces an equivalence of the attached small global compositions. Such a small global composition can be a configuration consisting of a chord interpretation and a melodic unit, for example. So here, the connotative levels are constructed by successively enriched small global compositions within the total global composition.

## 48.3 Structure and Process

**Summary.** The poietic process is confronted with the resulting 'score' structure in order to recall a more general approach to the score concept.

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We have stressed the global composition as a target of compositional strategies. However, the making of this composition can be more relevant than the result. In other words, the general score concept which we discussed in section 33.3.2 may be specialized to the interior score known from jazz: Instead of a more or less fixed score structure in form of a global composition, the processes that lead to the effectively played material are stressed, while the resulting structure is quite secondary (albeit not irrelevant, as some fundamental critics of free jazz have argued) in that it is one possible variant of those processes, and another variant—as an exemplification of the operating processes—would do as well. The situation is comparable to algorithms for complex shapes, such as L-systems or fractals. They are defined as processes, and the resulting variants are all an expression of the same basic process type, as with biological phenotypical expression of the genotype[1].

It is a very interesting research area to classify process score types and to implement corresponding tools in music composition software. An example of such an approach (the composition software OpenMusic) is given in chapter 51. The following chapter 49 describes a software (the composition software *presto*®) which is mainly based on paradigmatic strategies. As a matter of fact, such strategies are very difficult to handle because a paradigm is, by definition, in absentia to the text, whereas syntagmatic structures are in praesentia, which means that a software that is built on the LEGO-like juxtaposition of bricks will have an easier acceptance than the paradigmatic one. However, if the bricks are no longer concrete musical material, but abstract units of processual character, the ease of the LEGO approach is no longer valid.

[1] *Phenotype* is the "outward, physical manifestation" of the organism, while *genotype* is the "internally coded, inheritable information" (written in the genetic code) carried by all living organisms.

# Chapter 49

# The Paradigmatic Discourse on *presto*®

*Damit können Sie mich eine ganze Nacht lang allein lassen!*
Herbert von Karajan to Guerino Mazzola
on the occasion of the presentation of the
*presto*® prototype in Salzburg 1984

**Summary.** The *presto*® composition software was developed as a commercial implementation and operationalization of mathematical music theory. Several large compositions have been successfully realized on this software [48, 49, 338]. One of them will be discussed in chapter 50. We describe the overall architecture and functionality of *presto*®. This is mainly driven by a local/global paradigmatic perspective. The paradigmata of transformation and deformation are realized on (1) the level of modular affine transformations in the four-dimensional space of onset, pitch, duration, and loudness, and (2) the level of variational deformations. Both paradigmata are discussed and exemplified. We conclude the chapter with a remark on the problem of abstraction in paradigmatic composition, since composers tend to have major difficulties to get familiar with abstract paradigmatic structures.

– Σ –

Although the software was implemented in ANSI-C on now historic Atari®computers, we first describe its functional scheme since it represents a prototypical and still unique way of thinking paradigmatic composition strategies (to our knowledge, no similar strategy has been implemented on software to the present). The *presto*® software as well as the entire source code are GPL and can be downloaded—together with a manual, examples, and a concept presentation software `prestino.prg`—from the book's CD-ROM, see page xxx.

## 49.1 The *presto*® Functional Scheme

**Summary.** We describe the functional scheme of *presto*®. The scheme is centered around the

back bone of global score, score, and local score. This threefold architecture reflects an early stage of the global-local paradigm from mathematical music theory.

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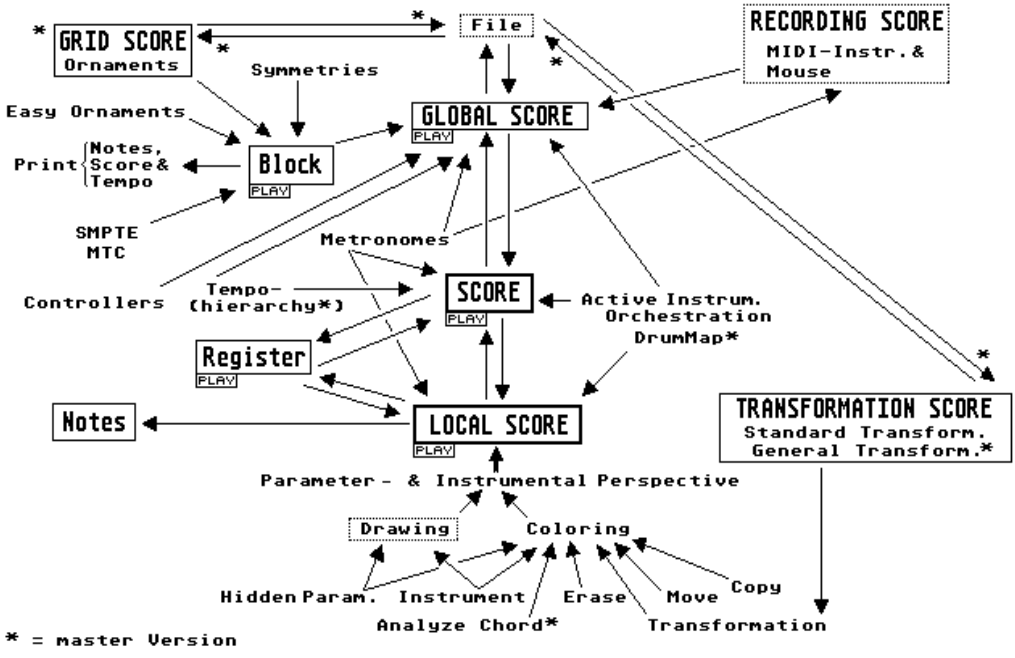


Figure 49.1: The functional scheme of the *presto*® software. It is centered around the back bone of global score, score, and local score. The graphics of this chapter are all from the original *presto*® manual [338].

The functional architecture of *presto*® consists of a series of so-called scores, see figure 49.1. The global score, the score, and the local score deal with geometric representation and editing of notes (mental note events) on global to local space levels. The recording score allows for mouse-driven and instrumental real-time input, the transformation score deals with graphical input of affine transformations, and the grid score allows for graphical input of ornamental grids. The input goes via recording via mouse or MIDI instrument, loading from MIDI or *presto*® data files, and painting with the mouse. Editing features split into affine transformations on charts of notes, building of ornaments, all-parametric variations according to ornamental attraction and repulsion fields, instrumental and parametric coloring of local charts, Boolean combination operations on groups of notes, and performance editing, especially construction of complex tempo hierarchies. For the latter, please refer to our corresponding discussion in section 38.2. Output is split into audio and SMPTE, saving is on MIDI and presto formats.

Figure 49.2 shows the main window of *presto*®. The rectangular score window in the middle shows onset (horizontal) and pitch (vertical, can also be set to loudness or duration),

and Schumann's Kinderszene "Haschemann". The black subrectangle marks a block which may be copied, cleared, moved, etc. Below the score, the tempo curve is visible. Above the score, the total composition space of the global score (narrow rectangle) is seen, below the tempo curve, six registers for provisional local parts of the score are placed. On top, we see 16 icons for instrumental colors (MIDI program change data), checked means the instrument is active; while editing, one may work on any checked subset. The different score and register windows

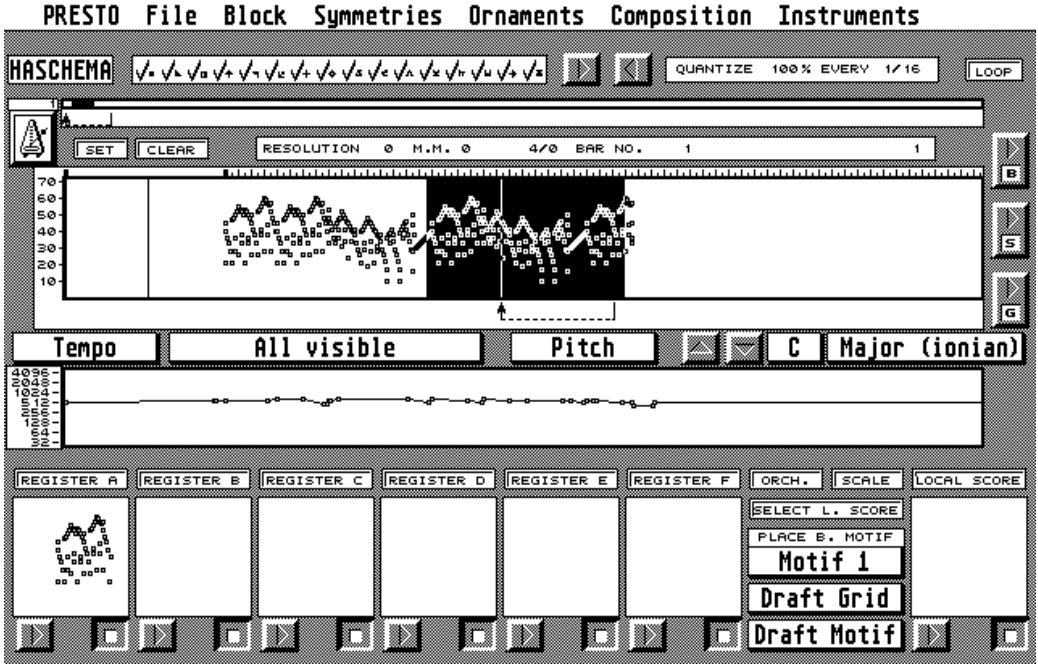


Figure 49.2: The main window of *presto*®. The rectangular score window in the middle shows onset (horizontal) and pitch (vertical, can also be set to loudness or duration), and Schumann's Kinderszene "Haschemann".

implement some chart types in the global composition strategy. One may perform any Boolean operations on such parts, e.g., on two registers. Evidently, modern object-oriented windowing techniques could vastly generalize this elementary implementation.

For a detailed editing of small composition parts, the local score window is available, see figure 49.3.

The local score can be opened by double-clicking on a selected position on the score; after editing it can be merged to the score. A miniature view of the local score's content is visible on the right lower corner of the main window in figure 49.2. The local global paradigm is realized on the local score via the feature of "coloring". On the local score, we view a four-dimensional cube, representing the four-dimensional discrete torus $\mathbb{Z}_{71}^4$, in any of the six relevant projections $Onset+Pitch, Onset+Loudness$, etc. onto two of the four parameters of pitch, onset, duration, and loudness. All parameters are integers modulo 71 and suitably calibrated: Pitch is a MIDI

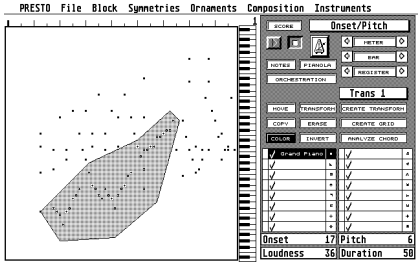


Figure 49.3: The local score is the interface for detailed editing of small portions of the actual composition. The darkened polygon marks a region where any operation, such as copy, erase, transform, change the instrumental icon, or simply play can be performed. New events can be defined by mouse drawing actions.

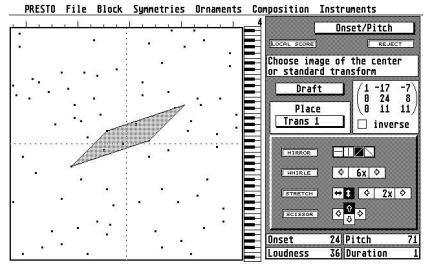


Figure 49.4: The transformation score is the interface for detailed editing of affine transformations of the torus $\mathbb{Z}_{71}^2$. The user's editing acts on a number of standard transformations (buttons to the right, middle), matrix text fields, and a graphical editing option by direct drawing of affine images of the unit square.

key number in the interval $[27, 97]$, loudness is a MIDI velocity value between velocity 0 and 127, equally distributed among 71 values, onset and duration are in integer units that can be set to be different fractions of a bar, depending on the time signature and the resolution preferences for each metrical unit.

The local-global paradigm is realized by the coloring feature: The user may define any closed polygon in a 2D projection, and thereby select all events that lie in that region with regard to the given projection. Such a coloring domain can be used to do many different things, such as moving around on $\mathbb{Z}_{71}^4$, copying, erasing, transforming, setting new instrumental icons, playing. By mouse-driven drawing actions, new events can be inserted. The parameter plane can be changed at any time. To the right, we see the orchestration (only piano in figure 49.3, see below for complex orchestras), where any instrumental icon can be unchecked for instrumentally specified editing options.

Of course selection of a toroidal representation and manipulation (for example for shifting operations, where events exit to the right and reenter from the left) is questionable, but once you are dealing with affine operations, such finiteness decisions have to be taken. The advantage of this selection will become clear in the next section 49.2.

## 49.2 Modular Affine Transformations

**Summary.** This section describes the mathematical framework of modular affine transformations in $\mathbb{Z}_{71}^2$, their four-dimensional extension and the combination options for such transformations, their action on selected local charts, and the graphical input features.

The selection of the torus $\mathbb{Z}_{71}^4$ was motivated by small number of pixels in those early screens, together with the need for icons in order to represent instrumental 'colors'[1]. Now, the selection of a prime number was not by case. In fact, the choice offers us a finite field where non-zero determinants of matrices automatically yield invertible matrices. So a large number of transformations becomes invertible, and therefore a large number of compositional processes become non-destructive. The software offers a transformation interface where any affine endomorphism in $\mathbb{Z}_{71}^4@\mathbb{Z}_{71}^4$ can be defined graphically as well as numerically see figure 49.4. We first use the fact (see section E.3.6, theorem 53) that $\mathbb{Z}_{71}^4@\mathbb{Z}_{71}^4$ is generated by its transformations which only move around two of the four dimensions. So the user may transform his/her material on the selected plane, then switch to another plane for further transformations, etc. The user may directly define a product of any number of plane transformations (for different planes) by a feature which memorizes a list of plane transformations. We have a total of

$$71^{16}.71^4 = 10'596'610'576'391'421'032'662'867'140'133'202'401 \approx 1.05966 \times 10^{37}$$

elements in $\mathbb{Z}_{71}^4@\mathbb{Z}_{71}^4$, and the number of

$$10'445'260'466'832'483'579'436'191'905'936'640'000 \approx 1.04453 \times 10^{37}$$

elements in $\overrightarrow{GL}(4, 71)$ according to the formulas in appendix C.3.5. The transformation score in figure 49.4 shows the basic input choices: matrix coefficients, standard transformations, and graphical editing by definition of the images of the three points: origin, head of horizontal unit vector, head of vertical unit vector. The parallelogram image of the unit square is visualized on the local score.

The geometric advantages of the prime number 71 are abundant, among others, we have these facts: Since the multiplicative group $71^\times$ is cyclic of order 70, i.e., isomorphic to $\mathbb{Z}_{70} \xrightarrow{\sim} \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_7$, we have fifth and seventh roots of unity in $\overrightarrow{GL}(4, 71)$. We also have $SO_2(71) \xrightarrow{\sim} \mathbb{Z}_{72}$ (see appendix C.3.5), and therefore, a generator $D$ of the special orthogonal group. Such a generator is

$$D = \begin{pmatrix} 30 & 33 \\ 38 & 30 \end{pmatrix}$$

which means that we may view $D$ as a rotation by $360° : 72 = 5°$. In the software, we have implemented the rotation by the triple, i.e., $D^3 \sim 15°$-rotation. Of course, such rotations in the modular torus can transform harmless compositions into very wild looking variants, but we have always experienced that the transformed version maintains certain regularities that were present in the original form.

## 49.3 Ornaments and Variations

**Summary.** Modular affine transformations are used to build ornaments by translation grids of "cells" of notes. Such ornaments are used directly as periodic note sets, for instance in drum patterns. They may also be used as "background" ornaments whose points act as centers of attraction or repulsion for other notes. The latter method is a multidimensional generalization

[1] Because of low resolution, color was not a good commercial option in 1988

of the well-known variational technique, in particular of the alteration of pitch sets (scales, chords, motives, etc.).

– Σ –

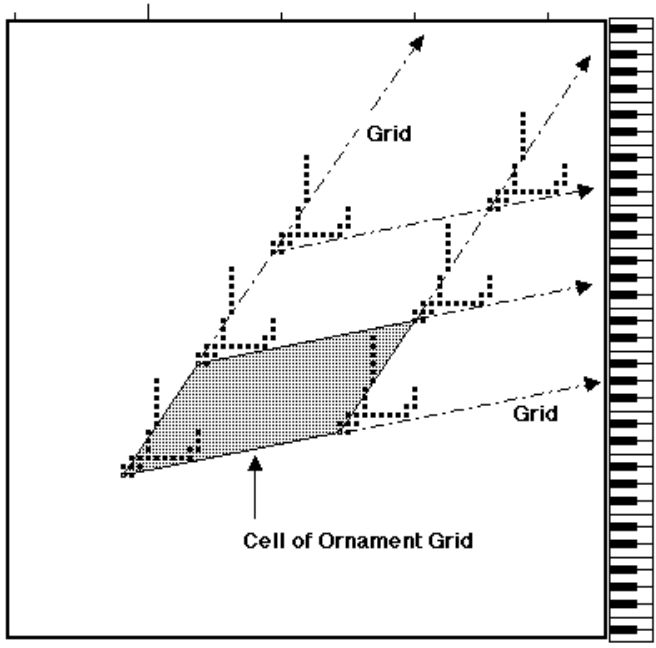


Figure 49.5: An ornament is defined by a motif, together with a translation grid and a range in each direction of the defining grid vectors. The grid cell is the parallelogram spanned by the generating vectors.

The software's module for ornaments is termed OrnaMagic. The idea is this: The user first defines a motif $M$, either a small one on the local score, or an arbitrary large one on the score. $M$ is just a local composition on the four-dimensional $EHLD$ space of the software. Next, two (usually linearly independent) translation vectors $g_h, g_v$ define the grid, i.e., the group $\langle e^{g_h}, e^{g_v} \rangle$. This group operates on the motif $M$ and yields a translated motif $M_{i,j} = e^{i.g_h + j.g_v}.M$ for each integer pair $(i, j)$, see figure 49.5.

The user defines a two-dimensional ornament of a special ornament window, the grid score, see figure 49.6. This gives us a 'grid of translations' $\bigcup_{a \le i \le b, c \le j \le d} M_{i,j}$. A second method allows the user to define also larger grid vectors on the score level, but the principle is the same.

In the composition "Synthesis" to be described in the subsequent chapter 50, we recognize a number of superimposed ornaments of drum sounds; here in the second movement, see figure 49.7. The compositional principles will be described in section 50.3.

Apart from this explicit usage of ornaments, there is a second truly paradigmatic, more precisely: topological usage by *two-dimensional alterations*. Here is the general procedure. We

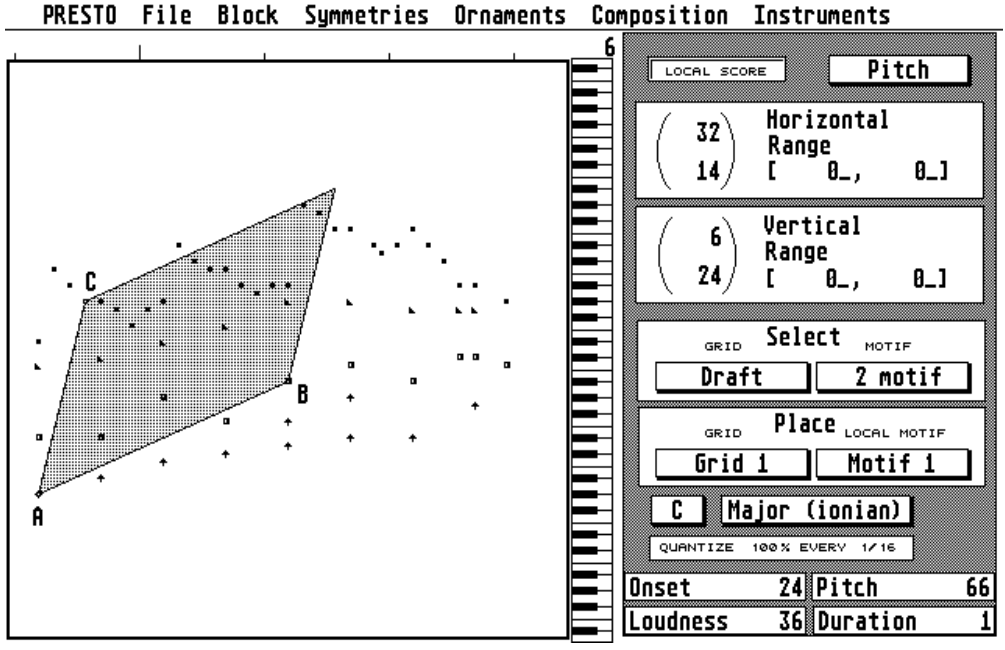


Figure 49.6: The grid score is the interface for defining an ornament via its grid vectors (so-called "horizontal" $g_h$ and "vertical" $g_v$) and the range $a \leq i \leq b, c \leq j \leq d$, i.e., the interval of horizontal/vertical translation $e^{i.g_h+j.g_v}$ to be effected on the motif $M$.

have two local compositions, $L, G$, where $G$ plays the role of the driving grid, whereas $L$ is the composition which we want to deform. Usually, $G$ is defined by an ornamental construction as described above, but this is not mandatory, i.e., one may also take a large motif $M$ and just apply the grid group on the zero range ($a = b = c = d = 0$).

Given these data, the alteration works as follows. Each event $x$ of $L$ is shifted towards or from $G$ according to a procedure which starts by the calculation of a 'nearest point' $G(x)$ to $x$ in $G$. This point is found starting on $x$, and then moving along a spiral outward around $x$, until the first point of $G$ is hit; this is $G(x)$. Following this first algorithm, several alteration strategies are offered. First, one may choose a degree of deformation, say $y\%$, a positive or negative real number. This means that the vector $G(x) - x$ is stretched to $y.(G(x) - x)$, and that the alteration of $x$ is set to $x+y.(G(x)-x)$. However, this is not the last word in alteration, one may also define its direction. Typically, this is vertical (conserving onset time), or horizontal (conserving pitch), etc. But the vector $G(x) - x$ is neither one nor the other. So we have to introduce the projection $p_t(G(x) - x)$ of the difference $G(x) - x$ to the direction $t$. This yields the final alteration

$$A_{t,y,G}(x) = p_t(y(G(x) - x)) + x$$

of $x$ according to the system variables $t, y, G$.

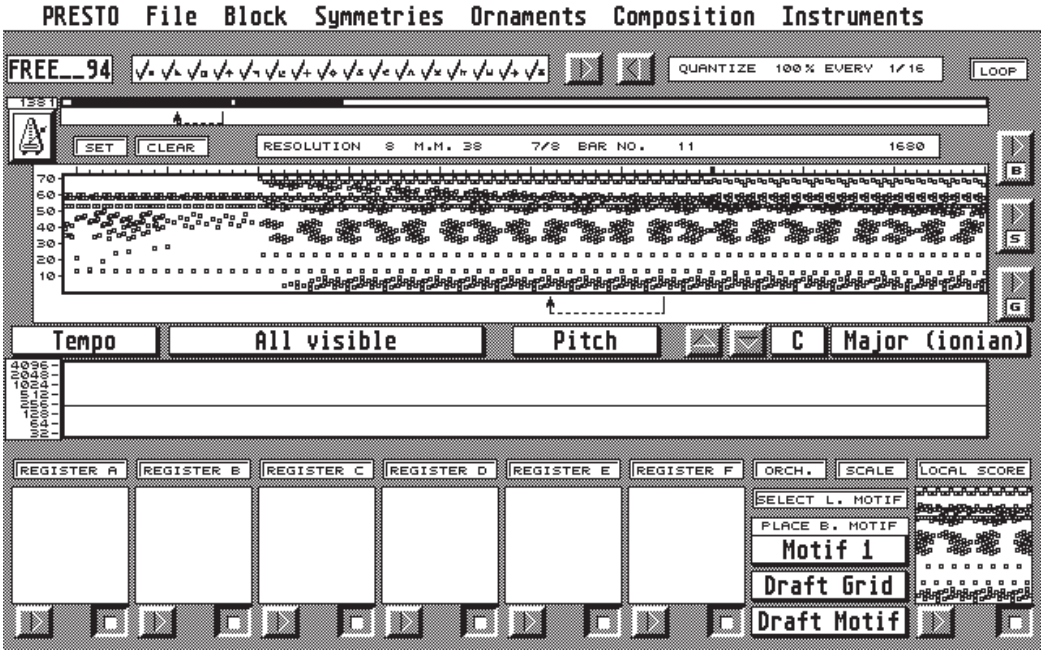


Figure 49.7: Drum patterns in the composition "Synthesis".

This is all implemented in the OrnaMagic module of *presto*®. Moreover, given a block with onset limits $u < v$, the user can define a successively increased alteration by setting the percentage to $y(u), y(v)$ and altering from $A_{t,y(u),G}$ to $A_{t,y(v),G}$ as onset moves from $u$ to $v$. Such procedures have been used in the composition "Mystery Child" in [49]. This piece can also be heard from the book's CD-ROM, see page xxx. The effect resembles a morphing operator which lets the background ornament $G$ act on $L$ with successive variation of its "alteration force field". As a special case, we have the classical *pitch alteration* that is driven by a background tonality $G$, as well as the onset alteration, i.e., better known as *groove effect* on sequencers, defined by an onset grid that is derived from a rhythmical onset configuration. These special effects are implemented as "easy alteration variants" in *presto*®.

**Example 60** A special application of pitch alteration is *tonal inversion*: If we want, for example, to apply tonal alteration of the C-major scale around e (leaving e fixed, exchanging d and f, c and g, b and a), we may first apply the real inversion $U_e$ and then apply the tonal alteration downwards with respect to C-major.

## 49.4 Problems of Abstraction

**Summary.** Whereas syntagmatic contiguity is a relation *in praesentia*, paradigmatic associa-

tivity is a relation *in absentia*. This difference has heavy consequences on the ease of managing paradigmatic composition tools. As a consequence, composers rarely transcend straightforward paradigmatics and prefer sticking to syntactical composition software, such as MAX, or common sequencer software.

– Σ –

The composition software *presto*® was designed as a tool for everybody, even for the non-experts in musical notation. The publicity and the product concept tried to invoke the statement "Beethoven's creativity at everyone's reach". The argument was that drawing music (see figure 49.8) is an interface that could give everybody access to the otherwise abstract and heavily codified music. Also, the simultaneous presentation in geometric coordinates of $EHLD$ space, together with the 2D projections, and the presentation in notes (see figure 49.9) was thought to be a big advantage against classical approaches to composition.

Now, at the center of this concept are evidently the TransforMaster (symmetries), the OrnaMagic (ornaments), and the AgoLogic (tempo curve hierarchies) modules. The first two of them are paradigmatic tools par excellence. They are powerful, but require strong abstraction capabilities. The *absentia* of the paradigm enforces a selection process of representatives, and a memorization of their relation to the paradigm. This reality switch is a major obstacle to a breakthrough of such a software concept. It seems to be a major task of future composition software development to build concepts that enable seamless transitions from abstract to concrete composition layers. In the limit, one should have very abstract objects, such as self-addressed compositions or even more generally addressed objects (why not play a functorial global composition, how would one do that?) at one's fingertips and handle them completely naturally as if they were normal notes or chords, and in fact: they *are* completely normal, only we did not learn to sonify them in appropriate shape. In this regard, Tom Johnson's compositorial approach [253] is one of the most promising in trying to morph mathematical objects into musical events.

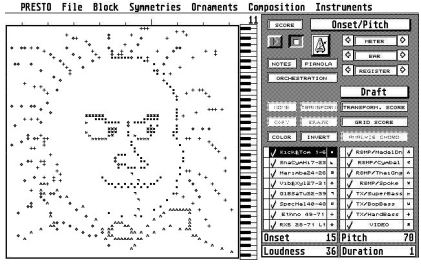


Figure 49.8: The *presto*® concept tried to merge Beethoven's creativity and ease of a graphically interactive interface for musical composition. The local score here shows a drawing of Beethoven in a geometric space, and using instrumental icons.

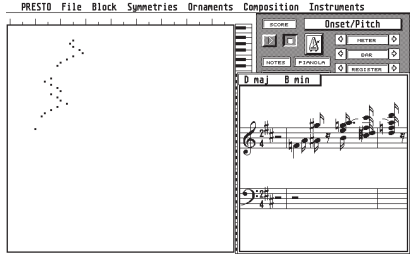


Figure 49.9: The geometric representation on the local score can also be viewed in classical score signs. One can immediately understand the geometric shape, whereas the common notation yields a completely cryptic object.

# Chapter 50

# Case Study I:"Synthesis" by Guerino Mazzola

*Habe die CD nun gründlich mir angehört und*
*frage mich, woher kommt bloss*
*die Kraft, die so was*
*Schönes schafft.*
Jazz Saxophonist Werner Lüdi on "Synthesis" [309]

**Summary.** "Synthesis" is a composition for piano, percussion and e-bass. Its global and local organization was driven by classification of local compositions and modulation theory on one hand, and by the *presto*® software tool on the other. We describe the overall organization and the four movements.

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"Synthesis" is documented on CD [339]. It is the result of a composition grant of the city of Zurich and was composed and recorded in 1990 using the *presto*® software, together with this hardware: one Atari®Mega ST4 computer, the synthesizers Roland R-8M (drums and percussion), Yamaha RX5 (drums, percussion, and special sounds), and Yamaha TX802 (bass), and a Steinway grand for the piano part played by the author.

The drum, percussion, and bass parts of the composition were written on *presto*® in four months and then completed by the piano part in an additional two months.

The music critics did not recognize on the CD (where no trace of the computerized music was given) that the whole composition's percussive and bass parts were synthetic. This is what Gérard Assayag rightly calls a *Turing test for the viability of computer-aided composition*, in particular since in the field of jazz this technology is thought to be an impossible tool.

The original presto-files (extender `.sto`) as well as the audio-files of "SYNTHESIS" are available on the book's CD-ROM, see page xxx.

## 50.1 The Overall Organization

**Summary.** We discuss the overall organization: its material, composition principles in the four movements, and the instrumentation strategy.

– Σ –

The overall organization is a four-fold view on a generic material: the 26 classes of three-element motives in $OnPiMod_{12,12}$ (see appendix M.3 for the class list). The multiplicity of views is given by classical forms:

- the sonata form,
- the cycle of variations,
- the scherzo, and
- a finale.

### 50.1.1 The Material: 26 Classes of Three-Element Motives

**Summary.** The entire composition is based on the 26 isomorphism classes of three-element motives in $OnPiMod_{12,12}$ (see section 11.3.8) and its specialization tree. We make precise the different usage modes which have been realized in "Synthesis".

– Σ –

After the inspiring analysis of the Schubert–Stolberg work in 11.6.2, the usage of the 26 classes was recommended. The classes were used in different contexts. We refer to section 11.3.8 for the theory and to appendix M.3 for the 26 classes, in particular figures M.1, M.2 for representatives and the specialization Hasse diagram, see section 12.2.2 for specialization.

To begin with, representatives of all classes were patched together to build the germinal melody already described in section 11.6.3, see especially figure 11.16. This melody appears explicitly in the introduction to movement four, see figure 50.1. The 26 motives also appear as percussive "phonemes" in the third movement, as already discussed in section 11.6.3. In the second movement, the germinal melody is altered according to ornamental deformation techniques and yields a sequence of characteristic melodies which are played by the bass and harmonically ornamented by the piano. Already the very first percussive motif in movement one (just after the piano solo intro) *is* the germinal melody, however played with pitches being encoded by percussive sounds. And in movement four, these motives are used to define ornaments and fractal refinements thereof. More schematically, this strategy is shown in figure 50.2.

### 50.1.2 Principles of the Four Movements and Instrumentation

**Summary.** The four movements include a sonata form, a variational sequence, a scherzo and a finale. The instrumentation issue is intimately related to the structural aspects. Each movement bears its specific constraints on the role of instrumentation.

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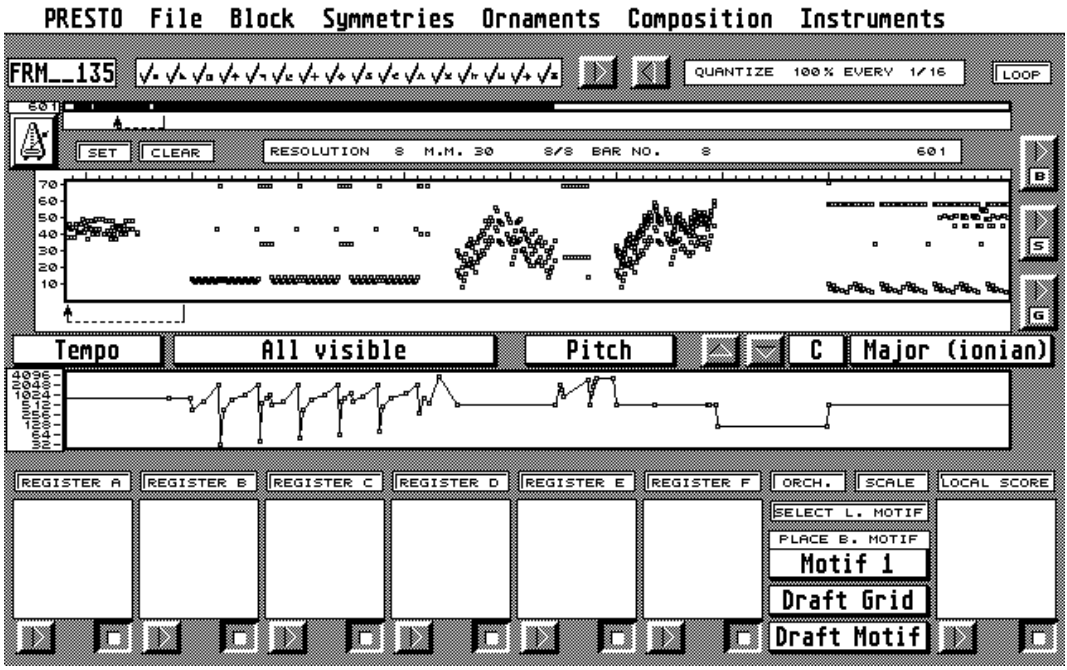



Figure 50.1: The beginning percussion rolls in movement four of "Synthesis", as extracted from *presto*®'s main window, together with the repeatedly accelerating tempo curve. The germinal melody comes twice just before the piano intro solo.

The principle of a four movement concert such as are encountered in the classical concert form is realized as follows, each movement with a total of 122 instruments:

**First movement: Earthquake/Full Force, duration: 10:46.** It is devoted to the Greek element of earth. It is a sonata form which exposes the ideas, followed by modulatory development, a reprise and ending by a coda.

Instrumentation: Yamaha RX5, TX802, Roland R-8M (PCM cards: Jazz, Jazzbrush, Ethnic Percussion).

**Second movement: Liquid Colours/Sea of Faces, duration: 14:05.** It is devoted to the Greek element of water. It is a cycle of variations. These variations are taken from the germinal melody and are concatenated in a paradigmatic way.

Instrumentation: Yamaha RX5, TX802, Roland R-8M (PCM cards: Jazzbrush, Ethnic Percussion, Mallets).

**Third movement: Poem of Wind/Fly! Fly! Fly!, duration: 09:27.** It is devoted to the Greek element of wind. It is a scherzo. The motivic alphabet of the 26 classes is produced in poetical arrangement of rhythmic patterns.

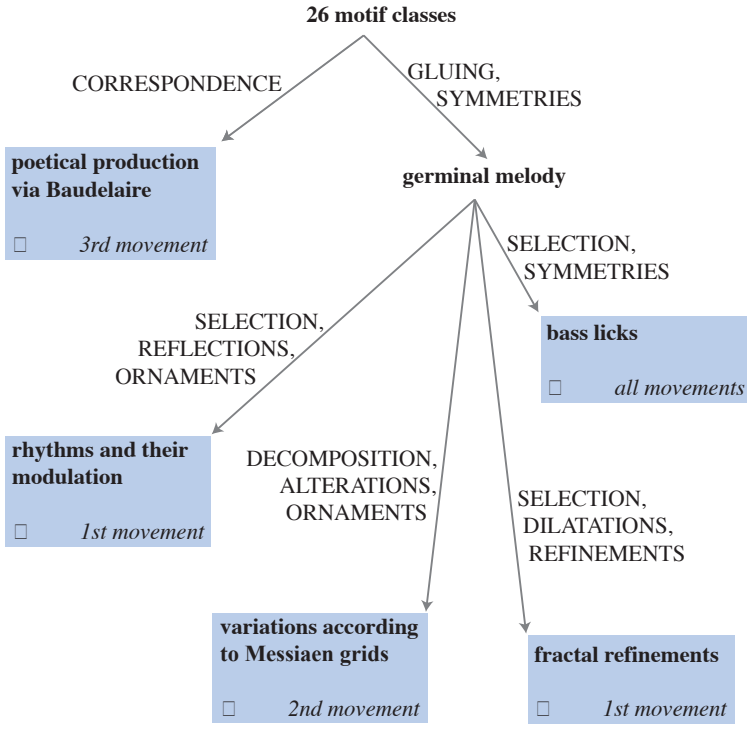



Figure 50.2: The overall strategy in "Synthesis" is driven by exploiting the 26 classes of three-element motives.

Instrumentation: Yamaha RX5, TX802, Roland R-8M (PCM cards: Jazz, Contemporary Percussion, Mallets).

**Fourth movement: Burning Spears/Interstellar Space, duration: 10:28.** It is devoted to the Greek element of fire. It is a finale. Self-similar repetitions and refinements of the germinal structure successively densify the musical material.

Instrumentation: Yamaha RX5, TX802, Roland R-8M (PCM cards: Jazz, Jazzbrush, Contemporary Percussion)

## 50.2 $1^{st}$ Movement: Sonata Form

**Summary.** The first movement "Earthquake" is a sonata form which uses the modulation theorem (see section 27.1.4) in time dimension.

– Σ –

This movement is a sonata form for rhythmical structures. After the piano intro, the rhythmical germinal melody is played on *toms* (pitches are associated with different toms) from

the RX5 synthesizer. During the exposition, the germinal subject is enriched by drum and percussion, multiplied and profiled by bass lines. The flight of the piano over the rhythmical carpet indicates an intense cadence of the exposition, ending on a fermata.

The development consists of three rhythmical modulations (see sections 28.3.1 and 28.3.2). This delicate process includes four different rhythmical macro-scales, which each consists of seven rhythmical three-tone motives from the 26 classes (see also figure 28.13). After the fermata, the first scale is built and cadenced by regular "falling drops" from the *ohkawa* instrument. Subsequently, its character is neutralized and transformed into a second scale. You hear the cadence of this new scale as being again marked by the ohkawa's regular "falling drops", this time enriched by a rain of light piano pearls.

After a further fermata, the second scale is altered into a third one, whose cadence coincides with an intermediate climax of the piano. When the piano finally recedes, the third scale is neutralized until a march-like turning point introduces the fourth scale, whose cadence terminates the development by another fermata.

The reprise follows in a slightly altered instrumentation. The finale starts after a short fermata. It is recognized on a heavy rock rhythm which is played in reduced tempo on the *kicks*.

## 50.3 $2^{nd}$ Movement: Variations

**Summary.** The second movement "Sea of Faces" follows a syntax of melodic variations of a fundamental melody which entails a particular "harmolodic" color in the vein of Ornette Coleman's music/theory.

– Σ –

This adagio is a labyrinthic wandering of *piano, bass, barafons, glockenspiel, sanzas, gender, tube bells* etc. through seven melodic variations of the germinal melody. Drums and percussion also obey these deformation forces, each harmonic, rhythmical, and melodic change is a specific expression of these forces.

Scales and rhythms are known to be periodic structures in pitch and onset. By the choice of two scales from the first three Messiaen scales, one for pitch, one for onset, we generate an ornament, a "harmonical-rhythmical" scale.

After certain rotations of such ornaments, the harmonic and rhythmical Messiaen components in onset and pitch are completely mixed. Here, the deformation forces of our germinal theme become manifest: Each tone event is displaced according to the ornament alteration algorithm described in 49.3.

The deformation uses the following grids (see figure 50.3): We take the Messiaen scales $M_1, M_2, M_3$ as described in example 13 of section 8.1.1. For each grid, we take a pair $(M_i, M_j)$, $i, j = 1, 2, 3$ and build the cartesian product $M_i \times M_j$ of two such scales, one in pitch, the other in onset direction, and each of them with a large range such that it acts on the germinal melody as if it were infinitely extended. The action is taken to be 100% without any directional constraints. This yields nine variations of the germinal melody, as shown in figure 50.4.

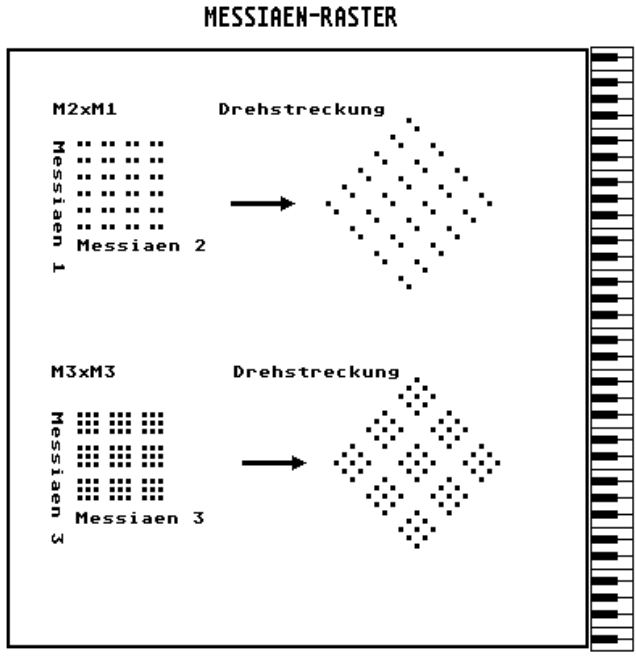
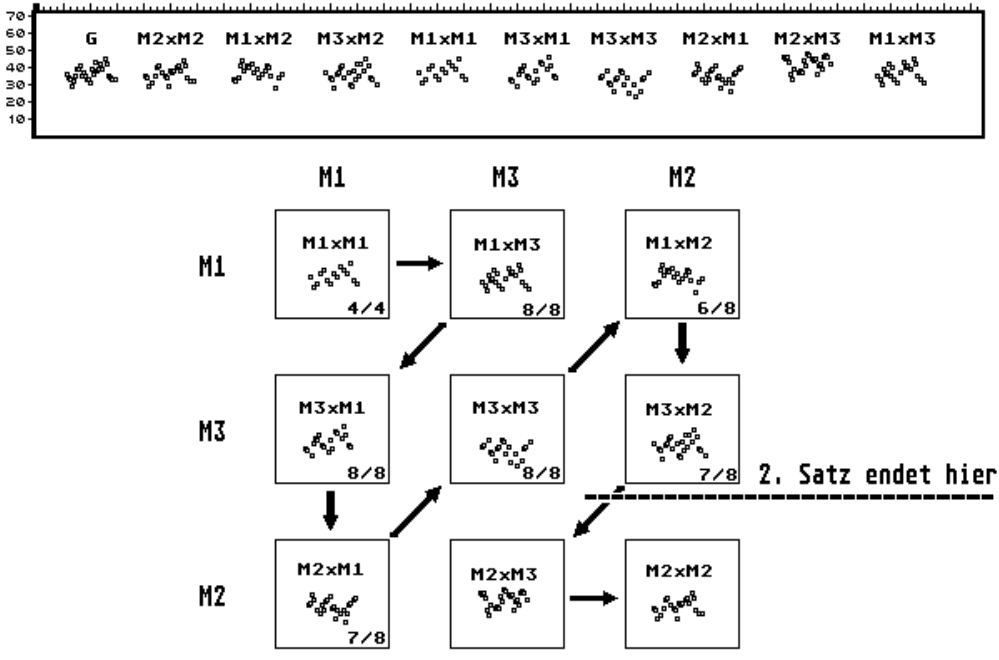



Figure 50.3: The original graphics for the grid construction. To the left two Messiaen grids, to the right their transformation via a rotation-dilatation which is used for rhythmical connections. The Messiaen grids are cartesian products of the Messiaen scales and rhythms.



In the second movement, seven of the nine variations are used in the order shown in figure 50.4, lower part. The bass plays these variations in a very extended slow gesture, and the piano ornaments these melodic lines by a harmonization work.

Each couple of such variations is connected by a rhythmically complex intermediate structure, see figure 50.5. The intermediate parts terminating in the $M_i \times M_j$-variation are constructed as follows: We take the Messiaen grid $M_i \times M_j$ and operate a dilatation-rotation by 45° on this object, see figure 50.3, right part, for two such transformations. From these transformed local compositions (or from a very similar skew transformation) we build an interpretation by a partition into five subcompositions which are again transformed as visualized in figure 50.6. These charts are then taken as motives of different ornaments which are generated by the grids as shown in figure 50.7. The upper part of figure 50.7 shows the superposition of the five ornaments generated from Messiaen grid $M_3 \times M_3$. This rhythmical construction is also seen in the right and lower part of figure 50.5.

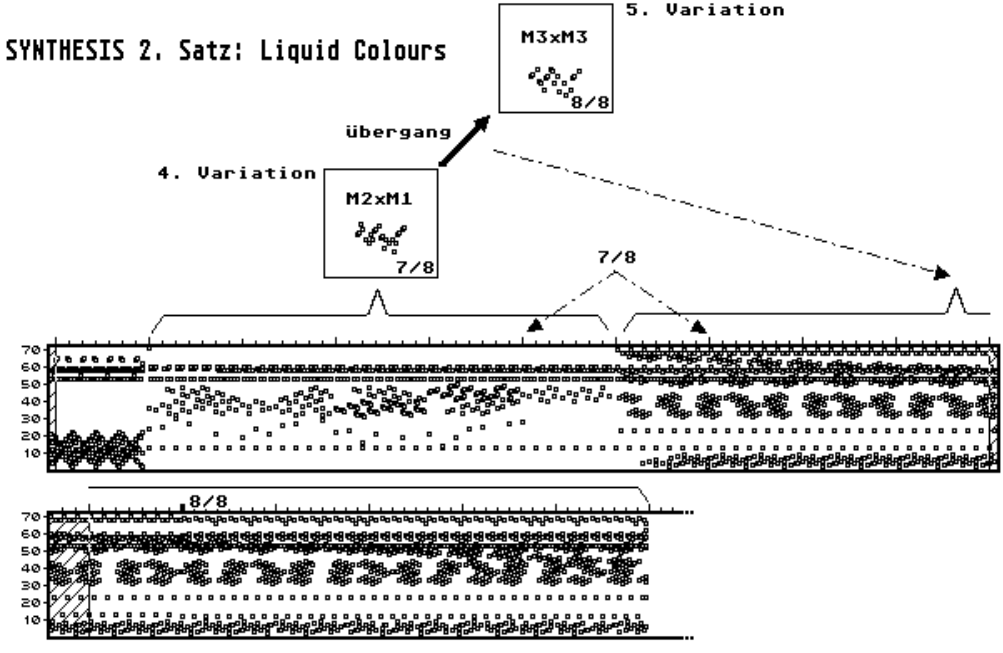



Figure 50.5: The original graphics for the intermediate structure between two successive variations, here connecting the $M_2 \times M_1$-variation to the $M_3 \times M_3$-variation. The connecting structure is a rhythmical construction that is deduced from the generating Messiaen grid $M_3 \times M_3$.

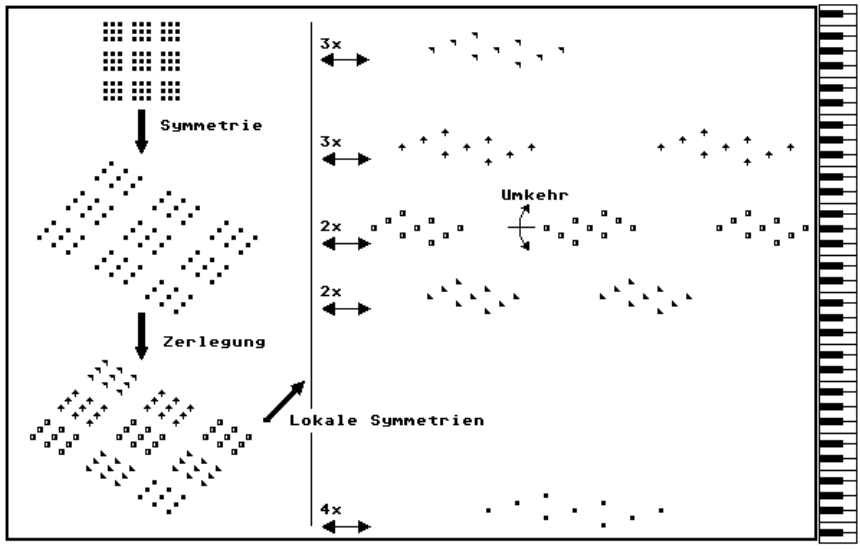



Figure 50.6: The original graphics for the interpretation of the rotated Messiaen grid and the symmetries applied to the partition's components.

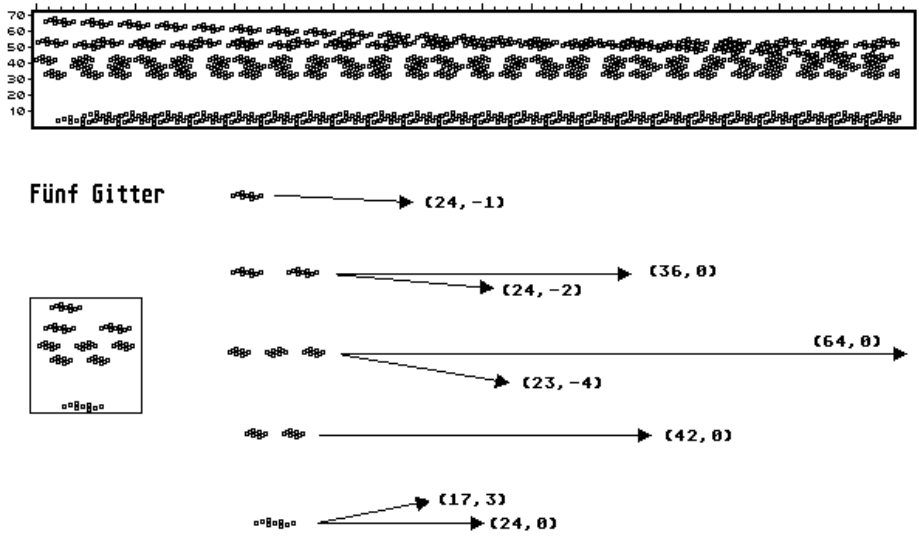



Figure 50.7: The original graphics for the ornaments that are generated by charts of a rotated Messiaen grid, following different grid translation vectors.

## 50.4 $3^{rd}$ Movement: Scherzo

**Summary.** The scherzo "Poem of Wind" is based on a transcription of Charles Baudelaire's last poem "La mort des artistes" in "Les fleurs du mal".

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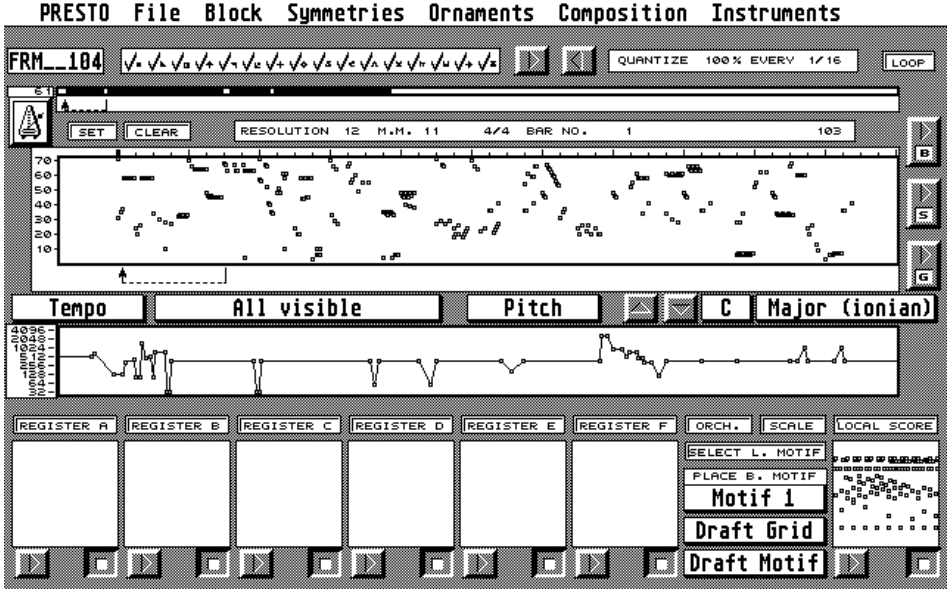


Figure 50.8: The beginning of the third movement with its breakneck changes in agogics.

This movement is in complete contrast to the quiet flowing of that adagio. It resembles Ernst Jandl's concrete poesy. A poem of wind: without firm ground or romantic sky. Here, the piano builds an expressive dialog with a pointillistic percussion.

The first two strophes of Charles Baudelaire's "La mort des artistes"is the last poem in his famous "Les fleurs du mal". The transformation of this poem was already discussed in section 11.6.3. This procedure transforms every word into a sequence of rhythmical motives, we hear it as a coherent sound sequence which is separated from its successor by a short fermata. At the end of each verse, longer interruptions are inserted.

From this raw material the final composition is constructed by an extremely refined agogical architecture, changing from breakneck accelerandi to stagnating ritardandi. Here, the third movement makes extensive use of the *presto*®-AgoLogic for tempo curves. The single motives are also enriched by echo-like variations and repetitions, such that their color engraves itself on ones mind.

The answer of the piano to this witty poesy is the albatross in the air, which flies, falls and plays with Cecil Taylor's florescence as if it were an old tale. Just one not too many salty swift—and goodbye.

## 50.5 $4^{th}$ Movement: Fractal Syntax

**Summary.** The fourth movement "Burning Spears" is constructed in a completely non-syntactic way from fractal principles.

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This finale is a steadily evolving rhythmical organism whose form could be compared to Alexander Scriabin's ecstatic sonata "Vers la flamme".

The movement gets off by a drum intro, followed by a bass intro which announces fanfare-like that something is to happen. It receives a promising answer from the low piano keys until a samba-like rhythm makes the final run.

This starting rhythm is again derived from the germinal melody. You can hear its essential elements from the drum intro. After two roll groups, the samba reappears, but now with augmented time values, whereas the now opened gaps are filled up by new micro rhythms (see figure 50.9 where the original motives are seen in the low and high regions, and figure 50.10, where the stretched motives are now also visible in these regions). This process of intensification is cadenced until a cutting bass line terminates the developmental part. The piano dances like an entranced sorcerer over this rhythmical lava.

In the reprise, the samba figure reappears, but with a higher tempo. It ends in a strong pulse which leads to a further time augmentation after a bass cadence. This time, the black dancing extension figure is refined by a cackling entwinment by *wood blocks* which is joined by a fast dialog with the piano.

The following earthy rhythm is introduced by a bass drive as if we should be warned that we are now changing the civilizations. Here too, the construction is an augmentation of rhythm and an intensification. What we are hearing is in fact a fractal repetition of self-similar time structures.

The last and extremely fast part expresses the force of a burning spear, which is thrown into the sound sky by a delirating dervish dancing on an exploding volcano... So the musical principle of this finale is not architectural, but a self-renewing process.

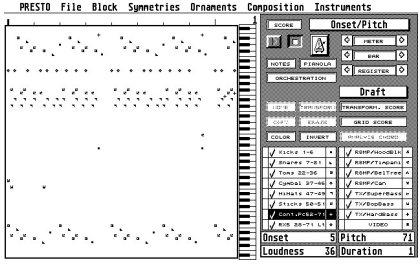


Figure 50.9: The original motives are seen in the low and high regions of the local score.

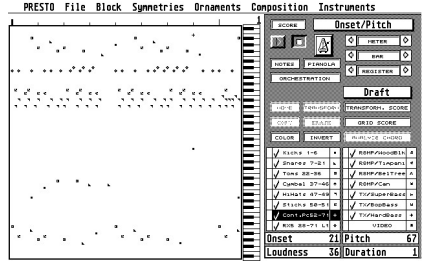


Figure 50.10: The stretched motives are also visible in the low and high regions.

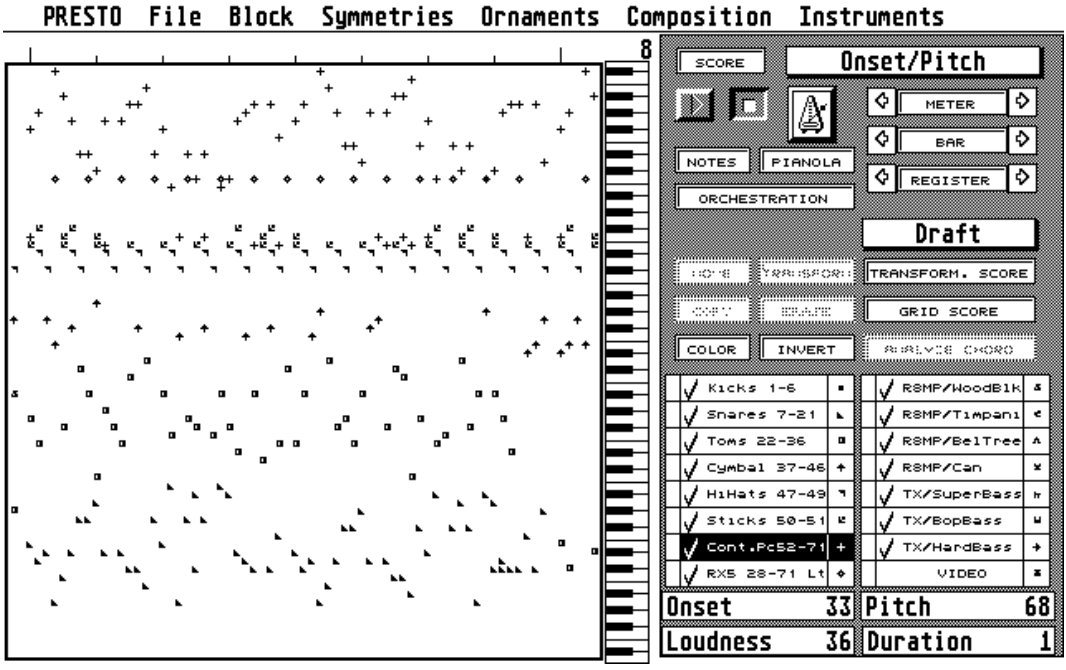


Figure 50.11: The end of the fourth movement shows a particularly dense and polyrhythmic percussive part.

# Chapter 51

# Object-Oriented Programming in OpenMusic

*Le problème que l'ordinateur pose au compositeur n'est pas d'abord d'ordre spécifiquement musical, mais avant tout culturel et philosophique. Il implique une refonte complète des rapports de l'abstrait et du concrete. Le musicien, le musicologue, l'auditeur doivent bien se rendre à l'evidence de ces changements sans aller pour autant jusqu'à la maladie de l'adaptation.*

Hugues Dufourt [130]

**Summary.** OpenMusic is a visual programming language for music composers. It was designed and implemented by the Musical Representation Team at Ircam-Centre Georges Pompidou. OpenMusic is based and implemented on CLOS (Common Lisp Object System) [510]. It shows several original features, such as reflexivity, meta-programming capacities, handling of the duality between musical and computational time, and provides a framework of predefined musical objects for handling sound, MIDI and musical notation. OpenMusic combines different technics of programming, e.g., functional programming, constraint programming and object-oriented programming. We will focus in this chapter on the last one. Object-oriented programming is crucially connected with the categorical approach. Category theory helps formalize in an original way concepts like inheritance, methods, classes, etc. More details on this relation can be found in [306] or in section 9.4.2 of this book.

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Although one may consider music composition to be an important issue in any computer music research or development, the term Computer Assisted Composition (CAC) has taken a specialized meaning during the past years. As opposed to the generation and processing of audio signal, by means of DSP hardware or software technologies, CAC systems such as OpenMusic focus on the formal structure of music. The software technology is rather based on symbolic computation, where the typical data structures (trees, graphs, sets, collections, associative memory, etc.) and algorithms (often issued from discrete mathematics) are suited to handle the complex structures involved in a compositional process. The great diversity of

esthetic, technical, formal (or anti-formal) models, implies that one cannot conceive an environment of CAC as a fixed application which provides a fixed collection of generative procedures and musical transformations. On the contrary, we conceive an environment as a programming language, helping each composer to constitute his personal universe. Of course, there is no sense in providing a traditional programming language, the control of which requires a great technical expertise. For this reason, our purpose is to build a programming language conceived especially for composers. This leads us to think about the various existing models of programming, intuitive graphical interfaces (which enable control of this programming) and the internal as well as external representations of musical structures that will be built and transformed when using this programming. Therefore the main goal is to implement a language containing the concept of notation of the result (a musical score) as well as the concept of notation of the process leading to this result (visual program).

## 51.1 Object-Oriented Language

**Summary.** In this section we describe the main entities of our object language. Such language entities are usually called *meta-objects*. We describe each meta-object in a graphical and formal way. Basic calculus for object-oriented programming inherits the approach imposed by the precursory language Simula and its successors. The attempts to formalize this family of object models used the concept of parametric polymorphism [80]. More recently, languages based on multiple-dispatching (methods that dispatch on a product of types rather than a single type) such as CLOS could be formalized using concepts of overloading or ad-hoc polymorphism, found in *λ&-calculus* [81]. OpenMusic, which is based on CLOS, may be formally described in this way. Although we do not give here a real formalization of OpenMusic, we use the *λ&-calculus* to give a general idea of each meta-object. From a visual point of view, the meta-objects of our calculus are represented as graphical entities called frames.

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Object-oriented programming is based on simple concepts. A program can be seen as a set of entities called *objects*. An object is made of data (*slots*) and operations applied on it (*methods*). Objects communicate in a specific way, usually called *message passing*. Most of the object-oriented languages implement the notion of *class* in order to abstract similar objects. New classes can be created from existing classes using the mechanism of *inheritance*. Inheritance allows the extension or partial modification of a class. If a class A inherits from a class B, A is called the *subclass* of B and B is called the *superclass* of A. Object-oriented programming offers in a natural way a dynamic management of resources, which means that we can create new objects at any time. The mechanism of object creation from a class is called *instantiation*. The new created object is called an *instance* of the class.

Meta-objects are represented either as composed frames or simple frames. Several frames (i.e., different point of views) may be produced for the same object. The simple frames which represent an object are called *views*. They generally appear as icons. The composed frames representing an object are called *editors*. For more details about formal and graphical description of OpenMusic see [11].

### 51.1.1 Patches

A *patch* is a meta-object specific to OpenMusic. It reifies the notion of program. A patch is the place where objects will be interconnected in order to specify musical algorithms. From a formal point of view, a patch can be seen as a $\lambda$-*function*. Patches are composed by boxes (icons) and connections between them. Boxes represent functional calls while connections represent functional composition. Figure 51.1 shows the *view* and the *editor* of a patch implementing an algorithm for the expression $x^2 + 2$. It can be formalized as the lambda function $\lambda x(x \cdot x + 2)$.

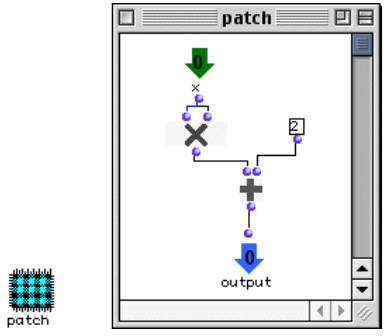


Figure 51.1: View and editor of a patch.

### 51.1.2 Objects

In the $\lambda\&$-*calculus*, objects are represented as registers. A register can be seen as a set of labeled fields $l = v$ where $l$ is called the label and $v$ is called the value. For instance, an object representing the note C3 can be written as: $note = \langle pitch = C, octave = 3\rangle$. Figure 51.2 shows the *view* and the *editor* of this note object.

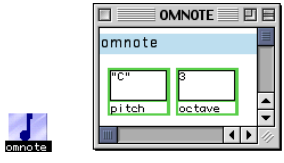


Figure 51.2: View and editor of an object.

The next two rules define the field selection and the field writing, respectively:

$$\langle l_1 = v_1, \ldots, l_n = v_n\rangle \circ l_i \mapsto v_i,$$
$$\langle l_1 = v_1, \ldots, l_i = v_i, \ldots, l_n = v_n\rangle[l_i \mid v] \mapsto \langle l_1 = v_1, \ldots, l_i = v, \ldots, l_n = v_n\rangle.$$

### 51.1.3 Classes

If objects are seen as registers, classes will then be seen as generators of registers. The editor for a class is an ordered collection of *views* representing *slots*. Slots contain information about their name, their type (a class icon), a default value and a flag that indicates if the slot is public or private. *View* and *editor* for the class of the note object defined in the previous section are shown in figure 51.3.

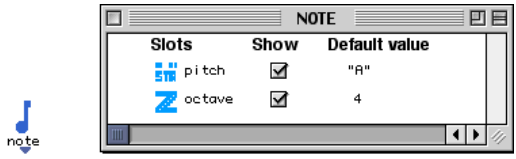


Figure 51.3: View and editor for a class.

### 51.1.4 Methods

Methods are simple functions ($\lambda$-*abstractions*) where arguments are typed by classes. The editor in figure 51.4 shows a method with two inputs *self* and *num* of type *Note* and *Integer* respectively.

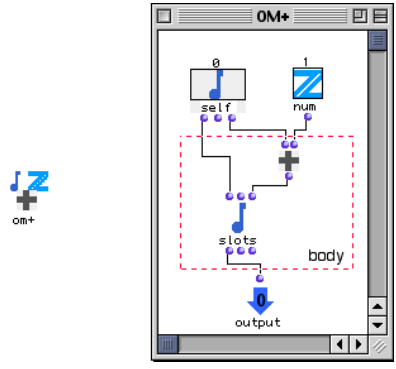


Figure 51.4: View and editor for a method.

The body of this method increments the value of the slot *octave* of *self* by *num* and returns the note object. This method is formalized by the expression:

$$\lambda self{:}note\ \lambda num{:}integer\ self[octave \mid self \circ octave + num].$$

### 51.1.5 Generic Functions

A generic function is a collection of methods $(\epsilon \& M_1 \& ... \& M_m)$ where $\epsilon$ is the empty method. The type of a generic function containing methods $M_i$ of type $U_i \rightarrow V_i$, $1 \leq i \leq m$, is $\{U_1 \rightarrow V_1, \ldots, U_m \rightarrow V_m\}$. However, not any set of methods can be seen as a generic function. A set of methods with types $U_i \rightarrow V_i$ is a generic function iff the two following conditions are satisfied for all i,j:

$$U_i \leq U_j \Rightarrow V_i \leq V_j,$$
$$U \textit{ is maximal in } LB(U_i, U_j) \Rightarrow \exists U_k, U_k = U$$

where $LB(U, V)$ indicates the set of common lower bounds[1] $V$ of types $U$. Figure 51.5 shows a generic function composed of two methods: the first one is the method described in the previous section; the second one is specialized for inputs of type *Number*.

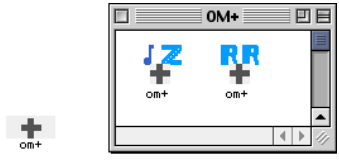


Figure 51.5: View and editor for a generic function.

### 51.1.6 Message Passing

Message passing is achieved by generic function invocation. We distinguish the application of a simple function from the application of a generic function, which will be indicated by the operator $\bullet$. The application of a generic function $G$ to arguments $N$ of type $U$ consists of two steps: selection of a method $M_j$ among the methods of $G$ and normal application of $M_j$ to $N$ .

$$(\epsilon \& M_1 \& ... \& M_m) \bullet N \mapsto M_j N.$$

Note that $U$ may not be contained in the set $U_i$ of input types of the generic function. In this case we select the method $M_j$ satisfying : $U_j = min_{i=1...m}\{U_i \mid U \leq U_i\}$. The *CPL* of a class is a linearization of its superclasses. The details of the linearization are not crucial for our purposes, see [510].

### 51.1.7 Inheritance

The inheritance mechanism is defined by the subtyping and the mechanism of method selection. Subtyping is defined by the following rules:

[1] The ordering $\leq$ is defined by the Class Precedence List (*CPL*) of the class $U$.

$$\frac{U_2 \leq U_1 \qquad V_1 \leq V_2}{U_1 \to V_1 \leq U_2 \to V_2} \qquad \frac{\forall i \in I,\ \exists j \in J,\ S_j \to T_j \leq U_i \to V_i}{\{S_j \to T_j\}_{j \in J} \leq \{U_i \to V_i\}_{i \in I}}$$

The rule on the left is the usual contravariant-covariant rule for arrow types. The other one states that an overloaded type is smaller than or equal to another overloaded type if for every branch in the latter, there is a branch in the former smaller than or equal to this one [81].

Graphically, a class inherits from another one when there exists an arrow from the superclass to the subclass, as is shown in figure 51.6. The class *rnote* extends the class *note* by adding a new slot called *rhythm* of type *string* with default value *quarter*.

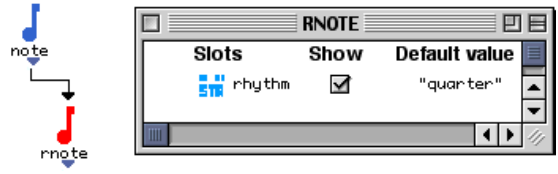


Figure 51.6: Graphic inheritance.

### 51.1.8 Boxes and Evaluation

Boxes are placed in patches and allow other meta-object to be involved in the calculus. Boxes are composed of an icon and an ordered set of *inlets* and *outlets*. There are different types of boxes depending on the referenced meta-object. The user may create boxes in a patch by dragging meta-objects into it. Figure 51.7 shows different types of boxes and their references.

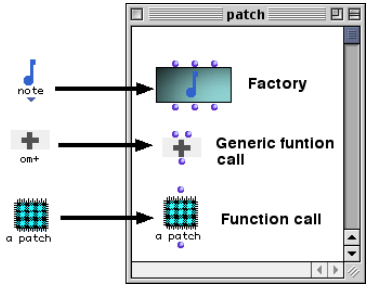


Figure 51.7: Boxes in a patch.

Inlets in functional call and generic function call boxes correspond to the function arguments. For factory boxes, inlets and outlets refer to the slots of the class. Boxes and connections in a patch can be seen as a graph of functional compositions. By clicking on the output of any box, an evaluation from the corresponding point into the graph is induced. A box evaluation can give rise to other box evaluations creating a chain corresponding to the execution of a program.

### 51.1.9 Instantiation

Users can create instances of a class with the aid of *factory boxes* (boxes built from a class). A factory contains a number of inputs corresponding to the public slots of the class. There are as many outputs as inputs. When evaluating an output, a new instance is created. The values returned by outlets are, from left to right, the instance itself, then the current value of its public slots. An instance can be visualized graphically as a box which can eventually be connected to other boxes (see figure 51.8).

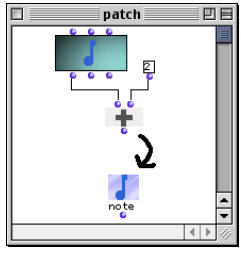


Figure 51.8: Graphical instantiation.

## 51.2 Musical Object Framework

**Summary.** A framework is a set of reusable classes that one can use as building blocks for a specific software [179]. This section describes the musical OpenMusic framework.

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OpenMusic offers a set of predefined classes and generic functions for musical representation and manipulation. This framework can be extended by using inheritance or by defining new methods in generic functions or by writing new generic functions. There are graphical editors for these definitions. In this section we will focus on the structure of musical objects (classes) rather than on their behavior (generic functions). The object-oriented concept of *encapsulation* can be defined as the separation between the internal representation and the interface of an object. These two aspects will be described in the next two sections.

### 51.2.1 Internal Representation

In OpenMusic any musical structure is a *container* which embeds other musical structures. All containers have a temporal extension ($e$) and an offset ($o$). $e$ and $o$ are expressed as a multiple of a rational unit $v = 1/u, u \in \mathbb{N}_+$, the set of positive integers, $v$ is considered a fraction of the quarter-note. The value $u$ can be redefined at each level of the embedded structure. From the temporal point of view, we have entities coming from different rational time scales. In order to express them in a homogeneous way, we have implemented a hierarchical unit system, which will be discussed briefly by showing an example. Figure 51.9 shows a music fragment and an integer hierarchy describing it.

**Exercise 82** Define the corresponding form whose denotators are these rhythmical hierarchies.

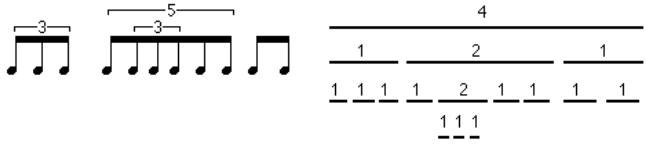


Figure 51.9: Hierarchical representation of a rhythm.

We can calculate the extension of each element $e_i$ by the recursive formula $d(e_i) = \frac{e_i}{S} d(F(e_i))$, where $S$ is the sum of $e_i$'s brothers including itself, and $F(e_i)$ is $e_i$'s father. The deepest triplet in our example has an extension given by: $\frac{1}{3}\frac{2}{5}\frac{2}{4}\frac{4}{1} = \frac{4}{15}$. For this reason, we set $u$ to 15 (the quarter note is equivalent to 15 units). In this scale the new extension of the triplet is 4. Following the idea of a variable scale adapted to each embedded level in a musical structure the rhythm of figure 51.9 can be represented as shown in figure 51.10.

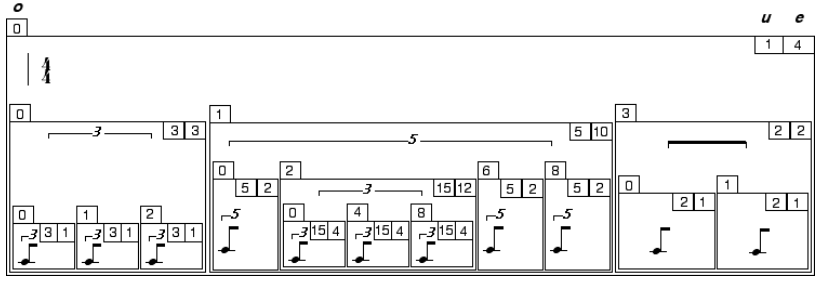


Figure 51.10: Temporal organization of musical instances.

Some basic operations for container manipulation are described below:

- $NewContainer$. Create an empty container ($u$ equal 1 by default).
- $AddTo(c1, c2, at)$. Set the container $c2$ into the container $c1$ at the position $at$ and calculate new values for $u$, $e$ and $o$ attributes.
- $RemoveFrom(c1, c2)$. Remove the container $c2$ from the container $c1$.
- $QReduce(c, [n])$. Computes $u' = ppcm(u_i)$, where $u_i$ are units for all subcontainers of $c$. The attributes $u$ of the container $c$ is set to $u'$. New values for parameters $u_i$, $o_i$ and $e_i$ are calculated too. This operation allows comparison between all parts of a container. If a value $n$ is given, then $u' = ppcm(u_i, n)$.
- $QNormalize(c)$. Sets $u$ value for $c$ and each of its subcontainers to optimal, i.e., the smallest possible integer number, considering the offset and extent values that have to be expressed as integer multiples of $1/u$.

The *QNormalize* operation sets all extensions to integer values. If we want to compare all values we need to choose for the whole measure one unit $u$ equal to $ppcm(2, 3, 15) = 30$. In this case the new values for $u$ are 30 for the last eighth note and 30 for the triplet, see figure 51.11. The choice of using only integers for our representation allows us an easier translation to music notation and a coexistence of objects described in hierarchical time and in continuous time. Moreover, in this way we avoid reversibility problems coming from the translation between reals and integers (continuous and discrete time).

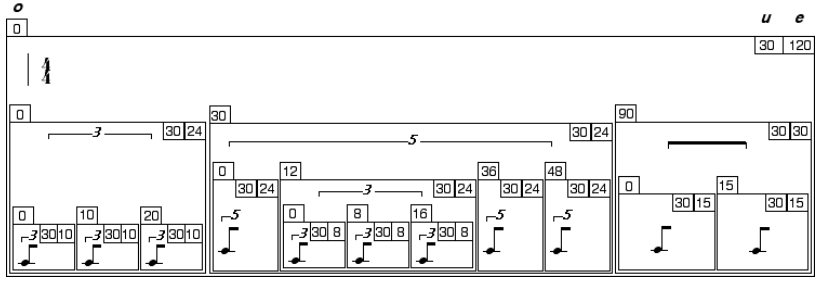


Figure 51.11: A normalized container.

This model of containers and their operations represents in a natural way traditional musical structures like polyphony, measures, chords, etc. We will see in the next section how these containers are passed to the composer in an object context.

### 51.2.2 Interface

A summary of the predefined musical classes available in OM is given in figure 51.12.

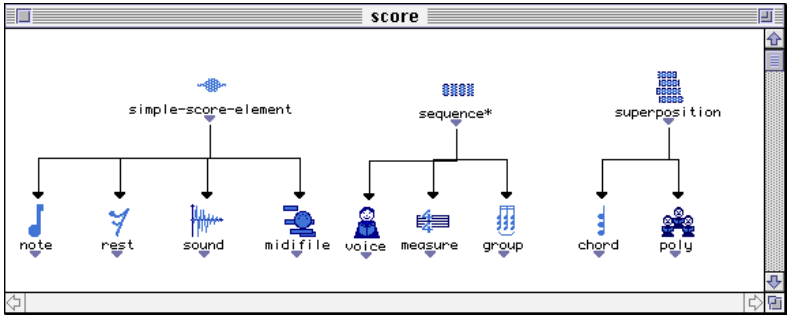


Figure 51.12: Musical framework.

There are three main classes of musical objects:

- *superposition*, chords and polyphonies are made of other objects placed in parallel.

- *sequence*, objects like voices, measures, etc are composed by other objects one after the other.
- *simple-score-element*, these objects are terminals (empty containers).

There is a predefined set of generic functions with methods for all musical classes. They are used to apply transformations to musical instances (i.e., transposition, inversion, etc.) or simply to play or visualize them. Editors of musical instances have been replaced by musical notation editors (figure 51.2 is replaced by figure 51.13). We use the paradigm *Model-View-Controller*, where the model is the musical instance, the view is its representation on the screen as a score and the controller is a user interface allowing us to change slot values.

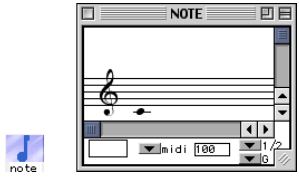


Figure 51.13: View and editor of a note instance.

As we have seen in the previous section, musical entities can be represented as containers. However it is not suitable for composers to have to build musical objects from containers. In order to create a musical instance we must provide to the composer a symbolic abstraction of the *real* musical object. For instance, *MIDIcents* are a good way to code pitch values. On the contrary, problems arise when the rhythmic content is taken into account. We propose in the next section a representation of rhythmic information adapted to the composer.

#### 51.2.2.1 Rhythmic Trees

Rhythmic Trees (*RT*s) are the base of the rhythmic representation in OpenMusic. They must be understood as an alternative description of symbolic rhythmic structures using traditional music notation, and an external specification for container objects.

- *Syntax* An *RT* is defined as a pair $(D\ S)$ where $D$ is an integer ratio $(> 0)$ and $S$ is a list of $n$ elements. Each element in $S$ can be either an integer or a *RT*. Here is an example corresponding to this syntax: (2 ((1 (1 1 1 1)) (1 (1 1 1 1))).

  **Remark 23** Solving the above exercise 82, we have here a nice example of a doubly circular form. To begin with, given a form $F$, a list form over $F$ is the circular form

  $$List(F) \underset{Id}{\longrightarrow} \mathbf{Colimit}(Item(F), Terminal)$$

  with the terminal form $Terminal \underset{Id}{\longrightarrow} \mathbf{Simple}(\mathbb{Z})$ in order to formalize the end of a finite list and to indicate its length with the integer of the coordinator module $\mathbb{Z}$. The form $Item(F)$ is a Cartesian product

  $$Item(F) \underset{Id}{\longrightarrow} \mathbf{Limit}(F, List(F)).$$

Building on this form, we define the form $RT$ by

$$RT \underset{Id}{\longrightarrow} \textbf{Limit}(DurationValue, List(RT))$$

and a simple duration form

$$DurationValue \underset{Id}{\longrightarrow} \textbf{Simple}(\mathbb{Q}).$$

It is useful to visualize this doubly circular form in its construction graph as drawn in figure 6.5.

- *Semantics* For a given $RT = (D, S)$, $D$ expresses a duration and $S$ defines a group of proportions of $D$. For instance, by taking as unity the quarter note we have for $RT = (1, (1, 1, 1, 1))$ the rhythm shown in figure 51.14.
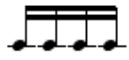
Figure 51.14: Rhythm for $RT = (1, (1, 1, 1, 1))$.

$RT$s allow us to represent, in a homogenous way, different types of musical objects. Polyphonies, voices, measures groups, etc. are expressed as $RT$s. When the value $D$ is at the measure level, we express it in whole note units. For example, the $RT$ for the next rhythm will be $(3/4, (2, 1))$.

Figure 51.15: $RT = (3/4, (2, 1))$.

As it was defined, $S$ represents a sequence of proportions of $D$. The example in figure

Figure 51.16: $RT = (4/4, (1, 2, 1))$.

51.16 shows the case for $RT = (4/4, (1, 2, 1))$.

Until now, we have presented $RT$s where $S$ was a list of integer elements, representing notes or beats. For an $RT = (4/4, (1, (2, (1, 1, 1)), (1, (1, 1, 1))))$, more complex elements appear as shown in figure 51.17. The $RT$s contained in $S$ represent what we call *(rhythmical) groups*. In general, groups are graphically represented as beamed notes. As well as measures, groups can contain either notes or other groups. Here is an example for the $RT = (4/4, (1, (1, (1, 1, (1, (1, 1, 1)), 1, 1)), 2))$:

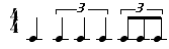

Figure 51.17: $RT = (4/4, (1, (2, (1, 1, 1)), (1, (1, 1, 1))))$.

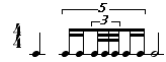

Figure 51.18: $RT = (4/4, (1, (1, (1, 1, (1, (1, 1, 1)), 1, 1))2))$.

We extend the $RT$ syntax in order to integrate rests and ties which will be represented respectively by negative numbers and floats[2]. For

$$RT = (4/4, ((1, (1, 1)), (2, (1.0, 1, -2, 1)), (1, (1.0, -1, 1))))$$

we obtain the following rhythm:

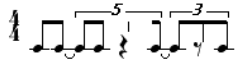

Figure 51.19: $RT = (4/4, ((1, (1, 1)), (2, (1.0, 1, -2, 1)), (1, (1.0, -1, 1))))$.

$RT$s may encode the metric intention in a rhythm. The two rhythms in the next figure are encoded by different $RT$s $RT1 = (2/4, (2, 2, 2, 2))$ and $RT2 = (2/4, ((1, (1, 1)), (1, (1, 1))))$.

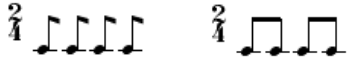

Figure 51.20: $RT$s $RT1 = (2/4, (2, 2, 2, 2))$ and $RT2 = (2/4, ((1, (1, 1)), (1, (1, 1))))$.

The symbolic representation is essentially based on the hierarchical musical content. This allows an overall readability of the hierarchical structure and a symbolic format, which can be controlled by algorithms and other transformations such as inversion, recursion, etc. This is made possible by the fact that the format is the abstraction of the object itself. $RT$s are automatically encoded into optimal container structures.

## 51.3 Maquettes: Objects in Time

**Summary.** The *maquette* is an OpenMusic *meta-object* aiming at representing, in a same object, patches (musical process) and containers (musical material). It is a new concept of score

[2]Perhaps, a better distinction of domains for ties would be to take imaginary rational numbers $i.x/y$ instead of floats, since rational numbers are not really different from floating point numbers. A more musical solution would be to take the form $TimeSig$ defined in formula (6.92) instead of $DurationValue$ for normal duration values (see also appendix A.2.1), to reserve $DurationValue$ for ties, and to replace $DurationValue$ by the colimit of $DurationValue$ and $TimeSig$ in the above definition.

where the static description of musical structures and the definition of dynamic computational processes seamlessly coexist. The user may go back and forth between these two metaphors by considering the maquette as a score (in traditional or graphical notation) or as a set of interconnected processes. As external objects like MIDIFiles or SoundFiles may also be imported, maquettes offer an original environment for music creation.

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The maquette is an original concept in OpenMusic which allows us to solve the problem of combining the design of high level hierarchical musical structures, the arrangement of musical material in time, and the specification of musical algorithms. Just like other meta-objects, maquettes may appear as a view or may be opened in a maquette editor, which is basically a 2-dimensional surface with time flowing along the x-axis. This surface contains several blocs that we call *temporal boxes*. In the maquettes the hierarchical imbrication of musical structures and their temporal order can be represented in an explicit visual way. In the musical sketch presented in figure 51.21, made by the composer Mikhail Malt, *temporal boxes* have been disposed on the maquette surface.

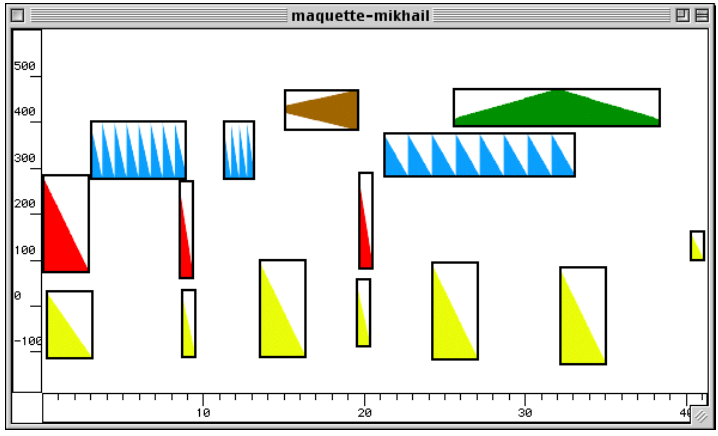


Figure 51.21: Temporal boxes in a maquette.

Horizontal temporal box positions correspond to onset values in 'absolute' (physical) time. Durations and intensities are given by their horizontal and vertical extensions. Pictures have been associated to the boxes in order to give an elementary musical semiotics. Thus, triangles correspond to chords whose resonance decrease quickly. Multiple triangles are associated with chords *ostinato*. Other figures are triangles whose intensity follows the geometrical contour of the picture.

In figure 51.22, connections between temporal boxes are shown. They represent a different kind of musical information. We can see that temporal boxes are inferred one from another by functional relations. For instance, the ostinatos are linked to the first chord. This level of information bears paradigmatic content, because analogies between any part of the structure can be derived.

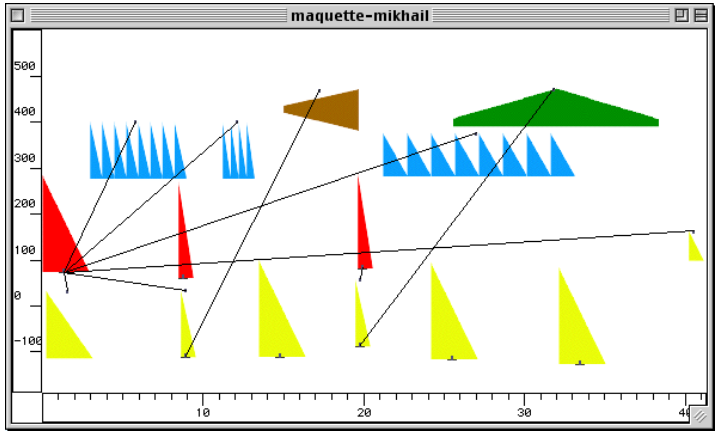


Figure 51.22: Functional relation of temporal boxes.

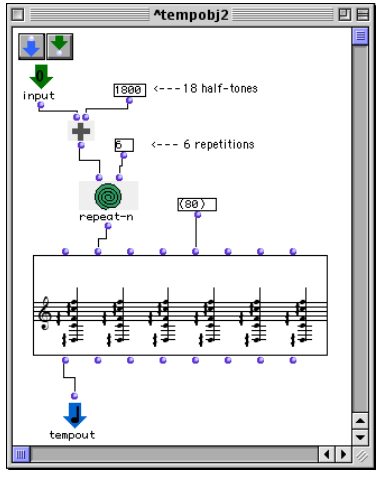


Figure 51.23: Patch calculating a temporal box.

If we open the editor of the third ostinato (figure 51.23), we can see that the chord coming from the first box (which is represented by the box called *input*) is transposed by an augmented-fourth (18 half-tons or 1800 MIDI-cents), then repeated six times and finally sent to a *factory* that builds an instance of the class chord-sequence. Elements of the musical material take place syntagmatically in the final result.

Let us take now the first chord that is provided as input for the ostinato. If we open its editor (figure 51.24), we can observe that the generating algorithm is reduced to a given data. This is the fourth level of information which concerns the basic music material representation.

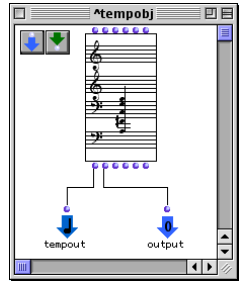


Figure 51.24: Patch with musical material for a temporal box.

Finally, we can toggle the visual representation of boxes by showing the traditional representation of the musical result, see figure 51.25.

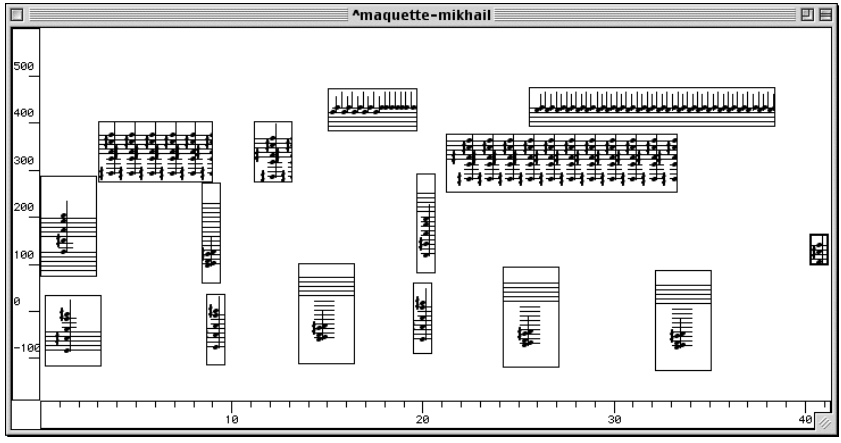


Figure 51.25: Musical notation of a maquette.

In the previous example, we tried to show how an OpenMusic object scheme enables us to structure the musical information at different levels:

- the static level of the form, allowing us to create visual semiotic markers;
- the dynamic and paradigmatic level of the form (i.e., its functional relations between the temporal boxes);
- the syntactical level, it is the calculus building the musical discourse inside the temporal boxes;
- the material level.

These four levels of information are obviously interconnected. The most important advantage in the maquette concept is to offer a visualization of this interaction and at the same time, interactive control of it. This produces a source of experimentation:

- **Recombination at the form level**: Temporal boxes are moved and stretched in time without changing the other three levels.
- **Modification of functional relations**: We do not change the position of the blocks but their causal relation.
- **Syntax modification**: Algorithms which build the material can be changed according to the compositional goals.
- **Change of the material**: The color of the piece may change while keeping all the formal organization.

When combining these procedures, sophisticated musical experiments can be carried out.

## 51.4 Meta-object Protocol

**Summary.** Meta-object protocols (MOP) provide an alternative framework that opens the language implementation to user's intervention. OpenMusic was implemented using meta-programming technics. We have extended CLOS meta-objects (methods, classes, generic functions, etc.) by adding visual counterparts. In the same way we extended CLOS as such by using the technique of meta-programming; the user can make extensions of the OpenMusic language thanks to the visual MOP. In this section we describe OpenMusic's graphical MOP, as well as musical applications.

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Originally the MOP was conceived for solving problems in the design and implementation of CLOS [262]. Usually the internal architecture of a programming language is not interesting for the programmer but only for language designers. For object-oriented languages, a MOP of the language is an interface to the language, presented as a framework. Classes, generic functions, patches, maquettes and other entities discussed in the previous sections are simply instances of special classes. These instances are called *meta-objects* and the classes *meta-object classes*. The set of these classes, as a framework, constitute the MOP. In addition, a language may support a MOP only if it has two characteristics: *reflection* and *reification*. The reflection is the ability of a program to inspect and modify its state at run time. In order to achieve this purpose it is necessary to have a mechanism to represent the program's data and state, this mechanism is called reification. A MOP is composed by a *static* and a *dynamic* part. The static part is given by the hierarchy of meta-object classes. The dynamic part is made of a set of generic functions that can be applied to meta-objects in order to control their behavior. Examples of such functions are:

- *Get-Elements*. We will consider that a meta-object is composed by a set of meta-objects together with a relation between them. For example, a class is composed by an ordered

list of slots, a generic function consists of a set of methods, a patch contains a list of boxes, etc. This function returns the list of elements of a meta-object.

- *Get-View* and *Get-Editor*. It returns the two possible graphical representations of a meta-object.
- *Open-Editor*. It shows the editor for a meta-object.
- *Add-Element* and *Remove-Element*. They allow editing of any basic object.
- *Box-Value*. It enables us to trigger the evaluation of a visual expression in a patch.

Figure 51.26 shows the graphical representation of the dynamic and static parts of the OpenMusic MOP. One may be surprised by the simplicity of the protocol. In general, a protocol which is too much specified is not very modular. It means that changes must be made in several places, and as consequence, modifications are not very reliable. On the other hand, a poor specification of the protocol makes it very difficult to find the place where modification should be included.

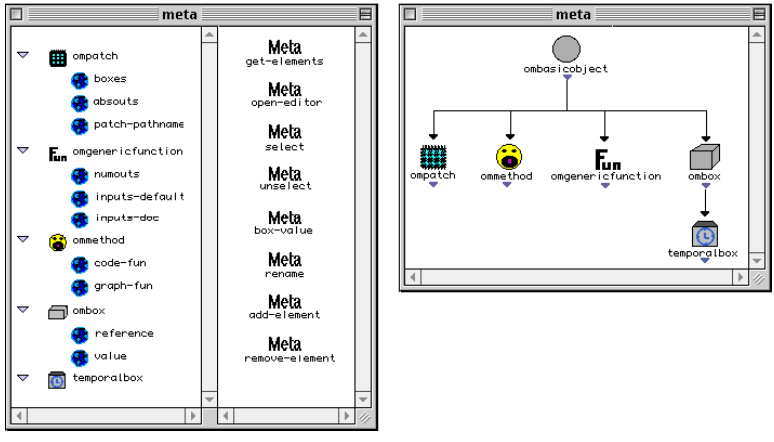


Figure 51.26: Dynamic and static part of the OpenMusic MOP.

The main tools for the meta-programmer are: subclassing inside the static class part and redefining functions in the dynamic protocol part. It is clear that the user can change some features of the language, but the most interesting point is that he can extend the language without loss of compatibility with old programs. Programmers can profit from customizing the language semantics. The following example in figure 51.27 shows the redefinition of the generic protocol function *Box-Value* which is called at each evaluation of a box. Graphical redefinition of these two methods changes the behavior of the language by introducing a visual trace of programs. The first method, which is defined with qualifier *before*, graphically selects any currently evaluated box before its execution. The second method, with qualifier *after*, deselects the box after its evaluation. These two simple modifications allow us to extend the language by tracing graphically and in an orthogonal way the functional composition.

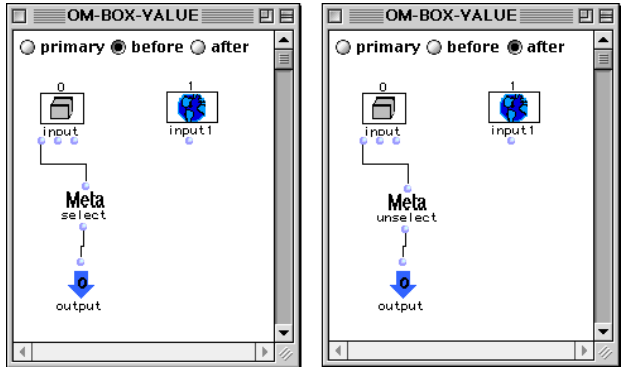


Figure 51.27: Changing a generic function of the MOP.

### 51.4.1 Reification of Temporal Boxes

Finally, this section shows how we have extended the temporal boxes in maquettes by applying meta-programming concepts. Indeed, thanks to the graphical reification of temporal boxes, users have access to the temporal boxes at the same level as other objects like notes, chords, etc. Figure 51.28 shows a first example of the new possibilities. By opening the editor of the temporal box, we can see a new box *self* (pointed by one arrow) that represents the temporal box itself. The three public slots—available as outlets—are from left to right: the meta-object itself, its offset and its extension. The process associated to this temporal box builds a major chord whose pitches are transposed taking into account the box's time offset in the maquette (which means that the start time of the box is directly proportional to the transposition interval).

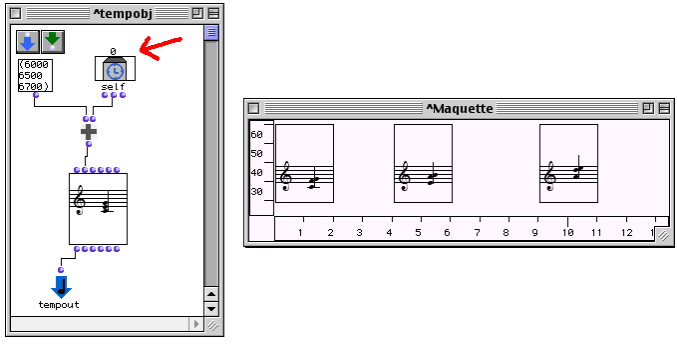


Figure 51.28: Mixing calculations and temporal relations.

As has been shown above, temporal boxes may send and receive data by using connections. By reifying temporal boxes, we allow them to have access to other temporal boxes belonging to the maquette. In figure 51.29 the upper temporal box is sent as a whole to the other one,

which sets its own offset to the same value as the previous. One evaluation of the maquette will produce two chords and will align them in time.

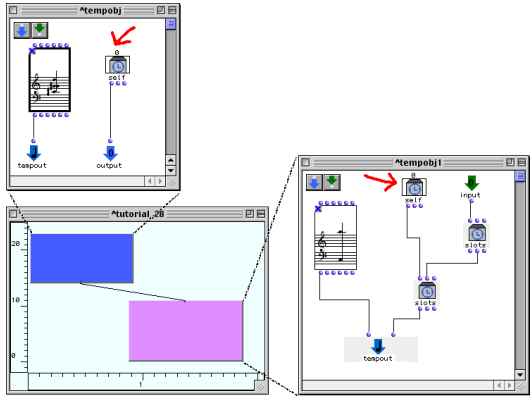


Figure 51.29: Mixing calculus and temporal relations again.

We stress two principal temporal box relations within maquettes: a temporal relation, given by the horizontal position, and a causal relation, established by functional connection between boxes. Examples in figures 51.28 and 51.29 expose how we have combined these relations by using the MOP. In the first example, the result of the calculation is affected by the position of the box. Informally, we can say that time changes calculation. In the second example, the evaluation of the functional composition of the boxes changes their position in time. In this case, the calculation changes the time organization.

We finish this chapter with a musical example of composition using OpenMusic.

## 51.5 A Musical Example

**Summary.** *Encore*, By Jean-Luc Hervé for two ensembles, live electronics, and two MIDI-controlled pianos (commissioned by IRCAM-EIC). Created April 9, 2000 by Ensemble Intercontemporain, Conductor : Patrick Davin. Frédéric Voisin was musical assistant and designed in OpenMusic the maquette described here.

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This piece is concluded by a cadence for two mechanical pianos (actually two MIDI-controlled pianos) followed by a short orchestral finale. The cadence is actually an OpenMusic maquette that is played through MIDI .

The piece is based on the concept of instrumental gesture. A gesture is a small musical unit with a typical energy profile, such as a glissando, a strong note preceded by a group of grace notes, a repeated note, etc. The gestures performed by the instruments may be continuous, such as a glissandi.

The piano cadence is prepared by a densification of orchestral gestures, soon imitated by the piano. At the end of the cadence, the instruments enter back again and the cadence progressively sinks into the orchestral mass.

The idea behind the cadence is to accumulate and superpose an enormous quantity of discrete gestures (played by the piano) into a well-defined architecture so that the total construction sounds as a quasi continuous sculpted sound shape. This is why it has been realized as a maquette: it was in fact impossible to write it manually in detail. Rather, the architecture is specified by the hierarchical inclusion of maquettes within maquettes (three levels).

At the deeper level, one finds the elementary gestures. Elementary gestures are temporal boxes that contain all the algorithmics necessary to generate them.

Groups of gestures inside a submaquette are linked by two kinds of relations. Firstly, temporal logic relations force them to begin and end simultaneously, even if they are moved or stretched in time by the composer. Secondly, they have inputs that are fed by links coming from a special block. This block is not a true temporal block in the sense that it is here only for computing harmonic material from which the gestures are colored. But this block does not generate any in-time music by itself. We will not detail the algorithmics behind gesture generation, we only want to make precise the relation between the computation of music material and the time information. Whenever the composer stretches a temporal block in order to experiment with duration, the note durations inside the block are not extended proportionally, as the default maquette behavior would enforce it. Rather, the composer's intention is to keep the same density of notes within a different duration. This is achieved by using the 'extent' outlet of the 'self' reflexive box inside a temporal block. By connecting this outlet to the right place in the visual algorithm, a gesture possessing the same overall profile is generated by inserting more notes picked out from the harmonic reservoirs.

The whole process is illustrated in figures 51.30 to 51.36.

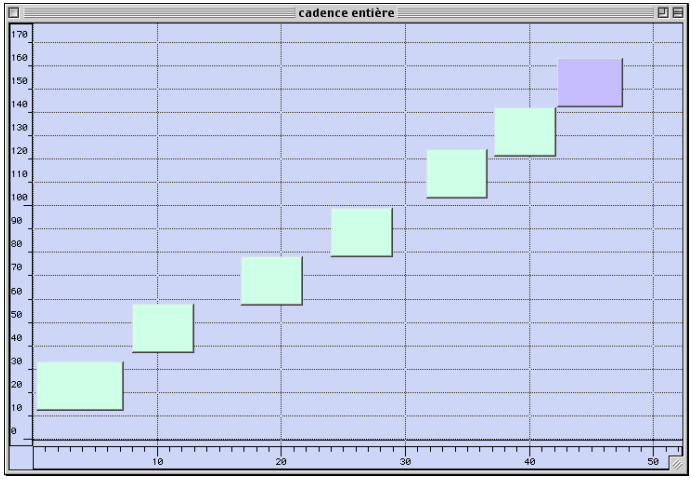


Figure 51.30: The piano cadence maquette of “Encore”.

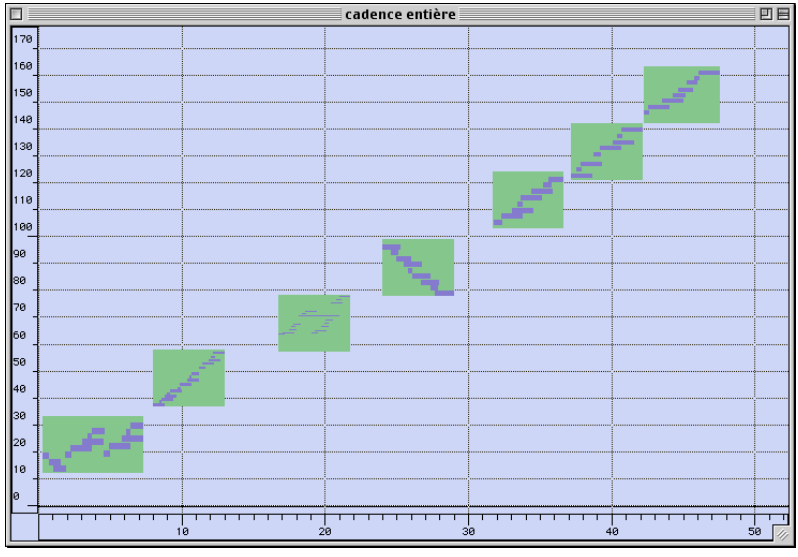


Figure 51.31: The second maquette level revealed.

