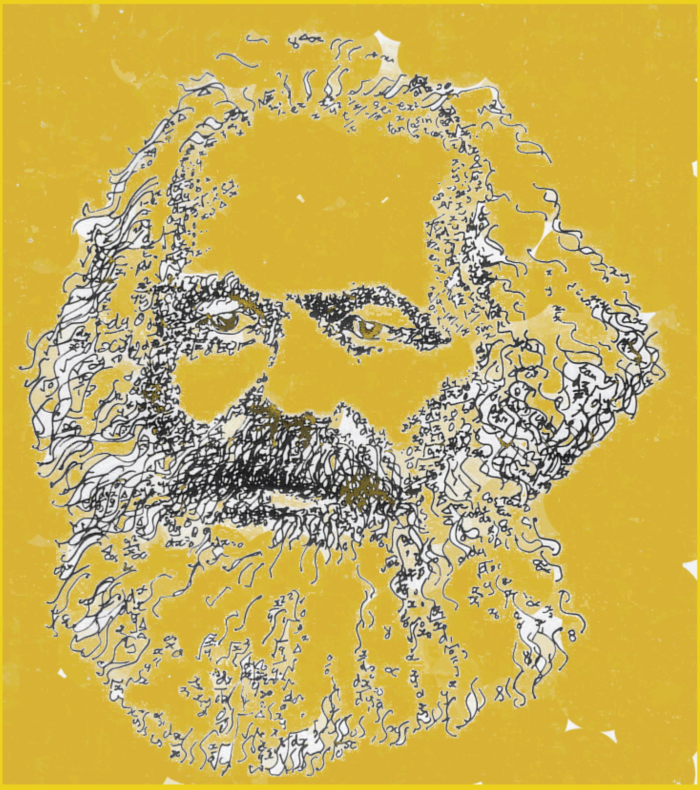
Paulus Gerdes

The philosophic-mathematical manuscripts of Karl Marx on differential calculus



An introduction

www.lulu.com
http://stores.lulu.com/pgerdes
http://www.lulu.com/spotlight/pgerdes

Paulus Gerdes

The philosophic-mathematical manuscripts of Karl Marx on differential calculus: An introduction.

Author: Paulus Gerdes

Title of the new English-language edition (2014):

The philosophic-mathematical manuscripts of Karl Marx on differential calculus: An introduction

First English-language edition (1985):

Marx demystifies calculus, Studies in Marxism, Vol. 16

MEP Publications, University of Minnesota, Minneapolis, USA

Translation into English from the Portuguese: Beatrice Lumpkin;

Bibliographic research for the translation: Gordon Welty;

Cover design: Peggy Lipschutz.

Original Portuguese-language edition (1983):

Karl Marx: "Arrancar o véu misterioso à matemática"

TLANU-Brochure No. 5, Department of Mathematics and Physics, Faculty of Education, Eduardo Mondlane University, Maputo, Mozambique

Latest Portuguese-language edition (2008):

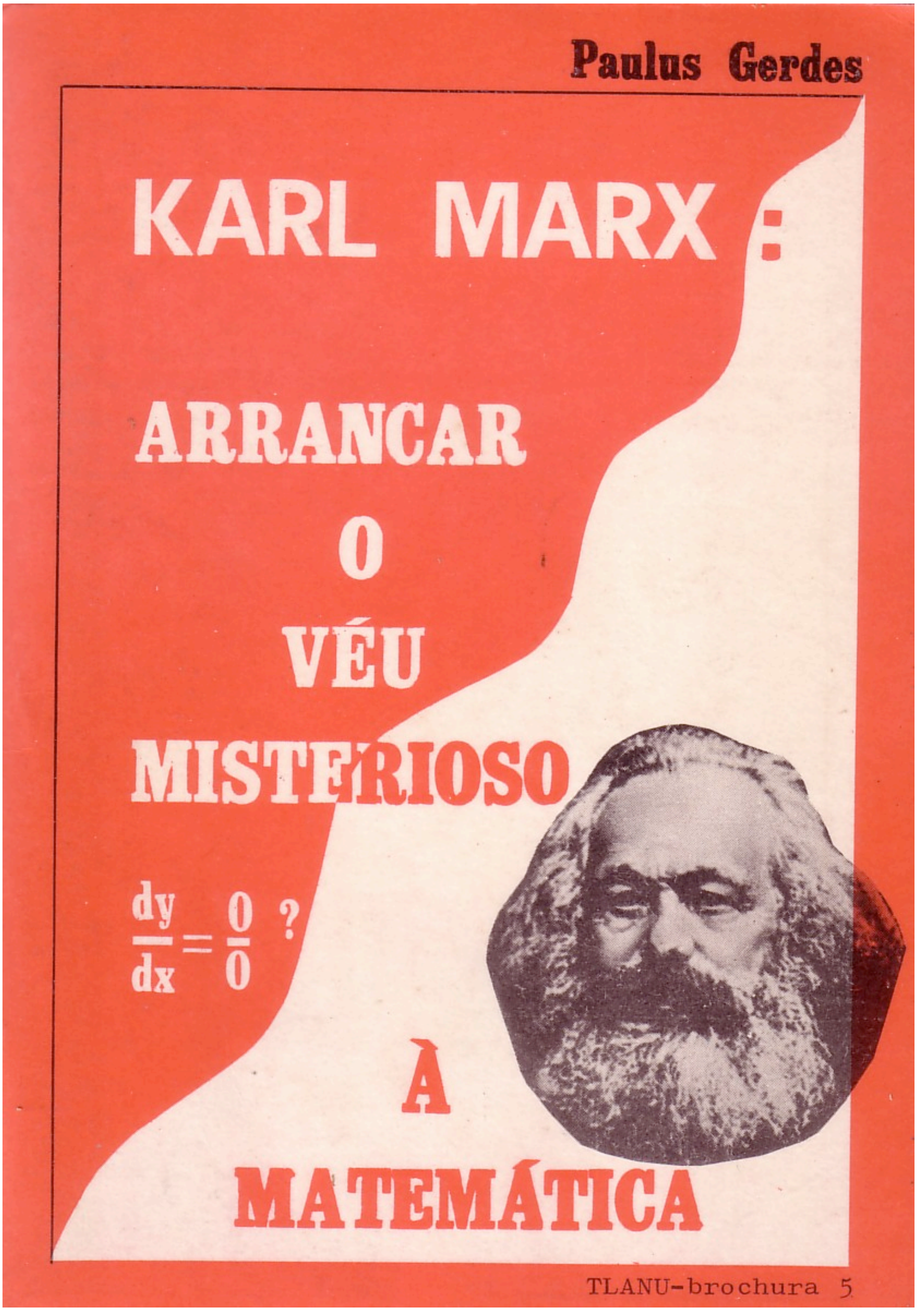
Os manuscritos filosófico-matemáticos de Karl Marx sobre o cálculo diferencial. Uma introdução

Distribution: http://stores.lulu.com/pgerdes

Or search 'Paulus Gerdes' on the webpage www.lulu.com

Paulus.Gerdes@gmail.com

CONTENTS



Cover of the original Portuguese-language edition of 1983

PRESENTATION

In March 1983 the Eduardo Mondlane University organized a seminar on the then significance of Karl Marx's works to celebrate the centennial of his death. It took place in Maputo, the capital of Mozambique. I had the honor to present, in the amphitheater of the Faculty of Medicine in the city center, some reflections of Karl Marx about mathematics, in particular, his considerations about and preoccupations with the foundations of differential calculus. My presentation included an introduction to the reasons for and contents, methods, and meaning of the mathematical manuscripts of Marx. In the same year, the journal TLANU, published by the Department of Mathematics and Physics of the Eduardo Mondlane University, published an extended version of my lecture as a book, entitled *Karl Marx: "Arrancar o véu misterioso à matemática."* In that version I also included a reflection on "negation of the negation" in mathematics education.

The book generated interest both in Mozambique and elsewhere. Two years later, the *Marxist Educational Press* published, in the USA, an English-language edition, based on the translation by Beatrice Lumpkin.

Already more than thirty years have passed. During these years I have dedicated my research mostly to a field nowadays called *ethnomathematics*, the study of the development of mathematical and mathematic-educational ideas and practices in diverse cultural, social, political and historical contexts. One may ask if there exists some relationship between ethnomathematics and the reflection on the mathematical manuscripts of Karl Marx. An answer may be found in the words of Arthur B. Powell and Marilyn Frankenstein in their book

Ethnomathematics: Challenging Eurocentrism in Mathematics Education (1997). In the section concerning interconnections between culture and mathematical knowledge, they include the paper "Marx and Mathematics" (1948) by Dirk Struik (1894-2000). In their introduction to Struik's paper, they state that Marx tried to analyze differential calculus as rooted in a cultural praxis – the conceptual and mathematical description of dynamics, of motion and of change – in the light of another cultural construction – *dialectics* – that was part of the philosophical and ideological perspective of a identifiable cultural group (Powell & Frankenstein, 1997, p. 124). The same geometer and historian Struik wrote, already 104 years old, the foreword to the English language edition of my book *Awakening of Geometrical Thought in Early Culture*, where he stresses my *dynamic* approximation of the historic process of acquiring mathematical ideas, in general, and geometric ideas, in particular.

More than thirty years have passed since the first edition of this book. The book belongs to my youth; today I would have written it most probably in a different way... Both the global historic context changed significantly as I matured in my research, reflections, and experiences. I did not try to rewrite the book, as there are other, more urgent challenges. The new edition of the book consists of the reproduction of the translation of 1985. I introduced some minor changes to the translation. The epilogue has been shortened. Some bibliographic references have been added.

Paulus Gerdes

February 27, 2014

References

Gerdes, Paulus (2003), *Awakening of Geometrical Thought in Early Culture*, MEP Publications, Minneapolis, 184 pp. (original: 1985) (Preface: Dirk J. Struik) (New edition: *Ethnogeometry: Awakening of Geometrical Thought in Early Culture*, ISTEG, Boane [distributed by Lulu Publishers]).

Powell, Arthur B. (1986), Marx and Mathematics in Mozambique. Review: Paulus Gerdes, Marx demystifies mathematics, *Science and Nature*, Nos. 7/8, 119-123.

Powell, Arthur B. & Frankenstein, Marilyn (Eds.) (1997), *Ethnomathematics: Challenging Eurocentrism in Mathematics Education*, SUNY, New York, 440 pp.

Struik, Dirk (1948), Marx and Mathematics, *Science and Society*, Vol. 12, No. 1, pp. 181-196 [reproduced in: Powell, A. B. & Frankenstein, M. (Eds.) (1997), pp. 173-192].

Acknowledgements (1983)

I thank colleagues Horst-Eckart Gross (Bielefeld, Germany), António Mussino (Roma, Italy), Ronald Schlauch (Magdenburg, Germany), Dirk J. Struik (Belmont, USA), Chrit van Ewijk (Utrecht, Netherlands), Hans Wussing (Leipzig, Germany), and Leonid Zimianin (Minsk, USSR), for the information and documents they sent me.

Acknowledgements (1985)

My thanks go to Beatrice Lumpkin (Chicago, USA), who translated this work from Portuguese to English and to Erwin Marquit (Minneapolis, USA) for editing the English-language edition. I am also indebted to Walter Purkert (Leipzig, Germany) for his helpful suggestions.

Translator's note (1985)

Bibliographic research for this translation was contributed by Gordon Welty, who located many of the English language sources for the Marxist classics quoted by Gerdes. In some cases there are slight differences in Gerdes's rendition of quotations from Marx's *Mathematical Manuscripts* and those appearing in the English translation by C. Aronson and M. Meo (London: New Park Publications, 1983). Such differences are due in part to Gerdes's use of the original German whereas Aronson and Meo worked from the Russian translation.

Gerdes's use of "provisional derivative" instead of "preliminary derivative" for the German "vorläufig Abgeleitete," however, is a matter of choice, to emphasize the true development that takes place during the process of differentiation. Gerdes's choice of subscripts is also retained here with x_0 and x_1 rather then the x and x_0 used by Marx.

Material appearing inside square brackets has been inserted either by the author – (P.G.), the translator – (B.L.), or the bibliographer – (G.W.)

Beatrice Lumpkin

Chapter 1

MARX THE MATHEMATICIAN?

It was the winter of 1974. In one of those small, progressive bookstores – *The Old Mole* – that I customarily frequented as a student in Nijmegen (The Netherlands), I had just noticed a new book. Only one copy was on the shelf. There, all by itself, was the book *Mathematical Manuscripts* by Marx – by Karl Marx, the founder of ‘scientific socialism’?

To what extent did this great fighter for the cause of liberation of the proletariat dedicate himself to mathematics, “the queen of the sciences”? Karl Marx? No! That was not possible! Maybe it was another person of the same name, a coincidence, like the Marx Brothers of comedy. I was ready to leave the store, into the icy wind, but I hesitated.

Wait! Marx, mathematics...?

Then I remembered that some months ago, in studying the *Pre-capitalist Forms of Production*, I had felt that Marx’s reasoning was as clear and as logical as mathematics itself. Also, I remembered that much of the structure of the first volume of *Capital* was deductive.

I returned to leaf through the book, and read the preface attentively. Yes, it was none other.

It was he, the founder of historic and dialectic materialism. But how is it that he also contributed toward overcoming the crisis in the foundations of differential calculus? Was his solution of the crisis the method we associate with the names of Cauchy, Dedekind, Weierstrass, Cantor...? And why had I not learned about it. I had already finished my university studies in the foundations of mathematics and logic...?

I bought the book and left *The Old Mole*, warmed by the fire of a series of new questions: Why did Marx dedicate himself to study and research in mathematics? What role did his analysis play in mathematics, in philosophy, in ...?

This was my first contact with the *Mathematical Manuscripts* of Karl Marx, published in the so-called “free world” only 100 years after it was written...

I will never forget that winter of 1974.

Chapter 2

WHY DID MARX STUDY MATHEMATICS

2.1 *To avoid computational errors*

After the defeat of the 1848-1849 revolutions, Marx, without interrupting his political work, devoted his attention mainly to theoretical development, in particular, economics. During the process of working out the theory of surplus value in detail, he sighed and exclaimed:

> "Diese verfluchten falschen Rechnungen soll der Teufel holen. Aber never mind. Começons de nouveau." [1]

Translating this mixture of "linguistic internationalism" (German, English, French) is almost impossible – he so forcefully intoned the beauty of each tongue. However, let us try:

> "These damn errors of calculation, to the devil with them. But never mind. We must begin again."

It was precisely to enable to deepen his research in economics that, in the 1850s, Marx studied not only the history of technology, agronomy, geology, ... ,[2] but also mathematics. In a letter dated January 11, 1858, Marx wrote to his devoted friend Frederick Engels (1820-1895):

> "In elaborating the principles of economics, I have

1 Marx, 1974a, p. 280.

2 Stepanova, 1979, p. 63. See also Fedoseyev *et al.*, 1974, p. 260.

> been so damnably held up by errors in calculation that in despair I have applied myself to a rapid review of algebra." [3]

This is the reason why Marx undertook a systematic study of mathematics while exiled in London, more than 20 years after he had finished secondary school with a grade of "good" in mathematics, at Trier, his native city in Germany. [4] Studying algebra in the same year of 1858, Marx developed some original ideas about possible generalization of the concept of powers, described in some of his notebooks for preparation of *Critique of Political Economy*. [5] The study of algebra was followed by analytic geometry and differential calculus. [6]

2.2 *Soothing effect of mathematics*

In one of the first biographies of Karl Marx, Franz Mehring (1846-1919) wrote:

> "Marx also sought intellectual recreation on quite a different field, namely mathematics. Particularly in times of mental anguish and other sufferings, he would seek consolation in mathematics, which exercised a soothing effect on him." [7]

In his letter to Engels, dated November 23, 1860, when his wife was suffering from smallpox, Marx exclaimed:

> "Writing articles is almost out of the question for me. The only activity, which can help me maintain the necessary peace of mind is mathematics". [8]

---

3 MEW, 1961, vol. 29, p. 256; Marx & Engels, 1983, vol. 40, p. 244.

4 Struik, 1948a, p. 181; 1975, p. 139; Kennedy, 1977, p. 305.

5 Yanovskaya, 1969, p. 21.

6 MEW, 1961, Vol. 20, p. xvi.

7 Mehring, 1962, p. 505.

8 MEW, 1961, Vol. 30, p. 113.

In another letter to Engels, dated July 6, 1863, we read:

> "My free time is dedicated to differential and integral calculus. By the way – I have a superabundance of publications on this subject and will send you some if you would like to begin a study of this discipline. I think it's almost necessary for your military studies. Besides, this constitutes a much easier part of mathematics with regard to the purely technical aspects, easier than, for example higher algebra." [9]

In the 1860s, Marx was deeply involved in the organization of working people. [10] In 1864 he founded the International Working Men's Association, the first international organization of the proletariat. At the same time he finished the first volume of *Capital*, in which he demonstrated that the fall of capitalism was inevitable. But his physique could not always sustain such titanic theoretical and practical work. Many times sickness kept him in bed, since he was living under very difficult material conditions. More than once, for example, in a letter of May 20, 1865, Marx pointed out the of mathematics as a diversion:

> "In the hours in-between – when I can't write constantly – I dedicate myself to differential calculus, $\frac{dy}{dx}$. I don't have patience to read anything else. All other reading makes me return to the table to write." [11]

2.3 ***Twofold purpose***

In the last 15 years of the life of Marx, his scientific work became continuously more extensive and encyclopedic. He continued his preparatory studies for Volumes II and III of *Capital*. He was

---

9 MEW, 1961, Vol. 30, p. 362.

10 See, for example, Stepanova, 1979, p. 85, or Fedoseyev *et al.*, 1974, pp. 360-62.

11 MEW, 1961, Vol. 31, p. 122.

studying not only all new publications on capitalist economy, but also the problems of agriculture, the agrarian question in Russia, and the impetuous development of capitalism in the United States of America. He dedicated himself to the history of the people of the world, geology, physiology, physics, and, in particular, mathematics. [12]

About this dedication to mathematics, Paul Labérenne observed:

> "Toward the end of his life, Marx made a very deep study of higher mathematics. His two-fold objective was to put into algebraic form the economic laws he had enunciated in *Capital*, and to study some of the modes of argument of mathematical analysis from the viewpoint of dialectics". [13]

In the opinion of Sofia Yanovskaya (1896-1966), the main reason for Marx's continued attention to mathematics was his desire to deepen his analysis of political economy. [14] Even so, he always was led to higher mathematics. For example, in 1869, in analyzing the circulation of capital, Marx made a number of purely mathematical commentaries in his summaries of the book by Feller and Odermann, *All About Commercial Arithmetic*.

The French socialist Paul Lafargue (1842-1911), son-in-law of Marx, emphasized that Karl Marx:

> "... found in higher mathematics dialectical movement in its most logical form, and at the same time, its simplest form. In his opinion, a science was really developed only when it successfully made use of mathematics". [15]

In this way, Marx consciously anticipated the possibility of applying mathematics to raise the scientific level of political economy.

---

12 MEW, 1961, Vol. 19, p. vi.

13 Labérenne, 1971, p. 58.

14 Yanovskaya, 1969, p. 21.

15 Cited by Rieske & Schenk, 1972, pp. 481-82; see also Molodschi, 1977, p. 85; and Thiel, 1975, p. 74.

[16] In a letter written to Engels, dated May 31, 1873, Marx gave an example of this line of thought:

> "You know those tables that show prices, discount rates, etc., etc. Their changes during the year, etc. are represented by zigzags, ups and downs. In analyzing the crises, I tried several times to calculate these up and down movements as irregular curves. And I thought it was possible to determine, in this way, the principal laws of crises mathematically (and I still think it's possible with sufficiently sifted material)". [17]

16 Cf. Yanovskaya, 1969, p. 22; Endemann "Einleitung" (Introduction) in Marx, 1974b, p. 15; and Ponzio in Marx, 1975, p. 33.

17 MEW, 1961, Vol. 33, p. 82.

Chapter 3

STILL SO LITTLE KNOWN

3.1 *Best known works*

What are the best known works of Marx? No doubt, *Capital* is the best known, or more exactly, the first volume of *Capital*. Only the first volume of *Capital* was published during the lifetime of Marx, in 1867. The manuscripts for the other volumes remained unfinished because illness frequently kept Marx from his worktable. Frederick Engels managed to guarantee the necessary supplementary material and edited the second and third volumes in 1885 and 1894, respectively. The fourth volume, *Theories of Surplus Value*, was published only in 1905 under the editorship of Karl Kautsky (1854-1938).

Aside from *Capital*, the works of Marx initially better known were probably those published during his lifetime, such as *The Holy Family* in 1845 and the *Communist Manifesto* in 1848, both written together with Engels, and his books *The Philosophy of Poverty or the Poverty of Philosophy* and *Critique of the Gotha Programme*, which were published in 1847 and 1875, respectively.

Other works appeared only after his death. When *The German Ideology* was written in 1846, no publisher could be found … (It was first published in 1932). Even less known are the unfinished manuscripts of Marx. He was prevented from finishing the second part of *Contribution to the Critique of Political Economy* by developments on the international scene, especially the Austro-Italian-French war of 1859. This intense political work was not the only factor. His extremely high standards as a scientist allowed him only to publish when he was certain of the quality and depth of his analysis. His personal suffering – poverty, misery, death of three children in the first years in London – did not allow him to finish many of the manuscripts. Included in this group of unfinished manuscripts are his mathematical manuscripts. [18]

In the preface of the second edition of *Anti-Dühring* (1885), Engels writes:

> "Since Karl Marx's death, however, my time has been requisitioned for more urgent duties, and I have been compelled to lay aside my work. For the present I must … wait to find some later opportunity to put together and publish the results which I have arrived at, perhaps in conjunction with the extremely important mathematical manuscripts left by Marx." [19]

Engels never got the chance. [20] The results of which he spoke were brought together and published under the title *Dialectics of Nature*, only in 1925 in Russian, in German in 1927, and in English in 1940. Nor did he succeed in editing the *Mathematical Manuscripts*.