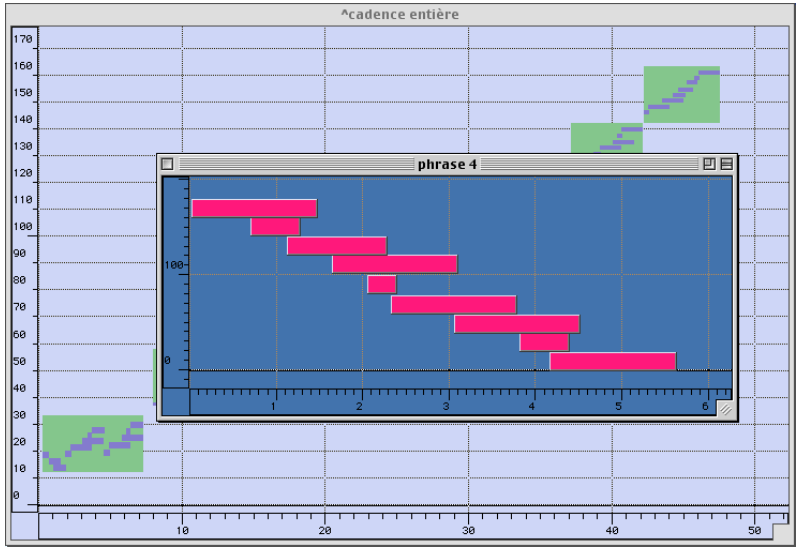


Figure 51.32: The fourth submaquette (starting left) opened.

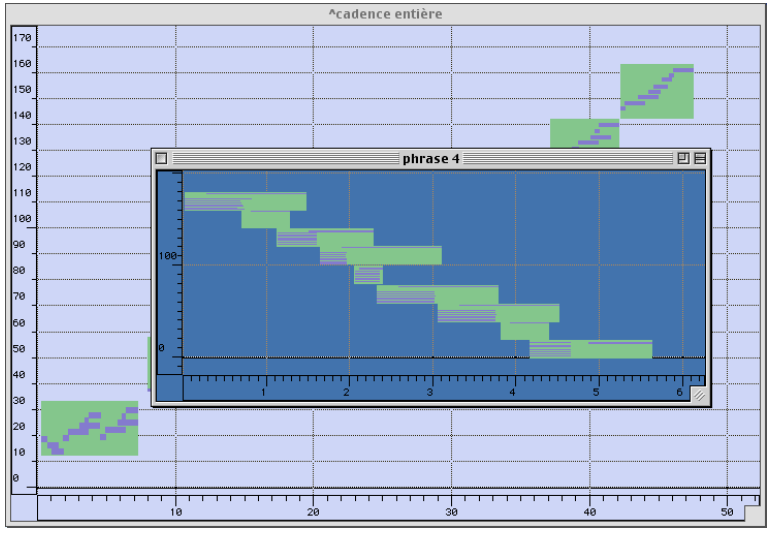


Figure 51.33: The third maquette level revealed.

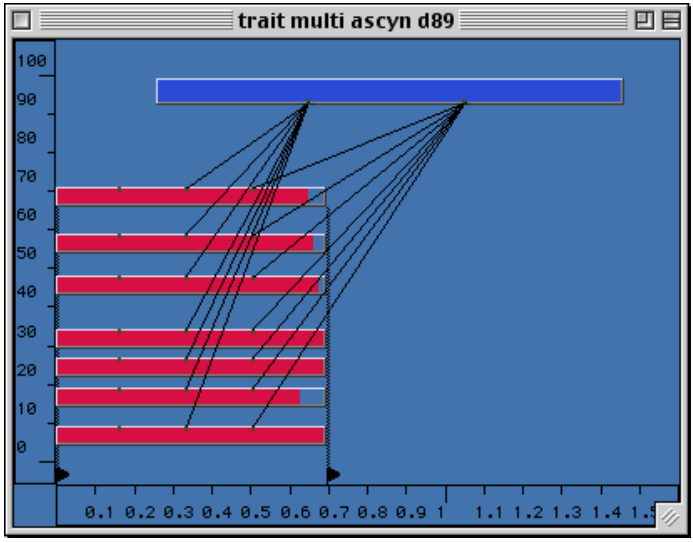


Figure 51.34: The fifth block (starting left) opened. The block on the top generates the harmonic material. The seventh block underneath are gestures. The vertical lines are temporal logic constraints.

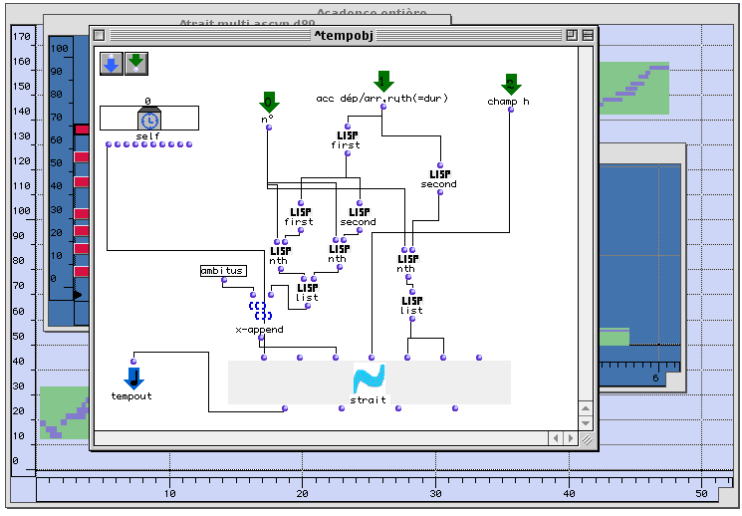


Figure 51.35: A gesture opened. The algorithmics generate in-time music material. The link coming from the 'self' box informs the algorithm about time conditions.

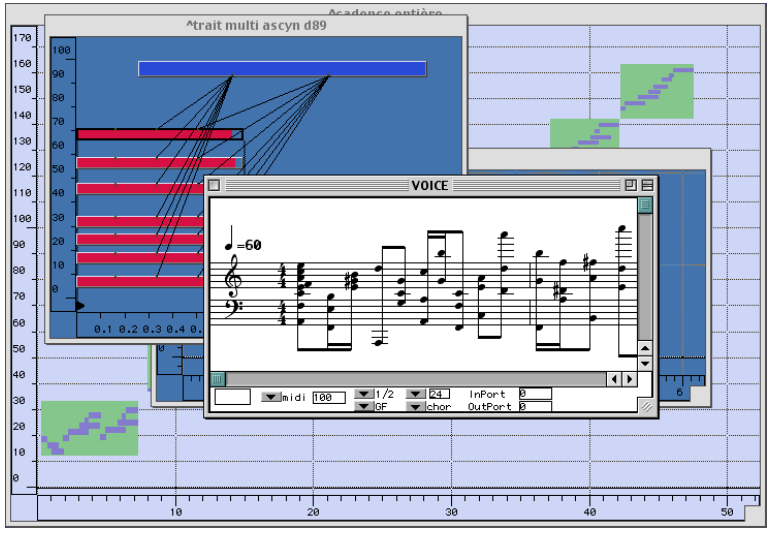


Figure 51.36: A gesture opened in music notation mode. Manual modifications can be achieved at that time.

# Part XIV

# String Quartet Theory

# Chapter 52

# Historical and Theoretical Prerequisites

*Bei der nämlichen Gelegenheit fragte ich Haydn,*
*warum er nie ein Violinquintett geschrieben habe,*
*und erhielt die lakonische Antwort, er habe*
*mit vier Stimmen genug gehabt.*
Ferdinand Ries [458, p.287]

**Summary.** This chapter introduces the best evolved theoretical part of instrumentation: string quartet theory. It starts with a short historic synopsis and then reviews Ludwig Finscher's work [151] on string quartet theory. We then focus on the technical core subject: the violin family as an instrumentation paradigm for the string quartet. The chapter concludes with a general discussion of semantics of sound colors.

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The scope of this part is the elaboration of a systematic foundation of the distinguished role of the string quartet at the end of the eighteenth century. This part is a synthesis of methods and results from modulation theory (chapter 27), classification (chapter 15), and counterpoint theory (part VII), combined with knowledge about the nature of sound parameters (see also appendix A). It is not astonishing that precisely the classical string quartet—one of the absolute highlights of instrumental music—needs a complex theoretical background for its comprehension. But since we deal with a systematically as well as historically founded phenomenon, it is adequate to prepend some remarks on the theory and history of the string quartet, remarks which we shall orient towards Ludwig Finscher's pioneering work [151].

This art form, whose instruments include two violins, one viola and one violoncello, can be characterized in the following way [374, p.409]:"In the string quartet, individuality and character of the single players are combined to a harmonic whole, where each one finds himself with an added profile. This reaches from a single complex chord which—in contrast to the piano—is not played and nuanced by the hand of one player, but by four players, to the entire performance. In the quality as a whole lies the specificity of the string quartet. In its harmonically contrasting

togetherness, this chamber music corresponds to the ideal world view and to the high humanistic vision of the classical epoch."

## 52.1 History

**Summary.** The history of the string quartet is exceptionally short since this instrumental species almost instantly appeared around 1760 with the works of Luigi Boccherini and Joseph Haydn. It almost instantly installed a *leading* stream of sophisticated instrumental expression. We discuss the historical background of this phenomenon.

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The *prehistory of the string quartet* is more complicated than that of any other instrumental art form of the eighteenth century. It cannot be causally deduced from any single one of the threads of tradition from where it comes. To a certain degree it is the creative act, the invention out of a moment of the delicate historic equilibrium, the *kairos* in the sense of ancient Greek thinking. The prehistory dates only from about 1720 to 1760 when *Luigi Boccherini* and *Joseph Haydn* independently invented the string quartet. In 1761, Boccherini wrote his first quartets in northern Italy, they were published 1767–68 in Paris under the name of "quatuor concertant". Probably Haydn had written quartet "divertimenti" already in the 1750s in Vienna, they were however only well-known in 1760. The sparse regional, instrumental, and stylistic rootedness in the string quartet's prehistory, from which this new art form has quite spontaneously emerged, provokes the question whether *beyond historical rationales* a more *systematic understanding* could better enlighten the 'string quartet phenomenon'. The problem is to question *this precise date* (1760) of the rise of *this precise instrumental art form* (the string quartet) in the context of the European music from the systematic point of view. In this question Dahlhaus [104, p.105,p.119] is fairly right in stating that

> *(...) erst die systematische Konstruktion den Blick dafür öffnet, welche Tatsachen einer Geschichte angehören, die zu erzählen lohnend scheint. (...) Daß etwa das Ausmaß in dem die Besetzung von Instrumentalmusik im 18. Jahrhundert gattungsprägend wurde, mit dem Grad ästhetischer Autonomie, mit der Herausbildung musikalischen 'Formdenkens' und mit der Festigung der Institution des öffentlichen Konzerts eng zusammenhing, ist keineswegs nur eine geschichtliche Tatsache, die sich empirisch feststellen läßt, sondern erscheint auch als Sinnzusammenhang, der sich einer phänomenologischen — also 'systematischen' — Analyse erschließt.*

## 52.2 Theory of the String Quartet Following Ludwig Finscher

**Summary.** In his habilitation thesis [151] Ludwig Finscher investigated the theory of the string quartet and exhibited three perspectives: the texture of four parts, the topos of conversation among cultivated humanists, and the family of violins. We discuss this threefold theory.

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If we have decided to turn the string quartet into the subject of a formally valid investigation, this is because the theoretical reflections on the string quartet have reached a scientifically valid status. Finscher remarks [151, p.279]:

> *Das Streichquartett ist die einzige Gattung der neueren Instrumentalmusik, die eine solche an einem einzigen künstlerischen Modell entwickelte, vergleichsweise genau und detailliert ausformulierte und als allgemeinverbindlich akzeptierte Theorie ausgebildet hat.*

This theory is based on two fundamentals:

- the four part texture;
- the topos of a conversation of four humanistically educated persons.

### 52.2.1 Four Part Texture

**Summary.** The texture of four parts is a basic structural prerequisite for the string quartet theory. We review its implications.

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The four part texture was the ideal type of structured polyphony which was oriented on the counterpoint with its long tradition. This is the formal, or better: formalized element of string quartet theory. We have to take it in the full conceptual ambiguity, i.e., on the one hand the texture "note against note" in its linear temporal progression in the sense of classical counterpoint. On the other, it is a texture of vertical units—charts in the terminology of global compositions—as an expression of harmonic relations. In the radically harmonic thinking, which is realized in the work of August Kollmann [268] from 1796, it is even possible to reverse the tradition in that the following thesis is proposed: Counterpoint should not start from the intervallic two-part texture, but from the four-part texture, since "a complete harmony" is four-part and not two- or three-part. It seems that the formation of theories was already influenced by Haydn's success with his famous "Russian" string quartets from 1782.

It is interesting to observe that this ideal type of an instrumental art form was fixed to exactly four voices, not one more. Especially with Haydn one could imagine that he could have added a fifth voice to "enrich the texture". But it is reported that he 'failed' on several occasions with this 'experiment'. Ries reports 1838 [151, p.287]:

> *Bei der nämlichen Gelegenheit fragte ich Haydn, warum er nie ein Violinquintett geschrieben habe, und erhielt die lakonische Antwort, er habe immer mit vier Stimmen genug gehabt. Man hatte mir nämlich gesagt, es seien drei Quintette von Haydn begehrt worden, die er aber nie hätte komponieren können, weil er sich in den Quartettstil so hineingeschrieben habe, daß er die fünfte Stimme nicht finden könne; er habe angefangen, es sei aber aus einem Versuche am Ende ein Quartett, aus dem anderen eine Sonate geworden.*

Presently, Haydn's argumentation that he has "enough" with the four voices, cannot be understood. We come back to this point at the end of our discussion in section 54.3.

### 52.2.2 The Topos of Conversation Among Four Humanists

**Summary.** String quartet tradition is intimately related to non-verbal humanistic conversation. This gives the species a *rhetoric* characteristic which has important consequences for the instrumentation problem.

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On the one hand, this is a topos which stems from the analogy of contrapuntal texture to the conversation and argumentation of humans. For example, Mattheson [151, p.285] says on imitation:

> *...daß eine Stimme die andere gleichsam gesprächsweise unterhalte.*

This topos must be cautiously distinguished from the well-known topos of a "Klangrede"[1], i.e., from the similarity of musical expression or semantics to the common language. In the case of the string quartet, the more important thing than speaking is the dialog, a fact that becomes more evident in the French expression "quatuor dialogué" for "string quartet" (in fact the invention of a publisher).

The association of a discourse to the string quartet was initiated by the musician Johann Friedrich Reichhardt in 1777 [151, p.287]:

> *Bei dem Quartett habe ich die Idee eines Gesprächs unter vier Personen gehabt.*

Like Haydn, Reichhardt also views the number of four as being the upper limit for a good dialog. He tries to add a fifth person to the quartet. But he fails:

> *Die fünfte Person ist hier ebensowenig zur Mannigfaltigkeit des Gesprächs nothwendig, als zur Vollstimmigkeit der Harmonie; und in jenem verwirrt sie nur und bringt Undeutlickeiten in's Stück.*

The same happens to Schumann [151, p.289] in a discussion about a viola that was added in a quintet:

> *Man sollte kaum glauben, wie die einzige hinzugekommene Bratsche die Wirkung der Saiteninstrumente, wie sie sich im Quartett äußert, auf einmal verändert, wie der Charakter des Quintetts ein ganz anderer ist, als der des Quartetts. Die Mitteltinten haben mehr Kraft und Leben; die einzelnen Instrumente wirken mehr als Massen zusammen; hat man im Quartett vier einzelne Menschen gehört, so glaubt man jetzt eine Versammlung vor sich zu haben.*

The quartet discourse as a dialog is very well suited to communicate understanding within music. This is the sense of the dialog. Goethe stresses this aspect in an enlightening comment on a concert by Niccolò Paganini [151, p.288], [185]:

[1] German for "sound speech", however, difficult to translate.

> *Mir fehlte zu Dem, was man Genuß nennt und was bei mir immer zwischen Sinnlichkeit und Verstand schwebt, eine Basis zu dieser Flammen- und Wolkensäule. Wäre ich in Berlin, so würde ich die Möserschen Quartettabende selten versäumen. Dieser Art Exhibitionen waren mir von je her von der Instrumentalmusik das Verständlichste: man hört vier vernünftige Leute sich untereinander unterhalten, glaubt ihren Diskursen etwas abzugewinnen und die Eigentümlichkeiten der Instrumente kennen zu lernen. Für diesmal fehlte mir in Geist und Ohr ein solches Fundament; ich hörte nur etwas Meteorisches und wußte mir weiter davon keine Rechenschaft zu geben.*

The connection between the dialogical discourse and the communication of understanding which Goethe indicates means a valuation of the dialog, its qualification in function of communication of understanding. This valuation in turn is related to the instance of varying competition by Karl Popper [420]: "The value of a dialog depends above all from the manifold of competing opinions." But this *manifold of opinions*, always related to a given subject, is nothing else than a variation of points of view, of the perspectives of the participants. Therefore the *dialogical principle* of the string quartet turns out to be an instance of the *Yoneda philosophy* discussed in section 9.3: *Understanding or classification, respectively, by a variation of the point of view.* It is not astonishing that this phrase is thoroughly indebted to humanism with which the string quartet is deeply associated.

**Result 9** *Summarizing, one root of the string quartet, the four part texture, appears as a form which, in the sense of Hanslick, is a carrier of musical spirit. This carrier in turn is made accessible to our understanding by the second root: the dialogical discourse of four violin personalities, by means of a variation of perspectives.*

At this point of our discussion the question arises, why the string quartet and the end of the eighteenth century are related to each other, more precisely: *What is the connection between the four-ness within the family of violins (and its above all violins, not violinists who speak!) and the paradigm of four part texture at Haydn's time?*—This question leads to a mathematically tractable apparatus in the manifold of violin sounds in defined parameter spaces.

### 52.2.3 The Family of Violins

**Summary.** We discuss the exceptional role of the family of violins in the building process of the string quartet species. In Finscher's work, other instrumental families are compared to the violins, we comment on the results of this study.

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The formation of the string quartet would not have been thinkable without the collaboration of the homogeneous sound of the instruments of the violin family, which in the eighteenth century was discovered as an ideal type of four-part music and also perfected on the artisanal level after the violin's creation from local variants of string instruments in the fifteenth century.

The high quality of violins is guaranteed by a technical standard which through its fineness, sometimes also imponderableness such as the choice of woods and the varnish covering, contradicts any normalization. The individual sound color of every good violin is characteristic

to the family. But the family of violins is also strongly differentiated from other families of string instruments. Finscher describes [151, p.124/125] the characteristic of violins as compared to the gambas, which were the preferred solo instruments in the seventeenth century, as follows:

> *Die Violinen hatten gegenüber den Gamben jedoch noch eine weitere zukunftsträchtige Eigenart: Sie gliederten den Tonraum, der ehemals in Analogie zu den menschlichen Stimmengattungen gebildet, nun aber in der Tiefe wie in der Höhe längst kräftig erweitert worden war, klarer und sinnfälliger, mit deutlicherer Individualisierung ihrer jeweiligen Tonbereiche. (...) Für das klassische Streichquartett, das die Beweglichkeit, den Lagenwechsel, den Kontrast- und Farbreichtum des symphonischen Streichersatzes mit der größtmöglichen Annäherung an eine streng auskomponierte Vierstimigkeiit zu verbinden suchte, bot sich das vierstimmige Ensemble aus Gliedern der Violinfamilie als das ideale Instrument an.*

It will be a task of the following chapter to access the specificities of the sound of the violin family via a mathematical description in the frame of parameter spaces.

# Chapter 53

# Estimation of Resolution Parameters

*Si tous les instrumets jouent l'accord staccato,*
*je supprime ce moyen naturel d'analyse,*
*et la perception ne peut plus discerner,*
*à l'interieur du bloc sonore,*
*de quelle combinaison il s'agit.*
*L'identification dépend, das ce cas,*
*de la présence ou de l'absence des*
*éléments essentiels pour la perception.*
Pierre Boulez [61, p.547]

**Summary.** This chapter is a technical account of the variety of sound parameters which intervene for the family of violins.

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Based on the preceding historical, systematic, and philosophical reflection we argue that the classification theory of global compositions (chapters 15 through 17) is an excellent candidate to be applied to the theory of the string quartet. From classification theory it follows that the resolution of a global composition is a formalization and optimization of the process of understanding this composition. We shall equally apply this approach to string quartets, which means that we have to think of the composition as being a global composition $G^I$, whose resolution $\Delta_{G^I}$ one wants to represent. In chapter 54, we shall specify the global composition associated with a string quartet composition. Our approach to the resolution is essentially realized via a variation of the perspective of the resolution parameters, i.e., by a differentiated change of their dichotomy into essential and accessory parameters. Within the string quartet, this dynamics of weighting of performance parameters is a subtle tool for communication of understanding and articulation as a vehicle of dialog. It is as if you walk around an object—which is the resolution in our case—and observe it from one, and then from another perspective.

If we now envisage the resolution of a string quartet composition, we first have to investigate the possibility of parametrization by concrete physical performance parameters as they intervene on the arsenal of the violin family.

## 53.1 Parameter Spaces for Violins

**Summary.** We describe the denotator types for violins, including generic geometric parameters, sound color parameters, and technical parameters, in particular those related to vibrato.

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As we already did in classification theory, we want to work over $\mathbb{Q}$, which is a reasonable postulate for string quartet theory since we are dealing with physical parameters.

Here, we envisage a fundamental problem of mathematics since we have to deal with a huge number of independent numerical parameters. If, for example, a sound color has to be represented by means of amplitudes and phases of the Fourier representation, and by the envelope, we easily add up hundreds of numerical parameters. Even though locally, the structure of music thinking may concern a small number of parameters, the techniques of classification theory, by their requirement of points in general position, enforce a number of geometric degrees of freedom that might possibly be far larger. We shall have to work with charts in modules $\mathbb{Q}^n$ with large dimension $n$. This should however not prevent us from *visualizing essential aspects of our reflections in three space* $\mathbb{Q}^3$. The only point here is to build mathematically representative analogies in three space.

What is the shape of the physical parameter space of a violin sound? To begin with, we dispose of parameters which we call *geometric*[1], i.e.,

- Onset
- Duration
- Pitch

The amplitude (loudness) is omitted in this presentation since the poietic violin parameters are coupled to the way of generation of the sound color aspect.

The other attributes of the violin sound span the *color space*. To begin with, this aspect includes the following ingredients:

- Envelope
- Amplitude
- Fourier spectrum

What is the range of variability of these color aspects of a violin sound? Regarding this question, we refer to [371]. For example, we consider the amplitude spectrum for pitch $g$ and $g_\sharp$ of a "Guarneri del Gesù" as compared to an F horn, see figure 53.1. One recognizes that the spectra

[1] See appendix A.1.2.1.

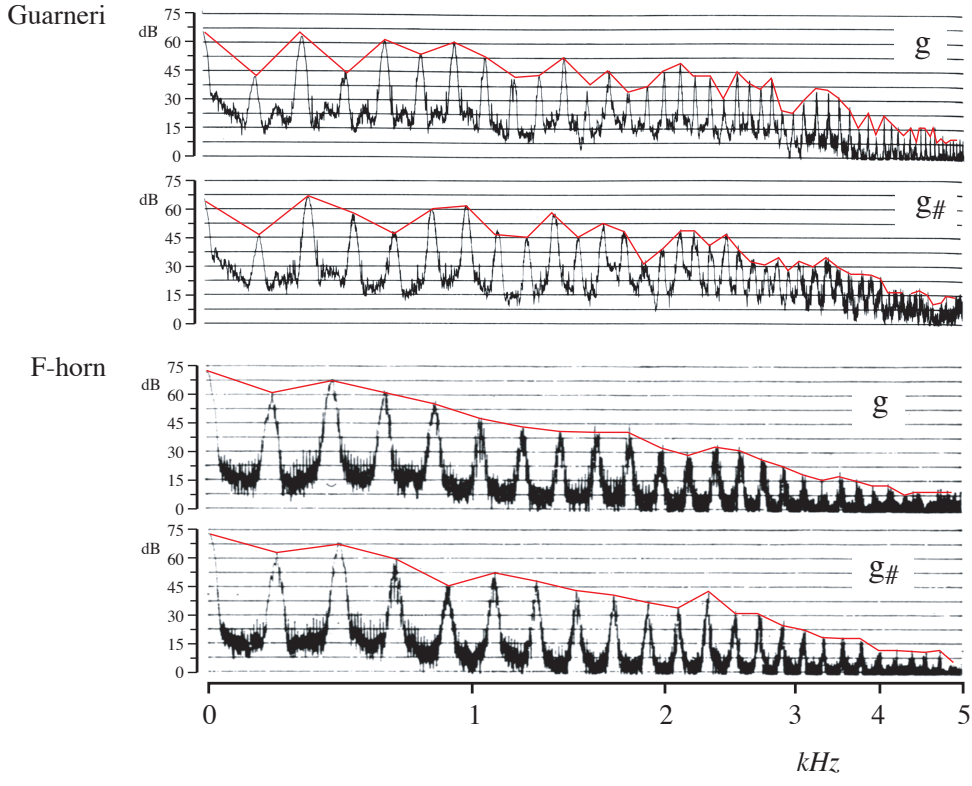


Figure 53.1: We consider the amplitude spectrum for pitch $g$ and $g_\sharp$ of a "Guarneri del Gesù" (top) as compared to a F horn (bottom). One recognizes that the spectra from $g$ to $g_\sharp$ are significantly more different for the "Guarneri" than for the F horn.

between $g$ and $g_\sharp$ are significantly more different for the "Guarneri" than for the F horn. In the representation of the amplitude spectrum as a vector in the color space, one may say that the change from $g$ to $g_\sharp$ for the F horn is essentially a dilatation, whereas the corresponding change for the "Guarneri" violin also includes a change of direction (figure 53.2).

In other words: The $g$ and $g_\sharp$ spectral vectors of the "Guarneri" span a plane in the color space, whereas the corresponding vectors for the F horn lie on a line. This statement evidently is not meant in a strictly physical sense, but in the sense of valence theory (see appendix B.2): For the F horn, the $g$- and $g_\sharp$-spectral vectors are indistinguishable in the auditory perception from a pair of linearly dependent vectors[2].

This instrumental difference is justified by the fact that for winds, the sound color is

[2]This discussion is somewhat speculative since precise measurements should be made and relations to valence theory should be investigated in a more quantitative way. However, the means for such an investigation depend on the insight in the basic problem setup.

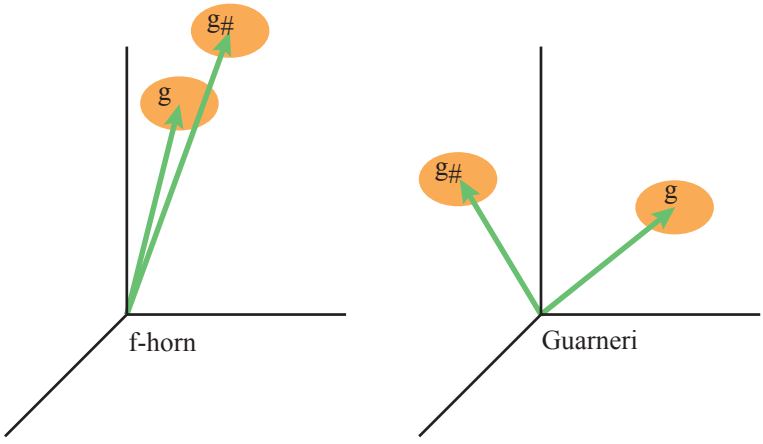


Figure 53.2: In the representation of the amplitude spectra as vectors (represented in 3D space here), the "Guarneri" vector differs beyond the valence limit when comparing the $g$ sound to the $g_\sharp$ sound.

essentially built from an air column, but not from the material. In contrast, for the violins, sound color is an essential function of the material, i.e., of the corpus' resonance properties. Moreover, the spectrum depends on the string on which a fixed pitch is intonated (to this end, the spectral envelopes for different strings are compared [371]).

All this turns the sound color of violins substantially into a function of the individual construction: from the material through the artisanal manufacturing to the individual history of the particular instrument. This variability is out of the question with wind instruments. For the same reasons, string pianos are inferior to violins. Figure 53.3 shows the principal configuration of the spectral vector for three violins as opposed to three such vectors for string pianos.

To these instrumental parameter properties, central techniques of instrumental practice of sound shaping are added, techniques which for several other instruments do not even exist. The string player has the following possibilities to vary parameters:

- Bow pressure
- Bow velocity
- Contact point of bow and string
- Bow angle

Further, the string player may shape his/her *vibrato* according to the following points of view:

- Delay time with respect to the tone's onset

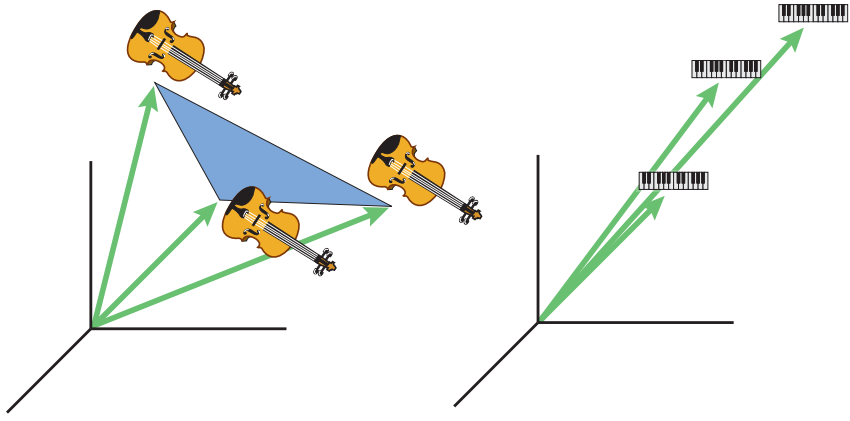

Figure 53.3: The instrumental sound parameters of violins are more variable individually than these parameters are for pianos. Here the amplitude spectra of three violins against three pianos are visualized schematically in 3D space. In contrast to the piano spectra, those of violins lie in general position.

- Modulation frequency (frequency of the finger's movement)
- Pitch modulation (Range of finger movement on the string)
- Amplitude modulation (Contact point of the finger-tip)

Compared to the color attributes envelope, amplitude, and spectrum, these *vibrato parameters* are new. They enable the violinist to realize sound in still larger spaces. Together with the four *bow parameters*, the four vibrato parameters define an additional eight-dimensional space. These eight parameters which emerge in contrast to the instrumental parameters are called *technical parameters*. Figure 53.4 shows the effect of bow pressure and contact point variation while bow velocity and sound color remain constant.

## 53.2 Estimation

**Summary.** This section is devoted to a theorem giving an estimation of the maximal possible chart dimension $ch(n)$ within a global composition, which can be produced by an orchestra consisting of $n$ individual strings (in the violin family). We make use of the resolution theory for classification of global compositions (see chapter 15.2).

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In the last step of our parameter analysis, we deal with differentiation within the technical parameters. Let us recall that we are searching for parameter spaces, i.e., coordinate functions which are suited for charts of global compositions. But this signifies that one has to distinguish

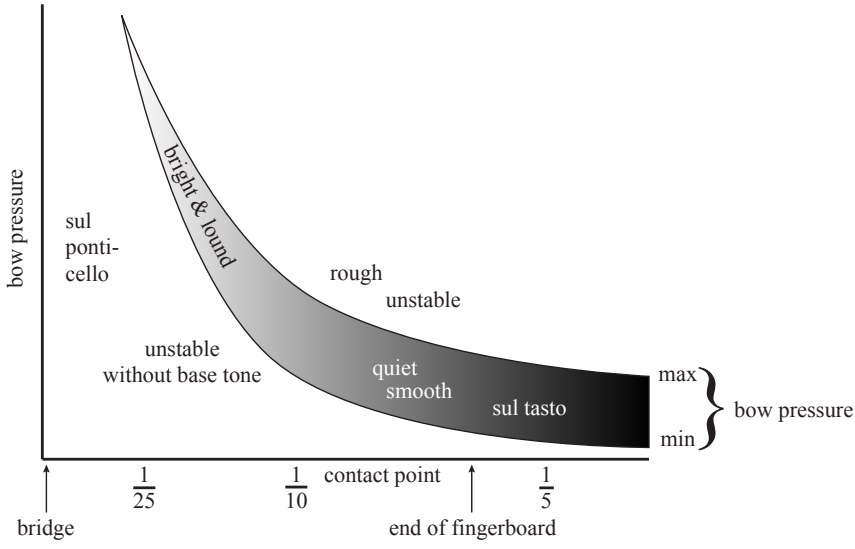



Figure 53.4: The effect of a variation of bow pressure and contact point while bow velocity and sound color remain constant.

parameters which can be varied independently of each other in short parts of the composition, in fact on charts, from those parameters which may very well vary from player to player or from situation to situation within a larger composition, but which are relatively constant on local regions. The latter, which may strongly depend on the player's personality, include:

- the vibrato parameters. Even for a professional violinist they can scarcely be separated from the personality and are difficult to control;
- The bow angle and contact point are relatively inert parameters, therefore not suited for extremely local purposes.

The local variability is therefore distributed on two dimensions:

- Bow pressure which above all acts on the amplitude. (This is the reason why we did not add amplitude to the geometric parameters here: it is only an aspect of the action of bow pressure!)
- Bow velocity which above all acts on the dynamics of partials.

Evidently the bow parameters are coupled with each other in their action; we only indicated the main actions.

Summarizing, we have found three types of color parameters:

1. *Instrumental parameters* (Violin type, choice of strings, performance conditions),

2. *global technical parameters* (vibrato, bow angle, contact point), which are a strong function of the individual player,

3. *local technical parameters* (bow pressure, bow velocity), which can be steered quite independently of each other.

Therefore, the *local* technical parameters define a plane in the color space. Since these are under general control and can be steered objectively, we may assume that these planes $H_1, H_2, \ldots$ for player 1, player 2, ... are one and the same player-independent plane $H$ up to an individual translation (figure 53.5). Observe that our hypothesis is made in the context of the widespread

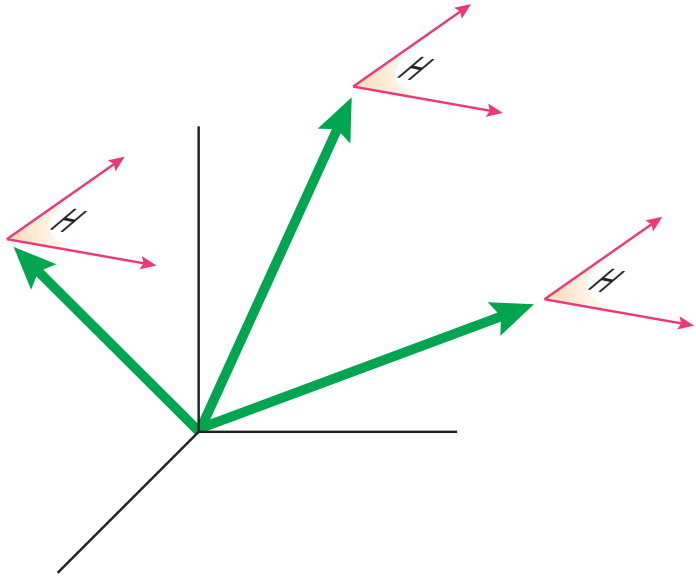



Figure 53.5: In the color space of violins a plane $H$ is appended to the heads of individual vectors $v_s$ of instrumental and global (inert) color parameters and yields $H_s = v_s + H$ for player $s$.

string quartet at the end of the eighteenth century, and that in fact the signification of the string quartet is a pronounced reality of music sociology with regard to normal music practice in bourgeois saloons. It would be unrealistic to model our theory upon elite ensembles in this context.

Based on the above observations about the individuality of colors and the violinists' personalities, we may set forth the following hypothesis in the spirit of string quartet theory:

**Assumption 3** *The $n$ "instrumental vectors" $v_1, v_2, \ldots$ which are spanned by the instrumental and global technical parameters can be chosen to be linearly independent of each other and of the common plane $H$.*

This means that the submodule is spanned by $v_1, \dots v_n$ and $H$ has dimension $n+2$. In other words: If $H$ is spanned by two vectors $h, k$, the zero vector and the heads of the vectors $v_1, \dots v_n, h, k$ are in general position[3].

If we add the three-dimensional module $G \xrightarrow{\sim} \mathbb{Q}^3$ of geometric parameters to the color space (direct sum), we obtain a *total parameter space*, where the $n$ instrumental vectors are positioned, of which to each is attached a $2+3 = 5$-dimensional space $H+G$ of local parameters, see figure 53.6.

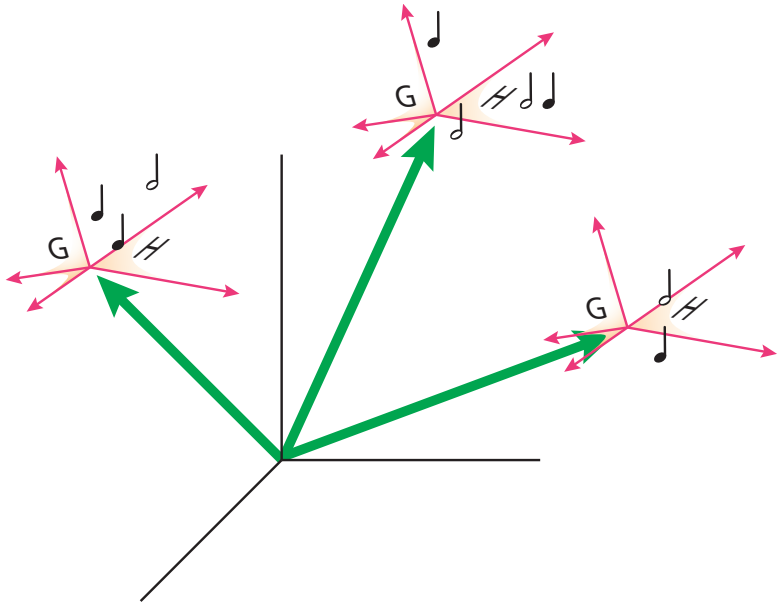



Figure 53.6: Viewed as points for the $n$ instruments of the violin family, the sounds are distributed on $n$ affine subspaces $H+G+v_s, s = 1, \dots n$, in the total attribute space. Here $G$ is the three-dimensional space of sound geometry (pitch, onset, duration), $H$ is the plane of the local technical parameters bow pressure and bow velocity, and $v_s$ is the $s^{th}$ instrument vector. One may assume that the $\mathbb{Q}$-module spanned by $G, H, v_s, s = 1, \dots n$ has dimension $n+5$.

In view of classification theory our question is how many points in the total parameter space can maximally be distributed in general position on the $n$ affine subspaces $H+G+v_s, s = 1, \dots n$. So every such point has the form $x = v_s + h + g, h \in H, g \in G$, see figure 53.6. It can be shown (appendix E.2.1, theorem 47) that there are maximally $n+5$ points, where 5 means the dimension of $G+H$. The musicological meaning of this result for classification theory is this:

**Theorem 36** *With $n$ string players from the violin family, charts of global compositions in local parameters can be defined with maximally $n+5$ points in general position.*

[3]See appendix E.3.4.

# Chapter 54

# The Case of Counterpoint and Harmony

*Au fur et à mesure que l'orchestre s'agrandit,*
*que le rôle de l'instrument devient,*
*je ne dirais pas flou, mais ductile, multiple,*
*les formes, elles aussi, s'amplifient.*
Pierre Boulez [61, p.544]

**Summary.** This final chapter on string quartet theory deals with the analytical conditions on global compositions which are powerful enough to comprehend the structural richness of central European music in the epoch of Boccherini and Haydn. These structures are—essentially—counterpoint and harmony. As a germ for a systematic theory of instrumentation we propose an estimation of maximal necessary chart dimension for Fuxian counterpoint and traditional harmony (including cadence and modulation).

– Σ –

As the last member of our model for the string quartet, we need the initially announced information to solve the question of how a string quartet should be defined as a global composition. To this end we imagine that a score, written by Haydn, say, is given as a local composition. From this one would like to construct an adequate interpretation (in the technical sense of interpretable global compositions). And it is here where the historical moment comes into the game. The interpretation should be such that the European structural music thinking in the four part texture at the end of the eighteenth century is expressed in its essential features. In our presentation of the roots of the string quartet theory, the structural basis was first presented: the four part texture with its polysemic meaning as a contrapuntal as well as harmonic setup. These two main components are therefore to be investigated for the construction of an atlas.

## 54.1 Counterpoint

**Summary.** In this section, we calculate the upper limit of chart dimensions for counterpoint.

We make use of the counterpoint model exposed in part VII.

– Σ –

In this section we shall refer to the discussion of the core theory of counterpoint, "note against note", as it was presented in chapters 29 through 31. For intervals which are represented by arrows in counterpoint theory, we need two-element charts (admitting that what is physically played are not arrows, but their heads and tails). For the cantus firmus and the discant steps we also need two-element charts, one for each. For the consideration of a progression from interval to interval, one needs four-element charts. It is advantageous to include also the succession of two interval steps, for example regarding hidden/composed tritones, we need six-element charts. This is however *not* the statement of the rules of counterpoint! As we know, this would be much more complex. But we have defined the "cartographical" setup, and that is what we need. Therefore:

**Result 10** *The classical contrapuntal texture as it was codified by Johann Joseph Fux in 1725, the early days of the prehistory of the string quartet, requires in its core structure maximally 6-element charts. All more complex configurations can be reduced to this core structure: Thereby charts are glued together, but not enlarged.*

## 54.2 Harmony

**Summary.** In this section, we calculate the upper limit of chart dimensions for harmony. We make use of the cadence (chapter 26) and modulation (chapter 27) models.

– Σ –

The degree theory of chords as vertical structures interprets a chord as being a subset of a triadic covering (see chapter 25), i.e., by three-element charts.

A cadence (see chapter 26) can be thought within the scheme consisting of three degrees (typically: IV-V-I), where the minimal cadential sets (here: IV,V) are completed by the first degree or else by the tonic note of the given tonality. This produces an interpretation of maximal nine tones per chart, consisting of one, three or nine points.

The most complex situation in the harmony at the end of the eighteenth century is crystallized in the modulation process which we have formalized in the modulation model that is based on triadic degrees, see chapter 27. In a modulation which is presented as a sequence of neutral degree, modulation degree, cadence degree, we may recognize these three charts, with three elements each. Moreover, like with a cadence, the entire process is collected in a nine-element chart that contains the three triadic degrees. The cadence as such, which we described above as a nine-element interpretation, intersects our modulation in the cadence degree and concludes it as a process. Therefore:

**Result 11** *Harmony has a maximum of nine points for the relevant local charts.*

## 54.3 Effective Selection

**Summary.** As a result of the instrumental parameter estimation made in chapter 53 and the structural parameter estimation from this chapter, we obtain a global theorem estimating the *minimal number four of string instruments* (in the violin family) which is needed to express analytical music structures in the compositions of the classical epoch of Boccherini and Haydn. This theorem makes essential use of the resolution theory for global compositions, as exposed in chapter 15.2.

– Σ –

Finally, we are in a state of indicating the minimal number of instruments from the violin family in order to provide the resolution of an interpretation of a score under the structure as preconized by harmony and counterpoint at the end of the eighteenth century by a sufficient number of free parameters.

In sections 54.1 and 54.2 we have seen that a maximal number of nine points per chart are present in such an interpretation. Since in a resolution, all charts have their points in general position, the number $n$ of instruments must, according to theorem 36 at the end of section 53.2, suffice the inequality

$$n + 5 \geq 9. \tag{54.1}$$

We therefore need at least four instruments for these scopes. As announced, our model of the string quartet has been deduced from the information about parameters, modulation theory, counterpoint, and the classification technique. From this point of view, it is not astonishing that classical string quartet composers such as Haydn did not see any sense in the accumulation of instruments: For a purely economical point of view, they were superfluous. Four string players were perfectly sufficient in order to provide the textural structure with a profiled representation in its resolution.

# Part XV

# Appendix: Sound

# Appendix A

# Common Parameter Spaces

This appendix chapter is an overview, not an exhaustive treatise of spaces which parametrize sound objects. These spaces where sounds are positioned always define an aspect, never the totality of music thinking, and every attempt to define a preferred space will narrow the music thinking, not the music. The best that can occur is that we offer an encompassing or at least a representative ensemble of parameter spaces which are interrelated by a precise relation. To this end, it is recommended to distinguish topographic positions, above all in their realities and communicative perspectives. This will also entail the corresponding mathematics.

We start by the physical descriptions, turn over to more mathematical abstractions and the describe more symbolic viewpoints which we call interpretative since they are not just a new mathematical rephrasing of a priori equivalent physical description, but express abstraction with some mental background constructions.

## A.1 Physical Spaces

As a physical object, a *sound*[1] is a more or less regular variation of normal air pressure[2] as a function of time. Starting at a determined *onset time e sec*, it starts from a source $Q$ at position $q = q_Q\ m$ in the ordinary physical space and expands as a wave. At a location $x$ and time $t$, the *pressure variation* (the difference from the normal pressure) $p_x^Q(t)\ Nm^{-2}$ is perceived as a longitudinal air wave, i.e., with a pressure front perpendicular to the waves expanding direction, see figure A.1.

For a punctual sound source, however, the wave front at $x \neq q$ is a spheric surface; we can write

$$p_x^Q(t) = |x-q|^{-1} p_q(t - |x-q|/v), \tag{A.1}$$

where $v$ is the expansion velocity of the wave[3]. It is calculated by the formula $v = C.\sqrt{T}$, where $T\ K$ is the absolute temperature in Kelvin degrees, and $C$ is a constant with value $C = 20.1$

[1] German: "Klang".

[2] At the zero height above sea and zero degrees Celsius, this is $\approx 1.1013.10^5 Nm^{-2}$.

[3] It is known [462] that the square of the pression variation is proportional to the intensity, i.e., energy flow per surface and time unit, and the latter, by energy conservation, decreases proportionally to the square of the distance $|x-q|$, whence the formula.

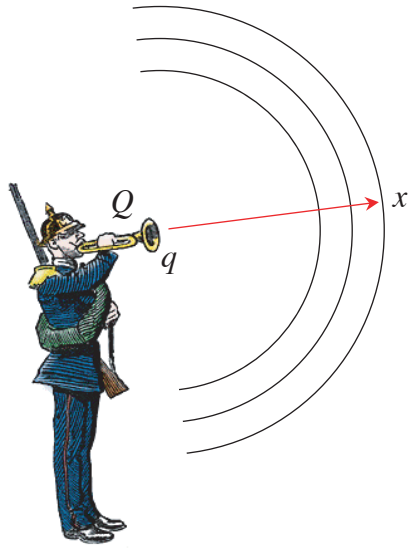



Figure A.1: The prototypical punctual sound source and the spherical sound wave.

for the normal pressure. For normal conditions, we have $v \approx 343\ msec^{-1}$. But with complex sound sources and room-specific reflection and refractions, the pressure variation of a sound may be an overlapping of different spheric wave components. If several sound sources $Q_1, \dots Q_s$ are given, the resulting pressure variation sums up to $p_x^{Q\cdot}(t) = \sum_{1 \le i \le s} p_x^{Q_i}(t)$, and it is the integral $\int dp_x^{dQ}(t)$ of a family of infinitesimal point sources $dQ$. In general, from the knowledge of $p_x^{Q\cdot}(t)$, one cannot infer the original source functions. This is like with painting where in general one perspectivic image does not allow us to infer the original object configuration: The esthesic position is not sufficient to reconstruct poiesis[4].

The fundamental problem of musical acoustics is that the neutral data, the objectively measurable pressure values, are far from what is intended by musicians, i.e., the neutral data is not what is interesting and what is the message. So one of the most important tasks of musical acoustics is the interpretation of the neutral data, in other words, of what is behind the data, what could be the hidden parameters of the audible phenomenena. And it is one of the worst tragedies of traditional music-acoustics and psycho-acoustics that the fundamental difference of neutral and poietic levels is ignored and disregarded.

### A.1.1 Neutral Data

In order to describe the neutral sound data, let us first concentrate on the information at the source location $q = q_Q$ of a point source $Q$, in an idealized model of a single instrument. We shall come back to room acoustics in the next section A.1.1.1. The source sound (variation)

[4]See chapter 2 for the concepts of music topography, such as "esthesic", "neutral", or "poietic".

event $p_q(t)$ is usually a finite event, starting at time $e$, and ending after the duration $d$. The variation between these time limits is also limited by the maximal amplitude $Am$ of the total pressure variation. So the function $p_q(t)$ is the affine image of a normalized function $p_q^0(t)$ which starts at time $e = 0$, has duration $d = 1$, and amplitude $A = 1$. More precisely:

$$p_q(t) = Support(A, e, d)(p_q^0)(t) = A.p_q^0((t - e)/d). \tag{A.2}$$

The operator $Support(A, e, d)$ reduces the unknown sound event to a normalized event. What happens between the normalized unit supports is however completely arbitrary. It may be a percussive sound or the sound of a Stradivari violin. The normalization by the support operator is completely harmless, but not much more than this data can be traced on the neutral level. The theory of all the rest is far from neutral; we are going to deal with this in section A.1.2.

#### A.1.1.1 Room Acoustics

The room is an important part in the information chain from the information source (instrument, speaker, public address system) to the receiver (listener, director, artist). Room sizes vary from small living rooms to huge cathedrals or concert halls. In this section, we want to tackle only basic features, reverberation time and acoustical power. For a large, irregular room, we can visualize the acoustical conditions by imagining a wave traveling inside the room. This wave travels in a straight line until it strikes a surface. It is reflected off the surface at an angle equal to the angle of incidence and travels in this direction until it strikes another surface. Because sound travels about 343 $msec^{-1}$, many reflections will occur within a small time span.

**Absorption.** After a wave has undergone a reflection from a wall that is absorbing, its intensity will be less during its next traverse of the room. In a large, irregular room, the number of waves traveling are so numerous that at each surface all directions of incident flow are equally probable. The sound absorption coefficient $\alpha$ is therefore taken to be averaged for all angles of incidence. All materials have absorption coefficients that are different at different frequencies[5] [54]. In the frequency range of interest between 250 $Hz$ and 4 $kHz$, plain walls and floors, as well as closed windows, have absorption indices below 0.2. Higher absorption can be achieved with acoustic tiles and, at least in the upper frequency range, with thick carpets and draperies. The total absorption $A$ of a room is the sum of the product of surfaces $A_i$ $m^2$ and absorption coefficients $\alpha_i$: $A = \Sigma_i \alpha_i A_i$.

If there is an open window in the room, all the energy incident on its area will pass outdoors and none will be reflected.[6] The absorption of an area of acoustical material in a room can therefore be expressed in terms of the equivalent area of an open window. For this reason, the total absorption $A$ of a room can be characterized by its equivalent "open window surface".

**Critical Distance.** The sound field which builds up in a room is fundamentally different from the free field situation. Let us first assume a sound source in a free field, i.e., a loudspeaker on a high post emitting sound in all directions. If the sound radiation is unidirectional

[5] See section A.1.2 for the discussion of the frequency concept.

[6] This statement is strictly true only if the window is several wavelengths wide and high, otherwise diffraction will occur.

and no reflections occur, the source will emit a spherical wave and sound pressure will decrease inversely proportional with the distance from the source.

For loss-less reflecting walls on all sides around the source, the sound waves will be reflected over and over. In the case of no absorption in the air, the sound pressure will now rise and rise, as no energy loss occurs. Small absorption will cause an equilibrium. However, the sound field in the room will no longer be a directional spherical field because the reflected waves will by far dominate over the direct sound wave. The sound field will be diffuse, reflections will arrive from all directions with equal probability. Only in the close vicinity of the small sound source is the sound field directional, because there the sound pressure of the unreflected direct sound wave will dominate over the diffuse sound field. In summary, the sound field in a room will be directional only close to the sound source $Q$. There, as we know from the above, at point $x$, the sound pressure falls with $1/|x-q|$. The sound field at a large distance from the source is diffuse. Sound pressure is almost constant and much higher compared to a free sound field. The distance from the source, where the transition between these distinct regions, occurs is called *critical distance* or *diffuse field distance* $r_H$. If the average absorption $\bar{\alpha} \leq 0.4$, the critical distance for the absorption $A$ can be calculated with a precision of about 10% [54]:

$$r_H = \sqrt{A/16\pi} \approx 0.14\sqrt{A}. \tag{A.3}$$

By the inverse distance law, the sound *pressure decrease factor* $\rho_D$ in the diffuse field of a room can be estimated by the formula:

$$\rho_D = 1/r_H \approx \frac{7.1}{\sqrt{A}}. \tag{A.4}$$

**Reverberation Time.** Temporal effects during the onset and the decay of sound are essential features in rooms and are not present in a free sound field. In large rooms, such as a cathedral, long decay times are apparent.The human auditory system can distinguish whether a sound source is located in a large room, a small room or even whether it is not in a room at all, but outside. The human ear is obviously able to extract information about the room size from the temporal structure of sound. We will therefore discuss the onset and decay of sound.

When the sound leaves its source, the direct sound reaches the receiver first, its delay is determined by the distance sound has to travel divided by the velocity of sound. With further delay, reflected sound waves with only a slightly longer travel distance arrive, and later, also reflections with longer routes and multiple reflections arrive. The sound pressure is increasing until it reaches its steady state value. Only the direct sound gives information about the location of the source, which is exploited by our hearing system for sound localization. The build-up of the reverberation is only perceived in highly reverberant rooms. The reverberation is much more perceptible after the source is muted, and the echo may be still noticeable after seconds. The direct sound ceases after the propagation time from the source to the receiver, however, all the reflected sound waves still arrive. Their intensity will be reduced by factor $1-\bar{\alpha}$ after each reflection on a wall. Therefore, the sound pressure will decrease exponentially.

The *reverberation time* $T$ is defined as the time required for the sound to decay by a sound pressure level[7] of 60 $dB$. Let $\alpha_L\ m^{-1}$ be the absorption in air (a highly frequency- and humidity-dependent variable), and let $V\ m^3$ be the volume of the room. Then the reverberation time $T$ can be estimated with *Sabine's* formula [595]:

$$T \approx \frac{0.161V}{A + 0.46\alpha_L V}\ sec. \tag{A.5}$$

In the frequency range below 4 $kHz$, sound absorption in air is usually neglected, whereas for frequencies above 4 $kHz$, the reverberation time $T$ is mainly determined by the absorption in air $\alpha_L$.

The reverberation time is the most important variable to describe the acoustics of rooms. In rooms with a long reverberation time, sources with a relatively low level yield a high sound intensity, however, speech intelligibility is decreased due to increased temporal masking. As a compromise, the reverberation time must be "appropriate" to the room size. For speech, reverberation time should be between 0.5 *sec* and 1 *sec* (increasing with room size), for music presentations 1 *sec* to 2 *sec* are acceptable.

Beyond the global description of a room by the reverberation time, the temporal fine-structure of the reverberation, the temporal incidence of the reflections, is of interest. From the first reflections, the human hearing system is able to extract information about the size of the room. If the first reflections occur very early (1 *msec* to 10 *msec* after the direct sound wave), the sound color—especially for music recordings—is altered. Reflections in the time span from 10 *msec* to 50 *msec* increase the perceived loudness. Single echoes, arriving with a delay of more than about 100 *msec*, are perceived as echoes. Very disturbing are periodic echoes, which are generated between parallel walls, for example. For good room acoustics, the reflections should be homogeneous and the intensity should decrease with time. Single echoes should not be larger than 5 $dB$ compared with their temporal vicinity. In concert halls, the reverberation time is measured as a function of frequency and additional sound absorbers and reflectors are placed to improve the acoustics.

**Remark 24** This sketchy discussion shows that room acoustics is a complex topic, therefore acoustic experts should be consulted already in the planning of music rooms. This need is documented by plenty of examples, where the acoustics of rooms built for the purpose of audio presentations is so bad that speech intelligibility is severely hampered. Thoroughly planned room acoustics is essential not only for concert and lecture halls, but also for most other rooms such as offices and even hallways and production areas, to keep noise levels down and achieve an environment which is pleasing to the ear.

**Remark 25** We have also included this discussion since it makes plausible that there is no chance to integrate a poor theory of room acoustics in a valid music-theoretic framework, and that great efforts should be made to lift this status in order to give the compositions with room acoustical specifications a firm background.

[7] See section A.2.2 for the definition of loudness.

### A.1.2 Sound Analysis and Synthesis

This central subject of acoustics relates to the poietic (synthesis) and esthesic (analysis) aspects of neutral sound data, more specifically, of the sound pressure variation function $p(t) = p_q^0(t)$ at a source location $q$ and cast in a determined standard support (see section A.1.1). More precisely, we are considering a time function $p$ which is defined for all times, vanishes outside a finite time interval, e.g., $[0, 1]$, and has a finite absolute amplitude supremum $sup(|p|)$, e.g., $sup(|p|) = 1$.

Sound synthesis means that we have exhibited an operator $\sigma$ which for each sequence of its finite or infinite number of numeric (mostly real-valued) parameters $x_1, x_2, \ldots$ yields a function $p = \sigma(x_1, x_2, \ldots)$ of the described type. Analysis then means that we are given $p$ and the operator $\sigma$ and would like to determine a sequence $x_1, x_2, \ldots$ of arguments such that $p = \sigma(x_1, x_2, \ldots)$. The general map $\sigma : (x_1, x_2, \ldots) \mapsto \sigma(x_1, x_2, \ldots)$ will be neither surjective nor injective. So synthesis is neither a synthesis of any imaginable $p$, nor is analysis unambiguous, i.e., the fiber of $\sigma$ could be a large set of parameters. Moreover, one may have a synthesis operator $\sigma_1$ and an analysis operator $\sigma_2$ such that the analysis of a tame synthesis may become pathological.

In the following sections, we shall discuss four such operators: Fourier, frequency modulation, wavelets, and physical modeling. We shall however not deal with mixed synthesis/analysis problems which, mathematically speaking, are wild ones—let alone the associated technological problems.

#### A.1.2.1 Fourier

The Fourier approach deals with periodic functions, in our case periodic pressure variations $p(t)$ as functions of time $t$. This however is not the direct approach to produce a pressure function which is inserted into the support operator, since such a $p$ has an infinite support. In order to turn a periodic function $f$ into one with a standard support $Support(1, 0, 1)$, say, it is usually multiplied by an *envelope* function $H$, see figure A.2. This is a continuous, piecewise differentiable[8] non-negative function on $\mathbb{R}$ which fits in the standard support $Support(1, 0, 1)$, i.e., $H(t) = 0$ outside $[0, 1]$ and $\|H\|_\infty = Max(H) = 1$. In most technological applications, $H$ is even a spline function (often even a linear, i.e., polygonal spline) modeling the attack and decay of a sound event. Then, the pressure function is given by $p = w.H$. Observe that this is already a source of poietic ambiguities: neither $w$ (not even its frequency), nor $H$ are uniquely determined by $p$. We then say that the standardized pressure function $p$ is defined by the envelope $H$ and the *wave* $w$.

Fourier's theorem deals with periodic wave functions $w$ which are *piecewise smooth*[9] waves. For any piecewise smooth function $w : \mathbb{R} \to \mathbb{R}$, the additive group $\mathbb{R}$ acts by translations: $(e^P.w)(t) = w(t + P)$. The group of periods $Periods_w$ of $w$ is the isotropy group of $w$ under this action. For any non-zero period $P$, the inverse $f_P = 1/P$ is called a *frequency of* $w$, its unit is *Hertz*, $Hz$. If $P_w = inf(\{P \in Periods_w, 0 < P\}) = 0$, $w$ is evidently constant, otherwise,

[8] Differentiable except for a finite set of points

[9] Continuous, except for a finite set of discontinuities (it need not be defined in these points, but the left and right limits of the functions exist in these points), with a continuous derivative, except for a finite set of points (where the derivative is not continuous or even not defined, but the left and right limits exist in all these singularities). Many examples are plain $C^1$ functions, but the saw-tooth function is not.

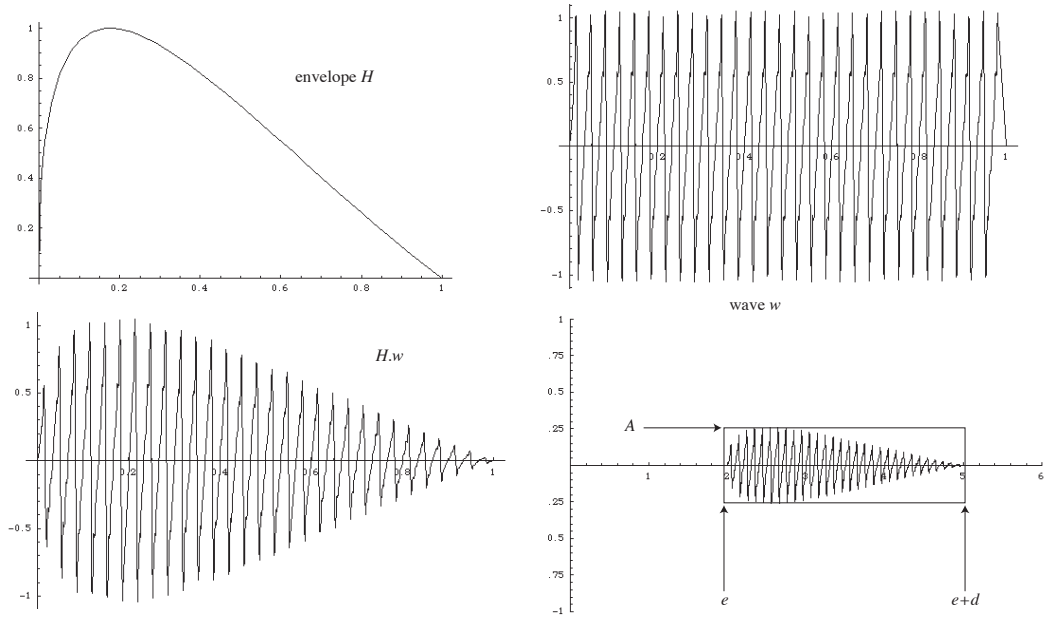



Figure A.2: The envelope $H$ (top left), the wave $w$ (top right), and its combination $H.w$ (bottom left), as well as the affine deformation by the support operator $Support(A, e, d)$ (bottom right).

$Periods_w = \langle P_w \rangle$ is the discrete group generated by the smallest positive period $P_w$. To avoid ambiguities in the periods or frequencies of a wave, one addresses this smallest period $P_w$ or frequency $f_w = 1/P_w$ if one speaks about the "fundamental period" or the "fundamental frequency" of $w$ (otherwise, not even the period or frequency of a wave would be uniquely determined). Fourier's theorem is this (for a proof, see [276]):

**Theorem 37** *If $w$ is a piecewise smooth wave function and $P \in Periods_w$ is a positive period, with $f = 1/P$ the corresponding frequency, then there are two sequences $(A_n, Ph_{n+1})_{n=0,1,2,...}$ of real numbers such that*

$$w(t) = A_o + \sum_{n=1,2,3,...} A_n \sin(2\pi n f t + Ph_n), \tag{A.6}$$

*i.e., the infinite series converges and represents the wave for every time $t$ for all points where the function is continuous. For the given period, the coefficients $A_n$ and $Ph_n$ are uniquely*

*determined and can be calculated as follows:*

$$
\begin{aligned}
A_0 &= f \int_{-P/2}^{P/2} w(t)dt, \\
a_n &= 2f \int_{-P/2}^{P/2} w(t)\cos(2\pi nft)dt, 0 < n, \\
b_n &= 2f \int_{-P/2}^{P/2} w(t)\sin(2\pi nft)dt, 0 < n, \\
A_n &= \sqrt{a_n^2 + b_n^2}, 0 < n, \\
Ph_n &= \arcsin(a_n/A_n) \text{ if } A_n \neq 0 \text{ and } Ph_n = 0 \text{ else.}
\end{aligned}
$$

*The $A_n$ is called the nth* amplitude*, whereas $Ph_n$ is called the nth* phase *of the wave with respect to the selected period. The sequence $(A_n)_n$ is called the* amplitude spectrum*, $(A_n^2)_n$ is called the* energy spectrum *since the energy of a wave is proportional to the square of the amplitude, and the sequence $(Ph_n)_n$ is called the* phase spectrum*. If the period/frequency is the fundamental period/frequency, one omits these specifications.*

An equivalent representation (with coefficients $a_n, b_n$, unique for a given period) is obtained for the explication of the sinoidal components via the goniometric formula

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

and yields

$$w(t) = A_o + \sum_{n=1,2,3,\ldots} a_n \cos(2\pi nft) + b_n \sin(2\pi nft). \tag{A.7}$$

**Remark 26** It is well known [307, Thm. 6.7.2], that the function sequences $(\sin(2\pi nft)_{n=1,2,3,\ldots}$, $(\cos(2\pi nft)_{n=0,1,2,\ldots}$ form an orthogonal basis of the pre-Hilbert space[10] $C^0[-P/2, P/2]$ of the continuous functions on $[-P/2, P/2]$ for the 2-norm (see appendix I.1.2), where $f = 1/P$. This follows in particular from the trigonometric orthogonality relations of the defining scalar product $(f, g) = \int_{-P/2}^{P/2} f(t)g(t)dt$ of functions $f, g \in C^0[-P/2, P/2]$, i.e.,

$$(\sin(2\pi nft), \cos(2\pi mft)) = (\sin(2\pi nft), \sin(2\pi nft)) = (\cos(2\pi nft), \cos(2\pi mft)) = 0 \tag{A.8}$$

for $n \neq m$. There is an infinity of such orthogonal bases for $C^0[-P/2, P/2]$, and mathematically, nothing distinguishes the sinoidal basis chosen by Fourier from the other orthogonal bases. Moreover, sinoidal functions are all but elementary. Mathematically, they are very complex, as is evident from Euler's identity $\cos(x) + i.\sin(x) = e^{ix}$. A justification for using sinoidal waves lies in the fact that simple mechanical differential equations, such as the spring equation $m.\ddot{x} = -k.x$, have sinoidal functions as their solutions. But this is a physical argument which must be coupled with a dynamical system of this equational type in order to give these functions any preference.

[10] A normed real vector space whose norm is defined by a positive definite symmetric bilinear form.

In order to meet the requirement for a unit amplitude, the coefficients of the Fourier representation can be dilated by a common factor, and we are done with the periodic wave. A common generalization of the Fourier representation (A.7) is defined if the frequency and coefficients are also functions of time: $f = f(t), A_n = A_n(t), Ph_n = Ph_n(t)$, a situation which is also needed to represent sounds of physical instruments with glissandi, crescendi, and their natural damping effects.

This construction is the poietic perspective. The esthesic one deals with the problem of constructing an envelope $H$, a periodic wave $w$ and its Fourier representation (A.7) for a given sound function $p$. As was already mentioned above, the wave and the envelope cannot be reconstructed unambiguously in general. Even if the wave is known, the envelope is not reconstructible, although a number of obvious candidates can be calculated, e.g., a polygonal envelope defined by the local maxima and minima of the enveloped wave. As to the wave, one candidate for such can be guessed by the analysis of a time window $[t_1, t_2]$ of $p$ within the supporting duration, such that the local maxima of $p$ are relatively constant (neither at the initial, nor at the decay phase of the sound). One can then take a multiple of the period as a time window, and calculate the Fourier representation of this time window which is interpreted as a finite interval of a really periodic function, i.e., prolongation of this window to infinity. Although this period will not be the fundamental period of the wave, the Fourier representation will yield the right coefficients modulo a multiple of the fundamental frequency. If the fundamental period is small relatively to the total duration of the sound, there is a chance to calculate the underlying wave. In general, this is a highly ambiguous situation.

Once the wave is reconstructed, the Fourier coefficients are uniquely determined by the Fourier theorem, and we are done. But this is only the last phase of a highly ambiguous situation. It is of course always possible to find an underlying wave, it suffices to take the total duration of the support and to set it to the wave's period, i.e., prolongation to infinity of the sound by adding copies of itself to the left and to the right of the sound support. For the relation of these reconstructions to what is heard, see below appendix B.

**Remark 27** A final remark on the terminology of sound frequencies in music. Usually, when we deal with "the frequency of a sound", we do not mean that this frequency is a neutral property of the sound (although it could happen that the sound really has a fundamental frequency), but the fundamental frequency of a wave that is used in the standard representation of the sound as a product of its (periodic) wave and a deformed envelope. *This is a poietic definition, and this is what we will use because a neutral definition does not exist for general sounds.* In this setup, we have the sound function $p = Support(A, e, d)(w_{A.,Ph.}(f).H)$. Here, we may assume that $w_{A.,Ph.}$ is the formal representation of the trigonometric sum by the amplitude and phase spectra, and such that the total amplitude maximum is 1; the frequency is given as an additional argument, and the envelope $H$ is given in its standard support. We call the 4-vector $(e, f, d, A)$ the *geometric coordinates* of the sound, and the pair $(w_{A.,Ph.}, H)$ the *color coordinates* since they are responsible for the sound color (timbre) in this representation. For other poieses of sound which may also use the frequency coordinate (for example those to be discussed in the following sections), this one would also be referred to, but there is no neutral access to this concept. *A word of caution: The way humans "detect" sound frequencies is not neutral, this is an esthesic psycho-physiological system whose function is far from understood, so do never mix up neutral facts with poietic or esthesic facts when dealing with sound attributes!*

#### A.1.2.2 Frequency Modulation

In this section, we have a similar decomposition as discussed before: the sound $p$ is written as a product $p = H.w$, where $H$ is the envelope, and $w$ is a "wave" function. However, this time we do not use the additive combination of sinoidal functions of Fourier synthesis to build $w$. The combination is rather a functional concatenation of such functions, i.e., sinoidal functions have in their arguments other sinoidal functions, and so on. This synthesis operator was introduced by John Chowning [86] and implemented first in the legendary Yamaha's DX7 synthesizers. The formal definition of frequency modulation (FM) functions in terms of circular denotators is given in section 6.7, example 3, and yields this type of expressions:

$$FMsound(myFMObject)(t) = \sum_{i}^{n} A_i \sin(2\pi F_i t + Ph_i + FMsound(myModulator_i)(t))$$

where $myModulator_i$ is the FM-Object factor of the limit type denotator $Knot_i$. In the terminology of FM synthesis, the (respective) interior functions $FMsound(myModulator_i)(t)$ are called the *modulators* with respect to the (respective) exterior sinoidal functions $A_i \sin(2\pi F_i t + Ph_i)$ which are named the *carriers*. We symbolize this relation by an arrow $myModulator_i \Rightarrow myFMObject$. So the FM functions start as a sum of carriers where modulators are inserted, and these modulators are again of this nature, etc., until no modulator appears and the recursion terminates. In FM synthesis it is also allowed to have circular denotators in the sense that a modulator can be a carrier in one and the same function! Observe that the existence of a denotator $myFMObject$ describing such a "self-modulating" function does not imply the existence of the corresponding function $FMsound(myFMObject)$. However in digital sound synthesis, one often resolves this problem by taking the time argument of the modulator one digital step before the time argument of the carrier, i.e.,

$$FMsound(myFMObject)(t_n) = \sum_{i}^{n} A_i \sin(2\pi F_i t_n + Ph_i + FMsound(myModulator_i)(t_{n-1})).$$

In the DX7 implementation, the FM recursion scheme is presented in the graphical block diagram format of a so-called *algorithm* (not an adequate wording, though); figure A.3 shows Yamaha's 32 algorithms.

Each sinoidal carrier component is written as a block, whereas the modulators of a given carrier are those blocks which are above the carrier and are connected by a line with this carrier. Each of the building blocks can be specified in the respective parameters.

The point of the FM synthesis is that it needs a small number of sinoidal functions—in fact only six sinoidal oscillators are needed for the DX7 algorithms—to simulate complex instrumental sounds, which is a theoretical and technological advantage.

But the elegant FM synthesis has also drawbacks concerning the uniqueness question: How many different denotators $myFMObject$ do yield the same function $FMsound(myFMObject)$? A general answer seems difficult, we have two partial solutions. The first regards a tower of modulators:

**Lemma 54** *Let $G : \mathbb{R} \to \mathbb{R}$ be a bounded $C^1$ function. Then $G$ is determined by its value $G(0)$ and by any function $F : \mathbb{R} \to \mathbb{R}$ of the shape $F(t) = a + b\sin(ct + G(t))$, with $b \neq 0$.*

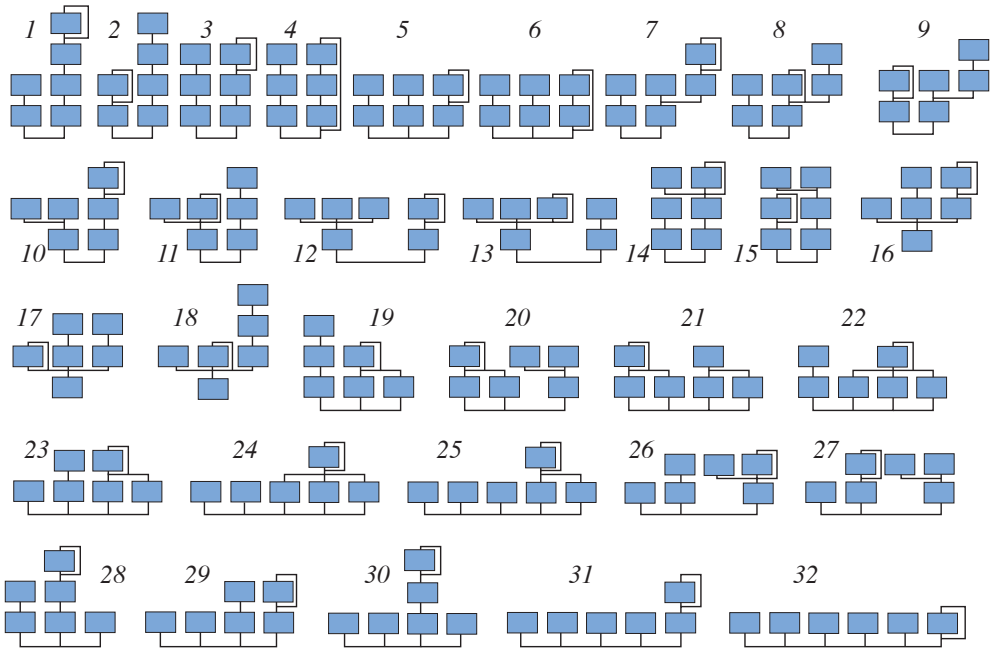



Figure A.3: Yamaha's 32 algorithms for FM synthesis. Only six sinoidal oscillators are required to generate a variety of more or less natural sounds.

Proof. Evidently, by the boundedness of $G$, $a+b = max(F), a-b = min(F)$, so $a = (max(F)+min(F))/2, b = (max(F)-min(F))/2$, are determined by $F$. Moreover, the continuous function $G' = \frac{bF'}{\sqrt{b^2-(F-a)^2}} - c$ is a function of $F$, and the value $G(0)$, together with the integral of $G'$ which is a function of $F$, completely determine $G$, QED.

**Proposition 61** *If $M = (M_n \Rightarrow M_{n-1} \Rightarrow M_{n-2} \Rightarrow \ldots M_0)$ is an FM denotator defined by a sequence of relative modulators for the functions $M_i = A_i \sin(2\pi f_i t+?)$, then the resulting sound function $FMsound(M)$ determines all the modulators.*

Proof. This follows from lemma 54 by recursion on the modulators, starting with total function $FMsound(M)(t) = A_0 \sin(2\pi f_0 t + FMsound(M_1))$, where $M_1$ is the denotator starting from $M_1$ instead of $M_0$. In the lemma, take $F = FMsound(M), G = FMsound(M_1)$, and observe that $G(0) = 0$, QED.

The second solution regards a flat sequence of unrelated carriers:

**Proposition 62** *Let $F(t) = \sum_{i=1,\ldots k} a_i \sin(b_i t)$ with $0 < b_1 < \ldots b_k, 0 < a_i$ be the function associated with a flat FM denotator. Then the coefficients, and therefore the denotator, are all uniquely determined by $F$.*

Proof. We have $(-1)^n F^{2n+1}(0) = \sum_{i=1,\ldots k} a_i b_i^{2n+1}$ for all $n \geqq 1$. Consider the function $H(a.,b.)(x) = \sum_{i=1,\ldots k} a_i b_i^x$ of the real variable $x$. Suppose we have two sequences $(a_i)_{i=1,\ldots k}$, $(b_i)_{i=1,\ldots k}$ and $(a_i')_{i=1,\ldots k}, (b_i')_{i=1,\ldots k}$ such that they yield the same function $F$. Then we have $H(a.,b.)(x) = H(a'.,b'.)(x)$ for all $x = 2n+1, n \geqq 1$. Write $b_i = e^{\beta_i}, b_i' = e^{\beta_i'}$, and suppose $b_k < b_k'$. Then

$$\frac{H(a.,b.)(x)}{H(a'.,b'.)(x)} = \frac{\sum_{i=1,\ldots k} a_i e^{\beta_i x}}{\sum_{i=1,\ldots k} a_i' e^{\beta_i' x}} = \frac{\sum_{i=1,\ldots k} a_i e^{(\beta_i-\beta_k)x}}{\sum_{i=1,\ldots k} a_i' e^{(\beta_i'-\beta_k)x}} = \frac{\sum_{i=1,\ldots k-1} a_i e^{(\beta_i-\beta_k)x} + a_k}{\sum_{i=1,\ldots k} a_i' e^{(\beta_i'-\beta_k)x}}.$$

But for $x \to \infty$, the denominator goes to 0 whereas the numerator goes to $a_k$, contradicting the fact that this quotient is 1 for all $x = 2n+1, n \geqq 1$. Therefore $b_k = b_k'$, so in the above quotient, the limit for $x \to \infty$ is $a_k/a_k'$ which is also 1, and we have equal coefficients for the index $k$. Therefore, we may proceed by induction to $k-1$ and we are done, QED.

These very special results show that given the tower or the flat FM schemes, the functions determine their coefficients (the background denotators) uniquely, but for general FM schemes, there is no such a result. Moreover, if the FM scheme is not known, we have no idea of how the scheme should be determined from the function. This is a drawback compared to the Fourier operator, where the coefficients are always uniquely determined once the fundamental frequency is fixed. In other words, FM synthesis is much more efficient than Fourier synthesis, but one has to pay for this when turning to the respective analyses.

The switch between Fourier and FM operators is essentially managed by Bessel functions. These are defined directly from a core situation from FM synthesis, i.e.,

**Definition 111** *Let $z$ be a real number. Then the Fourier expansion of the $2\pi$-periodic function $\sin(z\sin(t))$ of $t$ is*

$$\sin(z\sin(t)) = 2\sum_{n=0,1,2,\ldots} J_{2n+1}(z) sin((2n+1)t), \tag{A.9}$$

*whereas the Fourier expansion of the $2\pi$-periodic function $\cos(z\sin(t))$ of $t$ is*

$$\cos(z\sin(t)) = J_0(z) + 2\sum_{n=1,2,3,\ldots} J_{2n}(z) cos(2nt). \tag{A.10}$$

*The functions $J_m, m = 0,1,2,\ldots$ are called the $m$th Bessel functions.*

The above definition relies on the Fourier representation of the respective functions and on their properties as odd or even functions. An alternative definition of Bessel functions is $J_m(z) = \frac{1}{\pi}\int_0^\pi \cos(mt - z\sin(t))dt$.

From definition 111, one obtains the following fundamental equations for Bessel functions:

$$\sin(r + z\sin(s)) = \sum_{-\infty}^{\infty} J_n(z)\sin(r+ns),$$

$$\cos(r + z\sin(s)) = \sum_{-\infty}^{\infty} J_n(z)\cos(r+ns).$$

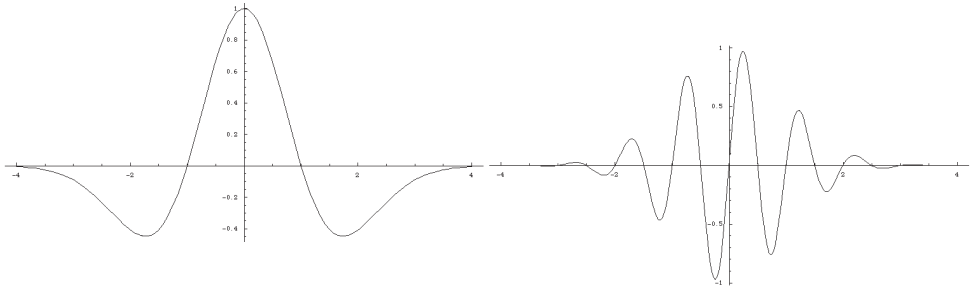
In particular, with $s = 2\pi ft, z = I, t = 2\pi gt$, we have

$$\sin(2\pi ft + I\sin(2\pi gt)) = \sum_{-\infty}^{\infty} J_n(I)\sin(2\pi(f + n.2\pi g)t),$$

which is a Fourier type linearization, however, it is not a proper Fourier representation since the so-called "$n$th side band" frequencies $f + n.2\pi g$ are not a multiple of a fundamental frequency in general. This is rather a reduction of a FM concatenation to a flat FM configuration. Conversely, every (finite) Fourier decomposition is evidently a flat FM configuration.

#### A.1.2.3 Wavelets

Although the FM operator is much more efficient than the Fourier operator, it is still an operator which produces functions with an infinite support, a property which no real sound shares, and we have in fact added an envelope to cope with this requirement for Fourier and FM operators. From this point of view, wavelets are fundamentally better suited for handling finite sound objects without any envelope casting. Refer to [308] and [279] for the wavelet theory and its applications to sound and music. Let $f$ be a square integrable function (element of $L^2(\mathbb{R})$, i.e.,



(1 - t^2)Exp[-t^2/2] Sin[2 Pi t]Exp[-(t)^2/2]

Figure A.4: Two wavelets: Murenzi's Mexican hat [387] (left) and Morlet's wavelet (imaginary part) [192] deduced from the sinoidal function (right).

$\int_{\mathbb{R}} |f(x)|^2 dx < \infty$). Then its Fourier transform is defined and is the function

$$\hat{f}(\omega) = (2\pi)^{-1} \int_{\mathbb{R}} f(x)e^{-ix\omega}dx. \tag{A.11}$$

**Definition 112** *A square integrable function $\psi$ is called a* wavelet *if* $0 < c_\psi = \int_{\mathbb{R}} \frac{|\hat{\psi}|^2}{|\psi|} d\psi < \infty$. *For a wavelet $\psi$, the* wavelet-transformed *of a square integrable function $f$ is the function (of two variables $a, b$)*

$$L_\psi f(a,b) = c_\psi^{-1/2}|a|^{-1/2} \int_{\mathbb{R}} f(t)\psi(\frac{t-b}{a})dt, \tag{A.12}$$

*with $a \in \mathbb{R} - \{0\}, b \in \mathbb{R}$.*

Figure A.4 shows two typical examples of wavelets. The wavelet transform is a function of two variables defined on every couple $(a, b) \in (\mathbb{R} - \{0\}) \times \mathbb{R}$. The point of this representation is that it is a kind of system of coefficients $(L_\psi f(a, b))_{(a,b)\in\mathbb{R}-\{0\}\times\mathbb{R}}$ which is parametrized by two real numbers, corresponding to a scalar product

$$c_\psi^{-1/2}(f, Support(|a|^{-1/2}, b, a)(\psi)) \tag{A.13}$$

of the function $f$ with an affinely deformed version[11] $Support(|a|^{-1/2}, b, a)(\psi)$ of the "mother" wavelet $\psi$. This deformation $Support(|a|^{-1/2}, b, a)$ is an isometry on the square integrable functions, see also figure A.5 for some deformed wavelets. By the following formula, the wavelet

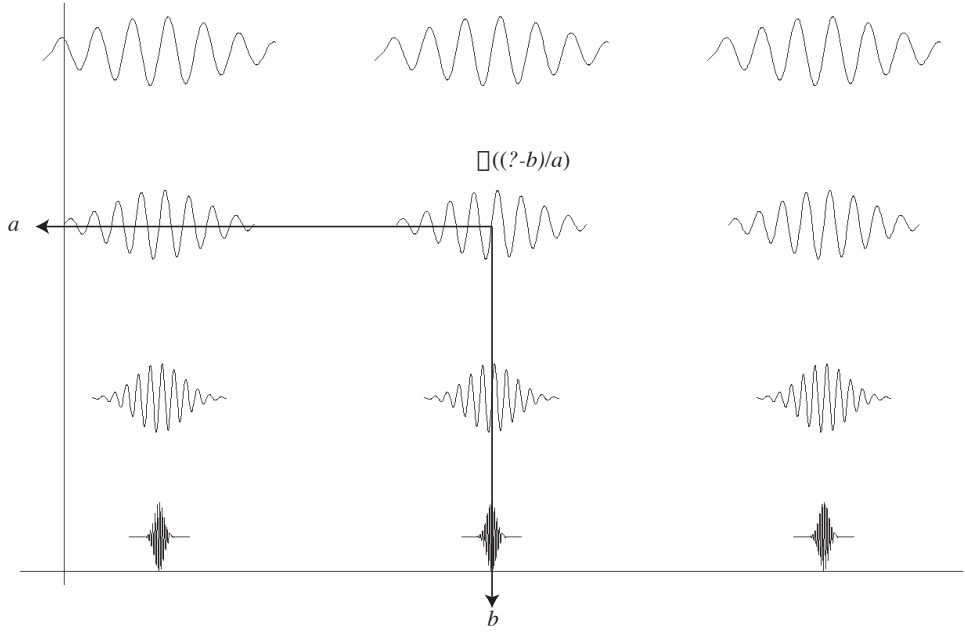



Figure A.5: Various affine deformations of the mother wavelet $\psi$.

transformations $L_\psi f(a, b)$ redetermine the original function $f$:

$$f(t) = c_\psi^{-1/2} \int_{\mathbb{R}^2} L_\psi f(a, b) Support(|a|^{-1/2}, b, a)(\psi)(t) \frac{da\ db}{a^2}. \tag{A.14}$$

For selected wavelets, it is possible to generate an orthonormal basis of $L^2(\mathbb{R})$ which is defined as a so-called *frame*. For $a_0 > 1, b_0 > 0$, and for the Meyer wavelet[12] $\psi$ we consider the frame $(Support(a_0^{-m/2}, nb_0a_0^m, a_0^m)(\psi))_{m,n\in\mathbb{Z}}$ of deformed versions of $\psi$. Then this is an

[11] Observe that the operator $Support(|a|^{-1/2}, b, a)$ defines a linear action $Support : \overrightarrow{GL}(\mathbb{R}) \to GL(L^2(\mathbb{R}))$ : $e^b a \mapsto Support(|a|^{-1/2}, b, a)$ of the affine group $\overrightarrow{GL}(\mathbb{R})$ on the space $L^2(\mathbb{R})$ of square integrable functions.

[12] See [308, 2.1.25].

orthonormal basis of $L^2(\mathbb{R})$ for $a_0 = 2, b_0 = 1$. This is an analogous situation as encountered for Fourier series and their sinoidal bases with the known trigonometric orthogonality relations in formulas (A.8).

For a comparison of Fourier and wavelet analysis, see [560].

#### A.1.2.4 Some Remarks on Physical Modeling

The previous operators were directly acting on the production of a sound function. Their poietic nature was a mathematical one, just to construct a time function $p(t)$ by a mathematical procedure from a certain type of "atomic", i.e., basis functions. In contrast, physical modeling is one step more poietic in that it does not directly deal with sound, but with a physical system that produces sound. On the one hand, this is a strong restriction since sound does not care for the physical system that evokes that sound. It seems however that this drawback is compensated by the fact that musical expressivity is largely determined by the physical device which the artist manipulates when interpreting or improvising music. It is also an important argument that the simulation of the physical instrument and then a possible canonical extension could yield more interesting sounds than just "abstract nonsense" procedures. For a general survey on physical modeling, see [460, chapter 7], we restrict our discussion of this extensive topic to some systematic remarks.

The idea is that one considers a physical model of a sound production device and then implements this model as a software which—if sufficient calculation power is available—calculates the physical output on the level of the sound wave that is emitted by the modeled instrument. At present, there are three methodologies for such a modeling: mass-spring, modal synthesis, and waveguide. The mass-spring paradigm just models a physical instrument (a string, a drum) by a finite space configuration of point masses that are related by springs and damping effects [225]. The modeling is built upon the classical mechanics of Newton's law and the corresponding dynamic behavior that eventually terminates in the air's vibration.

The modal synthesis paradigm [77] reduces the vibrating physical system to a system of vibrating substructures, usually very small in number compared with the mass-spring components. These substructures are characterized by their frequencies, damping coefficients and parameters for the vibrating mode's shape. This adds up to a sum of modal vibrations. Whereas in simple configurations these data can be obtained from classical literature in equations for vibrating systems, the complex data must be extracted from experimental results. A prototypical implementation of this paradigm is MOSAIC, developed by Jean-Marie Adrien and Joseph Morrison, see [460, p. 276 ff.].

The waveguide paradigm has been implemented in commercial physical modeling synthesizers by YAMAHA and KORG , see [460, p. 282 ff.]. It has mainly been developed by Julius O Smith III and collaborators [494, 495]. The waveguide model implements the traveling wave along a medium, such as a tube or a string. See [496] for an update of physical modeling strategies.

Although physical modeling is a successful approach in the simulation of musical instrumental sounds, it is a step back from the neutral sound objects to their generators. This has not only been a technological requirement for performance theory (where physical modeling is a core approach), it is also a consequence of the failure in the understanding of the topological semantics of sound objects as exposed in section 12.3. In particular, it is an open problem to

relate the physical modeling theory and technology to the other operators, such as Fourier, FM, and wavelets. The deeper question here is whether the neutral sound objects are really the most relevant ingredients of musical performance, i.e., how strongly the gestural components influence and characterize the sounding reality. It is not clear how deeply a sound conveys the generating gestures in its autonomous structure. To our knowledge, sound classification has not been directed towards a gestural coordinate in the neutral sound description, except, perhaps, in the straightforward envelope component.

## A.2 Mathematical and Symbolic Spaces

The physical description of sounds is not what can be used for music theories. This is based on (1) the way humans perceive sounds, (2) the shape of music thinking, and (3) available instrumental technologies. Therefore physical parameter spaces must be transformed into spaces which essentially encode the same information, but do so in a way which is more adapted to music. Basically, we shall present mathematical structures where physical parameters are represented. Based upon these spaces, we shall explain a derived set of representations which encode different music-topographic aspects. Our discussion regards the geometric coordinate pairs (basis coordinate plus associated pianola coordinate) onset and duration (A.2.1), amplitude and crescendo (A.2.2), and frequency and glissando (A.2.3).

### A.2.1 Onset and Duration

In music, one speaks about tempo, metronome, quavers, semiquavers, triplets, 3/4 meter, etc. The relation to the physical time parameters is as follows. To begin with, the *physical* onset time $e$ *sec* and duration $d$ *sec* are opposed to a *musical* onset time $E$ *note* and duration $D$ *note*, usually also in real values, and in units such as "*note*", meaning a whole note. However, in many practical contexts, the rational number field $\mathbb{Q}$ will do. In this latter context, we have the ratios of integer numbers $E = w/n, D = z/n, n > 0$, and the denominator is of the form $n = 2^r 3^s 5^t 7^u$ with natural exponents $r, s, t, u$, whereas the numerators $w, z$ are integers, i.e., we are working in the ring localization[13] $\mathbb{Z}[1/2, 1/3, 1/5, 1/7]$ at the primes $2, 3, 5, 7$. Musically, this means that one is allowed to add, subtract, multiply such numbers at will without leaving the domain, and also division of such numbers by $2, 3, 5, 7$ leaves the domain invariant, i.e., the construction of duplets, triplets, quintuplets, septuplets is possible without any restriction.

The musical time shares a mental reality and should not be confused with physical time. Genetically, musical time is an abstraction from physical time, but they are by no means equivalent. This abstraction is also a creation of an autonomous time quality where mental constructions such as a score can be positioned. The relation between these two time qualities is defined by the tempo, usually encoded as a metronomic indication of $x$ quarters per minute according to Mälzel, meaning a difference quotient (velocity) $\Delta E/\Delta e$ of musical time per physical time. This presupposes that we are given a one-to-one performance mapping $E \mapsto e(E)$ from musical time to physical time. *It is common in mathematical music theory to write physical parameters in lower case letters, whereas musical parameters are written in upper case letters.*

[13] Make the integers $2, 3, 5, 7$ invertible, i.e., admitting fractions $1/2, 1/3, 1/5, 1/7$. But see appendix E.4.1 for a formal definition.

Let $x, y$ be positive integers. Then every onset $E$ can be written uniquely as $E = \delta + \tau.x/y$, where $0 \leq \delta < x/y$ is in $\mathbb{Q}$, and $\tau \in \mathbb{Z}$. If $\delta = q/(ny)$, we say that *in $x/y$ time*[14]*, $E$ is on the $(q+1)$st ny-tuplet in bar $\tau + 1$*. The additive group $\mathbb{Z}.1/y$ is called the *meter* of the $x/y$ time. Evidently, this initializes a score at time $E = 0$, but this is pure convention. Pay attention *not* to view the symbol $x/y$ as a plain fraction, but as a pair $x, y$ giving rise to a fraction. In other words, in music notation, the symbol $x/y$ is the mathematical fraction $x/y$ *plus* the meter.

**Example 61** Let $x = 3, y = 4, E = 15.375$. Then we have $E = 3/8 + 20.3/4$, i.e., $E$ is on the fourth quaver of bar 21.

If we work in $\mathbb{Z}[1/2, 1/3, 1/5, 1/7]$, then with $E$ and $1/y$, the remainder $\delta$ and the meter are automatically in this domain. If we are given a duration $D$, then durations of form $D/n$ are also called $n$-tuplets (with respect to $D$).

The reason why the concept of onset time is not common in music lies in the fact that a note's onset can be deduced from its position in the score. The time signature, bar-lines, and simultaneous or preceding notes help establish the time context of each note. This helps calculate onsets algorithmically by recursion from the first score onset. This algorithm is derived from the linear syntagm of written language, but it sometimes leads to ambiguities.

### A.2.2 Amplitude and Crescendo

The amplitude $A$ of a sound relates to the loudness sensation, i.e., to musical dynamics. However, the chain of transformations to the physiologically relevant measures is quite complex [462]. The first member of this chain is the transformation which associates $A$ with the *sound pressure level* (SPL)

$$l(A) = 20. \log_{10}(A/A_{threshold})\ dB. \tag{A.15}$$

Here, $A_{threshold} = 2.10^5\ Nm^{-2}$ is close to the amplitude of the threshold pressure variation for a 1 $kHz$ sound in a young, normal hearing human subject. The unit $dB$ for $l$ is *Dezibel*. Of course, these constants are pure convention. From the auditory physiology, the shape $l(A) = a\ln(A) + b$ is essential, i.e., $l(A)$ is a conventionally normed linear function of the logarithm of the amplitude. The physio-psychological motivation for this approach lies in the Weber-Fechner law according to which the sensation of a difference of sensory stimuli is proportional to the stimuli [242], yielding the logarithmic representation as an adequate encoding of this sensorial modality. Figure A.6 shows some environmental SPL values (the musical units *ppp, pp*, etc. will be discussed below). For example, the SPL $\approx 120\ dB$ of an air jet is one million times the threshold SPL.

Just as duration is the time interval from the onset time to the "offset" time of a sound event, the *crescendo* parameter $c$ is the difference between the onset SPL and the offset SPL. In normal piano sounds, this vanishes, but for violins, trombones, etc., this is a relevant quantity. In this context, one supposes that the sound representation $p = Support(A, e, d)(w_{A.,Ph.}(f).H)$ has a time-dependent amplitude $A = A(t)$. At the offset time $e + d$, the amplitude has changed by the amount $A(e+d)/A(e)$, or in terms of loudness, $c = l(A(e+d)) - l(A(e))$. As with onset and duration, it is not specified what happens in the sound process between the onset and the offset, it's just the difference that matters. This difference $c$ is termed the (physical) *crescendo*.

[14] We stick to the continental terminology since "meter" will be reserved for a different concept.

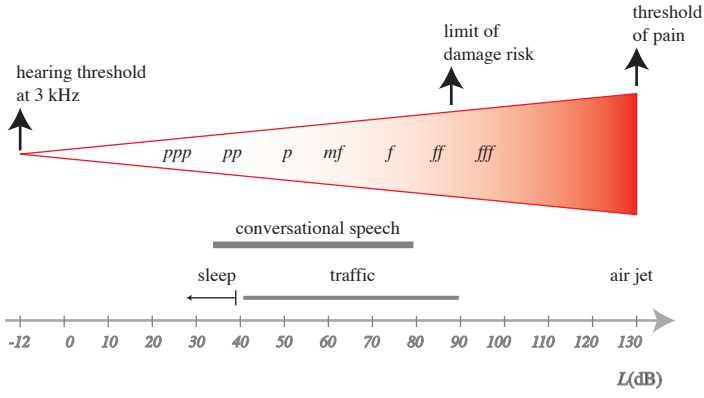



Figure A.6: Some environmental sound pressure level values.

In the musical abstraction $L$ for loudness, one also works in the field $\mathbb{R}$ of real numbers, but the common score notation is not well defined for all real values. More precisely, the first still physically motivated codification is a conventional calibration such as, for example, $l : \mathbb{Z} \to \mathbb{R}$ with $l(L) = 10\ dB.L + 60\ dB$ such that the values

$$\begin{array}{ll} & 0 \mapsto mf \\ 1 \mapsto f & -1 \mapsto p \\ 2 \mapsto ff & -2 \mapsto pp \\ 3 \mapsto fff & -3 \mapsto ppp \end{array}$$

are arranged symmetrically around the *mezzoforte* sign. This looks like an identification of the score symbols with precise physical values. But this is wrong, it is a transformation from mental to physical reality. In fact, the affine transformation $l(L)$ is only a "default" assignment of a mental quantity symbolized by a dynamic symbol (which is codified by an integer). This fact is more evident if one recalls the velocity parameter in the MIDI code. This quantity is an integer in the interval $[0, 127]$, but the physical meaning of a velocity value depends on the assignment by specific technological calibrations, in particular by the output chain where the loudspeakers can take values that have nothing to do with the velocities. Units in the mental level of loudness are not standard, but we could, for example, take *vel* for MIDI's velocity parameter.

The extension of integer values to real or rational values (depending on the specific usage) is a reasonable procedure, for instance preconized by MIDI velocity codification. It is also useful for finer loudness data management with relative loudness signs such as "crescendo", "diminuendo" which make sense in a mental crescendo parameter which we shall denote by $C$. See also our discussion of performance transformations and primavista weights in section 39.2.

### A.2.3 Frequency and Glissando

Sounds which share a frequency parameter $f$ are musically important, at least in the European tradition. However, the corresponding mathematics is a bit more involved than for time and amplitude. This is an expression of an intense discussion of traditional harmonies and interval theories with an evolving instrumental technology.

To begin with, $f$ behaves like amplitude: For reasons of auditory physiology (see also Appendix B), $f$ is transformed via the logarithmic formula $h(f) = u\ln(f) + v$, yielding the *physical pitch* $h$ of frequency $f$. The common unit of pitch is the *Cent* $Ct$, it corresponds to the logarithm of a relative frequency increase by the factor $2^{1/1200}$, i.e., one percent of a well-tempered semitone (see below for tuning types); this entails $u = 1200/\ln(2)$, but the constants $u, v$ are purely conventional. Presently, and in the Western framework, the relevant frequencies have the shape $f = 132\ Hz.2^p.3^s.5^t$, where $p, s, r \in \mathbb{Q}$. It is based on the *chamber pitch* $440\ Hz$ of the one-line $a$, as fixed in London 1939. Instead, we have chosen the (unlined) $c$ with the frequency $132\ Hz = 440\ Hz.2^{-1}.3.5^{-1}$ as a starting frequency in order to relate the examples with ease to $c$. Mathematically, the restriction to the first three primes $2, 3, 5$ is not essential. One could as well take any sequence $p_1, p_2, \ldots$ of mutually prime natural numbers (larger than 1), the rational powers of which are multiplied by $132\ Hz$.

The natural logarithm of $f$ is

$$\ln(f) = p.\ln(2) + s.\ln(3) + r.\ln(5) + \ln(132).$$

But music theory is rather interested in the relative pitch, i.e.,

$$\ln(f) - \ln(132) = p.\ln(2) + s.\ln(3) + r.\ln(5).$$

Moreover, the passage to another logarithmic basis $b$ could be desirable, i.e.,

$$\ln(b)^{-1}.\ln(f) - \log_b(132) = p.\log_b(2) + s.\log_b(3) + r.\log_b(5),$$

so that we are given the linear function

$$u.X + v = \ln(b)^{-1}.X - \log_b(132)$$

for $X = ln(f)$. For music theory, the restriction to rational exponents is essential, since this hypothesis enables an unambiguous representation, see Appendix E.2.1. Every frequency $f$ can be replaced by a point $x = (p, s, r) \in \mathbb{Q}^3$. Such a point represents the frequency $f(x) = 132\ Hz.2^p.3^s.5^r$, and from $f(x) = f(y)$, we conclude $x = y$. With this interpretation, $\mathbb{Q}^3$ is called the *Euler space* and a point $x = (p, s, r) \in \mathbb{Q}^3$ is called an *Euler point*. This means that a real number $p.\log_b(2) + s.\log_b(3) + r.\log_b(5)$ is viewed as a vector which is a rational linear combination of linearly independent vectors $\log_b(2), \log_b(3), \log_b(5)$.

The choice of the three first primes stems from the tradition of just tuning, where for two frequencies $f, g$, we have:

$(O) f/g = 2/1$ : $f$ is the *octave* (frequency) for $g$.
$(Q) f/g = 3/2$ : $f$ is the (just) *fifth* (frequency) for $g$.
$(T) f/g = 5/4$ : $f$ is the (just) *major third* (frequency) for $g$.

This is why 2 is associated with the octave, 3 with the fifth, and 5 with the major third. We therefore call $p$ the *octave coordinate*, $r$ the *fifth coordinate*, and $s$ the *third coordinate* (of the sound with frequency $f$ or of the corresponding Euler point. The Euler point $o = (1, 0, 0)$ is called the *octave point*, the point $q = (0, 1, 0)$ is called the *fifth point*, and $t = (0, 0, 1)$ is called the *third point*. Observe that fifth and third points are not the fifth and octave. The structural meaning of these points is explained in section 6.4.1. The minor just third frequency 6/5 does not add a new prime number.

For a fixed chamber pitch, the *2-3-5 just tuning* (short: just tuning) is the set of frequencies which are represented by integer coordinates, i.e., the subgroup $\mathbb{Z}^3 \subset \mathbb{Q}^3$ of the Euler space. This is the three-dimensional grid which was introduced by Leonhard Euler [143]. The group of those grid points whose third coordinate vanishes is the *Pythagorean tuning*.

The Euler space is derived from the logarithmic representation of pitch, but the coefficients are beyond physical reality, this is why we view this space as a mental space. The same is valid if we consider the codomain space of pitch $h$, but interpreted with rational coefficients (written as $\mathbb{R}_{[\mathbb{Q}]}$ instead of the usual real line $\mathbb{R}$, the one-dimensional real vector space[15]). More precisely, one should name $H$ the pitch on the mental pitch space $Pitch$, and $h$ the pitch on the physical pitch space $PhysPitch$. Whereas the unit for physical pitch is $Cent$, the unit for mental pitch could be $Semitone$ or MIDI's $key$(number), but no standard exists here. In performance, we have a transformation $\wp : Pitch \to PhysPitch$. Both spaces have the real numbers as underlying sets, but the meaning of the spaces is different. The mental pitch space encodes pitch as it is symbolized on the score or as a key number in MIDI code. Ideally, performance transforms this abstract data into mathematical pitch $Math(H) = (p, s, r)$ in the Euler space, and this is transformed to the pitch $\ln(f) = p.\ln(2) + s.\ln(3) + r.\ln(5) + \ln(132)$ in $\mathbb{R}_{[\mathbb{Q}]}$. Only after forgetting about the coefficients, we are in $PhysPitch$. And this is by no means a formal play: It is a dramatic change of reality if one views the reals as an infinite-dimensional space with linearly independent octave, fifth, and third logarithm vectors, or as a line, where everything shares the same direction! In analogy to onset and loudness, one introduces physical glissando $g$ and its symbolic counterpart $G$.

In the common visualization of the Euler space, all grid cells are shown as cubes. This could suggest that angles and distances are relevant to this space. So far, this has however no musicological reason, and it is nonsense to argue with angles and distances as Arthur von Oettingen [406], Carl Eitz[137], and Martin Vogel [547] have done.

Evidently, the mathematical structure of just tuning is independent of the historical choice of the sequence 2,3,5. One could as well take any pairwise prime positive numbers $1 < p_1 < p_2 < p_3$ and would get a $p_1$-$p_2$-$p_3$ just tuning.

If the pitch range in Euler space is described by non-integer coefficients, one speaks of *tempered tunings*. The most well known are defined by a uniform construction mode. For a natural number $w > 1$ one considers all pitches whose octave coordinate is a fraction of shape $p = x/w$, whereas the other coordinates vanish. This is called the *$w$-tempered tuning*. This defines a grid $\mathbb{Z}.1/w.o \xrightarrow{\sim} \mathbb{Z}$ with step width $1/w$ of the octave point. By the same recipe, tempered tunings in fifth and third direction can be defined. The 12-tempered tuning is the famous "well-tempered" tuning. The 1200-tempered tuning is less interesting for conventional composers than for measurement techniques, where the unit step is the $Ct$ step defined above.

[15] See Appendix E.2.1.

More generally, one may define tunings which consist of tempered and just components. The procedure runs as follows. Take three positive integers $w_1, w_2, w_3$ and consider the grid

$$\mathbb{Z}.\frac{1}{w_1}.o + \mathbb{Z}.\frac{1}{w_2}.q + \mathbb{Z}.\frac{1}{w_3}.t$$

which specializes to the 2-3-5 just tuning as well as the tempered tunings as defined above. Call this construction the $w_1$-$w_2$-$w_3$ *just-tempered tuning*. Historically relevant is the *mediante* tuning, which is the 1-1-2 just-tempered tuning, and which includes the tempered whole-tone step in the major third.

With respect to the auditory psycho-physiology (see Appendix B), we should consider the distribution of the just tuning grid vectors $x = (p, s, r)$ in $\mathbb{Z}^3$ with respect to the mathematical pitch $H(x) = p.\log_b(2) + s.\log_b(3) + r.\log_b(5)$. More generally, take any vector $x \in \mathbb{R}^3$ and denote by $H_{prime} = (\log_b(2), \log_b(3), \log_b(5))$ the *prime vector*. This means that pitch is the usual scalar product

$$H(x) = (H_{prime}, x) \tag{A.16}$$

of the prime vector with the generalized Euler point $x$. Therefore,

**Proposition 63** *With the definition* (A.16) *two vectors* $x, x' \in \mathbb{R}^3$ *have the same pitch* $H(x) = H(x')$ *iff their difference is orthogonal to the prime vector* $H_{prime}$.

Call $E = H_{prime}^{\perp}$ the plane orthogonal to the prime vector. Then the proposition means that for any generalized Euler point $x$, $x + E$ is the set of points with same pitch as $x$. Now, according to what we know, $E$ lies so skew in $\mathbb{R}^3$ that for a point $x \in \mathbb{Q}^3$,

$$(x + E) \cap \mathbb{Q}^3 = \{x\}.$$

Nonetheless, every real number $\phi$ can be approached within any given error by points of the just tuning grid: For any positive bound $\delta$, there is $x \in \mathbb{Z}^3$ such that $|H(x) - \phi| < \delta$, see Appendix 73 for a proof. In particular, we have the following proposition which has dramatic consequences for theories of hearing of just tuned pitch (see section B.2):

**Proposition 64** *If* $\phi = H(x_0)$ *is a mathematical pitch of a grid point* $x_0$*, then for any positive bound* $\delta$*, there is an infinity of grid points* $x$ *such that* $|H(x) - H(x_0)| < \delta$.

# Appendix B

# Auditory Physiology and Psychology

"Music listening" is a metonymy of understanding music: For all participants, the ear functions as an interface for perceiving music between physical, psychological, and mathematical reality. But a metonymy is not the matter as such. This is what deaf Beethoven teaches us impressively: his innermost ear was an organ of imagination that was uncoupled from the material ear. And the physiology of the hearing process teaches us that the neural coupling of the ear to the respective cortical regions is extremely complex and still hardly understood. Section B.1 is written to give an overview on auditory physiology.

Beyond receptive processes hearing means also an active shaping according to templates of esthesis and poiesis. One of the difficult basic problems concerning the activity of hearing deals with the compatibility of these templates and the physical input, and in particular the notorious "straightened out hearing" (German: "Zurechthören") which keeps alive a lot of wishful thinking in music theory. Basically, the problem is that we do not perceive physical sounds, but classes of indistinguishable sounds, what Werner Meyer-Eppler coined "valences" in [372]. Valence theory, which is sketched in section B.2, has dramatic consequences for the relation between mathematical theories and their semantic potential for music.

Historically and materially, and in view of the genealogy of a mathematical theory, the subject pairing of "consonance-dissonance" is an excellent illustration of auditory physiology and psychology. We deal with the formal and some physiological aspects of this subject in part VII. In the following section B.3, we want to expose the stratification of the phenomenon of consonance and dissonance. Hereby, the problem setup as well as the approaches to its solution demonstrate a strong dependency of the addressed reality layer. Methodologically, section B.3 is important since it makes evident that known approaches of the mathematical argumentation in musicology turn out to be too narrow with respect to the existing music, and too dogmatic and scientifically unbased with respect to music thinking.

## B.1 Physiology: From the Auricle to Heschl's Gyri

Two fundamentally different regions of sound processing in the auditory system can be distinguished. In the peripheral region, mechanical preprocessing takes place, especially in the fluid-filled inner ear. The sensory cells encode the preprocessed mechanical oscillations into electrical nerve action potentials. In the second region of the hearing system, neural processing of the sound information is conducted in ascending nuclei, which finally leads to auditory sensation. The hearing system, especially in mammals, has pushed its bandwidth up in a frequency range, where the limits of neural processing in the time-domain are exceeded by far. Instead of processing high frequencies in the time domain, evolution has developed the so-called frequency-place principle. In the inner ear, the frequency contents of sound signals are separated in the spatial domain. The sensitivity of our hearing system is thereby remarkable, in its most sensitive region, the threshold of the hearing system is limited only by thermal noise.

### B.1.1 Outer Ear

Sound energy is collected by the outer ear and transmitted through the outer ear canal to the ear drum (figure B.1).

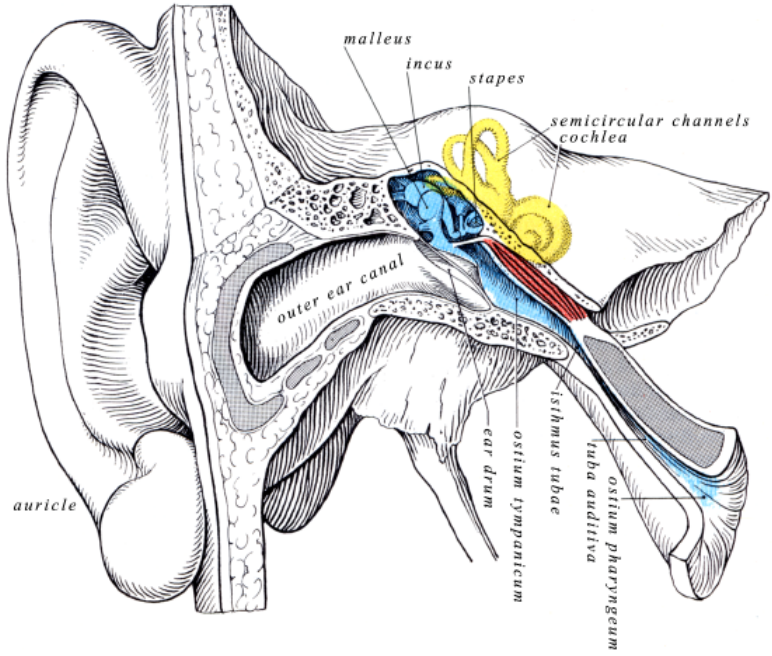



Figure B.1: Schematic view of the outer, middle and inner ear, modified from [594].

The delicate structures of our hearing system are well protected inside the skull. For the sound transmission, the outer ear canal acts like an open pipe with a length of about 20 to

30 $mm$. Its quarter-wave resonance is responsible for the high sensitivity of our hearing organ in this frequency range, indicated by the dip of the threshold in quiet around 4 kHz. This high sensitivity is however also the reason for high susceptibility to noise-induced damage in the region around 4 kHz.

### B.1.2 Middle Ear

The fluids of the inner ear must be excited by the sound-induced vibrations of the air particles in front of the ear drum. The light but sturdy funnel-shaped ear drum (tympanic membrane) operates over a wide frequency range as a pressure receiver. It is firmly attached to the long arm of the hammer (malleus) (figure B.1). The motions of the eardrum are so transmitted via the anvil (incus) to the stirrup (stapes). The stapes foot plate, together with a ring-shaped membrane called the oval window, forms the entrance to the inner ear. The middle ear optimizes the energy flow from air-borne sound in front of the ear drum to fluid motion in the inner ear by a mechanism called impedance transformation. One part of the impedance transformation is based on the lever ratio of about 1.5:1 produced by the different lengths of the arms of malleus and incus [72]. The lever ratio transforms oscillations of the ear drum with small forces into motions of the fluid with large forces. The law of energy conservation implies also that the tiny displacements of the ear drum are transformed into still smaller oscillations in fluid. An even larger transformation of the pressure is due to the ratio of the large ear drum to that of the small oval window. This ratio is about 17 [47]. Through the lever and area ratios, an almost perfect impedance match is reached in man in the middle frequency range between 1 and 4 kHz. It allows optimization of the energy flow into the inner ear, which otherwise would be reflected.

The middle ear operates normally when it is filled with air at atmospheric pressure. The Eustachian tube, which connects the middle ear cavity to the upper throat, normally opens and closes periodically, thereby insuring that the static pressure in the middle ear will remain the same as atmospheric pressure. We experience a pressure difference when the Eustachian tube fails to open during ascent or descent in an elevator. For an elevation of 8 $m$, the change in atmospheric pressure is 100 $Nm^{-2}$, corresponding to a sound pressure of 130 $dB$ relative to the 20 $\mu Nm^{-2}$ reference! This pressure causes a static deflection of the ear drum and increases the stiffness of the middle ear transmission system and sound transmission is attenuated.

### B.1.3 Inner Ear (Cochlea)

The shape of the cochlea resembles that of a snail shell with two and one-half turns (in humans) and hence its name (figure B.2). The central conical bony core of the cochlea is called the *modiolus*. The auditory nerve fibers run in this bone and exit the cochlea at its base. The outer wall of the modiolus forms the inner wall of a 30 $mm$ long canal which spirals the full two and one-half turns around the central core.

This canal is separated into three partitions called scales: Scala tympani is separated by the so-called cochlear partition, which is formed by a thin shelf of bone projecting from the modiolus (the ossesus spiral lamina), which is connected by the basilar membrane and the spiral ligament to the outer wall of the cochlea. The sensory organ of hearing, the organ of Corti, is located on top of the basilar membrane. As can be seen in the cross section of the cochlear spiral, the cochlear scalae become smaller and smaller in cross-sectional area as the apex is

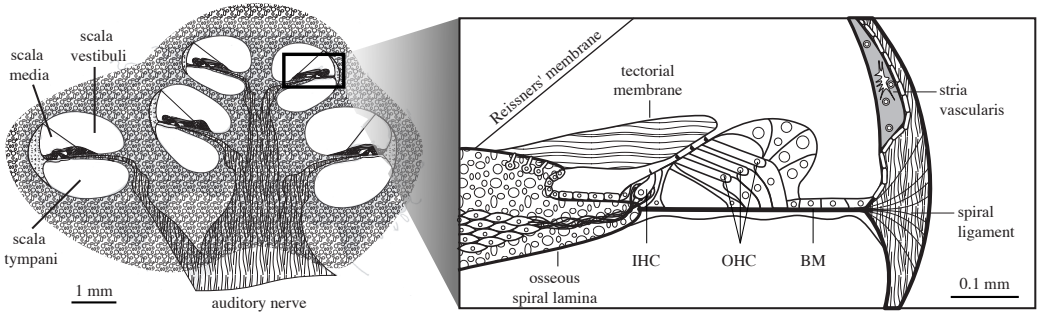



Figure B.2: Section through the human cochlea (left) and magnified view of the organ of Corti (right). IHC: inner hair cell, OHC: outer hair cell, BM: basilar membrane.

approached. Directly opposed, the basilar membrane becomes progressively wider towards the apex (figure B.2). This is because the osseous spiral lamina is broadest at the cochlear base where the basilar membrane is only about 0.16 $mm$ wide (in humans); at the apex the basilar membrane has broadened to about 0.52 $mm$.

Scala media and scala vestibuli are separated by a thin membrane, called Reissner's membrane (figure B.2). At the apical end of the cochlea, scala vestibuli and scala tympani are connected by an opening in the cochlear partition, called helicotrema. Scala tympani and scala vestibuli are filled with perilymph, which resembles in its chemical composition other extracellular fluids. Perilymph is characterized by high sodium ($Na^+$) concentration of about $140 - 150$ $mM$ and its low potassium ($K^+$) content of only around 5 $mM$. Scala media is filled with endolymph, which is unlike any other extracellular fluid found in the body. From its chemical composition, it resembles intracellular fluids. Its predominant cation is potassium with a concentration of about 157 $mM$; sodium is very low (1.3 $mM$). In addition to its special chemical composition, the endolymphatic space exhibits a considerable positive electrical potential within scala media of about $+80$ $mV$ relative to scala tympani and scala vestibuli, called the endocochlear potential. The chemical composition and the electrical potential of the endolymphatic space is sustained by active ion transport provided by the cellular layers of stria vascularis.

The organ of Corti lies just between the endolymphatic and perilymphatic spaces (figure B.2). Its surface is sealed by tight junctions to keep the fluids separate. The basilar membrane, on which the organ of Corti rests, is composed mainly of extracellular matrix material with embedded fibers. In contrast to the tight surface of the organ of Corti, the basilar membrane is thought to be permeable to perilymph. In the organ of Corti, two types of sensory cells, one row of inner hair cells and three to four rows of outer hair cells are embedded. In humans, there are approximately 3,500 inner- and 12,000 outer hair cells. The membrane potential of inner hair cells is about $-40$ $mV$, of outer hair cells even as low as $-70$ $mV$. Both types display "hair bundles" or stereocilia, which project into the endolymphatic space. The stereocilia are arranged in several rows, which are graded in size. Stereocilia from different rows are connected by a fine filament, called tip-link. The current theory of transduction assumes

that hair bundle deflection pulls on the tip-links, opening transducer channels which are close to the attachment points. This concept also explains that the hair bundle is only sensitive to mechanical stimulation in the direction of the tallest stereocilia. The transducer channel is a nonselective cation channel which is very impermeable to anions. Therefore, the transduction current is mainly carried by potassium ($K^+$) and calcium ($Ca^{2+}$) cations, driven by the large electrical potential between the endolymphatic space and the receptor cells. The driving potential sums up to 120 $mV$ for inner hair cells and to 150 $mV$ for outer hair cells. Potassium leaves the sensory cells via potassium channels present in the basolateral cell membrane and diffuses into scala tympani. It is interesting to notice that the intracellular concentration of potassium is as high as that of endolymph. Potassium is driven into the cells by the electrical potential and because of the concentration gradient, it can diffuse into scala tympani without requiring energy from the sensory cells.

The inner and outer pillar cells, the phalangeal processes of the Deiters' cells and the cylindrical bodies of the outer hair cells build a complex, three-dimensional truss (see figure B.3). One peculiarity is that outer hair cells are not in contact with other cells along their lateral

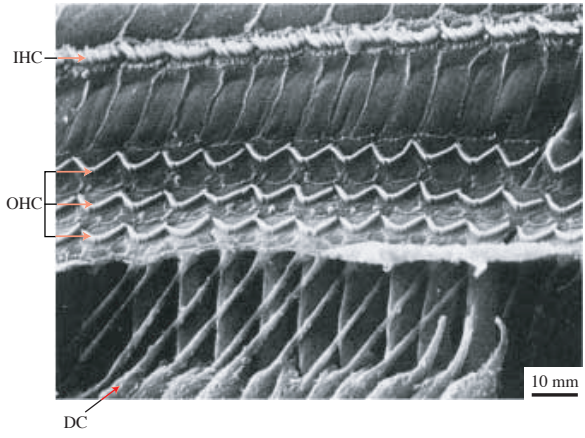



Figure B.3: Scanning electron micrograph of the three-dimensional arrangement of the organ of Corti. IHC: inner hair cells, OHC: outer hair cells, DC: Deiters cells [141].

surface but immersed in extracellular fluid. From figure B.3 also the different morphology of the hair bundles becomes apparent: Whereas the bundles of the inner hair cells are arranged in a straight line, these of the three rows of outer hair cells are W-shaped.

The hair bundles of outer hair cells are excited by a shearing motion between the surface of the organ of Corti and the so-called tectorial membrane. The tectorial membrane is a gel-like structure composed of extracellular matrix material and it is in direct contact with the longest row of outer hair cell stereocilia. In contrast, the hair bundles of the inner hair cells seem not to be in direct contact with the tectorial membrane. Their bundles are probably driven by fluid

forces. The transduction current flowing through the stereocilia is converted into a receptor potential in the cell body of the hair cell. At low frequencies, the receptor potential follows the stimulus cycle-by-cycle. Upon mechanical stimulation of the hair bundle in excitatory direction, tip-links are stretched and the transduction channels open. As positive $K^+$ ions are driven into the cell, the potential inside the cell becomes more positive. If the bundle is stimulated in the other direction, tip links relax and transduction channels close. However, as for inner hair cells at rest only 20% of transduction channels are open, the receptor potential is highly asymmetric when stimulated with a tone. On the basal pole of the inner hair cells about 10 to 30 afferent synapses are located, and upon depolarization of the cell membrane, voltage-sensitive $Ca^{2+}$-channels located in the basal pole of the cell membrane open, the increased $Ca^{2+}$ level causes vesicles filled with transmitter to fuse with the lateral cell membrane and release transmitter into the synaptic cleft. This transmitter release triggers an action potential in the afferent nerve, and these electrical spikes transmitted to the brain finally lead to the hearing sensations.

These mechanisms work well at low frequencies, where the events of neural processing can easily follow the sound stimulus. Because of its refractory period, a single auditory nerve fiber can not respond to each successive cycle of a high-frequency sound. This problem can be partially overcome by the fact that more than one nerve fiber contacts a single inner hair cell. Each fiber is incapable of responding to every cycle of the stimulus, but collectively, they can do so. The temporal structure of the sound is conserved, as each fiber responds in the depolarized state of the inner hair cell. We call this behavior "phase locking". Even if the firing rate of a nerve fiber is too slow to follow the stimulus, basic features of its temporal structure—like the phase—are still coded in the neural pulse train. Still, the effect of phase locking is limited by several factors. The lateral membrane has a time constant of about 1 $kHz$, and above this frequency, the AC-amplitude of the receptor potential decreases. Also the synaptic processes like vesicle release and the generation of the postsynaptic potential are limited in speed and accuracy. This leads to a gradual loss of phase locking starting above 1 $kHz$, and above 3 $kHz$, phase locking is completely lost in humans.

In the frequency region, where the receptor potential can no longer follow the stimulus on a cycle-by-cycle basis, we see a depolarization of the inner hair cells membrane potential while stimulated. This so-called DC-component of the receptor potential follows the stimulus *envelope*. This is because of the asymmetry of the hair-bundle transduction. The depolarizing currents dominate upon sinusoidal stimulation and the receptor potential, low-pass filtered by the cell membrane, cause a depolarization of the cell membrane. All the information about the sound's frequency is lost in this signal, therefore, a different way to code high frequencies had to be developed by evolution.

This task was achieved by "sorting" sounds by frequency. Sound signals are mechanically preprocessed in a way that they are separated spatially. This concept is well-known as the frequency-place principle where high frequencies are located in the basal part of the cochlea only and low frequencies in the apical part. Different sets of auditory nerve fibers elicit different auditory sensations by virtue of their central connections. We will examine the mechanical frequency separation process in the sec section B.1.4 and to do so, we will have to focus on the mechanical properties of the inner ear.

It is however not clear how specific excitations converge to yield well-defined pitch in general. There is increasing evidence that pitch is extracted in the time domain (periodicity analysis [286]). Moreover, pitch was found to be independent of the frequency-place mapping

of the components of complex tones [287].

### B.1.4 Cochlear Hydrodynamics: The Travelling Wave

The cochlea consists of three fluid-filled scalae, but from a mechanical point of view, the elastic properties of the thin Reissner membrane can be neglected compared to the stiffness of the basilar membrane. We can therefore simplify our mechanical investigations to a fluid model with two chambers separated by the cochlear partition (compare figure B.4). The basilar membrane contributes a large part of the elastic properties of the partition. Its width is increasing from base to apex from about 0.16 $\mu m$ to 0.52 $\mu m$ (in humans). Its thickness, on the other hand, decreases along the cochlea. Thus, its stiffness decreases greatly along the length of the cochlea.

At the basal end of the cochlea there are two openings to the cochlear ducts, one on each side of the cochlear partition, that are covered by membranes. One is called the oval window and, as we already mentioned, it is in contact with the stapes foot plate. The other, the round window, is just below the oval window. Because we can assume the cochlear fluids as incompressible, the round window has to move out of phase if the oval window is driven by stapes motion.

If we consider very slow stapes motions, the fluid is pushed along the entire length of scala vestibuli, through the helicotrema and back along scala tympani. Thus, the helicotrema provides a low-frequency shunt for extremely low frequencies. When the stapes moves into the cochlea, pressure builds up in the fluid, which deflects the cochlear partition. Pushing fluid requires overcoming inertial forces generated by the fluid mass. The elastic properties of the basilar membrane in combination with the mass of the surrounding fluid constitutes second-order resonators. As the stiffness of the basilar membrane is very high at the basal end of the cochlea and much lower at its apical extreme, the resonant frequency of the cochlear partition monotonically decreases from base to apex. Inversely, the time constant of each resonator increases from base to apex. However, the basilar membrane is not under tension, and it does not respond like a series of *independent* resonators, like the strings of a harp. Instead, each part of the cochlear partition is coupled to the next by the cochlear fluids, and due to the large inherent friction, they are highly damped. For a periodical motion of the stapes, the cochlear partition is first set into motion at the basal extreme, where the mechanical time-constant is smallest. Because the stiffness is very high, the deflection of the basilar membrane is fairly small. The deflection propagates in the form of a wave in apical direction. As the time-constants of the partition increase and the stiffness decreases, the response will be more and more delayed but its amplitude will grow. At the location of resonance, the wave will reach its maximum and lose its energy, its amplitude will drop very rapidly. The location of "cochlear resonance", the place where the displacement—and therefore the excitation—of the cochlear partition reaches its maximum, depends on the stimulus frequency. Low frequencies will travel along the basilar membrane and reach a maximum close to the cochlear apex, high frequency sounds will exhibit their maximum response close to the cochlea base and fade out.

The exact form of the vibration response of the cochlear partition was investigated by Georg von Békésy in human cadaver ears [47], a feat which earned him the Nobel Prize. He found that the deformation of the basilar membrane is a traveling wave. The wave starts at the cochlear base, where the basilar membrane is stiffest. It propagates toward the apex with a time delay that depends upon its own mechanical properties and the properties of the surrounding

fluid. Its vibration amplitude is increasing until it reaches a maximum, close to the location of cochlear resonance, and from then on, the wave diminishes rapidly. Because the stiffness gradient of the basilar membrane is approximately logarithmic, the peaks of the excitation patterns of sinoidal tones are located on the basilar membrane with a logarithmic frequency spacing. Figure B.4 schematically illustrates traveling waves elicited by a stimulus composed of three frequencies, together with the envelope of the peak displacement. The peaks of the

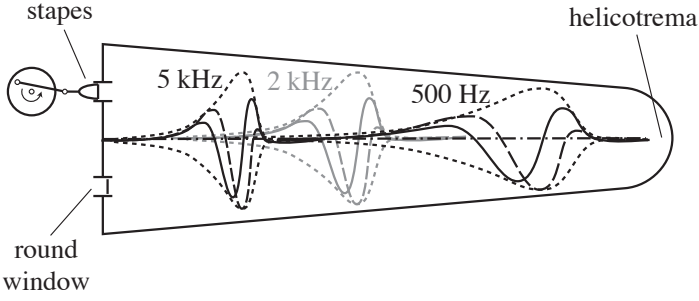



Figure B.4: Schematic illustration of the traveling waves elicited by three pure tones with frequencies of 500 Hz, 2 kHz and 5 kHz. The displacement of the basilar membrane is shown at the instant $T_0$ (solid line) and a quarter of a cycle later (dashed line). The dotted line indicates the envelope of the wave.

three waves are clearly separated along the cochlea, however, there is also considerable overlap between the waves and recordings from the auditory nerve have been found to be much more frequency selective, especially close to threshold, than the mechanical responses observed in cadaver ears. Therefore, a second mechanism is required to boost the frequency selectivity of the vibration responses.

### B.1.5 Active Amplification of the Traveling Wave Motion

We have so far neglected the function of the outer hair cells, the second group of receptor cells within the organ of Corti. Despite the at about three to four times higher number of outer hair cells, only 5% of the afferent fibers innervate outer hair cells. The fibers are so-called type II fibers, they are highly branched and each fiber innervates dozens of outer hair cells. Little is known about these fibers because they are small and unmyelinated, making it difficult to record their activity. They are expected to be less sharply tuned, since they innervate a broad region of the cochlea, and, because of their lack of myelination, their conduction velocity is likely to be very slow. Most of the frequency selective and time-critical auditory information must therefore be carried by the afferent fibers originating from the inner hair cells. Whereas inner hair cells are not directly innervated by the efferent system, myelinated fibers from the medial to the medial

superior olivary nuclei make direct synaptic contact with the outer hair cells. Outer hair cells seem to be under neural control, much more like muscles than sensory cells. A final indication of their "active" role was established when their ability to change their length upon electrical stimulation was detected. Modern research assumes that the outer hair cells indeed provide an active, mechanical amplification of the cochlear traveling wave. As neural processing is by far too slow to keep up with high frequency hearing, current concepts assume that amplification relies on a local mechanical feed-back process [107]:

> Outer hair cells sense motions of the cochlear partition by converting shearing motion between the surface of the organ of Corti and the tectorial membrane into an electrical receptor potential. Upon depolarization, the outer hair cell reacts with a contractile force, which is fed back into the motion of the basilar membrane.

From theoretical calculations derived from measurements in the inner ear, it is required that energy is pumped into the vibration of the cochlear partition in a region starting before the traveling wave reaches its maximum up to the place of cochlear resonance. The function of this amplification process is still unclear in its details, but from the observations of active cochlear mechanics it has wide reaching consequences. Measurements in the basal part of the cochlea indicate, that the amplification boosts cochlear sensitivity up to thousand-fold. The amplification is limited to a narrow frequency range, covering only about half an octave. The active traveling-wave response becomes very sharp in the region of cochlear resonance. The amplification of the vibration response is required to achieve the extraordinary sensitivity of the hearing system.

The second hallmark of amplification is non-linearity. The amplification process boosts only weak sounds, it saturates at increasing levels. This non-linearity greatly compresses the dynamic range of the mechanical responses. This is important, because the dynamic range of the inner hair cell receptor potential is limited to a range certainly not exceeding 60 $dB$. This non-linearity also has unwanted side-effects: If two sinoidal tones are presented, the non-linearities of the hearing-organ generate so-called distortion products, additional tones which we perceive under certain circumstances. In general, however, by virtue of its construction, artifacts due to the non-linearities of the inner ear are surprisingly small.

Figure B.5 shows the excitation pattern expected on the human basilar membrane when stimulated with a 3 $kHz$ sinoidal tone. For high sound levels (i.e., 100 $dB$), the feed-back amplifier is saturated and the traveling-wave is almost purely passive. It is highly damped and its envelope shows the characteristic shallow increase from base to apex and a sharp decay after the maximum is reached.[1] The threshold of the auditory nerve is somewhere between a minimal basilar-membrane velocity of 50 $\mu msec^{-1}$ and a displacement of 1 $nm$, exact values are still unknown. The broad traveling wave at high levels indicates that a large number of nerve fibers are stimulated, especially in the basal part of the cochlea. For faint sounds, the traveling-wave response becomes sharper and sharper and its envelope is almost symmetrical for levels below 40 $dB$. The location of maximum amplitude for a sinoidal tone at low levels is called its characteristic place. Only nerve fibers originating from a very narrow region around the characteristic place of the cochlea are stimulated. Note that for increasing levels, the maximum of the traveling

[1] The data in figure B.5 (dashed lines) indicates that further towards the apex, the wave does not die out completely in this experiment. It is still under debate, whether this remaining response is also present in the intact human cochlea.

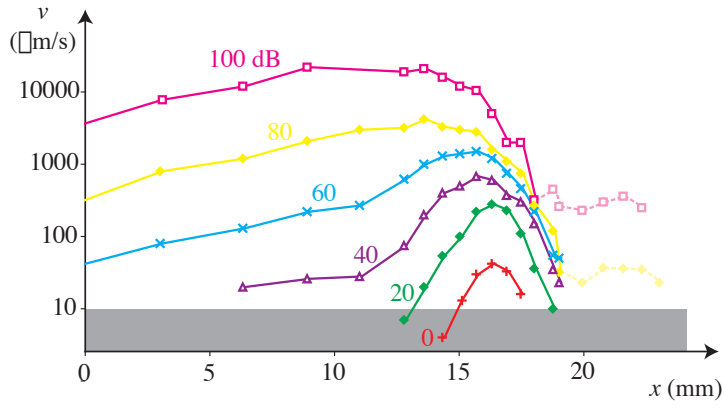



Figure B.5: Reconstruction of the excitation pattern in the human cochlea for a 3 $kHz$ sinoidal tone. Original measurements were recorded in a chinchilla cochlea at a distance of 3.5 $mm$ from its most basal extreme [392]. The characteristic frequency of this location was 9.5 $kHz$. Data has been converted assuming a frequency-place map of 8 $mm$/octave to illustrate the excitation pattern in a human cochlea. The shaded area indicates excitation below neural threshold, which is expected between 50 $\mu msec^{-1}$ and 1 $nm$.

wave shifts considerably in the basal direction. If we analyze the level-dependence of the responses at various locations of the cochlea, we clearly see the effects of the non-linearities of the amplification. At the characteristic place (about 16 $mm$ in figure B.5), the velocity amplitude increases from a value of 42 $\mu msec^{-1}$ at 0 $dB$ to 5 $msec^{-1}$ at 100 $dB$. Without amplification, the amplitude of the traveling-wave response would be expected to drop by a factor of $10^5$, or from 5 $mmsec^{-1}$ to 50 $nmsec^{-1}$ from 100 $dB$ to 0 $dB$! The amplification therefore is almost a factor of 1000 or 60 $dB$. In addition to enabling the detection of weak signals, the amplification therefore also compresses the dynamic range, again by almost 60 $dB$. The inner hair cells, at the location of the characteristic place, have to cope only with a stimulus ratio of a little bit more than 40 $dB$ to cover a 100 $dB$-level change of the sound stimulus.

### B.1.6 Neural Processing

The difficulty to understand neural processing of sound strongly stems from the extremely complex innervation from the auditory nerve to primary auditory cortex (Heschl's gyri) in the temporal lobe (see figure B.6). We have to stress that the image in this figure is considerably simplified, and that in particular, there are also connections from the cochlea to the ipsilateral auditory cortex, as well as efferent nerves from the auditory cortex down to the cochlea.

The auditory system includes at least five "relays stations", whence it is clear that any particular functional decomposition (like Fourier's) will not be transferred unchanged to the auditory cortex. For example, the tonotopy of the latter is a multiply distributed one, see figure B.7.

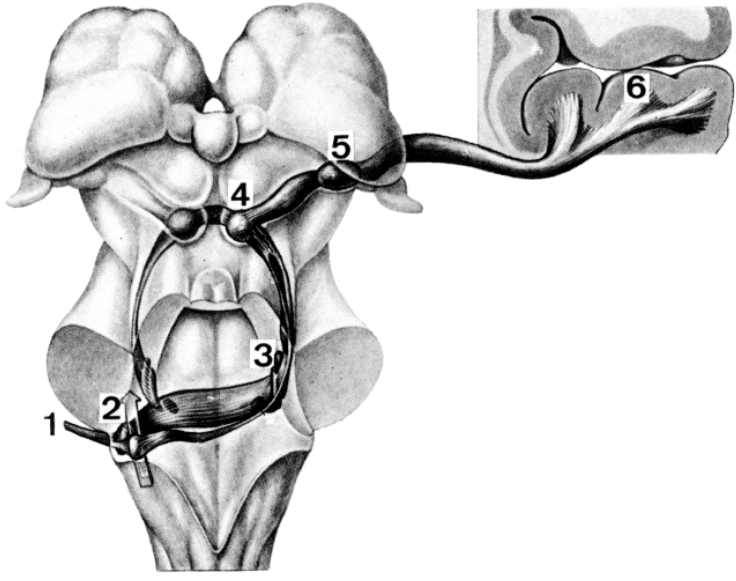



Figure B.6: (With kind permission of the Hallwag-Verlag) This simplified image shows the six relevant relais stations of the auditory path from the cochlea to the Heschl gyri: (1) Nervus cochlearis, (2) nucleus cochlearis, (3) nuclei superiores olivae, (4) colliculus inferior, (5) corpus geniculatum mediale, (6) gyri Heschl.

By tonotopy, one understands the spatial distribution of excitation patterns according to specific pitch. More recent research also demonstrates that outside the auditory cortex, i.e., in the limbic system (the hippocampal formation, to be precise, which plays an important role for emotional and memory tasks), one finds a refined processing of pitch information [336, 337, 570, 571].

The "template fitting model" of Julius Goldstein [187] shows how far we are from understanding the neural pitch processing. In this model, the mathematical principle of a neural "central pitch detector" is proposed, from which the fundamental frequency of a periodic wave can be extracted if its Fourier components are known. The central pitch detector is however charged with the solution of local minima problems for functions in two variables—a rather heroic task for a small neural population. It seems hopeless to identify within the neural network the physical realization of a dynamic system that solves these differential conditions. We presently have no chance to verify the model physiologically, since the human ethics excludes adequate experiments in humans.

In view of these facts, it is not only logically erroneous and experimentally very delicate to infer the higher sound processing from the superficial auditory physiology of the ears. And ethically, such an attempt is problematic since one runs the risk to "justify" discrimination of "degenerate music" against the acceptance of so-called "commonly accepted" music. It is also not clear how much such investigations reveal grown and trained configurations instead of

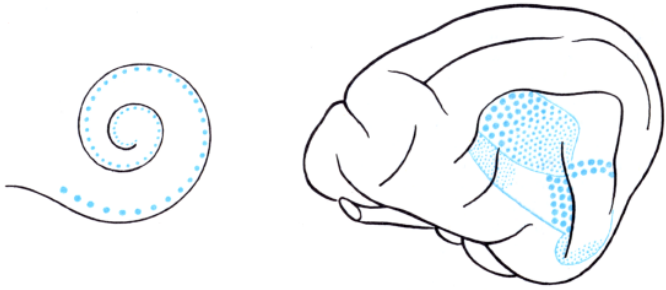


Figure B.7: (With kind permission of the Thieme-Verlag) Left: tonotopy in the spiral of the cochlea: high pitches are thick points, low pitches are thin points. Right: Corresponding multiple tonotopy in the auditory cortex of the cat.

biological inheritance.

## B.2 Discriminating Tones: Werner Meyer-Eppler's Valence Theory

From our everyday experience and from specific experiments it follows that we do not really hear the single tone events or chords, motives or rhythms as they have been parametrized on the level of physical or mathematical description. In fact, despite the very sensitive physiology of Corti's organ and its hair cells, we cannot distinguish all physically or mathematically distinct sound objects. For instance, sounds with frequency above $20,000\ Hz$ or below $0\ dB$ are indistinguishable since you cannot hear them. Or two sounds $p, q$ with a phase shift, i.e., $p(t + \Delta) = q(t)$ are indistinguishable. More important is that even under ideal condition we cannot distinguish sounds with arbitrary precision. Every singer or violinist who has to adapt his/her pitch to a prescribed context knows this. Investigations on variations of instrumental intonation show a remarkable bandwidth [57]. The same is valid for listening to time values, loudness degrees and instrumental colors.

It is mandatory but not easy to take into account these phenomena. Werner Meyer-Eppler [372, 373] has attempted to solve the problem by use of the concept of a "valence". According to this approach,

**Definition 113** *In a specific context, given two sound objects $s_1, s_2$, $s_1$ is* metamere *to $s_2$ with regard to a given predicate $P$ (short: $P$-metamere, in symbols: $s_1 \sim_P s_2$) iff the $s_1$ cannot be distinguished from $s_2$ with respect to $P$ by human listeners. The set $\sim_P s$ of sound objects which are $P$-metamere to a given sound $s$ are called the* valence of $s$.

Since usually the relation $\sim_P$ is symmetric, one calls two sound objects $s_1, s_2$ metamere if $s_1 \sim_P s_2$. If a sound object $s$ is defined by a sequence $P_1, \ldots P_k$ of predicates, one defines its valence as being the sequence of valences $\sim_{P_1}, \ldots \sim_{P_k}$, and we may then reduce the total valence to those components which do not include all sound objects since these predicates are

not relevant to the distinction of sound objects. The union $\bigcup_{\text{supporting valences}} \sim_{P_{i_j}}$ of supporting valences is in fact the valence of the conjunction predicate $\bigwedge_i P_i$. Predicates which are relevant in this sense are called *valence supporting* by Meyer-Eppler.

In practice, if we are given a sound whose sound color is described by partials with frequencies up to the limit frequency $f_0\ Hz$, and which last $d\ sec$, Meyer-Eppler deduces a maximal number of numerical predicates (dimensions) that are relevant to the valence, i.e., of valence supporting numerical predicates. This limit is called the *maximal structure content*, and its value is $K = 2.d.f_0$. It is however an open fundamental question of sound color theory, which and how many valence supporting predicates must be chosen in order to yield a differentiated perception of sound colors. Probably, these valences define a multiply connected topological space in the physical parameter space, but see also section 12.3.

Although Meyer-Eppler's conceptualization is plausible, it hides two delicate problems. The first concerns the context where valences take place, as related to the predicate in question. If the context's specification is neglected, the valence concept loses its meaning. Let us make two representative examples concerning pitch and onset. To begin with, we have to agree on who is accepted as a listener. In the sense of a statistical approach, Meyer-Eppler proposes that two pitches should be called metamere if at least 90% of the test subjects cannot distinguish them [373]. Moreover, one has to agree on the parameter(relations) of the test sounds. The result will depend essentially on the choice of instruments, flute, brass etc., and conditions upon the duration and the onset distances of test sounds. For example, the simultaneous presentation of sounds of several seconds duration will yield smaller valences because of beat effects, as compared to a comparison of non-overlapping sound events.

Besides the parametric conditions, the context can also depend on the chosen music. Let us discuss this on the onset parameter, for example. In [373], the duration valence density is indicated by $50 - 60$ per second. This means that sounds which are less than $1/50 - 1/60\ sec$ apart are perceived as being simultaneous. However, the musical context of such a claim is relevant. Within a very slow piece with a small number of instruments, the temporal variation of $1/8\ sec$ will scarcely be noticed, whereas in a rhythmically very dense and fast piece, $1/60\ sec$ is known to define a quite coarse grid.

The second problem is important for the theoretical significance of the valence concept. It relates to the fact that metamery is not an equivalence relation[2] in general. More precisely, any of the wanted properties: reflexivity, symmetry, and transitivity, can be violated. Reflexivity can be violated if the comparison of two sounds is temporally so separated, by several hours, say, that the human memory fails to recognize one and the same sound. Symmetry is violated if the order of appearance of sound objects is relevant, for example, a very loud sound, immediately followed by a very soft one, can mask the latter's properties. The most dramatic failure is the absence of transitivity: $s \sim_P t, t \sim_P u$ does not always imply $s \sim_P u$. For example, if three pitches are such that the first is perceived as being equal to the second, and the second being equal to the third pitch, this is not entailed by equality of first and third pitch! Therefore, pitch valences are not equivalence classes, they may overlap. This means that the attempt to define pitch by an esthesic position in music psychology must fail. The perceptional concept of pitch is a non-transitive relation among tones, and therefore is not an attribute of tones. You hear that two tones have the same pitch, but you do not hear the pitch. Therefore corresponding

[2]See appendix C.2.

attempts such as [394] must fail.

This has important consequences for musical practice and for theoretical aspects. In practice it is desirable to select grids of sounds such that their valences would not overlap. This needn't be a grid which is fixed once for ever, it might be a time-dependent construction, but it must yield a locally disjoint valence set.

With the common notation, such a grid is realized as an orientation device as well as an acoustic and performative scheme. The continuum of onsets, durations, pitches, and sound pressure values is quantized in the well-known way such that Boulez' "notched tone space" can be taken as a grid behind the *realiter* played or heard.

The reality layer of a grid is a mental or psychological one. The blurredness of hearing, as it is expressed in valences, has to be subjected to a cognitive interpretation. Semiologically speaking, the valence is the significant, the expression for a meaning which relates to our understanding of music. By use of the grid which is superposed to the valence perception, it is possible to associate the valences of perceived sound to objects of our imagination. The psychological quality of pitch then results from this mapping as a grid object in our imagination. The semiotic power of this signification depends upon the definition of the actual grid.

A music-theoretically fundamental grid is the selection of the pitch arsenal, the tuning, wherein tones may be played. Let us first look at the chromatic $w$-tempered scales (see section 7.2.1.1). If $w$ is not too large, valences of neighboring tones can be separated. From our experience with microtonal music, $w = 36$, i.e., tempered sixth-tone intervals, is not too large. So the postulate of an adjustment of pitch by inner grids is acceptable for $w$-tempered scales with $w \leq 36$.

The situation of just intonation (see section 7.2.1.2 and appendix A.2.3) is much more delicate. For each pitch $H(x_0)$ of a point $x_0$ in the Euler grid $\mathbb{Z}^3$ of just intonation, and for any positive real number $\epsilon$, there are infinitely many points $x$ such that $|H(x - x_0)| < \epsilon$, see proposition 64 in appendix A.2.3. In particular, infinitely many pitches fall into the valence of $x_0$. *In the valence semiology, just intonation is infinitely homonymic.* This problem can be solved by a restriction of the context, where just intonation music is played. As soon as the local context is a small region of the Euler space, for example, a small neighborhood of a determined finite portion of the chromatic scale, valences can be used to distinguish just grid points.

A second difficulty for just intonation relates to the perception of pitch differences. A classical argument of just music theory is Euler's substitution theory [144], according to which intervals are heard in a way such that the frequency relations of their tones form fractions $a/b$ (in reduced representation) with minimal numerator $a$ and denominator $b$. This is the basis of the classical consonance-dissonance theory which we address in the following section. In the grid of just intonation this means that the pitch difference of the interval is corrected/adapted to an interval which is realized by two points of the just intonation grid under the constraint that their (Euclidean or 1-norm) distance is minimal. Of course, this correction has to happen within a valence in order to be a reasonable process of human hearing. However, it is easy to see that in general, there are several solutions $a/b, a'/b', a"/b", \ldots$ with minimal distance. So Euler's substitution theory would have to impose a contextual restriction for single tones as well as for intervals.

These considerations should be taken for nothing more than they are: esthesic aspects of hearing. Nothing prevents us from doing music theory on neutral and poietic layers without bothering about valences. But then, one has to be conscious of the fact that highly differentiated

mathematical structures may be blurred by the semiology of valences from auditory psychology and physiology.

## B.3 Symbolic, Physiological, and Psychological Aspects of Consonance and Dissonance

For a long time, mathematical reflections in music were centered around the problematic concept couple of consonance and dissonance. This is based on the ancient Greek Pythagorean tradition where consonance and dissonance of intervals was laid in the involved frequency relation. Perfect consonant intervals corresponded to ratios like 2/1 for the octave, 3/2 for the fifth, and 4/3 for the fourth. This simple arithmetic corresponded to the philosophy of the metaphysical tetractys. See [330, 394] for a historical discussion of these roots.

Here, we simply want to recall that a unified mathematical foundation of musical thinking in the paradigm of simple consonant frequency ratios could not survive the differentiated development of theories in the contrapuntal setting [468], the psychological foundation of musical relations as introduced by René Descartes [126], and the discovery of physical partials by Marin Mersenne [310].

According to these more recent positions, the problem of consonances and dissonances changes as a function of the layer of reality where it is investigated—and on each layer it is not a minor one. In the present shorthand presentation, this result seems to be a provocative one since the conceptual unity seems violated. We want to make clear that we really are dealing with three different meanings of the sonance concept—Euler's gradus suavitatis on the mental layer, Helmholtz' beat model on the physical layer, and Plomp–Levelt's psychometrics on the psychological layer. In itself, each of these approaches is consistent, the problem only arises if one attempts to reduce one reality to another one. Following the knowledge about the neural processing of sounds (see the previous discussion in this chapter), it is hardly astonishing that psychological and physiological layers are not congruent: what the ear (in Helmholtz' model) does not "like" can very well be "agreeable" for the limbic system or the auditory cortex.

### B.3.1 Euler's Gradus Function

Being a number theorist, Euler was interested in prime numbers. A priori, his gradus function $\Gamma$ [143] is a purely number-theoretic function, it is defined as follows: According to the prime factorization of integers (see appendix D.2), a positive integer $a$ is the unique product $a = p_1^{e_1}.p_2^{e_2}.\ldots.p_n^{e_n}$ of positive powers of primes $p_1 < p_2 < \ldots p_n$ (with the singular case of zero factors and a product $= 1$ for $a = 1$). Euler's formula, the *gradus suavitatis*, is $\Gamma(a) = 1 + \sum_{1\leq k\leq n} e_k(p_k - 1)$, and more generally $\Gamma(x/y) = \Gamma(x.y)$ for a reduced fraction $x/y$.

In just intonation—where Euler's approach belongs—only intervals of tones are considered whose frequency ratio is a positive rational number $x/y$. The gradus function is defined for such intervals. Each interval evaluates to a positive integer. The frequency ratios with $\Gamma$-values $\leq 10$ are listed in table appendix J, see also figure B.8.

In just intonation, intervals can be read as differences $\Delta = x - y = (e, f, g)$ of Euler points $x$ and $y$ in the grid $\mathbb{Z}^3$. For the gradus function, we may restrict to non-negative coordinates

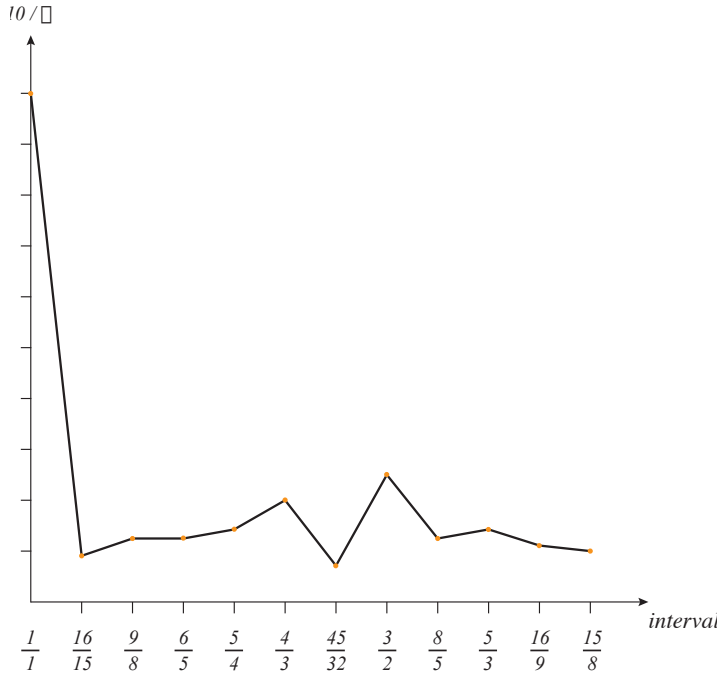



Figure B.8: The agreeableness of intervals within the octave in just intonation according to Vogel's chromatic [547], see section 7.2.1.2, is represented. It is reasonable to represent the reciprocal values $10/\Gamma$ instead of $\Gamma$ since the latter is rather a 'gradus dissuavitatis': small $\Gamma$ values are taken for the octave and the fifth, large ones for the second and tritone. The factor 10 is only a scaling constant. The order of Euler's valuation is this: Prime, fifth, fourth, major third/major sixth, minor third/minor sixth/major second, minor seventh, major seventh, minor second, tritone.

$e, f, g$. We then have

$$\Gamma(\Delta) = \Gamma(2^e.3^f.5^g) = 1 = 1 + (2-1)e + (3-1)f + (5-1)g \tag{B.1}$$

or as a scalar product

$$\Gamma(\Delta) = 1 + (\Phi, \Delta) \text{ with } \Phi = (1, 2, 4). \tag{B.2}$$

If we compare this formula with the pitch formula

$$H(\Delta) = (H_{prime}, \Delta)$$

where $H_{prime}$ is the prime vector from section A.2.3, then we observe a similar construction. The gradus function is something like a pitch function, but the 'direction' is $\Phi$ instead of $H_{prime}$, see figure B.9. The ranking of intervals by the gradus function was already criticized by Euler's contemporaries Mattheson, Mitzler, and Rameau [71]. But it is the merit of Euler to have defined a ranking by a linear expression which, together with pitch and octave coordinate

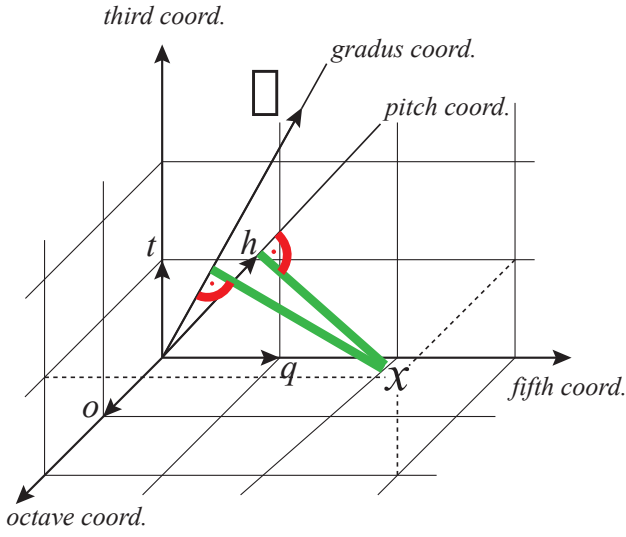



Figure B.9: The gradus function is something like a pitch function, but the 'direction' is $\Phi$ instead of $H_{prime}$, Both values are obtained from a scalar product of an Euler point $x$ (of a difference point $\Delta$, respectively) with $H_{prime}$ (with $\Phi$, respectively). However, the point is not uniquely determined by the 'gradus coordinate', in contrast to the 'pitch coordinate'.

defines a coordinate system for the Euler space on the one hand, and considers the consonances and dissonance, on the other. The disadvantage of Euler's approach is that it is based on the valence-theoretically invalid substitution hypothesis (see section B.2).

### B.3.2 von Helmholtz' Beat Model

Hermann von Helmholtz proceeds from the hypothesis that beats between partials of two tones is responsible for sonance phenomena. The fact that he uses partials is bound to Ohm's postulate that we have a cochlear Fourier analysis [217]. Hence Helmholtz' approach only regards the cochlear basis of music perception and not the higher limbic and cortical auditory processing. A *beat* is the periodic amplitude variation which results from the superposition of two sinoidal waves which have a frequency difference $\Delta = f - g$, the *beat frequency*[3] which is small with respect to their frequencies $f, g$.

Helmholtz calculates the *roughness*, i.e., the degree of dissonance of an interval which consists of two tones $p, q$ as the sum of the *beat intensities* $I_{n,m}$, which are associated with the $n$th partial of $p$ and the $m$th partial of $q$, where $I_{n,m}$ is supposed to have a strong maximum for beat frequency $\Delta_{n,m} = 33\ Hz$.

[3] One uses the trigonometric equation $\sin(x) + \sin(y) = 2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})$, see figure B.10.

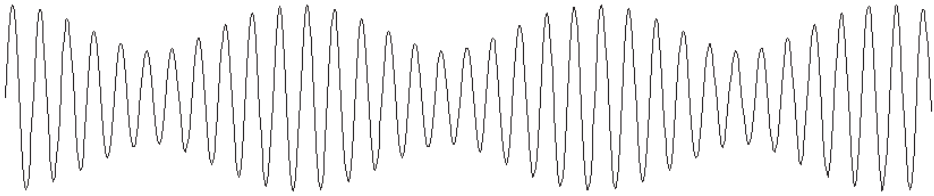


Figure B.10: A beat between two superposed sinoidal waves is characterized by a periodic amplitude variation whose frequency is twice (!) the difference of the given frequencies.

Therefore Helmholtz' dissonance concept depends on the pitches and the involved sound colors. On the example of the violin, Helmholtz obtained good coincidences with Euler's gradus function. This model is impressive since it explains the experience according to which consonance is a function of the instrument and the absolute pitch of the interval tones. Its experimental verification is somewhat problematic. It essentially depends on the measurability of the beat intensities in the cochlea. Since non-linear distortions on the sound's way to the cochlea change spectra, one would have to perform ethically problematic invasive cochlear measurements. Moreover, individual statistical variations of the non-linear distortions would make the experiments even less robust.

A fundamental doubt on the model's validity results from binaural experiments in hearing by Heinrich Husmann [242], where the interval tones are presented on separate left and right headphone inputs. In this case, no beats can intervene in the cochlea. Nonetheless, the experiments also revealed consonance as "happy moments of within the general (interval) disaster". The hypothesis that in the binaural experiments, Helmholtz beats must occur in the relays station of the corpus geniculatum mediale (see figure B.6) is speculative and demonstrates the limit of physiological models.

### B.3.3 Psychometric Investigations by Plomp and Levelt

In the psychological reality of interval perception, the judgment of interval qualities looks quite different. In their investigation of "pleasantness" of intervals, Reiner Plomp and Wilhelm Levelt [418] have presented pairs of sinoidal tones and asked for pleasantness as a function of the given interval. The experiment was intendedly performed with musically untrained individuals in order to avoid judgments as a function of musical knowledge. Figure B.11 shows the resulting valuation curve. It is quite different from Euler's function as shown in figure B.9.

Using this curve, Plomp and Levelt have tried to infer a description of Helmholtz' beat intensities, a procedure which was already recognized as being problematic in appendix B.3.2.

### B.3.4 Counterpoint

There is however another—quite remarkable—point of view of the consonance-dissonance phenomenon, which has been poorly recognized within the psychoacoustic discussion, namely the prominent meaning of the concept pairing in the contrapuntal tradition, which was elaborated

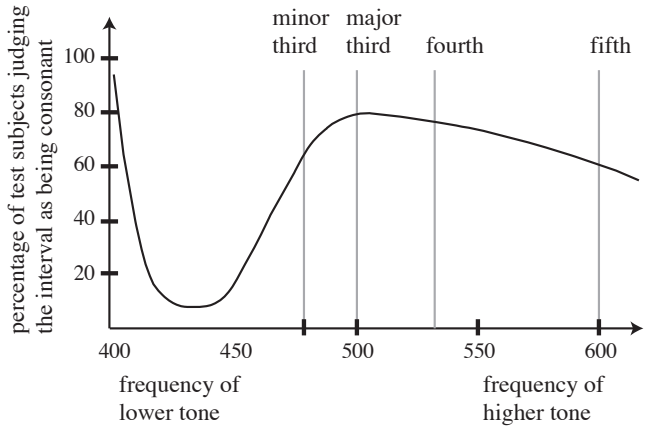



Figure B.11: (From [462],with permission of Springer-Verlag) The psychometric investigation [418] of Plomp and Levelt yields a valuation which differs significantly from Euler's Γ function and is based on sinoidal tones shown in figure B.9.

in the High and Late and Middle Ages, and which was encoded in an exemplary way by Johann Joseph Fux' *Gradus ad Parnassum* [174]. Carl Dahlhaus [99] has rightly pointed out that the textural function of the contrapuntal consonance concept is not yet fully understood. Interestingly, in the framework of the core theory of counterpoint, the interval of the fourth is dissonant, in contradiction to the other theories. It is inconceivable that the mathematical and physiological theories of counterpoint never included this perspective. A mathematical model of counterpoint is discussed in chapters 29 through 31.

### B.3.5 Consonance and Dissonance: A Conceptual Field

Even on the mental level, the concept of consonance and dissonance is multiply explained, without necessarily leading to contradictions. In fact, Euler's approach was a neutral mental (number-theoretic) one, whereas the contrapuntal approach is poietic. If we look at the status quo of the consonance-dissonance discussion and the fight for a valid final semantics, we are confronted with a disaster. The mathematically arbitrary ornaments which composers such as Klarenz Barlow [39] or neo-Pythagoreans such as Martin Vogel [549] add to Euler's formula does not interest psychoacousticians such as Ernst Terhardt [525], who would extrapolate cochlear findings into the auditory cortex, and the latter approach cannot shed light onto the esthetics of music.

What is common to all these positions adds to a concept of consonance and dissonance which is a conceptual field within the topography of music, a field with a quite ubiquitous presence. As a musical thought it results from a fundamental linear dynamic between polar extremals. It should be a main task of mathematical music theory to elaborate reliable and semantically reasonable, but esthetically undogmatic, models for such a way of thinking music.

# Part XVI

# Appendix: Mathematical Basics

# Appendix C

# Sets, Relations, Monoids, Groups

## C.1 Sets

The language of sets describes mathematical facts in a classical way. An alternative foundation to sets is the language of categories, see appendix G.

A *set* $M$ is an object which is defined as a collection of uniquely determined objects which are also sets. These objects are called the *elements* or *points* of $M$. Two sets are equal iff they have the same elements. Whenever we say that $M$ is a set, we mean that it is defined in a consistent way, i.e., without causing any contradiction. Existence of a mathematical object means that the object's definition causes no contradiction in classical logic (A is identical to A, (exclusive) either A or non A, and exclusion of a third). Then, for any set $m$ it is either an element of $M$ or it is not. One writes $m \in M$ for "$m$ is an element of $M$", or also "$m$ is a point of $M$". In order to define a set $M$ by its elements $m, m' \ldots$, one also writes $M = \{m, m', \ldots\}$. Observe that multiple enumeration does not change the set, for example $\{x, x, x, y\} = \{x, y\}$.

A set $N$ whose elements are all elements of $M$ is called a subset of $M$, in signs: $N \subseteq M$, also sometimes $N \subset M$ iff $N \subseteq M$ and $N \neq M$. Two sets are equal iff they are mutually subsets of each other. The *empty set* $\emptyset$ is defined as having no elements. It is a subset of any set. A set is called *finite* if is empty or its elements can be indexed[1] by a sequence[2] $0, 1, 2, 3, \ldots n$ of natural numbers. Otherwise it is called *infinite*.

[1] A mathematically correct definition of finiteness is this: A set if finite iff it is not in bijection with any proper subset.

[2] Recall that in this book, we make the logical, though not very common usage of the ellipsis symbol "...": it means that one has started with a sequence of symbol combinations which follows an evident law, such as $1, 2, \ldots n$, or $a_1 + a_2 + \ldots a_n$. The evidence is built upon the starting unit, such as "1," or "$a_1+$" in our examples, and then the following unit, such as "2," or "$a_2+$", and then inducing the following units to be denoted, such as "3," or "$a_3+$", "4," or "$a_4+$", etc., until the sequence is terminated by the last symbol, such as "$n$" or "$a_n$" in our examples. The ellipsis means that the building law is repeated, and as such, it is a meta-sign referring to the inductive offset. Therefore the more common notation $1, 2, \ldots, n$, or $a_1 + a_2 + \ldots + a_n$ is not correct. In the limit, for $n = 3$, it would imply a notation such as $1, 2, , 3$ or $a_1 + a_2 + +a_3$, which is nonsense. Moreover, in complicated indexing situation, the common notation would be overloaded.

### C.1.1 Examples of Sets

**Example 62** $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \pm n, \ldots\}$, the set of *integers*; $\mathbb{N} = \{n \in \mathbb{Z}, n \geq 0\}$, the set of *natural numbers*; the set $\mathbb{Q} = \{p/q, p, q \in \mathbb{Z}, q \neq 0\}$ of *rational numbers*; the set $\mathbb{R}$ of *decimal or real numbers*, e.g., $x = -741.76, \pi = 3.1415926\ldots$. We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, where integers $p$ are identified with rational numbers of form $p/1$, whereas rational numbers are identified with periodic real numbers.

**Example 63** If $M, N$ are sets, their *difference* $M - N$ or *the complement of* $N$ *in* $M$ is the set of points of $M$ which are not in $N$. If $V = (M_i)_i$ is a family of sets, we denote by $\bigcup V$ the *union of* $V$, whose elements are precisely the elements collected from any of the $M_i$, for finite families $V = M_1, \ldots M_n$, one also writes $\bigcup V = M_1 \cup \ldots M_n$. In particular, $\bigcup \emptyset = \emptyset$. A covering of a set $X$ is a family $V$ such that $\bigcup V = X$. The *intersection of a family* $V$ is the set $\bigcap V$ consisting exactly of those points which are points in any of the family's member $M_i$. For finite families $V = M_1, \ldots M_n$, one also writes $\bigcap V = M_1 \cap \ldots M_n$. In particular, $\bigcap \emptyset = AllSet$, the set whose elements are all existing sets. In less universal contexts, one only takes the intersection with regard to a large superset of the family's members. Observe that there is no reason why $AllSet$ should not exist, however, not any of its subcollections defined by predicates equally exists. For example, the subset of all sets not containing themselves as elements does not.

A *partition of a non-empty set* $X$ is a covering such that any two of its members are non-empty and *disjoint*, i.e., intersect in the empty set.

## C.2 Relations

**Definition 114** *If* $x, y$ *are two sets, the ordered pair* $(x, y)$ *is defined to be the set*

$$(x, y) = \begin{cases} \{\{x\}\} \text{ if } x = y, \\ \{\{x\}, \{x, y\}\} \text{ else.} \end{cases} \tag{C.1}$$

**Lemma 55** *For any four sets* $a, b, c, d$*, we have* $(a, b) = (c, d)$ *iff* $a = c, b = d$.

A triple $(a, b, c)$ is a pair of form $((a, b), c)$. Clearly, $(a, b, c) = (a', b', c')$ iff $a = a', b = b', c = c'$.

**Definition 115** *Given two sets* $A, B$ *their* Cartesian product *is the set*

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

*consisting of all ordered pairs* $(a, b)$*, with the* first coordinate *a an element in the* first factor *A, and the* second coordinate *b an element in the* second factor *B. A relation from A to B is a subset* $R \subseteq A \times B$*. One writes* $aRb$ *for* $(a, b) \in R$*, and, if the relation is clear, more simply* $a \sim b$*. The inverse of a relation is the set* $R^{-1} = \{(b, a) | aRb\}$*. If* $A = B$*, one also speaks of a relation on A.*

**Definition 116** *A* graph $f : A \to B$ from $A$ to $B$ *is a triple* $(A, B, f)$ *with* $f$ *a relation from A to B; it is called*

(i) total *iff every* $a \in A$ *is the first factor of a pair in* $f$*;*

(ii) functional *iff* $(a,b),(a,b') \in f$ *implies* $b = b'$*; one writes* $f(a)$ *instead of the uniquely determined* $b$*, or also* $f : a \mapsto b$*.*

(iii) *a* function *or* map *iff it is a total functional graph. The set* $A$ *is the* domain of $f$*, whereas* $B$ *is the function's* codomain*.*

If $f : A \to B$ and $g : B \to C$ are two functions, their *composition* or *concatenation* function $g \circ f : A \to C$ is defined by $g \circ f(a) = g(f(a))$. The function $f : A \to A$ with $f(a) = a$ for all $a \in A$ is called the identity on $A$ and is denoted by $Id_A$.

**Exercise 83** Verify that $g \circ f$ is indeed a function. If $h : C \to D$ is a third function, we have $(h \circ g) \circ f = h \circ (g \circ f)$ and therefore we write $h \circ g \circ f$. Show that $Id_B \circ f = f \circ Id_A = f$.

**Definition 117** *A function* $f$ *is called*

(i) injective *if* $f(a) = f(a')$ *always implies* $a = a'$*;*

(ii) surjective*, iff for every* $b \in B$*, there is* $a \in A$ *such that* $f(a) = b$*;*

(iii) bijective*, iff* $f$ *is injective and surjective.*

**Lemma 56** *For a function* $f : A \to B$*, the following conditions are equivalent:*

(i) *There is a function* $g : B \to A$ *such that* $g \circ f = Id_A$ *and* $f \circ g = Id_B$*.*

(ii) *The function* $f$ *is bijective.*

The $g$ in this lemma is uniquely determined by $f$ and is called the *inverse function of* $f$, it is denoted by $f^{-1}$. Bijections $f : A \to A$ are also called *permutations of* $A$. If there is a bijection $f : A \to B$ between two sets $A, B$, we say that *they have the same cardinality*, and write $card(A) = card(B)$. On $AllSet$, the cardinality relation is an equivalence relation, and one may define the *cardinality* $card(A)$ *of a set* $A$ as the equivalence class $[A]$ under this relation. A finite set is one whose cardinality is that of a set of form $\{1, 2, 3, \ldots n\}$, with a natural number $0 \leq n$, where we take the empty set for $n = 0$.

**Definition 118** *A binary relation* $\leq$ *on a set* $S$ *is said to be*

(i) reflexive *iff* $x \leq x$ *for all* $x \in S$*;*

(ii) transitive *iff* $x \leq y$ *and* $y \leq z$ *implies* $x \leq z$ *for all* $x, y, z \in S$*;*

(iii) symmetric *iff* $x \leq y$ *implies* $y \leq x$ *for all* $x, y \in S$*;*

(iv) antisymmetric *iff* $x \leq y$ *and* $x \neq y$ *excludes* $y \leq x$ *for all* $x, y \in S$*;*

(v) total *iff* $x \leq y$ *or* $y \leq x$ *for all* $x, y \in S$*.*

**Definition 119** *A binary relation $\leq$ on a set $S$ is called an* equivalence relation *iff it is reflexive, transitive, and symmetric. In this case, the relation is usually denoted by "$\sim$" instead of "$\leq$".*

**Lemma 57** *Let $\sim$ be an equivalence relation on $S$. Then the subsets $[s] = \{t | s \sim t\}$ are called* equivalence classes *of $\sim$. The set of equivalence classes is denoted by $S/\sim$. It defines a partition of $S$, i.e., it covers $S$, and for any two elements $s, t \in S$, either $[s] = [t]$ or $[s] \cap [t] = \emptyset$.*

**Definition 120** *A binary relation $\leq$ on a set $S$ is called a* partial ordering *iff it is reflexive, transitive, and antisymmetric. A partial ordering is called* linear *iff it is total. A linear ordering is called* well-ordered *iff every non-empty subset $T \subset S$ contains a minimal element.*

**Lemma 58** *Let $\leq$ be a binary relation on a set $S$. Denoting $x < y$ iff $x \leq y$ and $x \neq y$, the following two statements are equivalent:*

(i) *The relation $\leq$ is a partial ordering.*

(ii) *The relation $\leq$ is reflexive, the relation $<$ is transitive, and for all $x, y \in S$, $x < y$ excludes $y < x$.*

*If these equivalent properties hold, we have $x \leq y$ iff $x = y$ or else $x < y$. In particular, if we are given $<$ with the properties* (ii)*, and if we* define *$x \leq y$ by the preceding condition, then the latter relation is a partial ordering.*

Proof. (i) $\Rightarrow$ (ii) Clearly $\leq$ is reflexive. If $x < y$ and $y < z$, then $x \leq z$. If we had $x = z$, then we were in the second statement of (ii), and it suffices to prove this one. But $x < y$ and $y < x$ implies $x = y$ by the asymmetry of $\leq$, a contradiction. (ii) $\Rightarrow$ (i) Transitivity: If $x \leq y$ and $y \leq z$, and either $x = y$ or $y = z$, we are done. Otherwise $x < y$ and $y < z$, whence $x < z$, therefore $x \leq z$. Asymmetry: If $x \leq y$ and $y \leq x$, then $x \neq y$ is excluded by (ii) whence the claim. The last statement is clear, QED.

**Example 64** Suppose that $(I, <)$ is a linearly ordered "index set", and that we are given a family $((T_i, <_i, s_i))_I$ of linearly ordered sets $(T_i, <_i)$, each having a distinguished "constant" element $s_i$. Consider the set $W$ of all nearly constant "words" $(t_i)_i \in \prod_i T_i$, i.e., $t_i = s_i$ except for a finite set of indexes. Then $W$ has a canonical linear ordering, the *lexicographic ordering*, defined as follows. If we are given two different words, consider the smallest index where they differ. The smaller letter at that index defines the smaller word. Prove that this is a linear ordering.

In the usual lexicographic situation, the sets $T_i$ are all an alphabet with its alphabetic linear ordering—as expressed, for example, by the $ASCII$ or $UNICODE$ encoding by natural numbers—, and the empty space as constant element. The indexes are natural, $i = 0, 1, 2, \ldots$.

**Axiom 6** (Axiom of choice) *Let $V$ be a partition of a set $X$. Then there is a subset $Y \subseteq X$ such that for every member $M_i$ of the family $V$, the intersection $M_i \cap Y = \{m_i\}$ is a one-element set (a singleton).*

This is one of the many equivalent versions of the axiom of choice [281]. A particularly important variant is this:

**Theorem 38** (Zermelo) *If the axiom of choice holds there is a well-ordered relation for every set. Conversely, if every set can be well ordered, the axiom of choice holds.*

For a proof, see [281, p.261].

**Proposition 65** *Let $\leq$ be a partial ordering on a set $S$. Let $\preceq$ be the following binary relation on the set $Fin(S)$ of finite subsets of $S$. For $A, B \subset S$, let*

$$A \preceq B \begin{cases} A = B \text{ or} \\ A \neq B \text{ and for all } x \in A - B, \text{ there is } y \in B - A \text{ with } x < y. \end{cases} \tag{C.2}$$

*Then $\preceq$ is a partial ordering. If $\leq$ is linear, so is $\preceq$.*

Proof. We set $A \prec B$ as in lemma 58. If $\leq$ is total, then for any $A \neq B$ in $Fin(S)$, if both disjoint difference sets $A-B$ and $B-A$ are non-empty, they contain different maximal elements, and we are done. Next, we verify statement (ii) about $\prec$. Suppose that we have $A \prec B$ and $B \prec A$ for $A \neq B$. Hence, we may suppose $A - B \neq \emptyset$. Take a maximal $x \in A - B$. There is $y \in B - A$ with $x < y$. Then there is $z \in A - B$ with $y < z$, but by transitivity of $<$, $x < z$, a contradiction to maximality of $x$. To show transitivity, take $A \neq B \neq C$ in $Fin(S)$. Since $A - C = \emptyset$ is trivial, suppose that $x$ is a maximal element of $A - C$.

Suppose first that $x \in B$. Then $x \in B - C$, and there is $x < z$ with $z \in C - B$, take a maximal element $z$ of this type. If $z \in A$, then $z \in A - B$, hence there is $z < w$, $w \in B - A$. If $w \in C$, $w \in C - A$, and $x < z < w$ which was required. Else, $w \notin C$ whence $w \in B - C$, and there is $u \in C - B$ with $w < u$, but $z < w < u$ contradicts maximality of z in $C - B$. Now if $z \notin A$, we have $z \in C - A$, and we are also done.

Suppose now that $x \notin B$, and therefore $x \in A - B$. There is $z \in B - A$ with $x < z$, take a maximal such element in $B - A$. If $z \in C$, then $z \in C - A$ and there is $w \in C - B$ with $z < w$. If $w \notin A$, then $w \in C - A$ and $x < z < w$ which was required. If $w \in A$ we have $w \in A - B$, and there is $u \in B - A$ with $u < w$. But then $z < w < u$ contradicts maximality of $z$ in $B - A$, QED.

**Definition 121** *For a finite partial ordering $Rel$ on a set $X$, the* Hasse diagram *$Rel(X)$ is the relation whose pairs are the uniquely determined minimal set of generating relations of $Rel$.*

Proof of uniqueness. Suppose that two minimal generating sets $A, B$ are different. Then any relation of $A - B$ points at a decomposition with a factor in $B - A$, and so forth, vice versa, such that we obtain an infinite chain of relations, contradicting the finiteness of $X$.

For a finite partial ordering $Rel$ on $X$, the level function $lev : X \to \mathbb{N}$ is defined as follows. For minimal elements $x$ in the Hasse diagram $Rel(X)$, put $lev(x) = 0$, for any element $x \in X$, set $pre(x) = \{y |\ y < x \text{ in } Rel(X)\}$. Then we put $lev(x) = 1 + Max(lev(y), y \in pre(x))$.

### C.2.1 Universal Constructions

If $V = (M_i)_{i\in I}$ is a family of sets, the *product* set $\prod_I M_i$ is the set whose elements are the functions $f : I \to \bigcup V$ such that $f(i) \in M_i$ for all $i \in I$. Evidently, pairs, triples are special cases of such functions, but they are basic to the definition of functions, and therefore are treated separately. For each $j \in I$, we have the projection function $p_j : \prod_I M_i \to M_j$. Here is the *universal*[3] *property* of the product[4] set:

**Lemma 59** *With the above notation, if $(f_i : X \to M_i)_i$ is a family of functions, then there is exactly one function $f : X \to \prod_I M_i$ such that $f_i = p_i \circ f$ for all $i \in I$, i.e., $f(x)(i) = f_i(x)$.*

For the given family $V$, the *coproduct* set $\coprod_I M_i$ is the union of the family $V' = (\{i\} \times M_i)_i$. For each index $j$, we have a function $\iota_j : M_j \to \coprod_I M_i$ defined by $\iota_j(m) = (j, m)$. The *universal property* of the coproduct[5] is this:

**Lemma 60** *For every family $(f_i : M_i \to X)_i$ of functions, there is exactly one function $f : \coprod_I M_i \to X$ such that $f \circ \iota_i = f_i$, i.e., $f((i, m)) = f_i(m)$.*

**Lemma 61** *If $f : A \to B$ is a function, then the* image *$Im(f) = \{b | b \in B,$ there exists $a \in A$ such that $b = f(a)\}$, together with the inclusion function $i : Im(f) \to B$ and the surjective function $f' : A \to Im(f) : a \mapsto f(a)$ has the following* universal property*: We have $f = i \circ f'$, and for every factorization $f = u \circ v$, there is a unique factorization $u = i \circ h$, and $f' = h \circ v$.*

**Lemma 62** *Let $A^B$ be a* powerset*, i.e., the set of functions $f : B \to A$. Then there is a natural bijection of* adjunction *$ad : C^{A\times B} \to (C^B)^A$, defined by $ad(g)(a)(b) = g(a, b)$.*

For a function $f : A \to B$ and a subset $C \subseteq B$, we define by $f^{-1}(C) = \{x | x \in A, f(x) \in C\}$ the *inverse image of* $C$ under $f$; if $C = \{c\}$ is a singleton, we also write $f^{-1}(c)$ instead and call the set the *fiber of* $c$.

**Lemma 63** *Let $2 = \{0, 1\}, 0 = \emptyset, 1 = \{0\}$, and let $Sub(X)$ be the set of subsets of $X$ (also called the* powerset of $X$*). Then there is a natural bijection $\chi : Sub(X) \to 2^X$ defined by $\chi(Y)(z) = 0$ iff $z \in Y$, and $\chi(Y)(z) = 1$ iff $z \notin Y$. The function $\chi(Y)$ is called the* characteristic function of $Y$*. The inverse function maps $c : X \to 2$ to $Y_c = c^{-1}(0)$.*

### C.2.2 Graphs and Quivers

**Definition 122** *For a set $X$, denote by $P_2(X)$ the set of subsets of cardinality one or two in $X$, i.e., the set of singletons and unordered pairs in $X$. Then a* multigraph *is a triple $(L, V, G : L \to P_2(V))$. The elements of $L$ are called the* lines*, the elements of $V$ are called the* vertexes *of the multigraph. Often, we identify the multigraph with its map $G$. If $G$ is injective, the multigraph is called a* graph*. A multigraph is finite, iff all involved sets are finite.*

[3] This is a terminology from category theory, stating (in its most generic form) that a certain object in a category is final, but see appendix G.

[4] This is a special case of a limit set, see appendix G.2.1.

[5] This is a special case of a colimit set, see appendix G.2.1.

**Definition 123** *A* quiver *is a pair* $G = (\mathrm{head}, \mathrm{tail} : A \rightrightarrows V)$ *of set maps. The elements of* $V$ *are called* vertexes*, the elements of* $A$ *are called* arrows*. If every pair of vertexes is head and tail of at most one arrow, the quiver is called a* directed graph*.*

*A* path $p$ in $G$ *is either a vertex* $p = v$ *(a 'lazy' path) or else a sequence* $p = (a_0, a_1, \dots a_n)$ *of arrows with* $\mathrm{head}(a_i) = \mathrm{tail}(a_{i+1})$ *for each index* $i = 0, 1, \dots n-1$*. The length* $l(p)$ *of a path is 0 for a lazy path, and* $n$ *for a general path. A* closed path *is a path with* $\mathrm{head}(a_n) = \mathrm{tail}(a_0)$*. A lazy path is also a closed path of length 0. A closed path of length 1 is also called a* loop*. A* cycle *is an equivalence class of closed paths which differ from each other by their start/end point.*

*We also use the symbols* $\mathrm{head}(p), \mathrm{tail}(p)$ *for the head, or tail, respectively, of the last, or first arrow of* $p$*, respectively (or just the unique vertex* $v$ *for the lazy path).*

**Example 65** For a set $V$, a *complete* quiver is a quiver $G = (head, tail : A \rightrightarrows V)$ such that the map

$$f : A \to V^2 : a \mapsto (\mathrm{tail}(a), \mathrm{head}(a))$$

is a bijection.

For two paths $p, q$ with $\mathrm{head}(q) = \mathrm{tail}(p)$, we have a composed path $p.q$ which is the evident juxtaposition of arrow sequences or the respectively other path if one path is lazy. As with functions, the composition of paths is associative if it is defined.

## C.2.3 Monoids

**Definition 124** *A* semigroup *is a couple* $(M, \mu)$ *consisting of a set* $M$ *and an associative binary operation* $\mu : M \times M \to M : (m, n) \mapsto \mu(m, n) = m.n$*. The semigroup* $M$ *is a* monoid *if there exists a* neutral element $e$*, i.e.,* $e.m = m.e = m$ *for all* $m \in M$*. Since* $e$ *is uniquely determined by this property, it is called* the *neutral element. For a given subset* $S \subset M$ *of a semigroup, the semigroup* $\langle S \rangle$ *generated by* $S$ *is the smallest sub-semigroup of* $M$ *containing* $S$*. If* $M$ *is a monoid, the submonoid with neutral element* $e$ *generated by* $S$ *is denoted by* $\langle S \rangle_e$*. If* $m.n = n.m$ *for all* $m, n \in M$*, the semigroup is called* commutative*. An element* $x \in M$ *such that there is* $y$ *with* $x.y = e$ *is called invertible, its set* $M^\star$ *is a submonoid. A monoid* $(M, \mu)$ *is* finite*, iff the underlying set* $M$ *is so.*

*Given two monoids* $M, N$ *a* monoid homomorphism *is a set map* $f : M \to N$ *such that* $f(e_M) = e_N$ *and* $f(m.m') = f(m).f(m')$ *for all* $m, m' \in M$*. A* monoid isomorphism *is a bijective monoid homomorphism, its inverse set map is then automatically a monoid isomorphism.*

**Exercise 84** Show that the map $N \mapsto N_e = N \cup \{e\}$ defines a projection from the set of sub-semigroups of $M$ onto the set of sub-monoids of $M$ with $N_e = N$ iff $N$ is a submonoid and $N_e = N'_e$ iff $N = N'$ for sub-semigroups $N, N'$ which are not sub-monoids.

**Example 66** For a set[6] *Alphabet*, there is a monoid $FM(Alphabet)$ and an injection

$$i : Alphabet \to FM(Alphabet)$$

[6] E.g. $Alphabet = ASCII, UNICODE$.

such that for any set map $f : Alphabet \to M$ into a monoid $M$, there is exactly one monoid homomorphism $FM(f) : FM(Alphabet) \to M$ such that $FM(f) \circ i = f$. One calls the monoid $FM(Alphabet)$ the *free monoid over Alphabet* and the above property is the universal property that characterizes the free monoid up to isomorphism. The free monoid consists of all "word" expressions $b_1 \dots b_k$ for $b_i \in Alphabet, k > 0$, and the empty word (). The product is defined by juxtaposition of words, () is neutral. The free monoid is also called "word monoid over the alphabet *Alphabet*".

Consider the equivalence relation on $FM(Alphabet)$ with $b_1 \dots b_k \sim b_{\pi(1)} \dots b_{\pi(k)}$ for any permutation $\pi$, then the quotient space $FCM(Alphabet) = FM(Alphabet)/\sim$ with the induced multiplication is a commutative monoid and is called the *free commutative monoid over Alphabet*.

**Definition 125** *In a semigroup $M$, an* idempotent *is an element $x$ such that $x = x.x = x^2$. We denote $Idempot(M) = \{x|\ x \text{ is idempotent in } M\}$. For a subset $X \subset M$, we define its* radical *by*

$$\sqrt{X} = \{y|\exists n, 0 < n,\ \text{such that } y^n \in X\}.$$

*In particular, given an idempotent $x$, the radical $\sqrt{x}$ of $x$ is defined by*

$$\sqrt{x} = \{y|\exists n, 0 < n,\ \text{such that } x = y^n\}.$$

*Since for every $y \in \sqrt{x}$, $\langle y\rangle \subset \sqrt{x}$, we also have $\sqrt{x} = \bigcup_{y\in\sqrt{x}} \langle y\rangle$.*

**Example 67** For example, the neutral element in a monoid is idempotent.

**Example 68** If $Q$ is a left module[7] over a ring $R$, the set of idempotents $Idempot(End(Q))$ of the linear endomorphism ring of $Q$ is in bijection with the set $Dir(Q)$ of direct decompositions $Q = U \oplus V$ of $Q$. In fact, the bijection is set up by

$$Idempot(End(Q)) \xrightarrow{\sim} Dir(Q) : x \mapsto (Im(x), Im(Id_Q - x))$$

whose inverse defines the projection $p : Q \to U$ onto the factor $U$ of $Q = U \oplus V$. The image of an idempotent $x$ is a direct decomposition by the equation $Id_Q = x + (Id_Q - x)$ which yields $Q = Im(x) \oplus Im(Id_Q - x)$ with $x = Id_{Im(x)} \oplus 0_{Im(Id_Q - x)}$. Conversely, the projection $x = pr_U$ yields $U = Im(x), V = Im(Id_Q - x)$. So the above map is a bijection.

**Lemma 64** *The idempotents of the monoid $Q@Q$ of affine endomorphisms[8] are the elements $e^q.y$ such that $y \in Idempot(End(Q))$ and $q \in Im(Id_Q - y) = Ker(y)$.*

**Example 69** For example, the idempotents of $\mathbb{Z}_{12}$ are $0, 1, 4, 9$, and $Ker(0) = \mathbb{Z}_{12}, Ker(1) = 0, Ker(4) = 3.\mathbb{Z}_{12}, Ker(9) = 4.\mathbb{Z}_{12}$, so we have a total of 20 affine idempotents here.

**Lemma 65** *With the previous notation, for two idempotent elements $x, y$ of a semigroup $M$, $\sqrt{x} \cap \sqrt{y} \neq \emptyset$ iff $x = y$. Hence we have a partition of radicals in $M$, viz., $\sqrt{Idempot(M)} = \coprod_{x\in Idempot(M)} \sqrt{x}$, the* idempotent components[9] *of $M$.*

[7] See appendix E.
[8] See appendix E.
[9] Terminology of Noll, [400].

Clearly, if $f : M \xrightarrow{\sim} M$ is an automorphism of the semigroup $M$, we have $f(\sqrt{x}) = \sqrt{f(x)}$. In particular, if $M = Q@Q$ for a module $Q$, and if $y \in End(Q)$ is an idempotent linear endomorphism of $Q$, the conjugation by a translation $e^q, q \in Q$ gives the formula

$$e^q \sqrt{y} e^{-q} = \sqrt{e^{(1-y)q}.y} \tag{C.3}$$

which means that the translation exponent $(1-y)q$ is a general element of $Ker(y)$. Therefore:

**Proposition 66** *For a module $Q$, the orbits of idempotent components under the conjugation action of the translation group $e^Q$ are the sets of idempotent components associated with the linear idempotent endomorphisms of $Q$.*

**Lemma 66** *The canonical surjective linear factor projection $p : Q@Q \to End(Q)$ is compatible with the idempotent partitions, and the fiber of every idempotent component in $End(Q)$ is the orbit of the translation group action under conjugation (see proposition 66).*

**Proposition 67** *If in the above example the module $Q$ is finite, the respective idempotent components define partitions*

$$\sqrt{Idempot(End(Q))} = End(Q)$$

*and*

$$\sqrt{Idempot(Q@Q)} = Q@Q.$$

Proof. Let $y \in End(Q)$. By Fitting's lemma (appendix 77), there is a positive power $z = y^n$ and a direct decomposition $Q = U \oplus V$ such that $z|U \in GL(U), Ker(z) = V$. So by finiteness of $U$, there is a positive power $(z|U)^m = Id_U$. Therefore, $z^m = y^{nm}$ is an idempotent. As to the affine case, if $y = e^t.y_0 \in Q@Q$, we know by the preceding that a power $y_0^m = t$ is idempotent. So WLOG., we may suppose that $y_0$ is idempotent. Set $e^{t_k}.y_0 = (e^t.y_0)^k = y^k$; the positive powers of $y$ must have recurrent values, so take $y^u = y^{u+k}$, for positive $k$. Then $e^{t_u}.y_0 = e^{t_u}.y_0.e^{t_k}.y_0 = e^{t_u + y_0(t_k)}.y_0$, i.e., $t_k \in Ker(y_0)$, whence $y^k$ is an idempotent by lemma 64, QED.

**Lemma 67** *For the subgroup[10] $M^\star$ of invertible elements of the monoid $M$, we have the intersection formula*

$$M^\star \cap \sqrt{Idempot(M)} = \sqrt{1_M}.$$

Proof. Clearly, $\sqrt{1_M} \subset M^\star$. On the other hand, if $x \in \sqrt{t} \subset M^\star$, we have $x^k = t = t^2 = x^{2k}$, whence $1_M = x^k = t$.

In the special case of a total partition $M = \sqrt{Idempot(M)}$, we get

$$M^\star = \sqrt{1_M}.$$

This is the case for an affine endomorphism monoid $Q@Q$ of a finite module $Q$.

[10] See section C.3.

## C.3 Groups

**Definition 126** *A monoid $(M, \mu)$ with the neutral element $e$ such that every element $m$ has a left inverse $n$, i.e., $n.m = e$, is called a* group. *A left inverse of $m$ is also a right inverse, $m.n = e$, and the inverse is uniquely determined by $m$, it is denoted by $m^{-1}$. A subgroup of a group $M$ is a submonoid $N$ which is a group. A commutative group is also called* abelian. *If $n \in \mathbb{N}$, then we write $m^n$ for the $n$-fold product $m.m.\ldots.m$; if $n$ is a negative integer, then we write $m^n = (m^{-1})^{|n|}$, we also set $m^0 = e$. Evidently $m^n.m^l = m^{n+l}$.*

If a group is abelian, the product is usually written additively, i.e., $m + n$ instead of $m.n$, further $m - n$ instead of $m.n^{-1}$, and the neutral element is noted by 0 instead of $e$.

**Exercise 85** Observe that the neutral element $e$ of a group is the only idempotent element $x.x = x$. Therefore, a subgroup has necessarily the same neutral element as the supergroup.

### C.3.1 Homomorphisms of Groups

If $(G, \gamma)$, $(H, \eta)$ are two groups, a set map $f : G \to H$ is called a *group homomorphism* iff $f(g.g') = f(g).f(g')$ (products in the respective groups). We have $f(e_G) = e_H$ and $f(x^{-1}) = f(x)^{-1}$. Clearly, the set-theoretic composition of two group homomorphisms is also a group homomorphism. The set of group homomorphisms $f : G \to H$ is denoted by $Hom(G, H)$. If $f$ is a bijection, its inverse is also a group homomorphism; $f$ is then called a *group isomorphism*. $G, H$ are said to be *isomorphic* iff there is an isomorphism between them. Clearly, group isomorphisms are an equivalence relation, whence the term "isomorphism classes". The explicit description of isomorphism classes is the main task of group theory. For finite groups, this is essentially solved[11]. For finite commutative groups, the classification is described in appendix C.3.4.

The set $Aut(G)$ of isomorphisms of a group $G$ onto itself (the *automorphisms* of $G$) is a group under the composition of group homomorphisms, the identity $Id_G$ being the neutral element of $Aut(G)$. The group $\mathfrak{S}_M$ of permutations of a set $M$ is called the *symmetric group of $M$*.

**Example 70** A homomorphism $\mu : G \to \mathfrak{S}_M$ is called a *left action of $G$ on $M$*. Let $G^{opp}$ denote the *opposite group to* $G$ where products are interchanged: $g.^{opp}h = h.g$. Then a right action is a homomorphism $\mu : G^{opp} \to \mathfrak{S}_M$. A left action is equivalent to a map $\mu : G \times M \to M$ (same notation for $\mu$) with $\mu((e, m) = m, \mu(g(\mu(h, m)) = \mu(g.h, m)$, all $g, h \in G, m, n \in M$. If the action is clear, we write $g.m$ instead of $\mu(g, m)$. We usually mean *left* actions when we speak of actions.

A group action $\mu$ is called *effective* iff it is an injective homomorphism. For an element $m \in M$, the group $G_m = \{g \in G, g.m = m\}$ is called the *stabilizer* or *isotropy* or *fixpoint group of $m$*. If all stabilizers are trivial, the action is called *(fixpoint) free* or *faithful*.

For a group action $\mu$, an *orbit* is a set of form $G.m = \{g.m | g \in G\}$. The *orbit space* $G \setminus M$ defines a partition of $M$, with the canonical map $\pi : M \to G \setminus M : m \mapsto G.m$. The action is

[11] The classification of all finite simple groups is one of the main results in group theory of the $20^{th}$ century, see [188].

*transitive* iff the orbit space is a singleton. Clearly

$$card(G.m).card(G_m) = card(G).$$

If $\mu_1 : G \times M \to M, \mu_1 : G \times N \to N$ are two group actions of group $G$, and if $h : M \to N$ is a set map, we say that $h$ is *equivariant* iff it commutes with these group actions, i.e., if for all $m \in M, g \in G$, we have $h(\mu_1(g, m)) = \mu_2(g, h(m))$.

**Example 71** For a set $Alphabet$, there is a group $FG(Alphabet)$ and an injection $i : Alphabet \to FG(Alphabet)$ such that for any set map $f : Alphabet \to G$ into a group $G$, there is exactly one group homomorphism $FG(f) : FG(Alphabet) \to G$ such that $FG(f) \circ i = f$. The group $FG(Alphabet)$ is called the *free group over* $Alphabet$ and the above property is the universal property that characterizes the free group up to isomorphism. The free group consists of all "word" expressions $b_1^{n_1} \dots b_k^{n_k}$, where $b_i \in Alphabet, n_i \in \mathbb{Z}$ which are reduced, i.e., $b_i \neq b_{i+1}, n_i \neq 0$, and the empty word $()$. The product is defined by juxtaposition and cancelling of powers of adjacent letters.

**Lemma 68** *For a group homomorphism $f : G \to H$, $Im(f)$ is a subgroup of $H$. For a subgroup $I \subseteq H$, the inverse image $f^{-1}(I)$ is a subgroup of $G$. The inverse image of the trivial group (the singleton consisting of the neutral element $e_H$) in $H$ is called the* kernel of $f$ *and is denoted by $Ker(f)$*

**Example 72** For every element $g \in G$ of a group, we have a special group automorphism

$$Int_g : G \to G : h \mapsto g.h.g^{-1}$$

the *conjugation with* $g$. This yields a group homomorphism $Int : G \to Aut(G)$.

**Proposition 68** *For a group $G$, a subgroup $H$ and an element $g \in G$, we write $gH = \{g.h | h \in H\}, Hg = \{h.g | h \in H\}$ for the* left and right cosets *of $H$. The set of left, right cosets of $G$ is denoted by $G/H$, $H \setminus G$, respectively. Denoting $(G : H) = card(G/H)$, we have the Lagrange equation*[12]*:*

$$card(G) = card(H).(G : H)$$

*and in particular, if $G$ is finite, any subgroup cardinality divides the* order $card(G)$ of $G$.

**Proposition 69** *If $H$ is a subgroup of $G$, the following statements are equivalent:*

(i) *For all $g \in G$, $gH = Hg$.*

(ii) *There is a group homomorphism $f : G \to K$ with $Ker(f) = H$.*

If $H$ has the properties of proposition 69, it is called a *normal subgroup of* $G$, in symbols: $H \triangleleft G$. A *simple* group is one which has no normal subgroup except the trivial and the full subgroup. The group homomorphism in proposition 69 is constructed by the *quotient group* $G/H$ structure on $G/H$, where the product is defined by $gH.g'H = g.g'H$. The *canonical* homomorphism $f$ is defined by $K = G/H$, and $f(g) = gH$.

[12] Define $card(A).card(B) = card(A \times B)$.

**Proposition 70** *Let $H \triangleleft G$ be a normal subgroup, $\pi : G \to G/H$ the canonical homomorphism, and $K$ any group. Then*

$$Hom(G/H, K) \xrightarrow{\sim} \{f \in Hom(G,K) | H \subseteq Ker(f)\} : t \mapsto t \circ \pi$$

*is a canonical bijection. If $H = Ker(f)$, then the corresponding morphism $G/H \to K$ is an isomorphism onto $Im(f)$.*

**Exercise 86** If $f : G \to H$ is an isomorphism of groups, then the map

$$Int_f : Aut(G) \to Aut(H) : t \mapsto f.t.f^{-1}$$

is an isomorphism of groups.

Here are the basic isomorphism theorems:

**Theorem 39** (First isomorphism theorem) *Let $G, H, N$ be groups with $N \triangleleft G, H \subseteq G$. Then:*

(i) *The product set $H.N$ is a subgroup.*

(ii) *We have $N \triangleleft H.N$ and $N \cap H \triangleleft H$.*

(iii) *We have an isomorphism $H/N \cap H \xrightarrow{\sim} H.N/N : h(N \cap H) \mapsto hN$.*

**Theorem 40** (Second isomorphism theorem) *Let $N \triangleleft G$ be a normal subgroup.*

(i) *The normal subgroups $\bar{M} \triangleleft G/N$ are in bijection with the subgroups $M$ with $N \triangleleft M \triangleleft G$, and*

(ii) *we have an isomorphism $G/M \xrightarrow{\sim} (G/N)/(M/N) : gM \mapsto (gN).M/N$.*

## C.3.2 Direct, Semi-direct, and Wreath Products

Given a family $(G_i)_{i \in I}$ of groups, the set-theoretic product $\prod_I G_i$ becomes a *(direct) product group* via the coordinate-wise product, i.e., $(g_i).(h_i) = (g_i.h_i)$.

Given two groups $H, N$ and a group homomorphism $\phi : H \to Aut(N)$, the *semidirect product* $N \rtimes_\phi H$ is the group structure on the set $N \times H$ defined by

$$(n, g).(m, h) = (n.{}^g m, g.h)$$

where ${}^g m = \phi(g)(m)$. We have two group injections $i : N \to N \rtimes_\phi H : n \mapsto (n, e_H)$ and $j : H \to N \rtimes_\phi H : h \mapsto (e_N, h)$, and $i(N) \triangleleft N \rtimes_\phi H$. Further, $i(N) \cap j(H) = e$, $i(N).j(H) = N \rtimes_\phi H$, and the conjugation action identifies to the group homomorphism $\phi$, i.e., ${}^g m = Int_g(m)$. We therefore have the short exact sequence ($i$ is injective, $\pi$ is surjective, and $Ker(\pi) = Im(i)$):

$$e \longrightarrow N \xrightarrow{i} N \rtimes_\phi H \underset{\pi}{\overset{j}{\rightleftarrows}} H \longrightarrow e$$

which is split, i.e., $\pi \circ j = Id_H$. Conversely, any such split sequence

$$e \longrightarrow N \xrightarrow{i} G \underset{\pi}{\overset{j}{\rightleftarrows}} H \longrightarrow e$$

identifies the middle group $G$ to $N \rtimes_\phi H$.

**Example 73** For two groups $G, H$, we have the direct product group $G^H = \prod_H G$ and an action $\phi : H \to G^H$ via $\phi(h)(f)_k = f_{k.h}, k \in H$. Then the wreath product is $G \wr H = G^H \rtimes_\phi H$.

## C.3.3 Sylow Theorems on $p$-groups

**Proposition 71** (Sylow's proposition) *If $G$ is a finite group of order $n$, $p$ is prime, and $p^k | n$, then there is a subgroup $H \subseteq G$ with order $p^k$.*

A finite group $G$ has the order $card(G) = p^k$ for a prime $p$ iff all its elements have a power of $p$ as their orders. In this case, $G$ is called a *$p$-group*. A maximal $p$-subgroup in a group $G$ is called a *$p$-Sylow group* in $G$.

**Theorem 41** (Sylow's Theorem) *Let $G$ be a finite group, $p$ a prime. Then:*

(i) *The $p$-Sylow groups in $G$ are the $p$-subgroups $S$ with $p \nmid (G : S)$.*

(ii) *Any two $p$-Sylow groups are conjugate to each other.*

(iii) *Let $\sigma_p$ be the cardinality of the set of $p$-Sylow groups in $G$. Then we have $\sigma_p | card(G)$ and $\sigma_p \equiv 1 \pmod{p}$.*

See [21] for a proof.

## C.3.4 Classification of Groups

### C.3.4.1 Classification of Cyclic Groups

If $S \subset G$ is a subset of a group $G$, the smallest subgroup in $G$ containing $S$ is denoted by $\langle S \rangle$ and is called the *group generated by $S$*. A *finitely generated* group is one that admits a finite set of generators. A cyclic group is one that is generated by one element $G = \langle s \rangle$. For such a group, the group homomorphism $s^? : \mathbb{Z} \to G : n \mapsto s^n$ is a surjection with kernel $Ker(s^?) = O(s).\mathbb{Z}$, where $s^{O(s)} = e$ is the smallest positive power of $s$ which yields $e$, the *order of $s$*, or $O(s) = 0$ if no positive power of $s$ vanishes, in which case the order of $s$ is said to be infinite. This means that $\langle s \rangle \overset{\sim}{\to} \mathbb{Z}/O(s).\mathbb{Z}$, and we have classified all cyclic groups: They are isomorphic to the quotient groups $\mathbb{Z}_n = \mathbb{Z}/n.\mathbb{Z}$ with $card(\mathbb{Z}_n) = n$ for positive $n$, and $\infty$ for $\mathbb{Z}_0 = \mathbb{Z}$.

Among these groups, the groups $\mathbb{Z}_p$ of prime order $p$ are *simple*. For abelian groups, simplicity means being cyclic of prime order.

The number of generators of a finite cyclic group $\mathbb{Z}_n$ is the number of numbers $0 < t < n$ prime to $n$, i.e., the *Euler function* $\phi(n) = card(\mathbb{Z}_n^\times)$, where $\mathbb{Z}_n^\times$ denotes these numbers modulo $n$, this is in fact the group of invertible elements of the ring $\mathbb{Z}_n$, see appendix D.1.

**Proposition 72** *If $h = r.s$ is a factorization by natural numbers $1 < r, s$, and $(r, s) = 1$, then $\phi(h) = \phi(r).\phi(s)$. If $h = p^k$ is a positive power of a prime number $q$, then $\phi(p^k) = q^{k-1}(q-1)$. Therefore, if $h = \prod_i q_i^{k_i}$ is the prime decomposition of $h$, then*

$$\phi(h) = h.\prod_i (1 - 1/q_i).$$

### C.3.4.2 Classification of Finite Abelian Groups

Let $p$ be an prime integer ($p = 2, 3, 5, 7, 11, \ldots$). Take a weakly increasing sequence $u. = u_1 \leq u_2 \leq \ldots u_w$ of positive integers. We set

$$S(p, u.) = \mathbb{Z}_{p^{u_1}} \times \mathbb{Z}_{p^{u_2}} \times \ldots \mathbb{Z}_{p^{u_w}}.$$

If we are given an increasing sequence $p. = (p_1 < p_2 < \ldots p_t)$ of primes and for each such $p_i$ a sequence $u.^i$, i.e., a sequence of sequences $u.^{\cdot}$. Then we set

$$T(p., u.^{\cdot}) = \prod_i S(p_i, u.^i). \tag{C.4}$$

**Theorem 42** *For every finitely generated abelian group $G$, there is a natural number $f$ and a system of primes $p.$ and positive, weakly increasing sequences $u.^{\cdot}$ such that*

$$G \xrightarrow{\sim} \mathbb{Z}^f \times T(p., u.^{\cdot}).$$

*All the numbers $f$ (the torsion-free rank of $G$), and $p., u.^{\cdot}$ are uniquely determined. The length of the sequence $u.^i$ is called the $p_i$-*rank of $G$. *The image $T(G)$ of the factor $T(p., u.^{\cdot})$ in $G$ is the* torsion group *of $G$. The subgroups in $G$ corresponding to $S(p_i, u.^i)$ are called the $p_i$-*Sylow groups. *They are the subgroups of all elements with an order equal to a power of $p_i$.*

For a proof, see [540, Vol.II].

**Corollary 23** *If $G$ is a finite abelian group, and if $m|card(G)$, then there is a subgroup $H$ of $G$ with $card(H) = m$.*

## C.3.5 General Affine Groups

For a commutative ring $R$, we denote by $GL_n(R)$ the group $\mathbb{M}_{n,n}(R)^{\times}$ of invertible $n \times n$-matrices over $R$, see appendix D.1; we also denote $GL(n, p) = GL_n(\mathbb{Z}_p)$ and $\overrightarrow{GL}(n, p) = e^{\mathbb{Z}_p} \cdot GL(n, p)$, the affine automorphism group of $\mathbb{Z}_p^n$.

**Theorem 43** (Minkowski) *Suppose that $G$ is a finite subgroup of $GL_n(\mathbb{Z})$, and let $q \in \mathbb{N}$. Consider the canonical projection homomorphism*

$$\Phi : G \to GL_n(\mathbb{Z}_q).$$

*If $3 \leq q$, then $Ker(\Phi)$ is trivial. If $q = 2$, then there is $U \in GL_n(\mathbb{Z})$ with*

$$UNU^{-1} = Diag(\epsilon_1, \ldots \epsilon_n) \text{ diagonal matrix,}$$

*where $\epsilon_i = \pm 1$ for all $N \in Ker(\Phi)$. In particular, $card(Ker(\Phi))|w^n$.*

For a proof, see [264, Satz 5.1].

For a prime number $p$ and a power $n$, we have the following cardinalities:

$$card(GL(n,p)) = (p^n - 1)(p^{n-1} - 1)\dots(p-1)p^{n(n-1)/2}$$

and

$$card(\overrightarrow{GL}(n,p)) = p^n card(GL(n,p)).$$

## C.3.6 Permutation Groups

**Proposition 73** (Cayley) *Every group $G$ is isomorphic to a subgroup of permutations.*

Proof. In fact, the left regular representation map $l_? : G \to \mathfrak{S}_G$ with $l_?(g) = l_g : h \mapsto g \cdot h$ is such an embedding, QED.

A permutation group $G \subseteq \mathfrak{S}_P$ by definition acts on the underlying set $P$. Suppose that $P$ is finite. For $g \in G$, the orbits of $\langle g \rangle$ are called the cycles of $g$. Then these finite sets are arranged as sequences $C = (x, g.x, g^2.x, \dots g^{k(x)}.x)$ of pairwise different elements, i.e., $g^{k(x)+1}$ is the generator of the stabilizer of $x$. The permutation $g$ can be represented as sequence $(C_1, C_2, \dots C_r)$ of orbits in cycle representation.