3.2 *The history of the Mathematical Manuscripts*

The German social-democrats inherited the documents of Marx and Engels. One of them, Mehring, the biographer of Marx mentioned above, wrote:

> "Engels and Lafargue both contend that he made independent discoveries in this field [mathematics – P.G.], but this is beside the point here, and mathematicians who went through his manuscripts after his death are reported not to have endorsed this opinion." [21]

---

18 See, for example, Ivanov, 1979, p. 192.

19 Engels, 1962, p. 19.

20 See, for example, Stepanova, 1977, p. 194; also Ilyichov *et al.*, 1977, p. 308; and Fedoseyev *et al.*, p. 547.

21 Mehring, 1962, p. 505; Mehring, 1974, p. 275.

Already, in a letter to Engels dated November 22, 1882, Marx had observed that Samuel Moore, the jurist, who among his friends was the best versed in mathematics, did not understand it. [22] These German social-democrats were not capable of a good understanding of the role of dialectics in mathematics and nature. [23] And this is the main reason for such late publication of *Dialectics of Nature* – also of the *Mathematical manuscripts* – a lack of understanding of dialectics. [24]

This understanding first arose when some Russians indicated the fundamental significance of the philosophical works of Marx and Engels, in particular Vladimir Lenin (1870-1924), in his *Materialism and Empirio-Criticism* (1908).

At the International Congress on the History of Science and Technology that took place in London in 1931, Professor Ernst Kol'man (or Colman) (1893-1979), of the Institute of Mathematics and Mechanics of Moscow, [25] gave a list of the unpublished works of Karl Marx on mathematics, natural science and technology, and on the history of these disciplines. [26] In Kol'man's talk at this Congress analyzing the "Crisis in Mathematical Science," he declared:

> "The hitherto unpublished writings of Karl Marx, dealing with mathematics and its history, ... are of tremendous methodological importance." [27]

A first partial publication of the *Mathematical Manuscripts* appeared in 1933, in the Soviet journal, *Under the Banner of Marxism*, for the fiftieth anniversary of the death of Marx.

This publication immediately awoke the interest of specialists. In 1935, Valerii I. Glivenko (1897-1940) published a comparative

---

22 MEW, 1961, Vol. 35, p. 114.

23 Struik, 1948a, p. 195; Struik, 1975, p. 141; MEW, 1961, Vol. 20, p. xxii.

24 To understand some of the causes of this failure to comprehend dialectics, see Lenin, 1961, Vol. 38, pp. 180, 362.

25 Labérenne, 1971, p. 65.

26 See Kol'man, 1931b, pp. 233-35.

27 Kol'man, 1931a, p. 227.

analysis of the concepts of the differential in the works of Marx and those of the famous French mathematician, Jacques Hadamard (1865-1963). [28] In 1947, another Soviet, Levan P. Gokieli (1901-1975) published a monograph about the *Mathematical Manuscripts* of Marx. [29] The first author in the "West" to write about the mathematical manuscripts of Marx, is the North American mathematician of Dutch origin, Dirk-Jan Struik (1894-2000). [30] He wrote a paper entitled "Marx and Mathematics," which appeared in the journal *Science and Society*, in 1948.

At that time, the complete publication of the mathematical manuscripts was not yet possible. The manuscripts needed to be deciphered. It was necessary to put the papers in chronological order and to separate notes on books studied by Marx from his analysis and commentary. To succeed in this separation, it was imperative to procure and investigate the sources that Marx had used in his studies. This work of chronological ordering, deciphering, and classification was completed by a team of Soviet scientists A. S. Ryvkin, Konstantin A. Rybnikov (1913-2004) [31] and others, under the direction of Sofia A. Yanovskaya (1896-1966) (born in Poland), with the aid and advice of Academicians Andrey Kolmogorov (1903-1987) and Ivan Petrovsky (1901-1973). [32] This persistent labor of many years was crowned by the complete publication of the *Mathematical Manuscripts* of Marx in 1968, at the same time in German and Russian, for the commemoration of the 150th anniversary of the birth of Karl Marx.

Other editions followed. A German version was partially re-edited in 1974 in the Federal Republic of Germany. [33] In the same

---

28 Glivenko, 1934, pp. 79-85; cf. Struik, 1948a, p. 194; Struik, 1975, p. 155.

29 Gokieli, 1947. Cf. Struik, 1977, p. 242.

30 Kennedy, 1977, p. 304.

31 Ph.D. thesis (1954) on the mathematical research of Karl Marx. See, for example, Miller, 1969, p. 649.

32 Yanoyskaya, 1969, p. 35.

33 Marx, 1974b, edited by Endemann.

year, the University of Beijing put out a Chinese translation. [34] An Italian translation appeared in 1975. [35]

The publication of the *Mathematical Manuscripts* stimulated many articles of analysis and debate by, among others, Miller, Rieske, Schenk, Kennedy, Yanovskaya, Matarrese and Ponzio. In the Second Summer Conference on the History of Mathematics, held in 1978 in Liepaya in the Soviet Union, the philosopher-mathematician Vladimir Molodschi (1906-...) presented a communication entitled "The *Mathematical Manuscripts* of Marx and the Advances in the History of Mathematics in the USSR." In his paper, Molodschi pointed out that study of the *Mathematical Manuscripts* had an inspiring influence on the birth and development of the Soviet school on the history of mathematics. [36]

The *Mathematical Manuscripts* is still not well known to the general public because this work is still very "young," since little time has passed since its publication after a century of gestation.

---

34 Gu Jin-yong, 1976, p. 132.

35 Translated and edited by Matarrese and Ponzio, 1975.

36 Demidov & Volodarsky, 1980, p. 74.

Chapter 4

MANUSCRIPTS: CONTENTS AND BACKGROUND

4.1 Contents of the *Manuscripts*?

The mathematical manuscripts that Marx left to posterity consisted of thirty-one elaborate calculations, notes on arithmetic, algebra, analysis, and geometry, and nineteen sketches and independent mathematical studies, in a total of almost one thousand pages. In addition, there was preserved a series of applications of mathematics to problems of political economy: differential rent, the process of circulation, surplus value, profit, and the problem of crises. [37]

These studies varied from provisional notes to complete manuscripts ready to be published. They include the solution of algebraic equations of higher degree, series (especially divergent series), analytic geometry, and differential calculus. [38]

The major part of his studies was given to investigation of differential calculus. [39] As we will see, this was not by chance.

Once convinced of the immense applicability of differential and integral calculus and having collected all available material on the concepts and basic operations of differential calculus, Marx proceeded with his profound, critical analysis. He succeeded in completing two research studies “On the Concept of the Derivative” and “On the

[37] Kol’man, 1931a, p. 234.

[38] Struik, 1948b, p. 184; Struik, 1975, p. 142.

[39] Rieske & Schenk, 1972, p. 175.

Differential." His studies on the "Historical Development of Differential and Integral Calculus" remained unfinished.

I will limit myself here to a review of the manuscripts on the essence and history of differential calculus, since these are generally considered the most original and stimulating of the mathematical-philosophical writings of Marx.

In the following paragraphs I will describe in broad outline the development of differential calculus so that we can better appreciate the significance of Marx's contributions to the foundations of calculus and the period in which he worked.

4.2 *Why was Marx interested in the foundations of differential calculus?*

4.2.1 *Calculus did not drop out of the sky...*

In *Dialectics of Nature*, Engels praised the discovery of differential and integral calculus, or the calculus of infinitesimals, as follows:

> "Of all theoretical advances there is surely none that ranks so high as a triumph of the human mind as the discovery of the infinitesimal calculus in the last half of the seventeenth century." [40]

This triumph was no accident.

The invention of calculus, much as the birth of all modern science, followed closely the birth of capitalism. The great renaissance of commerce and industry in Europe, accompanied by the rise of the capitalist class in the fifteenth, sixteenth, and seventeenth centuries, began to exercise a tremendous influence on mathematics. With the discovery of analytic geometry and the function concept and the invention of calculus, mathematics was transformed from a science of constant quantities to the mathematics of varying quantities.

The introduction of mechanical tools of production, from windmills and cranes to water pumps and machines to drill stones, the

40 Engels, 1964, pp. 271-72; Engels, 1974b, p. 286.

development of oceanic navigation, new military techniques, and the natural sciences in general demanded new knowledge – necessitating means of analyzing and calculating *motions* (projectiles, free fall, planetary motion, accelerated motion, etc.). The mathematics of varying quantities constituted the mathematical response to this external stimulation, further enriched by the study of problems arising from the technical, inner development of mathematics, such as the study of abstract curves and surfaces, including the so-called tangent problem. [41] The mathematics of varying quantities represents the response of mathematics to a profound problem – the analysis of motion.

The socioeconomic pressure to discover adequate mathematical methods makes it easy for us to understand that the invention of calculus could not have been the work of one or another isolated genius. It should be pointed out that calculus was the culmination of the work of four generations of mathematicians: the Italians Federigo Commandino (1509-1575), Luc Valerio (1552-1618), Galileo Galilei (1564-1642), Bonaventura Cavalieri (1598-1647), and Evangelista Torricelli (1608- 1647); the German Johann Kepler (1571-1630); the Swiss Paul Guldin (1577-1643); the Belgian Grégoire de Saint-Vincent (1584-1667); the Dutchman Christian Huygens (1629-1695); the Frenchmen Antoine de Lalouvère (1600-1664), Giles de Roberval (1602-1675), and Pierre Fermat (1601-1665); the Englishmen John Wallis (1616-1703) and Isaac Barrow (1630-1677); the Scott James Gregory (1638-1675); the German-English Nicolaus Mercator (1620-1687); and outstandingly, the Englishman Isaac Newton (1643-1727) and the German Gottfried Leibniz (1646-1716). It was through joint work and mutual discussion [42] that they created the differential and integral calculus which, in the words of the physicist John D. Bernal (1901-1971):

> "... may be considered, as much as the telescope, an essential instrument of the new science." [43]

---

41 For an introduction to the relationships between the internal and external factors in the development of mathematics, see Gerdes, 1981, pp. 30ff.

42 See, for example, Wussing, 1979, pp. 161-62.

43 Bernal, 1971, Vol. 2, p. 484.

This mathematical "telescope" rapidly won new successes in astronomy and practical applications (however, still on a scale limited in accord with the interests of the absolutist, feudal state) such as artillery, construction of fortifications and hydraulics (water wheel, turbines, shape of ship hulls, etc.)

4.2.2 *A challenge ... "to strip away the veil of mystery"*

Karl Marx was convinced of the vast applicability of differential and integral calculus. He continued to be impressed by the notable successes referred to earlier. However, in studying the available manuals, written under the direct influence of the great mathematicians of the seventeenth and eighteenth centuries, particularly Newton, Leibniz, Euler, D'Alembert, and Lagrange, [44] Marx found the interpretations of the basic concepts of derivative and the differential mysterious, very diverse, and contradictory:

- Is the derivative based on the differential or vice versa?;
- Does the differential remain a small constant, or does it tend to zero, or is it equal to zero?

Marx became aware of the challenge. Referring to the need to place calculus on correct foundations, he said that it was necessary:

> "... to remove the veil of mystery from the methods of infinitesimals." [45]

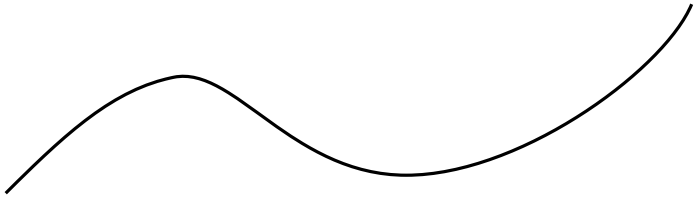
In the seventeenth and eighteenth centuries, work in the foundations of calculus was retarded in comparison with the headlong development of the content of mathematics. [46] The mathematicians of that period calculated the length of arcs of curves, considering them as the sum of an infinite number of infinitely small line segments:

---

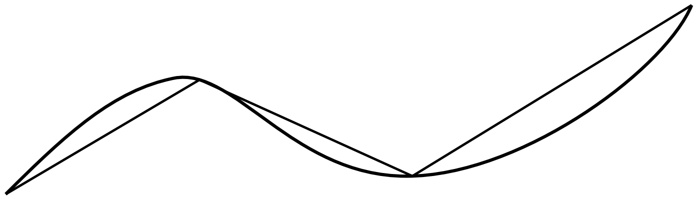
44 Struik, 1948b, p. 185; Struik, 1975, p. 143.

45 Marx, 1974b, p. 130; Marx, 1975, p. 153.

46 Molodschi, 1977, p. 162.



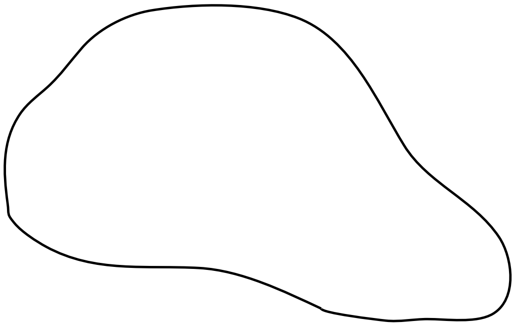
An arc of the curve



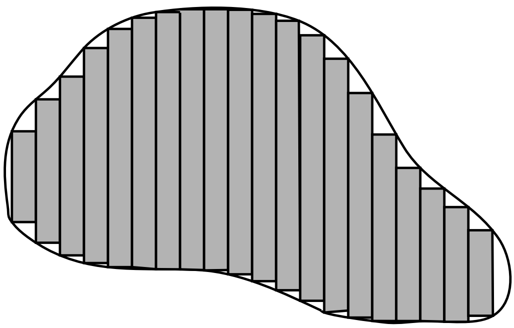
The arc approximated by line segments

The arc approximated by smaller and smaller line segments until there is an "infinitely great number of infinitely small line segments."

The same mathematicians succeeded in determining the area of the plane figure, interpreting it as the sum of an infinite number of infinitely small rectangles.



A plane figure



The plane figure approximated by smaller and smaller rectangles... etc. until there is the "sum of an infinite number of infinitely small rectangles."

These are "very fine examples of dialectics." [47] But what is their foundation? How can we interpret the infinitely large or the infinitely small?

These and many other fascinating successes of differential and integral calculus promoted the blind faith, which led Engels to observe in his *Anti-Dühring*:

> "With the introduction of variable magnitudes and the extension of their variability to the infinitely small and infinitely large, mathematics, usually so strictly ethical, fell from grace; it ate of the tree of knowledge,

[47] Labérenne, 1971, p. 67.

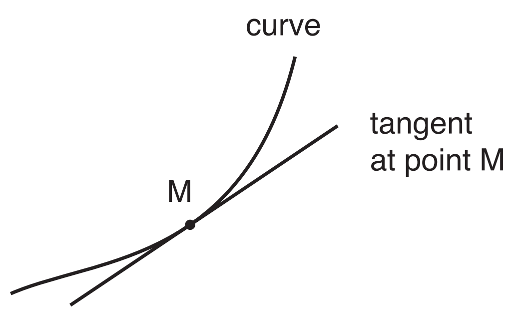
> which opened up to it the career of most colossal achievements, but at the same time the path of error. The virgin state of absolute validity and irrefutable proof of everything mathematical was gone forever; the realm of controversy was inaugurated, and we have reached the point where most people differentiate and integrate not because they understand what they are doing but from pure faith, because up to now it has always come out right." [48]

4.3 ***Marx's research on the history of mathematics***

4.3.1 ***The "mystical" calculus of Newton and Leibniz***

It was the analysis of the mechanics problem of velocity movement that brought Newton to the discovery of differential calculus. Leibniz arrived at the notion of derivative, the basic idea of all differential calculus, from geometric considerations. [49]

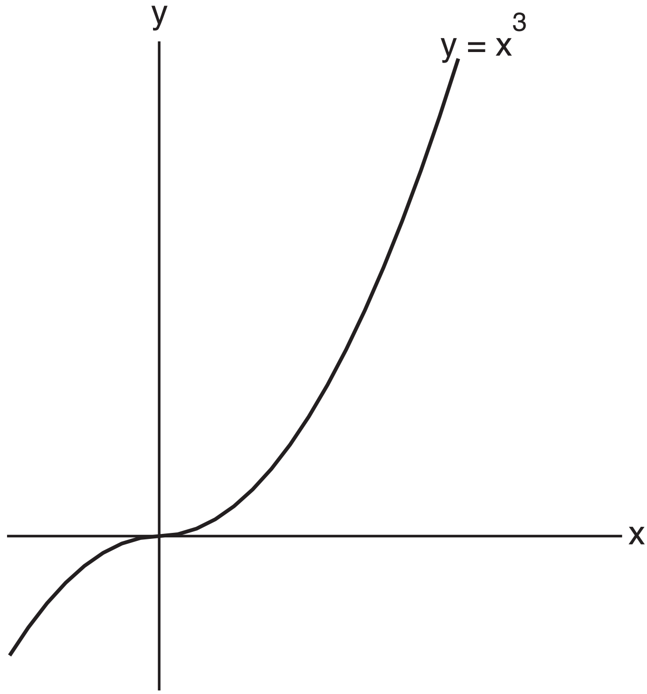
Let us follow the road of Leibniz: How can one construct the line tangent to a curve at the given point?



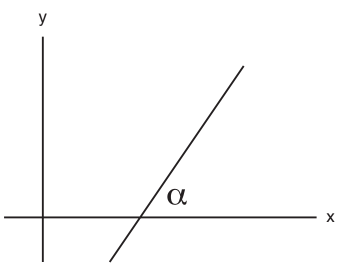
48 Engels, 1962, p. 123; Engels, 1974a, p. 112. Compare MEW, Vol. 20, p. 81 [See also Lenin, 1960, Vol. 38, p. 117: "Hitherto the justification (of the infinite in mathematics) has consisted only in the correctness of the results … and not in the clearness of the subject." – G.W.]

49 For the relation between these two points of departure, see, for instance, Piskounov, 1979, chap. 3.

When we know the slope of the required tangent line, [50] we can easily construct the tangent line. Now the question arises as to how one determines the slope. Let us look at a concrete example to see how Leibniz proceeded.



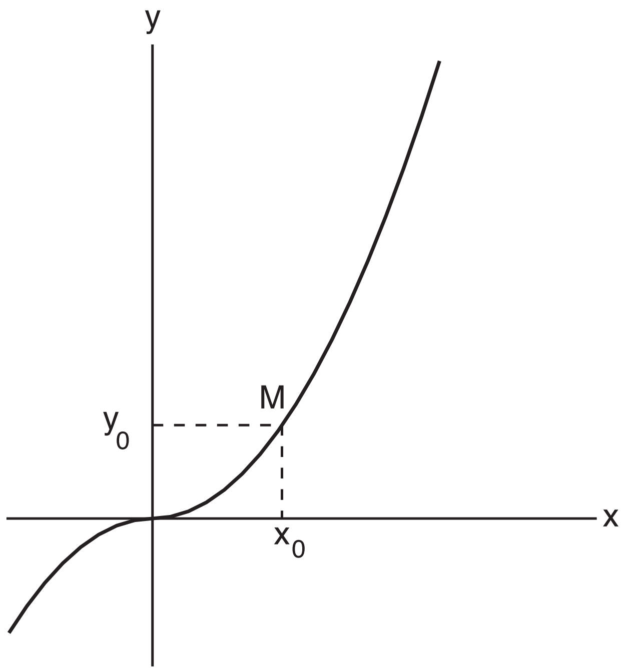
50 Given a system of Cartesian coordinates, then the slope of a line is tan α, where α is the angle between the positive x-axis and the line.



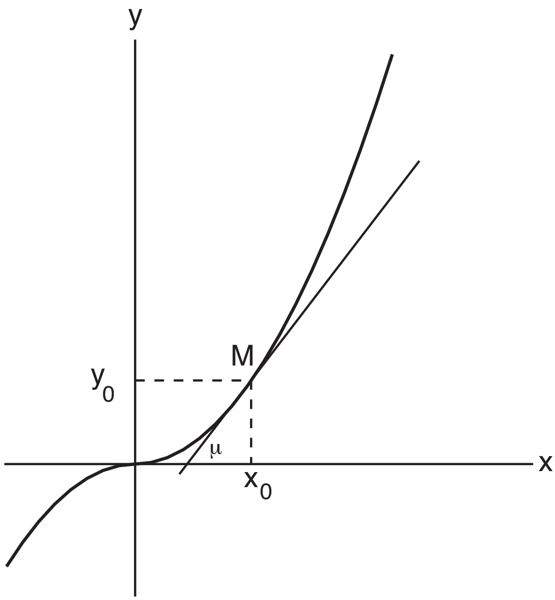
slope = tan α

Consider the function $y = x^3$, that is, $f(x) = x^3$, and the graph of this function in a system of Cartesian coordinates [This is an example used by Marx – B. L.].

Let M be the point on the curve that corresponds to a given value of x, let us say x_0.