**Definition 127** *If $G \subseteq \mathfrak{S}_P$ is a permutation group on a finite set $P$ of cardinality $p$, the* cycle index of an element $g \in G$ *is the polynomial*

$$X_{\cdot}^{cyc(g)} = X_1^{c_1} \cdot X_2^{c_2} \cdot \dots X_p^{c_p}$$

*with the* cycle type of $g$ $cyc(g) = (c_1, c_2, \dots c_p)$, *and* $c_i = card(\{C = \text{ cycle of } g, card(g) = i\})$.

Here is Fripertinger's cycle index formula [170]

$$Z(\overrightarrow{GL}(\mathbb{Z}_{12}^2)(1 + x, 1 + x^2, \dots 1 + x^{144})$$

for orbits of zero-addressed local compositions in $\mathbb{Z}_{12}^2$:

$x^{144} + x^{143} + 5x^{142} + 26x^{141} + 216x^{140} + 2\,024x^{139} + 27\,806x^{138} + 417\,209x^{137} +$
$6\,345\,735x^{136} + 90\,590\,713x^{135} + 1\,190\,322\,956x^{134} + 14\,303\,835\,837x^{133} + 157\,430\,569\,051x^{132} +$
$1\,592\,645\,620\,686x^{131} + 14\,873\,235\,105\,552x^{130} + 128\,762\,751\,824\,308x^{129} +$
$1\,037\,532\,923\,086\,353x^{128} + 7\,809\,413\,514\,931\,644x^{127} + 55\,089\,365\,597\,956\,206x^{126} +$
$365\,290\,003\,947\,963\,446x^{125} + 2\,282\,919\,558\,918\,081\,919x^{124} + 13\,479\,601\,808\,118\,798\,229x^{123} +$
$75\,361\,590\,622\,423\,713\,249x^{122} + 399\,738\,890\,367\,674\,230\,448x^{121} +$
$2\,015\,334\,387\,723\,540\,077\,262x^{120} + 9\,673\,558\,570\,858\,327\,142\,094x^{119} +$
$44\,275\,002\,111\,552\,677\,715\,575x^{118} + 193\,497\,799\,414\,541\,699\,555\,587x^{117} +$
$808\,543\,433\,959\,017\,353\,438\,195x^{116} + 3\,234\,171\,338\,137\,153\,259\,094\,292x^{115} +$
$12\,397\,650\,890\,304\,440\,505\,241\,198x^{114} + 45\,591\,347\,244\,850\,943\,472\,027\,532x^{113} +$
$160\,994\,412\,344\,908\,368\,725\,437\,163x^{112} + 546\,405\,205\,018\,625\,434\,948\,486\,100x^{111} +$
$1\,783\,852\,127\,215\,514\,388\,216\,575\,524x^{110} + 5\,606\,392\,061\,138\,587\,678\,507\,139\,578x^{109} +$
$16\,974\,908\,597\,922\,176\,404\,758\,662\,419x^{108} + 49\,548\,380\,452\,249\,950\,392\,015\,617\,673x^{107} +$
$139\,517\,805\,378\,058\,810\,895\,892\,716\,876x^{106} + 379\,202\,235\,047\,824\,659\,955\,968\,634\,895x^{105} +$

$$
\begin{aligned}
&995\,405\,857\,334\,028\,240\,446\,249\,995\,969x^{104} + 2\,524\,931\,913\,311\,378\,421\,460\,541\,875\,013x^{103} + \\
&6\,192\,094\,899\,403\,308\,142\,319\,324\,646\,830x^{102} + 14\,688\,225\,057\,065\,816\,000\,841\,247\,153\,422x^{101} + \\
&33\,716\,152\,882\,551\,682\,431\,054\,950\,635\,828x^{100} + 74\,924\,784\,036\,765\,597\,482\,162\,224\,697\,378x^{99} + \\
&161\,251\,165\,409\,134\,463\,248\,992\,354\,275\,261x^{98} + 336\,225\,833\,888\,858\,733\,322\,982\,932\,904\,265x^{97} + \\
&679\,456\,372\,086\,288\,422\,448\,712\,466\,252\,503x^{96} + \\
&1\,331\,179\,830\,182\,151\,403\,666\,404\,596\,530\,852x^{95} + \\
&2\,529\,241\,676\,111\,626\,447\,928\,668\,220\,456\,264x^{94} + \\
&4\,661\,739\,558\,127\,027\,290\,220\,867\,616\,981\,880x^{93} + \\
&8\,337\,341\,899\,567\,786\,249\,391\,103\,289\,453\,916x^{92} + \\
&14\,472\,367\,067\,576\,451\,752\,984\,797\,361\,008\,304x^{91} + \\
&24\,388\,618\,572\,337\,747\,341\,932\,969\,998\,362\,288x^{90} + \\
&39\,908\,648\,567\,034\,355\,259\,311\,114\,115\,744\,392x^{89} + \\
&63\,426\,245\,036\,529\,210\,051\,949\,169\,850\,308\,102x^{88} + \\
&97\,921\,220\,397\,909\,924\,969\,018\,620\,386\,852\,352x^{87} + \\
&146\,881\,830\,585\,458\,073\,270\,850\,321\,720\,445\,928x^{86} + \\
&214\,098\,939\,483\,879\,341\,610\,433\,150\,629\,060\,274x^{85} + \\
&303\,306\,830\,919\,747\,863\,651\,620\,555\,026\,700\,930x^{84} + \\
&417\,668\,422\,888\,061\,171\,460\,770\,548\,484\,103\,836x^{83} + \\
&559\,136\,759\,653\,084\,522\,330\,064\,385\,877\,590\,780x^{82} + \\
&727\,765\,306\,194\,069\,123\,565\,702\,210\,626\,823\,392x^{81} + \\
&921\,077\,965\,629\,957\,077\,012\,552\,741\,715\,036\,692x^{80} + \\
&1\,133\,634\,419\,214\,796\,834\,928\,853\,170\,296\,724\,314x^{79} + \\
&1\,356\,926\,047\,220\,511\,677\,349\,073\,201\,120\,481\,570x^{78} + \\
&1\,579\,704\,950\,475\,555\,411\,914\,967\,237\,903\,930\,342x^{77} + \\
&1\,788\,783\,546\,844\,376\,088\,722\,000\,995\,922\,467\,990x^{76} + \\
&1\,970\,254\,341\,437\,213\,013\,502\,048\,964\,983\,877\,090x^{75} + \\
&2\,110\,986\,794\,386\,177\,596\,749\,436\,553\,816\,924\,660x^{74} + \\
&2\,200\,183\,419\,494\,435\,885\,449\,671\,402\,432\,366\,956x^{73} + \\
&2\,230\,741\,522\,540\,743\,033\,415\,296\,821\,609\,381\,912x^{72} + \\
&2\,200\,183\,419\,494\,435\,885\,449\,671\,402\,432\,366\,956x^{71} + \\
&2\,110\,986\,794\,386\,177\,596\,749\,436\,553\,816\,924\,660x^{70} + \\
&1\,970\,254\,341\,437\,213\,013\,502\,048\,964\,983\,877\,090x^{69} + \\
&1\,788\,783\,546\,844\,376\,088\,722\,000\,995\,922\,467\,990x^{68} + \\
&1\,579\,704\,950\,475\,555\,411\,914\,967\,237\,903\,930\,342x^{67} + \\
&1\,356\,926\,047\,220\,511\,677\,349\,073\,201\,120\,481\,570x^{66} + \\
&1\,133\,634\,419\,214\,796\,834\,928\,853\,170\,296\,724\,314x^{65} + \\
&921\,077\,965\,629\,957\,077\,012\,552\,741\,715\,036\,692x^{64} + \\
&727\,765\,306\,194\,069\,123\,565\,702\,210\,626\,823\,392x^{63} + \\
&559\,136\,759\,653\,084\,522\,330\,064\,385\,877\,590\,780x^{62} + \\
&417\,668\,422\,888\,061\,171\,460\,770\,548\,484\,103\,836x^{61} + \\
&303\,306\,830\,919\,747\,863\,651\,620\,555\,026\,700\,930x^{60} + \\
&214\,098\,939\,483\,879\,341\,610\,433\,150\,629\,060\,274x^{59} + \\
&146\,881\,830\,585\,458\,073\,270\,850\,321\,720\,445\,928x^{58} + \\
&97\,921\,220\,397\,909\,924\,969\,018\,620\,386\,852\,352x^{57} + \\
&63\,426\,245\,036\,529\,210\,051\,949\,169\,850\,308\,102x^{56} +
\end{aligned}
$$

$$39\,908\,648\,567\,034\,355\,259\,311\,114\,115\,744\,392x^{55} +$$
$$24\,388\,618\,572\,337\,747\,341\,932\,969\,998\,362\,288x^{54} +$$
$$14\,472\,367\,067\,576\,451\,752\,984\,797\,361\,008\,304x^{53} +$$
$$8\,337\,341\,899\,567\,786\,249\,391\,103\,289\,453\,916x^{52} +$$
$$4\,661\,739\,558\,127\,027\,290\,220\,867\,616\,981\,880x^{51} +$$
$$2\,529\,241\,676\,111\,626\,447\,928\,668\,220\,456\,264x^{50} +$$
$$1\,331\,179\,830\,182\,151\,403\,666\,404\,596\,530\,852x^{49} +$$
$$679\,456\,372\,086\,288\,422\,448\,712\,466\,252\,503x^{48} + 336\,225\,833\,888\,858\,733\,322\,982\,932\,904\,265x^{47} +$$
$$161\,251\,165\,409\,134\,463\,248\,992\,354\,275\,261x^{46} + 74\,924\,784\,036\,765\,597\,482\,162\,224\,697\,378x^{45} +$$
$$33\,716\,152\,882\,551\,682\,431\,054\,950\,635\,828x^{44} + 14\,688\,225\,057\,065\,816\,000\,841\,247\,153\,422x^{43} +$$
$$6\,192\,094\,899\,403\,308\,142\,319\,324\,646\,830x^{42} + 2\,524\,931\,913\,311\,378\,421\,460\,541\,875\,013x^{41} +$$
$$995\,405\,857\,334\,028\,240\,446\,249\,995\,969x^{40} + 379\,202\,235\,047\,824\,659\,955\,968\,634\,895x^{39} +$$
$$139\,517\,805\,378\,058\,810\,895\,892\,716\,876x^{38} + 49\,548\,380\,452\,249\,950\,392\,015\,617\,673x^{37} +$$
$$16\,974\,908\,597\,922\,176\,404\,758\,662\,419x^{36} + 5\,606\,392\,061\,138\,587\,678\,507\,139\,578x^{35} +$$
$$1\,783\,852\,127\,215\,514\,388\,216\,575\,524x^{34} + 546\,405\,205\,018\,625\,434\,948\,486\,100x^{33} +$$
$$160\,994\,412\,344\,908\,368\,725\,437\,163x^{32} + 45\,591\,347\,244\,850\,943\,472\,027\,532x^{31} +$$
$$12\,397\,650\,890\,304\,440\,505\,241\,198x^{30} + 3\,234\,171\,338\,137\,153\,259\,094\,292x^{29} +$$
$$808\,543\,433\,959\,017\,353\,438\,195x^{28} + 193\,497\,799\,414\,541\,699\,555\,587x^{27} +$$
$$44\,275\,002\,111\,552\,677\,715\,575x^{26} + 9\,673\,558\,570\,858\,327\,142\,094x^{25} +$$
$$2\,015\,334\,387\,723\,540\,077\,262x^{24} + 399\,738\,890\,367\,674\,230\,448x^{23} +$$
$$75\,361\,590\,622\,423\,713\,249x^{22} + 13\,479\,601\,808\,118\,798\,229x^{21} + 2\,282\,919\,558\,918\,081\,919x^{20} +$$
$$365\,290\,003\,947\,963\,446x^{19} + 55\,089\,365\,597\,956\,206x^{18} + 7\,809\,413\,514\,931\,644x^{17} +$$
$$1\,037\,532\,923\,086\,353x^{16} + 128\,762\,751\,824\,308x^{15} + 14\,873\,235\,105\,552x^{14} +$$
$$1\,592\,645\,620\,686x^{13} + 157\,430\,569\,051x^{12} + 14\,303\,835\,837x^{11} + 1\,190\,322\,956x^{10} +$$
$$90\,590\,713x^9 + 6\,345\,735x^8 + 417\,209x^7 + 27\,806x^6 + 2\,024x^5 + 216x^4 + 26x^3 + 5x^2 + x + 1.$$

# Appendix D

# Rings and Algebras

## D.1 Basic Definitions and Constructions

**Definition 128** *A* (unitary) ring *is a triple* $(R, \alpha, \mu)$ *where* $(R, \alpha)$ *is an abelian group whose operation* $\alpha$ *is written additively* $(\alpha(r,s) = r+s)$ *with neutral element* $0_R$*, and* $(R, \mu)$ *is monoid, written multiplicatively* $(\mu(r,s) = r \cdot s)$ *with multiplicative neutral element* $1_R$ *such that these operations are coupled by* distributivity*, i.e.,* $(r+s)\cdot t = r\cdot t + s\cdot t, t\cdot(r+s) = t\cdot r + t\cdot s$ *for all* $r, s, t \in R$*. A ring is* commutative *iff its multiplicative monoid is commutative.*

*A set map* $f : R \to S$ *of rings* $R, S$ *is a* ring homomorphism *iff it is a homomorphism of the underlying additive groups and a homomorphism of the underlying multiplicative monoids. The set of ring homomorphisms from* $R$ *to* $S$ *is again denoted by* $Hom(R, S)$ *if no confusion is likely.*

*An element* $x$ *of a non-zero ring* $R$ *is called* invertible*, iff there is a multiplicative (left) inverse* $y$*, i.e.,* $y \cdot x = 1_R$*. The subset* $R^\times$ *of invertible elements is a multiplicative group. A* skew field *is a ring such that* $R^\times = R - \{0_R\}$*. A* (commutative) field *is a commutative ring which is a skew field.*

*The subring* $Z(R)$ *of all elements in a ring* $R$ *which commute with all of* $R$ *is called the center of* $R$*. For a commutative* $R$*, a ring homomorphism* $\varphi : R \to S$ *whose image is in* $Z(S)$ *is called an* $R$-algebra*. If* $\varphi$ *is clear, one says that "*$S$ *is an* $R$*-algebra", and* $\varphi$ *is called the "structural homomorphism". If* $\varphi : R \to S, \psi : R \to T$ *are two* $R$*-algebras, a ring homomorphism* $f : S \to T$ *is an* $R$*-algebra homomorphism, iff* $\psi = f \circ \varphi$*, i.e., iff the '*$R$*-elements are conserved under* $f$*'. One als often writes* $r$ *instead of* $\varphi(r)$ *if the algebra structure is clear.*

The set-theoretic composition $g \circ f$ of two ring homomorphisms $f : R \to S, g : S \to T$ is again a ring homomorphism. A bijective ring homomorphism automatically has an *inverse* ring homomorphism, i.e., this is a ring *isomorphism*, and a ring *endomorphism* is a homomorphism with domain equal to its codomain, whereas an *automorphism* is an endomorphism which is an isomorphism. The corresponding concepts for $R$-algebras are evident: The homomorphisms have to be algebra homomorphisms, so, for example, an $R$-algebra automorphism is an automorphism which conserves the structural homomorphism.

**Example 74** Classical examples of rings are the rings $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{H}$ of integers, rational numbers, real numbers, complex numbers and Hamilton quaternions. Except the integers, these rings are also skew fields, and $\mathbb{H}$ is not commutative. The conjugation $z = a + i.b \mapsto \bar{z} = a - i.b$ is an automorphism of the $\mathbb{R}$-algebra $\mathbb{C}$.

To every ring $R$, one has the opposite ring $R^{opp}$ which is the same additive group, but multiplication is defined by reversing to given multiplication, i.e., $r \cdot^{opp} s = s \cdot r$. An anti-homomorphism of rings is a homomorphism into the opposite codomain ring.

Every ring $R$ is a $\mathbb{Z}$-algebra in a unique way by $\varphi(z) = z.1_R$; the latter means that $z.1_R = 1_R + 1_R + \ldots 1_R$ if $z > 0$, it is $0_R$ if $z = 0$, and it is $-(-z).1_R$ if $z < 0$.

By the natural inclusions, $\mathbb{R}$ is a $\mathbb{Q}$-algebra, $\mathbb{C}$ is an $\mathbb{R}$-algebra, and $\mathbb{H}$ is an $\mathbb{R}$-algebra, but not a $\mathbb{C}$-algebra.

The $2 \times 2$ matrices over a commutative ring $R$ with coefficient-wise addition and usual matrix multiplication form a non-commutative ring. They are an $R$-algebra by the diagonal map $r \mapsto \left(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}\right)$.

**Proposition 74** *If $I$ is a subgroup of a ring $R$, the following statements are equivalent:*

(i) *For all $r \in R$, $r \cdot I \subseteq I$, i.e., $I$ is a* left ideal in $R$, *and $I \cdot r \subseteq I$,i.e., $I$ is a* right ideal in $R$.

(ii) *There is a ring homomorphism $f : R \to S$ with $Ker(f) = I$.*

If $I$ has the equivalent properties of proposition 74, we call it an *ideal* or a *two-sided ideal* in $R$. So ideals correspond to normal subgroups. The *quotient ring* $R/I$ (and the associated projection $\pi : R \to R/I$) is just the quotient group (with respect to the additive structure), together with the well-defined multiplication $(r + I)(s + I) = rs + I$, and we have $Ker(\pi) = I$. If $G$ is any subset of a ring $R$, the smallest ideal containing $G$ is denoted by $(G)$ and is called the *ideal generated by $G$*.

**Proposition 75** *Let $I$ be an ideal in $R$, $\pi : R \to R/I$ the canonical homomorphism, and $S$ any ring. Then*

$$Hom(R/I, S) \xrightarrow{\sim} \{f \in Hom(R, S) | I \subseteq Ker(f)\} : t \mapsto t \circ \pi$$

*is a canonical bijection. If $I = Ker(f)$, then the corresponding morphism $R/I \to I$ is an isomorphism onto $Im(f)$.*

**Example 75** By the Euclidean algorithm (D.3), every subgroup of $\mathbb{Z}$ is of the form $\langle n \rangle = n.\mathbb{Z}$ for a uniquely determined non-negative $n$. Such a subgroup is also an ideal. The quotient ring $\mathbb{Z}_n = \mathbb{Z}/n.\mathbb{Z}$ is the *ring of integers modulo $n$*. It is the so-called "prime field of characteristic $p$" iff the Euler function $\phi(n) = n - 1$, i.e., $n = p$ is a prime (see also proposition 72 in appendix C.3.4.1). For any skew field $F$, we have the unique $\mathbb{Z}$-algebra structure $p : \mathbb{Z} \to F$ whose kernel must be either $(0)$ or a prime ideal $p.\mathbb{Z}$. In the latter case, the image $Im(p)$ is the smallest subfield in $F$; in the former case, the smallest subfield of $F$ is evidently isomorphic to $\mathbb{Q}$. The prime field of a skew field is called its *prime field*, whereas the generator (zero or a positive prime) of $Ker(p)$ is called the characteristic $char(F)$ of the skew field.

A skew field $F$ has only two ideals: $(0), F$, and conversely, a ring which has only these ideals is called *simple*. The ring $\mathbb{M}_{n,n}(F)$ of $n \times n$-matrices (see E.1.1) over a skew field $F$ is simple (but not a skew field for $1 < n$).

### D.1.1 Universal Constructions

If $M$ is a multiplicative monoid and $R$ is a commutative ring, we have a ring $R\langle M\rangle$, the *monoid algebra* by the following construction. The underlying set is the set $R^{(M)}$ of functions $f : M \to R$ with $f(m) = 0$ except of a finite number of arguments. This is an additive group under the addition $(f + g)(m) = f(m) + g(m)$. The product is defined by $(f \cdot g)(m) = \sum_{n \cdot n' = m} f(n) \cdot g(n')$, which is reasonable since the summands which do not vanish are finite in number. The multiplicative neutral element $1_{R\langle M\rangle}$ is the function $1_{R\langle M\rangle}(e_M) = 1_R$ and zero else. The elements $f : M \to R$ of the monoid algebra are also written as a formal sum $\sum_{f(m)\neq 0} f(m)m$. The algebra structure is given by $r \mapsto re_M$. Here is the *universal property* of the monoid algebra:

**Proposition 76** *Every monoid homomorphism $\varphi : M \to (S, \mu)$ into the multiplicative monoid of an $R$-algebra $S$, can be extended in a unique way to a homomorphism $\Phi : S\langle M\rangle \to S$ of $R$-algebras. If $\sum f_m.m \in S\langle M\rangle$, its image is $\Phi(\sum f_m.m) = \sum f_m \cdot \varphi(m)$.*

A special case of a monoid algebra is for the free monoid $FM\langle Alphabet\rangle$ over *Alphabet*, see example 66 in appendix C.2.3. One usually writes $S\langle Alphabet\rangle$ instead of $S\langle FM(Alphabet)\rangle$. For example, if $Alphabet = \{X_1, X_2, \ldots X_n, \ldots\}$, we have the *algebra of non-commutative polynomials in the indeterminates* $X_1, X_2, \ldots X_n, \ldots$. If instead we take the free commutative monoid $FCM(Alphabet)$, we get the monoid algebra $S[Alphabet] = S\langle FCM(Alphabet)\rangle$ of commutative polynomials in the indeterminates $\{X_1, X_2, \ldots X_n, \ldots\}$ of *Alphabet*.

**Example 76** For the $R$-algebra of polynomials in one variable $R[X]$, the ideal $(X^2)$ defines the *$R$-algebra of dual numbers* $R[\varepsilon] = R[X]/(X^2)$ with $\varepsilon = X + (X^2)$ the class of $X$. An element of $R[\varepsilon]$ is uniquely described as a linear polynomial $a + \varepsilon.b$ in $\varepsilon$, and the multiplication is $(a + \varepsilon.b)(c + \varepsilon.d) = ac + \varepsilon.(ad + bc)$. The group $R[\varepsilon]^\times$ consists of those elements $a + \varepsilon.b$ with $a \in R^\times$.

**Definition 129** *Let $(R_i)_{i\in I}$ be a family of rings, then the* product $\prod_{i\in I} R_i$ *of this family is the following ring: As a set, it is the product of the underlying sets, addition and multiplication are defined coordinate-wise*[1]*, i.e.,*

$$(x_i) + (y_i) = (x_i + y_i) \text{ and}$$
$$(x_i) \cdot (y_i) = (x_i \cdot y_i),$$

*and the unity is the family $(1_i)$ of unities $1_i$ in the respective rings $R_i$. For each index $j \in I$, we have a canonical projection ring homomorphism*

$$p_j : \prod_{i\in I} R_i \to R_j$$

[1] Addition and multiplication are taken in the respective rings.

*defined by* $p_j((x_i)) = x_j$.

The product and its canonical projections shares the universal property of a product in the category of rings (see appendix G.2.1 for the definition of a product in a category):

**Lemma 69** *Let $S$ be a ring, and let $(R_i)_{i\in I}$ be a family of rings. For each family $(f_i : S \to R_i)_{i\in I}$ of ring homomorphisms, there is a unique ring homomorphism $f : S \to \prod_{i\in I} R_i$ such that $f_i = p_i \cdot f$ for all $i \in I$.*

Clearly, the map $f(s) = (f_i(x))$ solves the problem.

More generally, consider a diagram[2] $\mathbf{D}$ of rings with vertex set $I$. Then we have a subring $lim(\mathbf{D})$ of the product $\prod_{i\in I} R_i$ consisting of all families $(x_i)$ such that

$$f(x_j) = x_k \text{ for any homomorphism } f : R_j \to R_k$$

corresponding to a $\mathbf{D}$-arrow from vertex $j$ to vertex $k$.

**Proposition 77** *With the above notation, the ring of $lim(\mathbf{D})$, together with the induced canonical projections $p_i : lim(\mathbf{D}) \to R_i$, is the* limit *of the diagram $\mathbf{D}$ of rings, i.e., for any family $(f_i : S \to R_i)_{i\in I}$ of ring homomorphisms such that $f \cdot f_j = f_k$ whenever $f : R_j \to R_k$ is a homomorphism corresponding to a $\mathbf{D}$-arrow from vertex $j$ to vertex $k$, there is a unique ring homomorphism $g : S \to lim(\mathbf{D})$ such that $f_i = p_i \cdot g$, all $i \in I$.*

In particular, if we have a fiber product[3] diagram $A \underset{f}{\to} C \underset{g}{\leftarrow} B$ of rings, there is a limit, the fiber product of this diagram, usually denoted by $A \times_C B$ and inserted in the commutative "pullback" square

$$\begin{array}{ccc} A \times_C B & \xrightarrow{p_2} & B \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & C \end{array} \tag{D.1}$$

of ring homomorphisms.

For non-commutative rings, fiber sums do not exist in general. However, if the diagram's homomorphisms are algebras over a commutative ring, we have the following well-known result [63, ch.III, No.2]:

**Theorem 44** *Let $A \underset{f}{\leftarrow} C \underset{g}{\to} B$ be a fiber sum diagram of algebras over the commutative ring $C$. Then the tensor product $A \otimes_C B$ defines a fiber sum*

$$\begin{array}{ccc} C & \xrightarrow{f} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i_1} \\ B & \xrightarrow{i_2} & A \otimes_C B \end{array} \tag{D.2}$$

*of $C$-algebras with $i_1(a) = a \otimes 1_B$ and $i_2(b) = 1_A \otimes b$.*

[2] See definition 151, appendix G.1.2, for a formal definition of a diagram in a category.

[3] See appendix G.2.1 for the concept of a fiber product and the dual one of a fiber sum.

### D.1.1.1 Quiver Algebras

**Definition 130** *Suppose that we are given a quiver* $Q = (\text{head}, \text{tail} : A \rightrightarrows V)$ *(see appendix C.2.2, definition 123). The* path category *of* $Q$ *is the set* $P(Q)$ *of paths of* $Q$ *(the morphisms of the category, the lazy paths being the objects of the category), together with the composition* $p.q$ *of two paths* $p, q$*, defined if* $\text{head}(p) = \text{tail}(q)$*. In that case,* $p.q$ *is just the evident composed path of length* $l(p.q) = l(p) + l(q)$*.*

**Definition 131** *Suppose that we are given a finite quiver* $Q = (\text{head}, \text{tail} : A \rightrightarrows V)$ *and a commutative ring* $R$*. The quiver algebra* $R\langle Q\rangle$ *of* $Q$ *with coefficients in* $R$ *is the free left* $R$*-module whose basis is the path set* $P(Q)$*, together with* $R$*-bilinear extension of the composition in the path category, i.e., the product of two paths is their composition, if possible, and zero else. This means:*

$$\Big(\sum_i r_i.p_i\Big)\Big(\sum_j s_j.q_j\Big) = \sum_{i,j}(r_i.s_j)p_i.q_j.$$

*The unity*

$$1_{R\langle Q\rangle} = \sum_{v\in V} v$$

*is the sum of all vertexes.*

**Definition 132** *Let* $Q$ *be a quiver, and* $R\langle Q\rangle$ *its quiver algebra over the coefficient ring* $R$*. A sub-path of a path* $p$ *in* $R\langle Q\rangle$*, is a triple* $(u, w, v)$ *of paths such that* $p = u.w.v$*. If the external factors* $u, v$ *are clear, the sub-path is identified with the middle member* $w$*, and we write* $w \sqsubset p$*.*

For example, if $l(p) = 0$ (a vertex), the only sub-path of $p$ is $p$, in fact, $p = p.p.p$ is the only factorization of this kind. If $p = \circlearrowleft x$ is a loop, the sub-paths of $p$ are $x.x.p$, $p.x.x$, and $x.p.x$. So the middle member $x$ appears two times since it is in different positions in the factorization.

If $p = u_1.w_1.v_1 = u_2.w_2.v_2$ are two subpaths of $p$ with $l(w_1) = l(w_2)$, we write $(u_1, w_1, v_1) < (u_2, w_2, v_2)$ iff $l(u_1) < l(u_2)$. Clearly this relation is transitive and antisymmetric. Among all subpaths $(u, w, v)$ of $p$ with fixed middle length $l(w) = const.$, the relation corresponding to relation $\leq$ is an linear ordering. In fact, if $(u_1, w_1, v_1), (u_2, w_2, v_2)$ are two subpaths of $p$ such that $l(u_1) = l(u_2)$ and $l(w_1) = l(w_2)$, then they are identical, so the ordering relation is total. We shall use this total ordering to define linear endomorphisms, so-called *sub-path operators* of the quiver algebras as follows.

We shall define an endomorphism $\sqsubset_i^{\lambda^i}$ for each natural index $i = 0, 1, 2, \ldots$ and for each system $\lambda^i$ of coefficients in the following sense. We set

$$\lambda^i = (\lambda^i_{lj})_{i\leq l, 1\leq j\leq l-i+1}$$

with $\lambda^i_{lj} \in R$. The endomorphisms run as follows. Let $w$ be a path of length $l(w) = l$. Then

$$\sqsubset_i^{\lambda^i}(w) = \begin{cases} 0 & \text{if } l < i, \\ \sum_{v_j \sqsubset w, l(v_j)=i} \lambda^i_{lj} v_j & \text{else,} \end{cases}$$

with the indexes of the subpaths of $w$ referring to their total order defined above. In the musical applications, we shall encounter linear combinations $\Phi = \sum_i \mu_i \sqsubset_i^{\lambda^i}$ of such weighted sub-path operators.

## D.2 Prime Factorization

An integer $1 < p$ is called *prime* iff $p = u \cdot v$ with positive factors implies $u = 1$ or $v = 1$.

**Theorem 45** *Every non-zero integer $x$ is a product*

$$x = \pm 1 \cdot p_1^{n_1} \cdot p_2^{n_2} \cdot \ldots p_k^{n_k}$$

*with an increasing sequence of primes $p_1 < p_2 < \ldots p_k$ and positive exponents $n_i$ which are all uniquely determined.*

**Corollary 24** *Let $p_1 < p_2 < \ldots p_k$ be an increasing sequence of primes. For two sequences of rational numbers $q_1 < q_2 < \ldots q_k$ , $r_1 < r_2 < \ldots r_k$, the equation*

$$p_1^{q_1} \cdot p_2^{q_2} \cdot \ldots p_k^{q_k} = p_1^{r_1} \cdot p_2^{r_2} \cdot \ldots p_k^{r_k}$$

*implies the equality of the sequences of rational numbers.*

This implies that the logarithms $ln(p)$ of primes are linearly independent (see E.2.1) in the rational vector space $\mathbb{R}_{[\mathbb{Q}]}$ (see E.1.1), a central fact for the construction of the Euler module of pitch systems (see section A.2.3).

## D.3 Euclidean Algorithm

**Proposition 78** *Euclidean algorithm: Given a non-zero integer $d$, every integer $x$ has a unique representation $x = a.d + b, 0 \leq b < d$.*

**Lemma 70** *Let $n > 1$ be an integer. Then every positive integer $x$ has a representation*

$$x = \sum_{i=0}^{t} x_i.n^i$$

*with $0 \leq x_i < n$, and $x_t \neq 0$. The $x_i$ are uniquely determined. The representation is usually written as*

$$x = x_t x_{t-1} \ldots x_1 x_0$$

*and known as the $n$-adic representation.*

## D.4 Approximation of Real Numbers by Fractions

**Lemma 71** *Let $L = \log_2(3)$. Then for every real number $\delta > 0$, there is a pair $n, m$ of integers such that*

$$0 < n + m \cdot L < \delta.$$

Proof. We construct a sequence $(n_1, m_1), (n_2, m_2), \ldots (n_s, m_s), \ldots$ of pairs such that $0 < n_s + m_s \cdot L < 1/2^s$. We may start by $(n_1, m_1) = (2, -1)$ since $L \approx 1.58$. Suppose that we have found $(n_s, m_s)$ such that $0 < n_s + m_s \cdot L < 1/2^s$. There is a maximal positive integer $k$ such that $k \cdot (n_s + m_s \cdot L) < 1$. Then $(k+1) \cdot (n_s + m_s \cdot L) > 1$ since $L$ is not rational by corollary 24. For the same reason, either

$$(k+1) \cdot (n_s + m_s \cdot L) - 1 < 1/2 \cdot 1/2^s = 1/2^{s+1}$$

or

$$1 - k \cdot (n_s + m_s \cdot L) < 1/2^{s+1}.$$

Then either

$$(n_{s+1}, m_{s+1}) = ((k+1) \cdot n_s - 1, (k+1) \cdot m_s)$$

or

$$(n_{s+1}, m_{s+1}) = (1 - k \cdot n_s, -k \cdot m_s)$$

solves the problem, QED.

# D.5 Some Special Issues

## D.5.1 Integers, Rationals, and Real Numbers

**Definition 133** *The index function* $index : \mathbb{R} \to \mathbb{Z}$ *is defined by*

$$index(x) = \begin{cases} 1 & \text{if } 0 < x, \\ -1 & \text{if } 0 > x, \\ 0 & \text{if } 0 = x \end{cases} \tag{D.3}$$

*for* $x \in \mathbb{R}$.

A real number $x$ has unique representation $x = bottom(x) + x_+$ with $bottom(x) \in \mathbb{Z}, 0 \leq x_+ < 1$; we set $top(x) = floor(x) + 1$.

**Definition 134** *The rounding function* $round : \mathbb{R} \to \mathbb{Z}$ *is defined by*

$$round(x) = \begin{cases} floor(x) & \text{if } x_+ \leq 0.5, \\ top(x) & \text{else} \end{cases} \tag{D.4}$$

*for* $x \in \mathbb{R}$.

# Appendix E

# Modules, Linear, and Affine Transformations

## E.1 Modules and Linear Transformations

**Definition 135** *Let $R$ be a ring, then a* (left)[1] $R$-module *is a triple $(R, M, \mu : R \times M \to M)$ where $M$ is an additively written abelian group and $\mu$ is the* scalar multiplication*, usually written as $\mu(r, m) = r.m$ if $\mu$ is clear($R$ is also called the ring of* scalars*, and $M$ the group of* vectors*), with the properties:*

1. *We have $1_r.m = m$ for all $m \in M$.*
2. *For all $r, s \in R$ and $m, n \in M$, we have $(r + s).m = r.m + s.m$, $r.(m + n) = r.m + r.n$, and $r.(s.m) = (r \cdot s).m$.*

*If $(R, M, \mu : R \times M \to M), (R, N, \nu : R \times N \to N)$ are two $R$-modules, a group homomorphism $f : M \to N$ is called* $R$-linear (or a module homomorphism if the rest is clear) *iff it is "homogeneous", i.e., $f(r.m) = r.f(m)$, for all $r \in R, m \in M$, with the respective scalar multiplications. The set of $R$-linear homomorphisms from $M$ to $N$ is denoted by $Lin_R(M, N)$. It is an additive group under the pointwise addition $(f + g)(m) = f(m) + g(m)$. The set-theoretic composition $g \circ f$ of two module homomorphisms $f : M \to N, g : N \to L$ is also a module homomorphism, and we have distributivity, i.e., $(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$ for $g_i : N \to L, f : M \to N$ and $g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2$ for $f_i : M \to N, g : N \to L$.*

By the distributivity of composition of module homomorphisms, the group $End_R(M) = Lin_R(M, M)$ is a ring, the *endomorphism ring of* $M$, which contains the multiplicative *automorphism group* $Aut_R(M) = End_R(M)^\times$ *of* $M$.

An $R$-linear homomorphism $f : M \to N$ has a group-theoretic kernel $Ker(f)$ and an image $Im(f)$ which are also submodules. For a submodule $N \subseteq M$, the quotient group $M/N$ is also an $R$-module by the scalar multiplication $r.(m + N) = r.m + N$. The group-theoretic

[1] Right modules are defined in complete analogy, the scalar multiplication being written $m.r$ instead of $r.m$.

results proposition 70 and the isomorphism theorems 39, 40 in appendix C, are valid literally if we replace the respective groups by modules (the normality of subgroups is automatic here).

### E.1.1 Examples

Abelian groups $G$ are canonically identified to $\mathbb{Z}$-modules by $z.g = g + g + \dots g$, $z$ times for $z > 0$, $(-z).g = -(z.g)$, and $0.g = 0_G$.

If $M$ is an $R$-module, a submodule $N \subseteq M$ is a subgroup that is stable under scalar multiplication, i.e., $R.N = N$. If $S \subset M$ is a subset, the smallest submodule containing $S$ is denoted by $\langle S\rangle$ and consists of all *linear combinations* $\sum_i r_i.s_i, r_i \in R, s_i \in S$. If there is a finite set $S$ such that $\langle S\rangle = M$, $M$ is called *finitely generated*. Finitely generated $\mathbb{Z}$-modules are completely classified, see C.3.4.2. If $(M_i)_{i\in I}$ is a family of submodules of $M$, we denote by $\Sigma_I M_i$ the module $\langle \bigcup I M_i\rangle$. It consists of all finite sums $x_{i_1} + \dots x_{i_k}, x_{i_j} \in M_{i_j}$.

Every ring $R$ is a left $R$-module ${}_RR$ and a right $R$-module $R_R$ by the given multiplication. For commutative rings these structures coincide. A left, right ideal in $R$ identifies to a submodule of ${}_RR$, $R_R$, respectively.

If $\varphi : S \to R$ is a ring homomorphism and $M$ is an $R$-module, we have an $S$-module structure $M_{[\varphi]}$ on $M$ via $s.m = \varphi(s).m$, the *module defined by restriction of scalars*. If $\varphi$ is clear, one also writes $M_{[S]}$. For the $\mathbb{Z}$-algebra structure of every ring $R$, $M_{[\mathbb{Z}]}$ gives the underlying structure of an abelian group $M$ back. In particular, an $R$-algebra $S$ is an $R$ module via $({}_SS)_{[R]}$.

A *dilinear homomorphism* from an $S$-module $M$ to an $R$-module $N$ is a pair $(\varphi : S \to R, f : M \to N_{[\varphi]})$ consisting of a scalar restriction $\varphi$ and an $S$-linear homomorphism $f$. If $(\varphi : S \to R, f : M \to N_{[\varphi]}), (\psi : T \to S, g : L \to M_{[\psi]})$ are two dilinear homomorphisms, their composition is defined by $(\varphi \circ \psi : T \to R, f \circ g : L \to N[\varphi \circ \psi])$. The set of dilinear homomorphisms from $M$ to $N$ is denoted by $Dil(M, N)$. If the scalar restriction is fixed by $\varphi$, we denote the corresponding set by $Dil_\varphi(M, N)$, and the special case $Dil_{Id_R}(M, N)$ is just $Lin_R(M, N)$ as above.

For any family $(M_i)_{i\in I}$ of $R$-modules, we have the *product module* $\prod_I M_i$. This is the product of the underlying groups, together with coordinatewise scalar multiplication $r.(m_i) = (r.m_i)$ The submodule of those $(m_i)$ with only finitely many $m_i \neq 0$ is the *direct sum module* $\bigoplus_I M_i$. For every index $j$, one has the canonical (linear) projections $\pi_j : \prod_I M_i \to M_j$ and $\pi_j : \bigoplus_I M_i \to M_j$, via $\pi_j((m)_i)) \mapsto m_j$, as well as the canonical (linear) injections $\iota_j : M_j \to \bigoplus_I M_i$ with $\iota_j(m)$ having zero coordinates except for coordinate index $j$ where the value is $m$.

**Lemma 72** (Universal limit property of direct products of modules) *For every family $(f_i : X \to M_i)_i$ of linear homomorphisms there is exactly one linear homomorphism $f : X \to \prod_I M_i$ such that $f_j = \pi_j \circ f$ for all $j \in I$.*

(Universal colimit property of direct sums of modules) *For every family $(f_i : M_i \to X)$ of linear homomorphisms there is exactly one linear homomorphism $f : \bigoplus_I M_i \to X$ such that $f_j = \circ f \circ \iota_j$ for all $j \in I$.*

A sum $\Sigma_I M_i$ of submodules $M_i$ of a module $M$ is called (inner) direct, iff the linear homomorphism $\bigoplus_I M_i \to M$ which is induced by the inclusions $M_i \subseteq M$ is an isomorphism.

For two positive integers $m, n$ denote by $m \times n = [1, m] \times [1, n]$ the set of all pairs $(i, j), 1 \leq i \leq m, 1 \leq j \leq n$. For a ring $R$, we have the direct sum $\mathbb{M}_{m,n}(R) = \bigoplus_{m\times n} R$ whose elements

are written in the *matrix* notation

$$(r_{i,j}) = \begin{pmatrix} r_{1,1} & \dots & r_{1,n} \\ \dots & r_{i,j} & \dots \\ r_{m,1} & \dots & r_{m,n} \end{pmatrix}$$

whose *rows* or *columns* are the submatrices with constant first or second index, respectively.

If $(r_{i,j}) \in \mathbb{M}_{m,n}(R)$ and $(s_{j,k}) \in \mathbb{M}_{n,l}(R)$, we have the *matrix product* $(r_{i,j}) \cdot (s_{j,k}) = (t_{i,k}) \in \mathbb{M}_{m,l}(R)$ with $t_{i,k} = \sum_j r_{i,j} \cdot s_{j,k}$. Whenever defined, the product is associative. It is also distributive, i.e., $((r_{i,j}) + (r'_{i,j})) \cdot (s_{j,k}) = (r_{i,j}) \cdot (s_{j,k}) + (r'_{i,j}) \cdot (s_{j,k})$, and $(r_{i,j}) \cdot ((s_{j,k}) + (s'_{j,k})) = (r_{i,j}) \cdot (s_{j,k}) + (r_{i,j}) \cdot (s'_{j,k})$. For $m = n$, one has the identity matrix $E_m = (\delta_{ij})$ with the *Kronecker delta* $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$. With this identity and the matrix addition and multiplication, $\mathbb{M}_{m,m}(R)$ is a ring. With the matrix multiplication as scalar multiplication, $\mathbb{M}_{m,n}(R)$ becomes a left $\mathbb{M}_{m,m}(R)$-module and a right $\mathbb{M}_{n,n}(R)$-module.

If $R$ is commutative, $\mathbb{M}_{m,m}(R)$ is an $R$-algebra via $r \mapsto r.E_m$, i.e., the scalar multiplication of the $R$-module $\mathbb{M}_{m,m}(R)$ coincides with the multiplication with $R$-elements from the algebra embedding.

# E.2 Module Classification

## E.2.1 Dimension

For any set $C$ and ring $R$, we have the *free $R$-module* $\bigoplus_C = R^C$ *of rank* $card(C)$ which is the direct sum of $C$ copies of ${}_RR$ (for $C = \emptyset$, we take the zero module $0_R$). A *free module* $M$ is one that is isomorphic to a free module $R^C$. It is well known that the rank $card(C)$ is then uniquely determined and called the *dimension* $dim(M)$ of the free module $M$. If $R = F$ is a skew field, an $F$-module is called a *vector space* (over $F$), and we have the main fact of linear algebra:

**Theorem 46** *Every vector space $M$ over the skew field $F$ is free, and the dimensions are a complete system of invariants of isomorphism classes of vector spaces.*

The proof of this theorem is based on the concept of *linear (in)dependence* in a module. A family $(m_i)_i$ of elements $m_i \in M$ is called *linearly independent* iff any (finite) linear combination $0 = \sum_{j=1,\dots k} r_j.m_{i_j}$ implies $r_j = 0$, for all $j$. Otherwise the family is called *linearly dependent*. A *base* of a module is a family of linearly independent elements which generates the module. The main theorem 46 is proved by the *exchange theorem* which states that any family $(m_i)_i$ of linearly independent vectors can be inserted in a given basis by exchanging some of its elements with the $(m_i)_i$.

**Example 77** $dim(\mathbb{R}_{[\mathbb{Q}]}) = card(\mathbb{R}) = 2^{\aleph_0}$, and the sequence of $b$-logarithms $(\log_b(p))_{p=\text{ prime}}$ is linearly independent by corollary 24 in appendix D.2. This means that for any finite increasing sequence $p_\cdot = (p_1, p_2, \dots p_k)$ of primes and the corresponding sequence

$$H_{p_\cdot} = (\log_b(p_1), \log_b(p_2), \dots \log_b(p_k)),$$

the scalar product map

$$H : \mathbb{Q}^k \to \mathbb{R}_{[\mathbb{Q}]}$$

with $H(x) = (H_{p.}, x) = \sum_i \log_b(p_i)x_i$ is a linear injection. The special case of the first three primes and $H_{prime} = (\log_b(2), \log_b(3), \log_b(5))$ was discussed in section A.2.3. Here we can prove the result needed for proposition 64:

**Lemma 73** *For any positive real bound $\delta$ and every real number $\phi$, there is $x \in \mathbb{Z}^3$ such that $|H(x) - \phi| < \delta$*

Proof. WLOG, we can work with logarithm basis $b = 2$. We know from lemma 71 in appendix D.4 that there is $x' = (n, m, 0) \in \mathbb{Z}^3$ such that $0 < H(x') < \delta$. Clearly, there is an integer multiple $x = z.x'$ which does the job, QED.

**Corollary 25** (of theorem 46) *If $f : N \xrightarrow{\sim} R^n$ and $g : M \xrightarrow{\sim} R^m$ are two free modules of finite ranks $n, m$, isomorphic to the free modules $R^n, R^m$ via isomorphisms $f, g$, then the linear homomorphisms are described by matrices: If $h : N \to M$ is a linear homomorphism, there is a uniquely determined matrix $H \in \mathbb{M}_{m,n}(R)$ such that for $x \in N$, we have $h(x) = g^{-1}(H \cdot f(x))$, where $f(x)$ is written as a column matrix in $\mathbb{M}_{n,1}(R)$ which canonically identifies to $R^n$. And conversely, each such matrix defines a linear homomorphism. In other words, there is an isomorphism*

$$Lin_R(N, M) \xrightarrow{\sim} \mathbb{M}_{m,n}(R) \tag{E.1}$$

*of additive groups.*

If $R$ is a commutative ring, then an $R$-algebra $S$ is also an $R$-module. We have an injective homomorphism of $R$-algebras

$$\begin{aligned} \Lambda : S &\to End_R(S) \\ \Lambda(s)(s') &= s \cdot s' \end{aligned} \tag{E.2}$$

which is called the *left regular representation of $S$*. If $S$ is a free $R$-module of dimension $n$, then the isomorphism (E.1) induces the left regular representation in matrices:

$$\lambda : S \to \mathbb{M}_{n,n}(R). \tag{E.3}$$

More generally, a *linear representation* of an $R$-algebra $A$ is an algebra homomorphism $f : A \to End_R(M)$ into the $R$-algebra of endomorphisms of an $R$-module $M$.

**Definition 136** *The points of a $k$-element local composition $K$ in $\mathbb{R}^n$ is* in general position *iff $dim(\mathbb{R}.K) = k - 1$.*

**Theorem 47** *Let $v_1, v_2, \ldots v_n$ be $n$ vectors in a $\mathbb{Q}$-module, and take two submodules $G, H$ of dimensions $g = dim(G), h = dim(H)$. Suppose that the module which is generated by the vectors $v_i$, $G$, and $H$, has dimension $n + g + h$. Then we can have at most $n + g + h$ points in general position in the union $\bigcup_i v_i + G + H$.*

Proof. WLOG, one may suppose $v_1 = 0$ after a shift. Take bases $x_1, \dots x_g$, $y_1, \dots y_h$ of $G, H$, respectively. Then, the vectors $0, v_2, \dots v_n, x_1, \dots x_g, y_1, \dots y_h$ are in general position. Conversely, if the vectors $z_1, z_2, \dots z_m$ in the union $\bigcup_i v_i + G + H$ are in general position, then $dim(\mathbb{Q}.\{z_1, z_2, \dots z_m\}) = m-1$. But $\langle z_1, z_2, \dots z_m \rangle$ is contained in the module which is spanned by $v_2, \dots v_n$, $G$, and $H$, whose dimension is $n + g + h - 1$, i.e., $m - 1 \leq n + g + h - 1$, QED.

### E.2.2 Endomorphisms on Dual Numbers

For a commutative ring $R$, we have the commutative $R$-algebra $R[\varepsilon]$ of dual numbers (see example 76 in appendix D.1.1). As an $R$-module, it has dimension 2 and is isomorphic to $R^2$ under the map $a + \varepsilon.b \mapsto (a, b)$. By the above corollary 25, the $R$-linear endomorphism ring of $R[\varepsilon]$ identifies to the four-dimensional matrix ring $\mathbb{M}_{2,2}(R)$. In this situation, the left regular representation of $R[\varepsilon]$ is the homomorphism of $R$-algebras

$$\lambda : R[\varepsilon] \to \mathbb{M}_{2,2}(R)$$

with

$$\lambda(a + \varepsilon.b) = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix},$$

which represents the linear endomorphism of multiplication by $a+\varepsilon.b$. We have shown in section 29.6 that $\mathbb{M}_{2,2}(R)$ is generated by the four $R$-linear basis elements $\lambda(1_R), \lambda(\varepsilon), \alpha_+, \alpha_+ \cdot \lambda(\varepsilon)$, where $\alpha_+$ is the sweeping orientation

$$\alpha_+ = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

and that the $R$-algebra $\mathbb{M}_{2,2}(R)$ identifies to the quotient

$$R\langle \lambda(\varepsilon), \alpha_+ \rangle / (\lambda(\varepsilon)^2, \alpha_+^2 - \alpha_+, \alpha_+ \cdot \lambda(\varepsilon) + \lambda(\varepsilon) \cdot \alpha_+ - \lambda(1 + \varepsilon))$$

of the polynomial $R$-algebra in the two non-commuting variables $\lambda(\varepsilon), \alpha_+$.

### E.2.3 Semi-Simple Modules

A module $M \neq 0_R$ which has no submodules except $0_R, M$ is called *simple*. A module $M$ is called *semi-simple* iff it has the following equivalent properties:

**Lemma 74** *Let $M$ be an $R$-module. The following statements are equivalent:*

(i) *Every submodule of $M$ is a sum of simple submodules.*

(ii) *$M$ is the sum of simple submodules.*

(iii) *$M$ is the direct sum of simple submodules.*

(iv) *Every submodule $N$ of $M$ is a direct summand (i.e., there is a submodule $N'$ of $M$ such that $M = N \oplus N'$.*

The following is immediate:

**Lemma 75** *A linear homomorphism between simple $R$-modules is either an isomorphism or zero. Hence the endomorphism ring $End_R(M)$ of a simple module $M$ is a skew field.*

For example, the left $\mathbb{M}_{m,m}(F)$-module $\mathbb{M}_{m,n}(F)$ and the right $\mathbb{M}_{n,n}(F)$-module $\mathbb{M}_{m,n}(F)$ of $m \times n$-matrices over a skew field $F$ is semi-simple. For the left module, the $n$ columns are the simple submodules, whereas for the right module, the $m$ rows are the simple submodules. Moreover, the only left and right submodules of $\mathbb{M}_{m,n}(F)$ are the zero module and $\mathbb{M}_{m,n}(F)$. In particular, the only two-sided ideals in the ring $\mathbb{M}_{m,m}(F)$ are 0 and $\mathbb{M}_{m,m}(F)$, i.e., this ring is simple.

**Theorem 48** (Wedderburn) *The semi-simple rings are the finite products $\prod_i \mathbb{M}_{m_i,m_i}(F_i)$ of matrix rings $\mathbb{M}_{m_i,m_i}(F_i)$ over skew fields $F_i$.*

If $G$ is a finite group, and if $K$ is a commutative field, we have defined the monoid algebra $K\langle G\rangle$ in D.1.1, it is called the group algebra in this case. Here are the semi-simple group algebras over commutative fields:

**Theorem 49** (Maschke) *The group algebra $K\langle G\rangle$ is semi-simple iff $char(K) \nmid card(G)$.*

### E.2.4 Jacobson Radical and Socle

**Definition 137** *The intersection of all maximal submodules of an $R$-module $M \neq 0$ is called the* Jacobson radical of $M$, *it is denoted by $Rad(M)$.*

**Sorite 12** *let $M, N$ be $R$-modules. Then:*

(i) *For $f \in Lin_R(M,N)$, we have $f(Rad(M)) \subseteq Rad(N)$.*

(ii) *We have $Rad(M \oplus N) = Rad(M) \oplus Rad(N)$.*

(iii) *We have $Rad(M/Rad(M)) = 0$.*

(iv) *We have $Rad({}_RR).M \subseteq Rad(M)$.*

(v) *If $M$ is semi-simple, then $Rad(M) = 0$.*

(vi) *If a submodule $N$ of $M$ has $M/N$ semi-simple, then $Rad(M) \subseteq N$.*

For a ring $R$, one may look at its left radical $Rad({}_RR)$, or at its right radical $Rad(R_R)$. Fortunately, there is no difference in that:

**Proposition 79** *The left and right radicals of a ring $R$ coincide, $Rad({}_RR) = Rad(R_R)$, and this (two-sided) ideal $Rad(R)$ is the maximal ideal $I$ which annihilates every semi-simple module $M$, i.e., $I.M = 0$.*

For a ring $R$, we set

$$J_r = \{r \in R | 1_R \notin r \cdot R\}, J_l = \{r \in R | 1_R \notin R \cdot r\}.$$

**Proposition 80** *For a ring $R$ the following conditions are equivalent:*

(i) *The quotient ring $R/Rad(R)$ is a skew field.*

(ii) *We have $J_r = Rad(R)$.*

(iii) *We have $J_l = Rad(R)$.*

(iv) *The set $J_r$ is additively closed.*

(v) *The set $J_l$ is additively closed.*

**Definition 138** *A ring with the equivalent properties of proposition 80 is called* local. *In particular, a commutative ring $R$ is local iff it has a unique maximal ideal.*

The *length $l(M)$ of a module $M$* is the maximum length $l$ of finite chains $0 \subsetneqq N_1 \subsetneqq N_2 \subsetneqq \ldots N_l = M$ of submodules (if that maximum is $\infty$, we set $l(M) = \infty$). A module is called *indecomposable* iff it is not the direct sum of two proper submodules.

**Lemma 76** *If $M$ is an $R$-module with local endomorphism ring, then $M$ is indecomposable. Conversely, let $M$ be an indecomposable module of finite length $l$. Then $End_R(M)$ is a local ring, and the radical $S = Rad(End_R(M))$ is nilpotent, namely $S^l = 0$.*

**Proposition 81** *Let $X \subset M, Y \subset N$ be two non-zero submodules of indecomposable $R$-modules $M, N$ of finite lengths. If $F : M \to N, g : N \to M$ are linear maps which induce mutually inverse isomorphisms $f|X : X \tilde{\to} Y, g|Y = (f|X)^{-1} : Y \tilde{\to} X$, then $f, g$ are isomorphisms.*

Proof. Since $f \cdot f$ restricts to the identity of $X$, so does any of its positive powers. So non of them can be the zero endomorphisms of $M$. But from lemma 76 we know that $End(M)$ is local and that a non-nilpotent endomorphism must be invertible. A symmetric argument yields inversibility of $f \cdot g$ and therefore of both, $f$ and $g$, QED.

**Lemma 77** (Fitting's lemma) *Let $f : M \to M$ be a linear endomorphism of a module $M$ of finite length. Then there is a positive power $f^n$ and a direct decomposition $M = N \oplus Ker(F^n)$ such that $f^n$ is an automorphism on $N$.*

**Definition 139** *The socle $Soc(M)$ of a module $M$ is the sum of all its simple submodules.*

**Theorem 50** *For a module $M$ of finite length, the following three statements are equivalent:*

(i) *$M$ is semi-simple.*

(ii) *$Rad(M) = 0$.*

(iii) *$Soc(M) = M$.*

### E.2.5 Theorem of Krull–Remak–Schmidt

**Theorem 51** (Krull–Remak–Schmidt) *Let $M_1, \dots M_k, M'_1, \dots M'_l$ be modules with local endomorphism rings (in particular these modules are all non-zero). Suppose that the direct sums $\bigoplus_{i=1,\dots k} M_i$ and $\bigoplus_{j=1,\dots l} M_j$ are isomorphism of $R$-modules. Then $k = l$, and there is a permutation $\sigma$ of the indices such that we have isomorphisms $M_i \xrightarrow{\sim} M'_{\sigma(i)}$ for all $i = 1 \dots k$.*

**Corollary 26** *A module of finite length is a direct sum of indecomposable submodules in a unique way up to permutation and isomorphisms of the summands.*

This follows from theorem 51 in view of lemma 76, QED.

## E.3 Categories of Modules and Affine Transformations

See appendix G for a reference to category theory.

For an additive group $M$ and an element $m \in M$, the *translation by $m$* is the set map $e^m : M \to M : x \mapsto e^m(x) = m + x$. The exponential notation is chosen because $e^? : M \to \mathfrak{S}_M$ is an injective group homomorphism. We denote by $e^M$ the group of translations on $M$, a group which is isomorphic to $M$.

**Definition 140** *For two rings $R, S$, an $R$-module $M$ and an $S$-module $N$, a* diaffine homomorphism *$f$ is a map of form $e^n \cdot f_0$, where $e^n$ is a translation on $N$ and $f_0 \in Dil(M, N)$. The set of diaffine homomorphisms $f : M \to N$ is denoted by $M@N$. If we fix the underlying scalar restriction $\varphi : R \to S$ and only take $f_0 \in Dil_\varphi(M, N)$, the corresponding set is denoted by $M@_\varphi N$. In particular, if $\varphi = Id_R$, we write $M@_R N$ for the set of ($R$-)affine homomorphisms.*

**Sorite 13** *If $R, S, T$ are rings and $M, N, L$ are modules over these rings, respectively, we have the following facts:*

(i) *If $f = e^n \cdot f_0 \in M@N$, then $n = f(0)$, and $f_0 = e^{-n} \cdot f$. So the* translation part *$e^n$ and the* dilinear part *$f_0$ are uniquely determined.*

(ii) *If $f = e^n \cdot f_0 \in M@N, g = e^l \cdot g_0 \in N@L$, then the set-theoretic composition $g \cdot f \in M@L$ and*

$$g \cdot f = e^{l+g(n)} \cdot g_0 \cdot f_0. \tag{E.4}$$

(iii) *The diaffine $f = e^n \cdot f_0 \in M@N$ is an isomorphism iff its dilinear part $f_0$ is so, and then the inverse is $f^{-1} = e^{-f_0^{-1}(n)} \cdot f_0^{-1}$.*

(iv) *If $f, g \in M@_\varphi N$, then so is the pointwise difference $f - g$, hence $M@_\varphi N$ is an additive group. If $S$ is commutative, $M@_\varphi N$ is an $S$-module and also an $R$-module under $\varphi$.*

(v) *For $M = N, \varphi = Id_R$, the* general affine group *$\overrightarrow{GL}(M)$, i.e., the group of affine automorphisms is isomorphic to the semidirect product $M \rtimes_\phi Gl(M)$ of $M$ with the* general linear group *$GL(M)$ of linear automorphisms of $M$ under the group action $\phi = Id_{GL(M)}$.*

The category of modules and diaffine homomorphisms is denoted by **Mod**, whereas the subcategory of (left) $R$-modules and $R$-affine homomorphisms is denoted by $\mathbf{Mod}_R$. If $f : S \to R$ is a ring homomorphism, we have the scalar extension functor

$$S\otimes_R ? : \mathbf{Mod}_R \to \mathbf{Mod}_S : M \mapsto S \otimes_R M \tag{E.5}$$

which acts by scalar extension of the linear parts of morphisms and by the canonical map $M \to S\otimes_R : x \mapsto 1 \otimes x$ on the translation part. See [63, II.5.1] for details.

In **Mod**, we have to add an additional object $\emptyset_R$ for each ring $R$. This is the empty set plus the unique possible scalar multiplication. This is *not* a module in the usual sense since it is not even a group! But there are important category-theoretic reasons to introduce these objects. Observe that for an $S$-module $N$, $\emptyset_R@N$ is in bijection with the set of ring homomorphisms $Hom(R, S)$, whereas $N@\emptyset_R$ is empty if $N$ is not empty.

By $\mathbf{Mod}^{@}$, we denote the category of set-valued presheaves on **Mod**, i.e., the contravariant set-valued functors $F : \mathbf{Mod} \to \mathbf{Sets}$. In particular, the Yoneda embedding $\mathbf{Mod} \to \mathbf{Mod}^{@}$ yields the representable presheaf $@M$ for a module $M$, with $@M(X) = X@M$ for $X \in {}_0\mathbf{Mod}$. This is one reason why we also write $X@F$ for the evaluation $F(X)$, even if $F$ is not representable. By $M@$ we denote the covariant functor with $M@(X) = M@X$. In the context of presheaves, we often call a module $X$ that is an argument of such presheaves an *address*; the reasons for this wording are made explicit in the musicological chapter 6 on forms and denotators.

## E.3.1 Direct Sums

**Proposition 82** *Let $A$ be an $R$-module, and $n$ a natural number. Then there is a canonical isomorphism $@A^{\oplus n} \tilde{\to} (@A)^n$, i.e., $A^{\oplus n}$ represents the $n$-fold product functor.*

Proof. Let $X$ be any module. Then every affine homomorphism $f = e^t \cdot f_0 : X \to A^{\oplus n}$ projects to the $n$ factors $f_i = p_i \cdot f$ via the respective projections $p_i : A^{\oplus n} \to A$. Also, the dilinear part $f_0$ projects to the n dilinear factors $f_{0,i} : X \to A$. Let $t_i$ be the $i^{th}$ component of $t$. Then we have $f_i = e^{t_i} \cdot f_{0,i}$. This yields the desired bijection $X@A^{\oplus n} \tilde{\to} (X@A)^n$, and this is functorial in $X$. QED.

## E.3.2 Affine Forms and Tensors

In this section we suppose that all modules have a commutative coefficient ring $R$, i.e., we work in the category $\mathbf{Mod}_R$. Tensor products[2] are automatically taken over $R$. By $X^\star$ we denote the $R$-linear dual $Lin_R(X; R)$ of the $R$-module $X$.

For the $R$-module $A^\bigstar$ of affine forms on an $R$-module (address) $A$ and an $R$-module M, we have a canonical linear injection $M \rightarrowtail A^\bigstar \otimes M$. In fact, there is an $R$-linear isomorphism $A^\bigstar \tilde{\to} R \oplus A^\star : e^r \cdot x \mapsto r + x$, and we deduce an $R$-linear isomorphism $A^\bigstar \otimes M \tilde{\to} M \oplus A^\star \otimes M$, whence the above injection; it maps $m \in M$ to $e^{1_R}.0 \otimes m$.

With the above notation, fix an $R$-module $A$ (an address). We have the subfunctor

$$A@_R : \mathbf{Mod}_R \to \mathbf{Sets} : M \mapsto A@_R M \tag{E.6}$$

[2] See [63] for tensor products.

of $A@$, and induced on the subcategory $\mathbf{Mod}_R$ of $\mathbf{Mod}$. We further have the functor

$$A^\bigstar\otimes : \mathbf{Mod}_R \to \mathbf{Sets} : M \mapsto A^\bigstar \otimes M. \tag{E.7}$$

This functor acts as follows on affine morphisms $F = e^n \cdot F_0 : M \to N$. Identify $n$ with the canonically associated element of $A^\bigstar \otimes N$. Then, we define

$$A^\bigstar \otimes F = e^n \cdot A^\bigstar \otimes F_0, \tag{E.8}$$

in other words, $(A^\bigstar \otimes F)_0 = A^\bigstar \otimes F_0$. For the composition

$$e^{t+G_0(n)} \cdot G_0F_0 : M \xrightarrow{F} N \xrightarrow{G} T$$

of affine morphisms $F = e^n \cdot F_0$ and $G = e^t \cdot G_0$ this implies

$$\begin{aligned} A^\bigstar \otimes GF =& e^{t+G_0(n)} \cdot A^\bigstar \otimes G_0F_0 \\ =& e^t \cdot e^{G_0(n)} \cdot A^\bigstar \otimes G_0 \cdot A^\bigstar \otimes F_0 \\ =& e^t \cdot A^\bigstar \otimes G_0 \cdot e^n \cdot A^\bigstar \otimes F_0 \\ =& A^\bigstar \otimes G \cdot A^\bigstar \otimes F, \end{aligned}$$

whence the claimed functoriality.

**Lemma 78** *With the above notation, there is a natural transformation*

$$\theta : A^\bigstar\otimes \to A@_R. \tag{E.9}$$

*If $M$ is an $R$-module, it is defined by its action on pure tensors $x \otimes m \in A^\bigstar \otimes M$ by*

$$\theta(x \otimes m) : A \to M : a \mapsto x(a)m. \tag{E.10}$$

*The natural transformation $\theta$ is an isomorphism if $A$ is a finitely generated projective module.*

Proof. The formula (E.10) is an extension of a classical formula in the linear case, see [63, II.74]. In fact, write $A^\bigstar \otimes M \xrightarrow{\sim} M \oplus A^\star \otimes M$, and then $A@M \xrightarrow{\sim} M \oplus Lin_R(A, M)$. Then the classical formula $\theta_0 : A^\star \otimes M \to Lin_R(A, M) : x \otimes m \mapsto \theta_0(x \otimes m)$ with $\theta_0(x \otimes m)(a) = x(a)m$ extends to the linear map

$$\theta'(m + \lambda) = m + \theta_0(\lambda)$$

which for special pure tensor arguments $x \otimes m = e^r \cdot x_0 \otimes m$ yields

$$\begin{gathered} \theta'(rm + x_0 \otimes m) = rm + \theta_0(x_0 \otimes m), \text{ corresponding to} \\ e^{rm} \cdot \theta_0(x_0 \otimes m), \end{gathered}$$

and the latter evaluates to

$$(e^{rm} \cdot \theta_0(x_0 \otimes m))(a) = rm + x_0(a)m = x(a)m$$

which means $\theta' = \theta$. Let us then prove the naturality of $\theta$. Let $F = e^n \cdot F_0 : M \to N$ be an affine morphism. We have to show that the diagram

$$\begin{array}{ccc} A^\star \otimes M & \xrightarrow{\theta} & A@M \\ {\scriptstyle A^\star \otimes F}\downarrow & & \downarrow{\scriptstyle A@F} \\ A^\star \otimes N & \xrightarrow{\theta} & A@N \end{array} \tag{E.11}$$

is commutative. It suffices to verify it for pure tensors $x \otimes m \in A^\star \otimes M$. Take $a \in A$. Then

$$\begin{aligned} A@F(\theta(x \otimes m))(a) =& F(\theta(x \otimes m)(a)) \\ =& F(x(a)m) \\ =& n + F_0(x(a)m) \\ =& n + x(a)F_0(m) \\ =& \theta(n) + \theta(x \otimes F_0(m))(a) \\ =& \theta(n + x \otimes F_0(m))(a) \\ =& \theta((A^\star \otimes F)(x \otimes m))(a) \end{aligned}$$

and we are done with diagram (E.11). If $A$ is finitely generated projective, the classical linear map $\theta_0$ is iso, and hence so is $\theta$. QED.

Observe that the special case $A = 0$ of zero address is included in the lemma and that in this case, $\theta$ identifies to the identity transformation on the forgetful functor.

### E.3.3 Biaffine Maps

In this section we again suppose that all modules have a commutative coefficient ring $R$, i.e., we work in the category $\mathbf{Mod}_R$. In classical module theory, the tensor product is a universal construction relating to bilinear maps. The extension to biaffine maps runs as follows.

**Definition 141** *Let $U, V, W$ be modules in $\mathbf{Mod}_R$. A map*

$$f : U \times V \to W$$

*is called* biaffine *if it is affine in each variable, i.e., if*

$$\begin{aligned} &f_u : V \to W : v \mapsto f_u(v) = f(u, v) \text{ and} \\ &f^v : U \to W : u \mapsto f^v(u) = f(u, v) \end{aligned}$$

*are all affine, i.e., $f_u \in V @_R W$ and $f^v \in U @_R W$. The set of all biaffine maps $f$ is denoted by $\mathcal{A}_2(U, V; W)$.*

**Lemma 79** *For $R$-modules $U, V, W$ in $\mathbf{Mod}$, there is a canonical bijection*

$$\mathcal{A}_2(U, V; W) \xrightarrow{\sim} U @_R (V @_R W). \tag{E.12}$$

*If these sets are given their canonical structure of $R$-modules, bijection E.12 is an isomorphism of $R$-modules.*

Proof. Let $f : U \times V \to W$ be a biaffine map. Then the associated map $f_u, u \in U$ by definition stays in $V@_R W$. So we have a map $f_? : U \to V@_R W$. Let us show that this map is affine. Set $\lambda(v) = f(0, v)$, this affine map is the constant part of our candidate $f_?$, i.e., we claim that $f_u - \lambda$ is linear in $u$. But $f_u(v) - \lambda(v) = f^v(u) - \lambda(v) = f^v(u) - f^v(0)$ is linear in $u$ and we are done. Conversely, each $g \in U@_R(V@_R W)$ defines $\tilde{g}(u, v) = g(u)(v)$ with $\tilde{g} \in \mathcal{A}_2(U, V; W)$, and this clearly is an inverse to the map $f \mapsto f_?$. That we have a module isomorphism is clear. QED.

**Proposition 83** *Let $U, V, W$ be $R$-modules in* **Mod**, *and define the* affine tensor product $U \boxtimes V = U \otimes V \oplus U \oplus V$. *Then we have a canonical bijection*

$$\boxtimes : \mathcal{A}_2(U, V; W) \xrightarrow{\sim} (U \boxtimes V)@_R W, \tag{E.13}$$

*i.e., $U \boxtimes V$ is a universal object in the affine category* $\mathbf{Mod}_R$ *like the tensor product is for $R$-linear maps. If $f \in \mathcal{A}_2(U, V; W)$, then its image $\boxtimes f$ applies a typical element $u \otimes v + r + s$ to*

$$\boxtimes f(u \otimes v + r + s) = \square f(u \otimes v) + f^0(r) + f_0(s) - f(0, 0) \tag{E.14}$$

*where $\square f$ is the linear map associated with the bilinear map*

$$f_\square(u, v) = f(u, v) - f_0(v) - f^0(u) + f(0, 0). \tag{E.15}$$

*The universal map $i : U \times V \to U \boxtimes V$ is defined by $i(u, v) = u \otimes v + u + v$.*

Proof. The proposition follows directly from lemma 79, the definition of the affine tensor product and the universal property of the linear tensor product. We then have $\boxtimes f(i(u, v)) = \boxtimes f(u \otimes v + u + v) = \square f(u \otimes v) + f^0(u) + f_0(v) - f(0, 0) = f_\square(u, v) + f^0(u) + f_0(v) - f(0, 0) = f(u, v) - f_0(v) - f^0(u) + f(0, 0) + f^0(u) + f_0(v) - f(0, 0) = f(u, v)$. QED.

**Definition 142** *For modules $U, V, W, X$ in* $\mathbf{Mod}_R$ *and affine maps $f : U \to W$, $g : V \to X$, the affine tensor product map*

$$f \boxtimes g : U \boxtimes V \to W \boxtimes X \tag{E.16}$$

*is defined as the canonical affine map $\boxtimes h$ according to proposition 83 which is associated with the biaffine map*

$$h : U \times V \to W \boxtimes X : (u, v) \mapsto i(f(u), g(v)).$$

**Sorite 14** *For modules $U, V, W, X$ in* $\mathbf{Mod}_R$ *and with the notation of section E.3.8, we have:*

(i) $U \boxtimes V \xrightarrow{\sim} V \boxtimes U$.

(ii) $U \boxtimes 0_R \xrightarrow{\sim} 0_R \boxtimes U \xrightarrow{\sim} U$.

(iii) $U \boxtimes (V \boxtimes W) \xrightarrow{\sim} (U \boxtimes V) \boxtimes W$, *i.e., we can identify these products and write $U \boxtimes V \boxtimes W$.*

(iv) $(U \amalg V) \boxtimes W \xrightarrow{\sim} U \boxtimes W \amalg V \boxtimes W$.

For a module $M$ in $\mathbf{Mod}_R$, the functor $@_R M : \mathbf{Mod}_R \to \mathbf{Sets} : B \mapsto B@_R M$ contains redundant structure in $B$ since there are elements in $B$ which are annihilated by all linear maps into $M$. We want to reduce $B$ to a module where this annihilator set is the zero submodule.

**Definition 143** *With the above notation, set* $An(B,M) = \bigcap_{k\in B@M)} Ker(k_0)$, *denote* $B_{/M} = B/An(B,M)$ *and write* $/M : B \to B_{/M}$ *for the canonical projection. The module* $B_{/M}$ *is called the* $M$*-reduction of* $B$.

The following lemma is clear:

**Lemma 80** *The assignment* $?_{/M} : B \mapsto B_{/M}$ *defines a functor on* $\mathbf{Mod}_R$. *The projection* $/M : B \to B_{/M}$ *and the uniquely defined commutative diagrams*

$$\begin{array}{ccc} C & \xrightarrow{/M} & C_{/M} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_{/M}} \\ B & \xrightarrow{/M} & B_{/M} \end{array} \tag{E.17}$$

*which are associated with affine homomorphisms* $f : C \to B$ *define a natural transformation on* $Id_{\mathbf{Mod}_R} \to ?_{/M}$.

**Proposition 84** *Let* $@_R^{red} M = @_R M \cdot ?_{/M}$ *Then we have a natural isomorphism* $@_R^{red} M \xrightarrow{\sim} @_R M$.

Proof. In fact, the natural transformation $/M : B \to B_{/M}$ induces an isomorphism (of $R$-modules) $B@_R^{red} M \xrightarrow{\sim} B@_R M$. QED.

**Corollary 27** *With the above notation, the functors* $B\tilde{@}M$ *and* $B\tilde{@}^{red}M = B_{/M}\tilde{@}M$ *are canonically isomorphic.*

Proof. If $A \in {}_0\mathbf{Mod}_R$, we have

$$\begin{aligned} A@B\tilde{@}^{red}M &= A@B_{/M}\tilde{@}M \\ &= A \boxtimes B_{/M}@_R M \\ &\xrightarrow{\sim} A@_R(B_{/M}@_R M) \\ &\xrightarrow{\sim} A@_R(B@_R M) \\ &\xrightarrow{\sim} A \boxtimes B@_R M \\ &= A@B\tilde{@}M. \end{aligned}$$

Let $M, A$ be modules in $\mathbf{Mod}_R$. Then we have

**Lemma 81** *There is an isomorphism of* $R$*-modules*

$$Lin_R(M, A@_R R) \xrightarrow{\sim} A@_R M^\star \tag{E.18}$$

*which is functorial in both,* $A$ *and* $M$.

Proof. We have these functorial isomorphisms:

$$\begin{aligned} Lin_R(M, A@_R R) &\xrightarrow{\sim} Lin_R(M, A^\star \oplus R) \\ &\xrightarrow{\sim} Lin_R(M, A^\star) \oplus M^\star \\ &\xrightarrow{\sim} (M \otimes A)^\star \oplus M^\star \\ &\xrightarrow{\sim} (A \otimes M)^\star \oplus M^\star \\ &\xrightarrow{\sim} Lin_R(A, M^\star) \oplus M^\star \\ &\xrightarrow{\sim} A@_R M^\star. \end{aligned}$$

**Proposition 85** *Let $M$ be as above, and $A = R^n$, $0 \leq n$, then we have canonical $R$-linear maps*

$$u : A@_R M^\star \to (A@_R M)^\star, \tag{E.19}$$
$$d : A@_R M \to (A@_R M^\star)^\star, \tag{E.20}$$

*which are isomorphisms for $M$ finitely generated and projective.*

Proof. As to the first map, a linear map $v : Lin_R(A, M^\star) \to Lin_R(A, M)^\star$ is defined as follows: For $g : A \to M^\star$ and $f : A \to M$, we have the composition $f^\star \cdot g : A \to A^\star$, which is a bilinear form on $A$, and we may set $e(g)(f) = tr(f^\star \cdot g)$, a linear function of $g$, calculated in the canonical bases of $A$ and $A^\star$. This map is canonically extended by the identity on $M$ and we are done for $u$. For the second map, we have the canonical bidual linear map $l : A@_R M \to A@_R M^{\star\star}$, and we may apply the first map to the bidual of $M$. The statement concerning finitely generated projective modules is standard. QED

### E.3.4 Symmetries of the Affine Plane

We consider symmetries, i.e., affine transformations $D = e^t \cdot H$ on $\mathbb{R}^2$. From the geometric point of view, the set $\mathbb{R}^2@\mathbb{R}^2$ of these maps is described as follows. Fix a (zero-addressed) local composition $\Delta = \{u, v, w\}$ in the real plane $\mathbb{R}^2$, with three points in general position (see appendix E.2.1). Then we know from section 15.2.1 that the map

$$B(\Delta) : \mathbb{R}^2@\mathbb{R}^2 \to (\mathbb{R}^2)^3 : D \mapsto (D(u), D(v), D(w)) \tag{E.21}$$

is a bijection. We use this bijection to describe some special transformations:

**Shearings.** Let $G$ be a straight line in $\mathbb{R}^2$ (not necessarily through the origin), and let $u, v, w$ be in general position such that $u, v$ lie on $G$, whereas $w$ does not. A *shearing $S$ relating to* $G$ is a symmetry which leaves both, $u, v$, fixed and transforms $w$ into $w + r.(v - u), r \in \mathbb{R}$. Then $G$ remains fixed identically, and $w$ is shifted in parallel motion with respect to $G$. The $n^{th}$ power $S^n$ of $S$ is the shearing which fixes $G$ and transforms $w$ into $w + nr.(v - u)$.

**Dilatations.** For a point $u \in \mathbb{R}^2$, and two scalars $\delta, \sigma \in \mathbb{R}$, a *dilatation $D$ by factors $\delta, \sigma$* and centered in $u$ is defined by the prescription that $D(u) = u$, and that for two other points $v, w$ such that $u, v, w$ are in general position, we have $D(v) = u + \delta.(v - u), D(w) = u + \sigma.(w - u)$. A dilatation with $\delta = \sigma = -1$ is called *point reflection with center $u$*, it corresponds to a rotation by 180° around $u$.

**Glide Reflections.** Let $G$ be a straight line, and take $u, v, w$ as in the above paragraph E.3.4 about shearing. A *glide reflection $P$* is a symmetry, such that $P(u)$ lies on $G$, and $P(v) - P(u) = v - u, P(w) - P(u) = u - w$. Therefore, $P(v)$ lies also on $G$, and $P(w)$ lies on the "opposite side" of $w$ on the line through $w, u$, i.e., $P$ is a translation by $P(u) - u$, followed by a 'skew' reflection in $G$ in the direction of the line through $w$ and $u$. Especially for the diagonal $G = \mathbb{R}.(1, 1)$ and $u = P(u) = 0, w = (-1, 1)$, we obtain the exchange of coordinate axes: the *parameter exchange.*

## E.3.5 Symmetries on $\mathbb{Z}^2$

**Theorem 52** *Every symmetry $f \in \mathbb{Z}^2 @ \mathbb{Z}^2$ is the product of some of the following symmetries:*

1. *a translation $T = e^{(0,1)}$,*
2. *a shearing $S$ which leaves the first axis $\mathbb{Z}.(1, 0)$ fixed and transforms $(0, 1)$ to $(1, 1)$,*
3. *the parameter exchange $P$,*
4. *the reflection $K$ at the second axis,*
5. *the dilatations $D_m, 0 \leq m$ in the direction of the first axis by factor $m$.*

*The general affine group $\overrightarrow{GL}(\mathbb{Z}^2)$ is generated by $T, S, P, K$.*

Proof. The statement concerning the general affine group is immediate from the first part of the theorem. To show the latter, we say that a symmetry $X$ is "good" if $X$ can be written as a product of symmetries of the required shape. Here are the matrices for the generators:

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, D_m = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}.$$

We first show that all $2 \times 2$-matrices, i.e., all linear maps $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are products of symmetries of type $S, K, D_m, P$. Observe that $P^2 = K^2 = E_2$.

1. If $P \cdot X$ is good, then so is $X = P^2 \cdot X = P \cdot (P \cdot X)$. The same is valid for $X \cdot P$. Here, $P \cdot X$ is the exchange of rows in $X$, whereas $X \cdot P$ is the exchange of columns in $X$.
2. We have $S^{-1} = K \cdot S \cdot K = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Therefore the powers $S^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ are good for integer $n$. Further, $P \cdot S^{\pm 1} \cdot P = \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}$ and therefore $(P \cdot S \cdot P)^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ is good.

3. For $m \geq 0$, $P \cdot D_m \cdot P = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$ and $P \cdot K \cdot D_m \cdot P = \begin{pmatrix} 1 & 0 \\ 0 & -m \end{pmatrix}$ are good.

4. If a coefficient of $X$ vanishes, one can enforce $c = 0$ by a row or column exchange. Because of
$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$
and the preceding results, such an $X$ is also good.

5. If no coefficient of $X$ vanishes, one can apply $\pm E_2$ and exchange of columns that satisfy $c \geq d > 0$. Applying the Euclidean algorithm (section D.3), we can write $c = nd + r, 0 \leq r < d$. This yields
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix} = \begin{pmatrix} a - nb & b \\ c - nb & d \end{pmatrix} = \begin{pmatrix} a' & b \\ r & d \end{pmatrix}.$$
If $r = 0$, we are in case 4 above. Else, we have $d > r > 0$. Via column exchange we obtain $\begin{pmatrix} b & a' \\ d & r \end{pmatrix}$, and we may set $c' = d, d' = r$. But $d' < d$, so the algorithm leads to case 4 after a finite number of steps. This settles the linear maps.

For translations, observe the following identities:
$$e^{(1,0)} = P \cdot T \cdot P, e^{(-1,0)} = K \cdot e^{(1,0)} \cdot K, e^{(0,-1)} = P \cdot e^{(-1,0)} \cdot P.$$

Therefore all transpositions are good since we have natural numbers $x, y$ such that $e^{(\pm x, \pm y)} = (e^{(\pm 1,0)})^x \cdot (e^{(0,\pm 1)})^y$. This settles all the cases, QED.

### E.3.6 Symmetries on $\mathbb{Z}^n$

For integers $n \geq 2$ and $1 \leq i < j \leq n$, we have the *diagonal embedding* $\Delta_{i,j} : GL(\mathbb{Z}^2) \rightarrowtail GL(\mathbb{Z}^n)$ defined by

$$\Delta_{i,j}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (x_{u,v}) \text{ with}$$
$$x_{i,i} = a, x_{i,j} = b, x_{j,i} = c, x_{j,j} = d,$$
$$x_{u,u} = 1 \text{ for } u \neq i, j, \text{ and}$$
$$x_{u,v} = 0 \text{ else.}$$

**Theorem 53** *For an integer $n \geq 2$ the group $GL(\mathbb{Z}^n)$ is generated by the diagonal embedded groups $\Delta_{1,j} GL(\mathbb{Z}^2), 1 \leq j \leq n$.*

The proof goes by induction on $n$ and uses the Euclidean algorithm, we leave it as an exercise.

### E.3.7 Complements on the Module of a Local Composition

**Lemma 82** *Let $A$ be an address module over the commutative coefficient ring $R$ and $(K, A@M)$ a commutative local composition. Then:*

(i) $R.K \subset \langle K \rangle$.

(ii) $R.K = \langle K \rangle$ *iff* $K$ *is embedded.*

(iii) *If $f : K \to L$ is a morphism of embedded commutative local compositions*

$$(K, A@_R M), (L, A@_R N)$$

*at the same address $A$. Then any underlying symmetry $F : M \to N$ restricts to $R.K$ and $R.L$, i.e., $F(R.K) \subset R.L$, and this restriction is uniquely determined by $f$. We denote this affine map by $R@f : R.K \to R.L$.*

Proof. The first statement is clear since $R.K = \langle x - x_0 |\ x \in K \rangle$. If $R.K = \langle K \rangle$, then obviously $K \subset \langle K \rangle = R.K$, and $K$ is embedded. Conversely, if $K \subset R.K$, then also $\langle K \rangle \subset R.K$ and equality follows from (i). As to (iii), observe that we have a linear application $R.f : R.K \to R.L$ which is induced by the linear part $F_0$ of $F$, and which is only a function of $f$, by lemma 6 of chapter 8. Further, if $F = e^n \cdot F_0$, and if $k \in K$, we have $n = F(k) - F_0(k) = f(k) - R.f(k)$ since $K \subset R.K$; in other words, $n = n_f$ is only a function of $f$, and not of the underlying $F$. Therefore, since both, $f(k)$ and $R.f(k)$, are elements of $R.L$, $n_f \in R.L$. This means that for $x \in R.K$, $F(x) = n_f + F_o(x) = n_f + R.f(x) = R@f(x) \in R.L$, QED.

### E.3.8 Fiber Products and Fiber Sums in Mod

**Theorem 54** *The category* **Mod** *of modules and diaffine transformations has arbitrary fiber products.*

Proof. We are given a fiber product diagram

$$K \underset{f}{\to} M \underset{g}{\leftarrow} L \tag{E.22}$$

of modules over the fiber product diagram

$$A \underset{u}{\to} C \underset{v}{\leftarrow} B \tag{E.23}$$

of corresponding coefficient rings. If any of these modules $K$, $L$ or $M$ is empty, or if intersection $\mathrm{Im}(f) \cap \mathrm{Im}(g)$ is empty, then the empty module over the fiber product $A \times_C B$ of coefficient rings does the job. So we may suppose that neither of these four spaces is empty.

Consider the dilinear parts $f_0$ and $g_0$ of $f$ and $g$. Then we have the dilinear homomorphism $d : K \oplus L \to M : (k, l) \mapsto f_0(k) - g_0(l)$ with regard to the fiber product ring homomorphism $A \times_C B \to C$. Take any couple $(k, l) \in K \oplus L$ with $f(k) = g(l)$. Then the set-theoretic

fiber product $\Delta \subset K \oplus L$ equals $Ker(d) + (k,l)$. This implies that the diaffine embedding $e^{(k,l)} : Ker(d) \xrightarrow{\sim} \Delta \subset K \oplus L$, followed by the projections to $K$ and $L$ defines a fiber product

$$\begin{array}{ccc} Ker(d) & \xrightarrow{p_2 \cdot e^{(k,l)}} & L \\ {\scriptstyle p_1 \cdot e^{(k,l)}}\downarrow & & \downarrow{\scriptstyle g} \\ K & \xrightarrow{\quad f \quad} & M \end{array} \tag{E.24}$$

of modules and diaffine transformations. QED.

**Theorem 55** *The category* **Mod** *has fiber sums for all pushout diagrams of modules over a fixed coefficient ring, i.e., where the scalar restrictions are the identity.*

Proof. We are given a fiber sum diagram

$$K \underset{f}{\leftarrow} M \underset{g}{\rightarrow} L \tag{E.25}$$

of modules over coefficient ring $A$. If $M$ is empty we have to construct the sum $K \amalg L$, and we may suppose that both summands are nonvoid, the other cases being trivial. Consider the direct sum $S = K \oplus L \oplus A$ and the affine injections

$$\begin{aligned} &i_1 : K \rightarrowtail S : k \mapsto (k, 0, 1) \text{ and} \\ &i_2 : L \rightarrowtail S : l \mapsto (0, l, 0). \end{aligned}$$

Suppose we are given two diaffine transformations

$$\begin{aligned} &f : K \to X, \\ &g : L \to X, \end{aligned}$$

with factorizations $f = e^x \cdot f_0$ and $g = e^y \cdot g_0$ and scalar restriction $s : A \to B$. Define a dilinear map $h_0 : S \to X : (k,l,t) \mapsto f_0(k) + g_0(l) + s(t)(x-y)$. Then we have a diaffine transformation $h = e^y \cdot h_0$ which does the job, in fact, $h \cdot i_1 = f$, and $h \cdot i_2 = g$. Since $i_2$ is linear, we have $h(0) = h(i_2(0)) = g(0) = y$. Hence the affine part of $h$ is uniquely determined. If we had two candidates $h$ and $h^*$ for universal arrows, they would only differ in their dilinear parts $h_0$ and $h_0^*$. But then, their difference $d = h_0 - h_0^*$ would vanish on all elements of shape $(0,l,0)$, $l \in L$ and on all $(k,0,1)$, $k \in K$. The latter implies that $d(0,0,1) = 0$, and by dilinearity of $d$, $d(k,0,0) = d((k,0,1) - (0,0,1)) = 0$, whence the uniqueness of $h$.

On the other hand, if $M$ is non-empty, so are $K$ and $L$. We then have two arrows $u = i_1 \cdot f, v = i_2 \cdot g : M \rightrightarrows K \amalg L$ from the diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & K \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i_1} \\ L & \xrightarrow{i_2} & K \amalg L \end{array} \tag{E.26}$$

and we are done if we can show that there is a coequalizer[3] of the couple $u$ and $v$. If we have the factorizations $u = e^t \cdot u_0$ and $v = e^s \cdot v_0$, take the quotient module $E = K \amalg L/A(t-s)+\mathrm{Im}(u_0 - v_0)$. Clearly, the projection $p : K \amalg L \to E$ equalizes $u$ and $v$. If $r : K \amalg L \to X$ is any diaffine transformation with scalar restriction $s : A \to B$ and equalizing the couple $u$ and $v$, then $r$ has a unique factorization through $E$. In fact, we may suppose without loss of generality that $r$ is dilinear. In this case, $r$ has the required factorization since it annihilates $A(t-s)+\mathrm{Im}(u_0 - v_0)$.

With this construction we define $K \amalg_M L = E$ and obtain a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & K \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p\cdot u} \\ L & \xrightarrow{p\cdot v} & K \amalg_M L \end{array} \tag{E.27}$$

which is the required pushout diagram in **Mod**. Observe that this proof technique—build the sum and then the coequalizer—is a special case of the fact that existence of fiber sums is equivalent to existence of sums and coequalizers provided that we have an initial object, see appendix G.2.1. QED.

**Proposition 86** *A dilinear morphism $f : M \to N$ over scalar restriction $g : A \to B$ is mono iff $f$ is diinjective, i.e., iff $f, g$ are both injective.*

Proof. If both, $f, g$ are injective, then clearly the dilinear morphism is mono. If $f$ is not injective, there are two different affine morphisms $k_i : 0_{\mathbb{Z}} \to M, i = 1, 2$, which are equalized by $f$. If the scalar restriction $g$ is not injective, there are two different ring homomorphisms $r_1, r_2 : \mathbb{Z}[X] \to A$ on the polynomial ring $\mathbb{Z}[X]$ with $r_1(X) \in ker(g), i = 1, 2,$, and the zero morphism $= 0_{\mathbb{Z}[X]} \to M$ for these two scalar restrictions does the job.

# E.4 Complements of Commutative Algebra

In this section, all coefficient rings are commutative.

## E.4.1 Localization

See also [64, II] for concepts and facts described in this section.

Let $S$ be a multiplicative subset of a ring $A$, i.e., $st \in S$ for all $s, t \in S$, and $1 \in S$. The *localization* $S^{-1}A$ is the set of equivalence classes of $A \times S$ modulo the relation $(a, s) \sim (a', s')$ iff there is $t \in S$ such that $t(as' - a's) = 0$. The equivalence class of $(a, s)$ is denoted by the fraction $a/s$ or $\frac{a}{s}$. It is a ring by the well-defined addition $a/s + a'/s' = (as' + a's)/ss'$ and multiplication $a/s.a'/s' = aa'/ss'$. The canonical map $i_S : A \to S^{-1}A : a \mapsto a/1$ is a ring homomorphism with the universal property that for any ring homomorphism $f : A \to B$ such that $f(S) \subset B^\times$, there is a unique ring homomorphism $j : S^{-1}A \to B$ such that $f = j \circ i_S$. The ring $S^{-1}A$ is called the *localization of $A$ in $S$*. Classical example: $A$ is a domain (no zero divisors), $S = A - \{0\}$, whence $S^{-1}A$ is the classical *field $fr(A)$ of fractions over $A$*.

[3] See appendix G.2.1.

If $M$ is an $A$-module, the localization $S^{-1}M$ is the set $S^{-1}M$ of equivalence classes of $M \times S$ for the equivalence relation $(m,s) \sim (m',s')$ iff there is $t \in S$ such that $t(ms' - m's) = 0$. The addition $m/s + m'/s' = (ms' + m's)/ss'$ and the scalar multiplication $r/s.m/t = rm/st$ makes $S^{-1}M$ a $S^{-1}A$-module. One has the canonical dilinear homomorphism $i_M : M \to S^{-1}M : m \mapsto m/1$ with respect to the homomorphism $i_S$. It has this universal property: For every homomorphism $f : M \to N$ of $A$-modules, such that every dilatation $s? : N \to N : n \mapsto s.n$ is bijective, there is a unique homomorphism of $A$-modules $j : S^{-1}M \to N$ such that $f = j \circ i_M$. It is easily seen that the tensor product $S^{-1}A \otimes_A M$, together with the canonical homomorphism of $A$-modules $M \to S^{-1}A \otimes_A M : m \mapsto 1 \otimes m$ is isomorphic to the localization $S^{-1}M$.

For the multiplicative set $S_s = \{1, s, s^2, s^3, \ldots\}, s \in A$, one writes $A_s, M_s$ instead of $S_s^{-1}A, S^{-1}M$. For a prime ideal $q \subseteq A$, the complement $S = A - q$ is multiplicative by definition. And one then writes $A_q, M_q$ instead of $S^{-1}A, S^{-1}M$.

By the universal property of localization, if $f : A \to B$ is a ring homomorphism and if $S \subseteq A, T \subseteq B$ are multiplicative sets such that $f(S) \subseteq T$, then there is a canonical ring homomorphism $f_{S,T} : S^{-1}A \to T^{-1}B$ which extends $f$. If $g : M \to N$ is a dilinear homomorphism over $f$, it extends uniquely to a dilinear homomorphism $g_{S,T} : T^{-1}M \to T^{-1}N$ over $f_{S,T}$. For a multiplicative set $S \in A$ let $S' = \{t \in A | \text{there exists } a \in A, s \in S \text{ such that } s = at\}$ be the *saturation of* $S$. Then the canonical homomorphisms $S^{-1}A \to s'^{-1}A$ and $S^{-1}M \to s'^{-1}M$ are isomorphisms and we may identify the two corresponding localizations. In particular, if $s \in A$, we identify $A_s, M_s$ with $S_s'^{-1}A, S_s'^{-1}M$. So if for $s, t \in A$, one has $S_s' \subseteq S_t'$, one has canonical homomorphisms $A_s \to A_t, M_s \to M_t$.

**Proposition 87** *If $i_S : A \to S^{-1}A$ is the localization homomorphism, the inverse image $q \mapsto i_S^{-1}(q)$ is an order preserving bijection from the set of maximal (resp. prime) ideals in $S^{-1}A$ to the set of maximal (resp. prime) ideals in $A$ which are disjoint from $S$. In particular, if $S = A - p$ for a prime ideal $p$, the localization $A_p$ is a local ring with maximal ideal $m_p = p_p$ and the residue field $\kappa_p = A_p/m_p$ is isomorphic to the field of fractions $fr(A/p)$.*

## E.4.2 Projective Modules

**Definition 144** *An $A$-module $P$ is* projective *iff it is a direct summand of a free $A$-module. Equivalently, it is projective if for each pair of homomorphisms $u : P \to N$, $v : M \to N$, $v$ an epimorphism, there is a homomorphism $w : P \to M$ such that $u = v \circ w$.*

Let $U \mapsto U^{\star\star}$ be the bidual functor on $A$-modules. Then a direct summand $U \subset V$ is mapped into a direct summand $U^{\star\star}$ of $V^{\star\star}$. Now, if $U$ is projective, it is a direct summand of a free module $R^{(n)}$. It is easily seen that the bidual map $R^{(n)} \to (R^{(n)})^{\star\star}$ is injective. Therefore, if $U$ is projective, the bidual map $U \to U^{\star\star}$ is also injective. If $U$ is finitely generated and projective, the bidual map is an isomorphism.

Let $i : N \to M$ be the inclusion of a submodule $N$ of an $R$-module $M$. For any $x \in M$ and positive exponent $r$, let

$$\wedge^r x : \bigwedge^r N \to \bigwedge^{r+1} M : y \mapsto \bigwedge^r i(y) \wedge x$$

the linear map defined by the $r$th exterior power of $i$ and the wedge product with $x$ in the

exterior algebra of $M$. This defines a linear map

$$\wedge^r : M \to Lin_R(\bigwedge^r N, \bigwedge^{r+1} M) \tag{E.28}$$

which has the following property:

**Lemma 83** *With the above notation, if $N$ is a direct factor of $M$ which is locally free of rank $r$, then $Ker(\wedge^r) = N$.*

Proof. Let $x \in M$. In the special case where $N$ is free of rank $r$, if $\bigwedge^r N = R.u$ for a basis vector $u$ of $\bigwedge^r N$, the claim $x \in Ker(\wedge^r)$ is equivalent to $u \wedge x = 0$. But this follows from [63, ch.III, §7, no.9, Prop.13]. In the general case, clearly the condition is sufficient. Conversely, suppose $\wedge^r x = 0$. Recall from [64, ch.II, §5 no.3, Th.2] that an $R$-module is projective of rank $r \in \mathbb{N}$ iff it is locally free of rank $r$. Notice that localizing commutes with exterior powers ([64, ch.II, §2 no.8]), so we may localize in $f \in R$ such that for the localized element $x_f \in M_f$, we have $\wedge^r x_f \in N_f$ and $N_f$ is free of rank $r$, whence $x_f = 0$. As there is a cover of $\mathrm{Spec}(R)$ with basic open sets $D(f_i)$ associated with localizations $R_{f_i}$ such that $N_{f_i}$ is free, we deduce that $x \in N$. QED.

### E.4.3 Injective Modules

**Proposition 88** ([63, ch.II, §2, exercise 11]) *For an $R$-module $M$, the following properties are equivalent:*

(i) *The functor $Lin_R(?, M)$ is exact.*

(ii) *The functor $Lin_R(?, M)$ is exact on short exact sequences.*

(iii) *For every $R$-module $E$ and every linear injection $F \rightarrowtail E$, every linear map $F \to M$ extends to a linear map $E \to M$.*

(iv) *For every ideal $\mathfrak{a} \subset R$ and every linear map $f : \mathfrak{a} \to M$, there is $m \in M$ such that $f(a) = a.m$ for all $a \in R$.*

(v) *$M$ is a direct factor of every module which contains it.*

(vi) *For every $R$-module $E$ which is sum of $M$ and of a module $I$ with one generator, $M$ is a direct factor of $E$.*

**Definition 145** *An $R$-module $M$ is said to be* injective *iff it has the equivalent properties of proposition 88. The ring $R$ is said to* self-injective *iff it is injective as a (left) module over itself.*

**Exercise 87** Show the following statement: A direct sum of $R$-modules is injective iff each factor is.

**Example 78** Let $1 < N$ be a natural number. Then the Ring $\mathbb{Z}_N$ is self-injective. In fact, let $N = p_1^{n_1} \cdot \ldots p_r^{n_r}$ be its prime factorization (see appendix D.2). Then $\mathbb{Z}_N \xrightarrow{\sim} \mathbb{Z}_{p_1^{n_1}} \times \ldots \mathbb{Z}_{p_r^{n_r}}$, and injectivity can be checked on each factor, so suppose $N = p^n$. We apply criterion (iv) of proposition 88. An ideal $\mathfrak{a}$ in $\mathbb{Z}_{p^n}$ is generated by $p^m, m \leq n$, and we have the isomorphism of $\mathbb{Z}_{p^n}$-modules $\mathfrak{a} \xrightarrow{\sim} \mathbb{Z}_{p^{(n-m)}}$. Then a linear map $f : \mathbb{Z}_{p^{(n-m)}} \to \mathbb{Z}_{p^n}$ evaluates to $f(a) = f(a.1_{\mathbb{Z}_{p^{(n-m)}}}) = a.f(1_{\mathbb{Z}_{p^{(n-m)}}})$, and we are done.—In particular, every free $\mathbb{Z}_N$-module $\mathbb{Z}_N^r$ is injective.

The self-injectiveness of $\mathbb{Z}_{p^n}$ also follows from this:

**Proposition 89** ([139, proposition 21.5]) *Let $R$ be a zero-dimensional local ring. The following are equivalent:*

(i) *$R$ is self-injective.*

(ii) *The socle of $R$ is simple.*

In fact, the socle of $\mathbb{Z}_{p^n}$ is isomorphic to $\mathbb{Z}_p$, the simple group of order $p$ and it is well known that $\mathbb{Z}_{p^n}$ is zero-dimensional.

**Proposition 90** *A finitely generated $\mathbb{Z}_{p^n}$-module $M$ is injective iff it is free of finite rank.*

Proof. Clearly, by example 78, a free module of finite rank is injective. Conversely, by the main theorem on finitely generated abelian groups E.2 we have $M \xrightarrow{\sim} \mathbb{Z}_{p^{n_1}} \times \ldots \mathbb{Z}_{p^{n_k}}, n_i \leq n$. By exercise 87 above, it suffices to see that $\mathbb{Z}_{p^m}$ cannot be injective if $m < n$. In fact, if the injection $\mathbb{Z}_{p^m} \leftarrow \mathbb{Z}_{p^n} : x \mapsto p^{n-m}.x$ had a left inverse h, we would have $x = h(p^{n-m}.x) = p^{n-m}.x$ which only works for $x = 0$, a contradiction. QED.

### E.4.4 Lie Algebras

**Definition 146** *For a module $L$ over commutative ring $R$, a* Lie algebra *structure is an $R$-bilinear multiplication $[\ ] : L \times L \to L$, the* Lie bracket*, such that $[xx] = 0$ identically, and the* Jacobi identity

$$[x[yz]] + [y[zx]] + [z[xy]] = 0$$

*holds for all $x, y, z \in L$. A homomorphism of Lie algebras $f : L_1 \to L_2$ is a linear homomorphism such that $f([xy]) = [f(x)f(y)]$ for all $x, y \in L_1$. The corresponding category of Lie algebras over $R$ is denoted by* $\mathbf{Lie}_R$.

**Example 79** If $L \xrightarrow{\sim} R^n$ is free, and if $(x_i)$ is a basis, a Lie algebra structure on $L$ is defined by bilinearity and skew symmetry (which follows from $[xx] = 0$) of the Lie bracket if the Lie brackets $\sum_k a_{ij}^k x_k = [x_i, x_j], i < j$ are known. The condition for such a bracket to generate a Lie algebra is

$$a_{ii}^k = 0, \text{ all } i, k, \tag{E.29}$$

$$a_{ij}^k + a_{ji}^k = 0, \text{ all } i < j, k, \tag{E.30}$$

$$\sum_k a_{ij}^k a_{kl}^m + a_{jl}^k a_{ki}^m + a_{li}^k a_{kj}^m = 0, \text{ all } i, j, l, m. \tag{E.31}$$

The coefficients $a_{ij}^k$ are called *structural constants of the Lie algebra in the given basis.*

**Example 80** For any module $L$, the $R$-algebra of linear endomorphisms $End(L)$ becomes the *general linear algebra* $\mathfrak{gl}(L)$ by the bracket $[xy] = x \circ y - y \circ x$. A sub-Lie-algebra of a general linear algebra is called a *linear Lie algebra.* If $L \overset{\sim}{\to} R^n$ is free of rank $n$, we also write $\mathfrak{gl}(L) = \mathfrak{gl}(n, R)$. Its subalgebra of endomorphisms with vanishing trace (check that it is a sub-Lie-algebra!) is called the *special linear algebra* and denoted by $\mathfrak{sl}(L)$ or $\mathfrak{sl}(n, R)$.

**Example 81** Let $L$ be any $R$-module with a bilinear product $x \cdot y$ (no other conditions required). A derivation is a linear endomorphism $D : L \to L$ such that $D(x \cdot y) = x \cdot D(y) + D(x) \cdot y$. The set $Der(L)$ is a submodule of $End(L)$, and in fact a Lie subalgebra of the general linear algebra $\mathfrak{gl}(L)$.

In particular, if we take the Lie algebra structure on $L$, $Der(L)$ is another Lie algebra. Observe that for $x \in L$, the left multiplication $ad(x) = [x?] : y \mapsto [xy]$ is a derivation by the Jacobi identity. One has the Lie algebra homomorphism of adjunction

$$ad : L \to Der(L), \tag{E.32}$$

a representation of $L$ in the general linear algebra of $\mathfrak{gl}(L)$. A derivation $ad(x)$ is called *inner*, any other is called *outer* derivation.