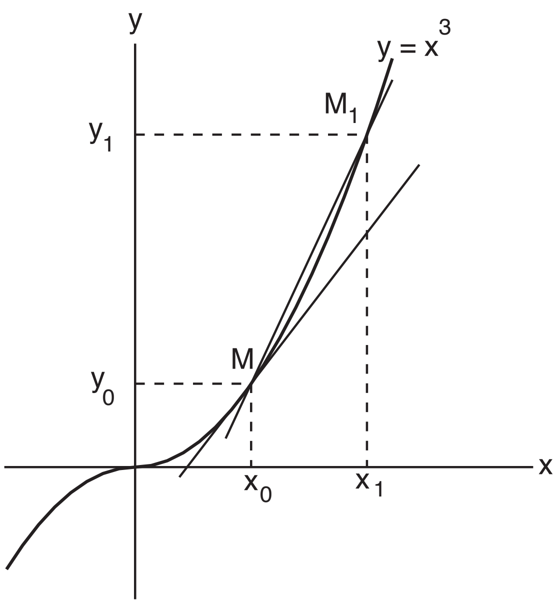


What will be the slope of the tangent at M? Or $\tan \mu = ...?$



Let M_1 be a point on the curve, very close to M. M_1 corresponds to a value of x, let us say x_1. We have:

$$y_1 = (x_1)^3$$

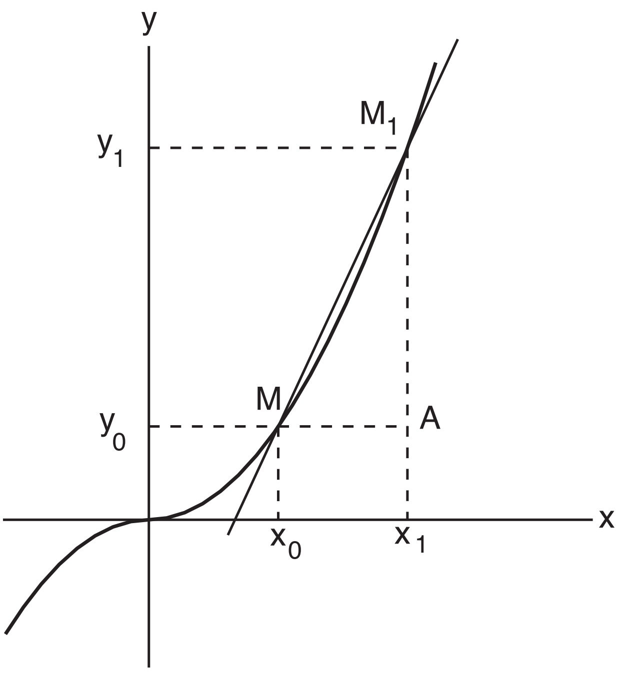


Thus, the secant MM_1 is very close to the tangent line at M. And the slope of secant MM_1 is also very close in value to the slope of the tangent line at M.

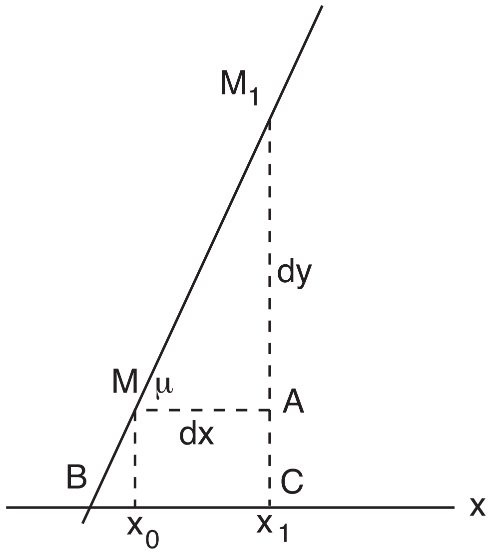
Mathematicians of the seventeenth century, such as Leibniz, advanced their argument in the following manner:

(1) Let M_1 be a point "infinitely close" to M.
Under this assumption, secant MM_1 coincides (!) with the tangent line at M and the slope of the tangent line is equal to the slope of secant MM_1.

Let us calculate the slope of secant MM_1. Let MA be parallel to the x-axis, as shown in the following figure.



Or enlarging one part of the figure:



Thus we have angle M_1MA = angle MBC = μ.

So tan μ = tan (∠ M_1MA) and:

$$\tan(\angle M_1MA) = \frac{|M_1A|}{|MA|}$$

$|M_1A|$ is equal to the "infinitely small difference" between y_1 and y_0, and for this reason Leibniz called it "**differential of y**" or briefly ***dy***. In the same manner, |MA| equals the "infinitely small difference" between x_1 and x_0 and was called the "**differential of x**" or just ***dx***. With these symbols, we have:

$$\tan\mu = \tan(\angle M_1MA) = \frac{|M_1A|}{|MA|} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{dy}{dx}$$

For M_1 to be "infinitely close" to M, we have x_1 "infinitely close" to x_0, and y_1 "infinitely close" to y_0. Or, to use the terminology of the seventeenth century, the differentials *dx* and *dy* are "**infinitely small**." But what is "infinitely small"?

Let us calculate the quotient of the differentials *dy* and *dx*.

We have $dx = x_1 - x_0$, which means $x_1 = x_0 + dx$, and $dy = y_1 - y_0$, where

(2)
$$y_1 = (x_1)^3 = (x_0 + dx)^3 = (x_0)^3 + 3(x_0)^2.dx + 3x_0.(dx)^2 + (dx)^3.$$

Taking into account $y_0 = (x_0)^3$, we can write:

$$y_1 = y_0 + 3(x_0)^2.dx + 3x_0.(dx)^2 + (dx)^3,$$

that is, $dy = y_1 - y_0 = + 3(x_0)^2.dx + 3x_0.(dx)^2 + (dx)^3$.

And, dividing both members by *dx* we get:

$$\frac{dy}{dx} = 3(x_0)^2 + 3x_0.dx + (dx)^2.$$

(3) Now, following Leibniz, we can suppress the terms on the right containing *dx* and $(dx)^2$ since they are "infinitely small." Thus,

$$\frac{dy}{dx} = 3(x_0)^2$$

or, for any value of *x*, $\frac{dy}{dx} = 3x^2$.

Based on the successes of applications of differential calculus, Leibniz and his successors knew that not only the final answer of $\frac{dy}{dx} = 3x^2$ of our example was correct, but also that the same method always resulted in correct formulas.

But this differential calculus, approached in this way, is very "*mysterious*", in the opinion of Marx:

> "... giving correct results by means of positively false mathematical procedure." [51]

---

51 Marx, 1974b, p. 119; Marx, 1975, p. 138.

To obtain the correct formulas of differential calculus, the "infinitely small quantities" are sometimes "*juggled away*" or "*forcibly suppressed*", [52] or, so to say, treated as zeros ($dx = 0$):

$$\frac{dy}{dx} = 3(x_0)^2 + 3x_0.dx + (dx)^2 = 3(x_0)^2 \quad \text{[see (3)]},$$

and sometimes treated as quantities different from zero ($dx \neq 0$):

$$y_1 = (x_1)^3 = (x_0 + dx)^3 = (x_0)^3 + 3(x_0)^2.dx + 3x_0.(dx)^2 + (dx)^3$$

[see (2)]. [53]

Note that if in (2) *dx* had been treated as equal to zero, this

"...leads to literally to nothing" [54] :

$$y_1 = (x_1)^3 = (x_0 + dx)^3 = (x_0 + 0)^3 = (x_0)^3$$

In this way:

$$dy = y_1 - y_0 = (x_1)^3 - (x_0)^3 = 0,$$

which has the result:

$$\frac{dy}{dx} = \frac{0}{0} \quad (!!???).$$

Because of this contradictory and mysterious treatment of "infinitely small quantities," Marx used the term "*mystical*" [55] in his historical analysis of the differential calculus of Newton and Leibniz, thus characterizing the "infantile disease of infinitesimal calculus." [56]

---

52 Marx, 1974b. p. 118; Marx, 1975, p. 136.

53 Cf. Ruzavin, 1977, p. 137.

54 Marx, 1974b, p. 53; Marx, 1975, p. 45; also Marx, 1983, p. 3 [This volume is C. Aronson and M. Meo's translation of part of Marx, 1968, with additional material. – G.W.]

55 Marx, 1974b, p. 117; Marx, 1975, p. 135.

56 Thiel, 1975, p. 73.

4.3.1.1 ***Refuting the attacks of idealists***

Obviously Marx was not the first to criticize the naïve opinions of Newton and Leibniz. The founders themselves already had their doubts.

Traditional mathematicians did not accept the new differential calculus. Philosophers and even poets, such as Jonathan Swift (l676-1745) in his *Gulliver's Travels*, [57] criticized the new calculus. Idealists did not hesitate to exploit the philosophical, logical-mathematical weaknesses in the foundations of differential calculus. [58] The Irish bishop-philosopher George Berkeley (1685-1753) perceived the indeterminate and ambiguous nature of the "infinitely small quantities," and in his pamphlet, *The Analyst*, launched an attack on the progressive mathematicians: Who believes in this "staggering" differential calculus has no reason at all for not believing in God ... [59]

Here he referred to the fact that many of those mathematicians who accepted differential calculus were atheists and interpreted mathematics in a spontaneous, materialist form as a science that describes the proprieties of real quantities that exist **outside** of human consciousness. These same mathematicians connected the concepts of "infinitely small" and "infinitely large" to the recognition of a material substance and its unlimited divisibility. Thus behind Berkeley's negative attitude toward differential calculus was his desire to refute atheism and materialism.

Others, already convinced of the practical value of calculus, tried to take advantage of its growing prestige. However, their objectives were similar to those of Berkeley.

---

57 Wussing, 1979, p. 199; see Swift, 1938, p. 199.

58 Ruzavin, 1977, p. 136.

59 Wussing, 1979, p. 200; Ruzavin, 1977, p. 136; Struik, 1948a, p. 197. [In his *Principles of Human Knowledge* (1710), Berkeley criticized Leibniz's doctrine of infinitesimals, especially as it was compatible with – if not conducive to – materialism and atheism; see George Berkeley, 1949, Vol. 2, pp. 102-03. In his *Analyst* (1734), he turned his criticism towards Newton's conception of fluxions; see Berkeley, 1949, Vol. 4. – G.W.]

Instead of ridiculing certain difficulties in the treatment of the "infinitely small" or "infinitely large," as Berkeley had done, the Italian priest, Father Guido Grandi (1671-1742), utilized these difficulties in a clever way. [60] For example, Grandi claimed that a paradox resulted from the summation of the following infinite series:

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots = \ldots ?$$

Blind application of the rules for finite sums gave on the one hand:

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots =$$
$$= (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + \ldots =$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + \ldots =$$
$$= 0,$$

and on the other hand:

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots =$$
$$= 1 - (1 - 1) - (1 - 1) - (1 - 1) - (1 - 1) - \ldots =$$
$$= 1 - 0 - 0 - 0 - 0 - 0 - \ldots =$$
$$= 1.$$

Thus, according to Grandi, we have 0 = 1. According to the priest, this demonstrates the possibility that God had created the world (=1) out of nothing (= 0)!

Knowing of these idealist attacks, [61] Karl Marx felt obliged to deepen his analysis of the "mystical differential calculus" of Leibniz and Newton in order to provide a materialist foundation for infinitesimal calculus.

---

60 Struik, 1948a, p. 126; Molodschi, 1977, p. 169; Wussing, 1979, p. 199.

61 Marx, 1983, pp. 91-94; Marx, 1974b, p. 119; Marx, 1975, p. 138.

4.3.1.2 ***Without artificial premises***

In his 1791 book, *Reflections on the Metaphysics of Infinitesimal Calculus*, [62] the French mathematician and revolutionary Lazare Carnot (1753-1823) explained the correctness of differential calculus as the result from mutual cancelling of errors made in the deduction of the theorems. It may appear that Marx agreed with Carnot:

> "... The sleight of hand [by filling in *dx* = 0 – P.G.] is unwittingly mathematically correct because it only juggles away errors of calculation arising from the original sleight-of-hand" [meaning *dx* as an infinitely small difference, in place of *Δx*, a finite difference. – P.G.] [63]

When *dx* is not an ordinary difference of two quantities, that is, when *dx* is not an ordinary number, how can we justify the use of the rules for ordinary numbers? This was a pertinent question, which Marx posed. In our example, how can we justify the application of the binomial expansion of Newton:

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \quad \text{in:}$$

$$(x_0 + dx)^3 = (x_0)^3 + 3(x_0)^2.dx + 3x_0.(dx)^2 + (dx)^3$$

if *dx* is not an ordinary number? [64]

Here Marx touches on a characteristic [65] common to all attempts to provide a rigorous foundation for differential calculus – a part of higher mathematics – in the seventeenth and eighteenth centuries. It is the transfer of principles and propositions of elementary mathematics

---

62 Wussing, 1979, p. 223 [See also Lenin, 1960, Vol. 38, p. 118: "Characteristic is the title – Carnot, *Reflections sur la Metaphysique du calcul infitesimal!*" – G.W.].

63 Marx, 1983, pp. 84, 92; Marx, 1974b, p. 118; Marx, 1975, p. 136. See Marx, 1974b, p. 111; Marx, 1975, p. 128.

64 Actually, we say, "If *dx* is not an *Archimedean* number."

65 Molodschi, 1977, p. 162.

[66] to new branches of mathematical research without argumentation or justification. Such an example is the treatment of the series

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots = \dots,$$

that we met in an earlier paragraph.

According to Marx, it is erroneous to treat, right from the start, the finite difference $x_1 - x_0$ or Δx, the increment of the variable x, as an "infinitely small quantity", dx.

> "The arbitrary supposition, $x_1 - x_0 = dx$ results in the need … to juggle away the terms in dx in the binomial expansion of $(x_0 + dx)^3$ in order to get the correct result." [67]

The question arises, why this juggling act?

> "Why the violent suppression of the terms standing in the way? That specifically assumes that one knows they stand in the way ..." [68]

How can we know which terms stand in the way? Marx concluded that this is possible only when we know in advance what the result should be – in our example $\frac{dy}{dx} = 3x^2$ – and simply look for some justification to make the result plausible. [69]

Summing up, we can say that Marx considered the differential calculus of Newton and Leibniz to be mystical because they had introduced the differentials *dx* and *dy* in a *metaphysical* manner, [70] that is to say, as infinitely small quantities, [71] without having clarified

---

66 In Engels's terminology, this is the "nieder" [i.e., "lower," "inferior"] mathematics; see Engels, 1964, p. 26; see also Engels, 1962, p. 167.

67 Marx, 1974b, p. 117; Marx, 1975, p. 135.

68 Marx, 1983, p. 92; Marx, 1974b, p. 118; Marx, 1975, p. 136.

69 Cf. Struik, 1948b, p. 188; Struik, 1975, p. 147.

70 Marx, 1983, p. 91; Marx, 1974b, p. 117; Marx, 1975, p. 135.

their origin and development nor having analyzed the nature of their specific properties.

Now how can the foundations of calculus be established without recourse to artificial assumptions, such as quantities, which appear ($dx \neq 0$) and disappear ($dx = 0$), and without sleight-of-hand? [72]

4.3.2 ***The "rational" differential calculus of D'Alembert and Euler***

The second important stage in the development of the methods of differential calculus, according to Karl Marx, was the differential calculus of the Frenchman Jean D'Alembert (1717-1783) and the Swiss Leonhard Euler (1707-1783). At the same time as we observe the hundredth anniversary (1983) of the death of Karl Marx, we could commemorate the bicentenary of the deaths of D'Alembert and Euler. It would not be in vain!

Euler, alone, left to posterity 886 books and articles. He was probably the most productive mathematician to date ... [73] D'Alembert and Euler achieved, in the words of Marx,

> "... enormous progress in removing the veil of mysticism from differential calculus." [74]

---

71 Initially, Marx himself still used terms such as "infinitely close," "infinitely small," etc. See his letter to Engels, written towards the end of 1865 or the beginning of 1866; MEW, 1961, Vol. 31, p. 165.

72 Some scientists explained the infinitesimals or infinitely small quantities in terms of the dialectical nature of opposites – at the same time equal to zero and different from zero. Yanovskaya called these scientists *pseudo-Marxists* because they forgot that dialectical materialism does not recognize *static contradictions* (= 0 and ≠0), but only contradictions connected with motion. See Kennedy, 1977, p. 310.

73 Struik, 1948a, p. 120.

74 Marx, 1974b, p. 122; Marx, 1975, p. 141; Struik, 1948b, p. 189.

For this success in "removing the veil of mysticism from differential calculus," Marx called the differential calculus of D'Alembert and Euler "**rational**." [75]

In what way was it rational? In what way was it a step forward?

4.3.2.1 ***How to remove the veil of mysticism from calculus?***

Newton and Leibniz took as their point of departure:

$$x_1 = x_0 + dx$$

where *dx* was an "infinitely small quantity" whose nature had not been clarified. D'Alembert immediately introduced a "*fundamental correction*," [76]

$$x_1 = x_0 + \Delta x$$

where *Δx* was any finite increment. Thus, *Δx* was an ordinary [Archimedean – B.L.] number and all the rules of algebra could be applied to it, in particular, the Newton binomial expansion. In this manner, continuing with our example, we have in the case of the function $y = x^3$:

$$y_1 = (x_1)^3 = (x_0 + \Delta x)^3 = (x_0)^3 + 3(x_0)^2 . \Delta x + 3x_0.(\Delta x)^2 + (\Delta x)^3.$$

Since *Δy* is the increment of the function *y*, corresponding to the increment *Δx* of the independent variable, we have:

$$\Delta y = y_1 - y_0 = \{ (x_0)^3 + 3(x_0)^2 . \Delta x + 3x_0.(\Delta x)^2 + (\Delta x)^3 \} - (x_0)^3 =$$

$$= 3(x_0)^2 . \Delta x + 3x_0.(\Delta x)^2 + (\Delta x)^3.$$

In forming the quotient of the increment of the function and the increment of the independent variable, we get:

$$\frac{\Delta y}{\Delta x} = 3(x_0)^2 + 3x_0.\Delta x + (\Delta x)^2.$$

---

75 MEW, 1961, Vol. 35, p. 114.

76 Marx, 1983, p. 121, Marx, 1974b, p. 119; Marx, 1975, p. 138; Struik, 1948b, p. 188-89.

Now substituting $x_1 = x_0$ in this equation, or $\Delta x = x_1 - x_0 = 0$ and consequently,

$$\Delta y = y_1 - y_0 = (x_1)^3 - (x_0)^3 = (x_0)^3 - (x_0)^3 = 0,$$

we obtain:

$$\frac{0}{0} = 3(x_0)^2.$$

Or, for any value of *x*: $\frac{0}{0} = 3x^2$.

Instead of $\frac{0}{0}$ here D'Alembert and Euler write $\frac{dy}{dx}$. Thus:

$$\frac{dy}{dx} = 3x^2.$$

The final result is the same as that obtained by applying the mystical method of Newton and Leibniz. However, as Marx observed, the conclusion $\frac{dy}{dx} = 3x^2$ is reached by a "correct mathematical operation", meaning that the terms

$$3x_0.\Delta x \quad \text{and} \quad (\Delta x)^2$$

were "removed without trickery." [77] For this reason, the conclusion is rational. [78]

Nevertheless, what interpretation can be given to the differentials, *dx* and *dy*, which appeared so suddenly at the end – exactly, to use an apt metaphor of Marx – "just before closing hour" ("knapp vor Torschluss" in German). [79] And what monster is this $\frac{0}{0}$?

---

77 Marx, 1974b, p. 121; Marx, 1975, p. 140.

78 D'Alembert's method, still based on Newton's vague limit concept, was improved at the end of the eighteenth century and the beginning of the nineteenth century by the Frenchmen J. Cousin and S. Lacroix, and by the Russians S. Gurjew and P. Rachanov. See, for example, Molodschi, 1977, p. 172.

79 Marx, 1974b, p. 120; Struik, 1948b, p. 189.

4.3.2.2 ***The zeros of Euler***

In accord with the appearance of $\frac{dy}{dx}$ in his method, Euler considered all the differentials equal to zero: [80]

$$dx = 0 \text{ and } dy = 0.$$

Then we have $\frac{dy}{dx} = \frac{0}{0}$. But how can we divide 0 by 0? How could this be possible? [81]

The expression $\frac{12}{4} = 3$ means that $12 = 4 \times 3$ and that there exists no number different from 3 such that 4 times that number equals 12.

In general, division of two numbers is defined in the following manner:

$\frac{a}{b} = c$ means that *c* is the unique number such that $a = b \times c$.

Taking this definition into account, we ask, what can be the significance of $\frac{0}{0}$. We conclude that $\frac{0}{0} = c$ means that *c* is the unique number such that $0 = 0 \times c$. But we have $0 \times 1 = 0$, $0 \times 2 = 0$, $0 \times 3 = 0$, $0 \times 4 = 0$, etc., or c=1, 2, 3, etc. In other words, the number *c* is not unique, as required by the definition of division. This means that $\frac{0}{0}$ does not exist, as every student learns in secondary school these days.

Yes, *arithmetically* 0 cannot be divided by 0, Leonhard Euler said, but …

Skillfully, in his book, *Differential Calculus* of 1755, he introduced other "zeros," "zeros in the *geometric* sense," [82] still speaking in terms of "infinitely small quantities."