**Proposition 91** *If $x$ is a nilpotent endomorphism in a linear algebra, then its adjoint $ad(x)$ is also a nilpotent endomorphism.*

See [239, p.12] for the easy proof.

Suppose that $L$ is a linear algebra in $End(V)$ for an $R$-module $V$, and that the conjugation $Int_e, e \in GL(V)$ leaves $L$ invariant. Then evidently the conjugation is an automorphism of $L$. Suppose now that $R$ is a $\mathbb{Q}$-algebra. If $ad(x)$ is nilpotent, then the exponential

$$exp(ad(x)) = 1 + ad(x) + ad(x)^2/2! + \dots ad(x)^k/k! + \dots \tag{E.33}$$

is defined. We have:

**Lemma 84** *If $ad(x)$ is nilpotent, then $edp(ad(x))$ is an automorphism of the Lie algebra $L$. Moreover, if $x$ is nilpotent, $exp(x)$ is defined and we have*

$$Int_{exp(x)} = exp(ad(x)).$$

See [239, p.9] for the proof.

# Appendix F

# Algebraic Geometry

For this chapter, we refer to [64, 123, 198, 199, 140].

## F.1 Locally Ringed Spaces

Given a topological space $X$, its system of open sets $Open_X$ is viewed as a category with inclusions as morphisms. If $f : X \to Y$ is a continuous map, the inverse image map $U \mapsto f^{-1}U$ defines a functor $Open_f : Open_Y \to Open_X$. This defines a functor $Open_? : \mathbf{Top} \to \mathbf{Cat}$ into the category of categories and functors[1].

Let $\mathbf{C}$ be a category of sets with some additional algebraic structure, such as the categories $\mathbf{Mod}, \mathbf{Mon}, \mathbf{Gr}, \mathbf{Ab}, \mathbf{Rings}, \mathbf{ComRings}$ of modules, monoids, groups, abelian groups, rings, or commutative rings, respectively. A contravariant functor $F : Open_X \to \mathbf{C}$ is called a $\mathbf{C}$-space (this is a presheaf plus the algebraic morphism conditions). For example, a ringed space is just a $\mathbf{Rings}$-space. In the present context, we always suppose that a ringed space is one with values in $\mathbf{ComRings}$, i.e., a commutatively ringed space. The set of $\mathbf{C}$-spaces on $X$ is denoted by $\mathbf{C}_X^{spaces}$. The contravariant functor $Open_?$ induces a functor $\mathbf{C}_?^{spaces} : X \mapsto \mathbf{C}_X^{spaces}$. It maps the continuous map $f : X \to Y$ to the set map $\mathbf{C}_f^{spaces} : F \mapsto F \circ Open_f$. The image $F \circ Open_f$ is denoted by $f_*F$ and is called the *direct image of* $F$. If $F : Open_X \to \mathbf{C}, G : Open_Y \to \mathbf{C}$ are $\mathbf{C}$-spaces and $f : X \to Y$ is continuous, then an *$f$-morphism* $h : F \to G$ is a natural transformation $h : G \to f_*F$. These morphisms define an evident category, the *category* $\mathbf{C}^{spaces}$ *of* $\mathbf{C}$*-spaces*.

Suppose that the category $\mathbf{C}$ has colimits for filtered diagrams[2], such as $\mathbf{Rings}, \mathbf{ComRings}$, $\mathbf{Mod}, \mathbf{Gr}, \mathbf{Ab}$, and take a $\mathbf{C}$-space $F$. For each point $x \in X$, the filtered system $Open_{X,x}$ of open neighborhoods of $x$ defines an object $F_x = colim_{U \in Open_{X,x}} F(U)$, the *stalk of* $F$ *at* $x$. So for a (commutatively) ringed space, this is a (commutative) ring. Let $h : F \to G$ be an $f$-morphism for $f : X \to Y$. For $x \in X$, we have the restriction $Open_{f,x} : Open_{Y,f(x)} \to Open_{X,x}$

[1] Restricted to a universe, if the limitless collection bothers the reader, or even the category of partially ordered sets with order preserving maps, to stick to reality.

[2] Meaning that for any two objects in the diagram quiver, there are two arrows with these domains targeting at a common codomain.

of $Open_f$ to the neighborhood systems $Open_{Y,f(x)}$ and $Open_{X,x}$. This induces a **C**-morphism $h_x : G_{f(x)} \to F_x$.

The subcategory **LocRgSpaces** of **ComRings**-spaces consists of all ringed spaces $F$ which have a local ring $F_x$ with maximal ideal $m_x$ in each point $x \in X$, and of those morphisms $h$ which induce local morphisms $h_x$ in all stalks, i.e., $h_x^{-1}(m_x) = m_y$. It is called the *category of locally ringed spaces*. The residue field in a point $x$ of such a space $F$ is the field $\kappa(x) = F_x/m_x$. For a section $s \in F(U)$ over the open set $U$, we denote by $s(x)$ the canonical image of $s$ in $\kappa(x)$.

For the category **ComMod** of modules over commutative rings and dilinear homomorphisms, a **ComMod**-space $F$ can also be described by the underlying ringed space $R$ and the abelian group space $F$ (same notation), together with a scalar multiplication $R(U) \times F(U) \to F(U)$ in each open set $U$, and the evident dilinear transition maps. One therefore also says that $F$ *is an $R$-module*. Same wording, mutatis mutandis, for an $R$-algebra or for an $R$-ideal.

If we are given a **C**-space $F$ on $X$ which is a sheaf, and if $\mathfrak{B}$ is a topological base for $X$, the restriction $F|\mathfrak{B}$ to this subcategory of $Open_X$ completely determines $F$. If $U \in Open_X$, $U = colim(B \in \mathfrak{B}|B \subseteq U)$, and

$$F(U) = lim(F(B), B \in \mathfrak{B}).$$

Conversely, if we are given a contravariant functor $F : \mathfrak{B} \to \mathbf{C}$, we obtain a **C**-space $F'$ by

$$F'(U) = lim(F(B), B \in \mathfrak{B})$$

and by the universally given transition morphisms. This presheaf is a sheaf if $F$ is a *sheaf on* $\mathfrak{B}$, i.e., if for every covering $(B_i)$ of $B \in \mathfrak{B}$ by elements of the base, the canonical application $F(B) \to \prod_i F(B_i)$ is a bijection $x \mapsto (x|B_i)$ onto the tuples $(x_i)$ such that for every $i, j$ and base element $B' \subseteq B_i \cap B_j$, we have $x_i|B' = x_j|B'$.

## F.2 Spectra of Commutative Rings

**Definition 147** *The (prime) spectrum is a contravariant functor*

$$Spec : \mathbf{ComRings} \to \mathbf{LocRgSpaces}$$

*which is defined as follows: Let $A, B$ be commutative rings, and let $f : A \to B$ be a ring homomorphism.*

1. *The topological space consists of the set $Spec(A) = \{p \text{ a prime ideal in } A\}$. The closed sets are the sets of the form $V(E) = \{p|E \subseteq p\}$ for a subset $E \subseteq A$. Equivalently, a base of open sets is given by the system $D_f = \{p|f \notin p\}, f \in A$, and we have $D_f \cap D_g = D_{fg}$.*

2. *For the base $\mathfrak{D} = \{D_f|f \in A\}$, we have a sheaf on $\mathfrak{D}$, which is defined by $D_f \mapsto A_f$, the localization at the saturated multiplicative set $S(f)$ defined by $f$, a well-defined setup since $D_f = D_g$ iff $S(f) = S(g)$ see* [198, I.1.3.2]. *This presheaf is a sheaf on $\mathfrak{D}$, and the associated sheaf on $Spec(A)$ is denoted by $\tilde{A}$ and called the* ring sheaf associated with $A$. *If $p \in Spec(A)$ is a prime ideal, we have $(\tilde{A})_p \xrightarrow{\sim} A_p$, i.e., $\tilde{A} \in$ **LocRgSpaces**.*

3. *For the homomorphism $f : A \to B$, the inverse image map on prime ideals $Spec(f) : Spec(B) \to Spec(A) : p \mapsto f^{-1}p$ is defined, and we have $Spec(f)^{-1}(D_g) = D_{f(g)}$, $Spec(f)^{-1}(V(E)) = V(f(E)$, i.e., $Spec(f)$ is continuous. Furthermore, we have a canonical map $f_g : A_g \to B_{f(g)}$ which is natural and therefore induces a morphism $Spec(f) : \tilde{A} \to \tilde{B}$ over the continuous (synonymous) map $Spec(f)$. The stalk homomorphism $f_p : \tilde{A}_p \to \tilde{B}_{Spec(f)(p)}$, colimit of the natural homomorphisms $f_g : A_g \to B_{f(g)}$, is local. One therefore has a contravariant functor Spec as announced, and one often denotes $Spec(A)$, when meaning the locally ringed space $\tilde{A}$ over $Spec(A)$.*

**Theorem 56** *The functor Spec is fully faithful, the inverse* global section functor $\Gamma$ *of a* **LocRgSpaces***-morphism $u : \tilde{B} \to \tilde{A}$ is given by the ring homomorphism $u(Spec(A)) : A = \tilde{A}(Spec(A)) \to \tilde{B}(Spec(B)) = B$.*

See [198, I.1.6.3] for a proof. Since one often writes $F(U) = \Gamma(U, F)$ and calls the elements section above $U$, the theorem's notation is justified by the global section notation $\Gamma(\tilde{A}) = \Gamma(Spec(A), \tilde{A})$. Let **Aff** be the full subcategory of **LocRgSpaces** consisting of the objects which are isomorphic to prime spectra. These spaces are called *affine schemes*. We therefore have that the map

$$Spec : \mathbf{ComRings} \to \mathbf{Aff}$$

is an equivalence of categories.

If $M$ is an $A$-module, we have a $\tilde{A}$-module $\tilde{M}$, whose sections on the base $\mathfrak{D}$ are defined by $\Gamma(D_g, \tilde{M}) = A_g \otimes M = M_g$, the localization of $M$ at the multiplicative set $S(g)$.

**Proposition 92** *The map $M \mapsto \tilde{M}$ is an exact*[3] *and fully faithful functor from the category of $A$-modules $\mathbf{Mod}_A$ to the category $\mathbf{Mod}_{\tilde{A}}$ of $\tilde{A}$-modules. It also commutes with colimits of modules, with tensor products, Hom-modules, with sums and intersections of submodules.*

*The inverse to this functor is the global section functor $\tilde{M} \mapsto \Gamma(Spec(A), \tilde{M})$, which is also exact.*

See [198, I.1.3] for a proof. The modules in $\mathbf{Mod}_{\tilde{A}}$, which are hit by this tilding process are the quasi-coherent ones: A module $\mathcal{M}$ over a ringed space $\mathcal{A}$ over a topological space $X$ is *quasi-coherent* iff there is a covering $X_i$ of $X$ such that each restriction $\mathcal{M}_i = \mathcal{M}|X_i$ is the cokernel of a homomorphism $f_i : \mathcal{A}_i^{I_i} \to \mathcal{A}_i^{J_i}$, where $\mathcal{A}_i = \mathcal{A}|X_i$.

**Theorem 57** *An $\tilde{A}$-module $\mathcal{M}$ is isomorphic to a module $\tilde{M}$ iff it is quasi-coherent.*

See [198, I.1.4.1] for a proof. This means that we have an equivalence of categories of quasi-coherent modules over $\tilde{A}$ and $\mathbf{Mod}_A$ (with linear homomorphisms).

For a ring element $f \in A$, one has $\mathrm{Spec}(A_f) \xrightarrow{\sim} D_f$. When restricting the associated ring and module sheaves $\tilde{A}, \tilde{M}$ to basic open sets $D_f$, this yields ring and module sheaves which are isomorphic to $\tilde{A}_f, \tilde{M}_f$.

**Theorem 58** *Let $M$ be an $A$-module. The following conditions are equivalent:*

[3] A sequence $K \xrightarrow{f} M \xrightarrow{g} L$ of linear homomorphisms of modules is exact in $M$ iff $Im(f) = Ker(g)$. Such exact sequences are preserved by the functor.

(i) *$M$ is projective*[4] *and finitely generated.*

(ii) *There is a finite family $(f_i)$ of elements of $A$ which generate the ideal $A$, i.e., $Spec(A) = \bigcup_i D_{f_i}$ such that the localizations $M_{f_i} = \Gamma(D_{f_i}\tilde{M})$ are free of finite rank over $A_{f_i}$.*

This is why one can also define a finitely generated projective module as being a locally free module of (locally defined) finite ranks. If the locally constant rank is constant $n$, the module is said to be *locally free of rank $n$.*

### F.2.1 Sober Spaces

A topological space $X$ is *irreducible* - iff every non-empty open subset is dense, or, equivalently, if any two non-empty open sets have a non-empty intersection. A subset of a topological space is called irreducible if it is so with its relative topology. A point $x$ of an irreducible space $X$ is said to be *generic* iff its (always irreducible) closure $\overline{\{x\}}$. We say that *a point $x$ dominates a point $y$, in signs $x > y$,* iff $\overline{\{y\}} \subseteq \overline{\{x\}}$. This is a partial order relation on $X$. An irreducible component of a space $X$ is a maximal irreducible subset.

**Sorite 15** *These are the sorite properties concerning irreducibility:*

(i) *A subset of a topological space is irreducible iff its closure is.*

(ii) *Irreducible components are closed.*

(iii) *Every irreducible subset is contained in an irreducible component, in particular, a topological space is the union of its irreducible components.*

(iv) *The image $f(E)$ of an irreducible subset $E \subseteq X$ under a continuous map $f : X \to Y$ is irreducible.*

**Definition 148** *A topological space $X$ is* sober *iff each closed irreducible subset has a unique generic point. Call* **Sob** *the full subcategory of the category* **Top** *of topological spaces consisting of sober spaces.*

If $A$ is a commutative ring, and if $E \subseteq Spec(A)$, then we denote $J(E) = \bigcap_{p\in E} p$, and $\overline{E} = V(J(E))$. This ideal is prime iff $E$ is irreducible. In this case, $\overline{E} = \overline{\{J(E)\}}$. In fact, for two points $p, q$ in $Spec(A)$, $p > q$ iff $p \subseteq q$. In particular, *$Spec(A)$ is a sober space.* Its irreducible components correspond to the minimal prime ideals.

**Proposition 93** *The canonical injection*

$$j : \mathbf{Sob} \to \mathbf{Top}$$

*has a left adjoint $?^s : \mathbf{Top} \to \mathbf{Sob}$.*

[4]See definition 144 in appendix E.4.2.

Proof idea. This adjoint associates with any $X$ a sober space $X^s$ which is defined as follows. Its points are the irreducible closed sets in $X$. The open sets are the sets $V^s = \{Y \in X^s | Y \cap V \neq \emptyset\}$, where $V$ varies over all open sets in $X$. Clearly, $(V \cap W)^s = V^s \cap W^s$ and $(\bigcup W_i)^s = \bigcup W_i^s$ for any family $(W_i)$ of open sets.

On continuous maps $f : X \to Y$, the functor acts via $f^s : X^s \to Y^s : E \mapsto \overline{f(E)}$. One has a canonical continuous map $q_X : X \to X^s : x \mapsto \overline{\{x\}}$ and a commutative diagram of continuous maps:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_Y} \\ X^s & \xrightarrow{f^s} & Y^s \end{array} \tag{F.1}$$

The map $q_X : X \to X^s$ is a homeomorphism if $X$ is sober. The adjunction is given by the mutually reciprocal maps $\mathbf{Top}(X, j(Y)) \to \mathbf{Sob}(X^s, Y) : f \mapsto q_Y^{-1} \circ f^s$ and $\mathbf{Sob}(X^s, Y) \to \mathbf{Top}(X, j(Y)) : g \mapsto g \circ q_X$.

**Lemma 85** *The canonical continuous map $q_X : X \to X^s$ is a quasi-homeomorphism,. i.e., the inverse image map $2^{X^s} \to 2^X$ is a bijection between the open sets of $X^s$ and those of $X$.*

## F.3 Schemes and Functors

A scheme $(X, \mathcal{O}_X)$ is a ringed space $\mathcal{O}_X$ on $X$ which locally is isomorphic to a spectrum of a commutative ring, i.e., there is an open covering $(X_i)$ of $X$ and a family $A_i$ of rings such that $(X_i, \mathcal{O}_X | X_i) \xrightarrow{\sim} Spec(A_i)$. The category **Schemes** of schemes is the subcategory of **LocRgSpaces** whose objects are schemes. By Yoneda, we have a fully faithful functor $Y : \mathbf{Schemes} \to \mathbf{Schemes}^{@}$.

**Proposition 94** *The restriction*

$$Y_{\mathbf{Aff}} : \mathbf{Schemes} \to \mathbf{Aff}^{@}$$

*is fully faithful. Equivalently, the corresponding functor*

$$Y_{\mathbf{ComRings}} : \ \mathbf{Schemes} \to \mathbf{ComRings}_{@}$$

*into the category* $\mathbf{ComRings}_{@}$ *of covariant set-valued functors on* **ComRings** *is fully faithful.*

This means that we may consider schemes as special covariant functors on the category of commutative rings. The functors which correspond to schemes are characterized by a sheaf condition:

**Property 3** *We are given a functor $G \in \mathbf{ComRings}_{@}$. For every ring $A \in \mathbf{ComRings}$, and every finite family $(f_i)$ of elements of $A$ which generate $A$ as an ideal, the diagram*

$$G(A) \to \prod_i G(A_{f_i}) \rightrightarrows \prod_{i,j} G(A_{f_i f_j})$$

*is exact, we say (by abuse of language, but theoretically justifiable) that $G$* is a sheaf.

Then the full subcategory of $\mathbf{ComRings}_{@}$ consisting of sheaves $G$ in the sense of property 3, together with the property that there is a family of rings $R_t$ and morphisms[5] $a_t : R_t@ \to G$ such that for every field $K$, $G(K) = \bigcup_t R_t@K$, comprises the functors which are isomorphic to images of schemes under the Yoneda map $Y_{\mathbf{ComRings}}$, see [198, I.2.3.6] and [140, VI.2] for details. This means that schemes are characterized without any reference to the geometry of ringed spaces. See also [314, III.3] for the relation of this setup to the systematic topos-theoretic restatement of the schemes in terms of the *Zariski site*.

The most important universal property of the category of schemes is that it has fiber products. In the affine case, we have $Spec(A) \times_{Spec(C)} Spec(B) \xrightarrow{\sim} Spec(A \otimes_C B)$.

## F.4 Algebraic and Geometric Structures on Schemes

If a scheme is viewed as a set-valued functor on rings, the sets may also be enriched by algebraic structures, such as groups, monoids, etc., to yield a category $\mathbf{C}$. We then view a scheme as a functor $S : \mathbf{ComRings} \to \mathbf{C}$, and say that $S$ is a $\mathbf{C}$-scheme, for example, an abelian group-scheme if $\mathbf{C} = \mathbf{Ab}$.

**Example 82** For $0 \leq n$, we have the additive group scheme $\mathbb{A}^n$ whose functor is $\mathbb{A}^n(R) = R^n$, with the canonical addition of this free module, and the canonical transitions $\mathbb{A}^n(R) \to \mathbb{A}^n(S)$ for a ring homomorphism $f : R \to S$.

**Example 83** The $n$-dimensional linear group scheme is given by the functor $R \mapsto GL(n, R) \subset \mathbb{M}_{n,n}(R) \xrightarrow{\sim} \mathbb{A}^{n^2}$, together with the canonical map $GL(n, R) \to GL(n, S)$ for a ring homomorphism $f : R \to S$. The set $GL(n, R)$ is defined as the set of $n \times n$-matrices $M$ with invertible determinant: $det(M) \in R^\times$. The functor is represented by the affine scheme $GL_n = Spec(\mathbb{Z}[X_{ij}, 1 \leq i, j \leq n]_{det})$, where $det = Det(X_{ij})$. The group structure is the multiplication of invertible matrices.

### F.4.1 The Zariski Tangent Space

For a field $K$, we are given a $K$-scheme $X$, i.e., a scheme $s : X \to Spec(K)$ in the comma category $\mathbf{Schemes}/Spec(K)$ (see section G.2.1). Suppose that we have a $K$-rational point $x : Spec(K) \to X$, i.e., a section of $s$. This means that the corresponding $K$-algebra $\mathcal{O}_{X,x}$ has an isomorphism $K \xrightarrow{\sim} \kappa_{X,x} = \mathcal{O}_{X,x}/m_{X,x}$. The *Zariski tangent space in* $x$ is the linear $K$-dual $T_{X,x} = (m_{X,x}/m_{X,x}^2)^*$. Consider the $K$-scheme $D_K = Spec(K[\varepsilon])$ over the dual numbers $K[\varepsilon]$, see example 76 in appendix D.1.1. It has the $K$-rational point $\varepsilon : Spec(K) \to D_K$ corresponding to the projection $K[\varepsilon] \to K : \varepsilon \mapsto 0$.

**Proposition 95** *With the above hypotheses and notation, there is a bijection of the Zariski elements $t$ of the tangent space $T_{X,x}$ and the morphisms $\tau : D_K \to X$ of $K$-schemes which map the $K$-rational point $\epsilon$ to the $K$-rational point $x$.*

[5] $R_t@$ is the covariant functor on rings, i.e., $R_t@R = Hom_{\mathbf{ComRings}}(R_t, R)$.

See [140, VI.1.3] for a proof. In particular, if the scheme $X$ is given by its functor on rings, this means that the tangents are special elements of the evaluation of the functor in dual numbers $X(K[\varepsilon])$. For example, if $X = \mathbb{A}^1_K = \mathbb{A}^1 \times_{\mathbb{Z}} K$, we have tangents $x + \varepsilon.\tau, \tau \in K$, over the rational point $x \in K \xrightarrow{\sim} \mathbb{A}^1_K(K)$.

## F.5 Grassmannians

A *subfunctor* $G \to F$ *in* $\mathbf{ComRings}_@$ *is open* iff for every morphism $a : R@ \to F$ (corresponding to an element $a \in F(R)$ via Yoneda), the fiber product projection $G \times_a R@ \to R@$ is isomorphic to the functor of an open subscheme of $Spec(R)$. Clearly, then, if $b : X@ \to F$ is a morphism from a representable functor $X@$ of a scheme $X$, then the projection $G \times_b X@ \to X@$ is isomorphic to an open subscheme of $X$. An *open covering of a functor* $F$ in $\mathbf{ComRings}_@$ is a family $(g_i : G_i \to F)$ of open subfunctors of $F$ such that the fiber product projections $G_i \times_b X@ \to X@$ for morphisms $b : X@ \to F$ from the representable functor of a scheme $X$ define an open covering of $X$. For example, the open subfunctors of an affine scheme $Spec(R)$ are the functors $F_I : \mathbf{ComRings} \to \mathbf{Sets}$ of form $F_I(S) = \{f : R \to S | f(I)S = S\}$, where $I$ is an ideal in $R$.

The Grassmann scheme $Grass_{r,n}$ is defined for any couple $0 \leq r \leq n$ of natural numbers by the functor

$$Grass_{r,n}(R) = \{V \subseteq R^n | R^n/V \text{ locally free of rank } r\},$$

which for a ring homomorphism $R \to S$ maps the exact sequence

$$0 \to V \to R^n \to R^n/V \to 0$$

to the exact sequence

$$0 \to Im(S \otimes_R V) \to S^n \to S \otimes_R (R^n/V) \to 0,$$

where the image of the tensorized space $S \otimes_R V$ is the image of $V$ under this map. The locally free quotient remains locally free since the localization on $R$ carries over to a localization over $S$: For $f \in R$, and its image $f' \in S$, we have $(S \otimes_R R^n/V)_{f'} \xrightarrow{\sim} S_{f'} \otimes_{R_f} (R^n/V)_f$.

The functor $Grass_{r,n}$ is covered by the following open subfunctors. Let $i_\cdot = i_1, i_2, \ldots i_r$ be an increasing subsequence $1 \leq i_1 < i_2 < \ldots i_r \leq n$. We have the affine open subfunctors $Grass_{n,r,i_\cdot}(R)$ of those submodules $V \subset R^n$ such that the factor $R^{i_\cdot} = \bigoplus_{j=1,\ldots r} R.e_{i_j}$ of $R^n$ projects isomorphically onto the quotient $R^n/V$. If $i'_\cdot$ denotes the complementary increasing sequence, $Grass_{n,r,i_\cdot}(R)$ identifies to the set of graphs $\Gamma f$ of linear maps in $Lin_R(R^{i'_\cdot}, R^{i_\cdot})$, i.e., to $n \times (n-r)$-matrices with columns $(e_{i'_k}, f(e_{i'_k}))^t$. In fact, the isomorphism $R^{i_\cdot} \xrightarrow{\sim} R^n/V$ corresponds to an isomorphism $V \xrightarrow{\sim} R^{i'_\cdot}$, and this makes $V$ a graph of a linear map in $Lin_R(R^{i'_\cdot}, R^{i_\cdot})$. The fact that these open subfunctors (represented by affine schemes $\mathbb{A}^{r\times(n-r)}$) cover the Grassmannian results from the situation over a field, where the covering is evident.

**Proposition 96** *Let $n$ be a positive natural number. Then the subfunctor*

$$B_n : R \mapsto \{x \in R^n |\ x \text{ is part of a basis of } R^n\} \tag{F.2}$$

*of the affine $n$-space $\mathbb{A}^n$ over $\mathbb{Z}$ is an open subscheme.*

Proof. Consider the open subscheme $GL_n$ of $\mathbb{A}^{n^2}$. Then $B_n$ is the image of $GL_n$ under the projection onto the first column $(n_{1j})$ which by [199, IV/2, 2.4.7] is (universally) open. QED.

**Lemma 86** *If $X$ is an $S$-scheme, $E$ a quasi-coherent $O_X$-module, then any section $s : S \to Grass_r(E)$ is a closed immersion.*

Proof. By [198, Proposition (9.7.7)], $Grass_r(E)$ separated over $S$, and by [198, Corollaire (5.2.4)], a section of such a structural morphism is a closed immersion. QED.

**Lemma 87** *If $R$ is a product of local rings of finite length, then for two elements $x, y$ of an $R$-module $M$, $R^\times x = R^\times y$ iff these elements generate the same space, i.e., $R.x = R.y$.*

Without loss of generality, we may suppose that $R$ is local with maximal nilpotent ideal $m$. Clearly, the condition is sufficient. Suppose now that $R^\times x \neq R^\times y$, and therefore $R^\times x \cap R^\times y = \emptyset$. Then, $R.x = R.x$ implies $R^\times x \subset m.y$, since $R = R^\times \cup m$. But then we have $x \in m.y$, and symmetrically $y \in m.x$, which gives $x \in m^k.y$ for all powers $k$, and $m$ being nilpotent yields $x = y = 0$, a contradiction. QED.

## F.6 Quotients

If $G$ is a finite group, and if $(X, \mathcal{O}_X)$ is a scheme, a group action of $G$ on $X$ can be given by a group homomorphism $\alpha : G \to Aut(X)$. This can also be seen as a morphism of schemes $\alpha' : G_{\mathbb{Z}} \times_{Spec(\mathbb{Z})} X \to X$ with the functorially described axioms of group actions associated with the functors of the schemes $G_{\mathbb{Z}}, X$, the scheme $G_{\mathbb{Z}} = Spec(\mathbb{Z}^G)$ is a group scheme whose multiplication is associated with the group multiplication $\mu : G \times G \to G$ via the ring homomorphism $\mu' : \mathbb{Z}^G \to \mathbb{Z}^{G\times G} \tilde{\to} \mathbb{Z}^G \otimes_{\mathbb{Z}} \mathbb{Z}^G$. The scheme $G_{\mathbb{Z}}$ is finite and locally free over $\mathbb{Z}$. The *set-theoretic orbits* of the action $\alpha$ are the equivalence classes defined by the relation on the product set $X \times X$, image of the set map $G \times X \to X \times X : (g, x) \mapsto (\alpha(g).x, x)$. If we use the schema-theoretic map $\alpha' : G_{\mathbb{Z}} \times_{Spec(\mathbb{Z})} X \to X$, the cokernel functor of the pair $pr_2, \alpha' : G_{\mathbb{Z}} \times_{Spec(\mathbb{Z})} X \rightrightarrows X$ of functor morphisms, if it exists, is called the *scheme functor of orbits of $X$ under the action of $G$*. We have this particular case of [123, III,2.6.1]:

**Theorem 59** *With the above notation, if $G$ is a finite group and $\alpha' : G_{\mathbb{Z}} \times_{Spec(\mathbb{Z})} X \to X$ the group action associated with an 'abstract' action $\alpha : G \to Aut(X)$ on the scheme $X$, such that every set-theoretic orbit is contained in an affine open subscheme of $X$, then there is a scheme-functor of orbits $Y = coker(pr_2, \alpha')$ and the associated diagram of schemes (qua locally ringed spaces)*

$$G \times_{Spec(\mathbb{Z})} X \rightrightarrows X \to Y$$

*is exact.*

# Appendix G

# Categories, Topoi, and Logic

For a comprehensive introduction to category theory, see [313]. For topos theory and sheaves see [314], for topos theory and logic, see [186].

## G.1 Categories Instead of Sets

One may rebuild mathematics from categories rather than from sets. In this framework, the most radical approach is the arrow-only definition of a category[1]:

**Definition 149** *A category* $\mathbf{C}$ *is a collection of objects* $f, g, h, \ldots$ *which are called* morphisms, *together with a partial composition* $f \circ g$ *which yields morphisms of* $\mathbf{C}$. *An identity is a morphism* $e$ *such that, whenever defined, we have* $e \circ f = f$ *and* $g \circ e = g$. *We have these axioms:*

1. *Whenever one of the two compositions* $(f \circ g) \circ h, f \circ (g \circ h)$ *is defined, both are defined and they are equal; we denote the resulting morphism by* $f \circ g \circ h$.
2. *If* $f \circ g, g \circ h$ *are both defined,* $(f \circ g) \circ h$ *is defined.*
3. *For every morphism* $f$ *there are two identities, a 'left' identity* $e_L$ *and a 'right' identity* $e_R$, *such that* $e_L \circ f, f \circ e_R$ *are defined (and necessarily equal to* $f$*).*

It is easily seen that two right (left) identities of a morphism $f$ are necessarily equal; they are called the *domain of* $f$ (*codomain of* $f$) and are denoted by $dom(f)$ ($codom(f)$). To make domain and codomain evident, one also writes $f : a \to b$ with $a = dom(f), b = codom(f)$ instead of $f$. For two morphisms $a, b$, the collection of those $f$ with $dom(f) = a, codom(f) = b$ is denoted by $Hom(a, b), Hom_{\mathbf{C}}(a, b), \mathbf{C}(a, b), \ldots$ according to the specific situation. Evidently, no morphism can be a member of $Hom(a, b)$ and of $Hom(a', b')$ if either $a \neq a'$ or $b \neq b'$, i.e., the $Hom$ collections form a partition of $\mathbf{C}$ (in the non-set-theoretic common sense).

[1] Mac Lane calls this type of set-less categories "metacategories", and reserves the proper term "category" for metacategories which are built upon sets. We do however preconize the foundational character of metacategories and therefore omit the "meta" prefix. However, we then should provide a germ for existing categories, in order to get off ground as with axiomatic set theory. See [314, VI.10] for a discussion of the foundation of mathematics via topoi.

**Exercise 88** Two identities $e, e'$ of $\mathbf{C}$ can be composed iff they are equal, and then $e \circ e = e$ (identities are idempotent).

In a more conservative understanding of categories, the identities are associated with the "objects" of a category, which are a second type of concepts, but do not enrich the category except in the way it is constructed. The identification of objects and identities is carried out as often as possible in our present text. In either case, the collection of identities (qua objects) is denoted by ${}_0\mathbf{C}$ or $Ob(\mathbf{C})$, whereas the morphisms are denoted by ${}_1\mathbf{C}$ or $Mor(\mathbf{C})$. To stress the morphic character of an identity $e$ (in contrast to the underlying object in the conservative understanding), one also writes $Id_e$ instead of $e$.

In a category, a morphism $f$ is *mono, a monomorphism*, iff for any two compositions $f \circ g, f \circ g'$, the equality $f \circ g = f \circ g'$ implies $g = g'$. The morphism $f$ is *epi, an epimorphism*, iff for any two compositions $g \circ f, g' \circ f$, the equality $g \circ f = g' \circ f$ implies $g = g'$. The morphism $f$ is called a *section* if there is a left inverse $g$, i.e., $g \circ f = dom(f)$; $f$ is called a *retraction* if it has a right inverse $h$, i.e., $f \circ h = dom(h)$. A morphism $f$ that is a section and a retraction is *iso, an isomorphism*. If $dom(f)) = codom(f)$, the morphism is called *endo, an endomorphism*. An endomorphism which is an isomorphism is called *auto, an automorphism*. The collection of endomorphisms for a domain $c$ is denoted by $End(c)$, whereas the collection of automorphisms for $c$ is denoted by $Aut(c)$. If these collections are sets, they define monoids $End(c)$ and groups $Aut(c)$ with the identity $Id_c$ as unit.

**Exercise 89** The composition of two monomorphisms, epimorphisms, isomorphisms, endomorphisms, and automorphisms, if defined, shares, each of these properties.

### G.1.1 Examples

**Example 84** The category **Sets** of all sets. The morphisms are the set maps between existing sets, and the composition is the usual composition of set maps.

**Remark 28** Usually, the delicate comprehension axiom which can cause contradictory constructions of sets, is avoided by a strong restriction of the available sets. One takes a very large set $U$, which has the properties of a "universe", i.e., it is stable in the following sense:

- If $x \in U$, then $x \subset U$;
- If $x, y \in U$, then $\{x, y\} \in U$;
- If $x \in U$, then $2^x \in U$ (the set of all subsets, the powerset);
- A set of all natural[2] numbers $\mathbb{N}$ is element of $U$;
- If $f : x \to y$ is a surjective function with $x \in U, y \subset U$, then $y \in U$

One then restricts the **Sets** objects to the elements of the universe $U$ and says that these are *small sets*. We denote such a category of small sets by $\mathbf{Sets}_U$.

[2]For example the set of finite ordinals $0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\}, \ldots n, n^+ = n \cup \{n\}, \ldots$

**Example 85** Given a quiver $Q = (\text{head}, \text{tail} : A \rightrightarrows V)$ (see section C.2.2), the *path category* $P(Q)$ has the paths as morphisms, the identities are the lazy paths, and the composition is the path composition. Here, the vertexes are separate concepts which can be identified (and in fact *are* identified in our construction) with the lazy paths. All paths are mono and epi, but only the identities are isomorphisms. The terminology "quiver" stems from algebra, in category theory, a quiver is more known as a "diagram scheme".

Relations among paths give rise to *quotient categories* as follows: Suppose that we are given any binary relation $\sim$ between some paths of equal domain and codomain. Consider the smallest equivalence relation $\sim'$ among paths which contains $\sim$ and is a 'two-sided ideal' in the sense that for $f \sim' g$ with $dom(f) = dom(g) = d, codom(f) = codom(g) = c$ and $h, k$ with $dom(h) = c, codom(k) = d$, we have $c \circ f \sim' c \circ g$ and $f \circ d \sim' g \circ d$. Then we obtain a new category, the quotient category $P(Q)/\sim$, and its morphisms are the equivalence classes of paths, while the composition is the composition of representatives of these classes. In the language of category theory, the relation $\sim$ is called a *commutativity relation* of the given diagram scheme.

**Example 86** Fix a ring $R$, the *matrix category over* $R$ is the collection $\mathbb{M}_R$ of all $m \times n$-matrices $M = (m_{i,j})$ with coefficients in $R$ and for any row and column numbers $m, n$, together with the usual matrix multiplication $M \cdot N$ as composition. The identities are all the identity matrices $E_n, n = 1, 2, \ldots$ (over $R$). We evidently have $Hom_{\mathbb{M}_R}(E_n, E_m) = \mathbb{M}_{m,n}(R)$. In particular, the vectors in $R^n$ are identified with the morphisms in $Hom_{\mathbb{M}_R}(E_1, E_m) = \mathbb{M}_{1,n}(R)$.

**Example 87** Given a category $\mathbf{C}$, the isomorphism classes of $\mathbf{C}$-objects define a *skeleton* category $\mathbf{C}/iso$: For each isomorphism class, select a representative and then consider the full subcategory[3] of $\mathbf{C}$ on these representative objects. Clearly $\mathbf{C}/iso$ is defined up to isomorphism of categories (see below G.1.2) and no two skeleton objects are isomorphic.

**Example 88** Common examples of categories are the categories **Mon** of monoids, **Gr** groups, **Rings** of rings with ring homomorphisms, $\mathbf{LinMod}_R$ $R$-modules with linear homomorphisms, **LinMod** modules with dilinear homomorphisms, $\mathbf{Mod}_R$ $R$-modules with affine homomorphisms, **Mod** of modules with diaffine homomorphisms, or **Top** of topological spaces with continuous maps.

**Example 89** For every category $\mathbf{C}$ we have the *opposite category* $\mathbf{C}^{opp}$. Its morphisms are the same, but composition works via $f \circ^{opp} g = g \circ f$, i.e., it is defined iff the composition with opposite factors is defined in $\mathbf{C}$. This opposite construction exchanges the domains and codomains of morphisms. Intuitively, an arrow $f : x \to y$ in $\mathbf{C}$ becomes a arrow $f : y \to x$ in $\mathbf{C}^{opp}$.

## G.1.2 Functors

Functors are the morphisms between categories:

**Definition 150** *If* $\mathbf{C}, \mathbf{D}$ *are categories, a functor* $F : \mathbf{C} \to \mathbf{D}$ *is a function which assigns to every morphism* $c$ *in* $\mathbf{C}$ *a morphism* $F(c)$ *in* $\mathbf{D}$ *such that*

[3] See example 90 below.

(i) *$F(c)$ is an identity if $c$ is so,*

(ii) *if $c \circ c'$ is defined in $\mathbf{C}$, then $F(c) \circ F(c')$ is defined and $F(c \circ c') = F(c) \circ F(c')$.*

In particular, functors carry isomorphisms to isomorphisms. Moreover, the composition $F \circ G : \mathbf{C} \to \mathbf{E}$ of two functors $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{E}$ is a functor. Two categories $\mathbf{C}, \mathbf{D}$ are called *isomorphic* if there exists a functor isomorphism, i.e., two functors $F : \mathbf{C} \stackrel{\sim}{\to} \mathbf{D}, F^{-1} : \mathbf{D} \stackrel{\sim}{\to} \mathbf{C}$ such that $F^{-1} \circ F = Id_{\mathbf{C}}, F \circ F^{-1} = Id_{\mathbf{C}}$. A functor is called *full* iff the $F(Hom_{\mathbf{C}}(x, y)) = Hom_{\mathbf{D}}(F(x), F(y))$ for all object pairs $c, d$. It is called *faithful* iff $F : Hom_{\mathbf{C}}(x, y) \to Hom_{\mathbf{D}}(F(x), F(y))$ is injective for all pairs $c, d$. It is called fully faithful iff it is full and faithful, i.e., the map $F : Hom_{\mathbf{C}}(x, y) \to Hom_D(F(x), F(y))$ is a bijection. Functors are also called "covariant" since they are opposed to functors $F : \mathbf{C}^{opp} \to \mathbf{D}$, which are called "contravariant" but then also denoted by $F : \mathbf{C} \to \mathbf{D}$.

One often considers systems of morphisms in a category $\mathbf{C}$, which are defined by a graphical approach: diagrams. Here is the precise definition.

**Definition 151** *A* diagram *in a category $\mathbf{C}$ is a functor $\Delta : P(Q) \to \mathbf{C}$, where $Q$ is a quiver. The diagram $\Delta$ is said to commute with respect to a relation $\sim$ among $Q$-paths, iff $\Delta$ factorizes through $P(Q)/\sim$. If the relation is* maximal (*it identifies all paths having common domain and codomain*)*, then the diagram is said to be* commutative *without specification of $\sim$.*

By the very definition of a path category, diagrams are given by systems of morphisms in $C$ which cope with the domain-codomain configuration in the underlying quiver (i.e., diagram scheme).

**Example 90** If $\mathbf{C}$ is a category, a subcategory is a sub-collection $\mathbf{C}'$ of $\mathbf{C}$ such that for each morphism $f$ in $\mathbf{C}'$, its domain and codomain are also in $\mathbf{C}'$, and such that for any two $f, g$ in $\mathbf{C}'$ such that $f \circ g$ is defined in $\mathbf{C}$, the composition is also a morphism in $\mathbf{C}'$. A category can be defined by an arbitrary selection of objects (identities) out of $\mathbf{C}$ and the full collections of morphisms having these identities as domains or codomains. Such a subcategory is called a *full* subcategory of $\mathbf{C}$. A subcategory obviously induces an *embedding* functor $\mathbf{C}' \to \mathbf{C}$ by the identity on the morphisms in $\mathbf{C}'$. For any collection $S$ of morphisms in $\mathbf{C}$, the smallest subcategory of $\mathbf{C}$ containing $S$ is denoted by $\langle S \rangle$ and called the subcategory *generated by $S$*.

**Example 91** If $\mathbf{C}, \mathbf{D}$ are two categories, the product category $\mathbf{C} \times \mathbf{D}$ consists of all ordered pairs $(c, d)$ of morphisms $c$ in $\mathbf{C}$ and $d$ in $\mathbf{D}$. The composition $(c, d) \circ (c', d')$ is possible iff it is possible in each component and then evaluates to $(c, d) \circ (c', d') = (c \circ c', d \circ d')$. One has the canonical projection functors $p_1 : \mathbf{C} \times \mathbf{D} \to \mathbf{C}, p_2 : \mathbf{C} \times \mathbf{D} \to \mathbf{D}$ with $p_1(c, d) = c, p_s(c, d) = d$. The same procedure allows the definition of any finite product of categories.

### G.1.3 Natural Transformations

Natural transformations are the morphisms between functors.

**Definition 152** *If $F, G : \mathbf{C} \to \mathbf{D}$ are two functors, a natural transformation $t : F \to G$ is a system of morphisms $t(c) : F(c) \to G(c)$ in $\mathbf{D}$, for each object $c$ in $\mathbf{C}$, such that for every*

*morphism $f : x \to y$ in $\mathbf{C}$, we have $G(f) \circ t(x) = t(y) \circ F(f)$. One can also rephrase this property by requiring the following commutative diagram in $\mathbf{D}$:*

$$\begin{array}{ccc} F(x) & \xrightarrow{t(x)} & G(x) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(y) & \xrightarrow{t(y)} & G(y) \end{array} \tag{G.1}$$

Natural transformations can be composed in an evident way, and the composition is associative. For every functor $F$ we have the natural identity $Id_F$.We therefore have the category $Func(\mathbf{C}, \mathbf{D})$ of functors $F : \mathbf{C} \to \mathbf{D}$ and natural transformations $Nat(F, G)$ between two functors $F, G : \mathbf{C} \to \mathbf{D}$. Properties between such functors are said to be *natural* if they relate to the category $Func(\mathbf{C}, \mathbf{D})$, for example, $F \xrightarrow{\sim} G$ is a natural isomorphism iff it is an isomorphism among the natural transformations from $F$ to $G$.

If two categories $\mathbf{C}, \mathbf{D}$ satisfy the following properties, they are called *equivalent*, equivalence is an equivalence relation which is weaker than isomorphism.

**Lemma 88** *For categories $\mathbf{C}, \mathbf{D}$ the following properties are equivalent:*

(i) *There are two functors $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}$ such that $G \circ F \xrightarrow{\sim} Id_{\mathbf{C}}$ and $F \circ G \xrightarrow{\sim} Id_{\mathbf{D}}$, where these isomorphisms are natural.*

(ii) *There is a functor $F : \mathbf{C} \to \mathbf{D}$ which is fully faithful and* essentially surjective*, i.e., every object (identity) in ${}_0\mathbf{D}$ is isomorphic to an image $F(c)$ of an object of $\mathbf{C}$.*

**Example 92** If $\mathbf{C}$ is a category with sets as hom collections $Hom(x, y)$, we have two types of hom functors as follows: For fixed object $x$, we have the functor $Hom(x, ?) : \mathbf{C} \to \mathbf{Sets} : y \mapsto Hom(x, y)$, which sends a morphism $f : y \to z$ to $Hom(x, f) : Hom(x, y) \to Hom(x, z) : u \mapsto f \circ u$. We further have the contravariant functor $Hom(?, y) : \mathbf{C}^{opp} \to \mathbf{Sets} : x \mapsto Hom(x, y)$, which sends a morphism $f : x \to z$ to $Hom(f, y) : Hom(z, y) \to Hom(x, y) : u \mapsto u \circ f$. The category $Func(\mathbf{C}^{opp}, \mathbf{Sets})$ of contravariant set-valued functors on $\mathbf{C}$ is denoted by $\mathbf{C}^{@}$; its elements are called (*set-valued*) *presheaves over $C$*. In the theory of denotators, one works with $\mathbf{Mod}^{@}$ and for a module $M$, we have the notation $Hom_{\mathbf{Mod}}(M, ?) = M@$, whereas the contravariant hom functor is $Hom_{\mathbf{Mod}}(?, M) = @M$.

**Example 93** For two categories $\mathbf{C}$ and $\mathbf{D}$ and an object $S$ of $\mathbf{D}$, we have the constant functor $[S] : \mathbf{C} \to \mathbf{D}$ with $[S](X) = S$ and $[S](f) = Id_S$ for all $X \in Ob(\mathbf{C})$ and all $f \in Mor(\mathbf{C})$. In particular, if $S$ is a set, then we write $[S]$ for the constant functor in $\mathbf{Mod}^{@}$ if the contrary is not stressed.

Given a quiver $G$, if we fix an object $c$ in a category $\mathbf{C}$, we have the constant diagram $\Delta_c = [c]$. It associates every vertex of $G$ with $c$ and every arrow with $Id_c$. For a diagram $\Delta$ in $\mathbf{C}$, a natural transformation $[c] \to \Delta$ is called a *cone* on $\Delta$, whereas a natural transformation $\Delta \to [c]$ is called a *cocone on* $\Delta$. In a cone, all arrows starting from $c$ must commute with the arrows of the diagram, whereas in a cocone all arrows arriving at $c$ must commute with the arrows of the diagram.

## G.2 The Yoneda Lemma

For a given category $\mathbf{C}$ with sets as hom collections $Hom(x, y)$, the *Yoneda embedding* $Y$ is the functor

$$Y : \mathbf{C} \to \mathbf{C}^{@} : x \mapsto Y(x) = Hom(?, x) \tag{G.2}$$

with the natural transformations $Y(f : x \to y) : Y(x) \to Y(y)$ being defined by $u \mapsto f \circ u$ for $u : z \to x \in Y(x)(z) = Hom(z, x)$. For $\mathbf{C} = \mathbf{Mod}$, we also write $Y = ?@$, i.e., $Y(M) = @M$ for a module $M$.

A functor $F$ in $\mathbf{C}^{@}$ is called *representable* iff there is an object $c$ in ${}_0\mathbf{C}$ such that $F \xrightarrow{\sim} Y(c)$. Yoneda's lemma states that the full subcategory of representable functors in $\mathbf{C}^{@}$ is equivalent to $\mathbf{C}$, and that such an equivalence is given by the Yoneda embedding. More precisely:

**Lemma 89** *For every functor $F$ in $\mathbf{C}^{@}$ and object $c$ in ${}_0\mathbf{C}$, the map*

$$\epsilon : Nat(Y(c), F) \to F(c) : h \mapsto h(c)(Id_c) \tag{G.3}$$

*is a bijection.*

The proof is an easy exercise, but see also [198, 313]. In particular, if $F = Y(d)$, we have a bijection $\epsilon : Nat(Y(c), Y(d) \xrightarrow{\sim} Hom(c, d)$. More precisely, this means that the Yoneda functor $Y$ is fully faithful, so we obtain an equivalence of categories as announced. For $\mathbf{C} = \mathbf{Mod}$, we also write $F(A) = A@F$, even if $F$ is not representable. We then have the bijection $Nat(@A, F) \xrightarrow{\sim} A@F$. This means that the evaluation of $F$ at "address" $A$ is the same as the calculation of the morphisms from $@A$ to $F$. This is a justification of the name "address" for the argument $A$: Evaluating $F$ at $A$ means "observing $F$ under all morphisms" when being "positioned on (the functor $@A$ of) $A$". And the Yoneda philosophy means that $F$ is known, when it is known while observed from all addresses.

### G.2.1 Universal Constructions: Adjoints, Limits, and Colimits

**Definition 153** *We suppose that for two categories $\mathbf{C}, \mathbf{D}$, the hom collections are sets. Given two functors $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}$, we say that $\mathbf{C}$ is* left adjoint to $\mathbf{D}$ *or (equivalently) that $G$* is right adjoint to $F$*, in signs $F \dashv G$ iff the functors $Hom_D(F(?), ?) : \mathbf{C}^{opp} \times \mathbf{D} \to \mathbf{Sets}$ and $Hom_{\mathbf{C}}(?, G(?)) : \mathbf{C}^{opp} \times \mathbf{D} \to \mathbf{Sets}$ are isomorphic. One also writes this fact in these symbols:*

$$\frac{c \to G(d)}{F(c) \to d}$$

*meaning that morphisms in the numerator correspond one-to-one to morphisms in the denominator.*

In particular, if we are given an adjoint pair of functors $F \dashv G$, when fixing the variable $d$ in $\mathbf{D}$, the adjointness isomorphism means that the contravariant functor $c \mapsto Hom_{\mathbf{D}}(F(c), d)$ is representable by the object $G(d)$.

**Example 94** For $\mathbf{C} = \mathbf{D} = \mathbf{Sets}$, fix a set $A$. We have the functors $A\times? : \mathbf{Sets} \to \mathbf{Sets} : X \mapsto A \times X$ and $?^A : \mathbf{Sets} \to \mathbf{Sets} : X \mapsto X^A$, which are an adjoint pair $A\times? \dashv ?^A$ via the isomorphism that sends $f : A\times X \to B$ to $f \dashv : X \to B^A : f \dashv (x)(a) = f(a, x)$. This adjointness property is crucial in the definition of exponential objects in topoi. The "exponential set" $B^A$ represents the functor $X \mapsto Hom(A \times X, B)$. See section G.3.2 for this subject.

A *terminal* object 1 in a category $\mathbf{C}$ is one that admits exactly one morphism, denoted by $! : x \to 1$ from each object $x$ of $\mathbf{C}$. An *initial* object 0 is a terminal object in the opposite category. For example, in **Sets**, every singleton, such as $1 = \{0\}$, is a terminal object, while the empty set 0 is initial.

**Example 95** A terminal object in $\mathbf{C}^{@}$ is defined by the constant $1_{\mathbf{C}^{@}} = [1]$ (of the set 1). For a presheaf $P \in \mathbf{C}^{@}$, a *global section* $\gamma$ is a natural transformation $\gamma : 1_{\mathbf{C}^{@}} \to P$. In other words, the global sections $\Gamma$ correspond to the hom functor $\Gamma(P) = Nat(1_{\mathbf{C}^{@}}, P)$. The global section functor $\Gamma : \mathbf{C}^{@} \to \mathbf{Sets}$ is right adjoint to the constant functor $[\ ] : \mathbf{Sets} \to \mathbf{C}^{@}$.

Universal objects in categories, such as limits and colimits, are related to terminal or initial objects as follows. Given a "basis" object $b$ of $\mathbf{C}$, the *comma category* $\mathbf{C}/b$ has all morphisms $f : x \to b$ as objects, and for two objects $f : x \to b, g : y \to b$, we have

$$Hom_{\mathbf{C}/b}(f, g) = \{u | g \circ u = f\},$$

the set of "commutative triangles above $b$" with the evident composition. The *cocomma category* $\mathbf{C}/^{opp}b$ is the comma category $(\mathbf{C}^{opp}/b)^{opp}$, in other words, for two objects $f : b \to x, g : b \to y$, we have

$$Hom_{\mathbf{C}/^{opp}b}(f, g) = \{u | u \circ f = g\}.$$

Given a quiver $Q$ and a category $\mathbf{C}$, we have in the category $Func(P(Q), \mathbf{C})$ of diagrams in $\mathbf{C}$, and given such a diagram $\Delta$, the comma category $Func(P(Q), \mathbf{C})/\Delta$. In this category, take the full subcategory $cones(\Delta)$ of cones $[c] \to \Delta$. Then a *limit of* $\Delta$ is a terminal object $lim(\Delta)$ in $cones(\Delta)$. Since terminal objects are evidently unique up to isomorphisms, a limit is also unique up to isomorphism. A *colimit* $colim(\Delta)$ *of a diagram* $\Delta$ is an initial object in the subcategory $cocones(\Delta)$ of cocones on $\Delta$ in the cocomma category $Func(P(Q), \mathbf{C})/^{opp}\Delta$.

If the diagram is a pair $f : a \to c, g : b \to c$, the limit is called the *fiber product* or *pullback* of $f, g$, or (more sloppily) of $a$ and $b$ if $f, g$ are clear; it is denoted by $a \times_c b$. If the diagram is a pair $f : c \to a, g : c \to b$, the colimit is called the *fiber sum* or *pushout* of $f, g$, or (more sloppily) of $a$ and $b$ if $f, g$ are clear; it is denoted by $a \sqcup_c b$. The limit of two isolated objects $a, b$ (discrete diagram with two points) is called the (*cartesian*) *product of* $a, b$ and denoted by $a \times b$. The colimit of two isolated objects $a, b$ is called the (*disjoint*) *sum of* $a, b$ and denoted by $a \sqcup b$.

**Theorem 60** *The category of sets* **Sets** *has arbitrary limits and colimits. For a category* $\mathbf{C}$ *with sets as hom collections, the category* $\mathbf{C}^{@}$ *of presheaves over* $\mathbf{C}$ *has arbitrary limits and colimits.*

Proof. If $\Delta$ is a diagram of set morphisms $f_{i,j,k} : X_i \to X_j$, the limit is the subset in $\prod_i X_i$ consisting of all families $(x_i)$ such that for any pair $(x_i, x_j) \in X_i \times X_j$ and any $f_{i,j,k}$, we have

$f_{i,j,k}(x_i) = x_j$. The projections $lim(\Delta) \to X_i$ are the restrictions of the canonical projections from the product to $X_i$. The colimit is the set $colim(\Delta)$ of equivalence classes $\coprod_i X_i/\sim$ defined by the equivalence relation generated by the relation $x_i \sim x_j$ iff there is $f_{i,j,k}(x_i) = x_j$. The morphisms $X_i \to colim(\Delta)$ are the injections $X_i \to \coprod_i X_i$, followed by the quotient map $\coprod_i X_i \to colim(\Delta)$. The universal properties are immediate and left as an exercise.

For a diagram $\Delta$ of presheaves $F_i$, we take for each argument $c$ in $C$ the set-theoretic limit or colimit, respectively, of the set diagram $\Delta(c)$ of the sets $F_i(c)$ and the corresponding maps to define the limit or colimit of $\Delta$, respectively, QED.

The following proposition makes sure that the category of presheaves over $\mathbf{C}$ is not too large with respect to its Yoneda embedding of $\mathbf{C}$:

**Proposition 97** *Every presheaf $F$ in $\mathbf{C}^{@}$ is a colimit of representable presheaves.*

See [314, pp.41/42] for a proof. The idea of this proof uses the so-called *category of elements* $\int_{\mathbf{C}} F$ of a functor $F$. Its objects are all pairs $(C, p)$ where $C$ is an object of $\mathbf{C}$, and $p \in C@F$. The morphisms $(C, p) \to (C', p')$ are the morphisms $u : C \to C'$ in $\mathbf{C}$ such that $p'.u = p$.

**Definition 154** *A category is called* ⟨finitely⟩ (co)complete *iff it has (co)limits for all ⟨finite⟩ diagrams ⟨diagrams with finitely many objects and arrows⟩.*

Proposition 97 turns out to make the Yoneda embedding into a universal device for making a category $\mathbf{C}$ cocomplete:

**Proposition 98** *For each functor $f : \mathbf{C} \to \mathbf{E}$ to a cocomplete category $\mathbf{E}$, there exists an essentially unique colimit preserving functor $L : \mathbf{C}^{@} \to \mathbf{E}$ such that $f = L \circ Y$.*

See [314, p.43] for a proof.

### G.2.2 Limit and Colimit Characterizations

**Proposition 99** *For any category $\mathbf{C}$, the following statements are equivalent:*

(i) $\mathbf{C}$ *is finitely complete.*

(ii) $\mathbf{C}$ *has finite products and equalizers*[4]*.*

(iii) $\mathbf{C}$ *has a terminal object and fiber products.*

For a proof, see [481, I, 7.8.8].

**Proposition 100** *For any category $\mathbf{C}$, the following statements are equivalent:*

(i) $\mathbf{C}$ *is finitely cocomplete.*

(ii) $\mathbf{C}$ *has finite sums and coequalizers*[5]*.*

(iii) $\mathbf{C}$ *has an initial object and fiber sums.*

[4] An equalizer is a limit of a pair $f, g : x \rightrightarrows y$ of arrows.
[5] A coequalizer is a colimit of a pair $f, g : x \rightrightarrows y$ of arrows.

This is just the dual statement of proposition 99.

**Proposition 101** *Let* $\mathbf{C}$ *be a finitely complete category. A morphism* $f : A \to B$ *in* $\mathbf{C}$ *is mono iff the canonical projections* $p_1$ *and* $p_2$ *in the pullback*

$$\begin{array}{ccc} X & \xrightarrow{p_1} & A \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array} \tag{G.4}$$

*coincide and are isomorphisms.*

Proof. Clearly, if $f$ is mono, then $X = A$ and $p_1 = p_2 = 1_A$ define a fiber product. Conversely, if $p_1 = p_2$ is an isomorphism, then any couple $u, v : Z \to A$ with $f \cdot u = f \cdot v$ creates factorizations $u = p_1 \cdot t$ and $v = p_2 \cdot t$ through $t : Z \to X$ which therefore also coincide. QED.

Therefore we have the dual result:

**Corollary 28** *Let* $\mathbf{C}$ *be a finitely cocomplete category. A morphism* $f : A \to B$ *in* $\mathbf{C}$ *is epi iff the canonical morphisms* $i_1$ *and* $i_2$ *in the pushout*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i_1} \\ B & \xrightarrow{i_2} & X \end{array} \tag{G.5}$$

*coincide and are isomorphisms.*

**Proposition 102** *For any category* $\mathbf{C}$*, let* $f : H \to G$ *be a morphism in* $\mathbf{C}^{@}$*. Then:*

(i) *The morphism* $f$ *is mono iff* $A@f : A@H \to A@G$ *is injective for all objects* $A$ *of* $\mathbf{C}$*.*

(ii) *The morphism* $f$ *is epi iff* $A@f : A@H \to A@G$ *is surjective for all objects* $A$ *of* $\mathbf{C}$*.*

(iii) *The morphism* $f$ *is iso iff it is mono and epi iff* $A@f : A@H \to A@G$ *is bijective for all objects* $A$ *of* $\mathbf{C}$*.*

Proof. Observe that $\mathbf{C}^{@}$ is finitely complete and cocomplete and that limits and colimits are calculated pointwise. Let us first look at point (iii). Clearly, $f$ is iso iff its evaluations $A@f : A@H \to A@G$ are all bijective. Further, we know from proposition 101 that $f$ is mono iff the fiber product projections $p_1$ and $p_2$ coincide and are iso. But with (iii) this is true iff this is true for all evaluations at objects $A$ of $\mathbf{C}$, i.e., iff this is true set-theoretically, and this means having an injection for every object $A$ of $\mathbf{C}$, and (i) is done; the dual argument shows (ii). Finally, iso always implies mono and epi; conversely, mono and epi means being in- and surjective, i.e., bijective at every object $A$ of $\mathbf{C}$, whence $f$ is iso. QED.

### G.2.2.1 Special Results for $\mathbf{Mod}^{@}$

**Lemma 90** *Let $H$ be a functor in $\mathbf{Mod}^{@}$, $M, N$ be two addresses, and $f : N \to M$ a morphism of addresses. Then the map*

$$M \mapsto M@Fin(H) := \{F \subset M@H, card(F) < \infty\}, \tag{G.6}$$

*together with the maps*

$$f@Fin(H) : M@Fin(H) \to N@Fin(H) : X \mapsto f@H(X), \tag{G.7}$$

*defines a functor $Fin(H)$ in $\mathbf{Mod}^{@}$.*

**Lemma 91** *For any functor $H$ in $\mathbf{Mod}^{@}$and address $M$, the maps*

$$sing_H(M) : M@H \to M@Fin(H) : x \mapsto \{x\} \tag{G.8}$$

*defines a monomorphism $sing_H : H \rightarrowtail Fin(H)$ of functors.*

**Lemma 92** *The map*

$$Fin : \mathbf{Mod}^{@} \to \mathbf{Mod}^{@} : H \mapsto Fin(H) \tag{G.9}$$

*defines an endofunctor on $\mathbf{Mod}^{@}$, and the monomorphism sing defines a natural transformation*

$$sing : Id_{\mathbf{Mod}^{@}} \rightarrowtail Fin. \tag{G.10}$$

**Lemma 93** *Let $\mathbf{D} = H_0 \underset{f_0}{\to} H_1 \underset{f_1}{\to} H_2...$ be a natural sequence diagram in $\mathbf{Mod}^{@}$. Then we have*

$$colim(Fin(\mathbf{D})) \xrightarrow{\sim} Fin(colim(\mathbf{D}). \tag{G.11}$$

This yields an important proposition for the construction of circular forms and denotators.

**Proposition 103** *Let $H$ be a functor in $\mathbf{Mod}^{@}$. Then there are functors $X$ and $Y$ in $\mathbf{Mod}^{@}$ such that*

$$X \xrightarrow{\sim} Fin(H \times X) \text{ and} \tag{G.12}$$

$$Y \xrightarrow{\sim} H \times Fin(Y). \tag{G.13}$$

Proof. For the first isomorphism, let $(X_n)_{0 \leq n}$ be the following sequence of functors in $\mathbf{Mod}^{@}$. We recursively define

$$X_n = \begin{cases} \emptyset & \text{for } n = 0, \\ Fin(H \times X_{n-1}) & \text{for } n > 0. \end{cases} \tag{G.14}$$

Then we have a diagram of subfunctors

$$f_n : X_n \hookrightarrow X_{n+1} \tag{G.15}$$

for all $0 \leq n$. In fact, clearly $X_0 \hookrightarrow X_1$. Now, let $0 < n$ and take an address $M$. We have $M@X_n = M@Fin(H \times X_{n-1}) = Fin(M@H \times M@X_{n-1})$. Since by induction $X_{n-1} \hookrightarrow X_n$, we have $M@X_{n-1} \subset M@X_n$ and hence $M@X_n \subset M@X_{n+1}$. Now, we know from [481] that the product commutes with the colimit over a sequence diagram. Taking the diagram $\mathbf{D} = X_0 \underset{f_0}{\rightarrow} X_1 \underset{f_1}{\rightarrow} X_2 ...$ and setting $X = colim(\mathbf{D})$, lemma 93 yields

$$
\begin{aligned}
& Fin(H \times X) = && \text{(G.16)}\\
& Fin(H \times colim(\mathbf{D})) \xrightarrow{\sim}\\
& Fin(colim(H \times \mathbf{D})) \xrightarrow{\sim}\\
& colim(Fin(H \times \mathbf{D})) \xrightarrow{\sim}\\
& colim(\mathbf{D}) =\\
& X && \text{(G.17)}
\end{aligned}
$$

and we are done for the first isomorphism. For the second, take

$$
Y_n = \begin{cases} \emptyset & \text{for } n = 0, \\ H \times Fin(Y_{n-1}) & \text{for } n > 0. \end{cases} \tag{G.18}
$$

We again have a diagram $\mathbf{E} = Y_0 \underset{g_0}{\rightarrow} Y_1 \underset{g_1}{\rightarrow} Y_2 ...$ of subfunctors

$$
g_n : Y_n \hookrightarrow Y_{n+1} \tag{G.19}
$$

for all $0 \leq n$. Setting $Y = colim(\mathbf{E})$, our second isomorphism results:

$$
\begin{aligned}
& H \times Fin(Y) = && \text{(G.20)}\\
& H \times Fin(colim(\mathbf{E})) \xrightarrow{\sim}\\
& H \times colim(Fin(\mathbf{E})) \xrightarrow{\sim}\\
& colim(H \times Fin(\mathbf{E})) \xrightarrow{\sim}\\
& colim(\mathbf{E}) =\\
& Y && \text{(G.21)}
\end{aligned}
$$

and we are done.

## G.3 Topoi

Topoi are special categories which imitate the crucial constructions of set theory, such as cartesian products, disjoint unions, power sets, and characteristic maps. In our context, topoi play two roles: (1) the role of basic mathematical realities which are instantiated to get off ground in denotator theory, i.e., to build compound concept spaces and their points; (2) the more technical role of topoi of sheaves associated with presheaves for Grothendieck topologies.

### G.3.1 Subobject Classifiers

**Definition 155** *Given a category* $\mathbf{C}$ *which is finitely complete, with the terminal object* $1$*, a monomorphism* $true : 1 \rightarrowtail \Omega$ *in* $\mathbf{C}$ *is called a* subobject classifier *iff given any monomorphism* $\sigma : S \rightarrowtail X$ *in* $\mathbf{C}$*, there is a unique morphism* $\chi_\sigma : X \to \Omega$ *such that the diagram*

$$\begin{array}{ccc} S & \xrightarrow{\sigma} & X \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \chi_\sigma} \\ 1 & \xrightarrow{true} & \Omega \end{array} \tag{G.22}$$

*is a pullback.*

Subobject classifiers are unique up to isomorphism. If a subobject classifier exists, the morphism $\chi$ must be the same if we replace $\sigma$ by $\sigma \circ q$ for any isomorphism $q : S' \xrightarrow{\sim} S$ since an isomorphic object to a pullback is also a pullback. A *subobject of* $X$ is an equivalence class of monomorphisms $\sigma : S \rightarrowtail X$ under the relation $\sigma \sim \sigma'$ iff there is an isomorphism $q$ such that $\sigma' = \sigma \circ q$. Suppose that the collection of subobjects of $X$ is a set $Sub_{\mathbf{C}}(X)$ for each object $X$ in $\mathbf{C}$. Then this is a presheaf in $\mathbf{C}^{@}$ by this map: take a morphism $f : Y \to X$. Then we define

$$Sub_{\mathbf{C}}(f) : Sub_{\mathbf{C}}(X) \to Sub_{\mathbf{C}}(Y) : \sigma \mapsto \sigma_f$$

where $\sigma_f : S \times_X Y \to Y$ is the canonical projection of the pullback under $f, \sigma$. It is straightforward that this is a monomorphism. Then we have:

**Proposition 104** *A category* $\mathbf{C}$ *which is finitely complete and such that the subobject presheaf* $Sub_{\mathbf{C}}$ *is defined, has a subobject classifier iff the* $Sub_{\mathbf{C}}$ *is representable,* $Sub_{\mathbf{C}}(X) \xrightarrow{\sim} Hom(X, \Omega)$ *for all* $X$*. If so, the subobject classifier can be set to the inverse image* $\epsilon \rightarrowtail \Omega$ *of* $Id_\Omega$ *in* $Sub_{\mathbf{C}}(\Omega)$*.*

See [314, p.33] for a proof.

**Example 96** In the category **Sets**, the ordinal number inclusion $true : 1 \rightarrowtail 2 = \{0, 1\} : 0 \mapsto 0$ is a subobject classifier. Since the subobjects of a set $X$ in **Sets** identify to the subsets $S \subseteq X$, we have the classical result that subsets $S$ of $X$ are characterized by their characteristic maps $\chi_S : X \to 2$, a fact that is also traced in the notation $2^X$ for the set of subsets, the powerset of $X$.

**Example 97** The equivalence classes of monomorphisms of presheaves $S \rightarrowtail X$ in $\mathbf{C}^{@}$ are defined by their images $Im(S) \subseteq X$ (take everything pointwise). So $sub_{\mathbf{C}^{@}}(X) \xrightarrow{\sim} \{S \subseteq X\}$, the set of subfunctors of $X$ (supposing that it exists as a set). By the Yoneda lemma, if a subobject classifier in $\mathbf{C}^{@}$ exists, we must have[6] $Sub_{\mathbf{C}^{@}}(@Y) \xrightarrow{\sim} Hom(@Y, \Omega) \xrightarrow{\sim} \Omega(Y)$. So the functor $Y \mapsto Sub_{\mathbf{C}^{@}}(@Y)$ is a canonical candidate for $\Omega$, and it in fact does the job, see [314, pp.37/38]. The final presheaf being the constant presheaf $1_{\mathbf{C}^{@}} : X \mapsto 1$, we get the *true* morphism (natural transformation) $true(0) = @Y$. A subfunctor of $@Y$ is called a *sieve in* $Y$, so a candidate for the *subobject classifier is the functor of sieves* (verify that it is a functor!).

[6] Writing the shorter $@Y$ instead of $Hom(?, Y)$.

**Exercise 90** The categories **Ab** of abelian groups or $\mathbf{Mod}_R$ of $R$-modules have no subobject classifiers.

In denotator theory, sieves and more general subfunctors replace local compositions (which are essentially subsets of ambient modules) in the functorial setup. This is also necessitated since module categories are no topoi (since they have no subobject classifiers, see definition 156 in appendix G.3.3), so the passage to the presheaves over modules, i.e., the category $\mathbf{Mod}^{@}$ is mandatory in order to recover the subobject classifier structure.

## G.3.2 Exponentiation

Recall example 94 in appendix G.2.1 of exponential sets. More generally, a category $\mathbf{C}$ is called *cartesian closed* iff it has finite products[7] and each element $A$ is *exponentiable*, which means that the functor $A\times?$ has a right adjoint $?^A$, i.e., we have an adjoint pair of functors $A\times? \dashv ?^A$.

**Example 98** The category of sets **Sets** is cartesian closed. And any product of cartesian closed categories is cartesian closed.

**Example 99** A category of presheaves $\mathbf{C}^{@}$ is cartesian closed by the following discussion. Again, we use the Yoneda lemma to find a canonical candidate of the exponentiation $X^Y$ of two presheaves $X, Y$. If the exponential $X^Y$ exists, we must have $U@X^Y \tilde{\to} Nat(@U, X^Y) \tilde{\to} Nat(@U \times Y, X)$. So one canonical definition must be

$$U@X^Y = Nat(@U \times Y, X) \tag{G.23}$$

for any object $U$ in $\mathbf{C}$, which is evidently a presheaf. The proof that this formula does the job is found in [314, p.47].

In every cartesian closed category $\mathbf{C}$ one has these standard formulas

$$1^X \tilde{\to} 1, X^1 \tilde{\to} X, (Y \times Z)^X \tilde{\to} Y^X \times Z^X, X^{Y\times Z} \tilde{\to} (X^Y)^Z,$$

which follow from the universal adjointness property of exponentiation.

## G.3.3 Definition of Topoi

There are several equivalent definitions of a topos, which we first summarize in the following proposition:

**Proposition 105** *For a category* $\mathbf{C}$*, the following group properties are equivalent:*

1. *(a)* $\mathbf{C}$ *is cartesian closed,*
   *(b)* $\mathbf{C}$ *has a subobject classifier* $1 \rightarrowtail \Omega$*.*
2. *(a)* $\mathbf{C}$ *is cartesian closed,*

[7] Equivalently: binary products and a terminal object.

(b) $\mathbf{C}$ *is finitely cocomplete,*

(c) $\mathbf{C}$ *has a subobject classifier* $1 \rightarrowtail \Omega$.

3. (a) $\mathbf{C}$ *has a terminal object and pullbacks,*

(b) $\mathbf{C}$ *has exponentials,*

(c) $\mathbf{C}$ *has a subobject classifier* $1 \rightarrowtail \Omega$.

4. (a) $\mathbf{C}$ *has a terminal object and pullbacks,*

(b) $\mathbf{C}$ *has an initial object and pushouts,*

(c) $\mathbf{C}$ *has exponentials,*

(d) $\mathbf{C}$ *has a subobject classifier* $1 \rightarrowtail \Omega$.

5. (a) $\mathbf{C}$ *is finitely complete,*

(b) $\mathbf{C}$ *has power objects*[8]*,*

**Definition 156** *A category* $\mathbf{C}$ *which has the equivalent groups of properties in proposition 105 is called a* (elementary) topos.

Here are some general properties and examples of topoi:

**Proposition 106** *For a topos* $\mathbf{C}$ *a comma category* $\mathbf{C}/b$ *is also a topos.*

**Proposition 107** *For a category* $\mathbf{C}$ *the presheaf category* $\mathbf{C}^{@}$ *is a topos.*

This is immediate from the previous discussion of the presheaf category.

**Proposition 108** *Let* $\mathbf{C}$ *be a topos. Then we have these properties:*

(i) *Every morphism* $f$ *has an image, i.e., factors as* $f = i \circ e$ *with* $i$ *mono and* $e$ *epi. For any two such factorizations* $f = i \circ e, f = i' \circ e'$*, there is an isomorphism* $t$ *such that* $e' = t \circ e, i' = i \circ t$.

(ii) *A morphism is iso iff it is mono and epi.*

(iii) *The pullback of an epi is an epi.*

(iv) *Every arrow* $X \to 0$ *is iso.*

(v) *Every arrow* $0 \to X$ *is mono.*

**Definition 157** Logical *morphisms between topoi are functors which preserve* (*up to isomorphism*) *finite limits, exponentials, and subobject classifiers.*

For example, the canonical base change functor $\mathbf{C}/b \to \mathbf{C}/c$ of comma topoi for a base change morphism $c \to b$ is logical, see [314, p.193].

[8] See [186, p.106] for this group of properties

## G.4 Grothendieck Topologies

Grothendieck topologies and associated topoi of sheaves are a classical example for the geometric aspects of topoi. Here is the context.

Given a finitely complete category (a small one for those who like universes) $\mathbf{C}$ with the subobject classifier functor of sieves $X@\Omega = Sub_{\mathbf{C}^@}(@X)$ (see example 97). Recall that given a morphism $f : Y \to X$ the functor maps a sieve $S \subseteq X$ to the pullback sieve $f^*(S) = S \times_X Y$.

**Definition 158** *A* Grothendieck topology *on a category* $\mathbf{C}$ *is a function* $J$ *which for each* $X$ *is a subset* $X@J \subseteq X@\Omega$ *of sieves in* $X$ *with these properties:*

(i) $@X \in X@J$,

(ii) *(Stability) If* $S \in X@J$, *then for* $f : Y \to X$, $f^*S \in Y@J$,

(iii) *(Transitivity) If* $S \in X@J$ *and* $R \in X@\Omega$ *with* $f^*R \in Y@J$ *for all* $f : Y \to X$ *in* $S$, *then* $R \in X@J$.

*A* site *is a pair* $(\mathbf{C}, J)$ *of a Grothendieck topology* $J$ *on a category* $\mathbf{C}$. *A sieve in* $X@J$ *is called a* covering sieve, *one also says that "it covers* $X$*".*

The first two requirements mean that $J$ is a subfunctor of $\Omega$ through which the *true* arrow factorizes. Very often, Grothendieck topologies are not given directly, but via a so-called basis:

**Definition 159** *For a finitely complete category* $\mathbf{C}$, *a* basis (for a Grothendieck topology) *is a function* $K$ *which assigns to each object* $X$ *a collection* $K(X)$ *of families of morphisms with codomain* $X$ *such that:*

(i) *For every isomorphism* $f : X' \xrightarrow{\sim} X$, *the singleton* $\{f\}$ *is in* $K(X)$;

(ii) *(Stability) If* $(f_i : X_i \to X) \in K(X)$, *and* $h : Y \to X$, *then* $(h^* f_i : X_i \times_X Y \to Y) \in K(Y)$;

(iii) *(Transitivity) If* $(f_i : X_i \to X) \in K(X)$ *and, for each index* $i$, $(f_{ij} : X_{ij} \to X_i) \in K(X_i)$, *then* $(f_i \circ f_{ij} : X_{ij} \to X) \in K(X)$.

*A pair* $(\mathbf{C}, K)$ *is again called a* site *(see below for a justification!); whereas the families in the sets* $K(X)$ *are called* covering families.

Here is the relation to Grothendieck topologies: Given a basis $K$ as above, one defines

$$J_K(X) = \{S | \text{ there is } R \in K(X) \text{ with } R \subseteq S\}, \tag{G.24}$$

where $R \subseteq S$ means that $R$ is in the union of the evaluations $\bigcup_{\mathbf{C}} Z@S$ of $S$.

And the converse: Given a family $R$ of morphisms with codomain $X$, we denote by $(R)$ the sieve generated by $R$, i.e., the smallest sieve in $X$ containing all arrows of $R$. Then a Grothendieck topology $J$ can be defined by the following basis $K$ which is this set at $X$:

$$K(X) = \{R \subseteq @X | (R) \in X@J\}. \tag{G.25}$$

### G.4.1 Sheaves

**Definition 160** *Given a site $(\mathbf{C}, J)$, a presheaf $P$ in $\mathbf{C}$ is a sheaf for $J$ iff for every covering sieve $S \subseteq @X$, the inclusion induces a bijection $Nat(@X, P) \xrightarrow{\sim} Nat(S, P)$.*

This condition can be rephrased for a basis $K$ of $J$ in a more effective and classical way. To this end recall that a (co)equalizer of a pair $f, g : x \rightrightarrows y$ of parallel arrows is the (co)limit of this diagram.

**Proposition 109** *A presheaf $P$ on $\mathbf{C}$ is a sheaf for the topology $J$, iff for any covering family $(f_i : X_i \to X) \in K(X)$, the canonical diagram*

$$P(X) \to \prod_i P(X_i) \rightrightarrows \prod_{i,j} P(X_i \times_X X_j) \tag{G.26}$$

*is an equalizer.*

Here the two arrows to the right stem from the two projections from $X_i \times_X X_j$ to $X_i$ and to $X_j$, whereas the arrow to the left stems from the covering morphisms $f_i$.

**Definition 161** *A* Grothendieck topos *is a category which is equivalent to the full subcategory $Sh(\mathbf{C}, J)$ of sheaves in $\mathbf{C}^{@}$*

The justification of this terminology lies in the following theorem:

**Theorem 61** *A Grothendieck topos is an elementary topos.*

The proof evidently splits in the verification of finite completeness, existence of exponentials and of a subobject classifier. For details, see [314, pp.128-144]. Finite completeness is easy since:

**Lemma 94** *Limits of sheaves are sheaves.*

**Lemma 95** *If $P$ is a presheaf and $F$ is a $J$-sheaf in $\mathbf{C}^{@}$, then the presheaf exponential $F^P$ is a sheaf and therefore an exponential in $Sh(\mathbf{C}, J)$.*

**Definition 162** *A sieve $L \subseteq @X$ is* closed *with respect to $(\mathbf{C}, J)$ iff $f : Y \to X$ with $f^*(L) \in Y@J$ implies $f \in Y@L$.*

**Proposition 110** *The function $X \mapsto X@\Omega_{Sh} = \{\text{closed sieves in } X\} \subset X@\Omega$ contains $@X$ defines a subpresheaf of the subobject classifier of $\mathbf{C}^{@}$ and is a sheaf. Together with the morphism $true : 1_{\mathbf{C}^{@}} \to \Omega_{Sh}$ it defines a subobject classifier of $Sh(\mathbf{C}, J)$.*

The topos $Sh(\mathbf{C}, J)$ is a subtopos of $\mathbf{C}^{@}$ with the natural inclusion $i : Sh(\mathbf{C}, J) \to \mathbf{C}^{@}$. This natural transformation has a left adjoint of sheafification which we shall discuss now. If $P$ is a presheaf over $\mathbf{C}$, the sheafification operator $P \mapsto P^+$ evaluates as follows. For an object $X \in \mathbf{C}$ and a sieve $S \in X@J$, consider the limit $Match_P(S) = \prod_{f:Y\to X \in S} Y@P$. Consider the diagram $(Match_P(S))_{S \in X@J}$ with canonical restriction maps $Match_P(S) \to Match_P(T)$ for

$T \subseteq S$, and define $X@P^+ = lim_{S \in X@J} Match_P(S)$. For a morphism $g : X_1 \to X_2$, we have a map $P^+(g) : X_2@P^+ \to X_1@P^+$. It takes a "matching family" $(x_f)_{f \in S}$ to the matching family $(x_{g.h})_{h \in g^* S}$. This evidently defines a presheaf, and we have a canonical morphism $\eta : P \to P^+$. Then:

**Theorem 62** *With the above notation, we have*

(i) *The presheaf $P^+$ is separated*[9]*.*

(ii) *The presheaf $P$ is separated iff $\eta$ is mono.*

(iii) *The presheaf $P$ is a sheaf iff $\eta$ is iso.*

For a proof, see [314, III.5]. In particular, the double application $\mathbf{a}P = ((P)^+)^+$ yields a sheaf and a natural presheaf morphism $P \to \mathbf{a}P$. We have

**Theorem 63** *The map $P \mapsto \mathbf{a}P$ defines a left adjoint of the inclusion $i$; $\mathbf{a} \dashv i$. The composition $\mathbf{a} \circ i$ is isomorphic to the identity on the sheaf category $Sh(\mathbf{C}, J)$.*

For a proof, see [314, III.5, Theorem 1] and [314, III.5, Corollary 6].

**Corollary 29** *If $f : F \to G$ is a morphism of sheaves, $f$ is mono iff it is an injection for each argument.*

The proof follows from the fact that this is true for presheaves, and that by the adjunction theorem 63, $i$ preserves and reflects[10] monomorphisms, QED.

# G.5 Formal Logic

Formal logic does not replace absolute logic which is built upon the non-formalizable theorem of identity (A is identical to A), of contradiction (A and non-A exclude each other), and of the excluded third (there is no third choice except A or non-A). It does however model the way a specific domain of knowledge can handle its formal truth mechanisms.

## G.5.1 Propositional Calculus

Sentences in propositional calculus are defined from a set $\Phi = \{\pi_0, \pi_1, \ldots\}$ of symbols, called *propositional variables*; a set $\Xi = \{!, \&, |, \ -> \}$ of logical connective symbols[11] (! for negation, & for conjunction, | for disjunction, and $->$ for implication); a set $\Delta = \{(, )\}$ of two brackets; mutually disjoint from each other. Let $EX = FM(\Phi \cup \Xi \cup \Delta)$ be the free monoid of word *expressions* above these symbols. Let $S(EX)$, the set of *sentences*, be the smallest subset of $EX$ with these properties:

[9] The left arrow in equation (G.26) for $P^+$ is injective.

[10] A functor $f$ reflects a property of morphisms if the property for the image morphism $f(x)$ implies the property for $x$.

[11] Our symbols are near to programming symbols, where the formalism is really needed.

**Property 4** *Given the symbol sets $\Phi, \Xi, \Delta$, we require:*

(i) $\Phi \subset S(EX)$;

(ii) *if* $\alpha \in S(EX)$, *then* $(!\alpha) \in S(EX)$;

(iii) *if* $\alpha, \beta \in S(EX)$, *then* $(\alpha)\&(\beta) \in S(EX)$;

(iv) *if* $\alpha, \beta \in S(EX)$, *then* $(\alpha)|(\beta) \in S(EX)$;

(v) *if* $\alpha, \beta \in S(EX)$, *then* $(\alpha) -> (\beta) \in S(EX)$.

Clearly, in $S(EX)$, the building blocks of a sentence are uniquely determined, so it makes sense to define set-valued functions $\epsilon : S(EX) \to A$ on such sentences by recursion of the building blocks. Suppose that $A$ is a lattice, i.e., a partially ordered set $(A, \leq)$ with a *join* operation $\vee : A \times A \to A$, a *meet* operation $\wedge : A \times A \to A$, minimum (False) $\bot$, a maximum (True) $\top$, further a unary *negation* operation $\neg : A \to A$, and a binary *implication* operation $\Rightarrow: A \times A \to A$. Call such an $A$ a *logical algebra*. Then any set function $\epsilon_0 : \Phi \to A$ extends in a unique way to the *evaluation* $\epsilon = \epsilon(\epsilon_0) : S(EX) \to A$ by these rules:

**Property 5** *For all sentences $\alpha, \beta$, we set*

(i) $\epsilon(!\alpha) = \neg\epsilon(\alpha)$;

(ii) $\epsilon((\alpha)\&(\beta)) = \epsilon(\alpha) \wedge \epsilon(\beta)$;

(iii) $\epsilon((\alpha)|(\beta)) = \epsilon(\alpha) \vee \epsilon(\beta)$;

(iv) $\epsilon((\alpha) -> (\beta)) = \epsilon(\alpha) \Rightarrow \epsilon(\beta)$.

Propositional calculus deals with the evaluation map on special logical algebras. A sentence $\alpha$ *is called A-valid*, in symbols: $A \vDash \alpha$, iff it $\epsilon(\alpha) = \top$ for all evaluations $\epsilon_0 : \Phi \to A$ on the propositional variables. It is called *classically valid* or a *tautology* iff it is 2-valid for the well-known Boolean algebra $2 = \{0, 1\}$ of classical truth values, where we set $\top = 1, \bot = 0$. The symbol for classical validity is $\vDash \alpha$. Here are typical classes of logical algebras:

**Boolean Algebras.** A Boolean algebra is a distributive logical algebra such that $x \vee \neg x = \top$ and $x \wedge \neg x = \bot$. Distributivity means that $x \wedge (y \vee z) = x \wedge y \vee x \wedge z$ and $x \vee (y \wedge z) = x \wedge y \vee x \wedge z$. Further, implication is defined by $x \Rightarrow x = \neg x \vee y$. In a Boolean algebra (BA), one has these properties: $\neg\neg x = x$, $x \wedge y = \bot$ iff $y \leq \neg x$, $x \leq y$ iff $\neg y \leq \neg x$, $\neg(x \wedge y) = \neg x \vee \neg y$, $\neg(x \vee y) = \neg x \wedge \neg y$.

**Heyting Algebras.** A Heyting algebra $A$ is a partially ordered set which, as a category whose morphisms $x \to y$ are the pairs $x \leq y$, has all finite products and coproducts, and which has exponentials, so it is cartesian closed. In other words, a Heyting algebra is a lattice with minimum $\bot$ and maximum $\top$ which has exponentials $x^y$. One writes the product as meet ($\wedge$) and the coproduct as join ($\vee$). The exponential $x^y$ is written as $y \Rightarrow x$, and the adjunction property of exponentiation reads

$$z \leq y \Rightarrow x \text{ iff } z \wedge y \leq x. \tag{G.27}$$

For a Heyting algebra, we define a negation $\neg x = x \Rightarrow \bot$, which is equivalent to $y \leq \neg x$ iff $y \wedge x = \bot$.

**Proposition 111** *For a Heyting algebra, we have these identities:* $x \leq \neg\neg x$, $x \leq y$ *implies* $\neg y \leq \neg x$, $\neg x = \neg\neg\neg x$, $\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y$, $(x \Rightarrow x) = \top$, $x \wedge (x \Rightarrow y) = x \wedge y$, $y \wedge (x \Rightarrow y) = y$, $x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z)$.

**Proposition 112** *A Heyting algebra is distributive, and it is Boolean iff* $x = \neg\neg x$ *for all* $x$*, or iff* $x \vee \neg x = \top$ *for all* $x$*.*

**Proposition 113** *For a presheaf category* $\mathbf{C}^{@}$*, the partially ordered set* $Sub_{\mathbf{C}^{@}}(P)$ *of an object* $P$ *is a Heyting algebra. The connectives are defined as follows. If* $S, T$ *are two subfunctors of* $P$*, then:*

(i) $X\,@(S \vee T) = X\,@S \cup X\,@T$;

(ii) $X\,@(S \wedge T) = X\,@S \cap X\,@T$;

(iii) $(S \Rightarrow T)(X) = \{x \in X\,@P |$ *for every morphism* $f : Y \to X$, *if* $x \cdot f \in Y\,@S$, *then* $x \cdot f \in Y\,@T\}$;

(iv) $X\,@(\neg S) = \{x \in X\,@P |$ *for every morphism* $f : Y \to X, x \cdot f \notin Y\,@S\}$;

More generally (see [314, IV.8] for details):

**Theorem 64** *For every topos* $\mathbf{C}$*, the partially ordered set* $Sub(X) \xrightarrow{\sim} Hom(X, \Omega)$ *of subobjects of* $X$ *is a Heyting algebra.*

To the left, this structure stems from the canonical Heyting algebra structure on the subobjects of $X$. To the right, this structure is induced by the following operations on $\Omega$:

1. Negation $\neg : \Omega \to \Omega$ is the characteristic map of the false arrow $false : 1 \rightarrowtail \Omega$, which is the characteristic map of the zero arrow $0 \rightarrowtail 1$.
2. Disjunction $\wedge : \Omega \times \Omega \to \Omega$ is the character of the diagonal morphism $\Delta(true, true) : 1 \to \Omega$.
3. Conjunction $\vee : \Omega \times \Omega \to \Omega$ is the character of the image of the universal morphism $a \sqcup b : \Omega \sqcup \Omega \to \Omega \times \Omega$, where $a = \Delta(Id_\Omega, true) : \Omega \to \Omega \times \Omega$ and $b = \Delta(true, Id_\Omega) : \Omega \to \Omega \times \Omega$.
4. Implication $\vee : \Omega \times \Omega \to \Omega$ is the character of the equalizer of $p_1, \wedge : \Omega \times \Omega \rightrightarrows \Omega$.

If a sentence $\alpha$ is valid for all Boolean algebras, we write $BA \vDash \alpha$. If it is valid for all Heyting algebras, we write $HA \vDash \alpha$. If it is valid in the Heyting algebra $1@\Omega$ of a topos $\mathbf{C}$, we write $\mathbf{C} \vDash \alpha$.

Validity is also described by a recursive construction process of valid sentences. One gives a set $AX$ of sentences, called *axioms*, and defines *theorems* as those sentences $s$ which are at the end of *proof chains*, i.e., finite sequences of sentences $(s_0, s_1, \ldots s_n, s)$ such that each member of this sequence is either an axiom or can be inferred from earlier members by a set $RULES$ of rules.

The classical setup is this. $AX$ consists of 12 types of sentences:

**Axiom 7** *Axioms of classical logic (CL):*

(i) $\alpha$ –> $(\alpha\&\alpha)$

(ii) $(\alpha\&\beta)$ –> $(\beta\&\alpha)$

(iii) $(\alpha$ –> $\beta)$ –> $((\alpha\&\gamma)$ –> $(\beta\&\gamma))$

(iv) $((\alpha$ –> $\beta)\&(\beta$ –> $\gamma))$ –> $(\alpha$ –> $\gamma)$

(v) $\beta$ –> $(\alpha$ –> $\beta)$

(vi) $(\alpha\&(\alpha$ –> $\beta))$ –> $\beta$

(vii) $\alpha$ –> $(\alpha|\beta)$

(viii) $(\alpha|\beta)$ –> $(\beta|\alpha)$

(ix) $((\alpha$ –> $\beta)\&(\beta$ –> $\gamma))$ –> $((\alpha|\beta)$ –> $\gamma)$

(x) $(!\alpha)$ –> $(\alpha$ –> $\beta)$

(xi) $((\alpha$ –> $\beta)\&(\alpha$ –> $(!\beta)))$ –> $(!\alpha)$

(xii) $\alpha|(!\alpha)$

The system CL has one single rule of inference:

**Principle 29** (Modus ponens) *From $\alpha$ and $\alpha$ –> $\beta$, $\beta$ may be derived.*

The property of a sentence $\alpha$ of being a CL-theorem is denoted by $\underset{CL}{|\!\!-\!\!-\!\!-} \alpha$. Following Heyting, the intuitionistic logic (IL) is the (CL) with the axiom (xii) omitted, and the same inference rule.

**Theorem 65** *The following validity statements are equivalent:*

(i) $\underset{CL}{|\!\!-\!\!-\!\!-} \alpha$.

(ii) $\vDash \alpha$.

(iii) *There exists a Boolean algebra $B$ such that $B \vDash \alpha$.*

(iv) $BA \vDash \alpha$.

See [186] for details. We have this weaker relation:

**Theorem 66** *The topos validity $\mathbf{C} \vDash \alpha$ implies classical validity $\underset{CL}{|\!\!-\!\!-\!\!-} \alpha$.*

**Definition 163** *A topos is Boolean iff the Heyting algebra $Sub(X)$ of each object $X$ is Boolean.*

**Theorem 67** *The following statements for a topos $\mathbf{C}$ are equivalent:*

(i) $\mathbf{C}$ *is Boolean.*

(ii) $Sub(\Omega)$ *is a Boolean algebra.*

(iii) $true : 1 \to \Omega$ *has a complement in* $Sub(\Omega)$.

(iv) $false : 1 \to \Omega$ *is the complement of* $true$ *in* $Sub(\Omega)$.

(v) $true \cup false = Id_\Omega$ *in* $Sub(\Omega)$.

(vi) $\mathbf{C}$ *is* classic, *i.e.,* $true \sqcup false \to \Omega$ *is iso.*

(vii) *The first inclusion* $\iota_1 : 1 \to 1 + 1$ *is a subobject classifier.*

For a proof, see [186, p.156 ff.].

**Theorem 68** *If the topos* $\mathbf{C}$ *is Boolean, then* $\mathbf{C} \vDash \alpha | ! \alpha$ *for all sentences* $\alpha$.

**Theorem 69** *For a topos* $\mathbf{C}$, *the following are equivalent:*

(i) $\mathbf{C} \vDash \alpha$ *iff* $\underset{CL}{|\!\!-\!\!-} \alpha$ *for all* $\alpha$.

(ii) $\mathbf{C} \vDash \alpha | ! \alpha$ *for all* $\alpha$.

(iii) $Sub(1)$ *is a Boolean algebra.*

**Theorem 70** *We have* $HA \vDash \alpha$ *iff* $\underset{IL}{|\!\!-\!\!-} \alpha$.

**Theorem 71** *For all topoi* $\mathbf{C}$, *the validity* $\underset{IL}{|\!\!-\!\!-} \alpha$ *implies* $\mathbf{C} \vDash \alpha$.

## G.5.2 Predicate Logic

Predicate calculus generates a richer set of sentences whose validity is a function of the interpretation of predicate variables and individual variables and not only of abstract propositional variables. We are given a set $\Upsilon = \{\iota_0, \iota_1, \ldots\}$ of *individual variables*, a set $\Xi = \{!, \&, |, \ ->, \exists, \forall\}$ of *predicate connectives*, a set $\Pi = \coprod_{i=0,1,2,\ldots} \Pi_i, \Pi_i = \{A^i, B^i, \ldots\}$ of *i-ary predicate variables*, and a set $\Delta = \{(,)\}$ of brackets as above. Within the free monoid $PEX = FM(\Upsilon \sqcup \Xi \sqcup \Pi \sqcup \Delta)$ of *predicate expressions*, one exhibits the subset $FO(PEX)$ of *formulae* as follows:

**Property 6** *Given the symbols* $\Upsilon, \Xi, \Pi, \Delta$, *we require*

(i) (Atomic formulae) $A^i \iota_{i_1} \iota_{i_2} \ldots \iota_{i_i} \in FO(PEX)$ *for any* $A^i \in \Pi_i$ *and* $\iota_{i_k} \in \Upsilon$, *and* 0*-ary predicate variables are constants.*

(ii) (Propositional formulae) *If* $\alpha, \beta \in FO(PEX)$, *then*

$$(!\alpha) \in FO(PEX),$$
$$(\alpha)\&(\beta) \in FO(PEX),$$
$$(\alpha)|(\beta) \in FO(PEX),$$
$$(\alpha) -> (\beta) \in FO(PEX).$$

(iii) (Quantifier formulae) *If* $\alpha \in FO(PEX), x \in \Upsilon$, *then*

$$(\forall x)\alpha \in FO(PEX),$$
$$(\exists x)\alpha \in FO(PEX).$$

An individual variable $x$ which appears in the formula after an expression of type $(\forall x)$ or $(\exists x)$ is *bound*, otherwise it is *free*.

A model $\mathfrak{M}$ of a predicate logic $FO(PEX)$ is a set $M$, together with $i$-ary relations $a^i, b^i \ldots \subseteq M^i$ (elements $a^0 \in M$ for constants). In such a model, the formal predicate expressions are interpreted via the interpretation of atomic formulae $A^i\iota_{i_1}\iota_{i_2}\ldots\iota_{i_i}$ by the truth value of $(x_{i_1}, x_{i_2}, \ldots x_{i_i}) \in a^i$. Whereas the interpretation of a quantifier formula $(\forall x)\alpha$ means "true" if the truth value of the interpretation of $\alpha$ is "true" for all valuations in $M$ of all occurrences of the variable $x$, and the interpretation of a quantifier formula $(\exists x)\alpha$ means "true" if the truth value of the interpretation of $\alpha$ is "true" for at least one value in $M$ of the occurrences of the variable $x$—of course, this is only decided if no free variables are left, in which case one calls the formula a *sentence*, otherwise, no truth value is defined and the formula is just a truth-value-function of the left free variables. We write $\mathfrak{M} \vDash \alpha[x]$ for truth value "true" for the evaluation $[x]$ of the free variables of $\alpha$.

The recursive calculation of truth values of compound formulas relates to the Boolean algebra $Sub(M)$ as follows: To begin with, if a logical combination of two formulae is considered, one may suppose that both variable sets of these formulas coincide by just taking their union if they do not coincide. If we fix such a variable set of cardinality $m$, say, the truth evaluation of a formula $\alpha$ with (at most) these $m$ free variables can be described by the inverse image $supp(\alpha) \subseteq M^m$ of *true*. Then evidently, $supp(!\alpha) = M^m - supp(\alpha)$, $supp((\alpha)\&(\beta)) = supp(\alpha) \cap supp(\beta)$, $supp((\alpha)|(\beta)) = supp(\alpha) \cup supp(\beta)$, $supp((\alpha) -> (\beta)) = supp(!\alpha) \cup supp(\beta)$. For the quantifiers, we have this situation: If a variable $x$ is bound by a quantifier, we have the support $supp(\alpha) \subseteq M^m$ of the given formula $\alpha$ and the support $supp((\forall x)\alpha) \subseteq M^{m-1}$ or $supp((\exists x)\alpha) \subseteq M^{m-1}$, respectively. Suppose that $x$ is the $i^{th}$ variable, then we have the projection $p_i : M^m \to M^{m-1}$, which omits the $i^{th}$ coordinate, and the inverse image map $p^* : 2^{M^{m-1}} \to 2^{M^m}$. If $S$ is a support of a formula $\alpha$ in $M^m$, then the support $(\forall x)(S)$ of $(\forall x)\alpha$ is the set $\{(y_1, \ldots, y_{m-1}) | (y_1, \ldots y_{i-1}, x, y_{i+1}, \ldots y_{m-1}) \in S \text{ for all } x \in M\}$, while the support of $(\exists x)(S)$ of $(\exists x)\alpha$ is the set $\{(y_1, \ldots y_{m-1}) | (y_1, \ldots y_{i-1}, x, y_{i+1}, \ldots y_{m-1}) \in S \text{ for at least one } x \in M\}$.

**Proposition 114** *The functor of partially ordered sets* $\forall x : 2^{M^m} \to 2^{M^{m-1}}$ *is a right adjoint of* $p^* : 2^{M^{m-1}} \to 2^{M^m}$, *while* $\exists x : 2^{M^m} \to 2^{M^{m-1}}$ *is its left adjoint, in other words,* $p^*(X) \subset Y$ *iff* $X \subset \forall(x)(Y)$ *and* $Y \subset p^*(X)$ *iff* $\exists(x)(Y) \subset X$.

The topos-theoretic generalization of this result is the following theorem; for a proof, see [314, p.209,p.206].

**Theorem 72** *If $f : A \to B$ is a morphism in the topos $\mathbf{C}$, then the functor $Sub(B) \to Sub(A)$ of Heyting algebras (which are viewed as categories via their partial orders as morphisms) associated with the natural morphism $\Omega^f : \Omega^B \to \Omega^A$ has a right adjoint functor $\forall_x$ and a left adjoint functor $\exists_x$.*

In order to rewrite the predicate calculus in general topoi, one uses the characteristic maps associated with supports of predicates as follows: If $M$ is a non-zero object of the topos $\mathbf{C}$, we consider the characters $\chi_{a^m} : M^m \to \Omega$ of the "supports" $a^m \subset M^m$ of $m$-ary predicates. Their recombination via logical constructions runs as follows: Using the morphisms of negation, conjunction, disjunction, and implication defined above in section G.5.1 for $\Omega$, one has the evident combination of supports of formulas via their characters. The new thing here is the definition of quantifier supports. Given a character $\chi_\alpha : M^m \to \Omega$, we have the adjoint morphism $ad_i(\chi_\alpha) : M^{m-1} \to \Omega^M$ with respect to the $i^{th}$ coordinate. Then we have two arrows $\forall_M, \exists_M : \Omega^M \to \Omega$. The first $\forall_M$ is the character of the adjoint of the composite $true \circ ! \circ pr_M : 1 \times M \to M \to \Omega$. The second $\exists_M$ is the character of the image of the composed map $pr_{\Omega^M} \circ \varepsilon_M : \varepsilon \rightarrowtail \Omega^M \times M \to \Omega^M$, where $\varepsilon_M : \varepsilon \rightarrowtail \Omega^M$ is the subobject whose character is the evaluation map $ev_M : \Omega^M \times M \to \Omega$ adjoint to the identity on $\Omega^M$. We then have these formulas:

$$\forall(x)\alpha \text{ has the character } \forall_M \circ ad_i(\chi_\alpha),$$
$$\exists(x)\alpha \text{ has the character } \exists_M \circ ad_i(\chi_\alpha).$$

### G.5.3 A Formal Setup for Consistent Domains of Forms

Since forms do not automatically exist if we allow circularity, it is important to set up a formal mathematical context in order to describe what a logically consistent domain of forms should be. This mathematical formalism turns out to be valid in an interesting general context. We have been working in the topos $\mathbf{Mod}^{@}$ of presheaves over the category **Mod**, where we have the Yoneda embedding $Y : \mathbf{Mod} \to \mathbf{Mod}^{@}$. Without loss of generality, we may identify **Mod** with the full subcategory of represented presheaves $@M, M$ a module.

More generally, we may consider *Yoneda pairs* $\mathcal{R} \subset \mathcal{E}$, where $\mathcal{R}$ is a full subcategory of a topos $\mathcal{E}$, $\mathcal{R}$ playing the role of represented modules (we also say that $\mathcal{R}$ is a *Yoneda subcategory*). This means that we require that the canonical Yoneda functor $\mathcal{E} \to \mathcal{E}^{@} \to \mathcal{R}^{@}$ be fully faithful. By Yoneda's lemma, we may identify the evaluation $M@F$ of a "presheaf" $F \in \mathcal{E}$ at a "module" $M \in \mathcal{R}$ by the morphism set from $M$ to $F$: $M@F = Hom_{\mathcal{E}}(M, F)$. This setup in particular includes the classical case of $\mathcal{E} = \mathbf{Sets}$ and $\mathcal{R}$ the one-element category consisting of a singleton 1 (the terminal object in **Sets**), say $1 = \{\emptyset\}$, and its unique identity morphism. In this case, we may identify $1@F$ and the set $F$.

To achieve the intended formalism, we consider the set $Mono(\mathcal{E})$ of monomorphisms in $\mathcal{E}$. We further consider the set

$$Types = \{\mathbf{Simple}, \mathbf{Syn}, \mathbf{Limit}, \mathbf{Colimit}, \mathbf{Power}\}$$

of form types. And we need the free monoid $Names = FM(UNICODE)$ over the $UNICODE$ alphabet (which is an extension of the $ASCII$ alphabet to non-European letters). We next need

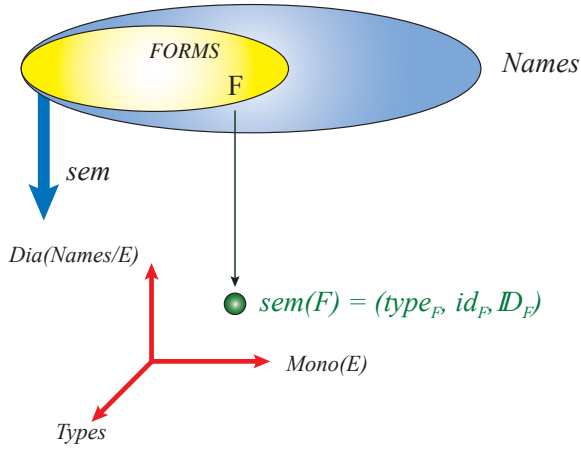



Figure G.1: The formal setup of a semiotic of $\mathcal{E}$-forms.

the set $Dia(Names)$ of all diagram schemes with vertexes in $Names$. More precisely, a diagram scheme over $Names$ is a finite directed multigraph whose vertexes are the elements of a subset of $Names$, and whose arrows $i : A \to B$ are triples $(i, A, B)$, with $i = 1, \ldots$ natural numbers to identify arrows for given vertexes.

Next, consider the set $Dia(Names/\mathcal{E})$ of diagrams on $Dia(Names)$ with values in $\mathcal{E}$. Such a diagram is a map

$$dia : D \to \mathcal{E}$$

which to every vertex of the diagram scheme $D$ associates an object of $\mathcal{E}$ and to every arrow associates a morphism in $\mathcal{E}$ between corresponding vertex objects. So $i : A \to B$ is mapped to the morphism $dia(i) : dia(A) \to dia(B)$. We also will identify two such diagrams iff their arrows for given names $A, B$ are permutations of each other, i.e., we only consider the orbits of diagrams modulo the permutation group of arrows on given names. Why? Because any construction of limits or colimits is invariant under this group since the limit condition is a logical conjunction which does not depend on the numbering of the arrows. So this identification will always be valid unless explicitly suspended. Observe further that a multiple appearance of a vertex in a diagram scheme is not allowed, so when constructing diagram schemes upon form names, one must add synonymous forms when multiple appearance of one and the same form in a diagram is desired. This is the advantage of form names: the annoying indexing of mathematical names can be absorbed by intrinsic renaming on the level of form names.

With these notation, we can define a *semiotic of $\mathcal{E}$-forms* as follows (see also figure G.1):

**Definition 164** *A* semiotic of $\mathcal{E}$-forms *is a set map*

$$sem : FORMS \to Types \times Mono(\mathcal{E}) \times Dia(Names/\mathcal{E})$$

*defined on a subset $FORMS \subset Names$ with the following properties* (i) to (iv)*. To ease language, we use the following notation and terminology:*

- *An element $F \in FORMS$ is called a* form name*, and the pair $(F, sem)$ a* form*,*
- $pr_1 \cdot sem(F) = t(F)$ *(=type of $F$),*
- $pr_2 \cdot sem(F) = id(F)$ *(= identifier of $F$),*
- $domain(id(F)) = fun(F)$ *(= topor[12] or "space" of $F$),*
- $codomain(id(F)) = frame(F)$ *(= frame or "frame space" of $F$),*
- $pr_3 \cdot sem(F) = coord(F)$ *(= coordinator of $F$).*

*Then these properties are required:*

(i) *The empty word $\emptyset$ is not a member of $FORMS$*

(ii) *For any vertex $X$ of the coordinator diagram $coord(F)$, if $X \in FORMS$, then we have*

$$coord(F)(X) = fun(X).$$

(iii) *If the type $t(F)$ is given, we have the following for the corresponding frames:*

- *For* **Syn** *and* **Power***, the coordinator has one vertex $G \in FORMS$ and no arrows, i.e., $coord(F) : G \to fun(G)$, which means that in these cases, the coordinator is determined by a form name $G$.*

  *Further, for* **Syn***, we have $frame(F) = fun(G)$, and for* **Power***, we have $frame(F) = \Omega^{fun(G)}$, if $coord(F) : G \to fun(G)$, as above.*
- *For* **Limit** *and* **Colimit***, the coordinator is any diagram $coord(F)$ whose names are all in $FORMS$. Further, for* **Limit***, we have $frame(F) = lim(coord(F))$, and for* **Colimit***, we have $frame(F) = colim(coord(F))$.*
- *For type* **Simple***, the coordinator has the unique vertex $\emptyset$, and a value $coord(F) : \emptyset \to M$ for a 'module' $M \in \mathcal{R}$ (i.e., a represented presheaf $M = @X$ in the case of presheaves over* **Mod***), or, in a more sloppy notation: $coord(F) = M$.*

Here, circular forms are evidently included via form names which refer to themselves in their diagrams or in deeper recursion structures. With this definition we may discuss the existence and size of form semiotics, i.e., the extent of the $FORMS$ set, maximal such sets, gluing such sets together along compatible intersections, etc. However, we shall not pursue this interesting and logically essential branch for simple reasons of space and time.

[12] The functor in the special case $\mathcal{E} = \mathbf{Mod}^{@}$.

#### G.5.3.1 Morphisms Between Semiotics of Forms

Although the theory of form semiotics is in its very beginnings, it is clear that two form semiotics with intersecting domains $FORM_1$ and $FORM_2$ nee not be contradictory even if the semiotic maps do not coincide on the intersection $FORM_1 \cap FORM_2$. In fact, it could happen that on this intersection, the maps are just "equivalent" semiotics. More generally, it could happen that two form semiotics have subsemiotics which are in complete correspondence and therefore we may glue them to a global semiotic structure. In other words: It is reasonable and feasible to consider morphisms and then categories of form semiotics and therefore isomorphisms of semiotics, which enables us to construct global semiotics just by gluing together local "charts" as usual.

Let us abbreviate $Sema(\mathcal{E}) = Types \times Mono(\mathcal{E}) \times Dia(Names/\mathcal{E})$, $Sema$ being an abbreviation for semantic target space. Suppose that we are given two form semiotics $sem_1 : FORMS_1 \to Sema(\mathcal{E}_1), sem_2 : FORMS_2 \to (\mathcal{E}_2)$. We correspondingly denote by

$$fun_1, fun_2, t_1, t_2, id_1, id_2, frame_1, frame_2, \text{ and } coord_1, coord_2$$

the respective maps. Consider pairs $(u, v)$ where $u : FORMS_1 \to FORMS_2$ is a set map, and where $v : \mathcal{E}_1 \to \mathcal{E}_2$ is a logical functor (see appendix G.3) sending $\mathcal{R}_1$ to $\mathcal{R}_2$. We say that the pair $(u, v)$ *is morphic* (for $FORMS_1, FORMS_2$) iff

1. We have $u(\emptyset) = \emptyset$.
2. The functors commute with $u, v$, i.e., we have $v \cdot fun_1 = fun_2 \cdot u$.
3. The type is invariant and $u$, i.e., $t_2 \cdot u = t_1$.

In particular, mono- and epimorphisms on $\mathcal{E}_1$ are preserved (see appendix G.2.2). Suppose that we are given a diagram scheme

$$C = coord_1(F) : D \to \mathcal{E}_1$$

(modulo permutations on the numberings of the arrows between fixed names, as announced!) associated with the form name $F \in FORM_1$. Let $|D|$ be the vertex names of $D$. We define a diagram scheme $E$ as follows. Its vertexes are the image $|E| = u(|D|)$. For every vertex pair $(X, Y)$ of $|E|$ we take all arrows $i : A \to B$ with $X = u(A), Y = u(B)$. By lexicographic order on the triples $(A, B, i)$, we can order all these arrows and index them with positive natural numbers $j = 1, \ldots n(X, Y)$. This defines a unique new diagram scheme. Secondly, we define a new diagram $C' : E \to \mathcal{E}_2$ as follows: If the arrow $i : A \to B$ gives arrow $j(i) : X \to Y$, the new diagram $C'$ maps this arrow to the morphism

$$v(C(i)) : fun_2(X) = v(fun_1(A)) \to v(fun_1(B)) = fun_2(Y).$$

Denote this diagram by $(u, v)(C)$. Clearly, since we only retain orbits of diagram schemes, we have functoriality, i.e., if $(u_1, v_1), (u_2, v_2)$ are two such morphic pairs for $FORMS_1, FORMS_2$, and $FORMS_2, FORMS_3$, respectively we have

$$(u_2, v_2)((u_1, v_1)(C)) = (u_2 \cdot u_1, v_2 \cdot v_1)(C). \tag{G.28}$$

**Definition 165** *A morphic pair $(u, v)$ for the pair $FORMS_1, FORMS_2$ is said to be a* morphism of form semiotics *$(u, v) : sem_1 \to sem_2$ if the following semiotic data with each given form name $F \in FORMS_1$ are verified:*

1. *Let $F$ be simple, i.e., $F \underset{Id}{\longrightarrow} \mathbf{Simple}(M)$. Then $u(F) \underset{Id'}{\longrightarrow} \mathbf{Simple}(N)$[13], and we require that*
   - *$N = v(M)$,*
   - *Id'=v(Id), i.e., the monomorphism of $Id'$ is the $v$-image of the monomorphism of $Id$—the domains and codomains are already the right ones, only the morphism (a monomorphism by the conservation of limits) has to fit—so*

$$Id' = v(Id) : fun_2(u(F)) = v(fun_1(F)) \rightarrowtail N = v(M).$$

2. *Let $F$ be synonymous, i.e., $F \underset{Id}{\longrightarrow} \mathbf{Syn}(G)$. Then $u(F) \underset{Id'}{\longrightarrow} \mathbf{Syn}(G')$, and we require that $G' = u(G)$ and $Id' = v(Id) = fun_2(u(F)) \rightarrowtail fun_2(u(G))$.*

3. *Let $F$ be of power type, i.e., $F \underset{Id}{\longrightarrow} \mathbf{Power}(G)$. Then $u(F) \underset{Id'}{\longrightarrow} \mathbf{Power}(G')$, and we require that $G' = u(G)$ and $Id' = v(Id) = fun_2(u(F)) \rightarrowtail v(\Omega^{fun_1(G)}) \xrightarrow{\sim} \Omega^{v(fun_1(G)} = \Omega^{fun_2(u(G))}$.*

4. *Let $F$ be of limit (resp. colimit) type, i.e., $F \underset{Id}{\longrightarrow} \mathbf{Limit}(C)$ (resp. $F \underset{Id}{\longrightarrow} \mathbf{Colimit}(C)$) with $C = coord(F)$. Then we have $u(F) \underset{Id'}{\longrightarrow} \mathbf{Limit}(C')$ (resp. $u(F) \underset{Id'}{\longrightarrow} \mathbf{Colimit}(C')$). We then require that $C' = (u, v)(C)$ and that $Id' = v(Id) = fun_2(u(F)) = v(fun_1(F)) \rightarrowtail v(lim(C)) \xrightarrow{\sim} lim(v \cdot C) \xrightarrow{\sim} lim((u, v)(C))$ (resp. the analogous expression with colimits).*

*In any case, the associated form $u(F)$ is related to its ingredients through the given functor $v$ and the recursive constructions on the coordinators via $u$.*

Clearly, the evident composition of two morphisms from formula (G.28) is again a morphism, and we obtain the category *ForSem* of form semiotics.

### G.5.3.2 Local and Global Form Semiotics

It is clear what one should understand by a *global form semiotic*: This is a set $G$, together with a covering $I$ and an atlas $f_i : I_i \xrightarrow{\sim} FORMS_i$ of bijections onto domains of form semiotics $semi_i : FORMS_i \to Sema$ such that all the induced bijections $u_{i,j} : FORMS_i|_j \xrightarrow{\sim} FORMS_j|_i$ extend to isomorphisms $(u_{i,j}, v_{i,j})$ of form semiotics. This means in particular that all intersections $FORMS_i|_j$ are form domains of form sub-semiotics in $sem_i$, and that the underlying functors on $\mathcal{E}_i$ are compatible. We leave the details to the interested reader.

[13] Observe that in the case $\mathcal{R} = \mathbf{Mod}, \mathcal{E} = \mathbf{Mod}^{@}$ of presheaves, we usually write the module as a coordinator, but we mean its represented functor $@M$.

#### G.5.3.3 Connotator From Semiotics

Denotator and form names were very simple word objects in the previous setup. But name spaces may also be required to encompass more articulated structures, in other words: we want names to be denotators as well, thereby turning the denotator concept into a 'connotator' concept. Here is the formal setup.

We again suppose given a Yoneda pair $\mathcal{R}, \mathcal{E}$. We also retain the set $Mono(\mathcal{E})$. We are given two sets $\mathcal{D}$ of denotators and $\mathcal{F}$ of forms, they are supposed to parametrize denotators and forms according to the following system of maps. We have three maps on $\mathcal{D}$:

$$\begin{aligned} &coordinate : \mathcal{D} \to \mathcal{E}, \\ &form : \mathcal{D} \to \mathcal{F}, \\ &denotatorName : \mathcal{D} \to \mathcal{D}. \end{aligned}$$

The coordinate $C$ of a denotator is supposed to be any morphism with domain $A$ within $\mathcal{R}$, which is called the denotator's address. We require that a denotator be uniquely determined by its coordinate $C$, form $F$, and denotatorName $N$. This is why denotators are also written as quadruples $N : A@F(C)$, where the address is denoted for comfort since it is important information.

The denotator's form mimics the space where the denotator lives. To this end, we need two more sets. The set of types is $\mathcal{T} = \{\mathbf{Limit}, \mathbf{Colimit}, \mathbf{Power}, \mathbf{Simple}\}$, it contains the basic constructors of objects in a topos. But we omit synonymy in this generic setup because it can be mimicked by a limit with just one vertex. We also need the set $Diagrams(\mathcal{D}/\mathcal{E})$ of finite diagrams whose vertexes are denotators, and whose arrows are numbered by $1, 2, 3, \ldots$ as above. This means that the diagram schemes are these symbols, and that the evaluation of the diagram scheme yields objects and morphisms in $\mathcal{E}$. Forms have, by hypothesis, uniquely determined values under these four maps:

$$\begin{aligned} &formName : \mathcal{F} \to \mathcal{D}, \\ &identifier : \mathcal{F} \to Mono(\mathcal{E}), \\ &diagram : \mathcal{F} \to Diagrams(\mathcal{D}, \mathcal{E}), \\ &type : \mathcal{F} \to \mathcal{T}. \end{aligned}$$

This means that a form can be written as $FN : Id.T(Dg)$, where $FN$ is the form's name denotator, $Id$ its identifier, $T$ its type, and $Dg$ its diagram.

We impose a small number of axioms for these structures. To this end, we call the domain $dom(Id)$ of a form $FN : Id.T(Dg)$ the form's *space*, whereas the codomain $cod(Id)$ is called its *frame space*. Accordingly, for a denotator $N : A@F(C)$, the composition $Id \circ C$ with its form's identifier $Id$ is called the *frame coordinate*, it uniquely determines the denotator's coordinate.

**Axiom 8** *Here are the conditions for this setup:*

(i) *The map formName is injective, i.e., the form's name is a key.*

(ii) *For all form diagrams, except for simple type, the vertex denotators of the diagram schemes are form names, and their values are the spaces of the respective forms.*

(iii) *If the form's type is* **Limit** *or* **Colimit***, its frame space is the limit or colimit of the diagram.*

(iv) *If the form's type is* **Power***, the diagram has just one vertex and no arrows, and the frame space is* $\Omega^S$*, where* $S$ *is the space of the vertex form.*

(v) *There is a denotator* $\emptyset$ *which is not a form name, and for a simple type form, the diagram has exactly the vertex* $\emptyset$*, no arrow, and the value is a 'representable' object* $X$ *in* $\mathcal{R}$*. In other words, the simple type frame space is just a representable object in disguise. Such a diagram is represented by* $\emptyset X$*.*

The language of forms and denotators has been encoded in an ASCII-based textual form, like TEX, which is therefore called *Denotex* and is available in BNF[14]. In RUBATO®, a Denotex parser is available for communication with Denotex files. Our present notation in this section, such as $FN : Id.T(Dg)$ for forms and $N : A@F(C)$ for denotators, is an illustration of the Denotex notation.

**Example 100** An elementary form for names can be set up as follows: The form $NF$ represented by $fn : Id.\mathbf{Simple}(Dg)$ is simple with the diagram $\emptyset\mathbb{Z}\langle UNICODE\rangle$. The identifier $Id$ is the identity on the representable presheaf $@\mathbb{Z}\langle UNICODE\rangle$, and the name $fn$ is a denotator $fn : 0@FN(C)$, whose coordinate $C$ is the zero-addressed homomorphism $C : 0 \to \mathbb{Z}\langle UNICODE\rangle$ with value $C(0) =$ "NameForm" with $denotatorName(fn : 0@NF(C)) = fn$, i.e., it is its proper name denotator. So its identification resides on its coordinate value "NameForm" and the form named $fn$. This identifies the entire $NF$ form. Then, general $UNICODE$ names $n$ may be defined by $n : 0@NF(Cn)$, where the value $Cn(0) =$ "anyName" is any $UNICODE$ string combination, such as "3.Violin+4.Piano", and which are their proper name denotators, i.e., $denotatorName(n : 0@NF(C)) = n$.

[14]Denotex was developed in collaboration with Thomas Noll, Jörg Garbers, Stefan Göller, and Stefan Müller.

# Appendix H

# Complements on General and Algebraic Topology

## H.1 Topology

Refer to [261] for general topology, and to [498] for algebraic topology.

### H.1.1 General

A topological space is a pair $(X, Open_X)$ of a set $X$ and a set $Open_X$ of *open* subsets of $X$ such that $X$ is open, $U \cap V$ is open if $U, V$ are so, and $\bigcup_i U_i$ is open for any family $(U_i)$ of open sets, in particular the union of the empty family, the empty set, is open. The complement $X - U$ of an open set $U$ is called *closed*. Therefore the collection $Closed_X$ of closed sets fits with the corresponding axioms: the union of any two closed sets is closed, the intersection of any family of closed sets is closed[1], and the empty set is closed. If we define the *closure* $\overline{Y}$ of any subset of $X$ as the intersection of all closed sets containing $Y$, then the topology is again defined by the axioms for the closure operator $\overline{\ } : 2^X \to 2^X$, i.e., $\overline{\emptyset} = \emptyset$, $\overline{\ }$ is idempotent, $Y \subseteq \overline{Y}$, and $\overline{Y \cup Z} = \overline{Y} \cup \overline{Z}$.

Given two topologies $Open_X, Open'_X$, one says that $Open_X$ *is coarser than* $Open'_X$ or that $Open'_X$ *is finer than* $Open_X$ iff $Open_X \subseteq Open'_X$. On any set $X$, the coarsest topology consists just of $X$ and of the empty set, it is called the *indiscrete topology*, whereas the finest topology is the powerset of $X$, it is called the *discrete topology*. The intersection of any family of topologies on $X$ is the finest topology which is coarser than each member of the family. Every set of subsets $\mathcal{S}$ of $X$ is contained in the intersection of all topologies containing this subset, a family containing at least the discrete topology. It consists of all unions of finite intersections (the empty intersection gives $X$) of members of $\mathcal{S}$ and is denoted by $Open(\mathcal{S})$.

A *neighborhood* $W$ of $x \in X$ is a subset containing an open set $U$ which contains $x$. Finite intersections of neighborhoods of $x$ are neighborhoods, supersets of neighborhoods are neighborhoods. An *accumulation point* of a subset $Y$ of $X$ is a point not in $Y$ which intersects

[1] The intersection of the empty family being defined as the total space $X$.

$Y$ in each of its neighborhoods. The closure of a subset $Y$ of $X$ is the union of $Y$ and of its accumulation points. The *interior* $U^o$ of a subset is the union of all open subsets of $U$. It is also the complement of the closure of its complement. The interior operator has an evident set of axioms corresponding to the closure axioms which also characterize the topology. The *boundary* $\partial U$ of a subset is the difference $\overline{U} - U^o$.

A subset $\mathcal{B}$ of open sets is called a *base for the topology* iff any open set is the union of a family of $\mathcal{B}$ members, or, equivalently, every neighborhood of a point contains a neighborhood from $\mathcal{B}$. The axioms for a set of subsets $\mathcal{B}$ of $X$ to be a base for a topology is that $X = \bigcup \mathcal{B}$, and that for any two $U, B \in \mathcal{B}$, $U \cap V$ is the union of members of $\mathcal{B}$. A *subbase for a topology* $Open_X$ on $X$ is a set $\mathcal{S}$ of subsets of $X$ such that $Open_X = Open(\mathcal{S})$.

## H.1.2 The Category of Topological Spaces

Suppose that $(X, Open_X), (Y, Open_Y)$ are topological spaces. A set map $X \to Y$ is *continuous* iff the inverse map $2^Y \to 2^X$ induces a map $Open_Y \to Open_X$. The set-theoretic composition of continuous maps is continuous, the identity map is so, and therefore, we have the *category* **Top** *of topological spaces and continuous maps*. An isomorphism of topological spaces is called a *homeomorphism*. Any subset $W$ of a topological space becomes a topological space by the coarsest topology $Open_W = Open_X|W$ such that the inclusion $W \subset X$ is continuous; its open sets are just the intersections of open sets of $X$ with $W$, this topology is called the *relative topology on* $W$. More generally, given any set map $f : X \to Y$ into a topological space $(Y, Open_Y)$ the coarsest topology $Open_Y|f$ (smallest set of open sets) on $X$ such that $f$ becomes continuous is given by the set of inverse images of open sets of $Y$, we also call it the *relative topology with respect to* $f$. Conversely, for a set map $f : X \to Y$, where $(X, Open_X)$ is a topological space, we have a finest topology such that $f$ becomes continuous, it is given by the set of all subsets of $Y$ such that their inverse image is open in $X$. This is the *quotient topology* $Open_X/f$. If $(X_i)$ is a family of topological spaces, the cartesian product $\prod_i X_i$ has the coarsest topology such that the projections to all factors become continuous. A base of this *product topology* is given by the products $\prod_i U_i$ of open sets $U_i \subseteq X_i$ with $U_i = X_i$ except for a finite number of indices. This is a limit in the category **Top**. The coarsest topology on the set-theoretical limit $lim(\mathbb{D})$ of a diagram of continuous maps is the limit in **Top**, a similar construction (this time with the finest topology) yields the colimit of a diagram of continuous maps.

If we are given a family $f_i : X_i \to X$ of set maps whose domains are topological spaces, there is a finest topology which makes these maps continuous. Its universal property is that with this topology on $X$, a map $g : X \to Y$ into a topological space $Y$ is continuous iff all compositions $g \circ f_i$ are so. This is a particular case of a quotient topology for the situation $\coprod_i X_i \to X$. This topology is called the *coinduced topology*. If the maps $f_i$ are inclusions of subspaces $X_i$ of a topological space $X$, the topology of $X$ is called *coherent* or *weak* if it is coinduced from the relative topologies on the spaces $X_i$.

If we are given a set $X$, together with a collection of subsets $C_i$ of $X$ which are topological spaces such that for all indexes $i, j$, the intersections $C_i \cap C_j$ have the same relative topology as inherited from $C_i$ or from $C_j$, and that these intersections are closed in both, $C_i$ and $C_j$. Then the coinduced topology is coherent with this family, in other words, the coinduced topology relativizes to the given topologies on all $C_i$.

### H.1.3 Uniform Spaces

Topologies are often defined by relations that stem from metrical distance functions. The axiomatics is as follows:

**Definition 166** *A* uniformity *on a set $X$ is a set $\mathcal{U}$ of* uniform sets $U \subseteq X^2$ *such that:*

(i) *Each uniform set contains the diagonal $\Delta$.*

(ii) *If $U$ is uniform, so is $U^{-1}$.*

(iii) *If $U$ is uniform, then there is a uniform $V$ such that $V \circ V$.*

(iv) *If $U, V$ are uniform, then so is $U \cap V$.*

(v) *If $U$ is uniform, then so is every superset in $X^2$.*

The prototype of a uniformity is given by a distance function, i.e., a pseudo-metric $d : X \times X \to \mathbb{R}$ as defined in definition 171 in appendix I.1.1. The uniformity contains all $U \subseteq X^2$ which contain a set of type $U_\epsilon = \{(x, y) | d(x, y) < \epsilon\}, \epsilon > 0$.

Each uniformity $\mathcal{U}$ gives rise to a *uniform topology* $Open(\mathcal{U})$ whose open sets are those $V$ such that for each $x \in V$, there is a uniform set $U$ with $U[x] \subset V$, where $U[x] = \{y | (x, y) \in U\}$. So the uniform topology imitates metrical neighborhoods.

### H.1.4 Special Issues

**Definition 167** *A topological space $X$ is said to be:*

(i) *$T_0$ iff for any two different points $x, y \in X$, at least one of them is not the specialization of the other;*

(ii) *$T_1$ iff every point is closed, i.e., no other point dominates it;*

(iii) *$T_2$ (Hausdorff) iff every two different points have disjoint neighborhoods.*

**Definition 168** *A subset $L \subset X$ of a topological space $X$ is said to be* locally closed *iff one of the equivalent properties holds:*

(i) *$L = O \cap C$, $O$ open, $C$ closed.*

(ii) *Every point $l \in L$ has an open neighborhood $U_l$ such that $U_l \cap L$ is closed in $U_l$.*

(iii) *$L$ is open in its closure in $X$.*

See [65, I,§3.3] for a proof.

**Definition 169** *A topological space $X$ is called* quasi-compact *iff every covering of $X$ by open sets admits a finite subcovering. A Hausdorff quasi-compact space is called* compact.

Typically, prime spectra of commutative rings are quasi-compact but not compact.

## H.2 Algebraic Topology

Refer to [498] for this section.

### H.2.1 Simplicial Complexes

A *simplicial complex* $K$ is a set $V$ of *vertexes*, together with a subset $K$ of $2^V$ whose elements are called *simplexes* such that (1) each singleton $\{v\}, v \in V$ is a simplex, (2) each non-empty subset of a simplex is a simplex. If for a simplex $s$ of $K$, $card(s) = q + 1$, one says that $s$ is a *q-simplex* or a *q-dimensional simplex*. A subsimplex $s' \subseteq s$ of a simplex $s$ is called a *face of* $s$; it is called a *q-face* if it is a $q$-dimensional simplex, we also write $s' \leq s$ instead of $s' \subseteq s$. Evidently, a simplicial complex is completely determined by its simplex set $K$ and may be identified with it.

**Example 101** Let $U$ be a covering of a set $X$ by non-empty subsets. The *nerve* $n(U)$ of $U$ is the simplicial complex with $V = U$, and the simplexes $s$ being those finite sets $s = \{u_0, u_1, \dots u_p\}$ in $U$ which have non-empty intersection $\cap s = \bigcap_i u_i$.

The *dimension* $dim(K)$ of a simplicial complex $K$ is the maximal dimension of its simplexes, including the special cases $dim(\emptyset) = -1$ $dim(K) = \infty$ if no maximal dimension exists. A simplicial map $f : K_1 \to K_2$ is a set map $f : V_1 \to V_2$ on the underlying vertex sets such that the induced map $2^f : 2^{V_1} \to 2^{V_2}$ carries simplexes to simplexes, i.e., restricts to a map $f : K_1 \to K_2$, meaning that if $s \in K_1$, then $f(s) \in K_2$. One may also say that it is a set map $F : K_1 \to K_2$ which is induced by a map $f$ on the underlying vertex sets. The simplicial complexes and their simplicial maps define the *category* **Simpl** *of simplicial complexes*.

A *subcomplex* $L$ of a simplicial complex $K$ is a subset of simplexes which is also a simplicial complex. $L$ is *full* iff a simplex of $K$ whose vertexes belong to $L$ is also in $L$. For example, given a simplicial complex $K$ and a natural number $k$, the $k$-dimensional skeleton $K|k$ is the subcomplex of all simplexes of dimension $\leq k$. For a covering $U$, the $k$-dimensional skeleton of its nerve is denoted by $n_k(U)$.

**Example 102** Let $Covens$ be the category of set coverings, whose objects are pairs $(X, I)$ of sets $X$ and coverings $I$ of $X$ by non-empty subsets. The morphisms are pairs $(f, \phi) : (X, I) \to (Y, J)$ with $f : X \to Y, \phi : I \to J$ two maps such that for all $i \in I$, $f(i) \subset \phi(i)$. We then have the nerve functor $n : Covens \to$ **Simpl** $: (X, I) \mapsto n(I)$.

### H.2.2 Geometric Realization of a Simplicial Complex

We have a functorially defined geometric representation of simplicial complexes $K$ by topological spaces $|K|$ as follows. The set $|K|$ is the subset of those functions $\alpha : V(K) \to I = [0, 1]$ into the real unit interval $I$ such that

1. the support $supp(\alpha) = \{v \in V(K) | \alpha(v) \neq 0\}$ of $\alpha$ is a simplex,
2. $\sum_{v \in V(K)} \alpha(v) = 1$.

The value $\alpha(v)$ is called the $v^{th}$ *barycentric coordinate* of $\alpha$. On the set $I^{(V(K))}$ of functions with finite support, one has the Euclidean metric $d(\alpha,\beta) = \|\alpha - \beta\|_2$. We induce this metric and its associated topology (see section H.1.3) on $|K|$ and denote it by $|K|_d$.

For a simplex $s \in K$, the closed simplex $|s|$ is defined by

$$|s| = \{\alpha \in K | supp((\alpha) \subset s\}.$$

Evidently, if $dim(s) = q$, there is a homeomorphism $|s|_d \overset{\sim}{\to} \Delta^q = \{x \in | \ I^{q+1} | \sum x_i = 1\}$ onto the "standard closed $q$-simplex". If $s,t \in K$, either $s \cap t = \emptyset$ or a common face, and then $|s \cap t| = |s| \cap |t|$, so $|s|_d \cap |t|_d$ is closed in both, $|s|_d, |t|_d$, and the relative topologies from $|s|_d, |t|_d$ coincide on the intersection. By the remarks on coinduced topologies in section H.1.2, we have the *coherent topology on* $|K|$ which is coinduced from the topologies on the closed simplexes. This means that

**Fact 19** *A subset $E \subseteq |K|$ is closed/open iff each intersection $E \cap |s|_d$ is closed/open.*

Therefore, a function $f : |K| \to X$ into a topological space $X$ is continuous iff its restrictions $f||s|$ are so for all simplexes $s$ of $K$. In particular, the identity $|K| \to |K|_d$ is continuous, therefore, $|K|$ is Hausdorff, it is also normal, see [498, 3.1, Th.17]. Also, $|K|$ is compact iff $K$ is finite. Call $K$ *locally finite*, iff every vertex belongs to a finite number of simplexes. Then

**Theorem 73** *For a simplicial complex $K$ the following statements are equivalent:*

(i) *$K$ is locally finite.*

(ii) *The identity $|K| \to |K|_d$ is a homeomorphism.*

(iii) *$|K|$ is metrizable, i.e., there is a metric whose topology is the coherent topology.*

See [498, 3.2, Th.8] for a proof.

If $f : K_1 \to K_2$ is a simplicial map, we have the continuous map

$$|f|(\alpha)(v) = \sum_{f(w)=v} \alpha(w)$$

which is continuous for both topologies on $|K|$. We are therefore given two functors $|?|, |?|_d$ : **Simpl** $\to$ **Top** and a natural transformation $Id : |?| \to |?|_d$.

A continuous map $f : |K| \to X \subset \mathbb{R}^n$ is said to be *linear* iff $f(\alpha) = \sum_{v \in V(K)} \alpha(v) f(v)$ for all $\alpha \in |K|$. Any function on the vertexes may uniquely be extended to a continuous linear map, this is the universal property of affine pointsets in general position. In particular, the maps $|f|$ associated with a simplicial map $f$ is linear.

**Definition 170** *A* geometric realization *of a simplicial complex $K$ in $\mathbb{R}^n$ is a linear embedding (injection) of $|K|$ in $\mathbb{R}^n$.*

**Theorem 74** *If a simplicial complex $K$ has a geometric realization in $\mathbb{R}^n$, then it is countable, locally finite and has dimension $\leq n$. Conversely, if it is countable, locally finite, and has dimension $\leq n$, then it has a geometric realization as a closed subset of $\mathbb{R}^{2n+1}$.*

**Example 103** For the nerve $n(U)$ of a finite covering $U$, we write $N(U)$ for the geometric realization $|n(U)|$, we also write $N_k(U)$ for $|n_k(U)|$.

### H.2.3 Contiguity

A *simplicial pair* is a couple $(K, L)$, where $K$ is a subcomplex of $L$. A simplicial map of pairs $f : (K_1, L_1) \to (K_2, L_2)$ is a simplicial map $f : L_1 \to L_2$ which induces a simplicial map on the respective subcomplexes. Two simplicial maps $f, f' : (K_1, L_1) \to (K_2, L_2)$ are called *contiguous* if for every simplex $s$ in $K_1$ or $L_1$, the union $f(s) \cup f'(s)$ is a simplex in $K_2$ or $L_s$. Contiguity is an equivalence relation and defines *contiguity classes* of simplicial maps.

Two continuous maps $f, g : X \rightrightarrows Y$ of topological spaces are called *homotopic* iff there is a continuous map (a homotopy) $F : X \times I \to Y$ such that $f = F(?, 0), g = F(?, 1)$; the homotopy relation is an equivalence relation. If $X' \subseteq X$ is a subspace, and if $f|X' = g|X'$, a homotopy is called *relative to this subspace*, iff $F|X' \times t = f|X' = g|X'$, all $t \in I$.

**Lemma 96** ([498, lemma 2, p.130]) *Contiguous simplicial maps which agree on a subcomplex define contiguous maps which are homotopic relative to the space of the subcomplex.*

## H.3 Simplicial Coefficient Systems

A simplicial complex $K$ can be viewed as a category whose objects are the simplexes $s$ of $K$, and whose morphisms are the inclusions $s \subseteq t$ of simplexes. For a commutative ring $R$, a *coefficient system* of $R$-modules is a covariant functor $M : K \to {}_R\mathbf{Mod}$ with values in the category ${}_R\mathbf{Mod}$ of $R$-modules and affine homomorphisms. Let $\Delta_q = \{0, 1, 2, \dots q\}$ be the standard simplex of dimension $q$. A *singular simplex of dimension* $q$ is a simplicial map $s : \Delta_q \to K$, i.e., a sequence $s_0, s_1, \dots s_q$ of points in $K$ which define a simplex. If we have any set map $f : \Delta_p \to \Delta_q$, we have the singular $p$-simplex $\overline{f}(s) = s \circ f : \Delta_p \to K$. For a singular simplex $s$, we denote $M(s) = M(Im(s))$. Clearly $Im(\overline{f}(s)) \subseteq Im(s)$. Therefore we have an affine homomorphism $f_s : M(\overline{f}(s)) \to M(s)$.

Denote by $S_n(K)$ the set of singular simplexes of dimension $n$ in $K$. Then we have a module $C^n(K; M) = \prod_{s \in S_n(K)} M(s)$, whose elements are called the *singular cochains of dimension* $n$. For a map $f : \Delta_p \to \Delta_q$, we have an affine map

$$\overline{f} : C^p(K; M) \to C^q(K; M) \tag{H.1}$$

which has $\overline{f}((a_s)_{s \in S_p(K)}) = (b_t)_{t \in S_q(K)}$ and $b_t = f_t(a_{\overline{f}(t)})$. In other words, $C^*(K; M) = (C^n(K; M))_n$ is a *simplicial cochain complex*.

### H.3.1 Cohomology

Suppose now that the simplicial cochain complex stems from a system of coefficients with linear maps. Then all the transition maps of equation (H.1) are linear. Consider now the strictly increasing $i^{th}$-face maps $F_n^i : \Delta_{n-1} \to \Delta_n$ leaving aside index $i$ in $\Delta_n$, i.e., mapping $\Delta_{n-1}$ onto the subset $\{0, 1, 2, \dots \hat{i} \dots n\}$. Then we have the coboundary map

$$\begin{aligned} &d_n : C^n(K; M) \to C^{n+1}(K; M), \\ &d_n(a) = \sum_{j=0}^{n+1} (-1)^j \overline{F_{n+1}^j}(a), \end{aligned} \tag{H.2}$$

and $d_{n+1} \circ d_n = 0$. This means that $Im(d_n) \subseteq Ker(d_{n+1})$, and we may consider the *cohomology groups*

$$H^n(K; M) = Ker(d_n)/Im(d_{n-1}) \tag{H.3}$$

for $n \geq 0$, with the trivial extension to $C^{-1}(K; M) = 0$.

# Appendix I

# Complements on Calculus

## I.1 Abstract on Calculus

### I.1.1 Norms and Metrics

**Definition 171** *A* pseudo-metric *on a set $V$ is a* (pseudo-distance) *function $d : V \times V \to \mathbb{R}$ such that:*

1. *(Positivity) $0 \leq d(x, y)$, and $d(x, x) = 0$ for all $(x, y) \in V \times V$;*
2. *(Symmetry) $d(x, y) = d(y, x)$ for all $(x, y) \in V \times V$;*
3. *(Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $(x, y, z) \in V \times V \times V$.*

*If conversely $d(x, y) = 0$ implies $x = y$, the pseudo-metric* (pseudo-distance function) *is called a* metric (distance function).

**Definition 172** *For a pseudo-metric space $(X, d)$, if $0 < r, x \in X$, the* open ball *of radius $r$ around $x$ is $B_r(x) = \{y | d(y, x) < r\}$. The system of open balls*

$$\{B_r(x) | 0 < r, x \in X\}$$

*is a base of a topology*[1]*, the* (uniform) topology associated with the pseudo-metric *$d$. Evidently, this topology is Hausdorff iff the pseudo-metric is a metric.*

*A map $f : V \to V$ of a pseudo-metric space $V$ is called an* isometry *iff $d(f(x), f(y)) = d(x, y)$, for all $(x, y) \in V \times V$.*

**Lemma 97** ([73, Lemma 4]) *Given an action $\mu : G \times V \to V$ of a group $G$ on a pseudo-metric space $(V, d)$ by isometries, then*

$$\inf_{g \in G} d(g.x, y) = \inf_{g \in G} d(g.x', y) \quad \text{whenever } G.x = G.x', \tag{I.1}$$

$$\inf_{g \in G} d(g.x, y) = \inf_{g \in G} d(g.x, y') \quad \text{whenever } G.y = G.y'. \tag{I.2}$$

[1] In fact, the system $\{B_r = \{(x, y) \in V^2 | d(x, y) < r\} | 0 < r\}$ is a base of a uniformity.

With the above notation, we may define

$$d^*(G.x, G.y) = \inf_{g \in G} d(g.x, y), \tag{I.3}$$

and lemma 97 guarantees that this is a well-defined function $d^* : G\backslash V \times G\backslash V \to \mathbb{R}$.

**Definition 173** *If $d(x, y)$ is a pseudo-metric on a set $V$, and if we have a group action $G \times V \to V$, we say that $g$ acts by isometries, iff each map $g. : V \to V$ is an isometry, i.e., iff $d(g.x, g.y) = d(x, y)$, for all $x, y \in V, g \in G$.*

**Lemma 98** ([73, Lemma 5]) *Let $d$ be a pseudo-metric on $V$, and $\mu : G \times V \to V$ a group action by isometries. Then the function $d^*$ defined in (I.3) is a pseudo-distance on the orbit space $G\backslash V$.*

### I.1.2 Completeness

A *Cauchy sequence* in a uniform space $(X,\mathcal{U})$ is a sequence $(x_i)_{i=0,1,2,\ldots}$ of elements in $X$ such that for every uniform set $U \in \mathcal{U}$, there is an index $t$ such that $(x_i, x_j) \in U$ for all $i, j > t$. A uniform space is (sequentially) *complete* iff every Cauchy sequence converges.

**Lemma 99** *A closed subspace of a complete uniform (in particular: a metric space) space is complete.*

**Definition 174** *A* norm *on real vector space $X$ is a function $\| \ \| : X \to \mathbb{R}$ such that for all $(x, y) \in V \times V$:*

1. *(Positivity) $0 \leq \|x\|$, and $\|x\| = 0$ iff $x = 0$;*
2. *(Homogeneity) $\|\lambda.x\| = |\lambda|.\|x\|$;*
3. *(Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$.*

*Every norm gives rise to an* associated metric *$d(x, y) = \|x - y\|$, and therefore to an* associated topology*. A normed vector space with a complete associated (uniform) topology is called a* Banach space*.*

**Example 104** On $\mathbb{R}^n$, we have three well-known norms. If $x = (x_1, \ldots x_n) \in \mathbb{R}^n$, then

1. the absolute or 1-norm is $\|x\|_1 = \sum_i |x_i|$,
2. the Euclidean norm is $\|x\|_2 = \sqrt{\sum_i x_i^2}$,
3. the uniform norm is $\|x\|_\infty = max\{|x_i|| \ i = 1, \ldots n\}$.

For real numbers $a < b$, we have the vector space $C^0[a, b]$ of continuous real-valued functions on the interval $[a, b]$. On $C^0[a, b]$, we have three well-known norms (corresponding to the above three norms). For $f \in C^0[a, b]$, we have:

1. the absolute or 1-norm is $\|f\|_1 = \int_a^b |f|$,
2. the Euclidean norm is $\|f\|_2 = (\int_a^b f^2)^{1/2}$,
3. the uniform norm is $\|f\|_\infty = Max_{[a,b]}|f|$.

Two norms $\| \ \|_1, \| \ \|_2$ on a real vector space $X$ are called equivalent iff there are two positive constants $a, b$ such that $\| \ \|_1 \leq a.\| \ \|_2, \| \ \|_2 \leq b.\| \ \|_1$. Equivalent norms give rise to the same associated uniformities and topologies, so they have the same Cauchy sequences.

**Theorem 75** *Any two norms on a finite-dimensional real vector space are equivalent.*

See [307, Th.3.4.1] for a proof. The theorem implies that every finite-dimensional normed real vector space is Banach, since the standard $\mathbb{R}^n$ is so under the Euclidean norm. We shall therefore mainly work in $\mathbb{R}^n$.

## I.1.3 Differentiation

We say that two functions $f, g : U \to \mathbb{R}^m$ that are defined in a neighborhood $U$ of $0 \in \mathbb{R}^n$ define the same *germ* iff they coincide on a common neighborhood of 0. (We are in fact considering the colimit of function spaces on the neighborhood system of 0.) The set of germs in 0 of functions $f$ with $f(0) = 0$ is denoted by $F_0$. Within this vector space, we have the vector subspace $DF_0$ of those $f$ with $f(0) = 0$ and $\|f(z)\|/\|z\| \to 0$ if $z \to 0$. We evidently have $Lin_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m) \cap DF_0 = \{0\}$.

**Definition 175** *A function $f : U \to \mathbb{R}^m$ which is defined in a neighborhood $U \subseteq \mathbb{R}^n$ of a point $x$ is* differentiable in $x$ *iff there is a linear map $D \in Lin_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$ such that $\Delta_x f - D \in DF_0$, where $\Delta_x f(z) = f(x+z) - f(x)$. By the above, $D$ is uniquely determined and is denoted by $Df_x$. The coefficient of row $i$ and column $j$ of the matrix of $Df_x$ in the canonical basis is denoted by $\partial f_i/\partial x_j$, whereas the matrix is called the* Jacobian of $f$ in $x$. *A function $f : O \to V$ on an open set $O \subseteq \mathbb{R}^n$ with values in an open set $V \subseteq \mathbb{R}^m$ is differentiable if it is differentiable in each point of its domain $O$.*

A differentiable function on $O$ defines its *derivative* $Df : O \to Lin_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m) \tilde{\to} \mathbb{R}^{nm}$, which may again be differentiated according to the norm on the space of linear maps. Inductively we define $D^{t+1}f = D(D^t f)$, if it exists. The function $f$ is $\mathcal{C}^r$ iff all derivatives $Df, D^2f, \ldots D^r f$ exist and are continuous, $\mathcal{C}^0$ denotes just the set of continuous functions. This definition is however not in the right shape for functorial behavior. One therefore adds the linear behavior to the function as follows: Let $TO = O \times \mathbb{R}^n$ be the *tangent bundle* of the open set $O$. Then we define $Tf : TO \to T\mathbb{R}^n$ by $Tf(x, u) = (f(x), Df_x(u))$. This implies that if $g : U \to \mathbb{R}^l$ is a second differentiable function on an open set $U \subseteq \mathbb{R}^m$, then $g \circ f$ is differentiable and

$$T(g \circ f) = Tg \circ Tf.$$

So we have a functor $T : f \mapsto Tf$ and the natural transformation $pr_1 : T \to Id$ of first projection. More generally, defining $T^{r+1}f = T(T^r f)$, we also have

$$T^r(g \circ f) = T^r g \circ T^r f.$$

Moreover, if we identify $Lin_{\mathbb{R}}(\mathbb{R}^n, Lin_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m))$ with $Bil_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m))$, etc. for higher multilinear maps, the higher derivatives $D^r f_x$ identify to $r$-linear maps $(\mathbb{R}^n)^r \to \mathbb{R}^m$.

**Proposition 115** *If $f$ is $\mathcal{C}^r$, then $D^r f_x$ is a symmetric $r$-linear matrix.*

The category of $r$ times differentiable or $\mathcal{C}^r$ functions has the property that the linear parts of the tangent maps compose as normal linear maps do, and this means that the Jacobians of isomorphisms are invertible quadratic matrices.

A curve in $\mathbb{R}^n$ is a $\mathcal{C}^1$-map $y : U \to \mathbb{R}^n$. Its derivative $Dy_t$ in a point $t \in U$ is a linear map $\mathbb{R} \to \mathbb{R}^n$ which identifies to the image of 1 in $Dy_t(1) \in \mathbb{R}^n$, meaning that the derivative can be identified with a continuous map $y' : U \to \mathbb{R}^n : t \mapsto y'(t) = Dy_t(1)$.

## I.2 Ordinary Differential Equations (ODEs)

Throughout this section, $D$ denotes an open set in $\mathbb{R}^n$, and $f : D \to \mathbb{R}^n$ denotes a continuous vector field (a function) with components $f_i, i = 1, \ldots n$.

**Definition 176** *Let $\zeta \in \mathbb{R}, \eta \in D$, $J(\zeta)$ an open interval containing $\zeta$, and $U(\eta) \subset D$ an open neighborhood of $\eta$ in $D$. Denote by $A(f, \zeta, \eta, J(\zeta), U(\eta))$ the set of all $C^1$-functions $y : J(\zeta) \to U(\eta)$ such that*

$$y' = f \circ y \text{ and } y(\zeta) = \eta. \tag{I.4}$$

*Denote by $B(f, \zeta, \eta, J(\zeta), U(\eta))$ the set of all $C^1$-functions $y : J(\zeta) \to U(\eta)$ such that*

$$y = \eta + \int_{\zeta}^{?} f \circ y. \tag{I.5}$$

**Lemma 100** *With the above definitions, we have*

$$A(f, \zeta, \eta, J(\zeta), U(\eta)) = B(f, \zeta, \eta, , J(\zeta), U(\eta)).$$

The easy proof is left to the reader.

### I.2.1 The Fundamental Theorem: Local Case

The following theorem is called the local case of the fundamental theorem of ordinary differential equations.

**Theorem 76** *With the preceding notation and definitions, suppose that $f$ is locally Lipschitz, i.e., for every $x \in D$, there is a neighborhood $U(x) \subset D$ and a positive number $L$ such that $x_1, x_2 \in U(x)$ implies $|f(x_1) - f(x_2)| \leq L.|x_1 - x_2|$. Then for any "initial condition" $\zeta \in \mathbb{R}, \eta \in D$, there is an open interval $J(\zeta)$ containing $\zeta$, and an open neighborhood $U(\eta)$ of $\eta$ such that $A(f, \zeta, \eta, J(\zeta), U(\eta))$ is a singleton. The element of $A$ is called the* local solution of the differential equation $y' = f \circ y$ *at* $J(\zeta), U(\eta)$.

The proof uses lemma 100 and refers to the set $B$. In fact, it is shown that the operator

$$T_{\zeta,\eta,f} = \eta + \int_{\zeta}^{?} f \circ y$$

is a contraction, and contractions have a unique fixpoint.

**Proposition 116** *Let $T : X \to X$ be a contraction on a complete metric space[2] $(X, d)$, i.e., there is a constant $0 < c < 1$ such that $d(T(x), T(y)) \leq c.d(x, y)$ for all $x, y \in X$. Then, $T$ has a unique fixpoint $z = T(z)$.*

Proof. It suffices to show that the sequence $(x_n = T^n(x))$ is Cauchy. In fact, setting $k = |n-m|$, we have $d(x_n, x_m) = d(T^n(x), T^m(x)) = c^{Min(n,m)}.d(x, T^k)$. But

$$\begin{aligned} d(x, T^k) &\leq d(x, T(x)) + d(T(x), T^2(x)) + \ldots d(x^{k-1}, T^k) \\ &\leq (1 + c + \ldots c^{k-1}) d(x, T(x)) \leq \frac{1}{1-c} d(x, T(x)). \end{aligned}$$

So this term is limited, while $c^{Min(n,m)}$ tends to zero as $n, m$ tend to infinity, QED.

**Corollary 30** *Let $X$ be a complete metric space, and $B = \overline{B_r(x)}$ the closed ball of radius $r > 0$ around $x$. Let $T : B \to X$ a contraction with $d(T(x), x) \leq (1-c)r, 0 < c < 1$. Then $T$ has a unique fixpoint in $B$.*

Proof. We know from lemma 99 that $B$ is complete. Further, for $y \in B$, we have $d(T(y), x) \leq d(T(y), T(x)) + d(T(x), x) \leq c.d(y, x) + (1-c)r \leq r$. Therefore, $T$ leaves $B$ invariant and the claim follows from proposition 116, QED.

**Corollary 31** *With the notation of corollary 30, suppose that $T : B_r(x) \to X$ is a contraction with $d(T(x), x) < (1-c)r, 0 < c < 1$. Then there is a unique fixpoint of $T$ in $B_r(x)$.*

Next, we need some auxiliary results concerning uniform convergence of continuous functions. Let $W$ be a Banach space (in our case $W = \mathbb{R}^n$), $A$ a set, then we set

$$B(A, W) = \{f : A \to W | \|f\|_\infty < \infty\}.$$

**Proposition 117** *The set $B(A, W)$ with the usual scalar multiplication and addition of functions is a Banach space.*

It is clear that $B(A, W)$ is a vector space. Let $(f_n)_n$ be a Cauchy sequence in $B(A, W)$. Since for any $x \in A$, $\|f_n(x) - f_m(x)\| \leq \|f_n - f_m\|$, and the right term converges to zero, the left term is also a Cauchy sequence in $W$ and converges to $lim_{n\to\infty} f_n(x) = f(x)$. We first show that $lim_{n\to\infty} f_n = f$. For $0 < \epsilon$, let $N$ be such that $n, m > N$ implies $\|f_m - f_n\| > \epsilon$. Then by definition, for all $x \in A$, $\|f_n(x) - f(x)\| = \|f_n(x) - lim_{m>N} f_m(x)\| = lim_{m>N} \|f_n(x) - f_m(x)\| \leq \epsilon$. Therefore $\|f_n - f\| \leq \epsilon$, and $f = (f - f_n) + f_n$ is a sum of two elements of $B(A, W)$ and therefore lives in $B(A, W)$, whereas $lim_{n\to\infty} f_n = f$, QED.

[2] See this appendix, section I.1.1.

**Theorem 77** *Let $A$ be a metric space, $W$ a Banach space, and let*

$$BC(A,W) = B(A,W) \cap C^0(A,W)$$

*be the set of continuous functions with limited norm. Then $BC(A,W) \subset B(A,W)$ is a closed sub-vector space, and therefore also Banach.*

Proof. It is clearly a sub-vector space. Let $(f_n)$ be a Cauchy sequence in $BC(A,W)$. It converges to $f$ in $B(A,W)$. We have to show that it is also continuous. In fact, given $0 < \epsilon$ select $n$ such that $\|f - f_n\| < \epsilon/3$. Let $a \in A$. Take $0 < \delta$ such that $d(x,a) < \delta$ implies $\|f_n(x) - f_n(a)\| < \epsilon/3$. Then $\|f(x) - f(a)\| \leq \|f(x) - f_n(x)\| + \|f_n(x) - f_n(a)\| + \|f_n(a) - f(a)\| < \epsilon/3 + \epsilon/3 + \epsilon/3$, QED.

We are now ready for the proof of the local theorem. Recall that we are given a locally Lipschitz vector field function $f : D \to \mathbb{R}^n$. Consider the Banach space $BC = BC(J(\zeta), \mathbb{R}^n)$ for an interval $J(\zeta)$ whose length $\delta$ will be determined in the course of the proof. Select $0 < r$ such that (1) the closed ball $B_r(\eta)^- \subset D$, and (2) $f|B_r(\eta)^-$ is Lipschitz with a constant $L$. Then $f$ is evidently limited on $B_r(\eta)^-$, let $m$ be an upper bound. Let $\bar{\eta} : J(\zeta) \to B_r(\eta)^- : t \mapsto \eta$ be the constant map. Consider the closed ball $B_r(\bar{\eta})^- \subset BC$ around $\bar{\eta}$. For every $g \in B_r(\bar{\eta})^-$, $f \circ g : J(\zeta) \to \mathbb{R}^n$ lives in $BC$.

We now show that the operator $T(g) = \eta + \int_\zeta^? f \circ g$ defines a contraction

$$T : B_r(\bar{\eta})^- \to BC$$

with contraction constant $c$ such that $d(T(\bar{\eta}), \bar{\eta}) < (1-c)r$. According to corollary 30, this will imply that $T$ has a unique fixpoint in $B_r(\bar{\eta})^-$ and we are done.

Evidently, $T(g)$ is continuous. Further, for any $x \in J(\zeta)$, we have $|T(g)(x)| \leq |\eta| + |\int_\zeta^x f \circ g| \leq |\eta| + |x - \zeta|.\|f \circ g\|_\infty$ which evidently is finite.

We are left with the contraction claims. We have

$$\begin{aligned} \|T(\bar{\eta}) - \bar{\eta}\|_\infty &= lub_{J(\zeta)} \|T(\bar{\eta})(t) - \bar{\eta}(t)\| = lub_{J(\zeta)} \left\| \int_\zeta^t f(\eta) \right\| \\ &= lub_{J(\zeta)} |t - \zeta|.|f(\eta)| \leq \delta.|f(\eta)| \leq \delta.m. \end{aligned}$$

For two functions $g_1, g_2 \in B_r(\bar{\eta})^-$, we have

$$\begin{aligned} \|T(g_1) - T(g_2)\| &= lub_{J(\zeta)} \left| \int_\zeta^t f(g_1) - f(g_2) \right| \leq \delta.\|f \circ g_1 - f \circ g_2\| \\ &= \delta.lub_{J(\zeta)} |f(g_1(s)) - f(g_2(s))| \leq \delta.L.lub_{J(\zeta)} |g_1(s) - g_2(s)| \\ &= \delta.L.\|g_1 - g_2\|. \end{aligned}$$

This means that $T$ is a contraction with $c = \delta.L$ if $\delta$ is such that $\delta.L < 1$. Further, we need $\delta.m < (1-c)r = (1 - \delta.L)r$, i.e., $\delta < \frac{r}{m + Lr}$ solves the problem, QED.

### I.2.2 The Fundamental Theorem: Global Case

The global fundamental theorem deals with maximal integral curves $y : J \to D$ for the differential equation $y' = f \circ y$.

**Definition 177** *We say that $u \sim v$ for $x, y \in D$ iff there is a curve $y : J \to D$, defined on an open interval $J$ for the differential equation $y' = f \circ y$, and such that $\{u, v\} \subset y(J)$.*

**Lemma 101** *The relation $\sim$ is an equivalence relation. The equivalence class of an element $x \in D$ is denoted by $[x]$.*

It is clearly reflexive and symmetric. It is transitive for the following reason. Let $y_i : J_i \to D, i = 1, 2$ be two integral curves such that $y_1(t_1) = x, y_1(t_2) = y, y_2(t_3) = y, y_2(t_4) = z$. By an evident parameter shift, we may suppose $t_2 = t_3$. We claim that

$$y_1|J_1 \cap J_2 = y_2|J_1 \cap J_2.$$

Suppose that $y_1(t) \neq y_2(t)$ for a $t > t_2$. Let $t_2 \leq t_0$ be the infimum of these $t$. Since our curves are continuous, we have $y_1(t_0) = y_2(t_0)$. But then, according to the local theorem 76, there is an $\epsilon$-ball $U_\epsilon(t_0)$ around $t_0$ and a neighborhood $U(y_1(t_0) = y_2(t_0))$ such that there is a unique integral curve $y : U_\epsilon(t_0) \to U(y_1(t_0))$. But we may suppose WLOG that $\epsilon$ is so small that both $y_1|U_\epsilon(t_0), y_2|U_\epsilon(t_0)$ have their codomains in $U(y_1(t_0))$. Evidently, these solutions must then coincide with the unique solution on the open interval $U_\epsilon(t_0)$, but this contradicts the choice of $t_0$. A symmetric argument holds for the supremum $s_0 \leq t_2$ of those arguments with $y_1(t) \neq y_2(t)$. Therefore, $y_1|J_1 \cap J_2 = y_2|J_1 \cap J_2$,and we may extend the integral curves $y_1, y_2$ to the domain $J_1 \cup J_2$, whence the transitivity of the $\sim$-relation, QED.

**Theorem 78** *Let $x \in D$. Then there is a unique integral curve $y : J \to D$ with $y(0) = x, y' = f \circ y$, and such that $J$ contains all domains of any integral curve $z, z(0) = x, z' = f \circ z$. We have $y(J) = [x]$ and write $\int_x f$ for this curve; it is called the* global solution through $x$.

Proof. Let $\Gamma = \{\Gamma_{y_i} \subset \mathbb{R} \times D | \Gamma_{y_i} =$ graph of solution $y_i$ of $y_i' = y_i \circ f, x = y_i(0)\}$. Since two solutions coincide on the intersection of their domains, the union $\bigcup \Gamma$ is functional, and the union of the domains is an open interval $J$. Further, the function $y$ of this graph is a solution of the differential equation $y' = f \circ f$ which reaches all elements equivalent to $x$, QED.

**Corollary 32** *Let $x_1 \sim x_2$ and $\int_{x_1} f(t_2) = x_2$. Then $\int_{x_2} f = \int_{x_1} f \circ e^{t_2}$ and $J_2 = e^{-t_2} J_1$.*

**Definition 178** *The quotient $D/\sim = \{\int_x f | x \in D\}$ is called* the phase portrait of the vector field $f$ *and denoted by $D/f$. An integral curve which is not an injective function of its parameter is called* a cycle of the field $f$.

**Proposition 118** *Let $\int_w f$ be a cycle with $\int_w f(t_1) = \int_w f(t_1 + T)$. Then the cycle's domain is $\mathbb{R}$ and $\int_w f$ is $T$-periodic.*

Proof. Let $y : J \to D$ be the cycle $\int_w f$ with $y(t_1) = z$. Consider the function $\hat{y} = y \circ e^T : J - T \to D$. Evidently, $\hat{y}(t_1) = y(t_1)$, and $\hat{y}$ also solves the differential equation since $\hat{y}'(t) = y'(t + T) = f \circ y(t + T) = f \circ \hat{y}(t)$. So, since $\hat{y}$ has a common value with $y$ at$t_1$, by maximality of $y$, we have $\hat{y} = y|J - T$, and $J - T \subset J$, whence $J = [-\infty, b[$. Symmetrically, exchanging $t_1$ with $t_1 + T$, and $T$ with $-T$, we obtain $J = [a, \infty[$, i.e., $J = \mathbb{R}$. Now, for any $t \in \mathbb{R}$, with $\hat{y} = y \circ e^T$, uniqueness guarantees $\hat{y} = y$, whence the periodicity of $y$, QED.

**Proposition 119** *Suppose that $D^-$ is compact (e.g.: $D$ is bounded), and that $f$ is locally Lipschitz on $D^-$. If the domain $J = ]a, b[$ of a maximal curve $\int_x f$ has finite upper bound $b$, then $t \to b$ implies $\int_x f(t) \to \partial D$*

Sketch of proof: Write $y = \int_x f$, and suppose that the closure of $y(J)$ were in $D$. Then, since $y(J)^-$ is compact, there is a convergent sequence $t_n \to b$ with a convergent image sequence $y(t_n) \to q, q \in D$. It can be shown that $q$ is uniquely determined, i.e., another such sequence yields the same limit. We then set $y(b) = lim_{t \to b} y(t) = q$, and $y$ may be extended to a local solution containing $b$, a contradiction, QED.

### I.2.3 Flows and Differential Equations

On an open set $O \subseteq \mathbb{R}^n$, a vector field (a $\mathcal{C}^1$-map) $f : O \to \mathbb{R}^n$ can also be viewed by its graph as a section $O \to TO : x \mapsto F(x) = (x, f(x))$. If $x \in O$, an *integral curve of $F$ in $x$* is a curve $y : U(0) \to O$ defined on an open neighborhood $U(0)$ of 0 such that $y(0) = x$ and $y' = f \circ y$. By the main theorem 78 of ODEs, there is a unique maximal integral curve $\int_x f$ for every point $x \in O$.

For a vector field $F$ on $O \subseteq \mathbb{R}^n$, a *flow box* is a triple $(U, a, W)$ where $U \subseteq O$ is open, $a$ is a positive real number of $\infty$, and $W : U \times ]-a, a[ \to O$ is $\mathcal{C}^1$ such that for all $x \in U$, $W_x : ]-a, a[ \to O : t \mapsto W(x, t)$ is an integral curve of $F$ at $x$. Two flow boxes $(U, a, W), (U', a', W')$ always coincide in their maps $W, W'$ on the intersection $(U \cap U') \times (]-a, a[ \cap ]-a', a'[)$ of their domains. For each point $x \in O$, there is a flow box $(U, a, W)$ with $x \in U$. Let $\mathcal{D}_F = \{(x, t) \in \mathbb{R} \times O |$ there is an integral curve $\int_x$ whose domain contains $t\}$. Then (1) $\mathcal{D}_F$ is open in $\mathbb{R} \times \mathbb{R}^n$; (2) there is a unique map $W_F : \mathcal{D}_F \to O$ such that $t \mapsto W_F(t, x)$ is an integral curve at $x$ for all $x \in O$.

### I.2.4 Vector Fields and Derivations

For a $\mathcal{C}^1$-function $f : O \to \mathbb{R}$, we have the derivative $Tf : TO \to T\mathbb{R}$, whose second component evaluates to linear forms on $\mathbb{R}^n$. This map $df = pr_2 \circ Tf = Df$ is called the *differential of $f$*. If $F : O \to TO$ is a vector field, the composition $L_F f = df \circ F : O \to \mathbb{R}$ is called the *Lie derivative of $f$ with respect to $F$*. If we denote by $grad(f)$ the differential of $f$ as a tangent vector $(\partial_{x_1} f, \ldots \partial_{x_n} f)$ (the old-fashioned gradient of $f$), the Lie derivative is just the scalar product of $grad(f)$ with the vector field. If $\mathcal{F}(O)$ denotes the real algebra[3] of $\mathcal{C}^1$-function on $O$, the map $L_F : \mathcal{F}(O) \to \mathcal{F}(O)$ is a *derivation* in the sense that:

*(i)* $L_F$ is linear;

*(ii)* for $f, g \in \mathcal{F}(O)$, we have $L_F(f.g) = f.L_F(g) + L_F(f).g$;

*(iii)* If $c \in \mathcal{F}(O)$ is constant, then $L_F c = 0$.

Therefore, we also have $d(f.g) = df.g + f.dg$ and $dc = 0$ for a constant $c$. Denote by $VF(O)$ the vector space of all $\mathcal{C}^1$-vector fields on $O$. Then:

[3]Multiplication goes pointwise.

**Theorem 79** *The Lie map*

$$L_? : VF(O) \to Der(\mathcal{F}(O)) : F \mapsto L_F$$

*is an isomorphism of vector spaces.*

See [2, Th.8.10] for a proof.

In particular, the Lie bracket $[L_F, L_G] = L_F \circ L_G - L_G \circ L_F$ which is a derivation, must be the Lie derivative of a unique vector field which is denoted by $[F, G]$, the *Lie bracket of the vector fields $F$ and $G$*. The Lie bracket makes the vector space $VF(O)$ into a real Lie algebra, see section E.4.4.

## I.3 Partial Differential Equations

For this section, refer to [252].

We only need a short review of quasi-linear first order partial differential equations (PDE). Recall that a PDE is an equation of type $E(x_1, x_2, \dots u, u_{x_1}, u_{x_2}, \dots u_{x_1x_1}, u_{x_1x_2}, \dots) = 0$ where $u$ is a function of the $n$ real variables $x_1, x_2, \dots x_n$, with its partial derivatives $u_{x_1}, \dots$, the higher partial derivatives $u_{x_1x_1} \dots$ etc. A solution is meant to be such a function $u$ which is defined in an open set $O$ of $\mathbb{R}^n$. Its *order* $m$ is the highest number of iterated partial derivatives, whereas $E$ is called *quasi-linear* iff it is an affine function of the derivatives of $u$ of highest order $m$, with coefficients that are functions of the variables $x_1, x_2, \dots u, u_{x_1} \dots$ until derivatives of order $m-1$.

A first-order quasi-linear PDE has the shape

$$\sum_i a_i(x_1, x_2, \dots u) u_{x_i} = c(x_1, x_2, \dots u)$$

and can be solved by a system of ODEs, this is the *method of characteristics*.

We illustrate the method for two variables, i.e., for the equation

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u). \tag{I.6}$$

The solution $u(x, y)$ is represented as a surface $z = u(x, y)$ in $\mathbb{R}^3$. Such a surface is called an *integral surface* of the equation (I.6). We have a vector field

$$F(x, y, z) = (a(x, y, z), b(x, y, z), c(x, y, z))$$

on the common domain $U$ of the three functions $a, b, c$. The tangent space of an integral curve at $x, y, u(x, y)$ is spanned by the vectors $X = (1, 0, u_x)$ and $Y = (0, 1, u_y)$. Their vector product $Y \wedge X = (u_x, u_y, -1)$ is the normal vector to the integral surface. Therefore equation (I.6) just means that the scalar product $(F(x, y, z), Y \wedge X)$ vanishes identically, i.e., the vector field $F$ is tangent to the integral surface. Clearly, only the direction of the vectors of the vector field $F$, the *characteristic directions* matter for the equation (I.6). It is easily seen that an integral curve of $F$, if it crosses a point of an integral surface, is entirely contained in this surface. Therefore an *integral surface is the union of integral curves of the directional vector field $F$*. An integral surface can be constructed by finding a curve $\Gamma$ which lies in an integral surface, and which

is never parallel to an integral curve of $F$. This is the *Cauchy problem for the equation* (I.6). Then, the parameter of $\Gamma$ and the curve parameter of a flow box (see section I.2.3) around $\Gamma$ describe the integral surface. Technically, the existence condition for $\Gamma(t) = (\Gamma_x(t), \Gamma_y(t), \Gamma_z(t))$ to generate a surface is that the projection $\Gamma_{xy}(t) = (\Gamma_x(t), \Gamma_y(t))$ is never parallel to the projection $F_{xy}$ of the directional field on the $xy$ plane. The existence of a curve $\Gamma$ is again guaranteed by the main theorem of ODEs, and we are done.

# Part XVII

# Appendix: Tables

# Appendix J

# Euler's Gradus Function

This table lists the rational numbers $x/y$ with Euler's gradus suavitatis $\Gamma(x/y) \leq 10$, see also [71].

| $\Gamma$ | Intervals |
|---|---|
| 2 | 1/2 |
| 3 | 1/3, 1/4 |
| 4 | 1/6,2/3,1/8 |
| 5 | 1/5,1/9,1/12,3/4,1/16 |
| 6 | 1/10,2/5,1/18,2/9,1/24,3/8,1/32 |
| 7 | 1/7,1/15,3/5,1/20,4/5,1/27,1/36,4/9,1/48,3/16,1/64 |
| 8 | 1/14,2/7,1/30,2/15,3/10,5/6,1/40,5/8,1/54,2/27,1/72,8/9,1/96,3/32,1/128 |
| 9 | 1/21,3/7,1/25,1/28,4/7,1/45,5/9,1/60,3/20,4/15,5/12,1/80,5/16,1/81,1/108, 4/27,1/144,9/16,1/192,3/64,1/256 |
| 10 | 1/42,2/21,3/14,6/7,1/50,2/25,1/56,7/8,1/90,2/45,5/18,9/10,1/120,3/40,5/24, 8/15,1/160,5/32,1/162,2/81,1/216,8/27,1/288,9/32,1/384,3/128,1/512 |

# Appendix K

# Just and Well-Tempered Tuning

This table lists the just coordinates of the just tuning intervals (with respect to c, second tone in first column) according to Vogel [547], see subsubsection 7.2.1.2, together with the value in Cents, and the deviation in % from the tempered tuning with 100, 200, 300, etc. Cents.

| Tone name | Frequency ratio | Octave coord. | Fifth coord. | Third coord. | Pitch ($Ct$) | % deviation |
|---|---|---|---|---|---|---|
| $c$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $d_\flat$ | 16/15 | 4 | -1 | -1 | 111.73 | +11.73 |
| $d$ | 9/8 | -3 | 2 | 0 | 203.91 | +1.96 |
| $e_\flat$ | 6/5 | 1 | 1 | -1 | 315.65 | +5.22 |
| $e$ | 5/4 | -2 | 0 | 1 | 386.31 | -3.42 |
| $f$ | 4/3 | 2 | -1 | 0 | 498.05 | -0.39 |
| $f_\sharp$ | 45/32 | -5 | 2 | 1 | 590.22 | -1.63 |
| $g$ | 3/2 | -1 | 1 | 0 | 701.96 | +0.28 |
| $a_\flat$ | 8/5 | 3 | 0 | -1 | 813.69 | +1.71 |
| $a$ | 5/3 | 0 | -1 | 1 | 884.36 | -1.74 |
| $b_\flat$ | 16/9 | 4 | -2 | 0 | 996.09 | -0.39 |
| $b$ | 15/8 | -3 | 1 | 1 | 1088.27 | -1.07 |

# Appendix L

# Chord and Third Chain Classes

## L.1 Chord Classes

This section contains the list of all isomorphism classes of zero-addressed chords in $PiMod_{12}$. The meanings of the column items are explained in subsection 11.3.7; here we give a short definition.

- *Class Nr.* is the number of the isomorphism class, numbers with extension ".1" indicate the class number for classification under symmetries from $\mathbb{Z}$ (no fifth or fourth transformations). Autocomplementary classes have a star after the number.
- *Representative* of Nr. without hat is the number's representative in full circles, the one with hat is the complementary chord.
- *Group of symmetries* is $Sym(Nr.)$. To keep notation readable, we use the notation with linear factor to the left.
- *Conj. Class* denotes the conjugacy class symbol of $Sym(Nr.)$ and refers to the numbering $1, 2, \ldots 19$ from [402].
- *Card. End. Cl. Nr.*$|\widehat{Nr.}$ is the pair of numbers of conjugacy classes of endomorphisms in *Nr.* and in its complement $\widehat{Nr.}$, respectively.

| **Chord Classes** | | | | |
|---|---|---|---|---|
| *Class Nr.* | *Representative* $Nr. = \bullet,\ \widehat{Nr.} = \circ$ | *Group of Symmetries* | *Conj. Class* | *♯ End.* $Nr.\vert\widehat{Nr.}$ |
| 1 | ●●●●●●●●●●●● | $\overrightarrow{GL}(\mathbb{Z}_{12})$ | 19 | 28\|28 |
| *One/Eleven Element* | | | | |
| 2 | ●○○○○○○○○○○○ | $\mathbb{Z}_{12}^{\times}$ | 8 | 1\|31 |
| *Two/Ten Elements* | | | | |
| 3 | ●●○○○○○○○○○○ | $\langle -1e^{-1} \rangle$ | 3 | 3\|23 |
| 3.1 | ●○○○○●○○○○○○ | | | |
| 4 | ●○●○○○○○○○○○ | $\{1, 7, -1e^{-2}, 5e^{-2}\}$ | 8 | 3\|25 |
| 5 | ●○○●○○○○○○○○ | $\{1, 5, 7e^{-3}, -1e^{-3}\}$ | 8 | 3\|19 |
| 6 | ●○○○●○○○○○○○ | $\{1, 7, 5e^{8}, -1e^{8}\}$ | 8 | 3\|31 |
| 7 | ●○○○○○●○○○○○ | $\mathbb{Z}_{12}^{\times} \ltimes e^{6\mathbb{Z}_{12}}$ | 13 | 3\|28 |
| *Three/Nine Elements* | | | | |
| 8 | ●●●○○○○○○○○○ | $\langle -1e^{-2} \rangle$ | 2 | 4\|14 |
| 8.1 | ●○●○○○○○●○○○ | | | |
| 9 | ●●○●○○○○○○○○ | $\{1\}$ | 1 | 4\|30 |
| 9.1 | ●○○○○○○○○○○○ | | | |
| 10 | ●●○○●○○○○○○○ | $\{1\}$ | 1 | 8\|36 |
| 10.1 | ●○○●○○○●○○○○ | | | |
| 11 | ●●○○○●○○○○○○ | $\langle 5 \rangle$ | 4 | 4\|20 |
| 12 | ●●○○○○●○○○○○ | $\langle 7e^{6} \rangle$ | 6 | 5\|29 |
| 13 | ●○●○●○○○○○○○ | $\{1, 7, -1e^{8}, 5e^{8}\}$ | 8 | 4\|18 |
| 14 | ●○●○○○●○○○○○ | $\langle 7 \rangle$ | 6 | 8\|31 |
| 15 | ●○○●○○●○○○○○ | $\{1, 5, -1e^{6}, 7e^{6}\}$ | 8 | 5\|32 |
| 16 | ●○○○●○○○●○○○ | $\mathbb{Z}_{12}^{\times} \ltimes e^{4\mathbb{Z}_{12}}$ | 15 | 4\|20 |
| *Four/Eight Elements* | | | | |
| 17 | ●●●●○○○○○○○○ | $\langle -1e^{-3} \rangle$ | 3 | 4\|8 |
| 17.1 | ●○●○○●○●○○○○ | | | |
| 18 | ●●●○●○○○○○○○ | $\{1\}$ | 1 | 5\|19 |
| 18.1 | ●○●○●○○●○○○○ | | | |
| 19 | ●●●○○●○○○○○○ | $\{1\}$ | 1 | 5\|19 |
| 19.1 | ●●○●○○○○●○○○ | | | |
| 20 | ●●●○○○●○○○○○ | $\{1\}$ | 1 | 7\|23 |

| **Chord Classes**—Continued | | | | |
|---|---|---|---|---|
| *Class Nr.* | *Representative* $Nr. = \bullet$, $\widehat{Nr.} = \circ$ | *Group of Symmetries* | *Conj. Class* | *♯ End.* $Nr.\|\widehat{Nr.}$ |
| 20.1 | ●●○○○●○●○○○○ | | | |
| 21 | ●●●○○○○●○○○○ | $\{1, 7, -1e^{-2}, 5e^{-2}\}$ | 9 | 7\|9 |
| 22 | ●●○●●○○○○○○○○ | $\langle -1e^{-4} \rangle$ | 2 | 6\|20 |
| 22.1 | ●○●○○●○○○●○○ | | | |
| 23 | ●●○●○●○○○○○○ | $\langle 5 \rangle$ | 4 | 5\|13 |
| 24 | ●●○●○○●○○○○○ | $\langle 7e^6 \rangle$ | 6 | 6\|17 |
| 25 | ●●○●○○○●○○○○ | $\{1\}$ | 1 | 0\|3 |
| 25.1 | ●●○●○●○○○○○○ | | | |
| 26 | ●●○●○○○○○●○○ | $\{1\}$ | 1 | 12\|31 |
| 26.1 | ●○●○○●○○●○○○ | | | |
| 27 | ●●○●○○○○○○●○ | $\{1, 7e^{-3}, 5e^2, -1e^{-1}\}$ | 11 | 5\|13 |
| 28 | ●●○○●●○○○○○○ | $\langle -1e^7 \rangle$ | 3 | 6\|14 |
| 28.1 | ●●○○○●○○●○○○ | | | |
| 29 | ●●○○●○○●○○○○ | $\langle 7 \rangle$ | 6 | 10\|23 |
| 30 | ●●○○●○○○●○○○ | $\langle 5e^4 \rangle$ | 4 | 11\|23 |
| 31 | ●●○○●○○○○●○○ | $\{1, -1e^{-1}, 5e^{-4}, 7e^3\}$ | 10 | 9\|19 |
| 32 | ●●○○○●●○○○○○ | $\{1, 5, -1e^6, 7e^6\}$ | 8 | 7\|15 |
| 33 | ●●○○○○●●○○○○ | $\{1, 7, -1e^{-1}, 5e^{-1}, e^6, 7e^6, 5e^5, -1e^5\}$ | 14 | 7\|14 |
| 34 | ●○●○●○●○○○○○ | $\{1, 7, -1e^6, 5e^6\}$ | 9 | 6\|17 |
| 35 | ●○●○●○○○●○○○ | $\{1, 7, -1e^4, 5e^4\}$ | 8 | 11\|19 |
| 36 | ●○●○○○●○●○○○ | $\{1, 7, -1e^{-2}, 5e^{-2}, e^6, 7e^6, 5e^4, -1e^4\}$ | 13 | 9\|28 |
| 37 | ●○○●○○●○○●○○ | $\mathbb{Z}_{12}^{\times} \ltimes e^{3\mathbb{Z}_{12}}$ | 17 | 7\|21 |
| *Five/Seven Elements* | | | | |
| 38 | ●●●●●○○○○○○○ | $\langle -1e^{-4} \rangle$ | 2 | 5\|7 |
| 38.1 | ●○●○●○○●○●○○ | | | |
| 39 | ●●●●○●○○○○○○ | $\{1\}$ | 1 | 6\|10 |
| 39.1 | ●●○●○●○○○○●○ | | | |
| 40 | ●●●●○○●○○○○○ | $\{1\}$ | 1 | 8\|12 |
| 40.1 | ●●○●○○●○●○○○ | | | |
| 41 | ●●●●○○○●○○○○ | $\{1\}$ | 1 | 8\|12 |

| **Chord Classes**—Continued | | | | |
|---|---|---|---|---|
| *Class Nr.* | *Representative* $Nr. = \bullet$, $\widehat{Nr.} = \circ$ | *Group of Symmetries* | *Conj. Class* | *♯ End.* $Nr.\|\widehat{Nr.}$ |
| 41.1 | ●●●○○●○●○○○○ | | | |
| 42 | ●●●○●●○○○○○○ | $\{1\}$ | 1 | 6\|16 |
| 42.1 | ●●○●○●○○●○○○ | | | |
| 43 | ●●●○●○●○○○○○ | $\{1\}$ | 1 | 8\|20 |
| 43.1 | ●●○●○●○●○○○○ | | | |
| 44 | ●●●○●○○●○○○○ | $\langle 7 \rangle$ | 6 | 7\|9 |
| 45 | ●●●○●○○○●○○○ | $\{1\}$ | 1 | 16\|22 |
| 45.1 | ●●○○●○●○●○○○ | | | |
| 46 | ●●●○●○○○○●○○ | $\langle 5e^{-4} \rangle$ | 4 | 5\|12 |
| 47 | ●●●○●○○○○○●○ | $\langle -1e^{-2} \rangle$ | 2 | 8\|14 |
| 47.1 | ●○●○●○●○○●○○ | | | |
| 48 | ●●●○○●●○○○○○ | $\{1\}$ | 1 | 8\|18 |
| 48.1 | ●●○●○○○●●○○○ | | | |
| 49 | ●●●○○●○○●○○○ | $\{1\}$ | 1 | 10\|18 |
| 49.1 | ●●○●○○○○●●○○ | | | |
| 50 | ●●●○○●○○○●○○ | $\langle -1e^{-2} \rangle$ | 2 | 9\|13 |
| 50.1 | ●●○●●○○○●○○○ | | | |
| 51 | ●●●○○○●●○○○○ | $\langle 7 \rangle$ | 6 | 9\|11 |
| 52 | ●●●○○○●○●○○○ | $\{1, -1e^{-2}, 7e^{6}, 5e^{4}\}$ | 8 | 7\|17 |
| 53 | ●●○●●○●○○○○○ | $\{1\}$ | 1 | 10\|20 |
| 53.1 | ●●○●○○●○○○●○ | | | |
| 54 | ●●○●●○○○●○○○ | $\{1\}$ | 1 | 14\|26 |
| 54.1 | ●●○○●○●○○●○○ | | | |
| 55 | ●●○●○●●○○○○○ | $\{1, 5, -1e^{6}, 7e^{6}\}$ | 8 | 8\|8 |
| 56 | ●●○●○●○○○●○○ | $\langle 5 \rangle$ | 4 | 16\|16 |
| 57 | ●●○●○○●●○○○○ | $\langle 7e^{6} \rangle$ | 6 | 12\|16 |
| 58 | ●●○●○○●○○●○○ | $\langle 7e^{6} \rangle$ | 6 | 18\|23 |
| 59 | ●●○●○○○●○●○○ | $\langle 7 \rangle$ | 6 | 13\|29 |
| 60 | ●●○○●●○○●○○○ | $\langle 5 \rangle$ | 4 | 11\|19 |
| 61 | ●●○○●○○●●○○○ | $\{1, 7, -1e^{4}, 5e^{4}\}$ | 8 | 14\|14 |
| 62 | ●○●○●○●○●○○○ | $\{1, 7, -1e^{4}, 7e^{4}\}$ | 8 | 11\|19 |

| **Chord Classes**—Continued | | | | |
|---|---|---|---|---|
| *Class Nr.* | *Representative* $Nr. = \bullet,\ \widehat{Nr.} = \circ$ | *Group of Symmetries* | *Conj. Class* | *♯ End.* $Nr.\vert\widehat{Nr.}$ |
| *Six/Six Elements* | | | | |
| 63* | ●●●●●●○○○○○○ | $\langle -1e^{-5} \rangle$ | 3 | 5\|5 |
| 63.1* | ●●○●○●○○●○●○ | | | |
| 64* | ●●●●●○●○○○○○ | $\{1\}$ | 1 | 9\|9 |
| 64.1* | ●●○●○●○●○○○●○ | | | |
| 65 | ●●●●●○○○●○○○○ | $\{1\}$ | 1 | 9\|9 |
| 65.1 | ●●●○●○○○●○●○○ | | | |
| 66 | ●●●●●○○○○●○○○ | $\langle -1e^{-4} \rangle$ | 2 | 12\|6 |
| 66.1 | ●●●○○●○●○●○○○ | | | |
| 67* | ●●●●○●○●○○○○ | $\langle 5e^{-2} \rangle$ | 5 | 6\|6 |
| 68* | ●●●●○●○○○●○○○ | $\{1\}$ | 1 | 9\|9 |
| 69 | ●●●●○●○○○○●○○ | $\{1\}$ | 1 | 15\|11 |
| 69.1 | ●●○●●○○●○●○○○ | | | |
| 70* | ●●●●○●○○○○○●○ | $\{1, 5, -1e^{-3}, 7e^{-3}\}$ | 10 | 6\|6 |
| 71* | ●●●●○○●●○○○○ | $\{1\}$ | 1 | 11\|11 |
| 71.1* | ●●●○○●○●●○○○ | | | |
| 72 | ●●●●○○●○●○○○ | $\langle 7e^{6} \rangle$ | 6 | 8\|10 |
| 73 | ●●●●○○●○○●○○ | $\langle -1e^{-3} \rangle$ | 3 | 13\|9 |
| 73.1 | ●●○●○○●○●●○○ | | | |
| 74* | ●●●●○○○●●○○○ | $\langle -1e^{-3} \rangle$ | 3 | 7\|7 |
| 75* | ●●●○●●○○●○○○ | $\{1\}$ | 1 | 17\|17 |
| 75.1* | ●●○○●●○○●○●○ | | | |
| 76* | ●●●○●●○○○●○○ | $\langle 5e^{-5} \rangle$ | 4 | 10\|10 |
| 77* | ●●●○●○●○●○○○ | $\langle 5e^{4} \rangle$ | 4 | 14\|14 |
| 78* | ●●●○●○●○○○●○ | $\{1\}$ | 1 | 23\|23 |
| 78.1* | ●●○●○●○●○●○○ | | | |
| 79 | ●●●○●○●●○○○○ | $\langle 7 \rangle$ | 6 | 18\|10 |
| 80 | ●●●○●○○●○○●○ | $\{1, 7, 5e^{-2}, -1e^{-2}\}$ | 9 | 15\|11 |
| 81* | ●●●○●○○○●●○○ | $\langle 5e^{-4} \rangle$ | 4 | 11\|11 |
| 82* | ●●●○○●●○○●○○ | $\{1\}$ | 1 | 17\|17 |
| 82.1* | ●●○●●○○○●●○○○ | | | |
| 83* | ●●●○○○●●●○○○ | $\{1, -1e^{-2}, 5e^{-2}, 7\}$ | 13 | 12\|12 |

| **Chord Classes**—Continued | | | | |
|---|---|---|---|---|
| *Class Nr.* | *Representative* $Nr. = \bullet,\ \widehat{Nr.} = \circ$ | *Group of Symmetries* | *Conj. Class* | ♯ *End.* $Nr.\vert\widehat{Nr.}$ |
| | | $\ltimes e^{6\mathbb{Z}_{12}}$ | | |
| 84* | ●●○●●○●○○●○○ | $\langle 7e^3 \rangle$ | 7 | 14\|14 |
| 85 | ●●○●●○○●○●○○ | $\{1, -1e^{-4}, 5e^{-4}, 7\}$ | 8 | 15\|23 |
| 86* | ●●○●○○●●○●○○ | $\langle 7 \rangle \ltimes e^{6\mathbb{Z}_{12}}$ | 12 | 20\|20 |
| 87* | ●●○●●○○●●○○○ | $\{1, e^4, e^8, 5, 5e^4, 5e^8, -1e^{-1}, -1e^3, -1e^7, 7e^{-1}, 7e^3, 7e^7\}$ | 16 | 12\|12 |
| 88* | ●○●○●○●○●○●○ | $\mathbb{Z}_{12}^{\times} \ltimes e^{\mathbb{Z}_{12}}$ | 18 | 12\|12 |

## L.2 Third Chain Classes

The following list of third chain translation classes shows the class number in the first column, where equivalence (∼) means that the same *pc* set is generated. The second column shows the pitch classes in the order of appearance along the third chain. The third column shows the third chain, the fourth column shows the chord class of the *pc* set, and the fifth column shows lead-sheet symbols as systematically derived in subsection 25.2.1.

| **Third Chains** | | | | |
|---|---|---|---|---|
| *Chain Nr.* ∼ *equiv.* | *Pitch Classes from 0* | *Third Chain* | *Chord Class* | *Lead-Sheet Symbols* |
| *Two Pitch Classes* | | | | |
| 1 | 0,3 | 3 | 5 | *trd* |
| 2 | 0,4 | 4 | 6 | *Trd* |
| *Three Pitch Classes* | | | | |
| 3 | 0,3,6 | 33 | 15 | $C0, Cm5\text{-}$ |
| 4 | 0,3,7 | 34 | 10.1 | $Cm$ |
| 5 | 0,4,7 | 43 | 10.1 | $C$ |
| 6 | 0,4,8 | 44 | 16 | $C+, C5+$ |
| *Four Pitch Classes* | | | | |
| 7 | 0,3,6,9 | 333 | 37 | $C07\text{-}$ |
| 8 | 0,3,6,10 | 334 | 26.1 | $C07$ |
| 9 | 0,3,7,10 | 343 | 22.1 | $Cm7$ |
| 10 | 0,3,7,11 | 344 | 30 | $Cm7+$ |
| 11 | 0,4,7,10 | 433 | 26.1 | $C7$ |
| 12 | 0,4,7,11 | 434 | 28.1 | $C7+$ |
| 13 | 0,4,8,11 | 443 | 30 | $C+7+$ |
| *Five Pitch Classes* | | | | |
| 14 | 0,3,6,9,1 | 3334 | 58 | $C07\text{-}/9\text{-}$ |
| 15 | 0,3,6,10,1 | 3343 | 53.1 | $C09\text{-}$ |
| 16 | 0,3,6,10,2 | 3344 | 56 | $C09$ |
| 17 | 0,3,7,10,1 | 3433 | 53.1 | $Cm9\text{-}$ |
| 18 | 0,3,7,10,2 | 3434 | 42.1 | $Cm9$ |
| 19 | 0,3,7,11,2 | 3443 | 59 | $Cm7+/9, Cmmaj7/9$ |
| 20 | 0,4,7,10,1 | 4333 | 58 | $C9\text{-}$ |
| 21 | 0,4,7,10,2 | 4334 | 47.1 | $C9$ |
| 22 | 0,4,7,11,2 | 4343 | 42.1 | $C7+/9, Cmaj7/9$ |
| 23 | 0,4,7,11,3 | 4344 | 60 | $C7+/9+$ |
| 24 | 0,4,8,11,2 | 4433 | 56 | $C+7+/9$ |

| **Third Chains**—Continued | | | | |
|---|---|---|---|---|
| *Chain Nr.* ∼ *equiv.* | *Pitch Classes from 0* | *Third Chain* | *Chord Class* | *Lead-Sheet Symbols* |
| 25 | 0,4,8,11,3 | 4434 | 60 | $C$+7+/9+ |
| *Six Pitch Classes* | | | | |
| 26 | 0,3,6,9,1,4 | 33343 | 84* | $C$07-/9-/11- |
| 27 | 0,3,6,9,1,5 | 33344 | 85^ | $C$07-/9-/11 |
| 28 | 0,3,6,10,1,4 | 33433 | 79 | $C$09-/11- |
| 29 | 0,3,6,10,1,5 | 33434 | 65.1^ | $C$09-/11 |
| 30 | 0,3,6,10,2,5 | 33443 | 69.1 | $C$011 |
| 31 | 0,3,7,10,1,4 | 34333 | 84* | $Cm$9-/11- |
| 32 | 0,3,7,10,1,5 | 34334 | 64.1* | $Cm$9-/11 |
| 33 | 0,3,7,10,2,5 | 34343 | 63.1* | $Cm$11 |
| 34 | 0,3,7,10,2,6 | 34344 | 75.1* | $Cm$11+ |
| 35 | 0,3,7,11,2,5 | 34433 | 69.1 | $Cm$7+/11 |
| 36 | 0,3,7,11,2,6 | 34434 | 82* | $Cm$7+/11+ |
| 37 | 0,4,7,10,1,5 | 43334 | 73.1 | $C$9-/11 |
| 38 | 0,4,7,10,2,5 | 43343 | 64.1* | $C$11 |
| 39 | 0,4,7,10,2,6 | 43344 | 78.1* | $C$11+ |
| 40 | 0,4,7,11,2,5 | 43433 | 65.1^ | $C$7+/11 |
| 41 | 0,4,7,11,2,6 | 43434 | 66.1^ | $C$7+/11+ |
| 42 | 0,4,7,11,3,6 | 43443 | 82* | $C$7+/9+/11+ |
| 43 | 0,4,8,11,2,5 | 44333 | 85^ | $C$+7+/11 |
| 44 | 0,4,8,11,2,6 | 44334 | 78.1* | $C$+7+/11+ |
| 45 | 0,4,8,11,3,6 | 44343 | 75.1* | $C$+7+/9+/11+ |
| 46 | 0,4,8,11,3,7 | 44344 | 87* | $C$+7+/9+/(11)/13- |
| *Seven Pitch Classes* | | | | |
| 47 | 0,3,6,9,1,4,7 | 333433 | 58^ | $C$07-/9-/11-/13- |
| 48 | 0,3,6,9,1,4,8 | 333434 | 54.1^ | $C$07-/9-/11-/13 |
| 49 | 0,3,6,9,1,5,8 | 333443 | 54.1^ | $C$07-/9-/13 |
| 50 | 0,3,6,10,1,4,7 | 334333 | 58^ | $C$09-/11-/13- |
| 51 | 0,3,6,10,1,4,8 | 334334 | 47.1 | $C$09-/11-/13 |
| 52 | 0,3,6,10,1,5,8 | 334343 | 38.1 | $C$09-/13 |
| 53 | 0,3,6,10,1,5,9 | 334344 | 54.1^ | $C$09-/13+ |
| 54 | 0,3,6,10,2,5,8 | 334433 | 47.1^ | $C$013 |
| 55 | 0,3,6,10,2,5,9 | 334434 | 54.1^ | $C$013+ |
| 56 | 0,3,7,10,1,4,8 | 343334 | 54.1^ | $Cm$9-/11-/13 |
| 57 | 0,3,7,10,1,5,8 | 343343 | 38.1^ | $Cm$9-/13 |

**Third Chains**—Continued

| *Chain Nr.* ∼ *equiv.* | *Pitch Classes from 0* | *Third Chain* | *Chord Class* | *Lead-Sheet Symbols* |
|---|---|---|---|---|
| 58 | 0,3,7,10,1,5,9 | 343344 | 47.1^ | $Cm$9-/13+ |
| 59 | 0,3,7,10,2,5,8 | 343433 | 38.1^ | $Cm$13 |
| 60 | 0,3,7,10,2,5,9 | 343434 | 38.1^ | $Cm$13+ |
| 61 | 0,3,7,10,2,6,9 | 343443 | 54.1^ | $Cm$11+/13+ |
| 62 | 0,3,7,11,2,5,8 | 344333 | 54.1^ | $Cm$7+/13 |
| 63 | 0,3,7,11,2,5,9 | 344334 | 47.1^ | $Cm$7+/13+ |
| 64 | 0,3,7,11,2,6,9 | 344343 | 54.1^ | $Cm$7+/11+/13+ |
| 65 | 0,3,7,11,2,6,10 | 344344 | 60^ | $Cm$7+/11+/(13)/15- |
| 66 | 0,4,7,10,1,5,8 | 433343 | 54.1^ | $C$9-/13 |
| 67 | 0,4,7,10,1,5,9 | 433344 | 54.1^ | $C$9-/13+ |
| 68 | 0,4,7,10,2,5,8 | 433433 | 47.1^ | $C$13 |
| 69 | 0,4,7,10,2,5,9 | 433434 | 38.1^ | $C$13+ |
| 70 | 0,4,7,10,2,6,9 | 433443 | 47.1^ | $C$11+/13+ |
| 71 | 0,4,7,11,2,5,8 | 434333 | 54.1^ | $C$7+/13 |
| 72 | 0,4,7,11,2,5,9 | 434334 | 38.1^ | $C$7+/13+ |
| 73 | 0,4,7,11,2,6,9 | 434343 | 38.1^ | $C$7+/11+/13+ |
| 74 | 0,4,7,11,2,6,10 | 434344 | 45.1^ | $C$7+/11+/(13)/15- |
| 75 | 0,4,7,11,3,6,9 | 434433 | 54.1^ | $C$7+/9+/11+/13+ |
| 76 | 0,4,7,11,3,6,10 | 434434 | 55^ | $C$7+/9+/11+/(13)/15- |
| 77 | 0,4,8,11,2,5,9 | 443334 | 54.1^ | $C$+7+/13+ |
| 78 | 0,4,8,11,2,6,9 | 443343 | 47.1^ | $C$+7+/11+/13+ |
| 79 | 0,4,8,11,2,6,10 | 443344 | 62^ | $C$+7+/11+/(13)/15- |
| 80 | 0,4,8,11,3,6,9 | 443433 | 54.1^ | $C$+7+/9+/11+/13+ |
| 81 | 0,4,8,11,3,6,10 | 443434 | 45.1^ | $C$+7+/9+/11+/(13)/15- |
| 82 | 0,4,8,11,3,7,10 | 443443 | 60^ | $C$+7+/9+/(11)/13-/15- |
| | *Eight Pitch Classes* | | | |
| 83 | 0,3,6,9,1,4,7,10 | 3334333 | 37^ | $C$07-/9-/11-/13-... |
| 84 | 0,3,6,9,1,4,7,11 | 3334334 | 26.1^ | $C$07-/9-/11-/13-... |
| 85 | 0,3,6,9,1,4,8,11 | 3334343 | 22.1^ | $C$07-/9-/11-/13... |
| 86 | 0,3,6,9,1,5,8,11 | 3334433 | 26.1^ | $C$07-/9-/13... |
| 87 | 0,3,6,10,1,4,7,11 | 3343334 | 29^ | $C$09-/11-/13-... |
| 88 | 0,3,6,10,1,4,8,11 | 3343343 | 18.1^ | $C$09-/11-/13... |
| 89 | 0,3,6,10,1,5,8,11 | 3343433 | 17.1^ | $C$09-/13... |
| 90 | 0,3,6,10,2,5,8,11 | 3344333 | 26.1^ | $C$013... |
| 91 | 0,3,6,10,2,5,9,1 | 3344344 | 31^ | $C$013+... |

| **Third Chains**—Continued | | | | |
|---|---|---|---|---|
| *Chain Nr.* ∼ *equiv.* | *Pitch Classes from 0* | *Third Chain* | *Chord Class* | *Lead-Sheet Symbols* |
| 92 | 0,3,7,10,1,4,8,11 | 3433343 | 31^ | $Cm9\text{-}/11\text{-}/13\ldots$ |
| 93 | 0,3,7,10,1,5,8,11 | 3433433 | 18.1^ | $Cm9\text{-}/13\ldots$ |
| 94 | 0,3,7,10,2,5,8,11 | 3434333 | 22.1^ | $Cm13\ldots$ |
| 95 | 0,3,7,10,2,5,9,1 | 3434344 | 18.1^ | $Cm13+\ldots$ |
| 96 | 0,3,7,10,2,6,9,1 | 3434434 | 29^ | $Cm11+/13+\ldots$ |
| 97 | 0,3,7,11,2,5,9,1 | 3443344 | 34^ | $Cm7+/13+\ldots$ |
| 98 | 0,3,7,11,2,6,9,1 | 3443434 | 29^ | $Cm7+/11+/13+\ldots$ |
| 99 | 0,3,7,11,2,6,10,1 | 3443443 | 28^ | $Cm7+/11+/(13)/15\text{-}\ldots$ |
| 100 | 0,4,7,10,1,5,8,11 | 4333433 | 29^ | $C9\text{-}/13\ldots$ |
| 101 | 0,4,7,10,2,5,8,11 | 4334333 | 18.1^ | $C15$ |
| 102 | 0,4,7,10,2,5,9,1 | 4334344 | 22.1^ | $C13+\ldots$ |
| 103 | 0,4,7,10,2,6,9,1 | 4334434 | 26.1^ | $C11+/13+\ldots$ |
| 104 | 0,4,7,11,2,5,9,1 | 4343344 | 18.1^ | $C7+/13+\ldots$ |
| 105 | 0,4,7,11,2,6,9,1 | 4343434 | 17.1^ | $C7+/11+/13+\ldots$ |
| 106 | 0,4,7,11,2,6,10,1 | 4343443 | 25.1^ | $C7+/11+/(13)/15\text{-}\ldots$ |
| 107 ∼ 84 | 0,4,7,11,3,6,9,1 | 4344334 | 26.1^ | $C7+/9+/11+/13+\ldots$ |
| 108 ∼ 87 | 0,4,7,11,3,6,10,1 | 4344343 | 29^ | $C7+/9+/11+/(13)/15\text{-}\ldots$ |
| 109 | 0,4,7,11,3,6,10,2 | 4344344 | 30^ | $C7+/9+/11+/(13)/15\text{-}\ldots$ |
| 110 | 0,4,8,11,2,5,9,1 | 4433344 | 31^ | $C+7+/13+\ldots$ |
| 111 | 0,4,8,11,2,6,9,1 | 4433434 | 18.1^ | $C+7+/11+/13+\ldots$ |
| 112 | 0,4,8,11,2,6,10,1 | 4433443 | 34^ | $C+7+/11+/(13)/15\text{-}\ldots$ |
| 113 ∼ 85 | 0,4,8,11,3,6,9,1 | 4434334 | 22.1^ | $C+7+/9+/11+/13+\ldots$ |
| 114 ∼ 88 | 0,4,8,11,3,6,10,1 | 4434343 | 18.1^ | $C+7+/9+/11+/(13)/15\text{-}\ldots$ |
| 115 | 0,4,8,11,3,6,10,2 | 4434344 | 35^ | $C+7+/9+/11+/(13)/15\text{-}\ldots$ |
| 116 ∼ 92 | 0,4,8,11,3,7,10,1 | 4434433 | 31^ | $C+7+/9+/(11)/13\text{-}/15\text{-}\ldots$ |
| 117 | 0,4,8,11,3,7,10,2 | 4434434 | 30^ | $C+7+/9+/(11)/13\text{-}/15\text{-}\ldots$ |
| *Nine Pitch Classes* | | | | |
| 118 | 0,3,6,9,1,4,7,10,2 | 33343334 | 15^ | $C07\text{-}/9\text{-}/11\text{-}/13\text{-}\ldots$ |
| 119 | 0,3,6,9,1,4,7,11,2 | 33343343 | 9.1^ | $C07\text{-}/9\text{-}/11\text{-}/13\text{-}\ldots$ |
| 120 | 0,3,6,9,1,4,8,11,2 | 33343433 | 9.1^ | $C07\text{-}/9\text{-}/11\text{-}/13\ldots$ |
| 121 | 0,3,6,9,1,5,8,11,2 | 33344333 | 15^ | $C07\text{-}/9\text{-}/13\ldots$ |
| 122 | 0,3,6,10,1,4,7,11,2 | 33433343 | 10^ | $C09\text{-}/11\text{-}/13\text{-}\ldots$ |
| 123 | 0,3,6,10,1,4,8,11,2 | 33433433 | 13^ | $C09\text{-}/11\text{-}/13\ldots$ |
| 124 | 0,3,6,10,1,5,8,11,2 | 33434333 | 9.1^ | $C09\text{-}/13\ldots$ |
| 125 | 0,3,6,10,2,5,9,1,4 | 33443443 | 10^ | $C013+\ldots$ |

| **Third Chains**—Continued | | | | |
|---|---|---|---|---|
| *Chain Nr.* ∼ *equiv.* | *Pitch Classes from 0* | *Third Chain* | *Chord Class* | *Lead-Sheet Symbols* |
| 126 | 0,3,7,10,1,4,8,11,2 | 34333433 | 10^ | *Cm*9-/11-/13 . . . |
| 127 | 0,3,7,10,1,5,8,11,2 | 34334333 | 9.1^ | *Cm*9-/13 . . . |
| 128 | 0,3,7,10,2,5,9,1,4 | 34343443 | 9.1^ | *Cm*13+ . . . |
| 129 | 0,3,7,10,2,6,9,1,4 | 34344343 | 15^ | *Cm*11+/13+ . . . |
| 130 | 0,3,7,11,2,5,9,1,4 | 34433443 | 13^ | *Cm*7+/13+ . . . |
| 131 ∼ 119 | 0,3,7,11,2,6,9,1,4 | 34434343 | 9.1^ | *Cm*7+/11+/13+ . . . |
| 132 | 0,3,7,11,2,6,9,1,5 | 34434344 | 14^ | *Cm*7+/11+/13+ . . . |
| 133 ∼ 122 | 0,3,7,11,2,6,10,1,4 | 34434433 | 10^ | *Cm*7+/11+/(13)/15- . . . |
| 134 | 0,3,7,11,2,6,10,1,5 | 34434434 | 11^ | *Cm*7+/11+/(13)/15- . . . |
| 135 | 0,4,7,10,1,5,8,11,2 | 43334333 | 15^ | *C*9-/13 . . . |
| 136 | 0,4,7,10,1,5,8,11,3 | 43334334 | 10.1^ | *C*9-/13 . . . |
| 137 | 0,4,7,10,2,5,8,11,3 | 43343334 | 10.1^ | *C*17 |
| 138 | 0,4,7,10,2,6,9,1,5 | 43344344 | 10.1^ | *C*11+/13+ . . . |
| 139 | 0,4,7,11,2,6,9,1,5 | 43434344 | 8.1^ | *C*7+/11+/13+ . . . |
| 140 | 0,4,7,11,2,6,10,1,5 | 43434434 | 12^ | *C*7+/11+/(13)/15- . . . |
| 141 | 0,4,7,11,3,6,9,1,5 | 43443344 | 14^ | *C*7+/9+/11+/13+ . . . |
| 142 | 0,4,7,11,3,6,10,1,5 | 43443434 | 12^ | *C*7+/9+/11+/(13)/15- . . . |
| 143 | 0,4,7,11,3,6,10,2,5 | 43443443 | 11^ | *C*7+/9+/11+/(13)/15- . . . |
| 144 | 0,4,8,11,2,6,9,1,5 | 44334344 | 10.1^ | *C*+7+/11+/13+ . . . |
| 145 | 0,4,8,11,2,6,10,1,5 | 44334434 | 14^ | *C*+7+/11+/(13)/15- . . . |
| 146 | 0,4,8,11,3,6,9,1,5 | 44343344 | 10.1^ | *C*+7+/9+/11+/13+ . . . |
| 147 | 0,4,8,11,3,6,10,1,5 | 44343434 | 8.1^ | *C*+7+/9+/11+/(13)/15- . . . |
| 148 | 0,4,8,11,3,6,10,2,5 | 44343443 | 14^ | *C*+7+/9+/11+/(13)/15- . . . |
| 149 ∼ 136 | 0,4,8,11,3,7,10,1,5 | 44344334 | 10.1^ | *C*+7+/9+/(11)/13-/15- . . . |
| 150 ∼ 137 | 0,4,8,11,3,7,10,2,5 | 44344343 | 10.1^ | *C*+7+/9+/(11)/13-/15- . . . |
| 151 | 0,4,8,11,3,7,10,2,6 | 44344344 | 16^ | *C*+7+/9+/(11)/13-/15- . . . |
| *Ten Pitch Classes* | | | | |
| 152 | 0,3,6,9,1,4,7,10,2,5 | 333433343 | 5^ | *C*07-/9-/11-/13- . . . |
| 153 | 0,3,6,9,1,4,7,11,2,5 | 333433433 | 4^ | *C*07-/9-/11-/13- . . . |
| 154 | 0,3,6,9,1,4,8,11,2,5 | 333434333 | 5^ | *C*07-/9-/11-/13 . . . |
| 155 | 0,3,6,10,1,4,7,11,2,5 | 334333433 | 3^ | *C*09-/11-/13- . . . |
| 156 | 0,3,6,10,1,4,8,11,2,5 | 334334333 | 4^ | *C*09-/11-/13 . . . |
| 157 ∼ 152 | 0,3,6,10,2,5,9,1,4,7 | 334434433 | 5^ | *C*013+ . . . |
| 158 | 0,3,6,10,2,5,9,1,4,8 | 334434434 | 6^ | *C*013+ . . . |
| 159 | 0,3,7,10,1,4,8,11,2,5 | 343334333 | 5^ | *Cm*9-/11-/13 . . . |

| **Third Chains**—Continued | | | | |
|---|---|---|---|---|
| *Chain Nr.* ∼ *equiv.* | *Pitch Classes from 0* | *Third Chain* | *Chord Class* | *Lead-Sheet Symbols* |
| 160 | 0,3,7,10,1,4,8,11,2,6 | 343334334 | 6^ | $Cm$9-/11-/13 . . . |
| 161 | 0,3,7,10,1,5,8,11,2,6 | 343343334 | 3.1^ | $Cm$9-/13 . . . |
| 162 | 0,3,7,10,2,5,9,1,4,8 | 343434434 | 3.1^ | $Cm$13+ . . . |
| 163 | 0,3,7,10,2,6,9,1,4,8 | 343443434 | 7^ | $Cm$11+/13+ . . . |
| 164 | 0,3,7,11,2,5,9,1,4,8 | 344334434 | 6^ | $Cm$7+/13+ . . . |
| 165 | 0,3,7,11,2,6,9,1,4,8 | 344343434 | 3.1^ | $Cm$7+/11+/13+ . . . |
| 166 | 0,3,7,11,2,6,9,1,5,8 | 344343443 | 7^ | $Cm$7+/11+/13+ . . . |
| 167 ∼ 160 | 0,3,7,11,2,6,10,1,4,8 | 344344334 | 6^ | $Cm$7+/11+/(13)/15- . . . |
| 168 ∼ 161 | 0,3,7,11,2,6,10,1,5,8 | 344344343 | 3.1^ | $Cm$7+/11+/(13)/15- . . . |
| 169 | 0,3,7,11,2,6,10,1,5,9 | 344344344 | 6^ | $Cm$7+/11+/(13)/15- . . . |
| 170 | 0,4,7,10,1,5,8,11,2,6 | 433343334 | 7^ | $C$9-/13 . . . |
| 171 | 0,4,7,10,1,5,8,11,3,6 | 433343343 | 3.1^ | $C$9-/13 . . . |
| 172 | 0,4,7,10,2,5,8,11,3,6 | 433433343 | 6^ | $C$19 |
| 173 | 0,4,7,10,2,6,9,1,5,8 | 433443443 | 6^ | $C$11+/13+ . . . |
| 174 | 0,4,7,11,2,6,9,1,5,8 | 434343443 | 3.1^ | $C$7+/11+/13+ . . . |
| 175 ∼ 170 | 0,4,7,11,2,6,10,1,5,8 | 434344343 | 7^ | $C$7+/11+/(13)/15- . . . |
| 176 | 0,4,7,11,2,6,10,1,5,9 | 434344344 | 3.1^ | $C$7+/11+/(13)/15- . . . |
| 177 ∼ 171 | 0,4,7,11,3,6,9,1,5,8 | 434433443 | 3.1^ | $C$7+/9+/11+/13+ . . . |
| 178 | 0,4,7,11,3,6,10,1,5,8 | 434434343 | 6^ | $C$7+/9+/11+/(13)/15- . . . |
| 179 | 0,4,7,11,3,6,10,1,5,9 | 434434344 | 7^ | $C$7+/9+/11+/(13)/15- . . . |
| 180 ∼ 172 | 0,4,7,11,3,6,10,2,5,8 | 434434433 | 6^ | $C$7+/9+/11+/(13)/15- . . . |
| 181 | 0,4,7,11,3,6,10,2,5,9 | 434434434 | 3.1^ | $C$7+/9+/11+/(13)/15- . . . |
| 182 | 0,4,8,11,2,6,10,1,5,9 | 443344344 | 6^ | $C$+7+/11+/(13)/15- . . . |
| 183 | 0,4,8,11,3,6,10,1,5,9 | 443434344 | 3.1^ | $C$+7+/9+/11+/(13)/15- . . . |
| 184 | 0,4,8,11,3,6,10,2,5,9 | 443434434 | 7^ | $C$+7+/9+/(11)/13-/15- . . . |
| 185 | 0,4,8,11,3,7,10,1,5,9 | 443443344 | 6^ | $C$+7+/9+/(11)/13-/15- . . . |
| 186 | 0,4,8,11,3,7,10,2,5,9 | 443443434 | 3.1^ | $C$+7+/9+/(11)/13-/15- . . . |
| 187 | 0,4,8,11,3,7,10,2,6,9 | 443443443 | 6^ | $C$+7+/9+/(11)/13-/15- . . . |
| *Eleven Pitch Classes* | | | | |
| 188 | 0,3,6,9,1,4,7,10,2,5,8 | 3334333433 | 2^ | $C$07-/9-/11-/13- . . . |
| 189 | 0,3,6,9,1,4,7,11,2,5,8 | 3334334333 | 2^ | $C$07-/9-/11-/13- . . . |
| 190 | 0,3,6,10,1,4,7,11,2,5,8 | 3343334333 | 2^ | $C$09-/11-/13- . . . |
| 191 | 0,3,6,10,1,4,7,11,2,5,9 | 3343334334 | 2^ | $C$09-/11-/13- . . . |
| 192 | 0,3,6,10,1,4,8,11,2,5,9 | 3343343334 | 2^ | $C$09-/11-/13 . . . |
| 193 ∼ 191 | 0,3,6,10,2,5,9,1,4,7,11 | 3344344334 | 2^ | $C$013+ . . . |

| **Third Chains**—Continued | | | | |
|---|---|---|---|---|
| *Chain Nr.* ∼ *equiv.* | *Pitch Classes from 0* | *Third Chain* | *Chord Class* | *Lead-Sheet Symbols* |
| 194 ∼ 192 | 0,3,6,10,2,5,9,1,4,8,11 | 3344344343 | 2^ | $C013+\ldots$ |
| 195 | 0,3,7,10,1,4,8,11,2,5,9 | 3433343334 | 2^ | $Cm9\text{-}/11\text{-}/13\ldots$ |
| 196 | 0,3,7,10,1,4,8,11,2,6,9 | 3433343343 | 2^ | $Cm9\text{-}/11\text{-}/13\ldots$ |
| 197 | 0,3,7,10,1,5,8,11,2,6,9 | 3433433343 | 2^ | $Cm9\text{-}/13\ldots$ |
| 198 ∼ 195 | 0,3,7,10,2,5,9,1,4,8,11 | 3434344343 | 2^ | $Cm13+\ldots$ |
| 199 | 0,4,7,10,1,5,8,11,2,6,9 | 4333433343 | 2^ | $C9\text{-}/13\ldots$ |
| 200 | 0,4,7,10,1,5,8,11,3,6,9 | 4333433433 | 2^ | $C9\text{-}/13\ldots$ |
| 201 | 0,4,7,10,2,5,8,11,3,6,9 | 4334333433 | 2^ | $C21$ |
| 202 ∼ 199 | 0,4,7,10,2,6,9,1,5,8,11 | 4334434433 | 2^ | $C11+/13+\ldots$ |
| 203 ∼ 191 | 0,4,7,11,3,6,10,2,5,9,1 | 4344344344 | 2^ | $C7+/9+/11+/(13)/15\text{-}\ldots$ |
| 204 ∼ 192 | 0,4,8,11,3,6,10,2,5,9,1 | 4434344344 | 2^ | $C+7+/9+/(11)/13\text{-}/15\text{-}\ldots$ |
| 205 ∼ 195 | 0,4,8,11,3,7,10,2,5,9,1 | 4434434344 | 2^ | $C+7+/9+/(11)/13\text{-}/15\text{-}\ldots$ |
| 206 ∼ 196 | 0,4,8,11,3,7,10,2,6,9,1 | 4434434434 | 2^ | $C+7+/9+/(11)/13\text{-}/15\text{-}\ldots$ |
| *Twelve Pitch Classes* | | | | |
| 207 | 0,3,6,9,1,4,7,10,2,5,8,11 | 33343334333 | 1^ | $C07\text{-}/9\text{-}/11\text{-}/13\text{-}\ldots$ |
| 208 ∼ 207 | 0,4,7,10,2,5,8,11,3,6,9,1 | 43343334334 | 1^ | $C23$ |
| 209 ∼ 207 | 0,4,7,10,2,6,9,1,5,8,11,3 | 43344344334 | 1^ | $C11+/13+\ldots$ |
| 210 ∼ 207 | 0,4,8,11,3,7,10,2,6,9,1,5 | 44344344344 | 1^ | $C+7+/9+/(11)/13\text{-}/15\text{-}\ldots$ |

# Appendix M

# Two, Three, and Four Tone Motif Classes

## M.1 Two Tone Motifs in $OnPiMod_{12,12}$

| *ClassNr.* | *Representative* |
|---|---|
| 1 | $(0,0),(0,1)$ |
| 2 | $(0,0),(0,2)$ |
| 3 | $(0,0),(0,3)$ |
| 4 | $(0,0),(0,4)$ |
| 5 | $(0,0),(0,6)$ |

## M.2 Two Tone Motifs in $OnPiMod_{5,12}$

| *ClassNr.* | *Representative* |
|---|---|
| 1 | $(0,0),(0,1)$ |
| 2 | $(0,0),(0,2)$ |
| 3 | $(0,0),(0,3)$ |
| 4 | $(0,0),(0,4)$ |
| 5 | $(0,0),(0,6)$ |
| 6 | $(0,0),(1,0)$ |
| 7 | $(0,0),(1,1)$ |
| 8 | $(0,0),(1,2)$ |
| 9 | $(0,0),(1,2)$ |
| 10 | $(0,0),(1,4)$ |
| 11 | $(0,0),(1,6)$ |

## M.3 Three Tone Motifs in $OnPiMod_{12,12}$

Refer to the discussion in subsection 11.3.8 for the entries of this table. The order of these representatives is a historical one. After this table, the representatives are also visualized on a $12 \times 12$ square in list M.1.