> "There exists an infinity of orders of infinitely small quantities, although they are all equal to zero. But

80 Cf. Rieske & Schenk, 1972, p. 478.

81 Cf. R. R. Struik, 1974.

82 Rieske & Schenk, 1972, p. 479.

> they must be distinguished among themselves when we observe them in their mutual proportions that can be explained by geometric ratios." [83]

However, what is meant by a "geometric ratio of two zeros" is not very clear in Euler's writings. [84] Because zeros are geometrically different, they must be distinctly symbolized as *dx*, *dy*, etc. Euler calculated with his "zeros," that is to say with his differentials, according to special rules:

$$a + dx = a,$$

$$dx + (dx)^2 = dx,$$

$$a\sqrt{dx} + b.dx = a\sqrt{dx}\text{ , etc.,}$$

where *a* and *b* are ordinary numbers.

Here Marx expressed a series of doubts. When these so-called zeros are characterized by their proportions, how can we speak of *dx* in isolation and calculate with it as a + dx = a, etc. [85]

> "In $\frac{dy}{dx} = \frac{0}{0}$, numerator and denominator are inseparably bound together." [86]

4.3.2.3 ***Marx's main criticism***

It was Marx's opinion that D'Alembert and Euler had not yet perceived the profound dialectics of the process differentiation. [87] Their method is formally correct. [88] However, the final result,

---

83 Cited in Struik, 1948a, p. 125.

84 In fact, D'Alembert's and Euler's method anticipates Cauchy's definition:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\ .$$

85 Marx, 1974b, pp. 62, 67; Marx, 1975, pp. 60, 66

86 Marx, 1974b, p. 98; Marx, 1975, p. 109.

87 Cf. Yanovskaya, 1969, p. 32.

88 Cf. Struik, 1948b, p. 192; Struik, 1975, p. 149.

$$\frac{dy}{dx} = 3x^2 \text{ in our example,}$$

is already known as the coefficient of Δx in the expansion of $(x + \Delta x)^3$ in powers of Δx:

$$(x + \Delta x)^3 = x^3 + 3x^2 . \Delta x + 3x.(\Delta x)^2 + (\Delta x)^3.$$

That is to say, $\frac{dy}{dx}$ is already present **before** of the differentiation, or before the calculation of $\frac{\Delta y}{\Delta x}$ and the substitution $\Delta x = 0$. Hence,

> "... the derivation is therefore essentially the same as that of Leibniz and Newton", [89]

but with the improvement that the derivative, in our example $3x^2$, is *liberated* [90] or *separated* ("losgewickelt" in the original German), from the other terms in a strictly algebraic form, without any juggling away. However, according to Marx, the derivative must be *developed* ("entwickelt" in German), as we will see in the paragraph about Marx's own contribution to the solution of this problem.

When $\frac{dy}{dx}$ is already known in advance, in our example, $3x^2$, as the coefficient of Δx in the expansion of, $(x+\Delta x)^3$ in powers of Δx, why not immediately define $\frac{dy}{dx}$ as the coefficient of Δx in that expansion?

4.3.3 *The "purely algebraic" differential calculus of Lagrange*

Why not immediately define $\frac{dy}{dx}$ as the coefficient of Δx in the expression of y_1 or $f(x_0 + \Delta x)$ as a sum of powers of Δx?

---

89 Marx, 1974b, p. 121; Marx, 1975, p. 140; Struik, 1948b, p. 189.

90 Marx, 1974b, p. 128; Marx, 1975, p. 149.

This is precisely what the Frenchman Joseph Lagrange (1736-1813) did in his famous handbook, *Theory of Analytic Functions*. The subtitle of his book already indicates Lagrange's objective:

> "Theory of analytic functions, including the principles of differential calculus, *freed* from any contemplation of *infinitely small quantities*, of quantities that disappear, of limits and fluxions, and reduced to *algebraic* analysis of finite quantities." [91]

Pursuing the example, $y = x^3$ or $f(x) = x^3$, we have

$$f(x+\Delta x) = (x + \Delta x)^3 = x^3 + 3x^2 . \Delta x + 3x.(\Delta x)^2 + (\Delta x)^3.$$

Lagrange called $3x^2$, the coefficient of Δx is this expansion, the *derivative*, labeling it as $f^1(x)$:

$$f^1(x) = 3x^2.$$

Introducing the derivative in this manner, Lagrange avoided the differentials *dy* and *dx*, and the quotient of differentials $\frac{dy}{dx}$. In Lagrange's opinion, an expression of the type $f(x+\Delta x)$ could be expanded (almost) always in a series of the type:

$$f(x) + p(x) . \Delta x + q(x) . (\Delta x)^2 + r(x) . (\Delta x)^3 + \ldots ,$$

where *p*, *q*, *r*, etc. are new functions in *x*, "derived" from the initial function f(x). [92]

Summing up Lagrange's argument as follows: [93]

$$f(x+\Delta x) - f(x) = 0, \text{ when } \Delta x = 0.$$

This implies that $f(x+\Delta x)-f(x)$, considered as a polynomial in Δx, is divisible by $\Delta x-0$, that is, by Δx (Bézout's theorem). Let $p(x+\Delta x)$ be the quotient. Then:

$$f(x+\Delta x) - f(x) = p(x+\Delta x). \Delta x \text{ , that is:}$$

$$f(x+\Delta x) = f(x) + p(x+\Delta x). \Delta x.$$

---

91 Wussing, 1979, p. 224.

92 Cf. Yanovskaya, 1969, p. 33.

93 Cf. Miller, 1969, p. 655.

In turn, repeating the same reasoning:

$$p(x+\Delta x) - p(x) = q(x+\Delta x).\ \Delta x,$$

$$q(x+\Delta x) - q(x) = r(x+\Delta x).\ \Delta x, \text{ etc.}$$

Substituting in

$$f(x+\Delta x) = f(x) + p(x+\Delta x).\ \Delta x,$$

we get

$$\begin{aligned} f(x+\Delta x) &= f(x) + \{p(x) + q(x+\Delta x).\ \Delta x\}.\Delta x = \\ &= f(x) + p(x).\Delta x + q(x+\Delta x).(\Delta x)^2 = \\ &= f(x) + p(x).\Delta x + \{q(x)+r(x+\Delta x).\ \Delta x\}.(\Delta x)^2 = \\ &= f(x) + p(x).\Delta x + q(x).(\Delta x)^2 + r(x+\Delta x).(\Delta x)^3 = \\ &= f(x) + p(x).\Delta x + q(x).(\Delta x)^2 + r(x).(\Delta x)^3 + s(x).(\Delta x)^4 + \ldots \end{aligned}$$

Lagrange called the function p(x) the first derivative function, using the notation $f^1(x)$, as in our example.

Lagrange's method constituted an improvement, according to Karl Marx, because, basing itself on algebraic operations, it

> "... freed itself from anything resembling metaphysical transcendence," [94]

meaning that it was free of infinitely small quantities, of quantities that appear ($dx \neq 0$) and disappear ($dx = 0$) and of $\frac{0}{0}$.

However, observed Marx, Lagrange did not prove that every function could be expanded in such a power series (called Taylor series):

$$f(x+\Delta x) = f(x) + p(x).\Delta x + q(x).(\Delta x)^2 + r(x).(\Delta x)^3 + s(x).(\Delta x)^4 + \ldots$$

How can we be sure that the sum, possibly infinite

$$f(x) + p(x).\Delta x + q(x).(\Delta x)^2 + r(x).(\Delta x)^3 + \ldots$$

94 Marx, 1974b, p. 135; Marx, 1975, p. 159; Kennedy, 1977, p. 308.

really exists? [95] To prove that the sum exists requires the concept of a limit, precisely what Lagrange wanted to avoid, a non-algebraic concept. [96]

For Marx, Lagrange's proof did not recognize the specific character of differential calculus and appeared

> "... to base itself on a delusion," [97]

substituting variable quantities for constants.

And, in the bargain, Lagrange continued to utilize in practice the method of Leibniz that he had criticized – for example, in the study of curves. [98]

For this reason, Lagrange did not succeed in his attempts to reduce differential calculus to algebra.

Lagrange, conscious that his method was inadequate, took the initiative in 1784, as president of the Academy of Berlin, to launch a contest for clarification of the following questions: [99]

1. How can one explain that so much important knowledge arises out of the contradictory, basic hypotheses of differential calculus?
2. How can one find a basic concept, both clear and correct, that could be substituted for the concepts of infinitely small and infinitely large, without requiring complicated and lengthy calculations?

Marx, probably unaware of the contest, accepted the challenge. He, himself, tried to provide a better foundation for differential

---

95 In other words, how can we be sure that this power series is summable?

96 See Boyer, 1960, pp. 533-34; Miller, 1969, p. 658.

97 Marx, 1974b, p. 137; Marx, 1975, p. 162.

98 Actually, as we know since Cauchy, there were the same mathematical shortcomings in Lagrange's attempt to prove expansibility. See, for example, Boyer, 1960, pp. 533-34; also Molodschi, 1977, p. 206.

99 Wussing, 1979, p. 223 etc.; Miller, 1969, p. 655.

calculus. In this attempt, as we shall see in the following chapter, he deepened his critique of the differential calculus – “mystical,” “rational” and “purely algebraic.”

Chapter 5

THE DIALECTICAL METHOD OF MARX

In one of his last letters to Marx, Engels characterized the principal distinction between the old methods for foundations of differential calculus and the method of Marx in the following manner:

> "... you let x_0 move to x_1, meaning you let it *really vary*, [100] while the others start with $x_0 + \Delta x$, which always represents strictly a sum of two quantities, but never a variation of a quantity." [101]

Let us see by use of concrete example $y = x^3$, or $f(x) = x^3$, how Marx did indeed let the quantity *x* vary.

When the independent variable *x* increases [or decreases] from x_0 to any value x_1, the dependent variable *y* varies from y_0 to y_1. Now taking the quotient of the finite differences $y_1 - y_0$ and $x_1 - x_0$, we have

$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{(x_1)^3 - (x_0)^3}{x_1 - x_0}$$

On factoring the numerator

$$(x_1)^3 - (x_0)^3 = (x_1 - x_0) . \{(x_1)^2 + x_1.x_0 + (x_0)^2\},$$

we see that:

100 The author's emphasis.

101 MEW, 1961, Vol. 35, p. 112.

$$\frac{\Delta y}{\Delta x} = \frac{(x_1 - x_0).\{(x_1)^2 + x_1.x_0 + (x_0)^2\}}{x_1 - x_0} = (x_1)^2 + x_1.x_0 + (x_0)^2,$$

and

$$\frac{\Delta y}{\Delta x} = (x_1)^2 + x_1.x_0 + (x_0)^2 .$$

The expression on the right, $(x_1)^2 + x_1.x_0 + (x_0)^2$, Marx called the *provisional derivative* (or *preliminary derivative*).

What happens when the variable x_1 goes back to x_0?

a) **To the right:**

The expression $(x_1)^2$ goes back to $(x_0)^2$; the expression $x_1.x_0$ goes to $x_0.x_0$, or $(x_0)^2$. Thus on setting $x_1 = x_0$, the provisional derivative is transformed from

$$(x_1)^2 + x_1.x_0 + (x_0)^2$$

to

$$(x_0)^2 + (x_0)^2 + (x_0)^2, \text{ that is, to } 3(x_0)^2.$$

The expression $3(x_0)^2$ is called the *definitive derivative*, abbreviated as $f^1(x_0)$.

Thus, the definitive derivative is

> "... the provisional derivative reduced to its absolute minimum value." [102]

What happens at the same time on the left side?

102 Marx, 1974b, p. 55; Marx, 1975, p. 49. Use of the term "minimum value" presupposes, in our example, that the variable *x* (positive) had been increased from x_0 to x_1. If it had decreased, we would have had to say, "maximum value."

b) **To the left:**

x_1 goes back to x_0, and finally becomes *equal* to x_0. After x_1 has reached x_0, we have:

$$x_1 - x_0 = 0 ,$$

which implies that

$$\Delta x = x_1 - x_0 = 0.$$

What about Δy?

$$\Delta y = y_1 - y_0 = (x_1)^3 - (x_0)^3 .$$

When x_1 becomes equal to x_0, we have $(x_1)^3 = (x_0)^3$.

Then $(x_1)^3 - (x_0)^3 = 0$, and $\Delta y = 0$.

Thus, the left member, $\frac{\Delta y}{\Delta x}$ is transformed in $\frac{0}{0}$. Again $\frac{0}{0}$?

> "Because, in the expression $\frac{0}{0}$, any vestige of its origin, and of its significance has disappeared, we substitute for it $\frac{dy}{dx}$, where the finite differences,
>
> $x_1 - x_0$, or Δx and $y_1 - y_0$, or Δy, appear as *symbolized*, [103] as *abolished* differences." [104]

We need not be afraid of the expression $\frac{dy}{dx}$, observed Karl Marx:

> "... the symbolic misfortune occurs only on the left-hand side, but it has already lost its terror, as it appears now only as the *expression of a process* that

---

103 Author's emphasis.

104 In German, "aufgehobene;" Marx, 1974b, p. 53 (Marx's emphasis); Marx, 1975, p. 4

already has shown its real content on the right-hand side of the equation." [105]

c) **Final result**

In this manner we get as the *well-grounded*, final result in the case of our example:

$$\frac{dy}{dx} = 3\,(x_0)^2 \text{, or more generally: } \frac{dy}{dx} = 3x^2.$$

For any function, we have:

$$\frac{dy}{dx} = \text{definitive derivative,}$$

$$\text{abbreviated as } \frac{dy}{dx} = f^1(x).$$

Now let us analyze, more closely, some aspects of Marx' method.

5.1 *A real development*

Leibniz and Newton used as their point of departure $x_1 = x_0 + dx$. D'Alembert, Euler and Lagrange began with $x_1 = x_0 + \Delta x$. [106] By starting with the *sum* $x_0 + dx$, or $x_0 + \Delta x$, they treated dx, or Δx, as a quantity distinct and separate from x_0, or, as Marx wrote, as a

"... fetus ..., *before* it had been fertilized." [107]

Why did he say, "before it had been fertilized"?

Let's return to our example $y = x^3$. In the old methods we have:

---

105 Marx, 1974b, p. 55; Marx, 1975, p. 50. See also Struik, 1948b, p. 191. Marx's method treated the differential quotient as a function of *two* variables x_0 and x_1 and defined the definitive derivative as the extension of this function at the critical point $x_1 = x_0$.

106 Frequently, they write *h* instead of Δx.

107 Marx, 1974b, p. 114; Marx, 1975, p. 131.

$$(x_1)^3 = (x_0 + \Delta x)^3 = (x_0)^3 + 3(x_0)^2 .\Delta x + 3\, x_0.(\Delta x)^2 + (\Delta x)^3 ,$$

where the definitive derivative $3(x_0)^2$ as we saw earlier has already appeared as the coefficient of Δx, that is, *before differentiation*, before the calculation of $y_1 - y_0$, $x_1 - x_0$, and of $\frac{\Delta y}{\Delta x}$, and the substitution, $\Delta x = 0$.

In the method of Marx, the definitive derivative $3(x_0)^2$ appears for the first time only *at the end* when x_1 has returned to x_0. Then, when $x_1 = x_0$ anew, the provisional derivative $(x_1)^2 + x_1.x_0 + (x_0)^2$ is transformed to $3(x_0)^2$. Here, the definitive derivative is the end result of the *process differentiation*.[108]

Consider another example.

Let $y = x^2 + 4x$. Let us compare the method of Marx with an earlier method, that of D'Alembert and Euler:[109]

D'Alembert's and Euler's Method

$$y_1 = f(x_0 + \Delta x) = (x_0 + \Delta x)^2 + 4(x_0 + \Delta x) =$$
$$= (x_0)^2 + 2x_0.\Delta x + (\Delta x)^2 + 4x_0 + 4\Delta x =$$
$$= (x_0)^2 + 4x_0 + (2x_0 + 4) . \Delta x + (\Delta x)^2.$$
$$\Delta y = y_1 - y_0 = \{(x_0)^2 + 4x_0 + (2x_0 + 4) . \Delta x + (\Delta x)^2\} - \{(x_0)^2 + 4x_0\}$$
$$= (2x_0 + 4) . \Delta x + (\Delta x)^2.$$
$$\frac{\Delta y}{\Delta x} = (2x_0 + 4) + \Delta x.$$

Set $\Delta x = 0$; then $\frac{0}{0} = 2x_0 + 4$, or $\frac{dy}{dx} = 2x_0 + 4$.

108 Or "differential process;" Marx, 1974b, pp. 52, 53, 56, etc.; Marx, 1975, p. 48 etc.

109 The others are essentially equal to this.

$y_1 = (x_1)^2 + 4x_1$ and $y_0 = (x_0)^2 + 4x_0$.

$\Delta y = y_1 - y_0 = \{(x_1)^2 + 4x_1\} - \{(x_0)^2 + 4x_0\} =$

$= \{(x_1)^2 - (x_0)^2\} + 4(x_1 - x_0) =$

$= (x_1 + x_0)(x_1 - x_0) + 4(x_1 - x_0) =$

$= (x_1 - x_0)(x_1 + x_0 + 4).$

$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{(x_1 - x_0)(x_1 + x_0 + 4)}{x_1 - x_0} = x_1 + x_0 + 4.$$

When $x_1 = x_0$, we obtain $\frac{0}{0} = x_0 + x_0 + 4 = 2x_0 + 4$, or

$$\frac{dy}{dx} = 2x_0 + 4.$$

In the first method, the derivative $2x_0 + 4$ appears immediately, from the outset, "the fetus before it was fertilized":

$$y_1 = (x_0)^2 + 4x_0 + (\mathbf{2x_0 + 4}) \cdot \Delta x + (\Delta x)^2.$$

However, in the second method [Marx's method – B.L.] it is "*really differentiated*." [110]

In the first method, there is only a *separating* out of the future definitive derivative from the other terms. In the second method, there takes place a true birth, verifying a true *development*.

In Marx's opinion, the line of thought of D'Alembert and the others did not correctly represent what happens when a function is differentiated, that is, when the derivative is determined. [111] It was

110 Marx, 1974b, p. 151; Marx, 1975, p. 181.

111 Cf. Struik, 1948b, p. 192; Struik, 1975, p. 152.

this essential shortcoming that Marx intended to overcome with his method.

5.2 *Negation of the negation*

Marx explained differentiation as a dialectical process, referring in particular to the negation of the negation.

In varying x, letting x increase (or decrease), from x_0 to x_1, whichever, x becomes different from x_0. In other words, we 'negate' x_0. This is the *first negation*.

In the concrete example $y = 5x^2$ we have:

$$\Delta y = y_1 - y_0 = 5(x_1)^2 - 5(x_0)^2 = 5\{(x_1)^2 - (x_0)^2\} =$$
$$= 5(x_1 + x_0)(x_1 - x_0).$$

And $\dfrac{\Delta y}{\Delta x} = \dfrac{5(x_1 + x_0)(x_1 - x_0)}{x_1 - x_0} = 5(x_1 + x_0)$

We have come to the provisional derivative $5(x_1 + x_0)$.

Now let x_1 return to x_0, that is, negate the fact of x_1 being different from x_0. Thus we obtain the definitive derivative:

$$f^1(x_0) = 5(x_0+x_0) = 5(2x_0) = 10x_0.$$

This phase of development represents the *second negation*, the negation of the first negation. [112]

When we negate a first negation, that is, when we put $x_1 = x_0$, we did not return to the starting point. We did not return to the original function $f(x) = 5x^2$, but arrived at a new function, derived function, $f^1(x) = 10x$. In it x_1 has not disappeared. As Marx pointed out:

> "The quantity, x_1, originally introduced by varying x, does not disappear; it is only reduced to its limit value", that is, to x_0. The quantity x_1 "... remains an element introduced in the original function that

[112] Cf. Kennedy, 1977, p. 311.

> furnished the definitive derivative by means of combinations, in part with itself, in part with x_0 ..." [113]

The first negation was not negated in just any manner.

Had the first negation been negated immediately, without taking into account the quotient of Δy and Δx, what result would have been obtained?

Let us consider the case of $y = 5x^2$.

We have $y_1 - y_0 = 5\{(x_1)^2 - (x_0)^2\}$.

Now when x_1 goes back to x_0, what happens? Put $x_1 = x_0$. On one side we get $x_1 - x_0 = 0$ and on the other side $y_1 = 5(x_1)^2 = 5(x_0)^2 = y_0$, or, $y_1 - y_0 = 0$.

Thus, $y_1 - y_0 = 5\{(x_1)^2 - (x_0)^2\}$ is transformed to $0 = 0$.

Negating the first negation in this manner does not lead to any result. Referring to this last type of simple procedure, Marx emphasized:

> "The entire difficulty in understanding the differential operation (as in that of any negation of the negation whatever) lies precisely in seeing how it differs from such a simple procedure and therefore leads to true results." [114]

---

113 Marx, 1974b, pp. 54-55; Marx, 1975, p. 49. Also, cf. Engels's letter to Marx, August 18, 1881: "After the function has passed through the process from x_0 to x_1 with all its consequences, x_1 can be quietly allowed to become x_0 again. It is no longer the old x_0, which was only variable in name; it has passed through *real change* and the result of the change remains, even if we liquidate it again … ." (MEW, 1961, Vol. 35, pp. 24-25). Cf. Struik, 1948b, p. 192.

114 Marx, 1974b, p. 51, Marx, 1975, p. 46. Cf. Kennedy, 1977, p. 309.

Later, in discussing the *Mathematical Manuscripts* of Karl Marx as a source of inspiration for new research, I will return to the question of how to negate, of how to negate the first negation.