**Three-Element Motif Classes in $OnPiMod_{12,12}$**

| *Class Nr.* | *Representative* | *Kernel* | *Class Weight* | *Volume* |
|---|---|---|---|---|
| 1 | $(0,0),(1,0),(2,0)$ | $\mathbb{Z}.(1,2) \times \mathbb{Z}.(1,1)$ | $(1,1,2)$ | 0 |
| 2 | $(0,0),(1,0),(3,0)$ | $\mathbb{Z}.(1,2) \times \mathbb{Z}.(0,1)$ | $(1,2,3)$ | 0 |
| 3 | $(0,0),(1,0),(4,0)$ | $\mathbb{Z}.(1,0) \times \mathbb{Z}.(0,1)$ | $(1,3,4)$ | 0 |
| 4 | $(0,0),(1,0),(5,0)$ | $\mathbb{Z}.(1,2) \times \mathbb{Z}.(1,1)$ | $(1,1,4)$ | 0 |
| 5 | $(0,0),(1,0),(6,0)$ | $\mathbb{Z}.(1,2) \times \mathbb{Z}.(1,0)$ | $(1,1,6)$ | 0 |
| 6 | $(0,0),(2,0),(4,0)$ | $(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times \mathbb{Z}.(1,1)$ | $(2,2,4)$ | 0 |
| 7 | $(0,0),(2,0),(6,0)$ | $(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times \mathbb{Z}.(0,1)$ | $(2,4,6)$ | 0 |
| 8 | $(0,0),(3,0),(6,0)$ | $\mathbb{Z}.(1,2) \times \mathbb{Z}_3^2$ | $(3,3,6)$ | 0 |
| 9 | $(0,0),(4,0),(8,0)$ | $(\mathbb{Z}_4^2) \times \mathbb{Z}.(1,1)$ | $(4,4,4)$ | 0 |
| 10 | $(0,0),(1,0),(0,1)$ | $0 \times 0$ | $(1,1,1)$ | 1 |
| 11 | $(0,0),(2,0),(0,1)$ | $\mathbb{Z}.(2,0) \times 0$ | $(1,1,2)$ | 2 |
| 12 | $(0,0),(3,0),(0,1)$ | $0 \times \mathbb{Z}.(1,0)$ | $(1,1,3)$ | 3 |
| 13 | $(0,1),(0,2),(3,0)$ | $0 \times \mathbb{Z}.(1,1)$ | $(1,1,1)$ | 3 |
| 14 | $(0,0),(0,1),(4,0)$ | $\mathbb{Z}.(1,0) \times 0$ | $(1,1,4)$ | 4 |
| 15 | $(0,0),(1,2),(2,0)$ | $\mathbb{Z}.(1,2) \times 0$ | $(1,1,2)$ | 4 |
| 16 | $(0,0),(2,0),(0,2)$ | $2\mathbb{Z}_4^2 \times 0$ | $(2,2,2)$ | 4 |
| 17 | $(0,0),(6,0),(0,1)$ | $\mathbb{Z}.(2,0) \times \mathbb{Z}.(1,0)$ | $(1,1,6)$ | 6 |
| 18 | $(0,0),(3,0),(0,2)$ | $\mathbb{Z}.(2,0) \times \mathbb{Z}.(0,1)$ | $(1,2,3)$ | 6 |
| 19 | $(0,0),(0,2),(3,1)$ | $\mathbb{Z}.(2,0) \times \mathbb{Z}.(1,1)$ | $(1,1,2)$ | 6 |
| 20 | $(0,0),(4,0),(0,2)$ | $(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times 0$ | $(2,2,4)$ | 4 |
| 21 | $(0,0),(4,0),(0,4)$ | $0 \times \mathbb{Z}_3^2$ | $(3,3,3)$ | 3 |
| 22 | $(0,0),(6,0),(0,2)$ | $2\mathbb{Z}_4^2 \times \mathbb{Z}.(1,0)$ | $(2,2,6)$ | 0 |
| 23 | $(0,2),(0,4),(6,0)$ | $2\mathbb{Z}_4^2 \times \mathbb{Z}.(1,1)$ | $(2,2,2)$ | 0 |
| 24 | $(0,0),(4,0),(0,4)$ | $\mathbb{Z}_4^2 \times 0$ | $(4,4,4)$ | 4 |
| 25 | $(0,0),(6,0),(0,3)$ | $\mathbb{Z}.(2,0) \times \mathbb{Z}_3^2$ | $(3,3,6)$ | 6 |
| 26 | $(0,0),(6,0),(0,6)$ | $2\mathbb{Z}_4^2 \times \mathbb{Z}_3^2$ | $(6,6,6)$ | 0 |

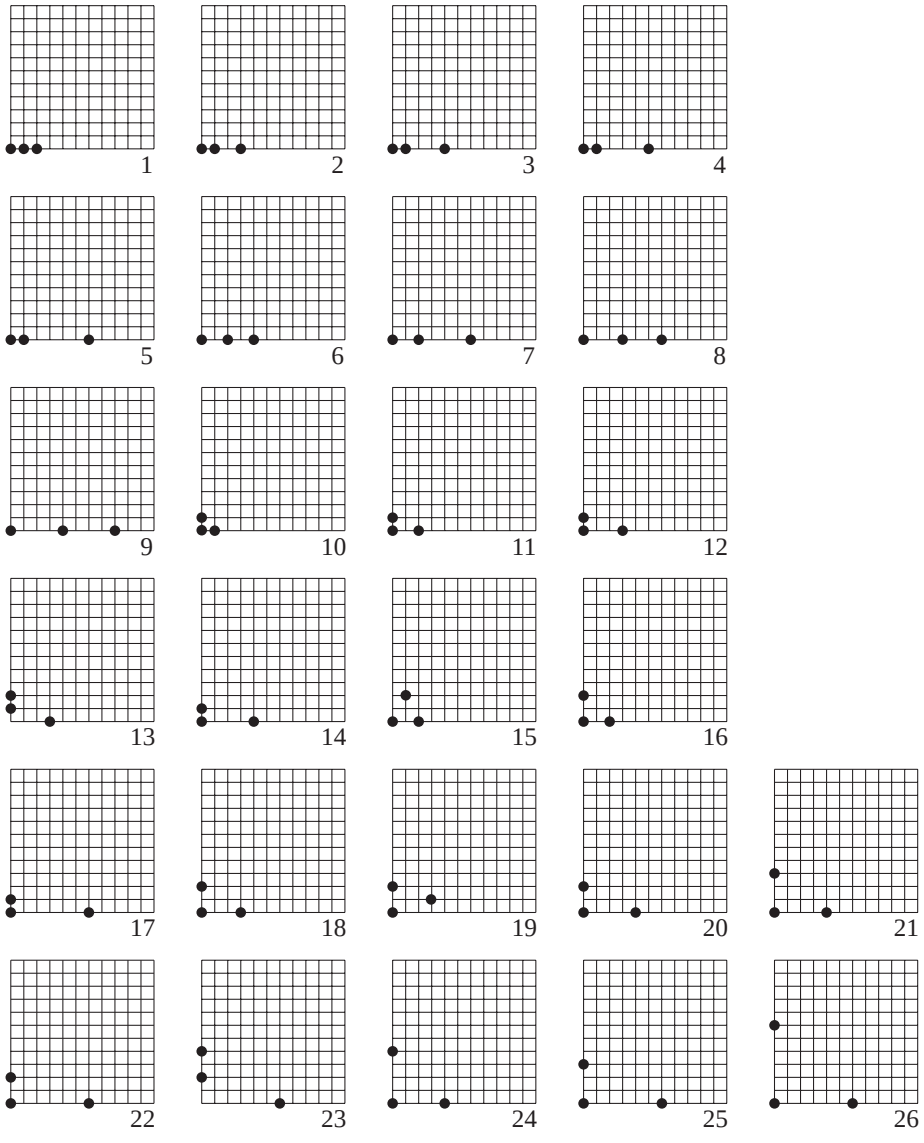



Figure M.1: Representatives of the 26 isomorphism classes of three-element motives in $OnPiMod_{12,12}$.

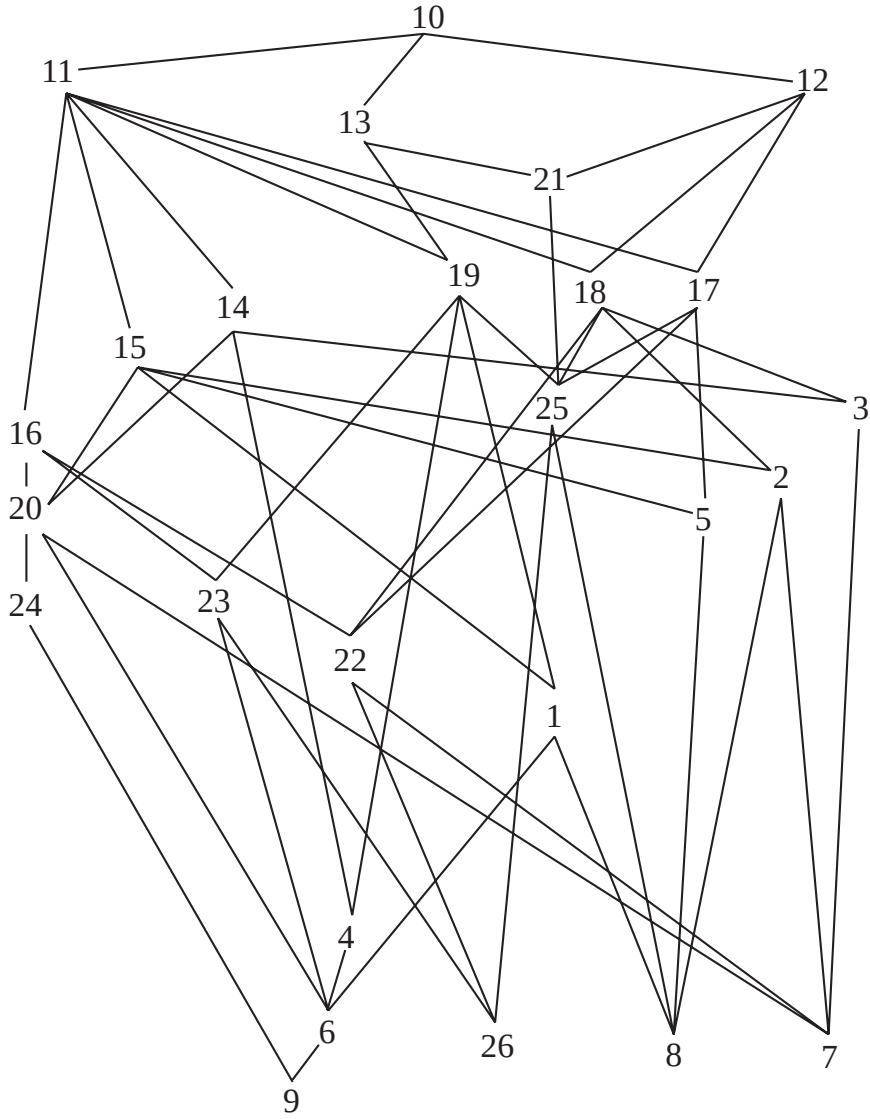



Figure M.2: Hasse diagram of dominance/specialization among the 26 isomorphism classes of motives in $OnPiMod_{12,12}$.

## M.4 Four Tone Motifs in $OnPiMod_{12,12}$

This list was calculated by Straub in [513], refer to subsection 11.3.8 for details. The list's numbering follows Straub's algorithm; * denotes classes which are not determined by volume and class weight.

| Four-Element Motif Classes | | | |
|---|---|---|---|
| *Class Nr.* | *Representative* | *Class Weight* | *Volume* |
| 0 | (0,0),(0,1),(0,2),(0,7) | (1,1,5,5) | 0 |
| 1 | (0,0),(0,1),(0,2),(0,3) | (1,1,2,2) | 0 |
| 2 | (0,0),(0,1),(0,2),(0,6) | (1,4,5,7) | 0 |
| 3 | (0,0),(0,1),(0,2),(0,5) | (1,4,2,3) | 0 |
| 4 | (0,0),(0,1),(0,2),(0,4) | (1,2,3,6) | 0 |
| 5 | (0,0),(1,0),(0,5),(0,6) | (4,4,5,5) | 0 |
| 6 | (0,0),(0,1),(0,4),(0,5) | (4,4,3,3) | 0 |
| 7 | (0,0),(0,1),(0,3),(0,5) | (4,2,2,6) | 0 |
| 8 | (0,0),(0,1),(0,4),(0,8) | (4,3,3,9) | 0 |
| 9 | (0,0),(0,1),(0,6),(0,7) | (5,5,5,5) | 0 |
| 10 | (0,0),(0,1),(0,3),(0,6) | (5,2,2,8) | 0 |
| 11 | (0,0),(0,1),(0,3),(0,7) | (5,2,3,7) | 0 |
| 12 | (0,0),(0,1),(0,4),(0,7) | (5,3,3,8) | 0 |
| 13 | (0,0),(0,1),(0,3),(0,10) | (2,2,2,2) | 0 |
| 14 | (0,0),(0,1),(0,3),(0,4) | (2,2,3,3) | 0 |
| 15 | (0,0),(0,1),(0,3),(0,9) | (2,3,7,8) | 0 |
| 16 | (0,0),(0,1),(0,4),(0,9) | (3,3,3,3) | 0 |
| 17 | (0,0),(0,2),(6,0),(6,10) | (23,23,22,22) | 0 |
| 18 | (0,0),(0,2),(0,4),(6,0) | (23,6,22,7) | 0 |
| 19 | (0,0),(0,2),(0,4),(6,2) | (6,6,22,22) | 0 |
| 20 | (0,0),(0,2),(0,4),(0,6) | (6,6,7,7) | 0 |
| 21 | (0,0),(0,2),0,4),(0,8) | (6,7,7,9) | 0 |
| 22 | (0,0),(0,2),(6,0),(6,2) | (22,22,22,22) | 0 |
| 23 | (0,0),(0,2),(6,0),(6,6) | (22,22,22,26) | 0 |
| 24 | (0,0),(0,2),(0,6),(6,2) | (22,22,7,7) | 0 |
| 25 | (0,0),(0,2),(0,6),(6,0) | (22,7,7,26) | 0 |
| 26 | (0,0),(0,2),(0,6),(0,8) | (7,7,7,7) | 0 |
| 27 | (0,0),(0,3),(0,6),(0,9) | (8,8,8,8) | 0 |
| 28 | (0,0),(0,6),(6,0),(6,6) | (26,26,26,26) | 0 |

| **Four-Element Motif Classes**—Continued | | | |
|---|---|---|---|
| *Class Nr.* | *Representative* | *Class Weight* | *Volume* |
| 29 | (0,0),(0,1),(0,2),(1,0) | (1,10,10,11) | 1 |
| 30 | (0,0),(0,1),(0,5),(1,0) | (4,10,10,14) | 1 |
| 31 | (0,0),(0,1),(0,6),(1,0) | (5,10,10,17) | 1 |
| 32 | (0,0),(0,1),(0,3),(1,0) | (2,10,11,12) | 1 |
| 33 | (0,0),(0,1),(0,4),(1,0) | (3,10,12,14) | 1 |
| 34* | (0,0),(0,1),(1,0),(1,5) | (10,10,10,10) | 1 |
| 35* | (0,0),(0,1),(1,0),(7,7) | (10,10,10,10) | 1 |
| 36* | (0,0),(0,1),(1,0),(1,1) | (10,10,10,10) | 1 |
| 37* | (0,0),(0,1),(1,0),(3,5) | (10,10,10,13) | 1 |
| 38* | (0,0),(0,1),(1,0),(3,11) | (10,10,10,13) | 1 |
| 39* | (0,0),(0,1),(1,0),(1,2) | (10,10,11,11) | 1 |
| 40* | (0,0),(0,1),(1,0),(5,10) | (10,10,11,11) | 1 |
| 41* | (0,0),(0,1),(1,0),(4,10) | (10,10,11,15) | 1 |
| 42* | (0,0),(0,1),(1,0),(2,4) | (10,10,11,15) | 1 |
| 43 | (0,0),(0,1),(1,0),(2,5) | (10,10,11,19) | 1 |
| 44* | (0,0),(0,1),(1,0),(1,3) | (10,10,12,12) | 1 |
| 45* | (0,0),(0,1),(1,0),(7,9) | (10,10,12,12) | 1 |
| 46* | (0,0),(0,1),(1,0),(3,3) | (10,10,12,12) | 1 |
| 47 | (0,0),(0,1),(1,0),(6,8) | (10,10,15,19) | 1 |
| 48* | (0,0),(0,1),(1,0),(1,4) | (10,10,14,14) | 1 |
| 49* | (0,0),(0,1),(1,0),(4,4) | (10,10,14,14) | 1 |
| 50 | (0,0),(0,1),(1,0),(1,6) | (10,10,17,17) | 1 |
| 51 | (0,0),(0,1),(1,0),(2,2) | (10,11,11,13) | 1 |
| 52 | (0,0),(0,1),(1,0),(2,3) | (10,11,13,15) | 1 |
| 53 | (0,0),(0,1),(1,0),(6,9) | (10,11,12,18) | 1 |
| 54 | (0,0),(0,1),(1,0),(8,8) | (10,13,14,14) | 1 |
| 55 | (0,0),(0,1),(1,0),(3,4) | (10,12,15,18) | 1 |
| 56 | (0,0),(0,1),(2,0),(3,1) | (11,11,12,12) | 1 |
| 57 | (0,0),(0,1),(2,0),(3,4) | (11,12,12,15) | 1 |
| 58 | (0,0),(0,1),(3,0),(4,1) | (12,12,14,14) | 1 |
| 59 | (0,0),(0,1),(0,2),(2,1) | (1,11,11,15) | 2 |
| 60 | (0,0),(0,1),(0,2),(2,0) | (1,11,11,16) | 2 |
| 61 | (0,0),(0,1),(0,5),(2,0) | (4,11,11,14) | 2 |
| 62 | (0,0),(0,1),(0,5),(2,1) | (4,11,11,20) | 2 |

| **Four-Element Motif Classes**—Continued | | | |
|---|---|---|---|
| *Class Nr.* | *Representative* | *Class Weight* | *Volume* |
| 63 | (0,0),(0,1),(0,6),(2,1) | (5,5,11,11) | 2 |
| 64 | (0,0),(0,1),(0,6),(2,0) | (5,22,11,11) | 2 |
| 65 | (0,0),(0,1),(0,3),(2,0) | (2,11,15,18) | 2 |
| 66 | (0,0),(0,1),(0,3),(2,1) | (2,11,16,18) | 2 |
| 67 | (0,0),(0,1),(0,4),(2,1) | (3,11,14,18) | 2 |
| 68 | (0,0),(0,1),(0,4),(2,0) | (3,11,20,18) | 2 |
| 69 | (0,0),(0,1),(2,0),(4,6) | (23,11,11,15) | 2 |
| 70 | (0,0),(0,1),(2,0),(4,0) | (6,11,11,14) | 2 |
| 71 | (0,0),(0,1),(2,0),(6,6) | (22,11,15,17) | 2 |
| 72 | (0,0),(0,1),(2,0),(6,0) | (7,11,14,17) | 2 |
| 73* | (0,0),(0,1),(2,0),(2,1) | (11,11,11,11) | 2 |
| 74* | (0,0),(0,1),(2,0),(2,5) | (11,11,11,11) | 2 |
| 75* | (0,0),(0,1),(2,0),(2,7) | (11,11,11,11) | 2 |
| 76* | (0,0),(0,1),(2,0),(2,11) | (11,11,11,11) | 2 |
| 77* | (0,0),(0,1),(2,0),(6,5) | (11,11,11,19) | 2 |
| 78* | (0,0),(0,1),(2,0),(6,11) | (11,11,11,19) | 2 |
| 79* | (0,0),(0,1),(2,0),(4,7) | (11,11,15,15) | 2 |
| 80* | (0,0),(0,1),(2,0),(8,11) | (11,11,15,15) | 2 |
| 81* | (0,0),(0,1),(2,0),(2,2) | (11,11,15,16) | 2 |
| 82* | (0,0),(0,1),(2,0),(8,10) | (11,11,15,16) | 2 |
| 83* | (0,0),(0,1),(2,0),(4,1) | (11,11,14,14) | 2 |
| 84* | (0,0),(0,1),(2,0),(8,5) | (11,11,14,14) | 2 |
| 85* | (0,0),(0,1),(2,0),(8,4) | (11,11,14,20) | 2 |
| 86* | (0,0),(0,1),(2,0),(2,4) | (11,11,14,20) | 2 |
| 87* | (0,0),(0,1),(2,0),(6,7) | (11,11,17,17) | 2 |
| 88* | (0,0),(0,1),(2,0),(6,1) | (11,11,17,17) | 2 |
| 89* | (0,0),(0,1),(2,0),(2,9) | (11,11,18,18) | 2 |
| 90* | (0,0),(0,1),(2,0),(2,3) | (11,11,18,18) | 2 |
| 91 | (0,0),(0,1),(2,0),(4,3) | (11,15,15,19) | 2 |
| 92 | (0,0),(0,1),(2,0),(4,2) | (11,15,16,19) | 2 |
| 93 | (0,0),(0,1),(2,0),(4,9) | (11,14,14,19) | 2 |
| 94 | (0,0),(0,1),(2,0),(4,8) | (11,14,20,19) | 2 |
| 95 | (0,0),(0,1),(4,2),(6,1) | (15,15,17,17) | 2 |
| 96 | (0,0),(0,1),(4,2),(6,4) | (15,16,18,18) | 2 |

| **Four-Element Motif Classes**—Continued | | | |
|---|---|---|---|
| *Class Nr.* | *Representative* | *Class Weight* | *Volume* |
| 97 | (0,0),(0,1),(4,0),(6,1) | (14,14,17,17) | 2 |
| 98 | (0,0),(0,1),(4,0),(6,4) | (14,20,18,18) | 2 |
| 99 | (0,0),(0,1),(0,2),(3,0) | (1,13,12,18) | 3 |
| 100 | (0,0),(0,1),(0,2),(3,1) | (1,12,12,19) | 3 |
| 101 | (0,0),(0,1),(0,5),(3,0) | (4,4,12,12) | 3 |
| 102 | (0,0),(0,1),(0,4),(3,2) | (4,3,13,12) | 3 |
| 103 | (0,0),(0,1),(0,6),(3,2) | (5,13,13,17) | 3 |
| 104 | (0,0),(0,1),(0,6),(3,1) | (5,12,12,17) | 3 |
| 105 | (0,0),(0,1),(0,6),(3,0) | (5,12,12,25) | 3 |
| 106 | (0,0),(0,1),(0,3),(3,2) | (2,13,12,19) | 3 |
| 107 | (0,0),(0,1),(0,3),(3,1) | (2,12,12,18) | 3 |
| 108 | (0,0),(0,1),(0,3),(3,0) | (2,12,21,18) | 3 |
| 109 | (0,0),(0,1),(0,4),(3,0) | (3,3,12,12) | 3 |
| 110 | (0,0),(0,1),(0,4),(3,1) | (3,3,12,21) | 3 |
| 111 | (0,0),(0,1),(3,0),(6,0) | (8,12,12,17) | 3 |
| 112 | (0,0),(0,3),(0,6),(3,0) | (8,21,21,25) | 3 |
| 113* | (0,0),(0,1),(3,0),(3,5) | (13,13,12,12) | 3 |
| 114* | (0,0),(0,1),(3,0),(3,11) | (13,13,12,12) | 3 |
| 115* | (0,0),(0,1),(3,0),(9,11) | (13,13,12,12) | 3 |
| 116 | (0,0),(0,1),(3,2),(3,8) | (13,13,17,17) | 3 |
| 117 | (0,0),(0,1),(3,0),(3,2) | (13,12,19,18) | 3 |
| 118* | (0,0),(0,1),(3,0),(9,7) | (12,12,12,12) | 3 |
| 119* | (0,0),(0,1),(3,0),(3,7) | (12,12,12,12) | 3 |
| 120* | (0,0),(0,1),(3,0),(3,1) | (12,12,12,12) | 3 |
| 121* | (0,0),(0,1),(3,0),(9,3) | (12,12,12,21) | 3 |
| 122* | (0,0),(0,1),(3,0),(3,3) | (12,12,12,21) | 3 |
| 123 | (0,0),(0,1),(3,0),(6,5) | (12,12,19,19) | 3 |
| 124 | (0,0),(0,1),(3,0),(6,1) | (12,12,17,17) | 3 |
| 125 | (0,0),(0,1),(3,0),(3,6) | (12,12,17,25) | 3 |
| 126 | (0,0),(0,1),(3,0),(3,10) | (12,12,18,18) | 3 |
| 127 | (0,0),(0,1),(3,0),(6,9) | (12,21,18,18) | 3 |
| 128* | (0,0),(0,3),(3,0),(3,3) | (21,21,21,21) | 3 |
| 129* | (0,0),(0,3),(3,0),(9,9) | (21,21,21,21) | 3 |
| 130 | (0,0),(0,3),(3,0),(3,6) | (21,21,25,25) | 3 |

| **Four-Element Motif Classes**—Continued | | | |
|---|---|---|---|
| *Class Nr.* | *Representative* | *Class Weight* | *Volume* |
| 131 | (0,0),(0,1),(0,2),(4,3) | (1,15,15,15) | 4 |
| 132 | (0,0),(0,1),(0,2),(4,1) | (1,15,14,14) | 4 |
| 133 | (0,0),(0,1),(0,2),(4,0) | (1,15,14,20) | 4 |
| 134 | (0,0),(0,1),(0,5),(4,2) | (4,15,15,14) | 4 |
| 135 | (0,0),(0,1),(0,5),(4,3) | (4,15,15,20) | 4 |
| 136 | (0,0),(0,1),(0,5),(4,0) | (4,14,14,14) | 4 |
| 137 | (0,0),(0,1),(0,5),(4,1) | (4,14,14,24) | 4 |
| 138 | (0,0),(0,1),(0,6),(4,3) | (5,5,15,15) | 4 |
| 139 | (0,0),(0,1),(0,6),(4,1) | (5,5,14,14) | 4 |
| 140 | (0,0),(0,1),(0,6),(4,0) | (5,7,15,14) | 4 |
| 141 | (0,0),(0,1),(0,3),(4,2) | (2,2,15,15) | 4 |
| 142 | (0,0),(0,1),(0,3),(4,1) | (2,2,14,20) | 4 |
| 143 | (0,0),(0,1),(0,3),(4,0) | (2,3,15,14) | 4 |
| 144 | (0,0),(0,1),(0,3),(4,3) | (2,3,15,20) | 4 |
| 145 | (0,0),(0,1),(0,4),(4,1) | (3,3,14,14) | 4 |
| 146 | (0,0),(0,1),(0,4),(4,0) | (3,3,14,24) | 4 |
| 147 | (0,0),(0,2),(2,0),(6,10) | (23,16,16,16) | 4 |
| 148 | (0,0),(0,2),(2,0),(4,4) | (23,16,20,20) | 4 |
| 149 | (0,0),(0,1),(4,0),(8,6) | (6,15,15,14) | 4 |
| 150 | (0,0),(0,2),(0,4),(2,0) | (6,16,16,20) | 4 |
| 151 | (0,0),(0,2),(0,4),(4,2) | (6,20,20,20) | 4 |
| 152 | (0,0),(0,2),(0,4),(4,0) | (6,20,20,24) | 4 |
| 153 | (0,0),(0,2),(2,0),(2,6) | (22,22,16,16) | 4 |
| 154 | (0,0),(0,2),(4,0),(6,2) | (22,22,20,20) | 4 |
| 155 | (0,0),(0,2),(0,6),(2,0) | (22,7,16,20) | 4 |
| 156 | (0,0),(0,2),(0,6),(4,0) | (7,7,20,20) | 4 |
| 157 | (0,0),(0,2),(0,6),(4,2) | (7,7,20,24) | 4 |
| 158 | (0,0),(0,1),(4,0),(8,0) | (9,14,14,14) | 4 |
| 159 | (0,0),(0,2),(4,0),(8,0) | (9,20,20,20) | 4 |
| 160 | (0,0),(0,4),(0,8),(4,0) | (9,24,24,24) | 4 |
| 161* | (0,0),(0,1),(4,2),(4,7) | (15,15,15,15) | 4 |
| 162* | (0,0),(0,1),(4,2),(4,3) | (15,15,15,15) | 4 |
| 163* | (0,0),(0,1),(4,0),(4,7) | (15,15,14,14) | 4 |
| 164* | (0,0),(0,1),(4,0),(4,11) | (15,15,14,14) | 4 |

| **Four-Element Motif Classes**—Continued | | | |
|---|---|---|---|
| *Class Nr.* | *Representative* | *Class Weight* | *Volume* |
| 165* | (0,0),(0,1),(4,0),(4,2) | (15,15,14,20) | 4 |
| 166* | (0,0),(0,1),(4,0),(4,10) | (15,15,14,20) | 4 |
| 167* | (0,0),(0,1),(4,0),(4,1) | (14,14,14,14) | 4 |
| 168* | (0,0),(0,1),(4,0),(4,5) | (14,14,14,14) | 4 |
| 169 | (0,0),(0,1),(4,0),(4,4) | (14,14,14,24) | 4 |
| 170 | (0,0),(0,2),(2,0),(2,2) | (16,16,16,16) | 4 |
| 171* | (0,0),(0,2),(2,0),(8,8) | (16,16,20,20) | 4 |
| 172* | (0,0),(0,2),(2,0),(2,4) | (16,16,20,20) | 4 |
| 173* | (0,0),(0,2),(4,0),(4,2) | (20,20,20,20) | 4 |
| 174* | (0,0),(0,2),(4,0),(4,10) | (20,20,20,20) | 4 |
| 175 | (0,0),(0,2),(4,0),(4,4) | (20,20,20,24) | 4 |
| 176 | (0,0),(0,4),(4,0),(4,4) | (24,24,24,24) | 4 |
| 177 | (0,0),(0,1),(0,2),(6,1) | (1,1,17,17) | 6 |
| 178 | (0,0),(0,1),(0,2),(6,3) | (1,2,19,18) | 6 |
| 179 | (0,0),(0,1),(0,2),(6,4) | (1,23,18,18) | 6 |
| 180 | (0,0),(0,1),(0,2),(6,0) | (1,22,19,17) | 6 |
| 181 | (0,0),(0,1),(0,5),(6,0) | (4,4,17,17) | 6 |
| 182 | (0,0),(0,1),(0,4),(6,5) | (4,3,19,18) | 6 |
| 183 | (0,0),(0,1),(0,5),(6,3) | (4,6,18,18) | 6 |
| 184 | (0,0),(0,1),(0,5),(6,1) | (4,7,19,17) | 6 |
| 185 | (0,0),(0,1),(0,6),(6,5) | (5,5,19,19) | 6 |
| 186 | (0,0),(0,1),(0,6),(6,1) | (5,5,17,17) | 6 |
| 187 | (0,0),(0,1),(0,6),(6,2) | (5,22,19,19) | 6 |
| 188 | (0,0),(0,1),(0,6),(6,4) | (5,22,18,18) | 6 |
| 189 | (0,0),(0,1),(0,6),(6,3) | (5,8,18,18) | 6 |
| 190 | (0,0),(0,1),(0,6),(6,0) | (5,26,17,17) | 6 |
| 191 | (0,0),(0,1),(0,3),(6,0) | (2,2,17,25) | 6 |
| 192 | (0,0),(0,1),(0,3),(6,4) | (2,2,18,18) | 6 |
| 193 | (0,0),(0,1),(0,3),(6,5) | (2,23,19,18) | 6 |
| 194 | (0,0),(0,1),(0,3),(6,1) | (2,22,17,18) | 6 |
| 195 | (0,0),(0,1),(0,3),(6,3) | (2,22,18,25) | 6 |
| 196 | (0,0),(0,1),(0,4),(6,1) | (3,3,17,25) | 6 |
| 197 | (0,0),(0,1),(0,4),(6,3) | (3,3,18,18) | 6 |
| 198 | (0,0),(0,1),(0,4),(6,2) | (3,6,19,18) | 6 |

| **Four-Element Motif Classes**—Continued | | | |
|---|---|---|---|
| *Class Nr.* | *Representative* | *Class Weight* | *Volume* |
| 199 | (0,0),(0,1),(0,4),(6,0) | (3,7,17,18) | 6 |
| 200 | (0,0),(0,1),(0,4),(6,4) | (3,7,18,25) | 6 |
| 201 | (0,0),(0,1),(6,3),(6,9) | (22,8,18,18) | 6 |
| 202 | (0,0),(0,3),(0,6),(6,3) | (8,8,25,25) | 6 |
| 203 | (0,0),(0,3),(0,6),(6,0) | (8,26,25,25) | 6 |
| 204* | (0,0),(0,1),(6,0),(6,5) | (19,19,17,17) | 6 |
| 205* | (0,0),(0,1),(6,0),(6,11) | (19,19,17,17) | 6 |
| 206* | (0,0),(0,1),(6,2),(6,3) | (19,19,18,18) | 6 |
| 207* | (0,0),(0,1),(6,2),(6,5) | (19,19,18,18) | 6 |
| 208* | (0,0),(0,1),(6,0),(6,1) | (17,17,17,17) | 6 |
| 209* | (0,0),(0,1),(6,0),(6,7) | (17,17,17,17) | 6 |
| 210* | (0,0),(0,1),(6,0),(6,3) | (17,18,18,25) | 6 |
| 211* | (0,0),(0,1),(6,0),(6,9) | (17,18,18,25) | 6 |
| 212* | (0,0),(0,1),(6,3),(6,4) | (18,18,18,18) | 6 |
| 213* | (0,0),(0,1),(6,3),(6,10) | (18,18,18,18) | 6 |
| 214* | (0,0),(0,3),(6,0),(6,3) | (25,25,25,25) | 6 |
| 215* | (0,0),(0,3),(6,0),(6,9) | (25,25,25,25) | 6 |

## M.5 Three Tone Motifs in $OnPiMod_{5,12}$

Refer to the discussion in subsection 11.3.8 for the entries of this table. The order of these representatives is a historical one.

| **Three-Element Motif Classes in $OnPiMod_{5,12}$** | | |
|---|---|---|
| *Cl. Nr.* | *Representative in* $\mathbb{Z}_5 \times \mathbb{Z}_{12}$ | *Kernel* |
| 1 | (0,0),(1,0),(0,4) | $\mathbb{Z}_4^2 \times (\mathbb{Z}_3 \times 0) \times (0 \times \mathbb{Z}_5)$ |
| 2 | (0,0),(1,0),(4,4) | $\mathbb{Z}_4^2 \times (\mathbb{Z}_3 \times 0) \times \mathbb{Z}_5.(1,1)$ |
| 3 | (0,0),(1,0),(3,4) | $\mathbb{Z}_4^2 \times (\mathbb{Z}_3 \times 0) \times \mathbb{Z}_5.(2,1)$ |
| 4 | (0,0),(0,4),(1,8) | $\mathbb{Z}_4^2 \times \mathbb{Z}_3.(1,1) \times (\mathbb{Z}_5 \times 0)$ |
| 5 | (0,0),(1,4),(4,8) | $\mathbb{Z}_4^2 \times \mathbb{Z}_3.(1,1) \times \mathbb{Z}_5.(1,1)$ |
| 6 | (0,0),(1,0),(0,10) | $(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (\mathbb{Z}_3 \times 0) \times (0 \times \mathbb{Z}_5)$ |
| 7 | (0,0),(1,0),(4,10) | $(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (\mathbb{Z}_3 \times 0) \times \mathbb{Z}_5.(1,1)$ |
| 8 | (0,0),(1,0),(3,10) | $(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (\mathbb{Z}_3 \times 0) \times \mathbb{Z}_5.(2,1)$ |
| 9 | (0,0),(0,4),(1,2) | $(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times \mathbb{Z}_3.(1,1) \times (\mathbb{Z}_5 \times 0)$ |
| 10 | (0,0),(1,4),(0,2) | $(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times \mathbb{Z}_3.(1,1) \times (0 \times \mathbb{Z}_5)$ |
| 11 | (0,0),(1,4),(4,2) | $(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times \mathbb{Z}_3.(1,1) \times \mathbb{Z}_5.(1,1)$ |
| 12 | (0,0),(1,4),(3,2) | $(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times \mathbb{Z}_3.(1,1) \times \mathbb{Z}_5.(2,1)$ |
| 13 | (0,0),(0,4),(1,6) | $(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (0 \times \mathbb{Z}_3) \times (\mathbb{Z}_5 \times 0)$ |
| 14 | (0,0),(1,4),(0,6) | $(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (0 \times \mathbb{Z}_3) \times (0 \times \mathbb{Z}_5)$ |
| 15 | (0,0),(1,4),(4,6) | $(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(1,1)$ |
| 16 | (0,0),(1,4),(3,6) | $(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(2,1)$ |
| 17 | (0,0),(3,4),(1,6) | $(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(1,2)$ |
| 18 | (0,0),(1,4),(1,6) | $(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(4,1)$ |
| 19 | (0,0),(1,0),(0,1) | $(\mathbb{Z}_4 \times 0) \times (\mathbb{Z}_3 \times 0) \times (0 \times \mathbb{Z}_5)$ |
| 20 | (0,0),(1,0),(4,1) | $(\mathbb{Z}_4 \times 0) \times (\mathbb{Z}_3 \times 0) \times \mathbb{Z}_5.(1,1)$ |
| 21 | (0,0),(1,0),(3,1) | $(\mathbb{Z}_4 \times 0) \times (\mathbb{Z}_3 \times 0) \times \mathbb{Z}_5.(2,1)$ |
| 22 | (0,0),(0,4),(1,5) | $(\mathbb{Z}_4 \times 0) \times \mathbb{Z}_3.(1,1) \times (\mathbb{Z}_5 \times 0)$ |
| 23 | (0,0),(1,4),(0,5) | $(\mathbb{Z}_4 \times 0) \times \mathbb{Z}_3.(1,1) \times (0 \times \mathbb{Z}_5)$ |
| 24 | (0,0),(1,4),(4,5) | $(\mathbb{Z}_4 \times 0) \times \mathbb{Z}_3.(1,1) \times \mathbb{Z}_5.(1,1)$ |
| 25 | (0,0),(1,4),(3,5) | $(\mathbb{Z}_4 \times 0) \times \mathbb{Z}_3.(1,1) \times \mathbb{Z}_5.(2,1)$ |
| 26 | (0,0),(0,4),(1,9) | $(\mathbb{Z}_4 \times 0) \times (0 \times \mathbb{Z}_3) \times (\mathbb{Z}_5 \times 0)$ |
| 27 | (0,0),(1,4),(0,9) | $(\mathbb{Z}_4 \times 0) \times (0 \times \mathbb{Z}_3) \times (0 \times \mathbb{Z}_5)$ |

| **Three-Element Motif Classes in** $OnPiMod_{5,12}$—continued | | |
|---|---|---|
| *Cl. Nr.* | *Representative in* $\mathbb{Z}_5 \times \mathbb{Z}_{12}$ | *Kernel* |
| 28 | (0,0),(1,4),(4,9) | $(\mathbb{Z}_4 \times 0) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(1,1)$ |
| 29 | (0,0),(1,4),(3,9) | $(\mathbb{Z}_4 \times 0) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(2,1)$ |
| 30 | (0,0),(3,4),(1,9) | $(\mathbb{Z}_4 \times 0) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(1,2)$ |
| 31 | (0,0),(1,4),(1,9) | $(\mathbb{Z}_4 \times 0) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(4,1)$ |
| 32 | (0,0),(0,6),(1,1) | $\mathbb{Z}_4.(1,2) \times (\mathbb{Z}_3 \times 0) \times (\mathbb{Z}_5 \times 0)$ |
| 33 | (0,0),(1,6),(0,1) | $\mathbb{Z}_4.(1,2) \times (\mathbb{Z}_3 \times 0) \times (0 \times \mathbb{Z}_5)$ |
| 34 | (0,0),(1,6),(4,1) | $\mathbb{Z}_4.(1,2) \times (\mathbb{Z}_3 \times 0) \times \mathbb{Z}_5.(1,1)$ |
| 35 | (0,0),(1,6),(3,1) | $\mathbb{Z}_4.(1,2) \times (\mathbb{Z}_3 \times 0) \times \mathbb{Z}_5.(2,1)$ |
| 36 | (0,0),(0,10),(1,5) | $\mathbb{Z}_4.(1,2) \times \mathbb{Z}_3.(1,1) \times (\mathbb{Z}_5 \times 0)$ |
| 37 | (0,0),(1,10),(0,5) | $\mathbb{Z}_4.(1,2) \times \mathbb{Z}_3.(1,1) \times (0 \times \mathbb{Z}_5)$ |
| 38 | (0,0),(1,10),(4,5) | $\mathbb{Z}_4.(1,2) \times \mathbb{Z}_3.(1,1) \times \mathbb{Z}_5.(1,1)$ |
| 39 | (0,0),(1,10),(3,5) | $\mathbb{Z}_4.(1,2) \times \mathbb{Z}_3.(1,1) \times \mathbb{Z}_5.(2,1)$ |
| 40 | (0,0),(0,10),(1,9) | $\mathbb{Z}_4.(1,2) \times (0 \times \mathbb{Z}_3) \times (\mathbb{Z}_5 \times 0)$ |
| 41 | (0,0),(1,10),(0,9) | $\mathbb{Z}_4.(1,2) \times (0 \times \mathbb{Z}_3) \times (0 \times \mathbb{Z}_5)$ |
| 42 | (0,0),(4,10),(1,9) | $\mathbb{Z}_4.(1,2) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(1,1)$ |
| 43 | (0,0),(3,10),(1,9) | $\mathbb{Z}_4.(1,2) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(1,2)$ |
| 44 | (0,0),(1,10),(3,9) | $\mathbb{Z}_4.(1,2) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(2,1)$ |
| 45 | (0,0),(1,10),(1,9) | $\mathbb{Z}_4.(1,2) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(4,1)$ |

# Appendix N

# Well-Tempered and Just Modulation Steps

## N.1 12-Tempered Modulation Steps

### N.1.1 Scale Orbits and Number of Quantized Modulations

In the following table, the exclamation sign (!) in column 6 means that quantization is not possible for every translation quantity $p$ in the notation of theorem 30.

| **Orbits and Number of Quantized Modulations** | | | |
|---|---|---|---|
| *Class* | *# Min. Cadence Sets* | *# Quanta* | *# Quant. Mod.* |
| 38 | 9 | 42 | 54 (!) |
| 38.1 | 5 | 20 | 26 |
| 47 | 6 | 28 | 30 |
| 47.1 | 15 | 66 | 114 |
| 50 | 7 | 34 | 42 |
| 50.1 | 6 | 36 | 46 |
| 52 | 5 | 24 | 24 (!) |
| 55 | 6 | 30 | 32 (!) |
| 61 | 10 | 38 | 62 |
| 62 | 5 | 24 | 24 (!) |
| 39 | 9 | 29 | 93 |
| 39.1 | 6 | 23 | 55 |
| 40 | 10 | 24 | 108 |
| 40.1 | 7 | 26 | 72 |

| **Orbits and Number of Quant. Mod.—Continued** | | | |
|---|---|---|---|
| *Class* | *# Min. Cadence Sets* | *# Quanta* | *# Quant. Mod.* |
| 41 | 7 | 25 | 75 |
| 41.1 | 6 | 21 | 53 |
| 42 | 6 | 22 | 54 |
| 42.1 | 7 | 28 | 74 |
| 43 | 6 | 22 | 57 |
| 43.1 | 7 | 26 | 72 |
| 44 | 9 | 23 | 89 |
| 45 | 7 | 21 | 63 |
| 45.1 | 10 | 21 | 105 |
| 46 | 6 | 26 | 56 |
| 48 | 10 | 23 | 109 |
| 48.1 | 7 | 28 | 68 |
| 49 | 7 | 21 | 71 |
| 49.1 | 7 | 26 | 74 |
| 51 | 9 | 13 | 86 |
| 53 | 7 | 27 | 67 |
| 53.1 | 9 | 25 | 91 |
| 54 | 7 | 32 | 71 |
| 54.1 | 21 | 32 | 226 |
| 56 | 7 | 24 | 70 |
| 57 | 8 | 21 | 71 |
| 58 | 18 | 17 | 185 |
| 59 | 11 | 22 | 101 |
| 60 | 6 | 21 | 60 |

### N.1.2 Quanta and Pivots for the Modulations Between Diatonic Major Scales (No.38.1)

| **Quanta and Pivots for the Modulations Between Diatonic Major Scales** | | | | |
|---|---|---|---|---|
| *Transl. p* | *Cadence* | *Quantum* | *Modulator* | *Pivots* |
| 1 | $\{II, V\}$ | ●○●●○●●●●●●● | $e^5 11$ | $\{II, III, V, VII\}$ |
| 1 | $\{II, III\}$ | ●○●●○●●●●●●● | $e^5 11$ | $\{II, III, V, VII\}$ |
| 2 | $\{VII\}$ | ○●●○●●○●○○○● | $e^6 11$ | $\{II, IV, VII\}$ |
| 2 | $\{II, V\}$ | ○●●○●●○●○●○● | $e^6 11$ | $\{II, IV, V, VII\}$ |
| 2 | $\{IV, V\}$ | ○●●○●●○●○●○● | $e^6 11$ | $\{II, IV, V, VII\}$ |
| 3 | $\{II, V\}$ | ●○●○○●○●●●●● | $e^7 11$ | $\{II, III, V, VII\}$ |
| 3 | $\{II, III\}$ | ●○●○○●○●●●●● | $e^7 11$ | $\{II, III, V, VII\}$ |
| 4 | $\{VII\}$ | ○○●●○●●○○●○● | $e^8 11$ | $\{II, IV, V, VII\}$ |
| 4 | $\{IV, V\}$ | ○●●●●●●●○●○● | $e^8 11$ | $\{II, III, V, VII\}$ |
| 4 | $\{II, III\}$ | ●●●●○●●●●●○● | $e^8 11$ | $\{V, VII\}$ |
| 5 | $\{VII\}$ | ○○●○●●○●○○●● | $e^9 11$ | $\{II, IV, VII\}$ |
| 6 | $\{II, III\}$ | ○●●●●●○●●●●● | $e^6$ | $\{II, III, V, VII\}$ |
| 6 | $\{IV, V\}$ | ○●●●●●●●●●○● | $e^{10} 11$ | $\{II, IV, V, VII\}$ |
| 6 | $\{IV, V\}$ | ●●●●○●●●●●○● | $e^6$ | $\{II, IV, V, VII\}$ |
| 6 | $\{II, III\}$ | ●●●●○●○●●●●● | $e^{10} 11$ | $\{II, III, V, VII\}$ |
| 7 | $\{VII\}$ | ●○●○○●●○○●○● | $e^{11} 11$ | $\{III, V, VII\}$ |
| 8 | $\{VII\}$ | ○●●○○●○●○○●● | $e^0 11$ | $\{II, VII\}$ |
| 8 | $\{IV, V\}$ | ○●●●●●○●●●●● | $e^0 11$ | $\{II, IV, V, VII\}$ |
| 8 | $\{II, III\}$ | ●●●●○●○●○●●● | $e^0 11$ | $\{II, III, V, VII\}$ |
| 9 | $\{II, V\}$ | ○○●○●●●●●●○● | $e^1 11$ | $\{II, IV, V, VII\}$ |
| 9 | $\{IV, V\}$ | ○○●○●●●●●●○● | $e^1 11$ | $\{II, IV, V, VII\}$ |
| 10 | $\{VII\}$ | ●○●●○●○○○●○● | $e^2 11$ | $\{III, V, VII\}$ |
| 10 | $\{II, V\}$ | ●○●●○●○●○●○● | $e^2 11$ | $\{II, III, V, VII\}$ |
| 10 | $\{II, III\}$ | ●○●●○●○●○●○● | $e^2 11$ | $\{II, III, V, VII\}$ |
| 11 | $\{II, V\}$ | ○●●○●●●●●●●● | $e^3 11$ | $\{II, IV, V, VII\}$ |
| 11 | $\{IV, V\}$ | ○●●○●●●●●●●● | $e^3 11$ | $\{II, IV, V, VII\}$ |

### N.1.3 Quanta and Pivots for the Modulations Between Melodic Minor Scales (No.47.1)

| Quanta and Pivots for the Modulations Between Melodic Minor Scales | | | | |
|---|---|---|---|---|
| *Transl. p* | *Cadence* | *Quantum* | *Modulator* | *Pivots* |
| 1 | $\{II, IV\}, \{IV, VII\}$ | ●●●●○●●○○●●○ | $e^3 11$ | $\{II, IV, VII\}$ |
| 1 | $\{III, VI\}, \{V, VI\}$ | ●●●●●●○●●○●● | $e^3 11$ | $\{I, III, V, VI\}$ |
| 1 | $\{III, VII\}$ | ●○○●●○●●●●○● | $e^3 11$ | $\{III, V, VII\}$ |
| 1 | $\{IV, V\}$ | ●●●●○●●●●●●○ | $e^3 11$ | $\{II, IV, V, VII\}$ |
| 1 | $\{II, III\}$ | ●○○●●●●●●●●● | $e^3 11$ | $\{II, III, V, VII\}$ |
| 1 | $\{I, VII\}$ | ●●●●●○●●●●○● | $e^3 11$ | $\{I, III, V, VII\}$ |
| 2 | $\{III, V\}, \{III, VII\}, \{II, III\}$ | ●●○●●●○●○●○● | $e^4 11$ | $\{II, III, V, VII\}$ |
| 2 | $\{II, IV\}, \{II, VI\}, \{I, II\}$ | ●○●○●●○●○●○● | $e^4 11$ | $\{I, II, IV, VI\}$ |
| 2 | $\{I, III\}, \{III, VI\}, \{III, IV\}$ | ○●●●○●○●○●○● | $e^4 11$ | $\{I, III, IV, VI\}$ |
| 3 | $\{III, V\}, \{III, VI\}, \{V, VI\}, \{I, V\}$ | ●○●●○●●●○○●● | $e^5 11$ | $\{I, III, V, VI\}$ |
| 3 | $\{III, V\}, \{III, VI\}, \{V, VI\}, \{I, V\}$ | ●○●●○●●●○○●● | $e^5 11$ | $\{I, III, V, VI\}$ |
| 3 | $\{III, V\}, \{III, VI\}, \{V, VI\}, \{I, V\}$ | ●○●●○●●●○○●● | $e^5 11$ | $\{I, III, V, VI\}$ |
| 3 | $\{III, V\}, \{III, VI\}, \{V, VI\}, \{I, V\}$ | ●○●●○●●●○○●● | $e^5 11$ | $\{I, III, V, VI\}$ |
| 4 | $\{III, V\}$ | ○●●○●●○○●●●○ | $e^6 11$ | $\{III, V\}$ |
| 4 | $\{I, III\}$ | ○○●●●○○●○○○● | $e^6 11$ | $\{I, III\}$ |
| 4 | $\{II, VI\}, \{I, II\}$ | ●●●○●●●●○●○● | $e^6 11$ | $\{I, II, IV, VI\}$ |
| 4 | $\{IV, VII\}$ | ●●●●●●●○○●○○ | $e^6 11$ | $\{II, IV, VII\}$ |
| 4 | $\{III, VI\}$ | ○●●●●●○●○○○● | $e^6 11$ | $\{I, III, VI\}$ |
| 4 | $\{III, VII\}$ | ●○○●○○●●○●○● | $e^6 11$ | $\{III, V, VII\}$ |
| 4 | $\{V, VI\}$ | ●●●●●●●●○○○● | $e^6 11$ | $\{I, III, V, VI\}$ |
| 4 | $\{III, IV\}$ | ○●●●●●○●○●○● | $e^6 11$ | $\{I, III, IV, VI\}$ |
| 4 | $\{II, III\}$ | ●●○●○●●●○●○● | $e^6 11$ | $\{II, III, V, VII\}$ |
| 4 | $\{I, VII\}$ | ●○●●●○●●○●○● | $e^6 11$ | $\{I, III, V, VII\}$ |
| 4 | $\{I, III\}$ | ●○○●●○○●●○○● | $e^4$ | $\{I, III\}$ |
| 4 | $\{III, V\}$ | ○○●●○○●●○○●● | $e^4$ | $\{III, V\}$ |
| 4 | $\{III, VII\}, \{II, III\}$ | ○●●●○●●●○●●● | $e^4$ | $\{II, III, V, VII\}$ |
| 4 | $\{III, VI\}, \{III, IV\}$ | ●●○●●●○●●●○● | $e^4$ | $\{I, III, IV, VI\}$ |
| 5 | $\{I, II\}, \{I, V\}, \{III, VI\}, \{V, VI\}$ | ●○●●●●○●●○○● | $e^7 11$ | $\{I, III, V, VI\}$ |
| 5 | $\{II, VI\}, \{I, II\}$ | ●○●○○●○●●●●● | $e^7 11$ | $\{I, II, IV, VI\}$ |

| **Quanta and Pivots for Melodic Minor Scales**—Continued | | | | |
|---|---|---|---|---|
| *Transl. p* | *Cadence* | *Quantum* | *Modulator* | *Pivots* |
| 5 | $\{IV, VII\}, \{IV, V\}$ | ●○●●●○●○●●○ | $e^7 11$ | $\{II, IV, V, VII\}$ |
| 5 | $\{III, VII\}$ | ●○○●●○○●●●●● | $e^7 11$ | $\{III, V, VII\}$ |
| 6 | $\{III, V\}, \{III, VII\}, \{II, III\}$ | ●●○●○●○●●●○● | $e^8 11$ | $\{II, III, V, VII\}$ |
| 6 | $\{I, III\}, \{III, VI\}, \{III, IV\}$ | ○●●●○●●●○●○● | $e^8 11$ | $\{I, III, IV, VI\}$ |
| 6 | $\{I, III\}, \{III, VI\}, \{III, IV\}$ | ●●○●○●●●○●○● | $e^6$ | $\{I, III, IV, VI\}$ |
| 6 | $\{III, V\}, \{III, VII\}, \{II, III\}$ | ○●●●○●○●●●○● | $e^6$ | $\{II, III, V, VII\}$ |
| 7 | $\{III, V\}, \{I, V\}, \{III, VII\}, \{I, VII\}$ | ●○●●○○●●○●●● | $e^9 11$ | $\{I, III, V, VII\}$ |
| 7 | $\{II, VI\}, \{I, II\}$ | ●○●○●●○●○●●● | $e^9 11$ | $\{I, II, IV, VI\}$ |
| 7 | $\{IV, VII\}, \{IV, V\}$ | ●○●●●●●●○●○○ | $e^9 11$ | $\{II, IV, V, VII\}$ |
| 7 | $\{III, VI\}$ | ○○●●●●●●○○●● | $e^9 11$ | $\{I, III, VI\}$ |
| 8 | $\{III, V\}$ | ●○○●○○○●○○●● | $e^{10} 11$ | $\{III, V\}$ |
| 8 | $\{I, III\}$ | ○○●●○○○●●○○● | $e^{10} 11$ | $\{I, III\}$ |
| 8 | $\{II, VI\}$ | ●●●○○●○○●●●● | $e^{10} 11$ | $\{II, IV, VI\}$ |
| 8 | $\{IV, VII\}, \{IV, V\}$ | ●●●●○●○●●●●○ | $e^{10} 11$ | $\{II, IV, V, VII\}$ |
| 8 | $\{III, VI\}$ | ○○●●○●○●●○○● | $e^{10} 11$ | $\{I, III, VI\}$ |
| 8 | $\{III, VII\}$ | ●●○●○○○●○●●● | $e^{10} 11$ | $\{III, V, VII\}$ |
| 8 | $\{V, VI\}$ | ●○●●○●○●●○●● | $e^{10} 11$ | $\{I, III, V, VI\}$ |
| 8 | $\{III, IV\}$ | ○●●●○●○●●●○● | $e^{10} 11$ | $\{I, III, IV, VI\}$ |
| 8 | $\{II, III\}$ | ●●○●○●○●○●●● | $e^{10} 11$ | $\{II, III, V, VII\}$ |
| 8 | $\{I, VII\}$ | ●●●●○○○●●●●● | $e^{10} 11$ | $\{I, III, V, VII\}$ |
| 8 | $\{I, III\}$ | ●○○●●○○●●○○● | $e^8$ | $\{I, III\}$ |
| 8 | $\{III, V\}$ | ○○●●○○●●○○●● | $e^8$ | $\{III, V\}$ |
| 8 | $\{III, VII\}, \{II, III\}$ | ○●●●○●●●○●●● | $e^8$ | $\{II, III, V, VII\}$ |
| 8 | $\{III, VI\}, \{II, IV\}$ | ●●○●●●○●●●○● | $e^8$ | $\{I, III, IV, VI\}$ |
| 9 | $\{III, V\}$ | ●○○●●○○●●○○● | $e^{11} 11$ | $\{III, V\}$ |
| 9 | $\{II, IV\}, \{II, VI\}$ | ●○●○○●●○○●○● | $e^{11} 11$ | $\{II, IV, VI\}$ |
| 9 | $\{I, III\}, \{I, V\}, \{III, VII\}, \{I, VII\}$ | ●○●●●○○●●●○● | $e^{11} 11$ | $\{I, III, V, VII\}$ |
| 9 | $\{I, II\}$ | ●○●○●●●●○●○● | $e^{11} 11$ | $\{I, III, IV, VI\}$ |
| 10 | $\{II, V\}, \{III, VII\}, \{II, III\}$ | ●●○●○●○●○●○● | $e^0 11$ | $\{II, III, V, VII\}$ |
| 10 | $\{II, IV\}, \{IV, VII\}, \{IV, V\}$ | ●○●●○●○●○●●○ | $e^0 11$ | $\{II, IV, V, VII\}$ |
| 10 | $\{I, III\}, \{III, VI\}, \{III, IV\}$ | ○●●●○●○●○●●● | $e^0 11$ | $\{I, III, IV, VI\}$ |

| **Quanta and Pivots for Melodic Minor Scales**—Continued | | | | |
|---|---|---|---|---|
| *Transl. p* | *Cadence* | *Quantum* | *Modulator* | *Pivots* |
| 11 | $\{II, IV\}, \{II, VI\}$ | ●●●○●●○○●●○● | $e^1 11$ | $\{I, IV, VI\}$ |
| 11 | $\{III, VI\}$ | ○○●●○●●●●○●● | $e^1 11$ | $\{I, III, VI\}$ |
| 11 | $\{III, VII\}, \{I, VII\}$ | ●●●●○●●○●●●● | $e^1 11$ | $\{I, III, V, VII\}$ |
| 11 | $\{V, VI\}$ | ●●●●○●●●●○●● | $e^1 11$ | $\{I, III, V, VI\}$ |
| 11 | $\{III, IV\}$ | ○○●●●●●●●●●● | $e^1 11$ | $\{I, III, IV, VI\}$ |
| 11 | $\{I, II\}$ | ●●●○●●●●●●○● | $e^1 11$ | $\{I, II, IV, VI\}$ |

### N.1.4 Quanta and Pivots for the Modulations Between Harmonic Minor Scales (No.54.1)

For this table, we need a numbering of the 21 minimal cadence sets:

$0 = \{II, VII\}$ $1 = \{I, III\}$ $2 = \{II, IV\}$ $3 = \{III, V\}$ $4 = \{IV, VI\}$
$5 = \{V, VII\}$ $6 = \{I, VI\}$ $7 = \{IV, VII\}$ $8 = \{I, V\}$ $9 = \{II, VI\}$
$10 = \{III, VII\}$ $11 = \{I, IV\}$ $12 = \{II, V\}$ $13 = \{III, VI\}$ $14 = \{VI, VII\}$
$15 = \{I, VII\}$ $16 = \{I, II\}$ $17 = \{II, III\}$ $18 = \{III, IV\}$ $19 = \{IV, V\}$
$20 = \{V, VI\}$

| **Quanta and Pivots for the Modulations Between Harmonic Minor Scales** | | | |
|---|---|---|---|
| *Transl. p* | *Cadence Nr.* | *Quantum* | *Pivots* |
| 3/9 | 1,3,6,8,10,11,15-20 | ○○○○○○○○○○○○ | $\{I, II, III, IV, V, VI, VII\}$ |
| 3/9 | 2,4,7,9,14 | ●○●●○●●○●●○● | $\{II, IV, VI, VII\}$ |
| 3/9 | 5,12 | ○●●○●●○●●○●● | $\{II, V, VII\}$ |
| 4/8 | 0,7,12,14-17,19 | ○○○○○○○○○○○○ | $\{I, II, III, IV, V, VI, VII\}$ |
| 4/8 | 1,6,13 | ●○○●●○○●●○○● | $\{I, II, VI\}$ |
| 4/8 | 2 | ●●●○●●●○●●●○ | $\{II, IV\}$ |
| 4/8 | 3 | ○○●●○○●●○○●● | $\{III, V\}$ |
| 4/8 | 4,11,18 | ●●○●●●○●●●○● | $\{I, III, IV, VI\}$ |
| 4/8 | 5,10 | ○●●●○●●●○●●● | $\{III, V, VII\}$ |
| 4/8 | 8,20 | ●○●●●○●●●○●● | $\{I, III, V, VI\}$ |
| 6 | 2,7 | ●○●○○●●○●○○● | $\{II, IV, VII\}$ |
| 6 | 3,10,17 | ○●●●○●○●●●○● | $\{II, III, V, VII\}$ |
| 6 | 4,9,14 | ●○●●○●●○●●○● | $\{II, IV, VI, VII\}$ |

| **Quanta and Pivots for Harmonic Minor Scales**—Continued | | | |
|---|---|---|---|
| *Transl. p* | *Cadence Nr.* | *Quantum* | *Pivots* |
| 6 | 5,12 | ○●●○○●○●●○○● | $\{II, V, VII\}$ |
| 6 | 8,11,13,15,16,18 | ●●●●○●●●●●○● | $\{I, II, II, IV, V, VI, VII\}$ |
| 6 | 19 | ●●●○○●●●●○○● | $\{II, IV, V, VII\}$ |
| 1,2,5,7,10,11 | any cadence set | ●●●●●●●●●●●● | $\{I, II, II, IV, V, VI, VII\}$ |

### N.1.5 Examples of 12-Tempered Modulations for all Fourth Relations

| **Examples of Modulations for all Fourth Relations** | | | |
|---|---|---|---|
| *Start* → *Target* | *Neutral* | *Pivots* | *Cadence* |
| $C \rightsquigarrow F$ | $I_C$ | $VII_F \cup II_F$ | $I_F, IV_F, V_F, I_F$ |
| $C \rightsquigarrow B_\flat$ | $I_C, V_C$ | $III_{B_\flat}, V_{B_\flat} \cup VII_{B_\flat}$ | $VII_{B_\flat}, I_{B_\flat}$ |
| $C \rightsquigarrow E_\flat$ | $I_C, V_C$ | $II_{E_\flat}, V_{E_\flat} \cup VII_{E_\flat}$ | $V_{E_\flat} \cup VII_{E_\flat}, I_{E_\flat}$ |
| $C \rightsquigarrow A_\flat$ | $I_C, IV_C$ | $II_{A_\flat} \cup VII_{A_\flat}$ | $IV_{A_\flat}, V_{A_\flat}, I_{A_\flat}$ |
| $C \rightsquigarrow D_\flat$ | $I_C$ | $III_{D_\flat}, II_{D_\flat} \cup VII_{D_\flat}$ | $II_{D_\flat}, V_{D_\flat}, I_{D_\flat}$ |
| $C \rightsquigarrow G_\flat$ | $I_C, IV_C \cup VI_C, VI_C \cup VII_C$ | $II_{G_\flat} \cup VII_{G_\flat}, II_{G_\flat}$ | $V_{G_\flat} \cup VII_{G_\flat}, I_{G_\flat}$ |
| $C \rightsquigarrow B$ | $I_C, V_C, II_C$ | $IV_B, II_B \cup VII_B$ | $IV_B, V_B, I_B$ |
| $C \rightsquigarrow E$ | $I_C, VI_C$ | $V_E \cup VII_E$ | $IV_E, V_E, I_E$ |
| $C \rightsquigarrow A$ | $I_C, V_C$ | $IV_A$ | $IV_A \cup VII_A, V_A, I_A$ |
| $C \rightsquigarrow D$ | $I_C, VI_C, V_C$ | $VII_D \cup V_D$ | $II_D, V_D, I_D$ |
| $C \rightsquigarrow G$ | $I_C, V_C$ | $III_G$ | $II_G, V_G, I_G$ |

## N.2 2-3-5-Just Modulation Steps

The following tables show data for modulations from $C$-tonic.

### N.2.1 Modulation Steps between Just Major Scales

Here, we have the two modulators

$$\Phi_1 = e^b, \tag{N.1}$$
$$\Phi_2 = e^b.A. \tag{N.2}$$

The numbering of minimal cadence sets is the one used in formula (26.2). The tonics $D^*$ and $B_\flat^*$ are the usual third comma shifted representatives of D and B-flat.

| Pivots for the Modulations Between Just Major Scales | | | | |
|---|---|---|---|---|
| *Translation* | *Target Tonic* | *Modulator* | *Cadence* | *Pivots* |
| $(1,0)$ | $G$ | $\Phi_2$ | 5 | $\{V, VII\}$ |
| $(-1,0)$ | $F$ | $\Phi_2$ | 1 | $\{II, IV\}$ |
| $(2,0)$ | $D$ | $\Phi_2$ | 1 | $\{II, IV\}$ |
| $(-2,0)$ | $B_\flat$ | $\Phi_2$ | 5 | $\{V, VII\}$ |
| $(0,1)$ | $E$ | $\Phi_2$ | 1 | $\{II, V, VII\}$ |
| $(0,1)$ | $E$ | $\Phi_1$ | 5 | $\{II, V, VII\}$ |
| $(0,-1)$ | $A_\flat$ | $\Phi_2$ | 5 | $\{II, IV, VII\}$ |
| $(0,-1)$ | $A_\flat$ | $\Phi_1$ | 5 | $\{II, V, VII\}$ |
| $(1,1)$ | $B$ | $\Phi_2$ | 1 | $\{II, V, VI\}$ |
| $(-1,-1)$ | $D_\flat$ | $\Phi_2$ | 5 | $\{II, V, VII\}$ |
| $(1,-1)$ | $E_\flat$ | $\Phi_2$ | 1 | $\{II, V, VII\}$ |
| $(1,-1)$ | $E_\flat$ | $\Phi_1$ | 1 | $\{II, IV, VII\}$ |
| $(-1,1)$ | $A$ | $\Phi_2$ | 5 | $\{II, IV, VII\}$ |
| $(-1,1)$ | $A$ | $\Phi_1$ | 1 | $\{II, IV, VII\}$ |
| $(-2,1)$ | $D^*$ | $\Phi_2$ | 5 | $\{III, V, VII\}$ |
| $(2,-1)$ | $B_\flat^*$ | $\Phi_2$ | 1 | $\{II, IV, VI\}$ |

### N.2.2 Modulation Steps between Natural Minor Scales

We have the two modulators

$$\begin{aligned} \Phi_1 &= e^b, \\ \Phi_2 &= e^b.A. \end{aligned}$$

The minimal cadence sets are these:

$$J_1 = \{VII\}, J_2 = \{III, VI\}, J_3 = \{V, VI\},$$
$$J_4 = \{IV, V\}, J_5 = \{II\}, J_6 = \{III, IV\}.$$

| Pivots for the Modulations Between Natural Minor Scales | | | | |
|---|---|---|---|---|
| *Translation* | *Target Tonic* | *Modulator* | *Cadence* | *Pivots* |
| $(-1,0)$ | $F$ | $\Phi_2$ | 5 | $\{II, IV\}$ |
| $(1,0)$ | $G$ | $\Phi_2$ | 1 | $\{V, VII\}$ |
| $(-2,0)$ | $B_\flat$ | $\Phi_2$ | 1 | $\{V, VII\}$ |
| $(2,0)$ | $D$ | $\Phi_2$ | 5 | $\{II, IV\}$ |

| **Pivots for the Modulations Between Natural Minor Scales**—Continued | | | | |
|---|---|---|---|---|
| *Translation* | *Target Tonic* | *Modulator* | *Cadence* | *Pivots* |
| $(0,-1)$ | $A_\flat$ | $\Phi_2$ | 1 | $\{II, IV, VII\}$ |
| $(0,-1)$ | $A_\flat$ | $\Phi_1$ | 5 | $\{II, IV, VII\}$ |
| $(0,1)$ | $E$ | $\Phi_1$ | 5 | $\{II, IV, VII\}$ |
| $(0,1)$ | $E$ | $\Phi_2$ | 5 | $\{II, V, VII\}$ |
| $(-1,-1)$ | $D_\flat$ | $\Phi_2$ | 1 | $\{III, V, VII\}$ |
| $(1,1)$ | $B$ | $\Phi_2$ | 5 | $\{II, IV, VI\}$ |
| $(1,-1)$ | $E_\flat$ | $\Phi_1$ | 1 | $\{II, V, VII\}$ |
| $(1,-1)$ | $E_\flat$ | $\Phi_2$ | 5 | $\{II, V, VII\}$ |
| $(-1,1)$ | $A$ | $\Phi_1$ | 1 | $\{II, V, VII\}$ |
| $(-1,1)$ | $A$ | $\Phi_2$ | 1 | $\{II, IV, VII\}$ |
| $(2,-1)$ | $B_\flat^*$ | $\Phi_2$ | 5 | $\{II, IV, VI\}$ |
| $(-2,1)$ | $D^*$ | $\Phi_2$ | 1 | $\{III, V, VII\}$ |

### N.2.3 Modulation Steps From Natural Minor to Major Scales

We have the two modulators

$$\begin{aligned}\Phi_1 &= e^b.A,\\ \Phi_2 &= e^b.B, B = \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}.\end{aligned}$$

The minimal cadence sets are the same as for major scales.

| **Pivots for the Modulations From Natural Minor to Major Scales** | | | | |
|---|---|---|---|---|
| *Translation* | *Target Tonic* | *Modulator* | *Cadence* | *Pivots* |
| $(-2,0)$ | $B_\flat$ | $\Phi_1$ | 5 | $\{V, VII\}$ |
| $(-1,0)$ | $F$ | $\Phi_1$ | 1 | $\{II, IV\}$ |
| $(1,0)$ | $G$ | $\Phi_1$ | 5 | $\{V, VII\}$ |
| $(2,0)$ | $D$ | $\Phi_1$ | 1 | $\{II, IV\}$ |
| $(-1,-1)$ | $D_\flat$ | $\Phi_1$ | 5 | $\{II, V, VII\}$ |
| $(0,-1)$ | $A_\flat$ | $\Phi_1$ | 5 | $\{II, IV, VII\}$ |
| $(1,-1)$ | $E_\flat$ | $\Phi_1$ | 1 | $\{II, V, VII\}$ |
| $(2,-1)$ | $B_\flat^*$ | $\Phi_1$ | 1 | $\{II, IV, VI\}$ |

### N.2.4 Modulation Steps From Major to Natural Minor Scales

We have the two modulators (same as above).

$$\begin{aligned}\Phi_1 &= e^b.A,\\ \Phi_2 &= e^b.B, B = \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}.\end{aligned}$$

The minimal cadence sets are those of minor scales.

**Pivots for the Modulations From Major to Natural Minor Scales**

| *Translation* | *Target Tonic* | *Modulator* | *Cadence* | *Pivots* |
|---|---|---|---|---|
| $(-2,0)$ | $B_\flat$ | $\Phi_1$ | 1 | $\{V, VII\}$ |
| $(-1,0)$ | $F$ | $\Phi_1$ | 5 | $\{II, IV\}$ |
| $(1,0)$ | $G$ | $\Phi_1$ | 1 | $\{V, VII\}$ |
| $(2,0)$ | $D$ | $\Phi_1$ | 5 | $\{II, IV\}$ |
| $(-2,1)$ | $D$ | $\Phi_1$ | 1 | $\{III, V, VII\}$ |
| $(-1,1)$ | $A$ | $\Phi_1$ | 1 | $\{II, IV, VII\}$ |
| $(0,1)$ | $E$ | $\Phi_1$ | 5 | $\{II, V, VII\}$ |
| $(1,1)$ | $B$ | $\Phi_1$ | 5 | $\{II, IV, VI\}$ |

### N.2.5 Modulation Steps Between Harmonic Minor Scales

We have the unique translation modulator

$$\Phi = e^b.$$

The minimal cadence sets are these:

$$J_1 = \{III\}, J_2 = \{II\}, J_3 = \{VII\}, J_4 = \{I, IV\}, J_5 = \{I, V\},$$
$$J_6 = \{I, VI\}, J_7 = \{IV, V\}, J_8 = \{IV, VI\}, J_9 = \{V, VI\}.$$

**Pivots for the Modulations Between Harmonic Minor Scales**

| *Translation* | *Target Tonic* | *Modulator* | *Cadence* | *Pivots* |
|---|---|---|---|---|
| $(3,0)$ | $A^*$ | $\Phi$ | 8 | $\{II, IV, VI\}$ |
| $(-3,0)$ | $E^*_\flat$ | $\Phi$ | 8 | $\{II, IV, VI\}$ |
| $(0,1)$ | $E$ | $\Phi$ | 2 | $\{II, IV\}$ |
| $(0,1)$ | $E$ | $\Phi$ | 4 | $\{I, III, IV, VI\}$ |
| $(0,1)$ | $E$ | $\Phi$ | 8 | $\{I, III, IV, VI\}$ |

| **Pivots for the Modulations Between Harmonic Minor Scales**—Continued | | | | |
|---|---|---|---|---|
| *Translation* | *Target Tonic* | *Modulator* | *Cadence* | *Pivots* |
| $(0,-1)$ | $A_\flat$ | $\Phi$ | 2 | $\{II, IV\}$ |
| $(0,-1)$ | $A_\flat$ | $\Phi$ | 4 | $\{I, III, IV, VI\}$ |
| $(0,-1)$ | $A_\flat$ | $\Phi$ | 8 | $\{I, III, IV, VI\}$ |
| $(2,1)$ | $f_\sharp$ | $\Phi$ | 8 | $\{II, IV, VI, VII\}$ |
| $(-2,-1)$ | $G_\flat$ | $\Phi$ | 8 | $\{II, IV, VI, VII\}$ |
| $(0,2)$ | $G_\sharp$ | $\Phi$ | 4 | $\{I, III, IV, VI\}$ |
| $(0,-2)$ | $F_\flat$ | $\Phi$ | 4 | $\{I, III, IV, VI\}$ |
| $(1,2)$ | $D_\sharp$ | $\Phi$ | 4 | $\{I, III, IV, VI\}$ |
| $(-1,-2)$ | $B_{\flat,\flat}$ | $\Phi$ | 4 | $\{I, III, IV, VI\}$ |
| $(1,0)$ | $G$ | $\Phi$ | 1,5,7,9 | $\{I \dots VII\}$ |
| $(-1,0)$ | $F$ | $\Phi$ | 1,5,7,9 | $\{I \dots VII\}$ |
| $(2,0)$ | $D$ | $\Phi$ | 9 | $\{I \dots VII\}$ |
| $(-2,0)$ | $B_\flat$ | $\Phi$ | 9 | $\{I \dots VII\}$ |
| $(3,0)$ | $A^*$ | $\Phi$ | 9 | $\{I \dots VII\}$ |
| $(-3,0)$ | $E^*_\flat$ | $\Phi$ | 9 | $\{I \dots VII\}$ |
| $(-2 \dots 1,1)$ | | $\Phi$ | 7 | $\{I \dots VII\}$ |
| $(-2 \dots -1,1)$ | | $\Phi$ | 9 | $\{I \dots VII\}$ |
| $(2,1)$ | $F_\sharp$ | $\Phi$ | 5,9 | $\{I \dots VII\}$ |
| $(1 \dots 2,1)$ | | $\Phi$ | 4 | $\{I \dots VII\}$ |
| $(1,1)$ | $B$ | $\Phi$ | 8 | $\{I \dots VII\}$ |
| $(-1 \dots 2,1)$ | | $\Phi$ | 7 | $\{I \dots VII\}$ |
| $(1 \dots 2,-1)$ | | $\Phi$ | 9 | $\{I \dots VII\}$ |
| $(-2,-1)$ | $G_\flat$ | $\Phi$ | 5,9 | $\{I \dots VII\}$ |
| $(-2 \dots -1,-1)$ | | $\Phi$ | 4 | $\{I \dots VII\}$ |
| $(-1,-1)$ | $D_\flat$ | $\Phi$ | 8 | $\{II, IV, VI\}$ |
| $(0,2)$ | $G_\sharp$ | $\Phi$ | 7 | $\{II, IV, VI\}$ |
| $(0,-2)$ | $F_\flat$ | $\Phi$ | 7 | $\{I \dots VII\}$ |

## N.2.6 Modulation Steps Between Melodic Minor Scales

We have the two modulators

$$\begin{aligned} \Phi_1 &= e^b, \\ \Phi_2 &= e^b.A. \end{aligned}$$

The minimal cadence sets are these:

$$J_1 = \{I\}, J_2 = \{II\}, J_3 = \{III\},$$
$$J_4 = \{I, IV\}, J_5 = \{VI\}, J_6 = \{VII\}.$$

**Pivots for the Modulations Between Melodic Minor Scales**

| *Translation* | *Target Tonic* | *Modulator* | *Cadence* | *Pivots* |
|---|---|---|---|---|
| $(1, 0)$ | $G$ | $\Phi_2$ | 6 | $\{V, VII\}$ |
| $(-1, 0)$ | $F$ | $\Phi_2$ | 2 | $\{II, IV\}$ |
| $(2, 0)$ | $D$ | $\Phi_2$ | 2 | $\{II, IV\}$ |
| $(2, 0)$ | $D$ | $\Phi_1$ | 5 | $\{II, IV, VII\}$ |
| $(-2, 0)$ | $B_\flat$ | $\Phi_1$ | 6 | $\{V, VII\}$ |
| $(-2, 0)$ | $B_\flat$ | $\Phi_1$ | 5 | $\{II, IV, VII\}$ |
| $(0, 1)$ | $E$ | $\Phi_2$ | 2 | $\{II, V, VII\}$ |
| $(0, 1)$ | $E$ | $\Phi_2$ | 3 | $\{I, III, IV, VI\}$ |
| $(0, 1)$ | $E$ | $\Phi_1$ | 5 | $\{I, III, IV, VI\}$ |
| $(0, 1)$ | $E$ | $\Phi_1$ | 6 | $\{II, III, V, VII\}$ |
| $(0, -1)$ | $A_\flat$ | $\Phi_1$ | 5 | $\{I, III, IV, VI\}$ |
| $(0, -1)$ | $A_\flat$ | $\Phi_2$ | 5 | $\{I, III, V, VI\}$ |
| $(0, -1)$ | $A_\flat$ | $\Phi_1$ | 6 | $\{II, III, V, VII\}$ |
| $(0, -1)$ | $A_\flat$ | $\Phi_2$ | 6 | $\{II, IV, VII\}$ |
| $(-1, 1)$ | $A$ | $\Phi_1$ | 2 | $\{II, IV, VI, VII\}$ |
| $(-1, 1)$ | $A$ | $\Phi_1$ | 3 | $\{I, III, V, VI\}$ |
| $(-1, 1)$ | $A$ | $\Phi_2$ | 5 | $\{I, III, V, VI\}$ |
| $(-1, 1)$ | $A$ | $\Phi_2$ | 6 | $\{II, IV, VII\}$ |
| $(1, -1)$ | $E_\flat$ | $\Phi_1$ | 2 | $\{II, IV, VI, VII\}$ |
| $(1, -1)$ | $E_\flat$ | $\Phi_1$ | 3 | $\{I, III, V, VI\}$ |
| $(-1, 1)$ | $E_\flat$ | $\Phi_2$ | 3 | $\{I, III, IV, VI\}$ |
| $(-1, 1)$ | $E_\flat$ | $\Phi_2$ | 2 | $\{II, V, VII\}$ |
| $(-2, 1)$ | $D^*$ | $\Phi_2$ | 5 | $\{III, V, VII\}$ |
| $(2, -1)$ | $B_\flat^*$ | $\Phi_2$ | 1 | $\{II, IV, VI\}$ |

### N.2.7 General Modulation Behaviour for 32 Alterated Scales

The following list refers to the 32 scales as defined in 27.1.6.1. Following Radl [429], we say that a scale type

- *has no modulations* if its modulation domain is empty (always excluding the start tonality!),
- *has infinite modulations* if its modulation domain is infinite,
- *has modulations* if its modulation domain is not empty (always excluding the start tonality!), *has limited modulations* if the transitive closure (all tonics which can be reached by successive modulations from relative modulation domains) of its modulation domain is not total space.

**Modulation Behaviour in 32 Alterated Scale Types**

| *No.* | *Scale Type* | *Behaviour* |
|---|---|---|
| 1 | $c, d, e, f, g, a, b$ | has modulations: see table N.2.1 |
| 2 | $c, d, e, f, g, a, b_\flat$ | has modulations: $\pm(1,0)$ with $III$, $\pm(0,1)$ with $V, VI$ |
| 3 | $c, d, e, f, g, a_\flat, b$ | has modulations: corresponds to No. 11 |
| 4 | $c, d, e, f, g, a_\flat, b_\flat$ | has modulations: corresponds to No. 9 |
| 5 | $c, d, e, f_\sharp, g, a, b$ | has modulations: see table N.2.2 |
| 6 | $c, d, e, f_\sharp, g, a, b_\flat$ | has modulations (special table in [429]) |
| 7 | $c, d, e, f_\sharp, g, a_\flat, b$ | has modulations: $\pm(1,0)$ with $II$, $\pm(0,1)$ with $III, VII$ |
| 8 | $c, d, e, f_\sharp, g, a_\flat, b_\flat$ | has limited modulations: see No. 25 |
| 9 | $c, d, e_\flat, f, g, a, b$ | has modulations: see table N.2.6 |
| 10 | $c, d, e_\flat, f, g, a, b_\flat$ | has modulations: corresponds to No. 2 |
| 11 | $c, d, e_\flat, f, g, a_\flat, b$ | has modulations: see table N.2.5 |
| 12 | $c, d, e_\flat, f, g, a_\flat, b_\flat$ | has modulations: see table N.2.2 |
| 13 | $c, d, e_\flat, f_\sharp, g, a, b$ | has modulations: $\pm(1,0)$ with $II$, $\pm(-1,1)$ with $I, VII$ |
| 14 | $c, d, e_\flat, f_\sharp, g, a, b_\flat$ | has modulations: $\pm(1,1)$ with $I, V$, $\pm(0,1)$ with $I, V$ |
| 15 | $c, d, e_\flat, f_\sharp, g, a_\flat, b$ | has no modulations |
| 16 | $c, d, e_\flat, f_\sharp, g, a_\flat, b_\flat$ | has modulations: corresponds to No. 7 |
| 17 | $c, d_\flat, e, f, g, a, b$ | has modulations: see No. 7 |
| 18 | $c, d_\flat, e, f, g, a, b_\flat$ | has modulations: corresponds to No. 14 |
| 19 | $c, d_\flat, e, f, g, a_\flat, b$ | has no modulations: see No. 15 |
| 20 | $c, d_\flat, e, f, g, a_\flat, b_\flat$ | has modulations: corresponds to No. 13 |
| 21 | $c, d_\flat, e, f_\sharp, g, a, b$ | has infinite modulations |
| 22 | $c, d_\flat, e, f_\sharp, g, a, b_\flat$ | has infinite modulations |
| 23 | $c, d_\flat, e, f_\sharp, g, a_\flat, b$ | has modulations: $\pm(1,0)$ with $V$, $\pm(0,1)$ with $II$ |
| 24 | $c, d_\flat, e, f_\sharp, g, a_\flat, b_\flat$ | has infinite modulations |
| 25 | $c, d_\flat, e_\flat, f, g, a, b$ | has limited modulations |

| **Modulation Behaviour in 32 Altered Scale Types**—Continued | | |
|---|---|---|
| *No.* | *Scale Type* | *Behaviour* |
| 26 | $c, d_\flat, e_\flat, f, g, a, b_\flat$ | has modulations: corresponds to No. 6 |
| 27 | $c, d_\flat, e_\flat, f, g, a_\flat, b$ | has modulations: see No. 16 |
| 28 | $c, d_\flat, e_\flat, f, g, a_\flat, b_\flat$ | has modulations: see table N.2.1 |
| 29 | $c, d_\flat, e_\flat, f_\sharp, g, a, b$ | has infinite modulations: corresponds to No. 24 |
| 30 | $c, d_\flat, e_\flat, f_\sharp, g, a, b_\flat$ | has infinite modulations: corresponds to No. 22 |
| 31 | $c, d_\flat, e_\flat, f_\sharp, g, a_\flat, b$ | has modulations: corresponds to No. 23 |
| 32 | $c, d_\flat, e_\flat, f_\sharp, g, a_\flat, b_\flat$ | has infinite modulations: corresponds to No. 21 |

# Appendix O

# Counterpoint Steps

## O.1 Contrapuntal Symmetries

All the following tables relate to representatives of strong dichotomies $(X/Y)$ which are indicated in the table after the counterpoint theorem 33 in subsection 31.3.3.