5.3 ***Symbolic equivalent*** $\frac{dy}{dx}$

In our example, $y = x^3$, or $f(x) = x^3$, we obtained as an end result:

$$\frac{dy}{dx} = 3x^2.$$

At the right, we have the definitive derivative $3x^2$.

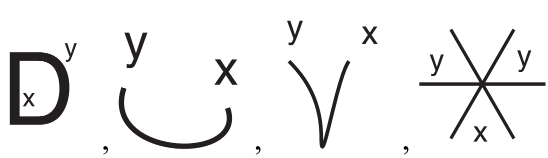
On the left of this definitive derivative appears

> "... its double, $\frac{0}{0}$, or $\frac{dy}{dx}$ as *symbolic equivalent*." [115]

On the other side, $\frac{0}{0}$ or $\frac{dy}{dx}$ has in $3x^2$ its

> "*real equivalent*." [116]

In place of $\frac{dy}{dx}$, we could have chosen another symbolic equivalent, such as y' and y'_x, which we find in differential calculus books, [117] or even



, etc.,

if we wish to invent new symbols.

The first reason to choose $\frac{dy}{dx}$ as our symbol is *historic*, continuing the tradition begun by Leibniz. This does not imply that

---

[115] Marx, 1974b, p. 61; Marx, 1975, p. 58.

[116] Marx, 1974b, p. 61; Marx, 1975, p. 58.

[117] For example, Piskounov, 1979, p. 74.

the interpretation of $\frac{dy}{dx}$ remains the same. Marx rejected equally Leibniz's concept of $\frac{dy}{dx}$ as the quotient od two "infinitely small quantities" and Euler's idea, to consider $\frac{dy}{dx}$ in the sense of a "geometric ratio of two zeros." Nothing mysterious, $\frac{dy}{dx}$ is only a symbolic notation for the definitive derivative.

Tradition could never constitute a sufficiently strong reason to continue the symbolic notation $\frac{dy}{dx}$. The fundamental reason is *practical.*[118] In practice, it was verified that the choice of the symbol $\frac{dy}{dx}$ in the form of a quotient greatly facilitates calculations. For example, in determining the derivative of a composite function y = f(u), where u = g(x), we use:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \text{ ,}$$ [119]

a rule which is analogous to the multiplication for fractions:

$$\frac{a}{b} = \frac{a}{c} \cdot \frac{c}{b}$$

and therefore easy to remember.

118 Cf. D'Ambrosio, 1975, p. 34: "This notation attributed to Leibniz, although much used and certainly practical, has the serious inconvenience of 'tempting' the beginning student of calculus to 'simplify' things and treat $\frac{dy}{dx}$ as *dy* divided by *dx*, which does not make sense!"

119 Marx, 1974b, p. 95; Marx, 1975, p. 105.

5.4 *Inversion of the method*

Marx was very interested in knowing just where the *specific* nature of differential calculus showed itself? In what respect did differential calculus differ from "algebra"? Where and how did the transition from algebra to differential calculus take place?

In other words, where did the *qualitative leap* from the mathematics of constant quantities to the mathematics of varying quantities take place?

We saw that Marx obtained $\frac{dy}{dx}$ = definitive derivative, or briefly, $\frac{dy}{dx} = f^1(x)$, as the final result of an algebraic process, that of differentiation. The differential symbol $\frac{dy}{dx}$ emerged as the symbolic equivalent of the definitive derivative.

Now, once $\frac{dy}{dx}$ remained the *well-grounded* result of a real change, it became subject to new calculations, those of differential calculus. [120] One of the first formulas of this new calculus that Marx deduced and analyzed is the formula for the calculation of the derivative of a function *y* that can be written as y = uz, as the product of two functions *u* and *z* in *x*:

$$\frac{dy}{dx} = z.\frac{du}{dx} + u.\frac{dz}{dx}$$

Let us look at a concrete example. Let $u = 2x^2+3x$ and $z = x^3 - x$. What is the derivative of the function y = uz, or

$$y = (2x^2+3x)(x^3-x)\ ?$$

Following the indicated formula, it is necessary to determine $\frac{du}{dx}$ and $\frac{dz}{dx}$ in order to find $\frac{dy}{dx}$ afterwards. This is a new situation. The

120 Cf. Struik, 1948b, p. 194; Struik, 1975, p. 154

"*symbolic values*" $\frac{du}{dx}$ and $\frac{dz}{dx}$ are given, and the task is to find their "*real values*." [121]

In our example we get:

$$\frac{du}{dx} = 4x+3, \text{ and } \quad \frac{dz}{dx} = 3x^2 - 1.$$

Substituting in

$$\frac{dy}{dx} = z.\frac{du}{dx} + u.\frac{dz}{dx},$$

we get $\frac{dy}{dx} = z(4x+3) + u(3x^2 - 1)$.

Noting that $u = 2x^2+3x$ and $z = x^3 - x$, we see that

$$\frac{dy}{dx} = (x^3-x)(4x+3) + (2x^2+3x)(3x^2 - 1) =$$

$$= 4x^4+3x^3-4x^2-3x+6x^4-2x^2+9x^3-3x =$$

$$= 10x^4 +12x^3-6x^2-6x.$$

Here $\frac{du}{dx}$ and $\frac{dz}{dx}$ serve as a starting point. They indicate the *operations* to be carried out with the functions *u* and *z*. Thus the symbols $\frac{du}{dx}$, $\frac{dz}{dx}$, etc., initially the result of the process of differentiation,

> "... become *inversely* ("umgekehrt") symbols of a process yet to be performed on the variables, thus, *operational symbols* ("Operationssymbolen"), which appear as points of departure rather than results, and this is their essential use ("Dienst") in differential calculus." [122]

To facilitate calculations, we can change the forms:

---

121 Marx, 1974b, p. 76; Marx, 1975, p. 80.

122 Marx, 1974b, p. 84; Marx, 1975, p. 91.

$$dy = f^1(x).dx \text{ in place of } \frac{dy}{dx} = f^1(x)$$

$$\text{and } d(uz) = z.du + u.dz \text{ in place of } \frac{d(uz)}{dx} = z.\frac{du}{dx} + u.\frac{dz}{dx},$$

Marx pointed out that these changes are only of form, not content: in $dy = f^1(x).dx$, the differentials *dy* and *dx* appear separated, but are indeed as inseparably connected as numerator and the denominator in $\frac{dy}{dx} = \frac{0}{0}$. [123]

Already we are in the terrain of the new calculus:

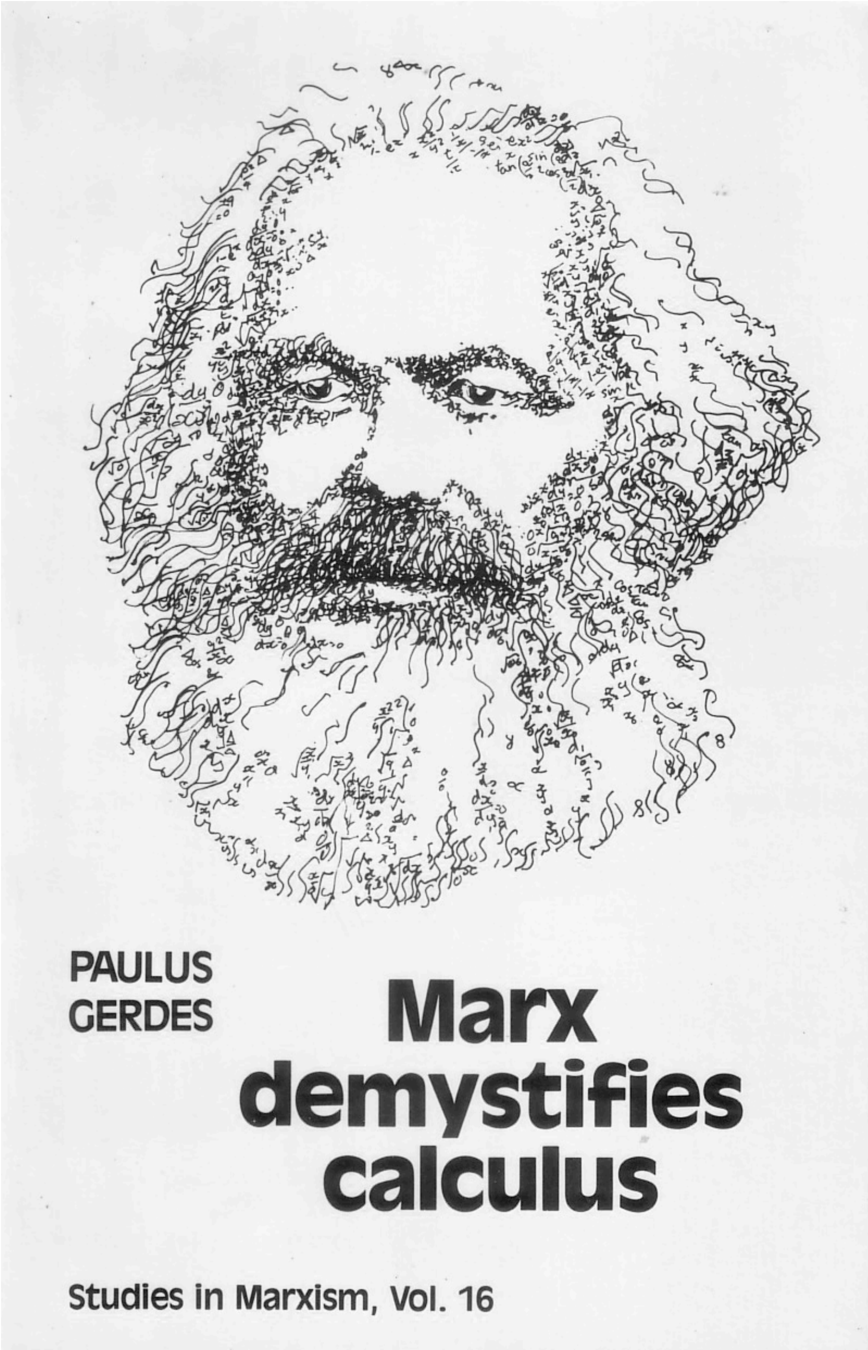
> "... to start out from the moment, in which the differential [*dx*, *dy*, *du*, etc.] functions as the point of departure of calculus, shows that the inversion of the algebraic method of differentiation is completed, and that differential calculus itself appears as a totally distinct and specific method, with values that vary." 124

Thus, Marx succeeded in indicating the exact moment of the *qualitative leap* from algebra (elementary mathematics) to differential calculus, or from mathematics of constant quantities to the mathematics of varying quantities.

---

123 Marx, 1974b, p. 98; Marx, 1975, p. 109.

124 Marx, 1974b, p. 69; Marx, 1975, p. 68.



Cover of the 1985 edition

Chapter 6

THE SIGNIFICANCE OF THE MATHEMATICAL MANUSCRIPTS

At the end of chapter 3, we had already referred to the influence of the *Mathematical Manuscripts* on the development of the history of mathematics as a scientific discipline. Now, what is the place of Marx in the history of mathematics itself?

6.1 ***Karl Marx and the development of mathematics***

At the end of the paragraph about the historical, mathematical investigations of Marx, we saw that Lagrange, conscious of the inadequacy of his foundations of differential calculus, launched a contest, in 1784, to obtain clarification of the basic concepts of infinitesimal calculus.

Marx accepted the challenge and ‘won.’ He succeeded in giving a dialectical foundation for the differentiation of a whole class of functions. Exiled in England, and out contact with professional mathematicians, it was difficult for him, if not impossible, to know of the work done by mathematicians on the European continent who had also accepted Lagrange’s challenge. [125]

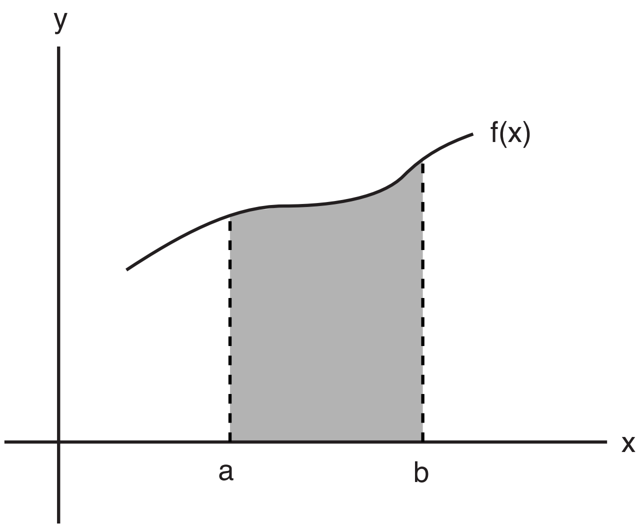
I will briefly summarize the main line of this research to enable us to evaluate better the role of Marx’s inquiries.

125 Struik, 1948b, p. 187; Struik, 1975, p. 144; Yanovskaya, 1969, p. 25.

6.1.1 *Refinements of the concepts of calculus in the nineteenth century*

The French Revolution (1789) and the Napoleonic period created extremely favorable conditions for the development of mathematics, particularly in France, where there was the greatest ideological break with the passed era. [126]

A whole series of new technical and scientific problems arose from the Industrial Revolution, such as the problem of construction of machine parts, transmission of force, friction, precision mechanics, and energy. This brought about a closer linkage between physicists and a number of mathematicians with material production. [127] At the same time, the concentration of workers in growing industrial cities gave rise to problems of supply of food, water, home heating materials, etc., and problems street lighting, construction of buildings, etc. The resolution of these and other problems – to service the process of capitalist production – obliged the natural sciences and mathematics to develop in a corresponding direction.



Newton-Leibniz Theorem: Barred area = F(b) – F(a),
where F(x) is a function for which f(x) is the derived function.

---

126 Struik, 1948a, p. 139.

127 See Wussing, 1979, p. 226.

6.1.1.1 ***Integrals, infinitely small quantities, and the limit***

Mathematicians of the seventeenth and eighteenth centuries almost always succeeded in calculating the integrals they came across, using the fundamental theorem of Newton-Leibniz. Integration was only the inverse process of differentiation. [128]

At the end of the eighteenth century, the situation began to change. The impetuous increase of steam-powered machinery opened new areas for application integral calculus. It was necessary to elaborate more general and more rigorous methods. The problem of the attraction of a material point by a body of given mass led to the concept of the *triple integral*. The problem of the spread of electric charges on the surface of a conductor led to the concept of *surface integral*. These new integrals could not be calculated in the same manner as those met earlier. Already in these examples integration was not the simple inversion of differentiation.

Thus, it became necessary to develop foundations for the concept of the integral [definite integral], independently of the concepts of the derivative and the differential. [129] The Frenchman Augustin Cauchy (1789-1857) was the first to succeed in defining the concept of the integral independently from differentiation. [130] The German Bernhard Riemann (1826-1866) and the Frenchman Henri Lebesgue (1875-1938) generalized the same concept. Cauchy had perceived that for such a definition it was necessary to consider an "infinitely small quantity," not – as the earlier mathematicians had done – as a fixed, but small number, or as zero – in the case of Euler – but yes, as a *variable*. In Cauchy's words:

> "On says that a variable quantity becomes infinitely small when its numerical value decreases indefinitely in such a way as to converge towards the limit zero." [131]

---

128 Perhaps this was the reason that Marx almost never applied himself to problems of integral calculus.

129 Molodschi, 1977, p. 192.

130 See, for example, Boyer, 1960, p. 564.

131 Cited in Boyer, 1960, p. 563.

In this context, Cauchy, and independently of him, the progressive Czech priest, Bernhard Bolzano (1781-1848), succeeded in sharpening the concept of the *limit*. Wrote Cauchy,

> "When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it as little as one wishes, this last is called the limit of all the others." [132]

The German Karl Weierstrass (1815-1897) perfected this definition, introducing the so-called delta-epsilon terminology [133] that is used today in our university courses:

> A function y = f(x) tends to the limit b (briefly $y \to b$) when x tends to a (briefly, $x \to a$), if for each positive number ε, no matter how small, one can indicate a positive number δ such that for all x different from a, where the inequality $|x - a| < \delta$ holds, then the inequality $|f(x) - b| < \varepsilon$ is satisfied. [134]

If b is the limit of the function f(x) when x tends to a, one writes:

$$\lim_{x \to a} f(x) = b.$$

In this way we have also gained a new foundation for the concept of the derivative: [135]

$$f^1(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

or $\dfrac{dy}{dx} = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$.

132 Cited in Boyer, 1960, p. 563.

133 Originally $\eta - \varepsilon$

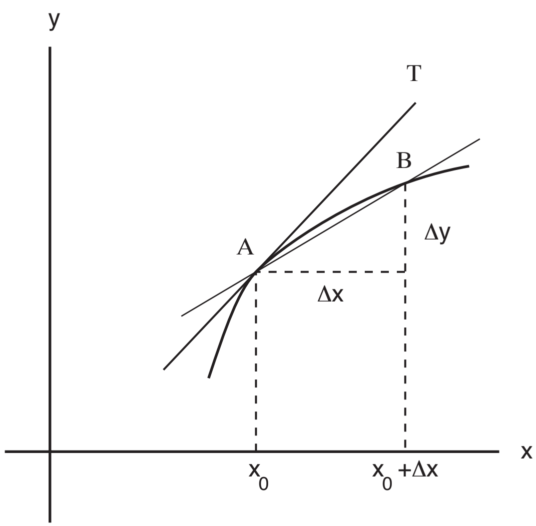
134 See, for example, Piskounov, 1979, Vol. 1, p. 37.

135 Also dialectics. See Yanovskaya, 1969, p. 29.

6.1.1.2 *Function, continuity, differentiability, real numbers*

The concepts of integral, of infinitely small quantities, and of the limit were not the only ones, which required greater precision and more rigorous definition in the nineteenth century.

For the mathematicians of the seventeenth and eighteenth centuries, motion without velocity and curves without tangents appeared unnatural. [136] Then, geometrically, it was obvious that



when Δx tends to zero, Δy also tends to zero; point B moves along the curve in the direction of A. And when Δx becomes equal to 0, secant AB coincides with the tangent AT.

They did not have a single doubt. Each function was continuous (without gaps or leaps), because it represented the motion of an object. Each function was differentiable (meaning at each point of its graph, a tangent line could be produced).

These ideas, apparently so natural, were surpassed in the nineteenth century.

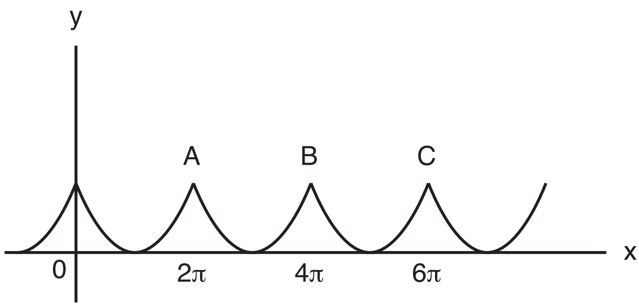
136 Molodschi, 1977, p. 183.

To solve the basic equations of his *Analytic Theory of Heat* (l822), the French mathematical physicist, Joseph Fourier (1768-1830), used series known as trigonometric series: [137]

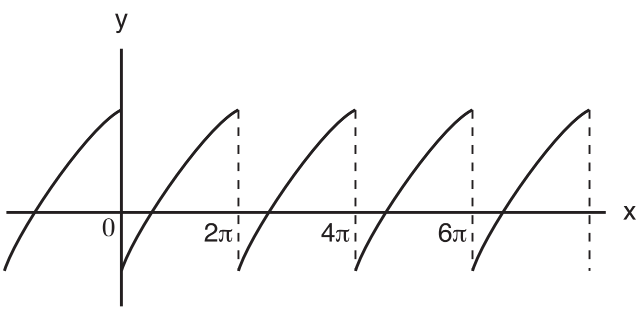
$$\frac{1}{2}a_0 + a_1\cos(x) + b_1\sin(x) + a_0\cos(2x) + b_2\sin(2x) + a_3\cos(3x) + \ldots$$

It was verified that not only the known continuous functions [138] could be represented by a trigonometric series, but also, for example, 'functions' corresponding to the following graphs:

At points A and B a unique tangent does not exist!



and



The function is discontinuous at $x=2\pi$, $x=4\pi$, etc.

The success of the Fourier series in physics brought mathematicians to acceptance of these new graphs as well, as graphs

137 See, for example, Molodschi, 1977, p. 196; Boyer, 1960, p. 598-99.

138 Periodic.

representing functions. These functions are not differentiable at some points (1st graph), or not continuous at some points (2nd graph).

In 1837, the Frenchman Lejeune Dirichlet (l805-1859) formulated a definition, which broadened the concept of function:

> "if a variable *y* is so related to a variable *x* of that whenever a numerical value is assigned to *x*, there is a rule according to which a unique value of *y* is determined, then *y* is said to be a function of the independent variable in *x*." [139]

Thus, the concept of function was liberated from the geometric and mechanical ideas of the seventeenth and eighteenth centuries, becoming an instrument more and more applicable to more branches of science. Independently of Dirichlet, the Russian Nikolai Lobatchevsky (1793-1856) reached a definition of the concept of function almost equal to that of Dirichlet. These definitions constituted only the first step in the generalization and extension of concept of function.