### O.1.1 Class Nr. 64

| $k$ | $g$ | $g.X[\varepsilon]$ | $g.X[\varepsilon] \cap X[\varepsilon]$ | $card(g.X[\varepsilon] \cap X[\varepsilon])$ |
|---|---|---|---|---|
| 2 | $e^6(1+\varepsilon.6)$ | $(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.e^6X$ | $z$ even: $z+\varepsilon.\{5,11\}$<br>$z$ odd: $z+\varepsilon.X$ | 48 |
| | $e^{\varepsilon.8}5$ | $\mathbb{Z}_{12} + \varepsilon.e^8 5X$ | $\mathbb{Z}_{12} + \varepsilon.\{4,5,7,9\}$ | |
| 4 | $e^6(1+\varepsilon.6)$ | see $k=2$ | see $k=2$ | 48 |
| 5 | $e^{\varepsilon.11}11$ | $\mathbb{Z}_{12} + \varepsilon.e^{11}11X$ | $\mathbb{Z}_{12} + \varepsilon.\{2,4,7,9\}$ | 48 |
| 7 | $e^6(1+\varepsilon.6)$ | see $k=2$ | see $k=2$ | 48 |
| 9 | $e^6(1+\varepsilon.6)$ | see $k=2$ | see $k=2$ | 48 |
| | $e^{\varepsilon.3}7$ | $\mathbb{Z}_{12} + \varepsilon.e^3 7X$ | $\mathbb{Z}_{12} + \varepsilon.\{2,4,5,7\}$ | |
| 11 | $e^{\varepsilon.3}7$ | see $k=9$ | see $k=9$ | 48 |
| | $e^{\varepsilon.11}11$ | see $k=5$ | see $k=5$ | |
| | $e^{\varepsilon.8}5$ | see $k=2$ | see $k=2$ | |

### O.1.2 Class Nr. 68

| $k$ | $g$ | $g.X[\varepsilon]$ | $g.X[\varepsilon] \cap X[\varepsilon]$ | $card(g.X[\varepsilon] \cap X[\varepsilon])$ |
|---|---|---|---|---|
| 0 | $e^{\varepsilon.6}7$ | $\mathbb{Z}_{12} + \varepsilon.e^6 7X$ | $\mathbb{Z}_{12} + \varepsilon.(X - \{0\})$ | 60 |
| 1 | $e^{\varepsilon.3}7$ | $\mathbb{Z}_{12} + \varepsilon.e^3 7X$ | $\mathbb{Z}_{12} + \varepsilon.\{0,2,3,5\}$ | 48 |
| | $e^{\varepsilon.6}(1+\varepsilon.6)$ | $(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.e^6 X$ | $z$ even: $z + \varepsilon.\{2,8\}$ | |
| | | | $z$ odd: $z + \varepsilon.X$ | |
| | $7 + \varepsilon.6$ | $(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.7X$ | $z$ even: $z + \varepsilon.\{0,2,8\}$ | |
| | | | $z$ odd: $z + \varepsilon.(X - \{0\})$ | |
| 2 | 5 | $\mathbb{Z}_{12} + \varepsilon.5X$ | $\mathbb{Z}_{12} + \varepsilon.\{0,1,3,5\}$ | 48 |
| 3 | $e^{\varepsilon.6}(1+\varepsilon.6)$ | $(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.e^6 X$ | $z$ even: $z + \varepsilon.\{2,8\}$ | 48 |
| | | | $z$ odd: $z + \varepsilon.X$ | |
| | $7 + \varepsilon.6$ | $(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.7X$ | $z$ even: $z + \varepsilon.\{0,2,8\}$ | |
| | | | $z$ odd: $z + \varepsilon.(X - \{0\})$ | |
| 5 | $e^{\varepsilon.3}11$ | $\mathbb{Z}_{12} + \varepsilon.e^3 X$ | $\mathbb{Z}_{12} + \varepsilon.\{0,1,2,3\}$ | 48 |
| | $e^{\varepsilon.6}(1+\varepsilon.6)$ | see $k = 1$ | see $k = 1$ | |
| | $7 + \varepsilon.6$ | see $k = 3$ | see $k = 3$ | |
| 8 | $e^{\varepsilon.3}11$ | see $k = 5$ | see $k = 5$ | 48 |
| | $e^{\varepsilon.3}7$ | see $k = 1$ | see $k = 1$ | |
| | 5 | see $k = 2$ | see $k = 2$ | |

### O.1.3 Class Nr. 71

| $k$ | $g$ | $g.X[\varepsilon]$ | $g.X[\varepsilon] \cap X[\varepsilon]$ | $card(g.X[\varepsilon] \cap X[\varepsilon])$ |
|---|---|---|---|---|
| 0 | $e^{\varepsilon.8}5$ | $\mathbb{Z}_{12} + \varepsilon.e^8 5X$ | $\mathbb{Z}_{12} + \varepsilon.\{1,2,6,7\}$ | 48 |
| | $e^{\varepsilon.8}(5+\varepsilon.6)$ | $(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.e^8 5X$ | $z$ even: $z + \varepsilon.\{1,2,6,7\}$ | |
| | | | $z$ odd: $z + \varepsilon.\{0,1,2,7\}$ | |
| 1 | $e^{\varepsilon.3}(7+\varepsilon.3)$ | $(1+\varepsilon.3)\mathbb{Z}_{12} + \varepsilon.e^3 7X$ | $z = 0,4,8 : z + \varepsilon.\{0,3\}$ | 42 |
| | | | $z = 1,5,9 : z + \varepsilon.\{0,1,3,6,7\}$ | |
| | | | $z = 2,6,10 : z + \varepsilon.\{3,6\}$ | |
| | | | $z = 3,7,11 : z + \varepsilon.\{0,1,2,6,7\}$ | |
| | $e^{\varepsilon.3}(7-\varepsilon.3)$ | $(1-\varepsilon.3)\mathbb{Z}_{12} + \varepsilon.e^3 7X$ | $z = 0,4,8 : z + \varepsilon.\{0,3\}$ | |
| | | | $z = 1,5,9 : z + \varepsilon.\{0,1,2,6,7\}$ | |
| | | | $z = 2,6,10 : z + \varepsilon.\{3,6\}$ | |
| | | | $z = 3,7,11 : z + \varepsilon.\{0,1,3,6,7\}$ | |
| | $e^{\varepsilon.9}(7+\varepsilon.3)$ | $(1+\varepsilon.3)\mathbb{Z}_{12} + \varepsilon.e^9 7X$ | $z = 0,4,8 : z + \varepsilon.\{3,6\}$ | |
| | | | $z = 1,5,9 : z + \varepsilon.\{0,1,2,6,7\}$ | |
| | | | $z = 2,6,10 : z + \varepsilon.\{0,3\}$ | |
| | | | $z = 3,7,11 : z + \varepsilon.\{0,1,3,6,7\}$ | |
| | $e^{\varepsilon.9}(7-\varepsilon.3)$ | $(1-\varepsilon.3)\mathbb{Z}_{12} + \varepsilon.e^9 7X$ | $z = 0,4,8 : z + \varepsilon.\{3,6\}$ | |
| | | | $z = 1,5,9 : z + \varepsilon.\{0,1,3,6,7\}$ | |
| | | | $z = 2,6,10 : z + \varepsilon.\{0,3\}$ | |
| | | | $z = 3,7,11 : z + \varepsilon.\{0,1,2,6,7\}$ | |
| 2 | $e^{\varepsilon.6}(1+\varepsilon.6)$ | $(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.e^6 X$ | $z$ even: $z + \varepsilon.\{0,1,6,7\}$ | 60 |
| | | | $z$ odd: $z + \varepsilon.X$ | |
| 3 | $e^{\varepsilon.6}(1+\varepsilon.6)$ | see $k = 2$ | see $k = 2$ | 60 |
| 6 | $e^{\varepsilon.2}5$ | $\mathbb{Z}_{12} + \varepsilon.e^2 5X$ | $\mathbb{Z}_{12} + \varepsilon.\{0,1,2,7\}$ | 48 |
| | $e^{\varepsilon.2}(5+\varepsilon.6)$ | $(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.e^2 5X$ | $z$ even: $z + \varepsilon.\{0,1,2,7\}$ | |
| | | | $z$ odd: $z + \varepsilon.\{1,2,6,7\}$ | |
| 7 | $e^{\varepsilon.9}(7+\varepsilon.3)$ | see $k = 1$ | see $k = 1$ | 42 |
| | $e^{\varepsilon.3}(7+\varepsilon.3)$ | see $k = 1$ | see $k = 1$ | |

### O.1.4 Class Nr. 75

| $k$ | $g$ | $g.X[\varepsilon]$ | $g.X[\varepsilon] \cap X[\varepsilon]$ | $card(g.X[\varepsilon] \cap X[\varepsilon])$ |
|---|---|---|---|---|
| 0 | $e^{\varepsilon.9}7$ | $\mathbb{Z}_{12} + \varepsilon.e^9 7X$ | $\mathbb{Z}_{12} + \varepsilon.\{1,4,5,8\}$ | 48 |
| | $e^{\varepsilon.9}(7+\varepsilon.4)$ | $(1+\varepsilon.4)\mathbb{Z}_{12} + \varepsilon.e^9 7X$ | $z = 0,3,6,9 : z + \varepsilon.\{1,4,5,8\}$ | |
| | | | $z = 1,4,7,10 : z + \varepsilon.\{0,1,5,8\}$ | |
| | | | $z = 2,5,8,11 : z + \varepsilon.\{0,1,4,5\}$ | |
| | $e^{\varepsilon.9}(7-\varepsilon.4)$ | $(1-\varepsilon.4)\mathbb{Z}_{12} + \varepsilon.e^9 7X$ | $z = 0,3,6,9 : z + \varepsilon.\{1,4,5,8\}$ | |
| | | | $z = 1,4,7,10 : z + \varepsilon.\{0,1,4,5\}$ | |
| | | | $z = 2,5,8,11 : z + \varepsilon.\{0,1,5,8\}$ | |
| | $e^{\varepsilon.6}(1+\varepsilon.6)$ | $(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.e^6 X$ | $z$ even: $z + \varepsilon.\{2,8\}$ | |
| | | | $z$ odd: $z + \varepsilon.X$ | |
| 1 | $e^{\varepsilon.6}(1+\varepsilon.6)$ | see $k = 0$ | see $k = 0$ | 48 |
| 2 | $e^{\varepsilon.8}(5+\varepsilon.4)$ | $(1+\varepsilon.4)\mathbb{Z}_{12} + \varepsilon.e^8 5X$ | $z = 0,3,6,9 : z + \varepsilon.\{0,1,4,8\}$ | 56 |
| | | | $z = 1,4,7,10 : z + \varepsilon.\{0,1,4,5,8\}$ | |
| | | | $z = 2,5,8,11 : z + \varepsilon.\{0,2,4,5,8\}$ | |
| | $e^{\varepsilon.8}(5-\varepsilon.4)$ | $(1-\varepsilon.4)\mathbb{Z}_{12} + \varepsilon.e^8 5X$ | $z = 0,3,6,9 : z + \varepsilon.\{0,1,4,8\}$ | |
| | | | $z = 1,4,7,10 : z + \varepsilon.\{0,2,4,5,8\}$ | |
| | | | $z = 2,5,8,11 : z + \varepsilon.\{0,1,4,5,8\}$ | |
| 4 | $e^{\varepsilon.6}(1+\varepsilon.6)$ | see $k = 1$ | see $k = 1$ | 48 |
| 5 | $e^{\varepsilon.8}(1 \pm \varepsilon.4)$ | see $k = 2$ | see $k = 2$ | 56 |
| 8 | $e^{\varepsilon.5}11$ | $\mathbb{Z}_{12} + \varepsilon.e^5 11X$ | $\mathbb{Z}_{12} + \varepsilon.\{0,1,4,5\}$ | 48 |
| | $e^{\varepsilon.5}(11+\varepsilon.4)$ | $(1+\varepsilon.4)\mathbb{Z}_{12} + \varepsilon.e^5 11X$ | $z = 0,3,6,9 : z + \varepsilon.\{0,1,4,5\}$ | |
| | | | $z = 1,4,7,10 : z + \varepsilon.\{1,4,5,8\}$ | |
| | | | $z = 2,5,8,11 : z + \varepsilon.\{0,1,5,8\}$ | |
| | $e^{\varepsilon.5}(11-\varepsilon.4)$ | $(1-\varepsilon.4)\mathbb{Z}_{12} + \varepsilon.e^5 11X$ | $z = 0,3,6,9 : z + \varepsilon.\{0,1,4,5\}$ | |
| | | | $z = 1,4,7,10 : z + \varepsilon.\{0,1,5,8\}$ | |
| | | | $z = 2,5,8,11 : z + \varepsilon.\{1,4,5,8\}$ | |

### O.1.5 Class Nr. 78

| $k$ | $g$ | $g.X[\varepsilon]$ | $g.X[\varepsilon] \cap X[\varepsilon]$ | $card(g.X[\varepsilon] \cap X[\varepsilon])$ |
|---|---|---|---|---|
| 0 | $e^{\varepsilon.9}(7+\varepsilon.3)$ | $(1+\varepsilon.3)\mathbb{Z}_{12}+\varepsilon.e^{9}7X$ | $z=0,4,8 : z+\varepsilon.\{1,4\}$ | 42 |
| | | | $z=1,5,9 : z+\varepsilon.\{0,2,4,6,10\}$ | |
| | | | $z=2,6,10 : z+\varepsilon.\{1,10\}$ | |
| | | | $z=3,7,11 : z+\varepsilon.\{0,1,4,6,10\}$ | |
| | $e^{\varepsilon.9}(7-\varepsilon.3)$ | $(1-\varepsilon.3)\mathbb{Z}_{12}+\varepsilon.e^{9}7X$ | $z=0,4,8 : z+\varepsilon.\{1,4\}$ | |
| | | | $z=1,5,9 : z+\varepsilon.\{0,1,4,6,10\}$ | |
| | | | $z=2,6,10 : z+\varepsilon.\{1,10\}$ | |
| | | | $z=3,7,11 : z+\varepsilon.\{0,2,4,6,10\}$ | |
| | $e^{\varepsilon.3}(7+\varepsilon.3)$ | $(1+\varepsilon.3)\mathbb{Z}_{12}+\varepsilon.e^{3}7X$ | $z=0,4,8 : z+\varepsilon.\{1,10\}$ | |
| | | | $z=1,5,9 : z+\varepsilon.\{0,1,4,6,10\}$ | |
| | | | $z=2,6,10 : z+\varepsilon.\{1,4\}$ | |
| | | | $z=3,7,11 : z+\varepsilon.\{0,2,4,6,10\}$ | |
| | $e^{\varepsilon.3}(7-\varepsilon.3)$ | $(1-\varepsilon.3)\mathbb{Z}_{12}+\varepsilon.e^{3}7X$ | $z=0,4,8 : z+\varepsilon.\{1,10\}$ | |
| | | | $z=1,5,9 : z+\varepsilon.\{0,2,4,6,10\}$ | |
| | | | $z=2,6,10 : z+\varepsilon.\{1,4\}$ | |
| | | | $z=3,7,11 : z+\varepsilon.\{0,1,4,6,10\}$ | |
| 1 | $e^{\varepsilon.6}(1+\varepsilon.6)$ | $(1+\varepsilon.6)\mathbb{Z}_{12}+\varepsilon.e^{6}X$ | $z$ even: $z+\varepsilon.\{0,4,6,10\}$ | 60 |
| | | | $z$ odd: $z+\varepsilon.X$ | |
| 2 | $e^{\varepsilon.6}(1+\varepsilon.6)$ | see $k=1$ | see $k=1$ | 60 |
| 4 | $5+\varepsilon.4$ | $(1+\varepsilon.4)\mathbb{Z}_{12}+\varepsilon.5X$ | $z=0,3,6,9 : z+\varepsilon.\{0,2,6,10\}$ | 56 |
| | | | $z=1,4,7,10 : z+\varepsilon.\{0,2,4,6,10\}$ | |
| | | | $z=2,5,8,11 : z+\varepsilon.\{1,2,4,6,10\}$ | |
| | $5-\varepsilon.4$ | $(1-\varepsilon.4)\mathbb{Z}_{12}+\varepsilon.5X$ | $z=0,3,6,9 : z+\varepsilon.\{0,2,6,10\}$ | |
| | | | $z=1,4,7,10 : z+\varepsilon.\{1,2,4,6,10\}$ | |
| | | | $z=2,5,8,11 : z+\varepsilon.\{0,2,4,6,10\}$ | |
| 6 | $e^{\varepsilon.9}(7\pm\varepsilon.3)$ | see $k=0$ | see $k=0$ | 42 |
| | $e^{\varepsilon.3}(7\pm\varepsilon.3)$ | see $k=0$ | see $k=0$ | |
| 10 | $e^{\varepsilon.6}(5+\varepsilon.2)$ | $(1+\varepsilon.2)\mathbb{Z}_{12}+\varepsilon.e^{6}5X$ | $z=0,6 : z+\varepsilon.\{0,2,4,6\}$ | 52 |
| | | | $z=1,7 : z+\varepsilon.\{1,2,4,6,10\}$ | |
| | | | $z=2,8 : z+\varepsilon.\{0,4,6,10\}$ | |
| | | | $z=3,9 : z+\varepsilon.\{0,2,6,10\}$ | |
| | | | $z=4,10 : z+\varepsilon.\{0,2,4,10\}$ | |
| | | | $z=5,11 : z+\varepsilon.\{0,2,4,6,10\}$ | |
| | $e^{\varepsilon.6}(5-\varepsilon.2)$ | $(1-\varepsilon.2)\mathbb{Z}_{12}+\varepsilon.e^{6}5X$ | $z=0,6 : z+\varepsilon.\{0,2,4,6\}$ | |
| | | | $z=1,7 : z+\varepsilon.\{0,2,4,6,10\}$ | |
| | | | $z=2,8 : z+\varepsilon.\{0,2,4,10\}$ | |
| | | | $z=3,9 : z+\varepsilon.\{0,2,6,10\}$ | |
| | | | $z=4,10 : z+\varepsilon.\{0,4,6,10\}$ | |
| | | | $z=5,11 : z+\varepsilon.\{1,2,4,6,10\}$ | |

### O.1.6 Class Nr. 82

| $k$ | $g$ | $g.X[\varepsilon]$ | $g.X[\varepsilon] \cap X[\varepsilon]$ | $card(g.X[\varepsilon] \cap X[\varepsilon])$ |
|---|---|---|---|---|
| 0 | $e^{\varepsilon.6}(1+\varepsilon.6)$ | $(1+\varepsilon.6)\mathbb{Z}_{12}+\varepsilon.e^{6}X$ | $z$ even: $z+\varepsilon.\{3,9\}$ | 48 |
| | | | $z$ odd: $z+\varepsilon.X$ | |
| | $e^{\varepsilon.6}(7+\varepsilon.6)$ | $(1+\varepsilon.6)\mathbb{Z}_{12}+\varepsilon.e^{6}7X$ | $z$ even: $z+\varepsilon.\{3,7,9\}$ | |
| | | | $z$ odd: $z+\varepsilon.(X-\{7\})$ | |
| | $e^{\varepsilon.11}(11-\varepsilon.4)$ | $(1+\varepsilon.4)\mathbb{Z}_{12}+\varepsilon.e^{11}11X$ | $z=0,3,6,9: z+\varepsilon.\{3,4,7,8\}$ | |
| | | | $z=1,4,7,10: z+\varepsilon.\{0,3,7,8\}$ | |
| | | | $z=2,5,8,11: z+\varepsilon.\{0,3,4,7\}$ | |
| | $e^{\varepsilon.11}(11+\varepsilon.4)$ | $(1-\varepsilon.4)\mathbb{Z}_{12}+\varepsilon.e^{11}11X$ | $z=0,3,6,9: z+\varepsilon.\{3,4,7,8\}$ | |
| | | | $z=1,4,7,10: z+\varepsilon.\{0,3,4,7\}$ | |
| | | | $z=2,5,8,11: z+\varepsilon.\{0,3,7,8\}$ | |
| | $e^{\varepsilon.11}11$ | $\mathbb{Z}_{12}+\varepsilon.e^{11}11X$ | $\mathbb{Z}_{12}+\varepsilon.\{3,4,7,8\}$ | |
| 3 | $e^{\varepsilon.8}(5-\varepsilon.4)$ | $(1+\varepsilon.4)\mathbb{Z}_{12}+\varepsilon.e^{8}5X$ | $z=0,3,6,9: z+\varepsilon.\{0,4,7,8\}$ | 56 |
| | | | $z=1,4,7,10: z+\varepsilon.(X-\{7\})$ | |
| | | | $z=2,5,8,11: z+\varepsilon.(X-\{9\})$ | |
| | $e^{\varepsilon.8}(5+\varepsilon.4)$ | $(1-\varepsilon.4)\mathbb{Z}_{12}+\varepsilon.e^{8}5X$ | $z=0,3,6,9: z+\varepsilon.\{0,4,7,8\}$ | |
| | | | $z=1,4,7,10: z+\varepsilon.(X-\{9\})$ | |
| | | | $z=2,5,8,11: z+\varepsilon.(X-\{7\})$ | |
| 4 | $e^{\varepsilon.6}(1+\varepsilon.6)$ | $(1+\varepsilon.6)\mathbb{Z}_{12}+\varepsilon.e^{6}X$ | see $k=0$ | 48 |
| | $e^{\varepsilon.6}(7+\varepsilon.6)$ | $(1+\varepsilon.6)\mathbb{Z}_{12}+\varepsilon.e^{6}7X$ | see $k=0$ | |
| 7 | 7 | $\mathbb{Z}_{12}+\varepsilon.7X$ | $\mathbb{Z}_{12}+\varepsilon.(X-\{7\})$ | 56 |
| 8 | $e^{\varepsilon.3}7$ | $\mathbb{Z}_{12}+\varepsilon.e^{3}7X$ | $\mathbb{Z}_{12}+\varepsilon.\{0,3,4,7\}$ | 48 |
| | $e^{\varepsilon.6}(1+\varepsilon.6)$ | $(1+\varepsilon.6)\mathbb{Z}_{12}+\varepsilon.e^{6}X$ | see $k=0$ | |
| | $e^{\varepsilon.6}(7+\varepsilon.6)$ | $(1+\varepsilon.6)\mathbb{Z}_{12}+\varepsilon.e^{6}7X$ | see $k=0$ | |
| | $e^{\varepsilon.3}(7+\varepsilon.4)$ | $(1+\varepsilon.4)\mathbb{Z}_{12}+\varepsilon.e^{3}7X$ | $z=0,3,6,9: z+\varepsilon.\{0,3,4,7\}$ | |
| | | | $z=1,4,7,10: z+\varepsilon.\{3,4,7,8\}$ | |
| | | | $z=2,5,8,11: z+\varepsilon.\{0,3,7,8\}$ | |
| | $e^{\varepsilon.3}(7-\varepsilon.4)$ | $(1-\varepsilon.4)\mathbb{Z}_{12}+\varepsilon.e^{3}7X$ | $z=0,3,6,9: z+\varepsilon.\{0,3,4,7\}$ | |
| | | | $z=1,4,7,10: z+\varepsilon.\{0,3,7,8\}$ | |
| | | | $z=2,5,8,11: z+\varepsilon.\{3,4,7,8\}$ | |

## O.2 Permitted Successors for the Major Scale

For the sweeping orientation, given a cantus firmus step $CF : x \mapsto y$, one is allowed to move from a consonance $c$ (i.e., $x + \varepsilon.c$) in the top row to a consonance $d$ (i.e., $y + \varepsilon.d$) in the right column iff there is a $*$ in the corresponding matrix entry.

1. Oblique Motion in Cantus Firmus

| $CF : 0 \mapsto 0$ | $CF : 2 \mapsto 2$ | $CF : 4 \mapsto 4$ | $CF : 5 \mapsto 5$ | $CF : 7 \mapsto 7$ | $CF : 9 \mapsto 9$ | $CF : 11 \mapsto 11$ |
|---|---|---|---|---|---|---|
| 0 4 7 9<br>* * 0<br>* * 4<br>* * 7<br>* * * 9 | 0 3 7 9<br>* * * 0<br>* * 3<br>* * 7<br>* * 9 | 0 3 7 8<br>* * * 0<br>* * * 3<br>* * 7<br>* * * 8 | 0 4 7 9<br>* * 0<br>* * 4<br>* * 7<br>* * * 9 | 0 4 7 9<br>* * 0<br>* * 4<br>* * 7<br>* * * 9 | 0 3 7 8<br>* * * 0<br>* * * 3<br>* * 7<br>* * * 8 | 0 3 8<br>* * 0<br>* * 3<br>* * 8 |

2. Minor Ascending Second in Cantus Firmus

| $CF : 4 \mapsto 5$ | $CF : 11 \mapsto 0$ | | | | | |
|---|---|---|---|---|---|---|
| 0 3 7 8<br>* * * * 0<br>* * * * 4<br>* * * 7<br>* * * * 9 | 0 3 8<br>* * * 0<br>* * * 4<br>* * * 7<br>* * * 9 | | | | | |

3. Minor Descending Second in Cantus Firmus

| $CF : 5 \mapsto 4$ | $CF : 0 \mapsto 11$ | | | | | |
|---|---|---|---|---|---|---|
| 0 4 7 9<br>* * * * 0<br>* * * * 3<br>* * * 7<br>* * * * 8 | 0 4 7 9<br>* * * * 0<br>* * * * 3<br>* * * * 8 | | | | | |

4. Major Ascending Second in Cantus Firmus

| $CF : 0 \mapsto 2$ | $CF : 2 \mapsto 4$ | $CF : 5 \mapsto 7$ | $CF : 7 \mapsto 9$ | $CF : 9 \mapsto 11$ | | |
|---|---|---|---|---|---|---|
| 0 4 7 9<br>* * 0<br>* * * * 3<br>* * * 7<br>* * * * 9 | 0 3 7 9<br>* * * * 0<br>* * * * 3<br>* * * 7<br>* * * * 8 | 0 4 7 8<br>* * 0<br>* * * 4<br>* * * 7<br>* * * * 9 | 0 4 7 9<br>* * 0<br>* * * * 3<br>* * * 7<br>* * 8 | 0 3 7 8<br>* * * * 0<br>* * * * 3<br>* * * * 8 | | |

5. Major Descending Second in Cantus Firmus

| $CF : 2 \mapsto 0$ | $CF : 4 \mapsto 2$ | $CF : 7 \mapsto 5$ | $CF : 9 \mapsto 7$ | $CF : 11 \mapsto 9$ | | |
|---|---|---|---|---|---|---|
| 0 3 7 9<br>* * 0<br>* * * * 4<br>* * * 7<br>* * * * 9 | 0 3 7 8<br>* * * * 0<br>* * * * 3<br>* * * 7<br>* * * * 9 | 0 4 7 9<br>* * 0<br>* * * 4<br>* * * 7<br>* * * * 9 | 0 3 7 8<br>* * 0<br>* * * * 4<br>* * * 7<br>* * 9 | 0 3 8<br>* * * 0<br>* * * 3<br>* * * 7<br>* * * 8 | | |

| 6. Minor Ascending Third in Cantus Firmus | | | | | |
|---|---|---|---|---|---|
| $CF : 2 \mapsto 5$ | $CF : 4 \mapsto 7$ | $CF : 9 \mapsto 0$ | $CF : 11 \mapsto 2$ | | |
| 0 3 7 9<br>* * 0<br>* * * * 4<br>* * * 7<br>* * 9 | 0 3 7 8<br>* * * * 0<br>* * * * 4<br>* * * 7<br>* * * 9 | 0 3 7 8<br>* * 0<br>* * * 4<br>* * * 7<br>* * * 9 | 0 3 8<br>* * * 0<br>* * 3<br>* * * 7<br>* * 9 | | |

| 7. Minor Descending Third in Cantus Firmus | | | | | |
|---|---|---|---|---|---|
| $CF : 5 \mapsto 2$ | $CF : 7 \mapsto 4$ | $CF : 0 \mapsto 9$ | $CF : 2 \mapsto 11$ | | |
| 0 4 7 9<br>* * * * 0<br>* * * 3<br>* * * 7<br>* * * 9 | 0 4 7 9<br>* * * * 0<br>* * * 3<br>* * * 7<br>* * * * 8 | 0 4 7 9<br>* * * * 0<br>* * * 3<br>* * * 7<br>* * * * 8 | 0 3 7 9<br>* * * * 0<br>* * * * 3<br>* * * * 8 | | |

| 8. Major Ascending Third in Cantus Firmus | | | | | |
|---|---|---|---|---|---|
| $CF : 0 \mapsto 4$ | $CF : 5 \mapsto 9$ | $CF : 7 \mapsto 11$ | | | |
| 0 4 7 9<br>* * 0<br>* * * * 3<br>* * * 7<br>* * * 8 | 0 4 7 9<br>* * 0<br>* * * * 3<br>* * * 7<br>* * * 8 | 0 4 7 9<br>* * 0<br>* * * * 3<br>* * 8 | | | |

| 9. Major Descending Third in Cantus Firmus | | | | | |
|---|---|---|---|---|---|
| $CF : 4 \mapsto 0$ | $CF : 9 \mapsto 5$ | $CF : 11 \mapsto 7$ | | | |
| 0 3 7 8<br>* * * * 0<br>* * * * 4<br>* * * 7<br>* * * * 9 | 0 3 7 8<br>* * * * 0<br>* * * * 4<br>* * * 7<br>* * * * 9 | 0 3 8<br>* * * 0<br>* * * 4<br>* * * 7<br>* * * 9 | | | |

| 10. Ascending Fourth in Cantus Firmus | | | | | |
|---|---|---|---|---|---|
| $CF : 0 \mapsto 5$ | $CF : 2 \mapsto 7$ | $CF : 4 \mapsto 9$ | $CF : 7 \mapsto 0$ | $CF : 9 \mapsto 2$ | $CF : 11 \mapsto 4$ |
| 0 4 7 9<br>* * * * 0<br>* * * * 4<br>* * * 7<br>* * * * 9 | 0 3 7 9<br>* * * * 0<br>* * * * 4<br>* * * 7<br>* * * * 9 | 0 3 7 8<br>* * * * 0<br>* * * * 3<br>* * * 7<br>* * * * 8 | 0 4 7 9<br>* * * * 0<br>* * * * 4<br>* * * 7<br>* * * * 9 | 0 3 7 8<br>* * * * 0<br>* * * * 3<br>* * * 7<br>* * * * 8 | 0 3 8<br>* * * 0<br>* * * 3<br>* * * 7<br>* * * 8 |

| 11. Descending Fourth in Cantus Firmus | | | | | |
|---|---|---|---|---|---|
| $CF : 5 \mapsto 0$ | $CF : 7 \mapsto 2$ | $CF : 9 \mapsto 4$ | $CF : 0 \mapsto 7$ | $CF : 2 \mapsto 9$ | $CF : 4 \mapsto 11$ |
| 0 4 7 9<br>* * * * 0<br>* * * * 4<br>* * * 7<br>* * * * 9 | 0 4 7 9<br>* * * * 0<br>* * * * 3<br>* * * 7<br>* * * * 9 | 0 3 7 8<br>* * * * 0<br>* * * * 3<br>* * * 7<br>* * * * 8 | 0 4 7 9<br>* * * * 0<br>* * * * 4<br>* * * 7<br>* * * * 9 | 0 3 7 9<br>* * * * 0<br>* * * * 3<br>* * * 7<br>* * * * 8 | 0 3 7 8<br>* * * * 0<br>* * * * 3<br>* * * * 8 |

| 12. Ascending Tritone in Cantus Firmus | | | | | |
|---|---|---|---|---|---|
| $CF : 5 \mapsto 11$ | $CF : 11 \mapsto 5$ | | | | |
| 0 4 7 9<br>* * 0<br>* * * 3<br>* * 8 | 0 3 8<br>* * 0<br>* * * 4<br>* * * 7<br>* * 9 | | | | |

# Part XVIII

# References

# Bibliography


[1] Abel V and Reiss P: MUTABOR II - Software Manual. Mutabor Soft, Darmstadt 1991

[2] Abraham R and Marsden J: Foundations of Mechanics. Benjamin, New York et al. 1967

[3] Ackermann Ph: Computer und Musik. Springer, Wien and New York 1991

[4] Aczel P: Non-well-founded Sets. No. 14 in CSLI Lecture Notes. Center for the Study of Language and Information, Stanford 1988

[5] Ackermann Ph: Developing Object-Oriented Multimedia Software. dpunkt, Heidelberg 1996

[6] Adorno Th W: Fragment über Musik und Sprache. Stuttgart, Jahresring 1956

[7] Adorno Th W: Der getreue Korrepetitor (1963). Gesammelte Schriften, Bd. 15, Suhrkamp, Frankfurt am Main 1976

[8] Agawu V K: Playing with Signs. Princeton University Press, Princeton 1991

[9] Agmon E: A Mathematical model of the diatonic system. JMT, 33, 1-25, 1989

[10] Agmon E: Coherent Tone-Systems: a study in the theory of diatonicism. JMT, 40(1), 39-59, 1996

[11] Agon A: OpenMusic: Un langage visuel pour la composition musicale assistée par ordinateur. PhD Dissertation, Université Paris VI, Paris 1998

[12] Akmajian A et al.: Linguistics. MIT Press, Cambridge MA 1995

[13] Alain: Alain Citation on Almada's painting in the Gulbenkian Foundation Center. Lisbon 1968

[14] D'Alembert J Le Rond: Einleitung zur Enzyklopädie (1751). (German Translation) Fischer, Frankfurt/Main 1989

[15] Amuedo J: Computational Description of Extended Tonality. Master Thesis, U Southern California, Los Angeles 1995

[16] Andreatta M: Group-theoretical Methods Applied to Music. Independent Study Dissertation, University of Sussex, Sussex 1997

[17] Andreatta M: La théorie mathématique de la musique de Guerino Mazzola et les canons rythmiques. Mémoire DEA, EHESS Paris IV & IRCAM, Paris 1999

[18] Ansermet E: Die Grundlagen der Musik im menschlichen Bewusstsein. 4th Ed. Piper, München and Zürich 1986

[19] Apfel E: Diskant und Kontrapunkt in der Musiktheorie des 12. bis 15. Jahrhunderts. Heinrichshofen, Wilhelmshafen 1982

[20] Aristoteles: Topik (Organon V, 345 b.C.). Rolfes E (German transl.), Meiner, Hamburg 1992

[21] Artin M: Algebra. Birkhäuser, Basel et al. 1993

[22] Artin M, Grothendieck A, Verdier J L: Théorie des Topos et Cohomologie Etale des Schémas (Tomes 1,2,3). Springer LN 269, 270, 305, Springer, New York et al. 1972-1973

[23] Askenfelt A et al.: Musical Performance. A Synthesis-by-Rule Approach. Computer Music J. 7/1, 1983

[24] Assayag G: Du calcul secret au calcul visuel. In: Delalande F and Vinet H (eds.): Interface homme-machine et création musicale. Hermes, Paris 1999

[25] Auroux S: La sémiotique des encyclopédistes. Payot, Paris 1979

[26] Babbitt M: Some Aspects of Twelve-Tone Composition. In: Hays W (ed.): The Score and IMA Magazine, 12, 53-61, 1955 (reprinted in "Twentieth Century Views of Music History", 364-371, Scribner, New York 1972)

[27] Babbitt M: Twelve-Tone Invariants as Compositional Determinants. Musical Quarterly, 46, 245-259, 1960

[28] Babbitt M: Set Structure as a Compositional Determinant. JMT, 5(2), 72-94, 1961

[29] Babbitt M: Twelve-Tone Rhythmic Structure and the Electronic Medium. PNM, 1(1), 49-79, 1962

[30] Babbitt M: The Structure and Function of Music Theory. College Music Symposium, Vol.5, 1965 (reprinted in Boretz and Cone, 1972, 10-21)

[31] Babbitt M: Words about Music. Dembski S and Straus J N (eds.), University of Wisconsin Press, Madison 1987

[32] Bach J S: Krebskanon. In: *Musikalisches Opfer*, 70. (BWV 1079), Neue Gesamtausg. sämtl. Werke Ser. VIII, Bd.1, Bärenreiter, Kassel 1978

[33] Bach J S: Choral No.6, In: *Himmelfahrtsoratorium* (BWV 11), Neue Gesamtausg. sämtl. Werke Ser. II, Bd.8, Bärenreiter, Kassel 1978

[34] Bacon F: De dignitate et augmentis scientiarum. 1st Ed. 1605, improved 1623

[35] Baker J, Beach D, and Bernard J: Music Theory in Concept and Practice. Eastman Studies in Music, University of Rochester Press, 1997

[36] Balzano G: The group-theoretic description of 12-fold and microtonal pitch systems. CMJ, 4, 66-84, 1980

[37] Bandemer H and Gottwald H: Fuzzy Sets, Fuzzy Logic, Fuzzy Methods. Wiley, New York et al. 1995

[38] Banter H: Akkord-Lexikon. Schott, Mainz 1982

[39] Barlow K: Über die Rationalisierung einer harmonisch irrationalen Tonhöhenmenge. Preprint, Köln 1985.

[40] Barrow, J and Tipler, F: The Anthropic Cosmological Principle: Oxford University Press, New York 1986

[41] Barthes R: Eléments de sémiologie. Communications 4/1964

[42] Basten A: Personal e-mail communciaction to G.M. November 22, 1996

[43] Bätschmann O: Einführung in die kunstgeschichtliche Hermeneutik. Wissenschaftliche Buchgemeinschaft Darmstadt, Darmstadt 1986

[44] Bauer M: Die Lieder Franz Schuberts. Breitkopf und Härtel, Leipzig 1915

[45] Bazelow A and Brickel F: A Partition Problem Posed by Milton Babbitt. PNM, 14(2), 15(1), 280-293, 1976

[46] Beethoven L van: *Grosse Sonate für das Hammerklavier* op.106 (1817-1818). Ed. Peters, Leipzig 1975

[47] Békésy G von: Experiments in Hearing. McGraw-Hill, New York 1960

[48] Beran J: *Cirri*. Centaur Records, 1991

[49] Beran J and Mazzola G: *Immaculate Concept*. SToA music 1002.92, Zürich 1992

[50] Beran J and Mazzola G: Analyzing Musical Structure and Performance—a Statistical Approach. Statistical Science. Vol. 14, No. 1, 47-79, 1999

[51] Beran J and Mazzola G: Visualizing the Relationship Between Two Time Series by Hierarchical Smoothing. Journal of Computational and Graphical Statistics, Vol. 8, No. 2, 213-238, 1999

[52] Beran J and Mazzola G: Timing Microstructure in Schumann's "Träumerei" as an Expression of Harmony, Rhythm, and Motivic Structure in Music Performance. Computers and Mathematics with Applications, Vol. 39, Issue 5/6, 99-130, 2000

[53] Beran J: Maximum likelihood estimation of the differencing parameter for invertible short and long-memory ARIMA models. J.R. Statist. Soc. B, 57, No.4, 659-672, 1995

[54] Beranek L L: Acoustics, 1954 -1993, The Acoustical Society of America, ISBN: 0-88318-494-X

[55] Berger M: Geometry I, II. Springer, Berlin et al. 1987

[56] Bernard J: Chord, Collection, and Set in Twentieth-Century Theory. In: Baker J et al. this bibliography, 11-52

[57] Biok H-R: Zur Intonationsbeurteilung kontextbezogener sukzessiver Intervalle. Bosse, Regensburg 1975

[58] Blood A, Zatorre J, Bermudez P, Evans A C: Emotional response to pleasant and unpleasant music correlates with activity in paralimbic brain regions. Nature Neuroscience, Vol. 2, No. 4, 382-387, April 1999

[59] Boretz B and Cone E T: Perspectives on Contemporary Music Theory. W.W. Norton and Company, New York 1972

[60] Boulez P: Musikdenken heute I,II; Darmstädter Beiträge V, VI. Schott, Mainz 1963, 1985

[61] Boulez P: Le timbre et l'écriture, le timbre et le langage. In: Bourgeois Chr (ed.): Le timbre, métaphore pour la composition. IRCAM, Collection Musique/Passé/Présent, Paris 1991

[62] Bourgeois Chr (ed.): Le timbre, métaphore pour la composition. IRCAM, Collection Musique/Passé/Présent, Paris 1991

[63] Bourbaki N: Eléments de Mathématique, Algèbre, Ch.1-9. Hermann, Paris 1970-1973

[64] Bourbaki N: Eléments de Mathématique, Algèbre Commutative, Ch.1-7. Hermann, Paris 1961-65

[65] Bourbaki N: Eléments de Mathématique, Topologie Générale, Ch.1-4. Hermann, Paris 1971

[66] Brändle L: Die "Wesentlichen Manieren" (Ornamente in der Musik). Oesterreichischer Bundesverlag, Wien 1987

[67] Brandt C and Roemer C: Standardized Chord Symbol Notation: Roerick Music, Sherman Oaks, CA 1976

[68] Brecht B et al.: Conductor Follower. In: ICMA (ed.): Proceedings of the ICMC 95, S. Francisco 1995

[69] Brockwell P J and Davis R A: Time Series: Theory and Methods. Springer, New York 1987

[70] Bruhn H: Harmonielehre als Grammatik der Musik. Psychologie Verlags Union, München et al. 1988

[71] Busch H R: Leonhard Eulers Beitrag zur Musiktheorie. Bosse, Regensburg 1970

[72] Buser P and Imbert M: Audition. Hermann, Paris 1987

[73] Buteau Ch: Motivic Topologies and their Signification in Musical Motivic Analysis. Masters Thesis, U Laval/Québec 1998

[74] Buteau Ch and Mazzola G: From Contour Similarity to Motivic Topologies. Musicae Scientiae, Vol. IV, No. 2, 125-149, 2000.

[75] Buteau Ch: Reciprocity Between Presence and Content Functions on a Gestalt Composition Space. Tatra Mt. Math. Publ. 23, 17-45, 2001

[76] Cabral B and Leedom L: Imaging vector fields using line integral convolution. Computer Graphics 27 (SIGGRAPH93 Proceedings), 263-272, 1993

[77] Calvet O et al.: Modal synthesis: compilation of mechanical sub-structure and acoustical sub-systems. In: Arnold S, Hair G, ICMA (eds.): Proceedings of the 1990 International Computer Music Conference. San Francisco 1990

[78] Camuri A et al.: Toward a cognitive model for the representation and reasoning on music and multimedia knowledge. In: Haus G and Pighi I (eds.): X Colloquio di Informatica Musicale. AIMI, LIM-Dsi, Milano 1993

[79] Carey N and Clampitt D: Aspects of well-formed scales. MTS, 11, 187-206, 1989

[80] Cardelli L and Wegner P: On Understanding types, data Abstraction and polymorphism. Computing Surveys, Vol. 17,4, 1985

[81] Castagna G: Foundations of Object-oriented programming. ETAPS, Lisbone 1998

[82] Castine P: Set Theory Objects. Lang, Frankfurt/Main et al. 1994

[83] CERN. Internet Information via http://www.cern.ch 1995

[84] Chatterjee S and Price B: Regression Analysis by Example. Wiley, 2nd ed., New York 1995

[85] Chomsky N and Halle M: The Sound Pattern of English. Harper and Row, New York 1968

[86] Chowning J: The Synthesis of Complex Audio Spectra by Means of Frequency Modulation. Journal of the Audio Engineering Society 21 (7), 1985

[87] Clarke E: Imitating and Evaluating Real and Transformed Musical Performances. Music Perception 10/3, 317-341, 1993

[88] Clough J and Myerson G: Variety and multiplicity in diatonic systems. JMT, 29, 249-270, 1985

[89] Clough J and Douthett J: Maximally even sets. JMT, 35, 93-173, 1991

[90] Clough J: Diatonic Interval Cycles and Hierarchical Structure. PNM, 32(1), 228-253, 1994

[91] Clynes M: Sentics. The Touch of Emotions. Anchor Doubleday, New York 1977

[92] Clynes M: Secrets of Life in Music. Analytica, Stockholm 1985

[93] Cohn R: Introduction to Neo-Riemannian Theory: A Survey and a Historical Perspective. JMT 42(2), 167-180, 1998

[94] Conen H: Formel-Komposition. Schott, Mainz et al. 1991

[95] Cooper K N D et al.: Handwritten Music-Manuscript Recognition. In: ICMA (ed.): Proceedings of the ICMC 96, S. Francisco 1996

[96] Couasnon B et al.: Using a Grammar For a Reliable Full Score Recognition System. In: ICMA (ed.): Proceedings of the ICMC 95, S. Francisco 1995

[97] Creutzfeldt O D: Cortex Cerebri. Springer, Berlin et al. 1983

[98] Czerny C: Pianoforte Schule. 1840

[99] Dahlhaus C: Zur Theorie des klassischen Kontrapunkts. Kirchenmusikalisches Jb 45, 1961

[100] Dahlhaus C: Über den Begriff der tonalen Funktion. In: Vogel M (ed.): Beiträge zur Musiktheorie des 19. Jahrhunderts. Bosse, Regensburg 1966

[101] Dahlhaus C and Eggebrecht H H: Was ist Musik? Heinrichshofen, Wilhelmshaven et al. 1985

[102] Dahlhaus C: Untersuchung über die Entstehung der harmonischen Tonalität. Bärenreiter, Kassel et al. 1967

[103] Dahlhaus C et al.: Neues Handbuch der Musikwissenschaft, Bd. 1-13: Athenaion and Laaber, Laaber 1980-1993

[104] Dahlhaus C and Mayer G: Musiksoziologische Reflexionen. In: Dahlhaus, C and de la Motte-Haber H (eds.): Neues Handbuch der Musikwissenschaft, Bd. 10: Systematische Musikwissenschaft. Laaber, Laaber 1982

[105] Dahlhaus C: Ludwig van Beethoven und seine Zeit. Laaber, Laaber 1987

[106] d'Alembert J Le Rond: Einleitung zur 'Enzyklopädie'. Fischer, Frankfurt/Main 1989

[107] Dallos P: The active cochlea. J. Neurosci. Dec;12(12):4575-85, 1992

[108] Dannenberg R B: An on-line algorithm for real-time accompaniment. In: ICMA (ed.): Proceedings of the ICMC 84, S. Francisco 1984

[109] Dannenberg R B et al.: Automatic Ensemble Performance. In: ICMA (ed.): Proceedings of the ICMC 94, S. Francisco 1994

[110] Danuser H et al. (eds.): Neues Handbuch der Musikwissenschaft, Bd. 11: Interpretation. Laaber, Laaber 1992

[111] de Bruijn N G: Pólya's Theory of Counting. In: Beckenbach E F (ed.): Applied Combinatorial Mathematics, Ch.5. Wiley, New York 1964

[112] de Bruijn N G: On the number of partition patterns of a set. Nederl. Akad. Wetensch. Proc. Ser. A 82 = Indag. Math. 41, 1979

[113] Debussy C: *Préludes*, Livre I (1907-1910). Henle, München 1986

[114] Dechelle F et al.: The Ircam Reall-Time Platform and Applications. In: ICMA (ed.): Proceedings of the ICMC 95, S. Francisco 1995

[115] Delalande F: La gestique de Gould. In: Guertin G (ed.): Glenn Gould — pluriel. Corteau, Verdun 1988

[116] Degazio B: A Computer-Based Editor For Lerdahl and Jackendoff's Rhythmic Structures. In: ICMA (ed.): Proceedings of the ICMC 96, S. Francisco 1996

[117] de la Motte D: Harmonielehre. Bärenreiter/dtv, Kassel 1976

[118] de la Motte-Haber H and Emons H: Filmmusik. Eine systematische Beschreibung. München, Hanser 1980

[119] de la Motte-Haber H: Handbuch der Musikpsychologie. Laaber-Verlag, 2.Ed., Laaber 1996

[120] de la Motte-Haber H: Rationalität und Affekt. In: Götze H and Wille R (eds.): Musik und Mathematik. Springer, Berlin et al. 1985

[121] de la Motte-Haber H: Musikalische Hermeneutik und empirische Forschung. In: Dahlhaus, C and de la Motte-Haber H (eds.): Neues Handbuch der Musikwissenschaft, Bd. 10: Systematische Musikwissenschaft. Laaber, Laaber 1982

[122] de la Motte-Haber H: Die Umwandlung der Interpretationsparameter in Struktureigenschaften. In: "Das Paradox musikalischer Interpretation", Symposion zum 80. Geburtstag von K. von Fischer, Univ. Zürich 1993

[123] Demazure M and Gabriel P: Groupes Algébriques. Masson & Cie./North-Holland, Paris/Amsterdam 1970

[124] Dennett, D C: Quining qualia. In Marcel A and Bisiach E (eds.): Consciousness in Contemporary Science. Oxford University Press, 1988

[125] Desain P and Honing H: The Quantization of Musical Time: A Connectionist Approach. Computer Music Journal 13 (3), 56-66, 1989

[126] Descartes R: Musicae Compendium. Herausgegeben und ins Deutsche übertragen als "Leitfaden der Musik" von J. Brockt, Wiss. Buchgesellschaft, Darmstadt 1978

[127] Dieudonné J: Foundations of Modern Analysis. Academic Press, New York et. al. 1960

[128] Döhl F: Webern - Weberns Beitrag zur Stilwende der Neuen Musik. Katzbichler, München et. al. 1976

[129] Dreiding A et al.: Classification of Mobile Molecules by Category Theory. In: Symmetries and Properties of Non-Rigid Molecules. Studies in Physical and Theoretical Chemistry, 23, 1983

[130] Dufourt H: Les difficultés d'une prise de conscience théorique. In: Le compositeur et l'ordinateur. 6-12, Ircam, Centre Georges Pompidou, Paris 1981

[131] Eberle G: "Absolute Harmonie" und "Ultrachromatik". In: Kolleritsch O (ed.): Alexander Skrjabin. Universal Edition, Graz 1980

[132] Eco U: Kunst und Schönheit im Mittelalter. Hanser, Wien 1991

[133] Eco U: Die Suche nach der vollkommenen Sprache. Beck, München 1994

[134] Eggebrecht H H: Interpretation. In: "Das Paradox musikalischer Interpretation", Symposion zum 80. Geburtstag von K. von Fischer, Univ. Zürich 1993

[135] Eggebrecht H H: Musik im Abendland. Piper, München and Zürich 1996

[136] Ehrenfels Chr von: Über Gestaltqualitäten. Vierteljahresschrift für wissenschaftliche Philosophie XIV, 1890

[137] Eitz C: Das mathematisch reine Tonsystem. Leipzig 1891

[138] Eimert H: Grundlagen der musikalischen Reihentechnik. Universal Edition, Wien 1964

[139] Eisenbud D: Commutative Algebra with a View Toward Algebraic Geometry. Springer New York et al. 1996

[140] Eisenbud D, and Harris J: The Geometry of Schemes. Springer, New York 2000

[141] Engström B: Stereocilia of sensory cells in normal and hearing impaired ears. Scand. Audiol. Suppl. 19, 1-34, 1983

[142] Essl K: Strukturgeneratoren. Beiträge zur elektronischen Musik 5, IEM, Graz 1996

[143] Euler L: Tentamen novae theoriae musicae (1739). In: Opera Omnia, Ser.III, Vol.1 (Ed. Bernoulli, E et al.). Teubner, Stuttgart 1926

[144] Euler L: Conjecture sur la raison de quelques dissonances générales reçues dans la musique (1764). In: Opera Omnia, Ser.III, Vol.1 (Ed. Bernoulli, E et al.). Teubner, Stuttgart 1926

[145] Euler L: De harmoniae veris principiis per speculum musicum representatis (1773). In: Opera Omnia, Ser.III, Vol.1 (Ed. Bernoulli, E et al.). Teubner, Stuttgart 1926

[146] Feldman J et al.: Force Dynamics of Tempo Change in Music. Music Perception, *10*, 1992

[147] Ferretti, R and Mazzola, G: Algebraic Varieties of Musical Performances. Tatra Mt. Math. Publ. 23, 59-69, 2001

[148] Feulner J et al.: MELONET: Neural Networks that Learn Harmony-Based Melodic Variations. In: ICMA (ed.): Proceedings of the ICMC 94, S. Francisco 1994

[149] Fichtner R: Die verborgene Geometrie in Raffaels "Schule von Athen". Oldenburg 1984

[150] Fink E: Grundlagen der Quantenmechanik. Akademische Verlagsgesellschaft, Leipzig 1968

[151] Finscher L: Studien zur Geschichte des Streichquartetts. Bärenreiter, Kassel 1974

[152] Finsler P: Über die Grundlegung der Mengenlehre. Erster Teil. Die Mengen und ihre Axiome. Math. Z. 25, 683-713, 1926

[153] Finsler P: Aufsätze zur Mengenlehre. Unger G (ed.), Wiss. Buchgesellschaft, Darmstadt 1975

[154] Fleischer A: Eine Analyse theoretischer Konzepte der Harmonielehre mit Hilfe des Computers. Magisterarbeit, MWS, HU Berlin 1996

[155] Fleischer A, Mazzola G, Noll Th: Zur Konzeption der Software RUBATO für musikalische Analyse und Performance. Musiktheorie, Heft 4, 314-325, 2000

[156] Forster M: Technik modaler Komposition bei Olivier Messiaen. Hänssler, Neuhausen-Stuttgart 1976

[157] Forte A: A Theory of Set-Complexes for Music. JMT, 8(2), 136-183, 1964

[158] Forte A: Structure of Atonal Music: Practical Aspects of a Computer-Oriented Research Project. In: Musicology and the Computer. Musicology 1966-2000. A Practical Program. Three Symposia. American Musicological Society, NY 1970

[159] Forte A: Structure of Atonal Music. Yale University Press, New Haven 1973

[160] Forte A: La Set-complex theory: Elevons les enjeux! Analyse musicale, 4e trimestre, 80-86, 1989

[161] Frank H: RUBATO® Broadcast. ORF2: Modern Times, Jan. 10, 1997

[162] Freedman D Z and Nieuwenhuizen P van: Supergravitation und die Einheit der Naturgesetze. In: Dosch H G (ed.): Teilchen, Felder und Symmetrien. Spektrum der Wissenschaft, Heidelberg 1984

[163] Friberg A: Generative Rules for Music Performance: A Formal Description of a Rule System. Computer Music Journal, Vol. 15, No. 2, 1991

[164] Friberg A et al.: Performance Rules for Computer-Controlled Contemporary Keyboard Music. Computer Music Journal, Vol. 15, No. 2, 1991

[165] Friberg A et al.: Recent Musical Performance Research at KTH. In: Sundberg J, (ed.): Generative Grammars for Music Performance. KTH, Stockholm 1994

[166] Friberg A: A Quantitative Rule System for Musical Performance. KTH PhD-Thesis, Stockholm 1995

[167] Fripertinger H: Enumeration in Musical Theory. Beiträge zur elektronischen Musik 1, Hochschule für Musik und Darstellende Kunst, Graz 1991

[168] Fripertinger H: Die Abzähltheory von Pólya. Diplomarbeit, Univ. Graz 1991

[169] Fripertinger H: Endliche Gruppenaktionen in Funktionenmengen—Das Lemma von Burnside—Repräsentantenkonstruktionen—Anwendungen in der Musiktheorie. Doctoral Thesis, Univ. Graz 1993

[170] Fripertinger H: Untersuchungen über die Anzahl verschiedener Intervalle, Akkorde, Tonreihen und anderer musikalischer Objekte in n-Ton Musik. Magisterarbeit, Hochschule für Musik und Darstellende Kunst, Graz 1993

[171] Fripertinger H: Anwendungen der Kombinatorik unter Gruppenaktionen zur Bestimmung der Anzahl "wesentlich" verschiedener Intervalle, Chorde, Tonreihen usw. Referat an der Univ. Innsbruck, Math. Institut d. Karl-Franzens-Univ., Graz 1996

[172] Fripertinger H: Enumeration of Mosaics. Discrete Mathematics, 199, 49-60, 1999

[173] Fripertinger H: Enumeration of Non-isomorphic Canons. Tatra Mt. Math. Publ. 23, 47-57, 2001

[174] Fux J J: Gradus ad Parnassum (1725). Dt. und kommentiert von L. Mitzler, Leipzig 1742

[175] Gabriel P: Personal Communication. Zürich 1979

[176] Gabrielsson A: Music Performance. In: Deutsch D (ed.): The Psychology of Music (2nd ed.). Academic Press, New York

[177] Gabrielsson A: Expressive Intention and Performance. In: Steinberg R (ed.): Music and the Mind Machine. Springer, Berlin et al. 1995

[178] Gamer C: Some combinatorial resources of equal-tempered systems. JMT, 11, 32-59, 1967

[179] Gamma E, Helm R, Johnson R, Vlissides J: Design Patterns, Elements of Reusable Object-Oriented Software. Addison-Wesley, Reading Mass. et al., 1994

[180] Gerwin Th: IDEAMA, Zentrum fur Kunst und Medientechnologie, Karlsruhe 1996

[181] Geweke John: A comparison of tests of independence of two covariance-stationary time series. J. Am. Statist. Assoc., 76, 363-373, 1981

[182] Giannitrapani D: The Electrophysiology of Intellectual Functions. Karger, Basel 1985

[183] Gilson E: Introduction aux arts du beau. Vrin, Paris 1963

[184] Godement R: Topologie algébrique et théorie des faisceaux. Hermann, Paris 1964

[185] Goethe J W von: Brief an Zelter. 9. Nov. 1829

[186] Goldblatt R: Topoi. North-Holland, Amsterdam et al. 1984

[187] Goldstein J L: An Optimum Processor Theory for the Central Formation of the Pitch of Complex Tones. J. Acoust. Soc. Am. 54, 1496 1973

[188] Gorenstein D: Classifying the finite simple groups. Bull. A.M.S. 14, 1-98, 1986

[189] Gottschewski H: Tempohierarchien. Musiktheorie, Heft 2, 1993

[190] Götze H and Wille R (eds.): Musik und Mathematik. Springer, Berlin et al. 1985

[191] Gould G: The Glenn Gould Reader. Alfred A. Knopf, New York 1984

[192] Goupillaud P, Grossmann A, Morlet J: Cycle-octave and related transforms in seismic signal analysis. Geoexploration, 23, 85-102, 1984-1985

[193] Grabusow N: Vielfalt akustischer Grundlagen der Tonarten und Zusammenklänge - Theorie der Polybasiertheit. Musiksektion des Staatsverlags, Moskau 1929

[194] Graeser W: Bachs "Kunst der Fuge". In: Bach-Jahrbuch 1924

[195] Greimas A J: Les actants, les acteurs et les figures. In: Chabrol C (ed.): Sémiotique narrative et textuelle. Larousse, Paris 1974

[196] Greub W: Linear Algebra. Springer, Berlin et al. 1967

[197] Gross D: A Set of Computer Programs to Aid in Music Analysis. Ph. Diss. Indiana Univ. 1975

[198] Grothendieck A and Dieudonné J: Eléments de Géométrie Algébrique I. Springer, Berlin et al. 1971

[199] Grothendieck A and Dieudonné J: Eléments de Géométrie Algébrique I-IV. Publ. Math IHES no. 4, 8, 11, 17, 20, 24, 28, 32, Bures-sur-Yvette 1960-1967

[200] Grothendieck A: Correspondence with G. Mazzola. April 1, 1990

[201] Guevara R C L et al.: A Modal Distribution Approach to Piano Analysis and Synthesis. In: ICMA (ed.): Proceedings of the ICMC 96, S. Francisco 1996

[202] Gurlitt et al. (eds.): Riemann Musiklexikon/Sachteil. Schott, Mainz 1967

[203] Habermann R: Elementary Applied PDEs, Prentice Hall 1983

[204] Halsey D and Hewitt E: Eine gruppentheoretische Methode in der Musiktheorie. Jahresber. d. Dt. Math.-Vereinigung 80, 1978

[205] Handschin J: Der Toncharakter. Zürich 1948

[206] Hanslick E: Vom Musikalisch-Schönen. Breitkopf und Härtel (1854), Wiesbaden 1980

[207] Hardt M: Zur Zahlenpoetik Dantes. In: Baum R. and Hirdt W (eds.): Dante Alighieri 1985. Stauffenburg, Tübingen 1985

[208] Harris C and Brinkman A R: A unified set of software tools for computer-assisted set-theoretic and serial analysis of contemporary music. Proc. ICMC 1986, ICA, San Francisco 1986

[209] Hartshorne R: Algebraic Geometry. Springer, New York et al. 1977

[210] Hashimoto S and Sawada H: Musical Performance Control Using Gesture: Towards Kansei Technology for Art. In: Kopiez R and Auhagen W (eds.): Controlling Creative Processes in Music. Peter Lang, Frankfurt am Main et al. 1998

[211] Haugh L D: Checking for independence of two covariance stationary time series: a univariate residual cross correlation approach. J. Am. Statist. Assoc., 71, 378-385, 1976

[212] Haus G et al.: Stazione di Lavoro Musicale Intelligente. In: Haus G and Pighi I (eds.): X Colloquio di Informatica Musicale. AIMI, LIM-Dsi, Milano 1993

[213] Hebb D O: Essay on mind. Hillsdale, New Jersey, Lawrence Erlbaum Associates, 1980

[214] Hegel G W F: Wissenschaft der Logik I (1812). Felix Meiner, Hamburg 1963

[215] Heiberg J L and Menge H (eds.): Euclidis opera omnia. 8 vol. & supplement, in Greek. Teubner, Leipzig 1883-1916.

[216] Heijink H, Desain P, Honing H, Windsor L: Make me a match: An evaluation of different approaches to score-performance matching. Computer Music Journal, 24(1), 43-56, 2000

[217] Helmholtz H von: Die Lehre von den Tonempfindungen als physiologische Grundlage der Musik (1863). Nachdr. Darmstadt 1968

[218] Henck H: Karlheinz Stockhausens Klavierstück IX. Verlag für systematische Musikwissenschaft, Bonn-Bad Godesberg 1978

[219] Hentoff N: Liner notes to Coltrane's last album "Expression". Impulse AS-9120, New York 1967

[220] Herbort H J: Keine Ausweispflicht für cis. Die Zeit Nr. 43, 21. Oktober 1988

[221] Hesse H: Das Glasperlenspiel (1943). Suhrkamp, Frankfurt/M. 1973

[222] Heussenstamm G: Norton Manual of Music Notation. Norton & Comp., New York 1987

[223] Hichert J: Verallgemeinerung des Kontrapunkttheorems für die Hierarchie aller starken Dichotomien in temperierter Stimmung. Diplomarbeit, TU Ilmenau 1993

[224] Hindemith P: Unterweisung im Tonsatz. Schott, Mainz 1940

[225] Hiller L and Ruiz P: Synthesizing sounds by solving the wave equation for vibration objects. J. of the Audio engineering Soc. 19: 463-470, 542-551, 1971

[226] Hjelmslev L: La Stratification du Langage. Minuit, Paris 1954

[227] Hjelmslev L: Prolégomènes à une théorie du langage. Minuit, Paris 1968-71

[228] Hjelmslev L: Nouveaux essays. PUF, Paris 1985

[229] Hofstädter D: Godel, Escher, Bach. New York: Basic Books, New York 1979

[230] Hong Y: Testing for independence between two covariance stationary time series. Biometrika, 83, No.3, 615-626, 1996

[231] Honing, H: Expresso, a strong and small editor for expression. In: ICMA (ed.): Proceedings of the ICMC 92, S. Francisco 1992

[232] Hooft G 't: Symmetrien in der Physik der Elementarteilchen. In: Dosch H G (ed.): Teilchen, Felder und Symmetrien. Spektrum der Wissenschaft, Heidelberg 1984

[233] Hörnl D et al.: Learning Musical Structure and Style by Recognition, Prediction and Evolution. In: ICMA (ed.): Proceedings of the ICMC 96, S. Francisco 1996

[234] Horry Y: A Graphical User Interface for MIDI Signal Generation and Sound Synthesis. In: ICMA (ed.): Proceedings of the ICMC 94, S. Francisco 1994

[235] Howe H: Some combinatorial properties of Pitch-Structures. PNM, 4(1), 45-61, 1965

[236] Hu S-T: Mathematical Theory of Switching Circuits and Automata. University of California Press, Berkeley and Los Angeles 1968

[237] Hudak P et al.: Haskore Music Notation - An Algebra of Music. J. Functional Programming Vol. 6 (3) 1996

[238] Hudspeth A J and Corey D P: Sensitivity, Polarity, and Conductance Change in the Response of Vertebrate Hair Cells to Controlled Mechanical Stimuli. Proc. Nat. Acad. Sci. Am. 74(6), 2407-2411, 1977

[239] Humphreys J E: Introduction to Lie Algebras and Representation Theory. Springer, New York et al. 1972

[240] Hung R et al.: The Analysis and Resynthesis of Sustained Musical Signals in the Time Domain. In: ICMA (ed.): Proceedings of the ICMC 96, S. Francisco 1996

[241] Hunziker E and Mazzola G: Ansichten eines Hirns. Birkhäuser, Basel 1990

[242] Husmann H: Einführung in die Musikwissenschaft. Heinrichshofen, Wilhelmshaven 1975

[243] Jackendoff R and Lerdahl F: A Generative Theory of Tonal Music. MIT Press, Cambridge MA, 1983

[244] Jakobson R and Halle M: Fundamentals of Language. Mouton, Le Hague 1957

[245] Jakobson R: Linguistics and Poetics. In: Seboek, TA (ed.): Style in Language. Wiley, New York 1960

[246] Jakobson R: Language in relation to other communication systems. In: Linguaggi nella società e nella tecnica. Edizioni die Communità, Milano 1960

[247] Jakobson R: Hölderlin, Klee, Brecht. Suhrkamp, Frankfurt/Main 1976

[248] Jakobson R and Pomorska K: Poesie und Grammatik. Dialoge. Suhrkamp, Frankfurt/Main 1982

[249] Jakobson R: Semiotik - Ausgewählte Texte 1919-1982. Holenstein E (ed.), Suhrkamp, Frankfurt/Main 1988

[250] Jauss H R: Rückschau auf die Rezeptionstheorie—Ad usum Musicae Scientiae. In: Danuser H and Krummacher F (Hsg.): Rezeptionsästhetik und Rezeptionsgeschichte in der Musikwissenschaft. Laaber, Laaber 1991

[251] Jeppesen K: Kontrapunkt. Breitkopf und Härtel, Wiesbaden 1952

[252] John F: Partial Differential Equations. Springer, Heidelberg et al. 1978

[253] Johnson T: See his web page at http://www.tom.johnson.org

[254] Julia G: Sur l'itération des fonctions rationnelles. J. de Math. Pure et Appl. 8, 1918

[255] Kagel M: Translation - Rotation. Die Reihe Bd.7, Universal Edition, Wien 1960

[256] Kahle W: Taschenatlas der Anatomie, Bd.3: Nervensystem und Sinnesorgane. Thieme/dtv, Stuttgart 1979

[257] Kaiser J: Beethovens 32 Klaviersonaten und ihre Interpreten. Fischer, Frankfurt/Main 1979

[258] Kant I: Kritik der reinen Vernunft. Meiner, Hamburg 1956

[259] Karg-Elert S: Polaristische Klang- und Tonalitätslehre (1931). Out of print, cf.: Schenk P.: Karg-Elerts polaristische Harmonielehre. In: Vogel, M (ed.) Beiträge zur Musiktheorie des 19. Jahrhunderts. Bosse, Regensburg 1966

[260] Katayose H et al.: Demonstration of Gesture Sensors for the Shakuhachi. In: ICMA (ed.): Proceedings of the ICMC 94, S. Francisco 1994

[261] Kelley J L: General Topology. Van Nostrand, Princeton et al. 1955

[262] Kiczales, G, Rivieres J, Bobrow D G: The Art of the Metaobject Protocol. The MIT Press, Boston 1991

[263] Kirsch E: Wesen und Aufbau der Lehre von den harmonischen Funktionen. Leipzig 1928

[264] Klemm M: Symmetrien von Ornamenten und Kristallen. Springer, Berlin et al. 1982

[265] Knapp J: Franz Liszt. Berlin 1909

[266] Köhler E: Brief an Guerino Mazzola. Hamburg 1988

[267] Koenig Th: Robert Schumanns *Kinderszenen* op.15. In: Metzger H-K und Riehn R (Hrg.): Robert Schumann II. edition text+kritik, München 1982

[268] Kollmann A: An Essay on Musical Harmony. London 1796

[269] Komparu K: The Noh Theatre. Weatherhill/Tankosha, New York et al. 1983

[270] Kopiez R and Langner J: Entwurf einer neuen Methode der Performanceanalyse auf Grundlage einer Theorie oszillierender Systeme. In: Behne K-E and de la Motte H (eds.): J.buch der D. Ges. für Musikpsychologie. **12** (1995), Wilhelmshaven 1996

[271] Langner J, Kopiez R, Feiten B: Perception ad Representation of Multiple Tempo Hierarchies in Musical Performance ad Composition: Perspectives from a New Theoretical Approach. In: Kopiez R and Auhagen W (eds.): Controlling Creative Processes in Music. Peter Lang, Frankfurt am Main et al. 1998

[272] Kopiez R: Aspekte der Performanceforschung. In: de la Motte H: Handbuch der Musikpsychologie. Laaber, Laaber 1996

[273] Kopiez R: Mensch - Musik - Maschine. Musica, 50 (1), 1996

[274] Kopiez R: "The most wanted song/The most unwanted song" – Klangfarbe als wahrnehmungsästhetische Kategorie. Musicology Conference, Halle 1998

[275] Kopiez R, Langner J, Stoffel Ch: Realtime analysis of dynamic shaping. Talk at the 6th International Conference on Music Perception and Cognition (ICMPC), 5.10. 8.2000, Keele, England

[276] Körner T W: Fourier Analysis. Cambridge University Press 1988, Cambridge

[277] Kostelanetz R: John Cage. Praeger, New York 1968

[278] Kouzes R T et al.: Collaboratories: Doing Science on the Internet. Computer, August 1996

[279] Kronland-Martinet R: The Wavelet Transform for Analysis, Synthesis, and Processing of Speech and Music Sounds. Computer Music Journal, 12 (4), 1988

[280] Kronman U and Sundberg J: Is the Musical Ritard an Allusion to Physical Motion? In: Gabrielsson A (ed.): Action and Perception in Rhythm and Meter. Bubl. of the Royal Swedish Acad. of Sci. 55, Stockholm 1987

[281] Kuratowski K and Mostowski A: Set Theory. North Holland, Amsterdam 1968

[282] Kühner H: Virtual Table. CG TOPICS 3/97

[283] Lang S: Introduction to Differentiable Manifolds. Interscience, New York et al. 1962

[284] Lang S: Elliptic Functions. Addison-Wesley, Reading, Mass. 1973

[285] Lang S: $SL_2(\mathbb{R})$. Addison-Wesley, Reading, Mass. 1975

[286] Langner G: Evidence for Neuronal Periodicity Detection in the Auditory System of Guinea Fowl: Implications for Pitch Analysis in the Time Domain. Exp. Brain Res. 52, 333-355 1983

[287] Langner G: Periodicity coding in the auditory system. Hear Res. Jul;60(2):115-42, 1992

[288] Langer S: Feeling and Form, Routledge and Kegan Paul, London 1953

[289] Langner J, Kopiez R, Stoffel Ch, Wilz M: Realtime analysis of dynamic shaping. In: Woods C et al. (eds.): Proceedings of the Sixth International Conference on Music Perception and Cognition, Keele, UK, 2000

[290] Lawvere F W: An elementary theory of the category of sets. Proc. Natl. Acad. Sci. 52, 1506-1511, 1964

[291] Leach J L: Towards a Universal Algorithmic System for Composition of Music and Audio-Visual Works. In: ICMA (ed.): Proceedings of the ICMC 96, S. Francisco 1996

[292] Leman M: Schema Theory. Springer, Berlin et al. 1996

[293] Lerdahl F: Timbral Hierarchies. Contemporary Music Review, vol.2, no.1, 1987

[294] Lévi-Strauss C: Le cru et le cuit: Mythologies I. Plon, Paris 1964

[295] Lewin D: Re: Intervallic Relations between Two Collections of Notes. JMT, 3(2), 298-301, 1959.

[296] Lewin D: The Intervallic Content of a Collection of Notes, Intervallic Relations between a Collection of Notes and Its Complement: An Application to Schoenberg's Hexachordal Pieces. JMT, 4(1), 98-101, 1960

[297] Lewin D: Forte's Interval Vector, My Interval Function, and Regener's Common-Note Function. JMT, 21(2), 194-237, 1977

[298] Lewin D: A Formal Theory of Generalized Tonal Functions. JMT 26(1), 32-60, 1982

[299] Lewin D: On Formal Intervals between Time-Spans. Music Perception, 1(4), 414-423, 1984

[300] Lewin D: Generalized Musical Intervals and Transformations. Yale University Press, New Haven and London 1987

[301] Lewin D: Musical Form and Transformation: 4 Analytic Essays. Yale University Press, New Haven and London 1993

[302] Lewis C I: Mind and Word Order. Dover, New York 1956

[303] Leyton M: Symmetry, Causality, Mind. MIT Press, Cambridge/MA and London 1992

[304] Leyton M: A Generative Theory of Shape. Springer, Berlin et al. 2001

[305] Lichtenhahn E: Romantik: Aussen- und Innenseiten der Musik. In: "Das Paradox musikalischer Interpretation", Symposion zum 80. Geburtstag von K. von Fischer, Univ. Zürich 1993

[306] Lippe E, and ter Hofstede A H M: A Category Theory Approach to Conceptual Data Modeling. Informatique Theorique et Applications, Vol 30, 1, 31-79, 1996

[307] Loomis L H and Sternberg S: Advanced Calculus. Addison-Wesley, Reading, Mass. 1968

[308] Louis A K et al.: Wavelets. Teubner, Stuttgart 1994

[309] Lüdi W: Fax to G Mazzola. Malans 01/23/1991

[310] Ludwig H: Marin Mersenne und seine Musiklehre. Olms, Hildesheim 1971

[311] Lussy M: Traité de l'expression musicale. Paris 1874

[312] Lutz R: Le poète dans son oeuvre. Seminar Jung Univ. Zürich, 1980

[313] Mac Lane S: Categories for the Working Mathematician. Springer, New York et al. 1971

[314] Mac Lane S and Moerdijk I: Sheaves in Geometry and Logic. Springer, New York et al. 1994

[315] MacLean P D: The triune brain, emotion, and scientific bias. In: Schmitt F O (ed.): The Neurosciences: Second Study Program, 336-348, Rockefeller Univ. Press, New York 1970

[316] Maeder R: Programming Mathematica. Addison-Wesely, Reading, Mass. 1991

[317] Marek C: Lehre des Klavierspiels. Atlantis, Zürich 1977

[318] Martinet A: Eléments de Linguistique Générale. Colin, Paris 1960

[319] Martino D: The Source Set and Its Aggregate Formations. JMT, 5(2), 224-273, 1961

[320] Marx A B: Die Lehre von der musikalischen Komposition. 4 Bde., Leipzig 1837-47

[321] Mason R M: Enumeration of Synthetic Musical Scales (...). J. of Music Theory 14, 1970

[322] Mathiesen Th J: Transmitting Text and Graphics in Online Database: The Thesaurus Musicarum Latinarum Model. In: Hewlett W B and Selfridge-Field E (eds.): Computing in Musicology Vol.9, CCARH, Menlo Park 1993-94

[323] Mattheson J: Der vollkommene Kapellmeister. Hamburg 1739

[324] Maxwell H J: An Expert System for Harmonizing Analysis of Tonal Music. In: Balaban M et al. (eds.): Understanding Music with AI: Perspectives on Music Cognition MIT Press. Cambridge MA 1992

[325] Mazzola G: *Akroasis—Beethoven's Hammerklavier-Sonate in Drehung (für Cecil Taylor).* LP, recorded August 30, 1979, WERGO/Schott, Mainz 1980

[326] Mazzola G: Musique et théorie des groupes. Conference at Institut de Mathématique, U Genève, January 31, 1980

[327] Mazzola G: Die gruppentheoretische Methode in der Musik. Lecture Notes, Notices by H. Gross, SS 1981, Mathematisches Institut der Universität, Zürich 1981

[328] Mazzola G: Gruppen und Kategorien in der Musik. Heldermann, Berlin 1985

[329] Mazzola G et al.: Rasterbild-Bildraster. Springer, Berlin et al. 1986

[330] Mazzola G: Die Rolle des Symmetriedenkens für die Entwicklungsgeschichte der europäischen Musik. In: Symmetrie, Katalogband Vol.1 zur Symmetrieausstellung, Mathildenhöhe, Darmstadt, 1986

[331] Mazzola G: Obertöne oder Symmetrie: Was ist Harmonie?. In: Herf F R (ed.): Mikrotöne-Helbling, Innsbruck 1986

[332] Mazzola G: Mathematische Betrachtungen in der Musik I,II. Lecture Notes, Univ. Zürich 1986/87

[333] Mazzola G: Der Kontrapunkt und die K/D-Dichotomie. Manuscript, University of Zürich 1987

[334] Mazzola G: Die Wahl der Zahl - eine systematische Betrachtung zum Streichquartett. In: Dissonanz 17, 1988

[335] Mazzola G and Hofmann G R: Der Music Designer MD-Z71 - Hardware und Software für die Mathematische Musiktheorie. In: Petsche H (ed.): Musik - Gehirn - Spiel, Beiträge zum 4. Herbert-von-Karajan-Symposion. Birkhäuser, Basel 1989

[336] Mazzola G et al.: A Symmetry-Oriented Mathematical Model of Classical Counterpoint and Related Neurophysiological Investigations by Depth-EEG. In: Hargittai I (ed.): Symmetry II, CAMWA, Pergamon, New York 1989

[337] Mazzola G et al.: Hirnelektrische Vorgänge im limbischen System bei konsonanten und dissonanten Klängen. In: Petsche H (ed.): Musik - Gehirn - Spiel, Beiträge zum 4. Herbert-von-Karajan-Symposion. Birkhäuser, Basel 1989

[338] Mazzola G: *presto* Software Manual. SToA music, Zürich 1989-1994

[339] Mazzola G: *Synthesis*. SToA 1001.90, Zürich 1990

[340] Mazzola G: Geometrie der Töne. Birkhäuser, Basel et al. 1990

[341] Mazzola G and Muzzulini D: Tempo- und Stimmungsfelder: Perspektiven künftiger Musikcomputer. In: Hesse H P (ed.): Mikrotöne III. Edition Helbling, Innsbruck 1990

[342] Mazzola G and Muzzulini D: Deduktion des Quintparallelenverbots aus der Konsonanz-Dissonanz-Dichotomie. Accepted for publication in: Musiktheorie, Laaber 1990

[343] Mazzola G: Mathematische Musiktheorie: Status quo 1990. Jber. d.Dt. Math.-Verein. 93, 6-29, 1991

[344] Mazzola G: Mathematical Music Theory—An Informal Survey. Edizioni Cerfim, Locarno 1993

[345] Mazzola G: RUBATO at SMAC KTH, Stockholm 1993

[346] Mazzola G and Zahorka O: Tempo Curves Revisited: Hierarchies of Performance Fields. Computer Music Journal **18**, No. 1, 1994

[347] Mazzola G and Zahorka O: The RUBATO Performance Workstation on NeXTSTEP. In: ICMA (ed.): Proceedings of the ICMC 94, S. Francisco 1994

[348] Mazzola G and Zahorka O: Geometry and Logic of Musical Performance I, II, III. SNSF Research Reports (469pp.), Universität Zürich, Zürich 1993-1995

[349] Mazzola G et al.: Analysis and Performance of a Dream. In: Sundberg J (ed.): Proceedings of the 1995 Symposium on Musical Performance. KTH, Stockholm 1995

[350] Mazzola G et al.: The RUBATO Platform. In: Hewlett W B and Selfridge-Field E (eds.): Computing in Musicology 10. CCARH, Menlo Park 1995

[351] Mazzola G and Zahorka O: The PrediBase Data Base System of RUBATO on NEXTSTEP. In: Selfridge-Field E (ed.): Handbook of Musical Codes. CCARH, Menlo Park 1995

[352] Mazzola G: Inverse Performance Theory. In: ICMA (ed.): Proceedings of the ICMC 95, S. Francisco 1995

[353] Mazzola G et al.: Neuronal Response in Limbic and Neocortical Structures During Perception of Consonances and Dissonances. In: Steinberg R (ed.): Music and the Mind Machine. Springer, Berlin et al. 1995

[354] Mazzola G and Zahorka O: RUBATO und der Einsatz von Hypermedien in der Musikforschung. Zeitschrift des Deutsches Bibliotheksinstituts, Berlin Jan. 1996

[355] Mazzola G and Zahorka O: Topologien gestalteter Motive in Kompositionen. To appear In: Auhagen W et al. (eds.): Festschrift zum 65. Geburtstag J.P. Fricke. Preliminary online version:
http://www.uni-koeln.de/phil-fak/muwi/publ/fs_fricke/festschrift.html

[356] Mazzola G: Towards Big Science. Geometry and Logic of Music and its Technology. In: Enders B and Knolle N (eds.): Symposionsband Klangart '95, Rasch, Osnabrück 1998

[357] Mazzola G et al.: The RUBATO Homepage.
http://www.rubato.org, Univ. Zűrich, since 1996

[358] Mazzola G: Objective C and Category Theory. Seminar Notes, Multimedia Lab, CS Department, U Zurich, Zurich 1996

[359] Mazzola G et al.: *Orbit*. Music & Arts CD-1015, Berkeley 1997

[360] Mazzola G and Beran J: Rational Composition of Performance. In: Kopiez R and Auhagen W (eds.): Proceedings of the Conference "Controlling Creative Processes in Music". Lang, Frankfurt and New York 1998

[361] Mazzola G: Semiotic Aspects of Musicology: Semiotics of Music. In: Posner R et al. (Eds.): A Handbook on the Sign-Theoretic Foundations of Nature and Culture. Walter de Gruyter, Berlin and New York 1998

[362] Mazzola G: Music@EncycloSpace. In: Enders B and Stange-Elbe J (ed.): Musik im virtuellen Raum (Proceedings of the klangart congress'98). Rasch, Osnabrück 2000

[363] Mazzola G: Humanities@EncycloSpace. FER-Studie Nr.XX. Schweizerischer Wissenschaftrat, Bern 1998
http://www.swr.ch

[364] Mazzola G: Die schöne Gefangene – Metaphorik und Komplexität in der Musikwissenschaft. In: Kopiez R et al. (eds.): Musikwissenschaft zwischen Kunst, Aesthetik und Experiment. Königshausen & Neumann, Würzburg 1998

[365] Mazzola G: The Topos Geometry of Musical Logic. Appears in the Proceedings of the Fourth Diderot Symposium of the European Math. Soc., Springer, Heidelberg et al. 2002

[366] Mazzola G: Degenerative Theory of Tonal Music. To appear in: Proceedings of the klangart congress'99. Universität Osnabrück 2002

[367] Mazzola G: Classifying Algebraic Schemes for Musical Manifolds. Tatra Mt. Math. Publ. 23, 71-90, 2001

[368] Mazzola G: L'Essence du Bleu (sonate pour piano). Acanthus, Rüttenen 2002

[369] Mesnage M: La Set-Complex Theory: de quels enjeux s'agit-il? Analyse musicale, 4e trimestre, 87-90, 1989

[370] Messiaen O: Technique de mon langage musical. Leduc, Paris 1944

[371] Meyer J: Akustik und musikalische Aufführungspraxis. Verlag Das Musikinstrument, Frankfurt/Main 1980

[372] Meyer-Eppler W: Grundlagen und Anwendungen der Informationstheorie. Springer, Berlin 1959

[373] Meyer-Eppler W: Zur Systematik der elektrischen Klangtransformationen. In: Darmstädter Beiträge III, Schott, Mainz 1960

[374] Michels U: dtv-Atlas zur Musik I,II. dtv/Bärenreiter, München/Kassel 1977, 1985

[375] Miklaszewski K: A case Study of a Pianist Preparing a Musical Performance. Psychology of Music, 17, 95-109, 1989

[376] Misch C and Wille R: Stimmungslogiken auf MUTABOR: Eine Programmiersprache. In: Herf F R (ed.): Mikrotöne II. Edition Helbling, Innsbruck 1988

[377] Molino J: Fait Musical et Sémiologie de la Musique. Musique en Jeu 17 1975

[378] Montiel Hernandez M: El Denotador: Su Estructura, construcción y Papel en la Teoría Matemática de la Musica. UNAM, Mexico City 1999

[379] Moog R: MIDI, Musical Instrument Digital Interface. Audio Eng. Soc. 34, Nr.5, 1986

[380] Morris R D: Composition with Pitch-Classes. Yale University Press, New Haven et al. 1987

[381] Morris R D: Compositional Spaces and Other Territories. PNM 33, 328-358, 1995

[382] Morris R D: K, Kh, and Beyond. In: Baker J et al. this bibliography, 275-306

[383] Mozart W A: *Die Zauberflöte*. (Klavierauszug) Ed. Peters o.J.

[384] Müller W: Darstellungstheorie von endlichen Gruppen. Teubner, Stuttgart 1980

[385] Mumford D: Lectures on Curves on an Algebraic Surface. Princeton University Press, Princeton 1966

[386] Mumford D and Suominen K: Introduction to the theory of moduli. In: Oort F (ed.): Algebraic Geometry Oslo 1970. Wolters-Noordhoff, Groningen 1972

[387] Murenzi R: Wavelets. Combes J M, Grossman A, Tchmitchian P (eds.), Springer Berlin et al., 1988

[388] MusicKit Online-Documentation. Version 4.0 1994

[389] Muzzulini D: Konsonanz und Dissonanz in Musiktheorie und Psychoakustik. Lizenziatsarbeit, MWS Univ. Zürich 1990

[390] Muzzulini D: Musical Modulation by Symmetries. J. for Music Theory 1995

[391] Muzzulini D: Tempo Modifications and Spline Functions. NF-Report 1993, Univ. Zürich 1993

[392] Narayan SS, Temchin AN, Recio A, Ruggero MA: Frequency tuning of basilar membrane and auditory nerve fibers in the same cochleae. Science. 282(5395):1882-4, Dec 4 1998

[393] Nattiez J-J: Fondements d'une Sémiologie de la Musique. Edition 10/18 Paris 1975

[394] Neumaier W: Was ist ein Tonsystem? Lang, Frankfurt/Main et al. 1986

[395] Neuwirth G: Josquin Desprez, "Erzählen von Zahlen". In: Musik-Konzepte 26/27, edition text+kritik, München 1982

[396] NeXTSTEP Online-Documentation; Version 3.3. NeXt Inc., Redwood City 1995

[397] Nieberle R: IRCAM Signal Processing Workstation. Keyboards, Dec. 1992

[398] Noether E: Hyperkomplexe Grössen und Darstellungstheorie. Math. Zeitschr., Vol. XXX 1929

[399] Noll J: Musik-Programmierung. Addison-Wesley, Bonn 1994

[400] Noll Th: Morphologische Grundlagen der abendländischen Harmonik. Doctoral Thesis, TU Berlin 1995

[401] Noll Th: Fractal Depth Structure of Tonal Harmony. In: ICMA (ed.): Proceedings of the ICMC 95, S. Francisco 1995

[402] Noll Th: http://www.cs.tu-berlin.de/ noll/ChordDictionary.sea.hqx, TU Berlin 1996

[403] Noll Th: Harmonische Morpheme. Musikometrika 8, 7-32, 1997

[404] Noll Th: The Consonance/Dissonance-Dichotomy Considered from a Morphological Point of View. In: Zannos I (ed.): Music and Signs. ASCO Publ., Bratislava 1999

[405] Nunn D et al.: Acoustic Quanta. In: ICMA (ed.): Proceedings of the ICMC 96, S. Francisco 1996

[406] Oettingen A von: Das duale Harmoniesystem. Leipzig 1913

[407] Okopenko A: Lexikon-Roman. Residenz Verlag, Salzburg 1970

[408] Opcode: MAX. http://www.opcode.com/products/max/, Opcode Systems Inc. 1997

[409] Osgood C E et al.: The Nature and Measurement of Meaning. Psychological Bulletin 49, 1952

[410] Parncutt R: The Perception of Pulse in Musical Rhythm. In: Gabrielsson A (ed.): Action and Perception in Rhythm and Music. Royal Swedish Adademy of Music, No.55

[411] Parncutt R: Recording Piano Fingering in Live Performance. In: Enders B, Knolle N (eds.): KlangArt-KOngress 1995, Rasch, Osnabrück 1998

[412] Parncutt R et al.: Interdependence of Right and Left Hands in Sight-read, Written, and Rehearsed Fingerings. Proc. Euro. Sco. Cog.Sci. Music, Uppsala 1997

[413] Parncutt R: Modeling Piano Performance: Physics and Cognition of a Virtual Pianist. In: ICMA (ed.): Proceedings of the ICMC 97, S. Francisco 1997

[414] Perle G: Serial Composition and Atonality: An Introduction to the Music of Schoenberg, Berg and Webern. 5th ed., revised, University of California Press, Berkeley 1981

[415] Petsche H et al.: EEG in Music Psychological Studies. In: Steinberg R (ed.): Music and the Mind Machine. Springer, Berlin et al. 1995

[416] Petsche H: Private correspondence. Vienna, March 2001

[417] Platon: Phaidron.

[418] Plomp R and Levelt W: Tonal Consonance and Critical Bandwidth. J.Acoust. Soc. Am. 38, 548, 1965

[419] Pope S T: Music Notations and the Representation of Musical Structure and Knowledge. Perspectives of New Music, Spring-Summer 1986

[420] Popper K R: Conjectures and Refutations. Routledge & Kegan Paul, London 1963

[421] Posner R: Strukturalismus in der Gedichtinterpretation. In: Blumensadth H (Ed.): Strukturalismus in der Literaturwissenschaft. Kiepenheuer & Witsch, Köln 1972

[422] Prame E: Measurements of the Vibrato Rate of 10 Singers. In: Friberg A et al. (eds.): Proceedings of the 1993 Stockholm Music Acoustic Conference. KTH, Stockholm 1994

[423] Priestley M B: Spectral Analysis of Time Series. Academic Press, London 1981

[424] Promies W: Symmetrie in der Literatur. In: Symmetrie, Katalogband Vol.1 zur Symmetrieausstellung. Mathildenhöhe, Darmstadt, 1986

[425] Promies W: Stolbergs Gedicht im Poesiegarten In: Symmetrie, Katalogband Vol.3 zur Symmetrieausstellung. Mathildenhöhe, Darmstadt, 1986

[426] Puckette M and Lippe C: Score Following in Practice. In: ICMA (ed.): Proceedings of the ICMC 92, S. Francisco 1992

[427] Pulkki V et al.: DSP Approach to Multichannel Audio Mixing. In: ICMA (ed.): Proceedings of the ICMC 96, S. Francisco 1996

[428] Quinn I: Fuzzy Extensions to the Theory of Contour. Music Theory Spectrum. Vol. 19/2, 1997

[429] Radl H: Versuch über die Modulationstheorie Mazzolas in reiner Stimmung. Diploma Thesis, U Augsburg, Augsburg 1998

[430] Rahn J: Basic Atonal Theory. Longman, New York 1980

[431] Rahn J: Review of D. Lewin's "Generalized Musical Intervals and Transformations". JMT, 31, 305-318, 1987

[432] Raffman D: Language, Music, and Mind. MIT Press, Cambridge et al. 1993

[433] Rameau J-Ph: Traité de l'Harmonie, Réduite à ses Principes Naturels. Paris 1722

[434] Ratz E: Einführung in die musikalische Formenlehre. Universal Edition, Wien 1973

[435] Read G: Music Notation. Crescendo Publ., Boston 1969

[436] Reeves H: Patience dans l'azur. L'évolution cosmique. Seuil, Paris 1981

[437] Reichardt J F: Vermischte Musikalien. Riga 1777

[438] Repp B: Diversity and Commonality in Music Performance: An Analysis of Timing Microstructure in Schumann's "Träumerei". J. Acoustic Soc. Am. **92**, 1992

[439] Repp B: e-mail communication of tempo data. Haskins Laboratories, New Haven, June 2, 1995

[440] Repp B: Patterns of note onset asynchronies in expressive piano performance. J. Acoustic Soc. Am. **100**, 1996

[441] Repp B: Pedal Timing and Tempo in Expressive Piano Performance: A Preliminary Investigation. Psychology of Music **24**, 1996

[442] Repp B: The Art of Inaccuracy: Why Pianists' Errors Are Difficult to Hear. Music Perception, *14*, 2 1997

[443] Repp B: Expressive Timing in a Debussy Prélude: A Comparison of Student and Expert Pianists. Musicae Scientiae 1, 1997

[444] Reti R: The Thematic Process in Music (1951). Greenwood Press, Westport 2nd ed. 1978

[445] Reti R, commented by Kopfermann M: Schumanns Kinderszenen: quasi Thema mit Variationen. In: Musik-Konzepte Sonderband Robert Schumann II, edition text + kritik, München 1982

[446] Rhode W S and Robles L: Evidence from Mössbauer Experiments for Nonlinear Vibration in the Cochlea. J. Acoust. Soc. Am. 48, 988, 1970

[447] Richter P et al.: How Consistent are Changes in EEG Coherence Patterns Elicited by Music Perception? In: Steinberg R (ed.): Music and the Mind Machine. Springer, Berlin et al. 1995

[448] Riemann B: Über die Hypothesen, welche der Geometrie zugrunde liegen (1854). Gött. Abh. No.13 (published in 1867)

[449] Riemann H: Musikalische Logik. Leipzig 1873

[450] Riemann H: Musikalische Syntaxis. Leipzig 1877

[451] Riemann H: Vereinfachte Harmonielehre oder die Lehre von den tonalen Funktionen der Akkorde. London 1893

[452] Riemann H: Über Agogik. In: Präludien und Studien II. Leipzig 1900

[453] Riemann H: System der musikalischen Rhythmik und Metrik. Breitkopf und Härtel, Leipzig 1903

[454] Riemann H: Handbuch der Harmonie- und Modulationslehre. Berlin o.J.

[455] Riemann H: Handbuch der Harmonielehre. Leipzig 6/1912

[456] Riemann H: Grundriss der Musikwissenschaft. Leipzig 1928

[457] Riemann Musiklexikon, Sachteil, 12. Auflage. Schott, Mainz 1967

[458] Ries F: Biographische Notizen über L. van Beethoven (1838). New edition by Kalischer A Ch, 1906

[459] Risset J-C: Real-time: Composition or performance? Reservations about real-time control in computer music and demonstration of a virtual piano partner. In: Kopiez R and Auhagen W (eds.): Controlling creative processes in music. Peter Lang, Frankfurt/Main 1998

[460] Roads C: The Computer Music Tutorial. MIT Press, Cambridge Mass. and London 1998

[461] Rodet X et al.: Xspect: a New X/Motif Signal Visualization, Analysis and Editing Program. In: ICMA (ed.): Proceedings of the ICMC 96, S. Francisco 1996

[462] Roederer J G: Physikalische und psychoakustische Grundlagen der Musik. Springer, Berlin 1977

[463] Rossi A: Die Geburt der modernen Wissenschaft in Europa. Beck, München 1997

[464] Rüetschi U-J: Denotative Geographical Modelling—an attempt at modelling geographical information with the Denotator system. Diploma thesis, University of Zürich, 2001

[465] Rufer J: Die Komposition mit zwölf Tönen. Bärenreiter, Kassel 1966

[466] Ruwet N: Langage, Musique, Poesie. Seuil, Paris 1972

[467] *S-Plus*. MathSoft Inc., Seattle, Washington 1995

[468] Sachs K-J: Der Contrapunctus im 14. und 15. Jahrhundert. AMW, Franz Steiner, Wiesbaden 1974

[469] Sattler D E (ed.): Friedrich Holderlin – Einhundert Gedichte. Luchterhand, Frankfurt/Main 1989

[470] Salzer F: Structural Hearing: Tonal Coherence in Music. (German: Strukturelles Hören. Noetzel, Wilhelmshaven 1960) Dover, New York 1962

[471] Saussure F de: Cours de Linguistique Générale. Payot, Paris 1922

[472] Saussure F de: Linguistik und Semiotik - Notizen aus dem Nachlass. Fehr J (ed.), Suhrkamp, Frankfurt/Main 1997

[473] Scarborough D L et al.: Connectionist Models for Tonal Analysis. Computer Music Journal Vol. 13, 1989

[474] Schenker H: Fünf Urlinien-Tafeln. Universal Edition, Wien 1932

[475] Schenker H: Theorien und Phantasien III: Der freie Satz. Universal Edition, Wien 1935

[476] Schmidt J (Hrsg.): Celibicache. Book and Movie PARS (ISBN 3-9803265-1-9), München 1992

[477] Schmidt-Biggemann W: Topica universalis. Meiner, Hamburg 1983

[478] Schönberg A: Harmonielehre (1911). Universal Edition, Wien 1966

[479] Schönberg A: Die Komposition mit zwölf Tönen. In: Style and Idea, New York 1950

[480] Schubert F: *Auf dem Wasser zu singen*, op. 72 (1823). Bärenreiter/Henle München/Kassel 1982

[481] Schubert H: Kategorien I, II. Springer, Berlin et al. 1970

[482] Schumann R: Kinderszenen, op. 15 (1839). Henle, München 1977

[483] Schumann R: Dritter Quartett-Morgen. NZ f. Musik 1838

[484] Schweitzer A: Johann Sebastian Bach. (1908) Breitkopf & Härtel, Wiesbaden 1979

[485] Serres M (ed.): Eléments d'histoire des sciences. Bordas, Paris 1989

[486] Shaffer L H: Musical Performance as Interpretation. Psychology of Music, Vol. 23, 1995

[487] Siron J: La partition intérieure. Outre mesure, Paris 1992

[488] Slawson W: The Musical Control of Sound Color. Canadian Univ. Music Review, No.3, 1982

[489] Slawson W: Sound Color. U. Cal. Press, Berkeley 1985

[490] Sloan D: From DARMS to SMDL, and back again. In: Haus G and Pighi I (eds.): X Colloquio di Informatica Musicale. AIMI, LIM-Dsi, Milano 1993

[491] Sloboda J: The Musical Mind: An Introduction to the Cognitive Psychology of Music. Calderon Press, Oxford 1985

[492] Sloterdijk P: Selbstexperiment. Hanser, Mnchen & Wien 1996

[493] SMDL committee: Standard Music Description Language Draft, Pittsford, NY 1995

[494] Smith III J O: Synthesis of bowed strings. In: Strawn J, Blum T, ICMA: Proceedings of the International Computer Music Conference, San Francisco 1982

[495] Smith III J O: Physical Modeling of Musical Instruments, part I. Computer Music Journal, 16 (4), 1992

[496] Smith III J O: Physical Modeling Synthesis Update. Computer Music Journal, 20 (2), 1996

[497] Sodomka A: KUNSTRADIO at documentaX. http://www.culture.net/orfkunstradio/BIOS/sodomkabio.html, Vienna 1997

[498] Spanier E H: Algebraic Topology. McGraw-Hill, New York et al. 1966

[499] Späth H: Eindimensionale Spline-Interpolations-Algorithmen. Oldenburg, München et al. 1990

[500] Späth H: Zweidimensionale Spline-Interpolations-Algorithmen. Oldenburg, München et al. 1991

[501] Squire R L and Butters N (eds.): Neuropsychology of Memory. D. Guildford Press, NewYork and London 1984

[502] Stahnke M: Struktur und Ästhetik bei Boulez. Wagner, Hamburg 1979

[503] Stange-Elbe J, Mazzola G: Cooking a Canon with RubatoPerformance Aspects of J.S. Bach's "Kunst der Fuge". In: ICMA (ed.): Proceedings of the ICMC 98, 179-186, San Francisco 1998.

[504] Stange-Elbe J: Analyse- und Interpretationsperspektiven zu J.S. Bachs "Kunst der Fuge" mit Werkzeugen der objektorientierten Informationstechnologie. Habilitationsschrift (manuscript, avaliable in several German libraries), Osnabrück 2000

[505] Stange-Elbe J: Instrumentaltechnische Voraussetzungen für eine computergestützte Interpretation. To appear in: Enders B (ed.): Proceedings of the klangart congress'01, Osnabrück

[506] Starr D and Morris R: A General Theory of Combinatoriality and the Aggregate. PNM, 16(1), 364-389, 16(2), 50-84, 1977-78

[507] Starr D: Sets, Invariance and Partitions. JMT, 22(1), 1-42, 1978

[508] Stein S and Sandor S: Algebra and Tiling: Homomorphisms in the Service of Geometry.: Math. Assoc. Amer., Washington, DC 1994

[509] Steinberg R (ed.): Music and the Mind Machine. Springer, Heidelberg 1995

[510] Steele G L: Common Lisp: The Language, 2nd Edition. Digital Press, 1990

[511] Stone P: Symbolic Composer. http:/www.xs4all.nl/∼psto/

[512] Stopper B: Gleichstufig temperierte Skalen unter Berücksichtigung von Oktavstreckungen. In: Hosp I (Hrsg.): Bozener Treffen 1991 — Skalen und Harmonien. Südtiroler Kulturinstitut, Bozen 1992

[513] Straub H: Beiträge zur modultheoretischen Klassifikation musikalischer Motive. Diplomarbeit ETH-Zürich, Zürich 1989

[514] Straub H: Kadenzielle Mengen beliebiger Stufendefinitionen. Unpublished manuscript, Zürich 1999/2000

[515] Straus J N: Voice Leading in Atonal Music. In: Baker J et al. this bibliography, 237-274

[516] Stumpf C: Tonpsychologie. Leipzig 1883-1890

[517] Sundberg J, Askenfelt A, Frydén L: Musical Performance: A Synthesis-by-Rule Approach, Computer Music Journal, 7, 37-43, 1983

[518] Sundberg J and Verillo V: On the Anatomy of the Retard: A Study of Timing in Music. J. Acoust. Soc. Am. **68**, 772-779, 1980

[519] Sundberg J: Music Performance Reseach. An Overview. In : Sundberg J, Nord L, Carlson R (eds.): Music language, Speech and Brain. London 1991

[520] Sundberg J (ed.): Generative Grammars for Music Performance. KTH, Stockholm 1994

[521] Sundin N-G: Musical Interpretation in Performance. Mirage, Stockholm 1984

[522] Suzuki T et al.: Musical Instrument Database with Multimedia. In: ICMA (ed.): Proceedings of the ICMC 96, S. Francisco 1996

[523] Suppan W: Zur Verwendung der Begriffe Gestalt, Struktur, Modell und Typus in der Musikethnologie. In: Stockmann D and Steszwski J (eds.): Analyse und Klassifikation von Volksmelodien. PWM Edition Krakau 1973

[524] Symmetrie. Katalogband Vol.1 zur Symmetrieausstellung, Mathildenhöhe, Darmstadt, 1986

[525] Terhardt E: Zur Tonhöhenwahrnehmung von Klängen. II. Ein Funktionsschema. Acustica 26, 187-199, 1972

[526] Thom R: Stabilité structurelle et morphogénèse. Benjamin, Reading MA 1972

[527] Tinctoris J: Opera Theoretica, Vol. 2: Liber de arte contapuncti. Seay A (ed.): Amer. Institute of Musicology 1975
gopher://IUBVM.UCS.INDIANA.EDU/00/tml/15th/tincon2.text

[528] Tittel E: Der neue Gradus. Doblinger, Wien-München 1959

[529] Todd N P M: A Model of Expressive Timing in Tonal Music. Music Percep. **3**, 33-58, 1985

[530] Todd N P M: Towards a Cognitive Theory of Expression: The Performance and Perception of Rubato. Contemporary Music Review **4**, 1989

[531] Todd N P M: A Computational Model of Rubato. Contemporary Music Review **3**, 1989

[532] Todd N P M: The Dynamics of Dynamics: A Model of Musical Expression. J. Acoustic Soc. Am. **91**, 3540-3550, 1992

[533] Todoroff T: A Real-Time Analysis and Resynthesis Instrument for Transformation of Sounds in the Frequency Domain. In: ICMA (ed.): Proceedings of the ICMC 96, S. Francisco 1996

[534] Uhde J: Beethovens Klaviermusik III. Reclam, Stuttgart 1974

[535] Uhde J and Wieland R: Denken und Spielen. Bärenreiter, Kassel et al. 1988

[536] van Dalen D: Logic and Structure. Springer, Berlin et al. 1997

[537] Valéry P: Leçon Inaugurale du Cours de Poétique au Collège de France. Gallimard, Paris 1945

[538] Valéry P: Cahiers I-IV (1894-1914). Celeyrette-Pietri N and Robinson-Valéry J (eds.), Gallimard, Paris 1987

[539] Valtieri S: La Scuola d'Athene. Mitteilungen des Konsthistorischen Instituts in Florenz 16, 1972

[540] van der Waerden B L: Algebra I, II. Springer, Berlin et al. 1966

[541] van der Waerden B L: Die Pythagoreer. Artemis, Zürich 1979

[542] Varèse E: Erinnerungen und Gedanken. In: Darmstädter Beiträge III. Schott, Mainz 1960

[543] Varèse E: Rückblick auf die Zukunft. edition text+kritik, München 1978

[544] Vercoe B: The Synthetic Performer in the Context of Live Performance. In: ICMA (ed.): Proceedings of the ICMC 84, S. Francisco 1984

[545] Vieru A: The Book of Modes. Editura Muzicală, Bucarest 1993

[546] Vinci A C: Die Notenschrift. Bärenreiter, Kassel 1988

[547] Vogel M: Die Lehre von den Tonbeziehungen. Verlag für systematische Musikwissenschaft, Bonn-Bad Godesberg 1975

[548] Vogel M: Arthur v. Oettingen und der harmonische Dualismus. In: Vogel M (ed.): Beiträge zu Musiktheorie des 19. Jahrhunderts. Bosse, Regensburg 1966

[549] Vogel M: Berechnung emmelischer und ekmelischer Mehrklänge. In: Herf F R (ed.): Mikrotöne II. Helbling, Innsbruck 1988

[550] Vogt H: Neue Musik seit 1945. Reclam, Stuttgart 1972

[551] Vossen G: Datenbankmodelle, Datenbanksprachen und Datenbank-Management-Systeme. Addison-Wesley, Bonn et al. 1994

[552] Vuza D T: Sur le rythme périodique. Revue Roumaine de Linguistique—Cahiers de Linguistique Théorique et Appliquée 22, no. 1 1985

[553] Vuza D T: Some Mathematical Aspects of David Lewin's Book *Generalized Musical Intervals and Transformations*. Perspectives of New Music, vol. 26, no. 1, 1988

[554] Vuza D T: Supplementary Sets and Regular Complementary Unending Canons (Part One). Perspectives of New Music, vol. 29, no. 2, 1991

[555] Vuza D T: Supplementary Sets and Regular Complementary Unending Canons (Part Two). Perspectives of New Music, vol. 30, no. 1, 1992

[556] Vuza D T: Supplementary Sets and Regular Complementary Unending Canons (Part Three). Perspectives of New Music, vol. 30, no. 2, 1992

[557] Vuza D T: Supplementary Sets and Regular Complementary Unending Canons (Part Four). Perspectives of New Music, vol. 31, no. 1, 1993

[558] Wanske H: Musiknotation. Schott, Mainz 1988

[559] Waldvogel M. et al.: *presto* source code. TRIMAX, Zürich 1986-1993

[560] Walker J S: Fourier Analysis and Wavelet Analysis. Notices of the AMS, vol.44, No.6, July 1997

[561] Waugh W A O'N: Music, probability, and statistics. In: Encyclopedia of Statistical Sciences, **6**, 134-137, 1985

[562] Webern A: *Variationen op.27* (mit Interpretationsanweisungen). UE 16845, Wien 1980

[563] Wegner P: Interactive Foundations of Object-Oriented Programming. Computer, IEEE Computer Soc., October 1995

[564] Weyl H: Symmetrie. Birkhäuser, Basel 1955

[565] Weinberger N M: Musica Research Notes. Music and Science Information Computer Archive, Center for Neurobiology of Learning and Memory, U Calfornia Irvine, Vol. III, Issue II, Fall 1996

[566] Wicinski A A: Psichologyiceskii analiz processa raboty pianista-ispolnitiela nad muzykalnym proizviedieniem. Izviestia Akademii Piedagogiceskich Nauc Vyp., 25 [Moscow], 171-215, 1950

[567] Widmer G: Learning Expression a Multiple Structural Levels. In: ICMA (ed.): Proceedings of the ICMC 94, S. Francisco 1994

[568] Widmer G: Modeling the Rational Basis of Musical Expression. Computer Music J. 18, 1994

[569] Wieser H-G: Electroclinical features of the psychomotor seizure. Fischer, Stuttgart and Butterworth, London 1983

[570] Wieser H-G and Mazzola G: Musical consonances and dissonances: are they distinguished independently by the right and left hippocampi? Neuropsychologia 24 (6):805-812, 1986

[571] Wieser H-G and Mazzola G: EEG responses to music in limbic and auditory cortices. In: Engel J Jr, Ojemann G A, Lüders H O, Williamson P D (eds.): Fundamental mechanisms of human function. Raven, New York 1987

[572] Wieser H-G: Musik und Gehirn. Revue Suisse Méd. 7, 153-162, 1987

[573] Wieser H-G and Moser S: Improved multipolar foramen ovale electrode monitoring. J Epilepsy 1: 13-22, 1986

[574] Wille R: Symmetrie — Versuch einer Begriffsbestimmung. In: Symmetrie, Katalogband Vol.1 zur Symmetrieausstellung, Mathildenhöhe, Darmstadt, 1986

[575] Wille R: Personal Communication. Darmstadt 1982

[576] Wille R: Musiktheorie und Mathematik. In: Götze H and Wille R (eds.): Musik und Mathematik. Springer, Berlin et al. 1985

[577] Wille R: Bedeutungen von Begriffsverbänden. Preprint Nr. 1058, TH Darmstadt 1987

[578] Wille R: Personal Communication. Darmstadt 1985

[579] Winson I: Brain and Psyche. The Biology of the Unconscious. Anchor Press/Doubleday, Garden City NY 1985

[580] Wittgenstein L: Tractatus Logico-Philosophicus (1918). Suhrkamp, Frankfurt/Main 1969

[581] Wolfram S: Software für Mathematik und Naturwissenschaften. In: Chaos und Fraktale. Spektrum der Wissenschaft Verlagsgesellschaft, Heidelberg 1989

[582] Xenakis I: Formalized Music. Indiana Univ. Press, Bloomington 1972

[583] Yaglom A M and Yaglom I M: Wahrscheinlichkeit und Information. Deutscher Verlag der Wissenschaften, Berlin, 1967

[584] Yoneda N: On the homology theory of modules. J. Fac. Sci. Univ. Tokyo, Sct. I,7, 1954

[585] Ystad S et al.: Of Parameters corresponding to a Propagative Synthesis Model Through the Analysis of Real Sounds. In: ICMA (ed.): Proceedings of the ICMC 96, S. Francisco 1996

[586] Yusuke A et al.: Hyperscore: A Design of a Hypertext Model for Musical Expression and Structure. J. of New Music Research, Vol. 24, 1995

[587] Zahorka O: Versuch einer Charakterisierung des altrömischen Melodiestils. Seminar Lütolf, MWS Univ. Zürich 1993

[588] Zahorka O: From Sign to Sound—Analysis and Performance of Musical Scores on RUBATO. In: Enders B (ed.): Symposionsband Klangart '95, Schott, Mainz 1997

[589] Zahorka O: PrediBase—Controlling Semantics of Symbolic Structures in Music. In: ICMA (ed.): Proceedings of the ICMC 95, S. Francisco 1995

[590] Zahorka O: RUBATO – Deep Blue in der Musik? Animato 97/3, 9-10, Zürich 1997

[591] Zarlino G: Istitutioni harmoniche. Venezia 1558

[592] Zekl G: Topos. Meiner, Hamburg 1990

[593] Zweifel P F: Generalized Diatonic and Pentatonic Scales: A Group-theoretic Approach. PNM, 34(1), 140-161, 1996

[594] Zwicker E and Fastl H: Psychoacoustics. Facts and Models. Springer, Berlin, et al. 1999

[595] Zwicker E and Zollner M: Elektroakustik (2nd edition). Springer, New York et al. 1987

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