In extension, the German mathematicians Hermann Hankel (1839-1873), Richard Dedekind (1831-1916), [140] and Georg Cantor (1845-1918); the English logicians, Augustus de Morgan (1816-1871), [141] and Charles Peirce (1839-1914); and the Italian, Giuseppe Peano (1858-1932), made fundamental contributions. [142]

In 1834, Bolzano discovered a continuous function that was not differentiable at any point. At none of the points of its graph does there exist a tangent line, incredible as this appears! Here it is interesting to note that Marx, without knowing about this work on the European continent, also reached the conclusion that it was necessary to separate differential calculus from geometric representation. [143]

---

139 Cited in Boyer, 1960, p. 600.

140 The concept of mapping.

141 The concept of relations.

142 See, for example, Wussing, 1979, pp. 229-30.

143 See Engels's letter dated November 22, 1882, in MEW, 1961, Vol. 35, p. 114.

This necessary separation obliged the mathematicians to think more clearly about the quantities they used in their theories, in particular, about different number systems. In the 1870s, in the same period in which Karl Marx elaborated the principal part of his mathematical manuscripts, the Frenchman Charles Méray (1835-1911) and the already mentioned Germans Weierstrass, Dedekind, and Cantor achieved great successes in the foundations of the theory of irrational and real numbers. In this context, Cantor developed the important theory of infinite sets.

6.1.2 *Original contributions*

I have already referred to the fact that the French Revolution and the Napoleonic period created extremely favorable conditions for the development of mathematics. Particularly, in France, and a little later in Germany, there took place major economic and political changes in the transition to a new capitalist structure. In England, although it was the heart of the Industrial Revolution, mathematics remained sterile for some decades. [144] Even in 1917, the well-known English specialist in number theory, Godfrey Hardy (1877-1947), wrote that he felt like a “missionary” on introducing the methods of Weierstrass, Dedekind, and Cantor in England. [145]

6.1.2.1 *Rediscoveries*

In this respect, the British island, with its Newtonian tradition deeply rooted in national chauvinism, was retarded in relation to the European continent. So it is understandable that it was practically impossible for Marx to have known modern tendencies in the differential and integral calculus. [146] And so it could happen that

144 Struik, 1948a, p. 139.

145 Yanovskaya, 1969, p. 25.

146 Cf. Rieske & Schenk, 1972, p. 476; Struik, 1948b, p. 187; Struik, 1975, p. 144.

some independent discoveries of Karl Marx constituted, in reality, rediscoveries.

Let us look at some examples.

(a) Like Fourier, Bolzano, etc., Marx perceived the necessity of separating calculus from its geometric representation.

(b) Marx discovered a dialectic method to lay the foundation for the differential calculus, as we saw in chapter 5. As to the purely mathematical aspect, the self-taught Englishman, John Landen, had developed a method similar to that of Marx in 1764. Marx wanted to study the work of Landen, [147] but could not find Landen's book in the libraries. Therefore, the discoveries were independent.

Marx's foundation for the derivative is valid for a whole class of functions. [148] The method cannot be used for all possible functions f(x). We now know that such general methods cannot exist because the method presupposes the possibility of really carrying out the division of $f(x_1) - f(x_0)$ by $x_1 - x_0$, which is often not possible.

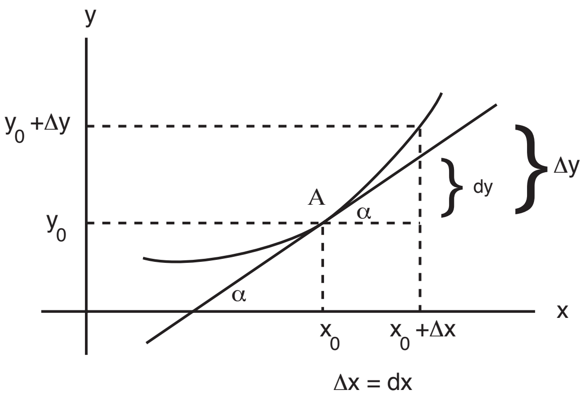
(c) Cauchy introduced the concept of "infinitely small" as a variable. With this concept, the differentials *dy* and *dx* become variables, with *dy* a variable that depends on the independent variable, *dx*. This idea is also expressed by Marx in his manuscript *On the differential*, [149] in which he interprets the differential *dy* as the principal (linear) part of the increment Δy, according to the Academician Andrei Kolmogorov (1903-1987).

Let us look at a diagram to get an initial, intuitive idea of what is meant by *dy* as principal part, or linear of the increment Δy:

[147] Marx, 1974b, p. 151; Marx, 1975, p. 181.

[148] See, for example, Yanovskaya, 1969, p. 26.

[149] Marx, 1983, pp. 14-33; Marx, 1974b, pp. 60-74; Marx, 1975, pp. 57-74, and, in particular, in one of his letters to Engels, MEW, 1961, Vol. 31, p. 165.



Tangent line at point A

$$\tan \alpha = \frac{dy}{\Delta x} = \frac{dy}{dx}$$

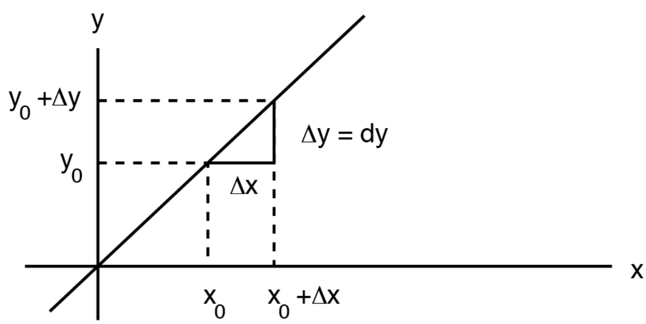
150

This interpretation of the differentials *dy* and *dx*, according to Kolmogorov, corresponds

> "completely to that stated in our modern textbooks and was absent from the texts studied by Marx." [151]

---

150 Consider the function y = x; the tangent coincides with the graph of the function itself.



Then in this case, dy=Δy. But y = x. Thus, it follows that dx = Δx. In general, we have $\tan \alpha = \dfrac{dy}{\Delta x}$.

Now with dx = Δx, it follows that $\tan \alpha = \dfrac{dy}{dx}$.

See, for example, Piskounov, 1979, pp. 118-123.

151 Cited in Kennedy, 1976, p. 492.

6.1.2.2 ***Discoveries***

In Karl Marx's method, $\frac{dy}{dx}$ appears as the expression of the process of differentiation; $\frac{dy}{dx}$ is the symbolic equivalent of the definitive derivative. Once it appears, $\frac{dy}{dx}$ can figure as a *operational symbol*, that is, to indicate what operation is to be performed with the function y = f(x). Kolmogorov, one of the greatest mathematicians of the twentieth century, founder of the axiomatics of probability theory, observed in 1954:

> "In an especially detailed manner, Marx analyzed the question of the content of the concept of the differential. He proposed the concept of the differential as a "operational symbol," anticipating an idea that came forward again only in the 20th century." [152]

Here, Marx surpassed the mathematicians of the nineteenth century. It was in 1927 that the Frenchman Jacques Hadamard (1865-1963), obviously unaware of the work of Marx, showed the operator role of the differential. [153]

His fellow countryman, Maurice Fréchet (1878-1956), extended this concept of the "differential operator," or "differentiation operator," [154] to "functional analysis." Functional analysis constitutes one of the principal branches of twentieth century mathematics. [155] K. A. Rybnikov pointed out, in the *Great Soviet Encyclopedia*, that:

> "The concept of the differential as an operational symbol, first discovered by Karl Marx, acquires ... a particular significance in the contemporary

---

152 Cited in Kennedy, 1976, p. 492.

153 Struik, 1948b, p. 194; Struik, 1975, p. 155.

154 For its properties, see, for example, D'Ambrosio, 1975, pp. 48, 51, etc.

155 For an introduction, see I. M. Gel'fand, 1977.

> generalizations of the concept of the differential in functional analysis." [156]

With the analysis of operational symbols by Marx, we are already entering the terrain of philosophy.

6.2 ***Philosophic problems of mathematics***

The German philosophers, Günter Rieske and Günter Schenk, point out that the *Mathematical Manuscripts* of Karl Marx contain:

> "… many observations on questions of the philosophy of mathematics that are still being discussed today." [157]

And

> "… a series of correct observations, solutions and embryonic solutions, methodological and philosophic analyses that have a lasting value for similar research in these days." [158]

6.2.1 ***Only one science***

Marx took a stand against the idea of "the mathematics" [as plural – B.L.]. Newton, Leibniz, and others had considered differential calculus with its special quantities as *separate* from algebra. For Marx, mathematics is a single science. [159] The autonomy of its branches is relative. The autonomy of differential calculus is displayed in its specific terminology and symbolization.

6.2.2 ***Symbolization and terminology***

Differential calculus is distinguished by its specific symbols and terminology, concepts such as "differential," "infinitely small quantities," symbols such as

---

[156] Cited in Kennedy, 1976, p. 492; see also Kennedy, 1977, p. 314.

[157] Rieske & Schenk, 1972, p. 475.

[158] Rieske & Schenk, 1972, p. 479.

[159] Cf. Rieske & Schenk, 1972, p. 481.

$$dx,\ dy,\ d^2y,\ d^ny,\ \frac{d^2y}{dx^2},\ \frac{dy}{dx},\ \frac{\partial z}{\partial x},\ \text{etc.}$$

In the past (Leibniz, Euler, etc.), the differentials were considered as quantities of a very particular nature. Presently, this is not so. The old symbols and terminology are maintained, although the corresponding concepts never have had any meaning. How could this have happened?

The best answer even today, according to the philosopher Yanovskaya, is found in the *Mathematical Manuscripts* of Karl Marx. [160] Marx understood that the essence of the problem is shown in the *operational role of the symbols* of differential calculus. He clearly showed why it is necessary to introduce new symbols:

* to avoid the continued repetition of the same process of calculation;
* to reduce a complicated problem to simpler problems – for example, by means of the formula:

$$\frac{d(uz)}{dx} = z\frac{du}{dx} + u\frac{dz}{dx}$$

as we saw.

6.2.3 *Algorithms*

The method of differentiation discovered by Marx indicates, in his own words, the "*actual process*" of determining the derived functions. In place of "actual process," we now use the term "*algorithm*." Thus in the opinion of Marx differential and integral calculus must be constructed on the base of the theory of algorithms. [161] This idea implicitly conveys a critique of the limit concept of Cauchy-Weierstrass, as not algorithmic. The definition of Cauchy-Weierstrass furnishes only a pragmatic criterion [162] to *verify* whether or not a given value is really the limit (of a function, of a sequence,

160 Yanovskaya, 1969, p. 26.

161 Cf. Yanovskaya, 1969, p. 27.

162 Struik, 1948a, p. 191; Struik, 1975, p. 153.

etc.). It does not supply any method for *determining* the referred value.

Here, in speaking of the significance of the algorithms for mathematics, we already touch on one of the great debates of the twentieth century between the diverse currents in the philosophy of mathematics (logicism, formalism, intuitionism, constructivism, etc.).[163] The development of the theory of algorithms, together with the theory of recursive functions, is fundamental for advances in the use of computers.

6.2.4 ***Reflection of the real world in mathematics***

In his method of foundations of differential calculus, Marx shows that in the transition of the provisional derivative

$$\frac{y_1 - y_0}{x_1 - x_0} = f^1(x_0)$$

to the definitive derivative

$$\frac{dy}{dx} = f^1(x_0)$$

we really have to substitute $x_1 = x_0$, and therefore:

> "$x_1 - x_0 = 0$ in the rigorous mathematical sense, without stories of a mere infinite approximation."[164]

In other words, for Marx x_1 does not tend to x_0, *Δx does not tend* to zero without knowing whether or not it reaches 0, but Δx becomes *equal* to 0. In this manner, Karl Marx takes a position contrary to those mathematicians and philosophers who interpret Cauchy's definition in terms of "Δx tends to 0," leaving open the question as to whether or not Δx becomes equal to 0.

---

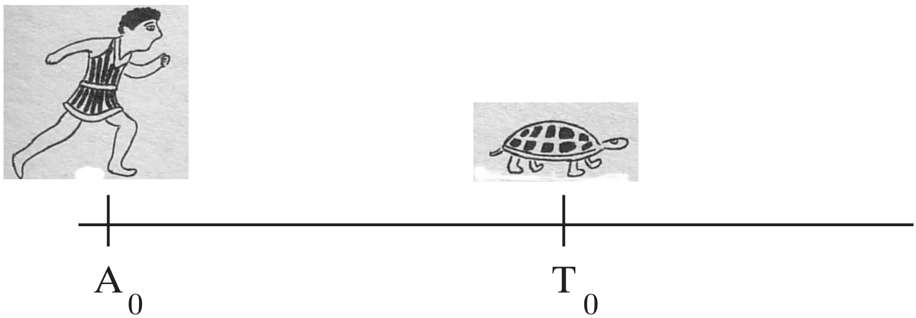
163 See, for example, Ruzavin, 1977, p. 139 and chapters 7, 8 and 9 in the book by Molodschi, 1977.

164 Marx, 1974b, p. 54; Marx, 1975, p. 49.

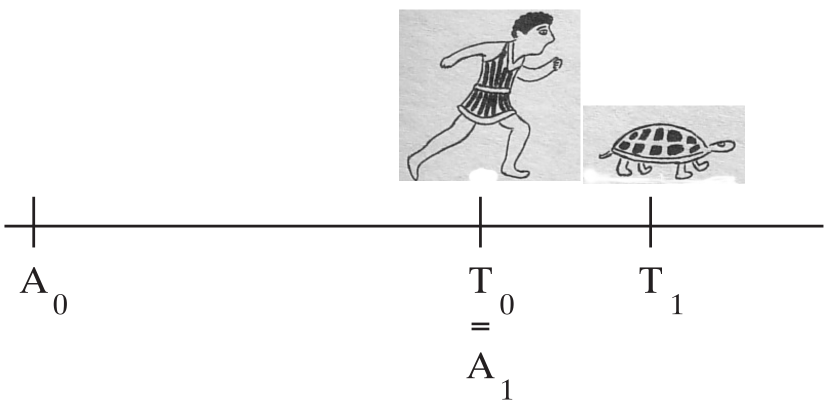
6.2.4.1 ***Achilles and the tortoise***

Here we are faced with one of the most important questions of the dialectical nature of motion, the focus of the debate since the Greek Zeno of Elea (490-430 B.C.) formulated his famous logical paradoxes, for example, that of Achilles and the tortoise.

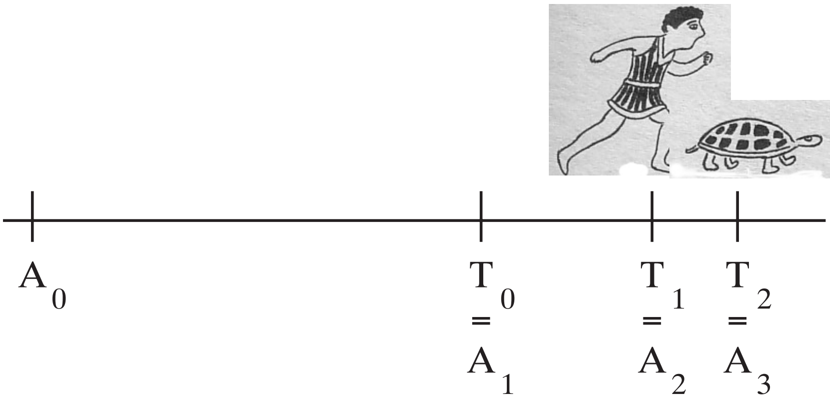
The hero Achilles, a very fast runner, bets on a race with a tortoise who is given a head start.



When Achilles, leaving from point A_0 arrives at the initial position of the tortoise T_0, the tortoise would already have advanced a bit, say, to point T_1.



When Achilles covers this distance, from A_1 to T_1 ($A_1 = T_0$, the initial position of the tortoise), the tortoise will have advanced a little more, say, to point T_2.



And so it continues, with the result that the speedy Achilles never could overtake the slow tortoise, according to Zeno.

The distance between Achilles and the tortoise, initially $|A_0T_0|$, later $|A_1T_1|$, $|A_2T_2|$, $|A_3T_3|$, $|A_4T_4|$, etc., becomes smaller each time, tending to zero. But does the distance ever become equal to zero?

The mathematicians and philosophers who base themselves on the Cauchy-Weierstrass definition of limit leave open the question whether or not Δx becomes equal to zero. For them it appears to be only a question of our *will* whether or not Achilles catches up to the tortoise. It is this hidden voluntarism that Marx implicitly criticizes.

In reality, Achilles is capable of overtaking the tortoise; the distance between the two will at one time be *equal* to zero. Therefore the limit will be reached. Or, in the case of the process of differentiation, the occurrence of $\Delta x = 0$ takes place objectively, and the limit $\frac{dy}{dx}$ is reached.

We can verify that Marx demanded maximum clarity of thought in interpreting the formal apparatus of the symbols (in this example, Δx only tends to 0, or becomes equal to zero), pointing out as a *materialist* that mathematics can be significant and relevant only when it reflects processes of the real world. [165]

Thus x_1 really has to become equal to x_0, meaning that $x_1 - x_0$, or Δx, is equal to 0, for this corresponds to motion in the real world.

With the paradox "Achilles and the tortoise" and others, Zeno tried to show that motion cannot be explained under the hypothesis of the infinite divisibility of space and time. This hypothesis, so important and necessary in the construction of mathematics (for example, in the construction of the set of real numbers), is an abstraction of such order that it cannot be justified by empirical experience. The philosopher Ruzavin points out:

> "The idea of infinite divisibility of objects and figures is an abstraction. It simplifies and schematizes real processes and thus contradicts experience." [166]

To resolve the paradox, because in reality Achilles is capable of overtaking the tortoise, we must find a method, also abstract, to calculate the distance traveled by Achilles to reach the tortoise, that is, to calculate the infinite sum of

$$|A_0A_1| + |A_1A_2| + |A_2A_3| \ldots$$

in order to know when Achilles reaches the tortoise (What is the finite distance to be covered by Achilles?)

In other words:

> "We have to eliminate an abstraction [the infinite divisibility – P.G.], with the aid of another [infinite sum – P.G.]," [167]

that is, with the aid of the abstraction of the sum of an infinite series to be able to reflect the real world in mathematics.

---

165 Cf. Struik, 1948b, p. 193; Struik, 1975, p. 154. Marx's thinking implies both a critique of "formalism" and of "empiricism" in the interpretation of mathematics. See, for example, Rieske & Schenk, 1972, p. 481.

166 Ruzavin, 1977, p. 131.

167 Ruzavin, 1977, p. 131. This is an excellent example in mathematics of the negation of the negation.

On this question, Lenin writes in his *Philosophical Notebooks*,

> "The question is not whether there is movement but how to express it in the logic of concepts." [168]

With his method of differentiation, Karl Marx succeeded in expressing fundamental aspects of motion in the real world. Similarly Lenin analyzed it later, reflecting on the arguments of Zeno:

> "Movement means to be in a given place and simultaneously not be in it. It is the unity of discontinuity and continuity [169] of space and time that makes motion possible." [170]

Marx understood this "to be in a given place and simultaneously not be in it," in his method of differentiation: x_1 moves away from x_0 ("not to be in a given place"), and x_1 goes back to x_0 ("to be in it").

The dialectical character of Marx's method, for example, the law of the negation of the negation, reflects in thought, the objective dialectics of motion in the real world. Yanovskaya says that the study of the *Mathematical Manuscripts* of Karl Marx is fundamental, to an understanding of the concept of a variable, [171] and how motion is introduced into mathematics.

Now we can better understand why the great theoretician and leader of the working class, Karl Marx, was so interested in the foundations of calculus, perceiving that it dealt with:

> "… the most profound kernel of the dialectical process, with the essence of change." [172]

How can we change the world, consciously transform the world, construct a new world free of exploitation of people by people, without understanding the essence of change?

---

168 Cited by Ruzavin, 1977, p. 132. Cf. Lenin, 1960, Vol. 38, p. 256.

169 This is the Law of the Unity of Opposites; see Lenin, 1960, Vol. 38, p. 258; also see Hegel, 1955, Vol. 1, p. 270.

170 Lenin, 1960, Vol. 38, p. 259; Hegel, 1955, Vol. 1, pp. 273-4; Cf. Rosental & Iudin, 1977, Vol. 5, p. 208.

171 Yanovskaya, 1969 , pp. 27-28.

172 Struik, 1948b, p. 185; Struik, 1975, p. 144.

Marx's analysis is pertinent, alive, and timely. This is why the Soviet academician, Boris W. Gnedenko, advised the philosophers and mathematicians to connect the study of the most recent developments in mathematics with the study of *Dialectics of Nature*, by Engels; the *Philosophic Notebooks* of Lenin; and the *Mathematical Manuscripts* of Marx, in order to be able to:

> "find new paths for the solution of a whole series of principal questions which are met on the frontier between mathematics and philosophy." [173]

Such questions include: the relation between mathematics and material reality; the role of the axiomatic method in mathematics; rigor in the foundations of mathematics; the content and significance of symbolic mathematics; the problem of infiniteness (actual, potential, or a unity the two?); the question of mathematical truth; the struggle of opposites: discrete and continuous, concrete and abstract, finite and infinite.

6.3 ***Influence of mathematical thought on other works of Marx***

The preoccupation of Marx

> "with mathematics exerted a lasting and profound influence on all his work," [174]

concluded the philosophers Rieske and Schenk. This is to be expected when one takes into account the profundity, clarity, and originality with which Marx investigated the foundations of differential calculus. However, exactly what influence did mathematical thought have on the other works of Marx? Here we are up against the problem of time.

> "Unfortunately, the influence of the 'mathematical mode of thought' on the work of Marx almost has not yet been investigated." [175]

---

173 Cited in Paul, 1976, p. 849. For an analysis of these questions, see, for example, the books by Heitsch, Molodschi, Ruzavin, and Thiel and the articles by Alexandrov, Labérenne, and Paul.

174 Rieske & Schenk, 1972, p. 475.

175 Rieske & Schenk, 1972, p. 482.

This can be explained by the "youth" of the *Mathematical Manuscripts* of Marx, published as a whole for the first time only in 1968.

In this context, the only studies, which the above-mentioned German philosophers refer to are those of J. Zeleny, and of V. Vazjulin on the logic of *Capital*. [176] The only article of Jindrich Zeleny, which I have managed to find to date is *The Concept of Science in Dialectical Materialism*, in which he analyzes the mathematical thought in *Capital*. This article shows that the author did not yet know the *Mathematical Manuscripts* of Marx. On the other hand, it appears that Rieske and Schenk did not know of the study, *On the So-Called Definition through Abstraction*, published in 1935 by Sofia Yanovskaya, the scholar who has already been mentioned as the principal editor of the *Mathematical Manuscripts* of Marx. This state of affairs shows, once more, that we are just at the beginning of research about the influence of "the mathematical mode of thought" on the other works of Marx.

However, Rieske and Schenk, Yanovskaya and Wolfgang Segeth have already arrived at a first, common conclusion: the concept of "*value*" in *Capital*, as well as the concept of the "*differential*" in the *Mathematical Manuscripts*, are developed by means of the so-called "definition by abstraction," based on an "equivalence relation," [177] such as the definition of the natural number concept by Cantor. [178]

---

176 Rieske & Schenk, 1972, p. 483 [See also Zeleny, 1980, pp. 100-02 – G.W.].

177 See Yanovskaya, 1980; Rieske & Schenk, 1972, p. 482; Segeth, in: Klaus & Buhr, 1976, Vol. 1, p. 452; also Marx, 1974b, p. 141; Marx, 1975, p. 170.

178 See Gerdes, 1980; Yanovskaya, 1980.

Chapter 7

SOURCE OF INSPIRATION

In *Dialectics of nature*, Friedrich Engels observed:

> "The turning point in mathematics was *Descartes' variable magnitude*. With that came *motion* and hence *dialectics* in mathematics, and at once, too, of necessity the differential and integral calculus..." [179]

Here, Engels conceived of dialectics as

> "... the science of the most general laws of *all* motion. This implies that its laws must be valid just as much for motion in nature and human history as for the motion of thought." [180]

Applying this conceptualization, we can interpret Engels's observation in the sense that mathematics, through differential and integral calculus, became capable for the first time of *reflecting*, in one of its methods, motion in nature.

I am of the opinion that Engels's observation should not be interpreted in the narrow sense that mathematical thought, in itself, was not dialectical before the discovery of differential calculus. Not so at all, because thought itself is a form of motion. With good reason, Engels also affirmed in *Anti-Dühring*:

---

179 Engels, 1964, p. 262; MEW, 1961, Vol. 20, p. 522; Engels, 1974b, p. 274; cf. Alexandrov, 1977, p. 51.

180 Engels, 1964, p. 271; MEW, 1961, Vol. 20, p. 530; Engels, 1974b, p. 286.

> "Men thought dialectically long before they knew what dialectics was, just as they spoke prose, long before the term prose existed." [181]

Immediately, Paul Labérenne added:

> "And what is true for people in general, is perhaps – in the case of dialectics – even truer of mathematicians." [182]

Now two apparently unconnected questions arise:

1) How to learn to think dialectically?;
2) What dialectical instances do we find in mathematics, including elementary mathematics?

These two questions, once fully analyzed – an analysis enriched by study of the *Mathematical Manuscripts* of Marx – will constitute an incentive to work out new methods of teaching mathematics. Application of dialectics may improve the quality of teaching, as I will try to show from my personal experience.

I must limit myself to my own personal experience since to date I have not found any publication about the relevance of the *Mathematical Manuscripts* of Marx to the teaching of mathematics.

7.1 ***On the negation of the negation in mathematics education***

In October of 1982, there took place in Paramaribo, capital of Suriname, the "Conference on Mathematics for the Benefit of the Caribbean People." At this conference, I gave a talk entitled *Mathematics in the Service of the People.* In the talk, I distinguished between individual and collective strategies to reinforce the self-confidence of students in their creative powers, a strategy in which dialectic reasoning played an extremely important role.

With the purpose of analyzing this role, I shall start with a dialogue between teacher and students.

---

[181] Engels, 1962, p. 195; MEW, 1961, Vol. 20, p. 133; Engels, 1974a, p. 179.

[182] Labérenne, 1971, p. 67.

7.1.1 *A student – teacher dialogue*

How can we calculate the derivative of the function

$$y = \sqrt{x}\ \text{, where } f : \Re^{+} \rightarrow \Re\ ?$$

Using the method of Marx, the students know that it is necessary to first find the provisional derivative:

$$\frac{y_1 - y_0}{x_1 - x_0} = ?$$

Observing that $y = \sqrt{x}$, we have $y_1 = \sqrt{x_1}$, and $y_0 = \sqrt{x_0}$.

Thus:

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{\sqrt{x_1} - \sqrt{x_0}}{x_1 - x_0}.$$

How can we divide $\sqrt{x_1} - \sqrt{x_0}$ by $x_1 - x_0$?

Caiado suggests: $\sqrt{x_1} - \sqrt{x_0} = \sqrt{x_1 - x_0}$ …

Is that true? $\sqrt{9} - \sqrt{4} = \sqrt{9 - 4}$, or, $3 – 2 = \sqrt{5}$ …?

No! No!

Another idea?

Pedro: Square it:

$$(\sqrt{x_1} - \sqrt{x_0})^2 = ...$$

That won't work? Why?

Luisa: Use different notation:

$$\sqrt{x_1} - \sqrt{x_0} = (x_1)^{\frac{1}{2}} - (x_0)^{\frac{1}{2}}$$

How should we continue?

Victor: We know $\frac{1}{2} = 2^{-1}$

Thus:

$$(x_1)^{\frac{1}{2}} - (x_0)^{\frac{1}{2}} = (x_1)^{2^{-1}} - (x_0)^{2^{-1}} = \left((x_1)^2 - (x_0)^2\right)^{-1}$$

Is that correct?

The last step is wrong too? What is the problem? ...

Why did Pedro wanted to work with squares?

$$(\sqrt{x_1})^2 - (\sqrt{x_0})^2 = x_1 - x_0 .$$

That's right.

Or, if we could transform the numerator

$$\sqrt{x_1} - \sqrt{x_0} \text{ to } (\sqrt{x_1})^2 - (\sqrt{x_0})^2,$$

it would be easy to carry out the division by $x_1 - x_0$.

Now how can we get

$$(\sqrt{x_1})^2 - (\sqrt{x_0})^2 \text{ ?}$$

In general, how can we go from $b - c$ to $b^2 - c^2$?

$$b^2 - c^2 = \ldots ?$$

$$b^2 - c^2 = (b - c) . \ \ldots ?$$

$$b^2 - c^2 = (b - c) . (b + c)$$

Can we use this?

Now each student tries on his or her own and gets:

$$\frac{\sqrt{x_1} - \sqrt{x_0}}{x_1 - x_0} = \frac{(\sqrt{x_1} - \sqrt{x_0}).(\sqrt{x_1} + \sqrt{x_0})}{(x_1 - x_0).(\sqrt{x_1} + \sqrt{x_0})} =$$

$$= \frac{(\sqrt{x_1})^2 - (\sqrt{x_0})^2}{(x_1 - x_0).(\sqrt{x_1} + \sqrt{x_0})} =$$

$$= \frac{x_1 - x_0}{(x_1 - x_0).(\sqrt{x_1} + \sqrt{x_0})} = \frac{1}{\sqrt{x_1} + \sqrt{x_0}} .$$

in this way we obtain the provisional derivative:

$$\frac{1}{\sqrt{x_1} + \sqrt{x_0}}$$

Now by moving x_1 back to x_0, the students easily find the derivative (putting $x_1 = x_0$)

$$f^1(x_0) = \frac{1}{\sqrt{x_0} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}$$

Or, in general:

$$f^1(x) = \frac{1}{2\sqrt{x}} .$$

These are only some parts of the dialogue between the teacher and students in the process of collective discovery. Let us analyze this discovery process more closely.

7.1.2 *What constitutes discovery?*

In our example, the discovery was *crystallized* at the moment when we wrote

$$\frac{\sqrt{x_1} - \sqrt{x_0}}{x_1 - x_0} = \frac{(\sqrt{x_1} - \sqrt{x_0}).(\sqrt{x_1} + \sqrt{x_0})}{(x_1 - x_0).(\sqrt{x_1} + \sqrt{x_0})}$$

To go from the right member to the left member of this equality is trivial. Just divide the numerator and the denominator by the common factor $\sqrt{x_1} + \sqrt{x_0}$. However, we did not go from the right to the left, rather from the left member to the right. And the transition from left to right was not at all trivial, not tautological. To go from left to right we multiplied numerator and denominator by $\sqrt{x_1} + \sqrt{x_0}$. Why by $\sqrt{x_1} + \sqrt{x_0}$ and not by some other factor?

By virtue of having multiplied both numerator and denominator by the same factor $\sqrt{x_1} + \sqrt{x_0}$, in reality we multiplied the whole quotient by 1:

$$\frac{\sqrt{x_1} - \sqrt{x_0}}{x_1 - x_0} . 1 = \frac{\sqrt{x_1} - \sqrt{x_0}}{x_1 - x_0} . \frac{\sqrt{x_1} + \sqrt{x_0}}{\sqrt{x_1} + \sqrt{x_0}} =$$

$$= \frac{(\sqrt{x_1} - \sqrt{x_0}).(\sqrt{x_1} + \sqrt{x_0})}{(x_1 - x_0).(\sqrt{x_1} + \sqrt{x_0})}$$

In other words, we wrote:

$$1 = \frac{\sqrt{x_1} + \sqrt{x_0}}{\sqrt{x_1} + \sqrt{x_0}}.$$

Again, the transition from right to left is trivial. But not from left to right – this *transformation* is not only *formal*, not just a change of form. For any number t different from 0 it is true that

$$1 = \frac{t}{t}$$

The discovery is realized, the creative moment is expressed, the decisive advance is achieved, exactly at the moment we chose

$$t = \sqrt{x_1} + \sqrt{x_0}.$$

With another choice for t we could not have moved forward easily! The choice for t reflects the understanding of the *content* and of the context:

Only by multiplying $\sqrt{x_1} - \sqrt{x_0}$ by $\sqrt{x_1} + \sqrt{x_0}$ is it possible to perform the division by $x_1 - x_0$ as we have seen in the above paragraph.

The form chosen for t reflects the understanding of the context, of the objective, which is possible to reach. In brief, it reflects an *understanding of the content of the process of transformation*.

On multiplying the numerator by $\sqrt{x_1} + \sqrt{x_0}$, we negate the value of the quotient

$$\frac{\sqrt{x_1} - \sqrt{x_0}}{x_1 - x_0}.$$

To reestablish equilibrium we must negate the first negation, multiplying the denominator also by $\sqrt{x_1} + \sqrt{x_0}$. Or, in abbreviated form, by writing,

$$1 = \frac{\sqrt{x_1} + \sqrt{x_0}}{\ldots}$$

we are *negating* the 1. Overcoming, ("aufhebend" in German), abolishing this first negation, we divide on the right by $\sqrt{x_1} + \sqrt{x_0}$:

$$1 = \frac{\sqrt{x_1} + \sqrt{x_0}}{\sqrt{x_1} + \sqrt{x_0}} .$$

Out of its context, the equality $1 = \frac{\sqrt{x_1} + \sqrt{x_0}}{\sqrt{x_1} + \sqrt{x_0}}$ is vacuous, sterile. But in its context, by reflecting the understanding by the mathematical subject of the contents of the process of transformation, the same equality is extremely fertile.

The first negation expresses a deep understanding of the transformation process: where it takes off and where – in what direction – we want to go. In this example, the second negation is an immediate consequence of the first, required algebraically to reestablish the equality.

Summarizing, we can reach the conclusion that the essence of the discovery lies in understanding the necessary process of algebraic transformation, a dialectical process that is characterized by the "negation of the negation." As Engels explains:

> "And so, what is the negation of the negation? An extremely general – and for this reason extremely far-reaching and important – law of development of nature, da history, and thought, a law which ... holds good in the animal and plant kingdoms, in geology, in *mathematics*, in history and in philosophy..." [183]

---

[183] MEW, 1961, Vol. 20, p. 131; Engels, 1974a, p. 177; Engels, 1962, p. 193. I adapted the translation (Engels, 1974a, p. 177). The original translation says, for example, that dialectics "has application … in mathematics," etc. This suggests that dialectics comes from *outside* nature, meaning ideas transplanted to nature. Understood thus, dialectics would be only subjective. "Has application to" hides an *idealist* point of view, conscious or unconscious of the translator.

7.1.3 *Other examples*

In this section I would like only to briefly indicate some other examples of the law of the negation of the negation in mathematical thought.

7.1.3.1 *In elementary algebra*

How can we obtain the general formula for the solution of second degree equations?

$$x^2 + px + q = 0.$$

Transform the left member so that it becomes a perfect square:

$$x^2 + px + ... = (x + ...)^2,$$

$$x^2 + px + \frac{p^2}{4} = (x + \frac{p}{2})^2.$$

By adding $\frac{p^2}{4}$ to the left, we are negating $x^2 + px$. In order not to violate the rules of algebra, we negate again, but this time, the $\frac{p^2}{4}$:

$$x^2 + px = x^2 + px + \frac{p^2}{4} - \frac{p^2}{4}.$$

Or briefly stated, $0 = \frac{p^2}{4} - \frac{p^2}{4}$. In this way, the equation

$$x^2 + px + q = 0,$$

is transformed to

$$x^2 + px + \frac{p^2}{4} - \frac{p^2}{4} + q = 0,$$

and:

$$(x + \frac{p}{2})^2 - \frac{p^2}{4} + q = 0,$$

or,

$$(x + \frac{p}{2})^2 = \frac{p^2}{4} - q.$$

And thus, with

$$x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{2} - q},$$

we obtain the solutions of the initial equation:

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{2} - q}.$$

The discovery was crystallized at: $0 = \frac{p^2}{4} - \frac{p^2}{4}$.

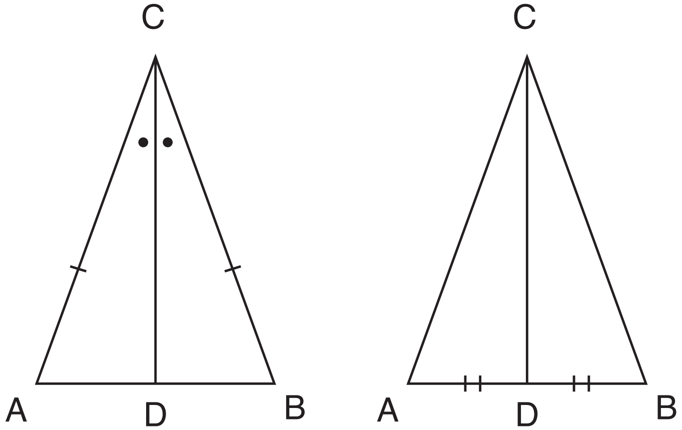
In general, we have 0 = k – k. The choice of $k = \frac{p^2}{4}$ reflects profound comprehension of the dialectical process through which the equation must pass in order to be solved.

7.1.3.2 ***In geometry***

How to construct the midpoint of a segment? Will it be a construction *within* the segment? Without leaving the segment?

What are the given facts we already know? Where does the concept of the midpoint arise? It does appear in the concept of median. What do we know about medians? The three medians of a triangle intersect at the unique point. Can we use this? No ...

What else do we know? In an isosceles triangle, the bisector of the vertex angle coincides with the median produced to the base:

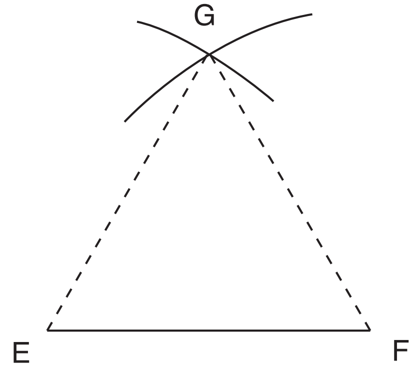


When AC = BC and $\angle ACD = \angle BCD$, then AD = BD.

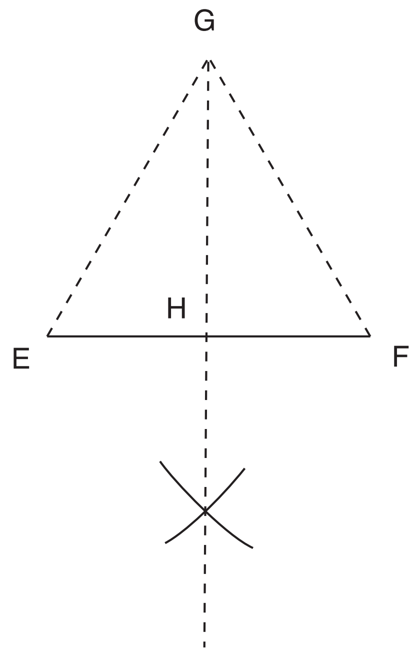
Is this information applicable to our problem?

Let's try. We have the segment EF

and construct an isosceles triangle, EFG, with this segment as base:



We construct GH, the bisector of angle EGF:



GH is also the median relative to base EF. Therefore H is the midpoint of EF.

By constructing the isosceles triangle EFG, we "left" the segment EF, meaning, we negated it. By returning to the segment EF, through the bisector GH (= median GH), we negated "having left" the segment EF, that is to say, negated the negation.

Neither the first negation nor the second were accidental. They reflect a profound understanding of the process needed to reach the objective. By this negation of the negation, we did not return to our point of departure, the segment EF in itself, but to its midpoint H, as we intended.

7.1.3.3 ***In trigonometry***

First example

How can we obtain a formula for the tangent of the sum of two angles, knowing the formulas for the sine and the cosine of the sum of two angles:

$$\tan(\alpha + \beta) = ...?$$

$$\tan(\alpha + \beta) = \frac{sin(\alpha + \beta)}{\cos(\alpha + \beta)} =$$

$$= \frac{sin(\alpha).\cos(\beta) + \cos(\alpha).sin(\beta)}{\cos(\alpha).\cos(\beta) - sin(\alpha).sin(\beta)}$$

Now, multiplying the quotient by 1 in the form $\frac{t}{t}$, where t = $\frac{1}{\cos(\alpha).\cos(\beta)}$, we easily get the formula:

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

Why do we choose

$$t = \frac{1}{\cos(\alpha).\cos(\beta)} ?$$

Did it fall from the sky? Clearly the choice of t = $\frac{1}{\cos(\alpha).\cos(\beta)}$ was based on an understanding of the process of negation of the negation needed to reach the objective of tan(α + β) expressed in terms of tan(α) and tan(β).

Second example

How do we get a formula for the tangent of half an angle?

$$\tan(\frac{\alpha}{2}) = \ldots ?$$

$$\tan(\frac{\alpha}{2}) = \frac{sin(\frac{\alpha}{2})}{\cos(\frac{\alpha}{2})}$$

At this point, multiply the quotient by 1 in the form $\frac{t}{t}$, where $t = 2\cos(\frac{\alpha}{2})$ [An alternative is $t = 2\,sin(\frac{\alpha}{2})$!], and obtain the intermediate result:

$$\tan(\frac{\alpha}{2}) = \frac{2sin(\frac{\alpha}{2}).\cos(\frac{\alpha}{2})}{2\cos^2(\frac{\alpha}{2})} = \frac{sin(\alpha)}{2\cos^2(\frac{\alpha}{2})} \quad .$$

In the denominator, $\frac{\alpha}{2}$ still appears. How can this be avoided?

Here we find that, in place of a process of transformation of the type $1 = \frac{t}{t}$, another type is needed. Knowing that

$$2\cos^2(\frac{\alpha}{2}) - 1 = \cos(\alpha)$$

we write:

$$2\cos^2(\frac{\alpha}{2}) - 1 + 1 \text{ in place of } 2\cos^2(\frac{\alpha}{2}) \, ,$$

In other words, we use the equality 0 = -1+1. This is the second time that we encounter a negation of the negation. After this we easily arrive at the result:

$$\tan(\frac{\alpha}{2}) = \frac{sin(\alpha)}{2\cos^2(\frac{\alpha}{2})} = \frac{sin(\alpha)}{2\cos^2(\frac{\alpha}{2})-1+1} = \frac{sin(\alpha)}{\cos(\alpha)+1}$$

7.1.3.4 *In calculation of limits*

How can we determine the following limit

$$\lim_{x \to 0} \frac{1-e^x}{sin(x)} \;?$$

Do we know limits, as x → 0, for any expressions with $1 - e^x$ or sin(x) ? A student remembering that

$$\lim_{x \to 0} \frac{sin(x)}{x} = 1,$$

can negate the denominator sin(x) with $\frac{sin(x)}{x}$, and afterwards negate the negation by multiplying the numerator $1 - e^x$ by $\frac{1}{x}$. Briefly, through multiplication by 1 in the form $\frac{\frac{1}{x}}{\frac{1}{x}}$, the expression $\frac{1-e^x}{sin(x)}$ is transformed to:

$$\frac{\frac{1-e^x}{x}}{\frac{sin(x)}{x}}.$$

Taking into account

$$\lim_{x \to 0} \frac{1-e^x}{x} = -1 \quad \text{and} \quad \lim_{x \to 0} \frac{sin(x)}{x} = 1,$$

we conclude that $\lim_{x \to 0} \frac{1-e^x}{sin(x)}$ exists and is given by:

$$\lim_{x \to 0} \frac{1-e^x}{sin(x)} = \frac{\lim_{x\to 0}\frac{1-e^x}{x}}{\lim_{x\to 0}\frac{sin(x)}{x}} = \frac{-1}{1} = -1.$$

Here it is interesting to note that there is another road, opposite to this one that students can take. If they happen to recall that

$$\lim_{x \to 0} \frac{1-e^x}{x} = -1$$

they can initiate the process from the numerator by taking $\frac{1-e^x}{x}$ as the first negation (multiplying by $\frac{1}{x}$), and then negate the negation by multiplying the denominator by $\frac{1}{x}$.

7.1.4 *A method for discovery of new results*

In the above paragraphs, I gave some examples of the law of negation of the negation in mathematical thought. Almost all the examples were selected from elementary mathematics to be more accessible to a larger public. In higher mathematics, the law of the negation of the negation appears perhaps even more frequently, and is more profound. But this is not the subject, here.

The academicians Kolmogorov, Alexandrov, and Schnirelman, recognized many times that they enjoyed advantages in making mathematical discoveries because of their knowledge of dialectics. [183] The advantages, which result from knowledge of dialectics must not and cannot be a right reserved for great mathematicians only.

It must not constitute an elitist right, but a right of all. True understanding of the law of the negation of the negation serves, according to the philosopher Günter Krober, as a *method of discovering new results*. [184] All students can master this powerful method! But this depends on the methods of teaching.

In the example I gave of a dialogue between the teacher and the students, the students learn to discover. They understand that "mathematics does not fall from the sky," but is the fruit of human labor. They understand the non-tautological nature of mathematical knowledge. By discovering collectively and reflecting on the

183 Labérenne, 1971, p. 67.

184 Klaus & Buhr, 1976, p. 857.

dialectical process of discovery, *all* students increase their creative capacities, individual and collective. All gain in self-confidence. [185]

The method of dialogue described here was born in the process of teaching itself, enriched by the inspirational study of the dialectical foundations of the differential calculus by Karl Marx. In conclusion, I would like to point out that the *Mathematical Manuscripts* of Marx constitute a source of inspiration to elevate the quality of mathematics education, to make the science of mathematics more accessible to all students – a source of inspiration still almost untouched.

[185] Here one can verify elaboration and convergence with a dominant tendency in progressive mathematics education – the discovery or reinvention method. See, for example, Gerdes, 1981b.

Chapter 8

THE SUN APPEARS ON THE HORIZON

> "Now the sun is rising. So the time has come for me to go for a walk. For the time being, I shall not continue to work in mathematics, but later, at an opportune moment, I will return to speak extensively about the distinct methods of differential calculus." 187

Thus, in that cold and cloudy winter, on November 22, 1882, Marx, full of so many new ideas, interrupted his letter to Engels. A few months later, on March 14, 1883, death deprived him of the opportunity to elaborate his thoughts about mathematics, deprived him of that opportunity to develop even further his creative potential.

He died.

No. Only interrupted – his unfinished mathematical work remains an extremely stimulating force for the furthering of new investigations in history, mathematics, philosophy, economics, and education, for new investigations, both necessary and original.

The mathematical work of Karl Marx constituted an integral part of his whole revolutionary work. He saw the necessity of understanding motion, just as much in society as in nature and thought. He understood that the class struggle is the motor force of historical motion, that is, of history. He understood that the struggle of opposites is also the motor of development of ideas, including mathematics.

---

[187] MEW, 1961, Vol. 35, p. 114.

Regarding mathematics, the negation of motion has been expressed since Plato in two interconnected ideas:

* Mathematics is ahistorical and eternal;
* Only an elite, an intellectual aristocracy, can “learn” mathematics.

By studying the mathematical investigations of Karl Marx, we can deepen our understanding of dialectical motion in mathematical theory, of its interconnections with the development of society. We can develop – and this is my main thesis – ideas about the dialectics of the process of learning mathematics, and in this way develop better methods to

“strip away the veil of mystery from mathematics.”

May 5, 1983

BIBLIOGRAPHY

Alexandrov, A. D. (1974) [Also spelled Aleksandrov – B.L.], Mathematik und Dialektik, in: M. Otte (Ed.), *Mathematiker über die Mathematik*, Springer Verlag, Berlin, pp. 47-63.

Alexandrov, A. D. (1977), A general view of mathematics, in: *Mathematics: Its Contents, Methods, and Meaning*, The MIT Press, Cambridge, Massachusetts, pp. 1-64.

Alexandrov, A. D., Kolmogorov, A. N., and Lavrent'ev, M. A. (Eds.) (1963), *Mathematics: Its Contents, Methods, and Meaning*, The MIT Press, Cambridge, Mass.

Berkeley, George (1949), *The Works of George Berkeley* (A. A. Luce and T. E. Jessop Eds.), Thomas Nelson and Sons, London (8 volumes).

Bernal, John D. (1971), *Science in History*, The MIT Press, Cambridge, Mass. (3 volumes)

Boyer, Carl (1960), *A History of Mathematics*, John Wiley & Sons, New York.

Colman, Ernst. See Kol'man, E.

D'Ambrosio, Ubiratan (1975), *Cálculo e introdução à análise*, Editora Nacional, São Paulo.

Demidov, S. S. & Volodarsky, A. I. (1980), Second Summer Conference on the history of mathematics in the Soviet Union, *Historia Mathematica*, New York, Vol. 7, 71-75.

Endemann, Wolfgang (1974), Einleitung, in: Marx (1974b), pp. 15-50.

Engels, Friedrich (1962), *Anti-Dühring*, FLPH, Moscow.

Engels, Friedrich (1964), *Dialectics of Nature*, Progress Publishers, Moscow.

Engels, Friedrich (1974a), *Anti-Dühring*, Edições Afrodite, Lisbon.

Engels, Friedrich (1974b), *Dialéctica da Natureza*; Editorial Presença, Lisbon.

Engels, Friedrich (1983), Elogio fúnebre a Karl Marx, *Notícias*, Maputo, 17/03/83, p. 3.

Fedoseyev, P. N., *et al.* (1974), *Karl Marx: A Biography*, Progress Publishers, Moscow.

Gelfand, I. M. (1977), Functional analysis, in: *Mathematics: Its Contents, Methods, and Meaning*; The MIT Press, Cambridge, Massachusetts, Vol. 3, pp. 227-262.

Gerdes, Paulus (1980), Sobre a origem histórica do conceito de número, *Ciência e Tecnologia*, Maputo, No. 1, 53-57.

Gerdes, Paulus (1981a), *A ciência matemática*, INDE, Maputo.

Gerdes, Paulus (1981b), Tendências no ensino da matemática, *Ciência e Tecnologia*, No. 3, 64-69.

Gerdes, Paulus (1982), *Mathematics for the Benefit of the People*, CARIMATHS, Paramaribo.

Glivenko, V. I. (1934), Ponyatie diferentsiala u Marksa i Adamara, *Pod znamenem markizma*, Vol. 5, 79-85.

Gokieli, L. P. (1947), *Matematischeskie rukopisi Karla Marksa i voprosy obosnovaniya matematiki*, AN gruzinkoj SSR, Tbilisi.

Gu, Jin-yong (1976), Der grosse Beitrag der altchinesischen Mathematik zur Weltkultur, in: W. Ruppert, *Mathematik in China*; China Forschungs Institut, Vienna, pp. 116-134.

Hegel, G. W. F. (1955), *Lectures on the History of Philosophy*, Routledge & Kegan Paul, London (3 volumes).

Heitsch, Wolfram (1978), *Mathematik und Weltanschauung*; Akademie-Verlag, Berlin.

Ilyichov, L. F., *et al.* (1977), *Frederick Engels: A Biography*, Progress Publishers, Moscow.

Ivanov, Nikolai; *Karl Marx, pequena biografia*; Nóvosti, Moscovo, 1979.

Janovskaja, S. A. See Yanovskaya, Sofia A.

Kennedy, Hubert (1976), Review. Karl Marx' Mathematische Manuskripte. Karl Marx' Manoscritti Matematici, *Historia Mathematica*, New York, Vol. 3, 490-494.

Kennedy, Hubert (1977), Karl Marx and the foundations of differential calculus, *Historia Mathematica*, New York, Vol. 4, 303-318.

Klaus, Georg, and Buhr, Manfred (1976) (Eds.), *Philosophisches Wörterbuch*, Bibliographisches Institut, Leipzig.

Kol'man, E. (1931a), The present crisis in the mathematical sciences and general outlines for their reconstruction, in: *Science at the crossroads* [Papers presented by the Delegates of the USSR to the International Congress of the History of Science and Technology, London, 29 June to 3 July 1931 – G.W.], Kniga, London, pp. 215-229.

Kol'man, E. (1931b), Short communication on the unpublished writings of Karl Marx dealing with mathematics, the natural sciences and technology and the history of these subjects, in: *Science at the crossroads*, Kniga, London, pp. 233-235.

Labérenne, Paul (1971), Mathematics and Marxism, in: F. Le Lionnais (Ed.), *Great currents of mathematical thought*, Dover Publications, New York, Vol. 2, pp. 58-68.

Lenin, V. I. (1960), *Collected Works*, Progress Publishers, Moscow, Vol. 38.

Lenin, V. I. (1975a), *Karl Marx e o desenvolvimento histórico do marxismo*, Edições Avante, Lisbon.

Lenin, V. I. (1975b), *Materialismo e empiriocriticismo*, Editorial Estampa, Lisbon.

Marx, Karl (1968), *Mathematischeskii Rukopisi*, Nauka, Moscow (Ed. Sofia A. Yanovskaya).

Marx, Karl (1972), *Pre-Capitalist Economic Formations*, International Publishers, New York.

Marx, Karl (1974a), *Grundrisse der Kritik der politischen Ökonomie*, Dietz Verlag, Berlin.

Marx, Karl (1974b), *Mathematische Manuskripte*, Scriptor Verlag, Kronberg (Ed. Wolfgang Endemann).

Karl Marx (1975), *Manoscritti Matematici*, Dedalo Libri, Bari (Trans. and ed. Francesco Matarrese and August Ponzio).

Marx, Karl (1983), *The Mathematical Manuscripts of Karl Marx*, New Park Publications, London, 280 pp. (Trans. C. Aronson and M. Meo)

(available at the webpage: www.marxists.org/archive/marx/works/1881/mathematical-manuscripts/index.htm).

(**MEW**) Marx, Karl and Engels, Friedrich (1961), *Marx-Engels Werke*, Dietz Verlag, Berlin, Vols. 19, 20, 30, 31, 33, 35.

Marx, Karl and Engels, Frederick (1983), *Collected Works*, International Publishers, New York.

Matarrese, Francesco (1975), Çalcolo, logjça formale e dialettica delle lorme, in: Marx (1975), pp. 5-22.

Mehring, Franz (1962), *Karl Marx: The Story of His Life*, University of Michigan Press, Ann Arbor.

Mehring, Franz (1974), *Karl Marx, vida e obra*, Editorial Presença, Lisbon.

Miller, Maximilian (1969), Karl Marx' Begründung der Differential-rechnung, *Wissenschaftliche Zeitschrift der Hochschule für Verkehrswesen Friedrich List*, Dresden, Vol. 16, 649-659.

Molodschi, W. N. (1977), *Studien zur philosophischen Problemen der Mathematik*, Deutscher Verlag der Wissenschaften, Berlin.

Mullin, A. A. (1979), Review of Ruzavin's "Die Natur der mathematischen Erkenntnis," *Mathematical Reviews*, 1979, Vol. 58, p. 712.

Paul, Siegfried (1976), Die Bearbeitung philosophischer Probleme der Mathematik in der Sowjetunion, *Deutsche Zeitschrift für Philosophie*, Vol. 8, 847-862.

Piskounov, N. (1979), *Çálculo diferencial e integral*, Edições Lopes da Silva, Porto.

Ponzio, Augusto (1975), Matematica, diallettica ed economia politica, in: Marx (1975), pp. 23-38.

Rieske, Günter and Schenk, Günter (1972), Marx und die Mathematik, *Deutsche Zeitschrift für Philosophie*, Vol. 4, 475-483.

Rosental, M. and Iudin, R. (1977) (Eds.), *Dicionário filosófico*, Editorial Estampa, Lisbon.

Ruzavin, G. I. (1977), *Die Natur der mathematischen Erkenntnis. Studien zur Methodologie der Mathematik*; Akademie-Verlag, Berlin.

Stepanova, Eugueniia (1977), *Friedrich Engels. Pequena biografia*; Edições Avante, Lisbon.

Stepanova, Eugueniia (1979), *Karl Marx. Pequena biografia*; Editorial Avante, Lisbon..

Struik, Dirk (1948a), *A concise history of mathematics*, Dover Publications, New York (Reprint 1967).

Struik, Dirk (1948b), Marx and Mathematics, *Science and Society*, Vol. 12, 181-196.

Struik, Dirk (1975), Marx und Mathematik, *Materialien zur Analyse der Berufspraxis des Mathematikers*, Vol. 16, 1975, 137-158 (German translation of 1948b).

Struik, Dirk (1977), *Geschiedenis van de wiskunde*, SUA, Amsterdam.

Struik, Dirk (1980), Why study the history of mathematics?, *The UMAP Journal*, Vol. 1, 3-28.

Struik, Ruth Rebekka (1974), Some remarks on the concept of limit, in: Cohen, Stachel, Wartofsky (Eds.), *For Dirk Struik, Scientific, historical and political essays in honor of Dirk J. Struik*, Reidel, Dordrecht.

Swift, Jonathan (1938), *Gulliver's Travels*, Ronald Press, New York (ed. A. E. Case).

Thiel, Rainer (1975), *Mathematik, Sprache, Dialektik*, Akademie Verlag, Berlin.

Wussing, Hans (1979), *Vorlesungen zur Geschichte der Mathematik*; Deutscher Verlag der Wissenschaften, Berlin.

Yanovskaya, Sofia A. (1969), Karl Marx' "Mathematische Manuskripte," *Sowjetwissenschaft, Gesellschaftwissenschaftliche Beiträge*, 20-35.

Yanovskaya, Sofia A. (1980), Über die sogenannte Definition durch Abstraktion, *Materialien zur Analyse der Berufspraxis des Mathematikers*, Bielefeld, Vol. 24, 155-204.

Zeleny, Jindrich (1973), Het wetenschapsbegrip van het dialekties materialisme, in: *Marxistiese kennistheorie*, Nijmegen, pp. 34-42.

Zeleny, Jindrich (1980), *The Logic of Marx*, Rowman and Littlefield, Totowa NJ.

Aditional bibliography (2008)

Alcouffe, Alain (1985), *Les Manuscrits Mathématiques de Marx*, Economica, Paris.

Alcouffe, Alain (1987), Le calcul différentiel, les mathématiques et les économistes du XIX°: K. Marx et H. Laurent, lecteurs de Boucharlat, *Revue Sciences et Techniques en Perspective*, Vol. XIII.

Blunden, Andy (1983), Review: The Mathematical Manuscripts of Karl Marx, *Labour Review*.

Dauben, Joseph W. (1998), Marx, Mao and Mathematics: The Politics of Infinitesimals, *Documenta Mathematica*, Extra Volume ICM III, 799-809.

Flores, Fernando & Natiello, Mário (2006), La filosofia matemática de Karl Marx en los manuscritos de 1881. Un esbozo, *Revista Brasileira da História da Matemática*, Rio Claro, Vol. 6, No. 12, 111-125.

Gerdes, Paulus (2003), *Awakening of Geometrical Thought in Early Culture*, MEP Publications, Minneapolis, 184 pp. (original in German: 1985) (Foreword: Dirk J. Struik) [New edition: *Ethnogeometry: Awakening of Geometrical Thought in Early Culture*, ISTEG, Boane & Lulu, Morrisville NC, 2014].

Glivenko, V. (1935), Der Differentialbegriff bei Marx und Hadamard, *Unter dem Banner des Marxismus*, 102-110.

Kline, George L. (1950), Review: L. P. Gokieli, On the Problem of the Axiomatization of logic; L. P. Gokieli, The Mathematical Manuscripts of Karl Marx and Problems of the Foundation of Mathematics, *Journal Symbolic Logic*, Vol. 14, No. 4, 243-244.

Kol'man, Ernst (1983), Karl Marx and Mathematics: on the 'Mathematical Manuscripts' of Marx, in: Marx (1983), pp. 217-234 (original in Russian, 1968).

Kol'man, Ernst & Yanovskaya, Sofia (1983), Hegel and Mathematics, in: Marx (1983), pp. 235-255 (original: 1931).

Marx, Karl (1987), *Manuscritos matemáticos*, Ediciόns Xerais de Galicia, Vigo, 140 pp. (Pres. and comments: Xenaro Garcia Suárez)

Marx, Karl (1994), *Mathematical Manuscripts* (in Bengali), Viswakos Parisad, Calcutta (Org. and trans. Pradip Baksi)

Matthews, Peter H. (2002), *The Dialectics of Differentiation: Marx's Mathematical Manuscripts and Their Relation to His Economics*, Middleburg College Working Paper Series.

Meo, M. (1986), Review: Paulus Gerdes, Marx demystifies mathematics, *Ganita Bhârati*, Vol. 9, No. 1-4, 78-79.

Molodschi, Vladimir (1982), The mathematical manuscripts of K. Marx and the evolution of the history of mathematics in the USSR (in Russian), *Istoriko-Matematiceskie Issledovanija*, Leningrad, Vol. 26, 9-17.

Olesko, Kathryn (1984), Review: The Mathematical Manuscripts of Karl Marx, *Isis*, Vol. 75, No. 1, 233-234.

Powell, Arthur B. (1986), Marx and Mathematics in Mozambique. Review: Paulus Gerdes, Marx Demystifies Mathematics, *Science and Nature*, Nos. 7 / 8, 119-123.

Powell, Arthur B. & Frankenstein, Marilyn (Eds.) (1997), *Ethnomathematics: Challenging Eurocentrism in Mathematics Education*, SUNY, New York.

Radice, Lúcio Lombardo (1972), Presentazione a Dai manoscritti matematici di Marx, *Critica marxista*, Quaderni, No. 6.

Ruccio, David F. (1991), The Merchant of Venice, or Marxism in the Mathematical Mode, *Social Scientist*, Vol. 19, Nos. 1 / 2, 18-46.

Smith, C. (1983), Hegel, Marx and the Calculus, in: Marx (1983), pp. 256-270.

Struik, Dirk (1948), Marx and Mathematics, *Science and Society*, Vol. 12, No. 1, 181-196 [reproduced in: Powell, A. B. & Frankenstein, M. (Eds.) (1997), pp. 173-192].

NAME INDEX

Books in English authored by Paulus Gerdes

Ethnogeometry: Awakening of Geometrical Thought in Early Culture, 220 pp. *

Geometry from Africa: Mathematical and Educational Explorations, The Mathematical Association of America, Washington DC, 210 pp.

Ethnomathematics and Education in Africa, 270 pp. *

Women, Culture and Geometry in Southern Africa, 276 pp. *

Otthava: Making Baskets and Doing Geometry in the Makhuwa Culture in the Northeast of Mozambique, 290 pp. (color) *

Lusona – Geometrical Recreations from Africa: Problems and Solutions, 216 pp. (color) *

Sona Geometry from Angola: Mathematics of an African Tradition, Polimetrica, Monza, 232 pp.

Sona Geometry from Angola: Volume 2: Educational and Mathematical Explorations of African Designs on the Sand, 220 pp. *

Drawings from Angola: Living Mathematics, 72 pp. (Children's book) (color) *

Mathematics in African History and Cultures. An annotated Bibliography (co-author Ahmed Djebbar), African Mathematical Union, 430 pp. *

History of Mathematics in Africa: AMUCHMA 25 Years (co-author: Ahmed Djebbar), African Mathematical Union (Volume 1: 1986-1999; Volume 2: 2000-2011), 924 pp. *

African Pythagoras: A Study in Culture and Mathematics Education, 124 pp. (color) *

---

* Distribution: www.lulu.com/spotlight/pgerdes (available in print and as eBooks)

Tinhlèlò, Interweaving Art and Mathematics: Colourful Circular Basket Trays from the South of Mozambique, 132 pp. (color) *

Hats from Mozambique, Associação Índico, Maputo, 52 pp. (color) *

Sipatsi: Basketry and Geometry in the Tonga Culture of Inhambane (Mozambique, Africa), 422 p. & *Sipatsi Images in Colour: A Supplement,* 56 pp. *

Adventures in the World of Matrices, Nova Science Publishers New York, 196 pp.

African Basketry: A Gallery of Twill-Plaited Designs and Patterns, 220 pp. *

Lunda Geometry: Mirror Curves, Designs, Knots, Polyominoes, Patterns, Symmetries, 198 pp. *

The philosophic-mathematical manuscripts of Karl Marx on differential calculus: An introduction, 116 pp. *

Enjoy puzzling with biLLies, 248 pp. *

More puzzle fun with biLLies, 76 pp. *

Puzzle fun with biLLies, 76 pp. *

The Bisos Game: Puzzles and Diversions, 72 pp. *

Geometry and Basketry of the Bora in the Peruvian Amazon, 176 pp. (color) *

1000 Doctoral Theses by Mozambicans or about Mozambique, 236 pp. *

African Doctorates in Mathematics: A Catalogue, African Mathematical Union, Maputo, 383 pp. *